More on the potential for the farthest-point distance function

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Abstract

For some particular cases in dimension 3 and higher we prove a conjecture of Laugesen and Pritsker [11] concerning the farthest-point distance function. Moreover, we consider some examples that provide more insight in the nature of the problem and allow us to simplify the proof of Gardiner and Netuka in two dimensions.

1 Introduction

For a compact set $E \subset \mathbb{R}^n$ the farthest-point distance function $d_E : \mathbb{R}^n \mapsto \mathbb{R}$ is defined by

$$d_E(x) = \max\{|x-y|\,; y \in E\}.$$

Notice that, unless E is singleton, $d_E(x)$ is everywhere positive. Next to the intrinsic motivation the function plays a role in several areas of analysis. See the papers by Gardiner and Netuka [7] and Wise [16]. Our main interest is the farthest-point distance function itself and a related probability measure σ_E. Following Boyd [3] for finite sets E, Pritsker in [12] showed that for a compact set $E \subset \mathbb{R}^2$, there exists a probability measure σ_E such that

$$\log(d_E(x)) = \int_{\mathbb{R}^2} \log|x-y|\, d\sigma_E(y). \tag{1}$$

Before going into details, let us first give a sketch of the level lines of d_E for two special cases in Figure 1.

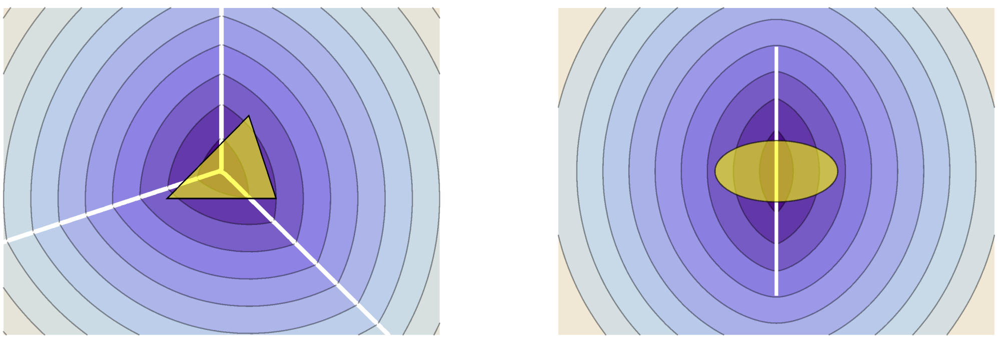

Figure 1: *Level lines of d_E for a triangle and an ellipse; the white lines show where d_E is not C^1 and where σ_E is not absolutely continuous with respect to the Lebesgue measure.*

Our main interest are dimensions $n \geq 3$. In those dimensions the formula that replaces (1), is as follows:

$$d_E(x)^{2-n} = \int_{\mathbb{R}^n} |x-y|^{2-n}\, d\sigma_E(y) \ \text{ for all } x \in \mathbb{R}^n. \tag{2}$$

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If E consist of a single point z, then $d_E(x) = |x-z|$ and the measure σ_E necessarily is the Dirac-delta distribution at z. In that case it obviously holds that $\sigma_E(E) = 1$. For compact sets $E \ni z$ one finds that $d_E(x) \simeq |x-z|$ as $|x| \to \infty$, which supports the claim that $\sigma_E(\mathbb{R}^n) = 1$. So, if σ_E is a positive measure, then $\sigma_E(\mathbb{R}^n) = 1$ implies that σ_E is a probability measure and obviously $\sigma_E(E) \le 1$. Surprisingly, except in the case that E is a single point, $\sigma_E(E)$ seems to be strictly less than 1. Laugesen and Pritsker noticed that for a ball $E = \{x \in \mathbb{R}^n; |x| \le r\}$ it holds that

$$\sigma_E(E) = 2^{1-n}$$

and they conjectured, that for all compact sets E, containing more than one point, it holds that

$$\sigma_E(E) \le 2^{1-n}. \tag{3}$$

The conjecture was shown to be true for $n = 2$ by Gardiner and Netuka [6] [7]. Our contribution to a better understanding of the conjecture will provide an alternative proof (see Theorem 3.1) in the case $n = 2$ as well as a proof for $n \ge 3$ in some special cases (sets of constant width and centrally symmetric sets) described in Theorems 4.1 and 5.1. Let us first recall some background material.

1.1 Potential theoretic intermezzo

For dimensions $n \ge 3$ the function $x \mapsto |x-y|^{2-n}$ is superharmonic and converges to 0 as $|x| \to \infty$. So, if E consists of finitely many points $\{y_1, \dots y_k\}$, then

$$x \mapsto d_E(x)^{2-n} = \min\left\{|x-y_i|^{2-n}; 1 \le i \le k\right\}$$

is superharmonic as the minimum of superharmonic functions. Similarly for a set E, which is not finite but still compact, one finds that the function

$$x \mapsto d_E(x)^{2-n} = \inf\left\{|x-z|^{2-n}; z \in E\right\}$$

as infimum of superharmonic functions is superharmonic. For a superharmonic function u the distribution $\mu_u := -\Delta u$ exists as a nonnegative Riesz measure. See [1, Corollary 4.3.3 and Definition 4.3.4]. So indeed, the distribution

$$\mu_E := -\Delta\left(d_E(\cdot)^{2-n}\right) \tag{4}$$

is a nonnegative Riesz measure, and since $\mathbb{R}^n$, with $n \ge 3$, is Greenian ([1, Theorem 4.1.2]) one finds by [1, Theorem 4.4.1] that

$$d_E(x)^{2-n} = \int_{\mathbb{R}^n} \Gamma_n(x-y)\, d\mu_E(y) \text{ with } \Gamma_n(x) = \frac{|x|^{2-n}}{(n-2)\, n\omega_n}. \tag{5}$$

Remark 1.1 *Here Γ_n is the fundamental solution for $-\Delta$ on $\mathbb{R}^n$ with $\omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ the volume of the unit ball. Remember that $\alpha_n = n\omega_n$ is the $n-1$-dimensional surface area of the unit ball.*

The measure σ_E mentioned above, is defined by

$$\sigma_E := \frac{1}{(n-2)\, n\omega_n}\mu_E = \frac{-\Delta\left(d_E(\cdot)^{2-n}\right)}{(n-2)\, n\omega_n}. \tag{6}$$

Indeed, since we have taken the appropriate normalization, we obtain (2).

For $n = 2$ the whole space $\mathbb{R}^2$ is not Greenian, but nevertheless (1) holds for the probability measure σ_E defined by

$$\sigma_E := \frac{1}{2\omega_2}\mu_E = \frac{-\Delta\left(-\log(d_E)\right)}{2\pi}. \tag{7}$$

Since $\lim\limits_{n\downarrow 2} \frac{t^{2-n}-1}{n-2} = -\ln t$ holds, the definition in (7) is the obvious extension of (6).

1.2 The farthest point envelope

One easily remarks that for a proof of the conjecture (3), it suffices to consider convex sets E. Indeed, suppose that E is not convex and let $co\,(E)$ be the convex hull of E. Since

$$|x-(\theta y_a+(1-\theta)\,y_b)|\leq \max\left(|x-y_a|\,,|x-y_b|\right) \text{ for all } \theta\in(0,1)\,,$$

it follows that

$$d_{co(E)}\,(x)=d_E\,(x) \text{ for all } x\in\mathbb{R}^n, \tag{8}$$

so $\sigma_E=\sigma_{co(E)}$ and hence

$$\sigma_E\,(E)=\sigma_{co(E)}\,(E)\leq\sigma_{co(E)}\,(co\,(E))\,. \tag{9}$$

Gardiner and Netuka in [6], [7] noticed that it even suffices to prove the inequality $\sigma_E\,(E)\leq 2^{1-n}$ for the farthest-point envelope E^* of a convex set E.

Definition 1.2 *The farthest-point envelope E^* of E is defined by*

$$E^*:=\bigcap_{x\in E}\overline{B\,(x,d_E(x))}$$

with $\overline{B\,(x,r)}=\{y\in\mathbb{R}^n;|x-y|\leq r\}$.

Let us state some features of the farthest-point envelope:

Lemma 1.3 *Suppose that E and E_0 are convex compact sets. Then the following holds:*

1. $d_{E^*}\,(x)=d_E\,(x)$ *for all* $x\in E$*;*
2. $E^*=(E^*)^*$*;*
3. $E^*=\bigcap\left\{\overline{B_{1/\varepsilon}\,(x)};\overline{B_{1/\varepsilon}\,(x)}\supset E^*\right\}$ *for* $\varepsilon\in(0,\varepsilon_0)$ *with* $\varepsilon_0>0$ *sufficiently small;*
4. *if* $E\subset E_0\subset E^*$*, then* $E_0^*\subset E^*$*.*

Remark 1.4 *In Figure 2 one finds an example of the last result with strict inclusions.*

Proof. 1. This first claim follows from the definition.

2. $E^*\subset(E^*)^*=\bigcap_{x\in E^*}\overline{B\,(x,d_{E^*}(x))}\subset\bigcap_{x\in E}\overline{B\,(x,d_{E^*}(x))}=\bigcap_{x\in E}\overline{B\,(x,d_E(x))}=E^*$.

3. Take $\varepsilon_0=(\max_{x\in E}d_E(x))^{-1}$.

4. $E_0^*=\bigcap_{x\in E_0}\overline{B\,(x,d_{E_0}(x))}\subset\bigcap_{x\in E}\overline{B\,(x,d_{E_0}(x))}\subset\bigcap_{x\in E}\overline{B\,(x,d_{E^*}(x))}=E^*$. ■

By the first item of Lemma 1.3 one finds $d_{E^*}\,(x)=d_E\,(x)$ for all $x\in E$, similar to (8) but now only on E. Hence $\sigma_{E^*}=\sigma_E$ on E^o. As noted in [7, page 42], in general $d_{E^*}\geq d_E$, and since $d_{E^*}=d_E$ on ∂E, it follows that $\left(\sigma_{E^*}-\sigma_E\right)_{|\partial E}\geq 0$ in the sense of measure. Hence

$$\sigma_E\,(E)\leq\sigma_{E^*}\,(E)\leq\sigma_{E^*}\,(E^*)\,. \tag{10}$$

So it will be sufficient to consider (3) for domains satisfying $E=E^*$.

Gardiner and Netuka also found for $n=2$ that equality in (3) holds for all sets E of constant width w_E. For a set of constant width w_E one has $E^*=E$ and

$$d_E(x)=w_E+d(x,\partial E)\quad \text{ in } \mathbb{R}^n\setminus E. \tag{11}$$

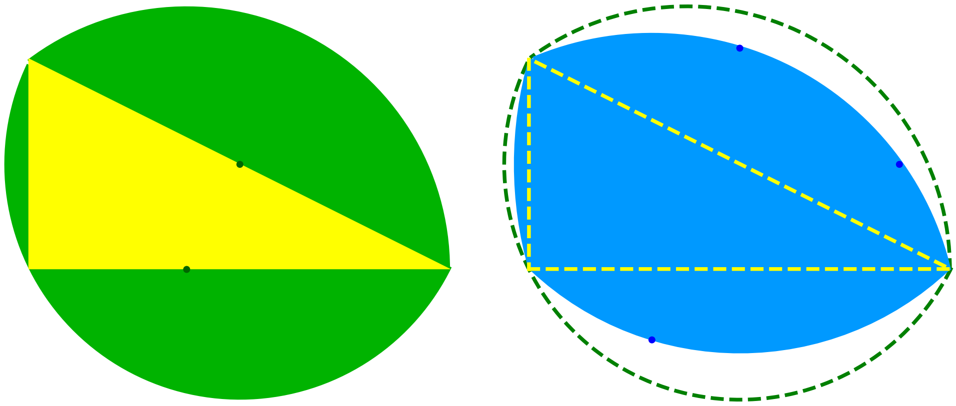

Figure 2: *On the left a triangle E_1 and its farthest-point envelope E_1^* . On the right a curvilinear domain E_2 that equals its farthest-point envelope E_2^*. One has $E_1 \subsetneq E_2 = E_2^* \subsetneq E_1^*$. The points show the centers of the disks that are used in constructing the farthest-point envelope.*

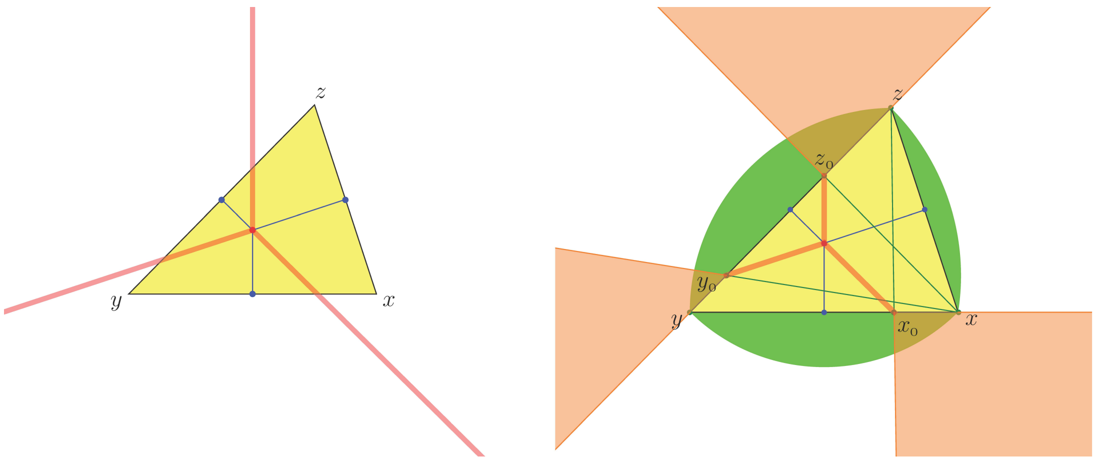

Figure 3: *On the left a triangle E and the support of σ_E as three half lines originating from the circumcenter. On the right the corresponding E^*, a curvilinear triangle, and the support of σ_{E^*}. Inside E one finds $\sigma_E = \sigma_{E^*}$. Outside E the measure σ_{E^*} is absolutely continuous with respect to the Lebesgue measure.*

Definition 1.5 *Let $E \subset \mathbb{R}^n$ be a compact set. We define the width in direction $\vartheta \in \mathbb{S}^{n-1} := \{\vartheta \in \mathbb{R}^n; |\vartheta| = 1\}$ by*

$$w_E(\vartheta) = \max\{(x-y)\cdot\vartheta; x, y \in E\}.$$

The set E is said to have constant width w_E, when $w_E(\vartheta) = w_E$ for all $\vartheta \in \mathbb{S}^{n-1}$.

While other examples are discussed below, we will show that in any dimension n, whenever E is convex, the value of $\sigma_E(E)$ can be estimated in terms of a boundary integral on ∂E involving intrinsic geometric quantities. By way of this estimate we first provide an alternative proof of conjecture (3) in dimension $n = 2$. Next we show that in dimension $n \geq 3$ the conjecture holds true both in the class of sets of constant width and in the class of centrosymmetric sets. In contrast to the 2d-case, however, equality $\sigma_E(E) = 2^{1-n}$ does no longer hold for any set of constant width other than a ball. For the particular case of the rotated Reuleaux-triangle in $\mathbb{R}^3$ this has also been observed by Wise in [16].

2 Main Tool

First we formulate an auxiliary result for convex compact sets E and apparently unrelated sets K. In a first and simpler looking version of this result K was identical to E, and its consequences are used in Lemma 2.4, but the more general version will turn out to be useful later in Remark 6.4. We generalize outside normals as follows:

Definition 2.1 *Let $E \subset \mathbb{R}^n$ be convex. For $x \in \partial E$ we call ν an outer normal at x when*

$$\nu \cdot (y - x) \leq 0 \text{ for all } y \in E.$$

Proposition 2.2 *Suppose $n \geq 2$. Let $K \subset \mathbb{R}^n$ be a compact set consisting of at least two points, and let $E \subset \mathbb{R}^n$ be compact, convex, and with nonempty interior. For $x \in \partial E$, let $\nu_E(x)$ denote an outer normal. Then the following holds.*

- *The outer normal $\nu_E(x)$ is uniquely defined for all $x \in \partial E$ with the exception of a set of $\mathcal{H}^{n-1}$ measure zero.*
- *For those $x \in \partial E$ where $\nu_E(x)$ is unique, the one-sided derivative of d_K at x in the direction $\nu_E(x)$ is well-defined:*

$$\partial^+_{\nu_E} d_K(x) = \lim_{\varepsilon \downarrow 0} \frac{d_K(x + \varepsilon\, \nu_E(x)) - d_K(x)}{\varepsilon}.$$

- *Most important, we have*

$$\sigma_K(E) \leq \frac{1}{n\omega_n} \int_{\partial E} d_K(x)^{1-n}\ \partial^+_{\nu_E} d_K(x)\ d\mathcal{H}^{n-1}(x). \tag{12}$$

As in Remark 1.1, ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Remark 2.3 *We call the points x of ∂E regular, when the outer normal to ∂E at x is uniquely defined. We write $\nu_E(x)$ for this normal. For convex E it is well-defined $\mathcal{H}^{n-1}$ a.e. $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.*

Proof of Proposition 2.2. Here we work out the proof for $n \geq 3$, the case $n = 2$ is similar.

• For the first claim, notice that since E has a nonempty interior, the convexity of E implies that the outside normal $\nu(x)$ is uniquely defined on ∂E except for a set of $\mathcal{H}^{n-1}$-measure zero.

• For the second claim we recall that d_K is a convex function. Indeed for all $\theta \in (0,1)$ we have

$$d_K(\theta x + (1-\theta) y) \leq \max_{z \in K} (\theta |x - z| + (1-\theta)|y - z|) \leq \theta d_K(x) + (1-\theta) d_K(y).$$

The one-sided derivative of a convex function is well-defined and hence $\partial^+_\eta d_K(x)$ is pointwise defined for all x in $\mathbb{R}^n$ and for all $\eta \in \mathbb{S}^{n-1}$.

• Since K contains at least two points, d_K is strictly positive and hence, using the first two items, the integral on the right-hand side of (12) is well defined. Moreover we will use the fact that $\partial^+_\eta d_K(x)$ is upper semicontinuous in $\mathbb{R}^n \times \mathbb{S}^{n-1}$ (see [14, Theorem 24.5]), in the sense that

$$\lim_{(y,\mu) \to (x,\eta)} \partial^+_\mu d_K(y) \leq \partial^+_\eta d_K(x) \qquad \text{for all } (x,\eta) \in \mathbb{R}^n \times \mathbb{S}^{n-1}.$$

For the estimate we consider $\sigma_K(E^c)$. Since σ_K is a probability measure, we have

$$\sigma_K(E) = 1 - \sigma_K(E^c). \tag{13}$$

Setting

$$\varphi_R(x) = \begin{cases} 1 & \text{for } |x| \leq R \\ 0 & \text{for } |x| \geq R+1 \\ 1 - |x| - R & \text{for } R < |x| < R+1 \end{cases}$$

and using (4) and (6) we obtain

$$\sigma_K(E^c) = \frac{-1}{n\,(n-2)\,\omega_n} \lim_{R\to\infty} \left\langle \Delta\left(d_K\,(\cdot)^{2-n}\right), \mathbf{1}_{E^c}\,(\cdot)\ \ \varphi_R(\cdot) \right\rangle, \tag{14}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between measures and continuous functions on the open set E^c. Since in the open set E^c, $\Delta(d_K\,(x)^{2-n})$ is a bounded measure as divergence of an L^∞ vectorfield and since φ_R is a Lipschitz function, it is possible to apply the divergence theorem (see [5]), and the pairing term on the righthand side of (14) becomes

$$\left\langle \Delta\left(d_K\,(\cdot)^{2-n}\right), \mathbf{1}_{E^c}\,(\cdot)\ \ \varphi_R(\cdot) \right\rangle = I - \int_{x\in B(0,R+1)\cap E^c} \nabla\left(d_K\,(x)^{2-n}\right) \nabla\varphi_R(x)dx, \tag{15}$$

with I a boundary term for ∂E.

Let us first consider this boundary term. Taking into account [5, Theorem 2.3] and the fact that $\varphi_{|\partial E} \equiv 1$, the term I in (15) can be expressed by the following limit:

$$I = -\operatorname{ess}\lim_{s\downarrow 0} \int_{\partial E_s} \nabla\left(d_K(x)^{2-n}\right) \cdot \nu_{E_s}(x)\, d\mathcal{H}^{n-1}, \tag{16}$$

where $\nu_{E_s}(\cdot)$ is the unit outer normal to ∂E_s and E_s is a one-parameter family of sets, with all sets homothetic to E, that we construct as follows. We fix y_0 in the interior of E, and define for $s > 0$ the set

$$E_s = \{y + t(y - y_0) : y \in E,\ 0 \leq t \leq s\}.$$

After the change of variables $x = y + s(y - y_0)$ the expression in (16) becomes

$$I = -\operatorname{ess}\lim_{s\downarrow 0} \int_{\partial E} \nabla\left(\left(d_K(y + s(y-y_0))\right)^{2-n}\right) \cdot \nu_E(y)\,(1+s)^{n-1}\, d\mathcal{H}^{n-1}(y).$$

Here we have used the homothety, which implies that $\nu_{E_s}(y + s(y - y_0)) = \nu_E(y)$ for all $s > 0$.

Observing that $\nabla d_K \cdot \nu_E = \partial^+_{\nu_E} d_K$ holds a.e., the upper semicontinuity of $\partial^+_{\nu_E} d_K(y+s(y-y_0))$ for $s \downarrow 0$ yields

$$I \leq (n-2) \int_{\partial E} d_K^{1-n}\,(x)\ \ \partial^+_{\nu_E} d_K\,(x)\ \ d\mathcal{H}^{n-1}\,(x). \tag{17}$$

Next we take care of the second term on the right hand side in (15). This term can be estimated as follows:

$$\begin{aligned} \int_{x\in B_{R+1}(0)\cap E^c} \nabla\left(d_K\,(x)^{2-n}\right) \nabla\varphi_R(x)dx &= -\int_{R<|x|<R+1} \nabla\left(d_K\,(x)^{2-n}\right) \cdot \frac{x}{|x|}dx \\ &= (n-2) \int_{R<|x|<R+1} d_K\,(x)^{1-n}\ \ \nabla d_K\,(x) \cdot \frac{x}{|x|}dx \\ &= (n-2)\, n\omega_n + o(1) \quad \text{as } R \to \infty. \end{aligned} \tag{18}$$

Combining (13), (15) and (18) we find

$$\sigma_K\,(E) = 1 - \sigma_K\,(E^c) = \frac{I}{n\,(n-2)\,\omega_n} + o(1) \quad \text{as } R \to \infty,$$

and the proof for $n \geq 3$ is complete by using (17). When $n = 2$ the very same proof works as well by using now (7) instead of (6). ■

For later reference we shall now study a useful companion of Proposition 2.2 in the case that $E = K$.

Lemma 2.4 *Let $E \subset \mathbb{R}^n$ be compact, convex, and with non empty interior. Suppose that $x \in \partial E$ is a regular point. Then the following holds:*

1. *For all $\tilde{x}$ in ∂E with $d_E(x) = |x - \tilde{x}|$ it holds that*

$$\partial^+_{\nu_E} d_E(x) \geq \lim_{\varepsilon \downarrow 0} \frac{|x + \varepsilon \nu_E(x) - \tilde{x}| - |x - \tilde{x}|}{\varepsilon} = \nu_E(x) \cdot \frac{x - \tilde{x}}{|x - \tilde{x}|}. \tag{19}$$

2. *There exists $\bar{x}$ on ∂E such that $d_E(x) = |x - \bar{x}|$ and*

$$\partial^+_{\nu_E} d_E(x) = \nu_E(x) \cdot \frac{x - \bar{x}}{|x - \bar{x}|}. \tag{20}$$

Proof. The first estimate follows from $d_E\left(x + \varepsilon\, \nu_E(x)\right) \geq |x + \varepsilon \nu_E(x) - \tilde{x}|$. In order to prove equality (20) we consider $x \in \partial E$ and a sequence x_n converging to x from the direction normal to ∂E, so that $\nu_E(x) = \frac{x_n - x}{|x_n - x|}$. We also consider a sequence

$$\bar{x}_n \in \operatorname{argmax}\left\{|x_n - y|\,; y \in E\right\}$$

which, possibly by considering a subsequence, converges to some $\bar{x} \in \partial E$. By continuity

$$\bar{x} \in \operatorname{argmax}\left\{|x - y|\,; y \in E\right\}$$

and hence $|x - \bar{x}| = d_E(x)$. We fix $\xi_n = \frac{x_n - \bar{x}_n}{|x_n - \bar{x}_n|}$, find that $\lim\limits_{n \to \infty} \xi_n = \xi := \frac{x - \bar{x}}{|x - \bar{x}|}$ and use

$$d_E(x_n) = |x_n - \bar{x}_n| = |x - \bar{x}_n| + (x_n - x) \cdot \xi_n + o\left(|x_n - x|\right)$$

to deduce that

$$\begin{aligned} \partial^+_{\nu_E} d_E(x) &= \lim_{n \to \infty} \frac{d_E(x_n) - d_E(x)}{|x_n - x|} = \lim_{n \to \infty} \frac{|x_n - \bar{x}_n| - |x - \bar{x}|}{|x_n - x|} \\ &\leq \lim_{n \to \infty} \frac{|x_n - \bar{x}_n| - |x - \bar{x}_n|}{|x_n - x|} = \lim_{n \to \infty} \nu_E\left(x\right) \cdot \xi_n = \nu_E(x) \cdot \xi. \end{aligned} \tag{21}$$

With the estimate from part 1 the result follows. ■

3 A 2-dimensional Result

Results from [6] and [7] can be combined to get to the following statement.

Theorem 3.1 (Gardiner - Netuka) *Let $E \subset \mathbb{R}^2$ be a compact set.*

1. *Suppose that E contains at least two points. Then*

$$\sigma_E\left(E\right) \leq \frac{1}{2}, \tag{22}$$

2. *Suppose that E is convex and that ∂E is of class C^1. Then $\sigma_E\left(E\right) = \frac{1}{2}$ implies that E is a set of constant width.*

Remark 3.2 *The C^1-smoothness of the boundary in the second item is necessary, as one might see from Example 3.8. It obviously implies also that E has a nonempty interior.*

By way of (12) with $K \equiv E$ we can provide a alternative proof of the theorem above. In case of strongly convex smooth domains the proof becomes rather elementary. In the case that E has corners or faces, the proof can be adjusted but the technicalities become cumbersome. For the first item strongly convex is not a real restriction since $\sigma_E(E) \leq \sigma_{E^*}(E^*)$. See (10). By strongly convex we mean that there exists $\varepsilon > 0$ such that $E = E_\varepsilon$, where

$$E_\varepsilon = \bigcap \left\{ \overline{B_{1/\varepsilon}(x)}; \overline{B_{1/\varepsilon}(x)} \supset E \right\}. \tag{23}$$

For smooth domains it means that the curvature of the boundary is larger than or equal to ε. Note that for $\varepsilon < w_E^{-1}$ one finds that $(E^*)_\varepsilon = E^*$, and hence that E^* is strongly convex.

Remark 3.3 *Concerning the first claim Gardiner and Netuka showed that it is sufficient to prove inequality (22) for the farthest point distance envelope E^*. So indeed, also for proving the first item we may assume that E has a nonempty interior.*

In our proof of Theorem 3.1 we will use a special parametrization of the boundary. For smooth, strongly convex E, the parametrization is straightforwardly defined in the next lemma.

Lemma 3.4 *Suppose $E \in \mathbb{R}^2$ is compact. If ∂E is C^2 and E is strongly convex, then there exists a bijective C^1 parametrization $\vartheta \mapsto x(\vartheta) : \mathbb{S}^1 \to \partial E$ such that for all $\vartheta \in \mathbb{S}^1$:*

1. $\nu_E(x(\vartheta)) = -\nu_E(x(-\vartheta))$ *and*
2. $\vartheta = \dfrac{x(\vartheta) - x(-\vartheta)}{|x(\vartheta) - x(-\vartheta)|}$.

Remark 3.5 $\mathbb{S}^1 = \left\{\vartheta \in \mathbb{R}^2; |\vartheta| = 1\right\}$. *We call $\vartheta \mapsto x(\vartheta) : \mathbb{S}^1 \to \partial E$ a parametrization, if $x(\cdot)$ is continuous, surjektiv and 'turns around left at most once'. Assuming $(0,0) \in E^o$ and $\arg(x(1,0)) = 0$, we mean by the last condition that $\theta \mapsto \arg(x(\cos\theta, \sin\theta)) : [0, 2\pi) \to [0, 2\pi)$ is increasing.*

Remark 3.6 *If one removes the assumption $\partial E \in C^2$ or replaces strong convexity by convexity, one may still show the existence of a parametrization with the properties as in 1 and 2. The parametrization will still be Lipschitz but not necessarily C^1. It would however still be appropriate for our approach. Even allowing ∂E to have corners, would still give such a Lipschitz parametrization. In that case such a parametrization $\vartheta \mapsto x(\vartheta) : \mathbb{S}^1 \to \partial E$ will still be 'increasing' but no longer necessarily 'strictly increasing'. A proof is rather technical and cumbersome. For example, in the case of opposite faces, a parametrization $x(\cdot)$ is not uniquely defined by the geometric properties 1 and 2. Instead one may define $x(\cdot)$ as the limit of $x_\varepsilon(\cdot) : \mathbb{S}^1 \to \partial(E_\varepsilon)$ with E_ε as in (23). Since it doesn't contribute to a better understanding we have skipped such a proof.*

Proof. We may assume that $(0,0)$ lies in the interior of E and that the $\omega \mapsto r(\omega)\omega : \mathbb{S}^1 \to \partial E$ with $r(\omega) \geq r_0 > 0$ is a C^2 parametrization of the boundary in polar coordinates. We write $X(\omega) = r(\omega)\omega$. Since the curvature is strictly bounded away from 0, the map $\omega \mapsto \nu_E(X(\omega))$ is a diffeomorphism homothetic to the identity, which maps $\omega = \frac{x}{|x|}$ for $x \in \partial E$ to the normal $\nu_E(x)$. Let $\Upsilon : \mathbb{S}^1 \to \mathbb{S}^1$ denote its inverse. Consider $\Psi : \mathbb{S}^1 \to \mathbb{S}^1$ defined by

$$\Psi(\xi) = \frac{X(\Upsilon(\xi)) - X(\Upsilon(-\xi))}{|X(\Upsilon(\xi)) - X(\Upsilon(-\xi))|}. \tag{24}$$

Since $\xi \mapsto \Upsilon(\xi) = \frac{X(\Upsilon(\xi))}{|X(\Upsilon(\xi))|}$ and $\xi \mapsto -\Upsilon(-\xi) = \frac{-X(\Upsilon(-\xi))}{|X(\Upsilon(-\xi))|}$ are diffeomorphisms on $\mathbb{S}^1$ homothetic to the identity, also a 'weighted' average in $\mathbb{S}^1$-sense, such as Ψ, is such a diffeomorphism on $\mathbb{S}^1$. Note that Ψ maps a normal direction $\xi \in \mathbb{S}^1$ to an angle $\Psi(\xi) = \vartheta = (\cos\theta, \sin\theta) \in \mathbb{S}^1$ and this is what we want to use from our parametrization. Indeed, the parametrization with the desired properties is the following:

$$\vartheta \mapsto x(\vartheta) = X\left(\Upsilon \circ \Psi^{inv}(\vartheta)\right) \text{ from } \vartheta \in \mathbb{S}^1 \text{ to } \partial E.$$

See Figure 4. ■

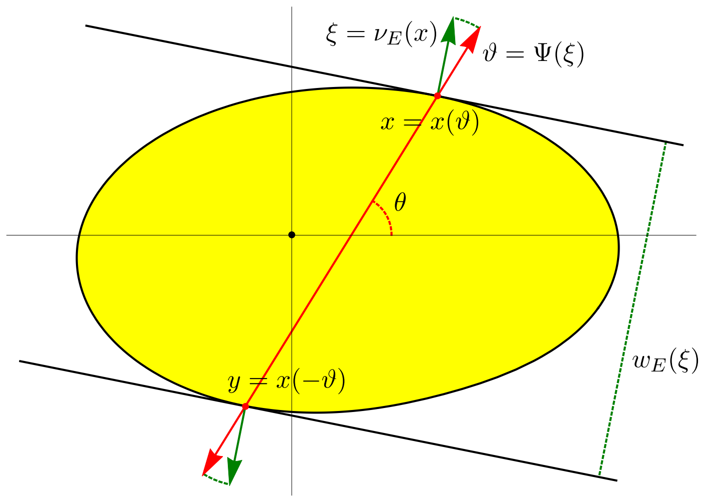

Figure 4: *Illustration for the parametrization in Lemma 3.4*

Proof of Theorem 3.1 assuming E to be strongly convex and $\partial E \in C^2$. Let $\vartheta \mapsto x(\vartheta) : \mathbb{S}^1 \to \partial E$ be the parametrization as in Lemma 3.4.

Since the outer normal $\nu_E(x)$ is uniquely defined for $x \in \partial E$, also the width at $x \in \partial E$ is well-defined by

$$\widetilde{w}(x) := w_E(\nu_E(x)),$$

where $w_E(\cdot)$ is as in Definition 1.5. Setting $x = x(\vartheta)$ and $y = x(-\vartheta)$ and using that x and y have opposite outer normals, we find

$$\widetilde{w}(x(\vartheta)) = (x - y) \cdot \nu_E(x) = (y - x) \cdot \nu_E(y) = \widetilde{w}(x(-\vartheta)). \tag{25}$$

Let $\tilde{x} \in E$ be such that $d_E(x) = |x - \tilde{x}|$. One finds, since $|x - y| \le d_E(x) = |x - \tilde{x}|$ and by Lemma 2.4, that

$$\partial^+_{\nu_E} d_E(x) = \nu_E(x) \cdot \frac{x - \tilde{x}}{|x - \tilde{x}|} \le \frac{\widetilde{w}(x)}{d_E(x)}$$

and one obtains

$$\widetilde{w}(x) \ge d_E(x)\ \partial^+_{\nu_E} d_E(x).$$

Using the result of Proposition 2.2, we find that

$$\sigma_E(E) \le \frac{1}{2\pi} \int_{\partial E} d_E^{-1}(x)\ \partial^+_{\nu_E} d_E(x)\ d\mathcal{H}^1(x) \le \frac{1}{2\pi} \int_{\partial E} d_E^{-2}(x)\ \widetilde{w}(x)\ d\mathcal{H}^1(x). \tag{26}$$

Using the parametrization mentioned above we continue with (26), identifying

$$\vartheta = (\cos\theta, \sin\theta) \text{ and } \boldsymbol{x}(\theta) := x(\cos\theta, \sin\theta) = x(\vartheta),$$

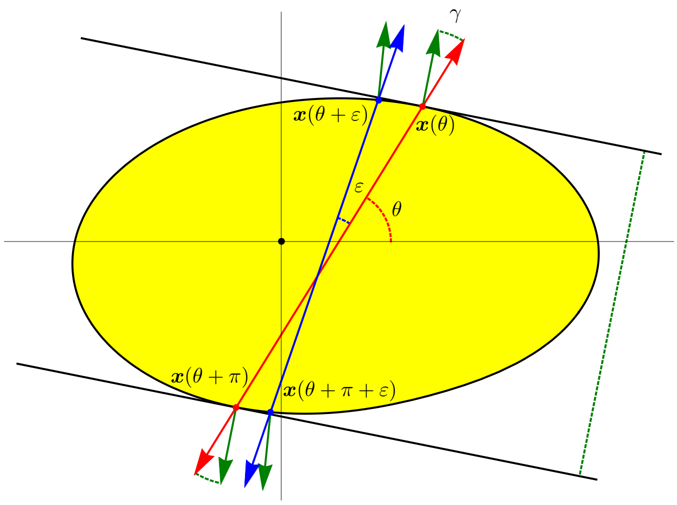

Figure 5: *Illustration for the proof of Theorem 3.1*

to find

$$\begin{aligned}\sigma_E(E) &\leq \frac{1}{2\pi}\int_{\partial E} d_E^{-2}(x)\ \widetilde{w}(x)\ d\mathcal{H}^1(x)\\ &= \frac{1}{2\pi}\int_0^{2\pi} d_E^{-2}(\boldsymbol{x}(\theta))\ \widetilde{w}(\boldsymbol{x}(\theta))\ \left|\boldsymbol{x}'(\theta)\right|\ d\theta\\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |\boldsymbol{x}(\theta)-\boldsymbol{x}(\theta+\pi)|^{-2}\ \widetilde{w}(\boldsymbol{x}(\theta))\ \left|\boldsymbol{x}'(\theta)\right|\ d\theta\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\boldsymbol{x}(\theta)-\boldsymbol{x}(\theta+\pi)|^{-2}\ \widetilde{w}(\boldsymbol{x}(\theta))\ \left|\boldsymbol{x}'(\theta+\pi)\right|\ d\theta. \end{aligned} \tag{27}$$

In the last step we used (25). Averaging the two last terms in (27) we obtain

$$\sigma_E(E) \leq \frac{1}{4\pi}\int_0^{2\pi} |\boldsymbol{x}(\theta)-\boldsymbol{x}(\theta+\pi)|^{-2}\ \widetilde{w}(\boldsymbol{x}(\theta))\ \left(\left|\boldsymbol{x}'(\theta)\right|+\left|\boldsymbol{x}'(\theta+\pi)\right|\right)\ d\theta, \tag{28}$$

which yields the desired inequality once we have shown that

$$|\boldsymbol{x}(\theta)-\boldsymbol{x}(\theta+\pi)|^{-2}\ \widetilde{w}(\boldsymbol{x}(\theta))\ \left(\left|\boldsymbol{x}'(\theta)\right|+\left|\boldsymbol{x}'(\theta+\pi)\right|\right) = 1 \qquad \text{for all } \theta\in[0,2\pi].$$

Indeed one has, with γ the angle from ϑ to $\nu_E(x(\vartheta))$ as in Figure 5, that

$$|\boldsymbol{x}(\theta+\varepsilon)-\boldsymbol{x}(\theta)|+|\boldsymbol{x}(\theta+\pi+\varepsilon)-\boldsymbol{x}(\theta+\pi)| = \widetilde{w}(\boldsymbol{x}(\theta))\ (\tan(\gamma+\varepsilon)-\tan(\gamma))+o(\varepsilon).$$

Dividing by ε and passing to the limit for $\alpha\to 0$ we find

$$\left|\boldsymbol{x}'(\theta)\right|+\left|\boldsymbol{x}'(\theta+\pi)\right| = \frac{\widetilde{w}(\boldsymbol{x}(\theta))}{\cos^2(\gamma)} = \frac{|\boldsymbol{x}(\theta)-\boldsymbol{x}(\theta+\pi)|^2}{\widetilde{w}(\boldsymbol{x}(\theta))}. \tag{29}$$

The equations in (29) and (28) imply $\sigma_E(E)\leq\frac{1}{2}$.

Concerning the second claim it follows from $\sigma_E(E)=\frac{1}{2}$ that it is necessary to have equality in (27). This means that

$$|x(\vartheta)-x(-\vartheta)| = d_E(x(\vartheta)) = d_E(x(-\vartheta)) \text{ for all } \vartheta\in\mathbb{S}^1,$$

which implies for $\partial E\in C^1$ that the normal to ∂E at $x(\vartheta)$ is given by

$$\nu_E(x(\vartheta)) = \vartheta = \frac{x(\vartheta)-x(-\vartheta)}{|x(\vartheta)-x(-\vartheta)|}.$$

Hence we find

$$\widetilde{w}\left(x\left(\vartheta\right)\right)=\left(x\left(\vartheta\right)-x\left(-\vartheta\right)\right)\cdot\nu_E\left(x\left(\vartheta\right)\right)=\left|x\left(\vartheta\right)-x\left(-\vartheta\right)\right|,$$

and

$$\frac{\partial}{\partial\theta}\widetilde{w}\left(\boldsymbol{x}\left(\theta\right)\right)=\frac{\boldsymbol{x}\left(\theta\right)-\boldsymbol{x}\left(\theta+\pi\right)}{\left|\boldsymbol{x}\left(\theta\right)-\boldsymbol{x}\left(\theta+\pi\right)\right|}\cdot\left(\boldsymbol{x}'\left(\theta\right)-\boldsymbol{x}'\left(\theta+\pi\right)\right)=\vartheta\cdot\left(\boldsymbol{x}'\left(\theta\right)-\boldsymbol{x}'\left(\theta+\pi\right)\right).$$

Through $\boldsymbol{x}'\left(\theta\right)\cdot\vartheta=0=\boldsymbol{x}'\left(\theta+\pi\right)\cdot\vartheta$ for $\vartheta=\nu_E\left(x\left(\vartheta\right)\right)=\nu_E\left(x\left(-\vartheta\right)\right)$ it follows that $\widetilde{w}\left(x\left(\vartheta\right)\right)$ is constant. In other words, E is a set of constant width. ■

Remark 3.7 *As we shall see in the following example, if the set E is not C^1, then equality in* (22) *can hold true also if E has non constant width. In this respect the C^1 assumption in the statement of Theorem 3.1 is optimal.*

Example 3.8 *This example is taken from [6, 7]. In $\mathbb{R}^2$ we consider three points $x_1=(0,1)$, $x_2=(\frac{1}{2}\sqrt{3},-\frac{1}{2})$, $x_3=(-\frac{1}{2}\sqrt{3},-\frac{1}{2})$, which are vertices of an equilateral triangle T. We define also the Steiner tree connecting these points*

$$E:=\bigcup_{i=1}^{3}\{tx_i : t\in[0,1]\}.$$

The farthest point envelope according to Definition 1.2 is given by $T^*=\bigcap_{x\in T}\overline{B(x,d_E(x))}$ and happens to coincide now with the Reuleaux triangle *$\bigcap_{i=1}^{3}\overline{B(x_i,\sqrt{3})}$. Indeed, writing $r\left(t\right):=d_E\left(tx_j\right)=\sqrt{t^2+t+1}$ for $t\geq 0$ and using that $\bigcap_{i=1}^{3}B\left(x_i,\sqrt{3}\right)\subset B\left(tx_j,r\left(t\right)\right)$ for all $t\in[0,1]$ and $j\in\{1,2,3\}$, we find*

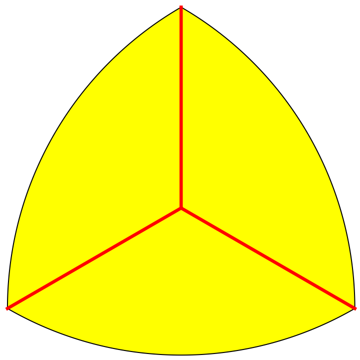

Figure 6: *Reuleaux Triangle.*

$$T^*=\bigcap\left\{\overline{B\left(tx_i,r\left(t\right)\right)};0\leq t\leq 1,\ i\in\{1,2,3\}\right\}=\bigcap_{i=1}^{3}\overline{B(x_i,\sqrt{3})}=\{x_1,x_2,x_3\}^*.$$

It is instructive to note that for the line segment $I=[-e_1,e_1]=co\{-e_1,e_1\}$ with e_1 denoting the first unit vector, I^* does not happen to coincide with $\{-e_1,e_1\}^*$:

$$I^*=\bigcap_{-1\leq\theta\leq 1}\overline{B\left(\theta e_1,1+|\theta|\right)}=\overline{B\left(0,1\right)}\subsetneqq\overline{B(-e_1,2)}\cap\overline{B(e_1,2)}=\{-e_1,e_1\}^*.$$

So the fact that E^* coincides with the Reuleaux triangle T^* is not an automatism.

Since it is proven in [6, 7], that for planar convex set of constant width the equality in (3) *is achieved, we have $\sigma_{T^*}(T^*)=\frac{1}{2}$. However, one finds directly that*

$$d_{T^*}(x)=\max|x-x_i|\ \textit{for all } x\in\mathbb{R}^2,$$

and therefore $\log d_{T^*}(x)$ is harmonic in $T^*\setminus E$ implying that σ_{T^*} vanishes there. Furthermore, σ_{T^*} is absolutely continuous on $\mathbb{R}^2\setminus E$ with respect to Lebesgue measure. We also have $d_{T^*}(x)=d_E(x)=d_T(x)$ on T^*, and we can deduce, that

$$\sigma_E(E)=\sigma_T(T)=\sigma_{T^*}(T^*)=\frac{1}{2}.$$

In fact, the set

$$A=\bigcap_{i=1}^{3}\overline{B(2x_i,\sqrt{7})}$$

is strictly convex and such that $T\subset A\subset T^*$. One may check that also $\sigma_A\left(A\right)=\frac{1}{2}$. So the C^1 condition in the second part of Theorem 3.1 cannot be removed.

4 Bodies of Constant Width in n Dimensions

We consider now the case of n-dimensional bodies of constant width. This class is particularly interesting because – as we already observed in the 2-dimensional case – it provides the optimal bound in the sense that for any given E of constant width $\sigma_E(E) = \frac{1}{2}$. In contrast to this we shall now prove the following result for $n \geq 3$.

Theorem 4.1 *Suppose that $n \geq 3$ and that $E \subset \mathbb{R}^n$ is a closed set of positive constant width. Then $\sigma_E(E) \leq 2^{1-n}$ and equality holds only for the ball.*

Proof. First we observe that for any given convex set E, inequality (21) implies $\partial^+_{\nu_E} d_E(x) \leq 1$ for all x on ∂E. Moreover for bodies of constant width d_E is constant on the boundary of E and equals the width w_E. From (12), with $K \equiv E$, we immediately have

$$\sigma_E(E) \leq \frac{1}{n\omega_n} w_E^{1-n} \ |\partial E|$$

It remains to estimate $|\partial E|$ in terms of w_E. We know of two approaches.

a) One can refer to [8], their estimate (7) with $\omega = d = w_E$ in Theorem 1 and observe that the upper bound on $|\partial E|$ given there is sharp for $n \geq 3$ only if E is a ball, and then $|\partial E| = n\omega_n (w_E/2)^{n-1}$. The authors of [8] seem to have overlooked the equality case (Barbier's Theorem) for sets of constant width in $n = 2$ dimensions. Barbier's theorem states that in $\mathbb{R}^2$ any set of constant width has perimeter π times its width, regardless of its shape.

b) Alternatively, one can refer to the Alexandrov-Fenchel inequalities in [15, Section 6.6 page 351] and observe that, when $n \geq 3$, among all sets of same mean-width only balls achieve maximal surface area. In either case the proof is complete. ■

5 Centro-symmetric Sets

Finally we present another class of sets for which the conjectured bound holds true. We recall that $E \subset \mathbb{R}^n$ is centro-symmetric with respect to the origin O, when $x \in E$ implies that $-x \in E$.

Theorem 5.1 *Suppose that $n \geq 3$ and that the set $E \subset \mathbb{R}^n$, which consists of at least two points, is compact and centrosymmetric. Then $\sigma_E(E) \leq 2^{1-n}$. If equality holds, then E is a ball.*

Proof. If E is centrosymmetric, we may assume that the center is the origin O. Furthermore, we assume that E is also convex and with nonempty interior. Otherwise we can replace E by its convex hull $co(E)$. This convex hull is still centrosymmetric and, as explained in section 1.2, satisfies $\sigma_E(E) \leq \sigma_{co(E)}(co(E))$. Moreover if $co(E)$ has empty interior then $\sigma_{co(E)}(co(E)) = 0$ (see Remark 6.4).

We now introduce the polar representation $\vartheta \mapsto x(\vartheta) : \mathbb{S}^{n-1} \to \mathbb{R}^n$ of the boundary ∂E. By Proposition 2.2

$$\sigma_E(E) \leq \frac{1}{n\omega_n} \int_{\partial E} d_E^{1-n} \ \partial^+_\nu d_E \ d\mathcal{H}^{n-1} = \frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} \frac{|x|^{n-1}}{d_E^{n-1}} \ \frac{\partial^+_\nu d_E}{\vartheta \cdot \nu} \ dS_\vartheta, \tag{30}$$

where we have used the change of variables $\vartheta = x/|x| \in \mathbb{S}^{n-1}$, and with a little abuse of notation we mean $x \equiv x(\vartheta)$, $d_E \equiv d_E(x(\vartheta))$ and $\nu \equiv \nu_E(x(\vartheta))$. One uses $d\mathcal{H}^{n-1} = \frac{|x(\vartheta)|^{n-1}}{\nu(\vartheta)\cdot\vartheta} dS_\vartheta$ and as usual dS_ϑ denotes the surface area element on $\mathbb{S}^{n-1}$.

Since the set E is centrosymmetric with respect to the origin O, one has $|x(\vartheta)| = |x(-\vartheta)|$, and we immediately have

$$|x(\vartheta)|^{n-1} = \left(\frac{|x(\vartheta)| + |x(-\vartheta)|}{2}\right)^{n-1} \leq 2^{1-n} d_E(\vartheta)^{n-1}.$$

Using the last inequality in (30) the proof is complete, once we show that

$$0 \leq \partial_\nu^+ d_E(x(\vartheta)) \leq \vartheta \cdot \nu_E(x(\vartheta)) \text{ for all } \vartheta \in \mathbb{S}^{n-1},$$

or equivalently, that we have $0 \leq \partial_\nu^+ d_E(x) \leq \frac{x}{|x|} \cdot \nu_E(x)$ for all $x \in \partial E$.

To this aim we use inequality (20) and notice that

$$\partial_\nu^+ d_E(x) \leq \max \left\{ \nu_E(x) \cdot \frac{x-y}{|x-y|};\ y \in \partial E \right\} = \frac{x}{|x|} \cdot \nu_E(x),$$

In the last inequality we have used the symmetry of E, which implies that $\nu_E(x) = -\nu_E(-x)$ and that the maximum is reached for $y = -x$.

It remains to consider the equality case. When a convex centrosymmetric set E with nonempty interior satisfies $\sigma_E(E) = 2^{1-n}$, then, following the previous steps, we necessarily have

$$d_E(\vartheta) = 2\,|x\,(\vartheta)| = |x(\vartheta) - x(-\vartheta)| \text{ for all } \vartheta \in \mathbb{S}^{n-1}.$$

This means that the vector θ is orthogonal to ∂E at $x(\vartheta)$ for all $\vartheta \in \mathbb{S}^{n-1}$ and therefore E is a ball. Hence, if $E \subset \mathbb{R}^n$ consists of at least two points, is compact and centrosymmetric, then $\sigma_E(E) = 2^{1-n}$ if and only if $co(E)$ is a closed ball. Let D denote such a closed ball. Obviously $\partial D \subseteq E$ and hence $\sigma_E = \sigma_{co(E)} = \sigma_D$ on $\mathbb{R}^n$. Since σ_D is absolutely continuous with respect to Lebesgue measure, $\sigma_E(E) = \sigma_E(D)$ implies $E = D$. ■

6 Examples and Remarks

The general conjecture of Laugesen and Pritsker [11] remains open.

Conjecture 6.1 *Let $n \geq 3$ and suppose that $E \subset \mathbb{R}^n$ is compact and consists of more than one point. Let σ_E be the probability measure given by (2). Then $\sigma_E(E) \leq 2^{1-n}$. Moreover equality holds only if E is a ball.*

Remark 6.2 *As mentioned before, if E consists of one point z, then $\sigma_E = \delta_z$, the Dirac measure at z, and $\sigma_E(E) = 1$.*

Example 6.3 *Wise in [16] considered E consisting of two points or the connecting interval. Similar results hold in higher dimensions. Set $e_1 = (1, 0, \ldots, 0)$ and consider $E = \{-e_1, e_1\}$ or $E = [-e_1, e_1]$. In both cases*

$$d_E(x) = \sqrt{1 + 2\,|x_1| + |x|^2}.$$

With $\delta_{x_1=0}$ the one-dimensional Dirac measure and λ' the $(n-1)$-dimensional Lebesgue-measure with respect to $x' = (x_2, \ldots, x_n)$ one finds

$$\sigma_E = \delta_{x_1=0} \times \frac{2}{n\omega_n \left(1 + |x'|^2\right)^{n/2}} \lambda'.$$

So this σ_E is not absolutely continuous with respect to the n-dimensional Lebesgue measure. Moreover, one checks directly that $\sigma_E(E) = \sigma_{co(E)}(co(E)) = 0$.

More generally we have the following property.

Remark 6.4 *If $F \subset \mathbb{R}^n$ is a compact subset of an $(n-1)$-dimensional hyperplane H and consists of at least two points, then $\sigma_F(F) = 0$. In other words, a "flat" set F has zero σ_F measure. Indeed, if H denotes the hyperplane, then there exists an $n-1$ dimensional closed ball $D \subset H$ so that $E \subset D$. In an orthogonal reference frame in which the first $(n-1)$ coordinate axes* *are parallel to the plane H, we consider the closed cylinder $\mathcal{C}_\varepsilon = D \times [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Obviously*

$$\sigma_F(F) \leq \lim_{\varepsilon \to 0} \sigma_F(\mathcal{C}_\varepsilon).$$

Using (12) *and identifying K with F and E with $\mathcal{C}_\varepsilon$ we can now estimate $\sigma_F(\mathcal{C}_\varepsilon)$. Denoting by $\hat{y}$ a unit vector normal to H and using the upper semicontinuity of $\partial^+_\eta d_F(x)$ for (x, η) varying in $\mathbb{R}^n \times \mathbb{S}^{n-1}$ (see [14, Theorem 24.5]), a direct computation gives*

$$\lim_{\varepsilon \to 0} \sigma_F(\mathcal{C}_\varepsilon) \leq \frac{1}{n\omega_n} \int_B d_F(x)^{1-n} \, \left(\partial^+_{\hat{y}} d_F(x) + \partial^+_{-\hat{y}} d_F(x)\right) \, d\mathcal{H}^{n-1}(x).$$

Finally we have to observe that $\partial^+_{\hat{y}} d_F(x) = \partial^+_{-\hat{y}} d_F(x) = 0$ on D because the set F is 'flat'. Therefore we can deduce $\sigma_F(F) = 0$.

Example 6.5 *For $E = \overline{B(0,1)}$, a closed* Ball in *$\mathbb{R}^n$ one finds $d_E(x) = |x| + 1$, and since for $n \geq 3$*

$$-\Delta \left(|x| + 1\right)^{2-n} = \frac{(n-2)(n-1)}{|x| \left(1 + |x|\right)^n}$$

it follows with $\omega_n = \frac{n\pi^{n/2}}{\Gamma(1+n/2)}$ and the fundamental solution for $-\Delta$, namely $F_n(x) = \frac{|x|^{2-n}}{n(n-2)\omega_n}$, that

$$\begin{aligned} \left(|x| + 1\right)^{2-n} &= \int_{y \in \mathbb{R}^n} \frac{|x-y|^{2-n}}{n(n-2)\omega_n} \frac{(n-2)(n-1)}{|y| \left(1 + |y|\right)^n} dy \\ &= \int_{y \in \mathbb{R}^n} |x-y|^{2-n} \frac{(n-1)}{n\omega_n |y| \left(1 + |y|\right)^n} dy. \end{aligned}$$

Hence

$$d\sigma_E(y) = \frac{(n-1)}{n\omega_n |y| \left(1 + |y|\right)^n} dy$$

which implies $\sigma_E(\mathbb{R}^n) = 1$ and

$$\sigma_E(E) = \int_E d\sigma_E(y) = \int_{r=0}^1 \frac{(n-1) r^{n-2}}{(1+r)^n} dr = 2^{1-n}.$$

This confirms part of Theorems 4.1 and 5.1 in another explicit way.

Example 6.6 *Recall that in Example 3.8 we saw in case $n = 2$ for the equilateral triangle T with side length ℓ as well as for the Reuleaux-triangle $T^* := \cap_{i=1,\dots,3} \overline{B(x_i, \ell)}$ that we attain the optimal bound $\sigma_T(T) = \sigma_{T^*}(T^*) = 2^{1-n}$.*

This observation does not extend to higher dimensions. In fact, let W denote a regular tetrahedron *in $\mathbb{R}^3$ with edge length ℓ and corners x_i $i = 1, \dots 4$ on the unit sphere. Then one can calculate (see Example 8 in [16]) that*

$$\sigma_W(W) = \frac{1}{\pi} \left(3 \arccos \frac{1}{3} - \pi\right) \approx 0.1755 < \frac{1}{4}.$$

Next consider the corresponding Reuleaux-tetrahedron *$\widetilde{W} := \cap_{i=1,\dots,4} \overline{B(x_i, \ell)}$ and notice that by the same reasoning as in Example 3.8 for the Reuleaux triangle we have*

$$\widetilde{W} = \bigcap_{i \in \{1,2,3,4\}} \overline{B\left(x_i, d_W(x_i)\right)} = \bigcap_{x \in W} \overline{B\left(x, d_W(x)\right)} = W^*.$$

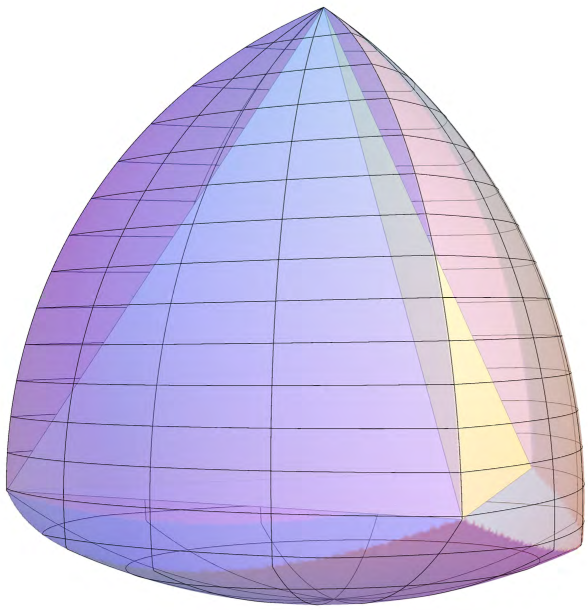

Figure 7: *A tetrahedron W and the corresponding Reuleaux-tetrahedron* W^*

Therefore (12) *yields now*

$$\sigma_{\widetilde{W}}\left(\widetilde{W}\right) = \sigma_{W^*}\left(W^*\right) \le \frac{1}{4\pi}\int_{\partial W^*} d_{W^*}^{-2}(x)\ \partial_\nu^+ d_{W^*}(x)\ d\sigma$$
$$\le \frac{1}{4\pi\ell^2}\left|\partial W^*\right| = \frac{1}{4\pi}\left(8\pi - 18\ \arccos\frac{1}{3}\right) \approx 0.2367 < \frac{1}{4}.$$

Example 6.7 *In case* $n = 3$ *the following two bodies of constant width can be explicitly studied. If* R *denotes the* ROTATED REULEAUX TRIANGLE, *then (see [16])*

$$\sigma_R(R) \approx 0.1042,$$

and if M_i, $i = 1, 2$ *denotes one of the* MEISSNER BODIES, *then (see [10])*

$$\sigma_{M_i}(M_i) = \frac{1}{4\pi\ell^2}|\partial M_i| = \frac{1}{4}\left(2 - \frac{\sqrt{3}}{2}\ \arccos\frac{1}{3}\right) \approx 0.2335.$$

Both are less than $1/4$ *as implied by Theorem 4.1.*

Remark 6.8 *The examples in this section can be ordered as* $W \subset M_i \subset W^* \subset D = \overline{B(0,1)}$. *The corresponding measures satisfy* $\sigma_W(W) < \sigma_{M_i}(M_i) < \sigma_{W^*}(W^*) < \sigma_D(D)$. *While this suggests that* $A \subset B$ *might imply* $\sigma_A(A) \le \sigma_B(B)$, *for general convex sets* A *and* B, *let us present a simple counterexample. If* W^* *is the Reuleaux-tetrahedron let* δ *denote its inradius. Then* $D_\delta := \overline{B(0,\delta)} \subset W^*$, *but* $\sigma_{W*}(W^*) < \sigma_{D_\delta}(D_\delta)) = 1/4$.

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