

The Tikhonov regularization for equilibrium problems and applications to quasi-hemivariational inequalities

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Abstract

In this paper, we deal with the Tikhonov regularization method for pseudo-monotone equilibrium problems. Under mild conditions of semicontinuity, we show that strictly pseudo-monotone bifunctions can be used as regularization bifunctions rather than strongly monotone bifunctions. We extend Berge's maximum theorem and establish the relationship between quasi-hemivariational inequalities and equilibrium problems. An application of the Tikhonov regularization method to quasi-hemivariational inequalities is also given.

Keywords: Equilibrium problem, quasi-hemivariational inequality, regularization method, multivalued mapping, semicontinuity, pseudo-monotonicity.

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1. Introduction

Let C be a nonempty, closed and convex subset of a real Banach space E and let $\Phi : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying $\Phi(x, x) = 0$, for every $x \in C$. Such a bifunction Φ is called an *equilibrium bifunction*.

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Recall that an *equilibrium problem* in the sense of Blum, Muu and Oettli (see Blum and Oettli (1994); Muu and Oettli (1992)) is a problem of the form:

$$\text{find } x^* \in C \text{ such that } \Phi(x^*, y) \geq 0 \quad \forall y \in C, \quad (\text{EP})$$

where its set of solutions is denoted by $SEP(C, \Phi)$.

Equilibrium problems encompass several problems including variational inequalities, mathematical programming, Nash equilibrium, Kakutani fixed points, optimization and many other problems arising in nonlinear analysis. Recently, some practical models of interest in engineering and economics have been formulated as an equilibrium problem of the form (EP), see for example Kassay (2010); Konnov (2001); Hadjisavvas and Schaible (2009) and the references therein.

Equilibrium problems also encompass quasi-hemivariational inequalities. Recall that if E is a real Banach space which is continuously embedded in $L^p(\Omega; \mathbb{R}^n)$, for some $1 < p < +\infty$ and $n \geq 1$, where Ω is a bounded domain in \mathbb{R}^m , $m \geq 1$, then a *quasi-hemivariational inequality* is a problem of the form:

$$\text{find } u \in E \text{ and } z \in A(u) \text{ such that}$$

$$\langle z, v \rangle + h(u) J^0(iu; iv) - \langle Fu, v \rangle \geq 0 \quad \forall v \in E,$$

where i is the canonical injection of E into $L^p(\Omega; \mathbb{R}^n)$, $A : E \rightrightarrows E^*$ is a nonlinear multivalued mapping, $F : E \rightarrow E^*$ is a nonlinear operator, $J : L^p(\Omega; \mathbb{R}^n) \rightarrow \mathbb{R}$ is a locally Lipschitz functional and $h : E \rightarrow \mathbb{R}$ is a given nonnegative functional. We denote by E^* the dual space of E and by $\langle \cdot, \cdot \rangle$ the duality pairing between E^* and E .

For technical reasons, we will consider the following quasi-hemivariational inequality:

$$\text{find } u \in C \text{ and } z \in A(u) \text{ such that}$$

$$\langle z, v - u \rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u \rangle \geq 0 \quad \forall v \in C, \quad (\text{QHVI})$$

where its set of solutions is denoted by $SQHVI(C, A)$. Note that in the special case when C is the whole space E , the above two formulations of quasi-hemivariational inequalities are one and the same.

Studies about inequality problems captured special attention in the last decades where one of the most recent and general type of inequalities is the *hemivariational inequalities* introduced in Panagiotopoulos (1983, 1985) as a variational formulation for several classes of mechanical problems with nonsmooth and nonconvex energy super-potentials. The theory of hemivariational inequalities has produced an abundance of important results both in pure and applied mathematics as well as in other domains such as mechanics and engineering sciences as it allowed mathematical formulations for new classes of interesting problems, see Naniewicz and Panagiotopoulos (1995); Rădulescu (2008); Wangkeeree and Preechasilp (2013); Panagiotopoulos (1983, 1985); Alleche and Rădulescu (2014); Costea and Rădulescu (2012) and the references therein.

When h is equal to zero in the quasi-hemivariational inequality (QHVI) corresponding to convex super-potentials, we obtain the standard case of *variational inequalities*, which were studied earlier by many authors, see Lions and Stampacchia (1967); Kinderlehrer and Stampacchia (1980). The setting corresponding to h equal to 1 describes the hemivariational inequalities. These inequality problems appear as a generalization of variational inequalities, but they are much more general than these ones, in the sense that they are not equivalent to minimum problems but give rise to substationarity problems. The general case when h is nonconstant corresponds to *quasi-hemivariational inequalities*, which were first studied in (Naniewicz and Panagiotopoulos, 1995, Section 4.5), in relationship with relevant models in mechanics and engineering. We refer to Rădulescu (2008); Wangkeeree and Preechasilp (2013); Costea and Rădulescu (2012) for recent contributions to the qualitative analysis of hemivariational and quasi-hemivariational inequalities. One can also consult Alleche and Rădulescu (2014); Alleche (2013, 2014) where some techniques on continuity of functions are introduced and new results in the field are obtained.

On the other hand, *regularization methods* that are widely used in convex optimization and variational inequalities have been also considered for equilibrium problems. The *proximal point method* as well as the *Tikhonov regularization method*, which are a fundamental regularization technique for han-

ding ill-posed problems, have been recently applied to equilibrium problems, see Moudafi (2010); Dinh and Muu (2011); Ding (2010); Mastroeni (2003); Hung and Muu (2011) and the references therein.

In this paper, we deal first with the Tikhonov regularization method for *pseudo-monotone equilibrium problems*. Under weakened conditions of upper semicontinuity of bifunctions in their first variable on a subset, we extend some results of Hung and Muu (2011); Dinh and Muu (2011) and prove that strictly pseudo-monotone bifunctions can be also used as regularization bifunctions instead of strongly monotone bifunctions. We extend Berge's maximum theorem and develop some results in the qualitative analysis of quasi-hemivariational inequalities to establish the relationship between quasi-hemivariational inequality problems and equilibrium problems. We also give examples and apply the Tikhonov regularization method to quasi-hemivariational inequalities.

2. Notations and preliminary results

Let X be Hausdorff topological space, $x \in X$ and $f : X \rightarrow \mathbb{R}$ be a function. Recall that f is said to be

1. *upper semicontinuous* at x if for every $\epsilon > 0$, there exists an open neighborhood U of x such that

$$f(y) \leq f(x) + \epsilon \quad \forall y \in U;$$

2. *lower semicontinuous* at x if for every $\epsilon > 0$, there exists an open neighborhood U of x such that

$$f(y) \geq f(x) - \epsilon \quad \forall y \in U.$$

It is well-known that if X is a metric space (or more generally, a Fréchet-Urysohn space, see Alleche and Calbrix (1999)), then f is upper (*resp.* lower) semicontinuous at $x \in X$ if and only if for every sequence $(x_n)_n$ in X converging to x , we have

$$f(x) \geq \limsup_{n \rightarrow +\infty} f(x_n) \quad (\text{resp. } f(x) \leq \liminf_{n \rightarrow +\infty} f(x_n)),$$

where $\limsup_{n \rightarrow +\infty} f(x_n) = \inf_n \sup_{k \geq n} f(x_k)$ and $\liminf_{n \rightarrow +\infty} f(x_n) = \sup_n \inf_{k \geq n} f(x_k)$.

We say that f is upper (*resp.* lower) semicontinuous on a subset S of X if it is upper (*resp.* lower) semicontinuous at every point of S .

The notion of upper (*resp.* lower) semicontinuous function on a subset generalizes the notion of upper (*resp.* lower) semicontinuous function at a point.

If X is a metric space, these notions of upper and lower semicontinuous functions on a subset coincide respectively with those of sequentially upper and sequentially lower semicontinuous functions on a subset considered in Alleche and Rădulescu (2014).

Clearly, if S is an open subset of X , then $f : X \rightarrow \mathbb{R}$ is upper (*resp.* lower) semicontinuous on S if and only if $f|_S$ is upper (*resp.* lower) semicontinuous on S , where $f|_S$ denotes the restriction of f on S . It is also not difficult to prove the following result which shows how easy is to construct upper (*resp.* lower) semicontinuous functions on subsets.

Proposition 2.1. *Let $f : X \rightarrow \mathbb{R}$ be a function and let S be a subset of X . If the restriction $f|_U$ of f on an open subset U containing S is upper (*resp.* lower) semicontinuous on S , then any extension of $f|_U$ to the whole space X is upper (*resp.* lower) semicontinuous on S .*

The following result provides us with some properties of upper (*resp.* lower) semicontinuous functions.

Proposition 2.2. *Let $f : X \rightarrow \mathbb{R}$ be a function, S a subset of X and $a \in \mathbb{R}$.*

1. *f is upper semicontinuous on S if and only if*

$$\overline{\{x \in X \mid f(x) \geq a\}} \cap S = \{x \in S \mid f(x) \geq a\}.$$

In particular, if f is upper semicontinuous on S , then the trace on S of any upper level set of f is closed in S .

2. *f is lower semicontinuous at S if and only if*

$$\overline{\{x \in X \mid f(x) \leq a\}} \cap S = \{x \in S \mid f(x) \leq a\}.$$

In particular, if f is lower semicontinuous on S , then the trace on S of any lower level set of f is closed in S .

Proof. The second statement being similar to the first one, we prove only the case of upper semicontinuity. Let

$$x^* \in \overline{\{x \in X \mid f(x) \geq a\}} \cap S.$$

Clearly, $x^* \in S$. To prove that $f(x^*) \geq a$, we argue by contradiction and assume that $f(x^*) < a$. Take $\epsilon > 0$ such that $f(x^*) + \epsilon < a$. By upper semicontinuity of f at x^* , let U be an open neighborhood of x^* such that $f(y) \leq f(x^*) + \epsilon$, for every $y \in U$. It follows that

$$U \cap \{x \in X \mid f(x) \geq a\} = \emptyset,$$

which is a contradiction.

Conversely, let $x^* \in S$, $\epsilon > 0$ and put $a = f(x^*) + \epsilon$. We have $f(x^*) < a$ and then

$$x^* \notin \overline{\{x \in X \mid f(x) \geq a\}}.$$

Let U be an open neighborhood of x^* such that $\{x \in X \mid f(x) \geq a\} \cap U = \emptyset$. It follows that

$$f(y) < a = f(x^*) + \epsilon \quad \forall y \in U.$$

Finally, we have

$$\{x \in X \mid f(x) \geq a\} \cap S = \{x \in S \mid f(x) \geq a\},$$

which yields that the trace on S of any upper level set of f is closed in S . \square

In the sequel, for $y \in C$, we define the following sets:

$$\Phi^+(y) = \{x \in C \mid \Phi(x, y) \geq 0\} \quad \text{and} \quad \Phi^-(y) = \{x \in C \mid \Phi(y, x) \leq 0\}.$$

Clearly, $x^* \in C$ is a solution of the equilibrium problem (EP) if and only if $x^* \in \bigcap_{y \in C} \Phi^+(y)$.

The following result of Alleche and Rădulescu (2014) is a generalization of the well-known Ky Fan's minimax inequality theorem (see Fan (1972); Kassay (2010)). We sketch the proof for the convenience of the reader.

Theorem 2.3. *Let $\Phi : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction and suppose the following assumptions hold:*

1. Φ is quasi-convex in its second variable on C ;
2. there exists a compact subset K of C and $y_0 \in K$ such that

$$\Phi(x, y_0) < 0 \quad \forall x \in C \setminus K;$$

3. Φ is upper semicontinuous in its first variable on K .

Then, the equilibrium problem (EP) has a solution and its set of solutions $SEP(C, \Phi)$ is a nonempty compact set.

Proof. Since Φ is an equilibrium bifunction, then $\overline{\Phi^+(y)}$ is nonempty and closed, for every $y \in C$.

By quasi-convexity of Φ in its second variable, the mapping $y \mapsto \Phi^+(y)$ is a KKM mapping (see for example, Alleche and Rădulescu (2014); Alleche (2014); Bianchi and Schaible (1996); Fan (1961, 1972); Kassay (2010)), and since $\Phi^+(y_0)$ is contained in the compact subset K , then by Ky Fan's lemma, we have

$$\bigcap_{y \in C} \overline{\Phi^+(y)} \neq \emptyset.$$

On the other hand, we have

$$\bigcap_{y \in C} \overline{\Phi^+(y)} = \bigcap_{y \in C} (\overline{\Phi^+(y)} \cap K).$$

By Proposition 2.2, we have

$$\overline{\Phi^+(y)} \cap K = \Phi^+(y) \cap K \quad \forall y \in C.$$

Thus,

$$\bigcap_{y \in C} \Phi^+(y) = \bigcap_{y \in C} \overline{\Phi^+(y)} \neq \emptyset.$$

The compactness of the set of solutions is obvious. □

Remark 1. The set K in the condition 2 of Theorem 2.3 is known in the literature under the name of the *set of coerciveness*.

The Minty lemma for equilibrium problems deals in particular with properties such as compactness and convexity of the set of solutions of equilibrium problems (see for example, Konnov (2001)). For more properties of the set of solutions of equilibrium problems, we need some additional concepts of monotonicity for bifunctions.

A bifunction $\Phi : C \times C \rightarrow \mathbb{R}$ is called

1. *strongly monotone* on C with modulus β if

$$\Phi(x, y) + \Phi(y, x) \leq -\beta \|x - y\|^2, \quad \forall x, y \in C;$$

2. *monotone* on C if

$$\Phi(x, y) + \Phi(y, x) \leq 0, \quad \forall x, y \in C;$$

3. *strictly pseudo-monotone* on C if

$$\Phi(x, y) \geq 0 \implies \Phi(y, x) < 0, \quad \forall x, y \in C, x \neq y;$$

4. *pseudo-monotone* on C if

$$\Phi(x, y) \geq 0 \implies \Phi(y, x) \leq 0, \quad \forall x, y \in C.$$

Every strongly monotone bifunction is both monotone and strictly pseudo-monotone and every strictly pseudo-monotone bifunction Φ is pseudo-monotone provided it is an equilibrium bifunction, that is, $\Phi(x, x) = 0, \forall x \in C$.

The following result extends (Bianchi and Schaible, 1996, Theorem 4.2) for equilibrium problems defined on non necessarily convex sets and its proof is elementary. We call such a problem, an *equilibrium-like problem*.

Proposition 2.4. *Let $\Phi : C \times C \rightarrow \mathbb{R}$ be a strictly pseudo-monotone bifunction. Then for every subset A of C , the following equilibrium-like problem*

$$\text{find } x^* \in A \text{ such that } \Phi(x^*, y) \geq 0 \quad \forall y \in A$$

has at most one solution.

We also need the following notions about convexity of functions. A function $f : C \rightarrow \mathbb{R}$ is said to be

1. *semistrictly quasi-convex* on C if, for every $x_1, x_2 \in C$ such that $f(x_1) \neq f(x_2)$, we have

$$f(\lambda x_1 + (1 - \lambda)x_2) < \max\{f(x_1), f(x_2)\} \quad \forall \lambda \in]0, 1[;$$

2. *explicitly quasi-convex* on C if it is quasi-convex and semistrictly quasi-convex (see Hadjisavvas and Schaible (1993); Konnov and Schaible (2000); Avriel et al. (1988)).

Note that there is not any inclusion relationship between the class of semistrictly quasi-convex functions and that of quasi-convex functions. However, if f is a lower semicontinuous and semistrictly quasi-convex function, then f is explicitly quasi-convex, see Cambini and Martein (2009).

Here, we obtain some additional properties of the set of solutions of equilibrium problems.

Theorem 2.5. *Under assumptions of Theorem 2.3 and suppose the following conditions hold:*

1. Φ is pseudo-monotone;
2. Φ is explicitly quasi-convex in its second variable on C .

Then the equilibrium problem (EP) has a solution and its set of solutions $SEP(C, \Phi)$ is nonempty compact set. If in addition, K is convex, then $SEP(C, \Phi)$ is convex.

Proof. The first part of this theorem being proved above, we prove the second part. By pseudo-monotonicity, we have $\Phi^+(y) \subset \Phi^-(y)$, for every $y \in C$. Since $\bigcap_{y \in C} \Phi^+(y) \subset K$, then

$$\bigcap_{y \in C} \Phi^+(y) \subset \left(\bigcap_{y \in C} \Phi^-(y) \right) \cap K.$$

Now, by explicit quasi-convexity (see (Alleche, 2014, Proposition 1.3)), we obtain

$$\left(\bigcap_{y \in C} \Phi^-(y) \right) \cap K \subset \bigcap_{y \in C} \Phi^+(y).$$

It follows that

$$\bigcap_{y \in C} \Phi^+(y) = \left(\bigcap_{y \in C} \Phi^-(y) \right) \cap K.$$

By quasi-convexity, the set $\Phi^-(y)$ is convex, for every y . Thus, the set of solutions $\text{SEP}(C, \theta)$ is convex whenever K is convex. \square

Note that Theorem 2.5 also holds if we replace upper semicontinuity of Φ in the first variable by upper hemicontinuity in the first variable and lower semicontinuity in the second variable. Recall that upper hemicontinuity is upper semicontinuity on line segments. The notion of upper hemicontinuity on a subset has been used in Alleche (2014); Alleche and Rădulescu (2014).

3. The Tikhonov regularization method for equilibrium problems

The *Tikhonov regularization method* (or *ridge regression* in statistics) (see Tikhonov (1963)) is a powerful tool in convex optimization to handle discrete or continuous ill-posed problems. In the framework of monotone variational inequalities, the basic idea of this method is to perturb the problem with a strongly monotone operator depending on a regularization parameter to the monotone cost operator to obtain a strongly monotone variational inequality. The optimal regularization parameter is usually unknown and usually in practical problems it is determined by various methods, such as the discrepancy principle, cross-validation, L -curve method, Bayesian interpretation, restricted maximum likelihood, and unbiased predictive risk estimator. The resulting regularized inequality problem has a unique solution that depends on the regularization parameter. Next, passing to the limit as the parameter goes to a suitable value, the unique solution of the regularized problem tends to a solution of the original problem. We point out that if the cost operator is pseudo-monotone rather than monotone, then the monotonicity of the regularized problem may fail.

3.1. Main result

In this section, we define a regularized equilibrium problem for the equilibrium problem (EP). Let $\theta : C \times C \rightarrow \mathbb{R}$ be an equilibrium bifunction that we call the *regularization equilibrium bifunction*. Then, for every $\epsilon > 0$, we define the equilibrium bifunction $\Phi_\epsilon : C \times C \rightarrow \mathbb{R}$ by

$$\Phi_\epsilon(x, y) = \Phi(x, y) + \epsilon\theta(x, y)$$

and we associate with the equilibrium problem (EP), the regularized equilibrium problem defined as follows:

$$\text{find } x_\epsilon^* \in C \text{ such that } \Phi_\epsilon(x_\epsilon^*, y) \geq 0 \quad \forall y \in C, \quad (\text{REP})$$

where its set of solutions is denoted by $\text{SREP}(C, \Phi_\epsilon)$.

Note that when Φ or θ is pseudo-monotone, the regularized equilibrium bifunction Φ_ϵ does not inherit any monotonicity property from Φ and θ in general.

The following result is an extension of (Dinh and Muu, 2011, Theorem 2.9) (see also (Hung and Muu, 2011, Theorem 3.2)) in which we avoid the strong monotonicity of the bifunction θ . We also avoid the lower semicontinuity of Φ and θ in their second variable on C .

Theorem 3.1. *Let $(\epsilon_n)_n$ be a sequence of positive numbers such that $\lim_{n \rightarrow +\infty} \epsilon_n = 0$ and suppose the following conditions hold:*

1. Φ and θ are pseudo-monotone on C ;
2. Φ and θ are quasi-convex in the second variable on C ;
3. there exist a compact subset K of C and $y_0 \in C$ such that $\Phi(x, y_0) < 0$, for every $x \in C \setminus K$;
4. Φ and θ are upper semicontinuous in the first variable on K .

Then any cluster point $x^ \in C$ of a sequence $(x_n)_n$ with $x_n \in \text{SREP}(C, \Phi_{\epsilon_n}) \cap K$ for every n , is a solution to the equilibrium-like problem:*

$$\text{find } x^* \in \text{SEP}(C, \Phi) \text{ such that } \theta(x^*, y) \geq 0 \quad \forall y \in \text{SEP}(C, \Phi). \quad (\text{E-LP})$$

Assume in addition that the following hypotheses hold:

1. θ is strictly pseudo-monotone on C ;
2. there exists $A \subset K$ such that $\theta(x, y_0) < 0$, for every $x \in C \setminus A$.

Then, the regularized equilibrium problem (REP) is solvable, for every n , and any sequence $(x_n)_n$ with $x_n \in \text{SREP}(C, \Phi_{\epsilon_n})$ for every n , converges to the unique solution of the equilibrium-like problem (E-LP).

Proof. Let $(x_n)_n$ be a sequence with $x_n \in \text{SREP}(C, \Phi_{\epsilon_n}) \cap K$ for every n , and admitting $x^* \in C$ as a cluster point. We have $x^* \in K$ and without loss of generality, we may assume that $(x_n)_n$ converges to x^* . Then, for every n

$$\Phi(x_n, y) + \epsilon_n \theta(x_n, y) \geq 0 \quad \forall y \in C.$$

By Theorem 2.3, we have $\text{SEP}(C, \Phi) \neq \emptyset$. Let $z \in \text{SEP}(C, \Phi)$. By pseudo-monotonicity of Φ , we have $\Phi(x_n, z) \leq 0$ and then

$$\epsilon_n \theta(x_n, z) \geq -\Phi(x_n, z) \geq 0,$$

which implies that $\theta(x_n, z) \geq 0$. Letting n go to $+\infty$, we obtain by upper semicontinuity of θ in its first variable on K that $\theta(x^*, z) \geq 0$. Thus,

$$\theta(x^*, z) \geq 0 \quad \forall z \in \text{SEP}(C, \Phi).$$

It remains now to prove that $x^* \in \text{SEP}(C, \Phi)$. Again by upper semicontinuity of Φ and θ in their first variable on K and since

$$\Phi(x_n, y) + \epsilon_n \theta(x_n, y) \geq 0 \quad \forall y \in C,$$

then by using the properties of the upper limits, we have

$$\Phi(x^*, y) \geq 0 \quad \forall y \in C.$$

Now, we prove the second part of the theorem. Note that for every n ,

$$\Phi(x_n, y_0) + \epsilon_n \theta(x_n, y_0) < 0 \quad \forall x \in C \setminus K.$$

By Theorem 2.3, the regularized equilibrium problem ($\text{REP}(C, \Phi_{\epsilon_n})$) is solvable and its set of solutions $\text{SREP}(C, \Phi_{\epsilon_n})$ is contained in K . Let $(x_n)_n$ be

a sequence such that $x_n \in \text{SREP}(C, \Phi_{\epsilon_n})$, for every n . Then the sequence $(x_n)_n$ has a cluster point $x^* \in K$ and by the first part of the theorem, x^* is a solution to the equilibrium-like problem (E-LP). Since θ is strictly pseudo-monotone, then by Proposition 2.4, the above equilibrium-like problem (E-LP) has a unique solution. It follows that every subsequence of the sequence $(x_n)_n$ admits this unique solution of the equilibrium-like problem (E-LP) as a cluster point. Thus, the sequence $(x_n)_n$ converges to the unique solution of the equilibrium-like problem (E-LP). \square

Remark 2. Note that when E is a finite dimensional real Banach space, θ is strongly monotone on C , θ and Φ are convex and lower semicontinuous in the second variable on C , then the sequence $(\Phi + \epsilon_n \theta)_n$ is uniformly coercive whenever Φ has a set of coerciveness, see (Dinh and Muu, 2011, Corollary 2.6). This means that Theorem 3.1 can also be considered as a generalization of (Dinh and Muu, 2011, Theorem 2.9) even when θ is strongly monotone on C . In this case, we choose θ such that θ and Φ are upper semicontinuous in their first variable on the subset of the uniform coerciveness.

3.2. Example of suitable bifunctions

We construct in what follows two bifunctions Φ and θ satisfying all the conditions of Theorem 3.1 without being upper semicontinuous in their first variable on the whole space C and where θ is strictly pseudo-monotone but not strongly monotone on C . This example is obtained by modification of some equilibrium bifunctions constructed in the literature, see Alleche (2013); Bianchi and Schaible (1996, 2004); Ansari et al. (2014).

Example 3.2. Let $E = C = \mathbb{R}$, $K = [-1, +1]$ and $y_0 = 0$.

- The bifunction $\theta : C \times C \rightarrow \mathbb{R}$ is defined by

$$\theta(x, y) = \begin{cases} \frac{y^4 - x^4}{65} & \text{if } (x, y) \in \{2\} \times [-3, 3], \\ y^4 - x^4 & \text{otherwise.} \end{cases}$$

1. Clearly $\theta(x, x) = 0$, for every $x \in C$ and $\theta(x, 0) < 0$, for every $x \notin K = [-1, +1]$. It is also easy to see that θ is strictly pseudo-monotone and not strongly monotone on C .
2. To see that θ is quasi-convex in its second variable, let $x \in C$ be fixed.
 - (a) if $x \neq 2$, then $\theta(x, y) = y^4 - x^4$, for every $y \in C$. The function $y \mapsto y^4 - x^4$ is convex on C .
 - (b) if $x = 2$, then

$$\theta(2, y) = \begin{cases} \frac{y^4 - 16}{65} & \text{if } y \in [-3, 3], \\ y^4 - 16 & \text{otherwise.} \end{cases}$$

If $y' \notin [-3, 3]$, then $\theta(2, y) \leq \theta(2, y')$, for every $y \in [-3, 3]$.

Thus, the function $y \mapsto \theta(2, y)$ is quasi-convex on C .

3. To see that θ is upper semicontinuous in its first variable on $[-1, +1]$, let $y \in C$ be fixed and denote by $f : C \rightarrow \mathbb{R}$ the function defined by

$$f(x) = \theta(x, y).$$

The restriction $f|_U$ of f on the open set $U =]-\infty, 2[$ containing $[-1, +1]$ is defined by $f|_U(x) = y^4 - x^4$ which is continuous on U and then by Proposition 2.1, f is upper semicontinuous on $[-1, +1]$.

4. Finally, the bifunction θ is not upper semicontinuous in its first variable on C . Indeed, consider $y = 3$ (or $y = -3$). Let $(x_n)_n$ be a converging sequence to 2 such that $x_n \neq 2$, for every n . We have

$$\theta(2, 3) = 1 < 65 = \limsup_{n \rightarrow +\infty} \theta(x_n, 3).$$

- The bifunction $\Phi : C \times C \rightarrow \mathbb{R}$ is defined by

$$\Phi(x, y) = \begin{cases} (x + 2)(y - x) & \text{if } x \in]-\infty, -2[, \\ (x + 1)(y - x) & \text{if } x \in [-2, -1[, \\ \max(x, 0)(y - x) & \text{otherwise.} \end{cases}$$

1. Clearly, $\Phi(x, x) = 0$, for every $x \in C$ and $\Phi(x, 0) < 0$, for every $x \notin [-1, +1]$.
2. To verify that Φ is pseudo-monotone on C , let $x, y \in C$ such that $\Phi(x, y) \geq 0$.
 - (a) If $x \in]-\infty, -2[$, then $\Phi(x, y) = (x + 2)(y - x)$. It follows that $y - x \leq 0$ and then, $y < -2$. Thus $\Phi(y, x) = (y + 2)(x - y) \leq 0$.
 - (b) If $x \in [-2, -1[$, then $y - x \leq 0$ and then, $y < -1$. If $y \in [-2, -1[$, then $\Phi(y, x) = (y + 1)(x - y) \leq 0$, and if $y \in]-\infty, -2[$, then $\Phi(y, x) = (y + 2)(x - y) \leq 0$.
 - (c) If $x \geq -1$, then $y \geq x$. It follows that $y \geq -1$ and then $\Phi(y, x) = \max(y, 0)(x - y) \leq 0$.
3. Clearly, Φ is quasi-convex in its second variable on C and upper semicontinuous in its first variable on $[-1, +1]$.
4. To see that Φ is not upper semicontinuous in its first variable on C , consider $y > -2$ and take a sequence $(x_n)_n$ in $]-\infty, -2[$ converging to -2 . We have

$$\Phi(-2, y) = -(y + 2) < 0 = \limsup_{n \rightarrow +\infty} (x_n + 2)(y + 2) = \limsup_{n \rightarrow +\infty} \Phi(x_n, y).$$

5. Note that in addition, Φ is not lower semicontinuous in its first variable on C . To see this fact, consider $y < -2$ and take a sequence $(x_n)_n$ in $]-\infty, -2[$ converging to -2 . We have

$$\Phi(-2, y) = -(y + 2) > 0 = \liminf_{n \rightarrow +\infty} (x_n + 2)(y + 2) = \liminf_{n \rightarrow +\infty} \Phi(x_n, y).$$

6. Finally, let us point out that Φ is not strictly pseudo-monotone on C since $\Phi(x, y) = \Phi(y, x) = 0$ whenever $x, y \in [-1, 0]$.

4. Applications to quasi-hemivariational inequalities

In this section we give some results on the relationship between equilibrium problems and quasi-hemivariational inequalities. We develop results in the qualitative analysis of quasi-hemivariational inequalities and give a generalization

to Berge's maximum theorem in order to apply the Tikhonov regularization for quasi-hemivariational inequalities.

Recall that a function $\phi : E \rightarrow \mathbb{R}$ is called *locally Lipschitzian* if for every $u \in E$, there exists a neighborhood U of u and a constant $L_u > 0$ such that

$$|\phi(w) - \phi(v)| \leq L_u \|w - v\|_X \quad \forall w \in U, \forall v \in U.$$

If $\phi : E \rightarrow \mathbb{R}$ is locally Lipschitzian near $u \in E$, then the *Clarke generalized directional derivative* of ϕ at u in the direction of $v \in E$, denoted by $\Phi^0(u, v)$, is defined by

$$\phi^0(u, v) = \limsup_{\substack{w \rightarrow u \\ \lambda \downarrow 0}} \frac{\phi(w + \lambda v) - \phi(w)}{\lambda}.$$

Among several important properties of the generalized directional derivative of locally Lipschitzian functions, we will make use in the present paper of the following properties (for proofs and related properties, we refer to (Clarke, 1990, Proposition 2.1.1)).

Suppose that $\phi : E \rightarrow \mathbb{R}$ is locally Lipschitzian near $u \in E$. Then,

1. the function $v \mapsto \phi^0(u, v)$ is finite, positively homogeneous and subadditive;
2. the function $(u, v) \mapsto \phi^0(u, v)$ is upper semicontinuous.

Remark 3. To avoid any confusion in the definition of semicontinuity on subsets, from now on and in all what follows, the functions h and F , and the multivalued mapping A will be considered from C rather than from E .

It is easily seen that any solution of the quasi-hemivariational inequality (QHVI) is a solution of the equilibrium problem (EP) where the equilibrium bifunction $\Phi : C \times C \rightarrow \mathbb{R}$ is defined by

$$\Phi(u, v) = \sup_{z \in A(u)} \langle z, v - u \rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u \rangle \quad \forall u, v \in C.$$

The converse needs some additional conditions on the multivalued mapping A and holds by a classical approach.

Theorem 4.1. *If A has nonempty, convex and weak* compact values, then any solution of the equilibrium problem (EP) is a solution of the quasi-hemivariational inequality problem (QHVI).*

Proof. Let $u^* \in C$ be such that $\Phi(u^*, v) \geq 0$, for every $v \in C$, and assume that there does not exist $z \in A(u^*)$ satisfying

$$\langle z, v - u^* \rangle + h(u^*) J^0(iu^*; iv - iu^*) - \langle Fu^*, v - u^* \rangle \geq 0 \quad \forall v \in C.$$

Clearly, for every $z \in A(u^*)$, there exists $v_z \in C$ such that

$$\langle z, v_z - u^* \rangle + h(u^*) J^0(iu^*; iv_z - iu^*) - \langle Fu^*, v_z - u^* \rangle < 0.$$

Since, for every $v \in C$, the mapping defined on E^* by

$$z \mapsto \langle z, v - u^* \rangle + h(u^*) J^0(iu^*; iv - iu^*) - \langle Fu^*, v - u^* \rangle$$

is weak* continue, and since $A(u^*)$ is weak* compact, there exist $\epsilon > 0$ and $v_j \in C$, $j = 1, \dots, n$, such that for all $z \in A(u^*)$

$$\min_{j=1, \dots, n} (\langle z, v_j - u^* \rangle + h(u^*) J^0(iu^*; iv_j - iu^*) - \langle Fu^*, v_j - u^* \rangle) < -\epsilon.$$

The Clarke generalized directional derivative being finite, then for every $j = 1, \dots, n$, the functions

$$z \mapsto \langle z, v_j - u^* \rangle + h(u^*) J^0(iu^*; iv_j - iu^*) - \langle Fu^*, v_j - u^* \rangle,$$

defined on the convex set $A(u^*)$ are concave and proper with domain containing $A(u^*)$, and therefore by a standard result of convex analysis (see (Rockafellar, 1970, Theorem 21.1)), there exist $\mu_j \geq 0$, $j = 1, \dots, n$, with $\sum_{j=1}^n \mu_j = 1$ such that for all $z \in A(u^*)$

$$\sum_{j=1}^n \mu_j (\langle z, v_j - u^* \rangle + h(u^*) J^0(iu^*; iv_j - iu^*) - \langle Fu^*, v_j - u^* \rangle) < -\epsilon.$$

Set $v^* = \sum_{j=1}^n \mu_j v_j$. Then $v^* \in C$ and by the positive homogeneity and the subadditivity of the Clarke generalized directional derivative in its second variable, we have

$$\langle z, v^* - u^* \rangle + h(u^*) J^0(iu^*; iv^* - iu^*) - \langle Fu^*, v^* - u^* \rangle < -\epsilon \quad \forall z \in A(u^*)$$

which implies that $\Phi(u^*, v^*) < 0$, a contradiction. \square

We turn now into studying the properties inherited by the equilibrium bifunctions defined from quasi-hemivariational inequalities.

Theorem 4.2. *The bifunction Φ is lower semicontinuous and convex in its second variable on C .*

Proof. From the positive homogeneity and the subadditivity of the Clarke generalized directional derivative in its second variable, the function

$$v \mapsto \langle z, v - u \rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u \rangle$$

is convex, for every $u \in C$ and every $z \in A(u^*)$. It is also lower semicontinuous since the Clarke generalized directional derivative is lower semicontinuous. The bifunction Φ being the superior envelope of a family of convex and lower semicontinuous functions, it is then convex and lower semicontinuous in its second variable on C . \square

The properties inherited by Φ in its first variable are more complicated and need additional conditions on the functions and multivalued mappings involved in the quasi-hemivariational inequalities.

Recall that a multivalued mapping T from a topological space X with values in the set of subsets of a topological space Y is called *upper semicontinuous* at a point $x \in X$ if whenever V an open subset containing $T(x)$, there exist an open neighborhood U of x such that $T(x') \subset V$, for every $x' \in U$. We say that T is upper semicontinuous on a subset S of X if T is upper semicontinuous at every point of S .

The following result is a generalization of the well-known Berge's maximum theorem, see (Papageorgiou and Kyritsi-Yiallourou, 2009, Theorem 6.1.18).

Theorem 4.3. *Let X and Y be two Hausdorff topological spaces, S a nonempty subset of X , U an open subset containing S , $T : X \rightrightarrows Y$ a multivalued mapping and $\psi : Y \times X \rightarrow \mathbb{R} \cup \{+\infty\}$ a function. Suppose that ψ is upper semicontinuous on $Y \times U$ and T is upper semicontinuous on S with nonempty compact values*

on U . Then the value function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x) = \sup_{y \in T(x)} \psi(y, x)$$

is upper semicontinuous on S .

Proof. By Proposition 2.1, it suffices to prove that the restriction $g = f|_U$ of f on U is upper semicontinuous on S . Let $a \in \mathbb{R}$ and by Proposition 2.2, we have to prove that

$$\overline{\{x \in U \mid g(x) \geq a\}} \cap S = \{x \in S \mid g(x) \geq a\},$$

where the closure is taken with respect to U . Let $x^* \in \overline{\{x \in U \mid g(x) \geq a\}} \cap S$ and choose a net $(x_\alpha)_{\alpha \in \Lambda}$ in $\{x \in U \mid g(x) \geq a\}$ converging in U to x^* . Since $x_\alpha \in U$, then the restriction of the function ψ on $Y \times \{x_\alpha\}$ is upper semicontinuous and therefore, by the Weierstrass theorem, it attains its maximum on the compact set $T(x_\alpha)$, for every $\alpha \in \Lambda$. Let $y_\alpha \in T(x_\alpha)$ be such that $g(x_\alpha) = \psi(y_\alpha, x_\alpha)$, for every $\alpha \in \Lambda$.

The net $(y_\alpha)_{\alpha \in \Lambda}$ has a cluster point in $T(x^*)$. Indeed, suppose the contrary holds. Then the compactness of $T(x^*)$ yields the existence of an open set V containing $T(x^*)$ and $\alpha_0 \in \Lambda$ such that $y_\alpha \notin V$, for every $\alpha \geq \alpha_0$. It follows by upper semicontinuity of T at x^* the existence of an open neighborhood W of x^* such that $T(x) \subset V$, for every $x \in W$. Let $\alpha_1 \in \Lambda$ be such that $x_\alpha \in W$, for every $\alpha \geq \alpha_1$. Thus $y_\alpha \in V$, for every $\alpha \geq \alpha_1$. Contradiction.

Take now $y^* \in T(x)$ and $(y_\alpha)_{\alpha \in \Gamma}$ a subnet of $(y_\alpha)_{\alpha \in \Lambda}$ converging to y^* . The net $((y_\alpha, x_\alpha))_{\alpha \in \Gamma}$ is in $Y \times U$, converging to (y^*, x^*) and satisfies

$$\psi(y_\alpha, x_\alpha) \geq a \quad \forall \alpha \in \Gamma.$$

By upper semicontinuity of ψ on $Y \times U$, it follows that $g(x^*) \geq \psi(y^*, x^*) \geq a$, which completes the proof. \square

We give in what follows a sufficient condition for the upper semicontinuity in its first variable of the equilibrium bifunction Φ .

Corollary 4.4. *Let K be a subset of C , U be an open subset containing K and suppose the following conditions hold:*

1. the nonlinear multivalued mapping A is upper semicontinuous on K with respect to the strong topology of E^* and has nonempty compact values on U ;
2. for every $v \in C$, the mapping $u \in C \mapsto h(u) J^0(iu; iv - iu)$ is upper semicontinuous on U ;
3. for every $v \in C$, the mapping $u \in C \mapsto \langle F(u), v - u \rangle$ is lower semicontinuous on U .

Then Φ is upper semicontinuous in its first variable on K .

Proof. Let $v \in C$ be fixed and define the function $\psi : E^* \times C \rightarrow \mathbb{R}$ by

$$\psi(z, u) = \langle z, v - u \rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u \rangle.$$

The function ψ being a sum of upper semicontinuous functions on $E^* \times U$, it is upper semicontinuous on $E^* \times U$, where E^* is equipped with the strong topology. It follows by Theorem 4.3 that the value function $u \mapsto \Phi(u, v)$ is upper semicontinuous on K . \square

Corollary 4.5. *Let K be a subset of C , U be an open subset containing K and suppose the following conditions hold:*

1. the nonlinear multivalued mapping A is upper semicontinuous on K with respect to the weak* topology of E^* and has nonempty weak* compact values on U ;
2. for every $v \in C$, the mapping

$$(z, u) \in E^* \times U \mapsto \langle z, v - u \rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u \rangle$$

is upper semicontinuous on $E^ \times U$.*

Then Φ is upper semicontinuous in its first variable on K .

Recall that a multivalued mapping $T : C \rightarrow 2^{E^*}$ is said to be:

1. *pseudo-monotone* on C if

$$\langle z, u - v \rangle \leq 0 \implies \langle t, v - u \rangle \geq 0 \quad \forall u, v \in C, \forall z \in A(u), \forall t \in A(v);$$

2. *strictly pseudo-monotone* if

$$\langle z, u - v \rangle \leq 0 \implies \langle t, v - u \rangle > 0 \quad \forall u, v \in C, \forall z \in A(u), \forall t \in A(v).$$

It is well-known that if T has weak* compact values, then T is pseudo-monotone (*resp.* strictly pseudo-monotone) if and only if the equilibrium bifunction θ is pseudo-monotone (*resp.* strictly pseudo-monotone) where $\theta : C \times C \rightarrow \mathbb{R}$ is defined by

$$\theta(u, v) = \sup_{z \in T(u)} \langle z, v - u \rangle.$$

This follows from the fact that for every $u, v \in C$, by the weak* compactness of the values of T , there exist $z \in T(u)$ and $t \in T(v)$ such that

$$\theta(u, v) = \langle z, v - u \rangle \quad \text{and} \quad \theta(v, u) = \langle t, u - v \rangle.$$

Now, to apply the Tikhonov regularization for quasi-hemivariational inequalities, first we take a multivalued function $G : C \rightarrow 2^{E^*}$ and $\epsilon > 0$, and define the multivalued function $A_\epsilon : C \rightarrow 2^{E^*}$ by

$$A_\epsilon(x) = A(x) + \epsilon G(x).$$

The regularized quasi-hemivariational inequality has the following form:

Find $u \in C$ and $z \in A_\epsilon(u)$ such that

$$\langle z, v - u \rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u \rangle \geq 0 \quad \forall v \in C. \quad (\text{RQHVI})$$

As previously, we denote its set of solutions by $\text{SRQHVI}(C, A_\epsilon)$.

We say that a quasi-hemivariational inequality (QHVI) is pseudo-monotone on C if the associated equilibrium bifunction Φ is pseudo-monotone on C .

Theorem 4.6. *Let K be a compact subset of C , U an open subset containing K and $(\epsilon_n)_n$ is a sequence of positive numbers such that $\lim_{n \rightarrow +\infty} \epsilon_n = 0$. Suppose that the following assumptions hold:*

1. G is pseudo-monotone on C , upper semicontinuous on K with respect to the strong topology of E^* and has nonempty, convex and compact values on C ;

2. the quasi-hemivariational inequality (QHVI) is pseudo-monotone on C ;
3. A is upper semicontinuous on K with respect to the strong topology of E^* and has nonempty, convex and compact values on C ;
4. for every $v \in C$, the mapping $u \in C \mapsto h(u) J^0(iu; iv - iu)$ is upper semicontinuous on U ;
5. for every $v \in C$, the mapping $u \in C \mapsto \langle F(u), v - u \rangle$ is lower semicontinuous on U .
6. there exists $v_0 \in C$ such that

$$\langle z, v_0 - u \rangle + h(u) J^0(iu; iv_0 - iu) - \langle Fu, v_0 - u \rangle < 0 \quad \forall u \in C \setminus K, \forall z \in A(u).$$

Then any cluster point $x^* \in C$ of a sequence $(x_n)_n$ with $x_n \in SRQHVI(C, A_{\epsilon_n}) \cap K$ for every n , is a solution to the multivalued variational inequality:

Find $u \in SQHVI(C, A)$ and $z \in G(u)$ such that

$$\langle z, v - u \rangle \geq 0 \quad \forall v \in SQHVI(C, A).$$

Assume in addition, that the following conditions hold:

1. G is strictly pseudo-monotone on C ;
2. there exists $K' \subset K$ such that $\langle z, v_0 - u \rangle < 0$, for every $u \in C \setminus K'$ and every $z \in G(u)$.

Then the regularized quasi-hemivariational inequality ($RQHVI(C, F_{\epsilon_n})$) is solvable, for every n , and any sequence $(x_n)_n$ with $x_n \in SRQHVI(C, F_{\epsilon_n})$ for every n , converges to the unique solution of the multivalued variational inequality problem:

Find $u \in SQHVI(C, A)$ and $z \in G(u)$ such that

$$\langle z, v - u \rangle \geq 0 \quad \forall v \in SQHVI(C, A).$$

Proof. Note that

$$\sup_{z \in A_\epsilon(x)} \langle z, y - x \rangle = \sup_{z \in A(x)} \langle z, y - x \rangle + \epsilon \sup_{z \in G(x)} \langle z, y - x \rangle \quad \forall x, y \in C.$$

The result holds now easily from the results developed above and by applying Theorem 3.1. \square

Remark 4. Under assumptions of Theorem 4.6, the set of solutions of the quasi-hemivariational inequality (QHVI) is nonempty and compact. It is also convex whenever K is convex.

Conclusions. In this work, we have proved that under weakened conditions of semicontinuity, the Tikhonov regularization method can be applied to pseudo-monotone equilibrium problems with strictly pseudo-monotone bifunctions as regularized equilibrium bifunctions instead of strongly monotone bifunctions. We have obtained a generalization of Berge's maximum theorem and developed new techniques in the qualitative analysis of quasi-hemivariational inequalities in order to establish the relationship between quasi-hemivariational inequalities and equilibrium problems. We have also applied the Tikhonov regularization method to quasi-hemivariational inequality problems.

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