NONDEGENERACY IN THE OBSTACLE PROBLEM WITH A DEGENERATE FORCE TERM

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Abstract. In this paper we prove the optimal nondegeneracy of the solution u of the obstacle problem

$$\triangle u = f\chi_{\{u>0\}}$$

in a bounded domain $D \subset \mathbb{R}^n$, where we only require f to have a nondegeneracy of the type $f(x) \geq \lambda|(x_1, \cdots, x_p)|^\alpha$ for some $\lambda > 0$, $1 \leq p \leq n$ (an integer) and $\alpha > 0$. We prove optimal uniform $(2+\alpha)$-th order and nonuniform quadratic nondegeneracy, more precisely we prove that there exists $C > 0$ (depending only on n, p and α) such that for x a free boundary point and $r > 0$ small enough we have

$$\sup_{\partial B_r(x)} u \geq C\lambda(r^{2+\alpha} + |(x_1, \cdots, x_p)|^\alpha r^2).$$

We also prove the optimal growth with the assumption $|f(x)| \leq \Lambda|(x_1, \cdots, x_p)|^\alpha$ for some $\Lambda \geq 0$ and the porosity of the free boundary.

1. Introduction

Let $n \geq 1$ be an integer and $D \subset \mathbb{R}^n$ a bounded domain. Let $f \in L^\infty(D)$, $g \in H^1(D)$ such that $g \geq 0$ on ∂D. Let $u \in H^1(D)$ be the unique minimiser (cf. [3]) of the functional

$$\int_D \big(|\nabla u|^2 + 2fu\big)dx$$

in the admissible set of functions

$$\big\{u \geq 0 \text{ a.e. in } D \text{ and } u = g \text{ on } \partial D\big\}.$$

It is known (cf. [4]) that we have $u \in W^{2,q}_{loc}(D)$ for all $1 < q < \infty$ and thus by the Sobolev imbeddings we have $u \in C^{1,\beta}_{loc}(D)$ for all $0 < \beta < 1$. Also we have

$$\triangle u = f\chi_{\{u>0\}} \text{ in } D \tag{1.1}$$

in the sense of distributions.

Let us denote by Ω the noncoincidence set and by Γ the free boundary, i.e.

$$\Omega = \big\{x \in D \;\big|\; u(x) > 0\big\} \qquad \text{and} \qquad \Gamma = D \cap \partial\Omega.$$

To study the structure and regularity of the free boundary Γ it is crucial to have an optimal nondegeneracy result of the solution. For example, using this nondegeneracy estimate one can rule out degenerate blow up limits with the correct scaling at the free boundary.

In [1, 4], the authors have studied the case when the force term is bounded away from zero by a positive constant, i.e. $\lambda \leq f$ for some constant $\lambda > 0$. In this case one obtains optimal quadratic nondegeneracy of the solution.

Date: May 21, 2014.

2010 *Mathematics Subject Classification.* Primary 35R35; Secondary 35J60.

Key words and phrases. Free boundary, Obstacle problem, Degenerate, Optimal growth, Optimal nondegeneracy, Porosity.

In this paper our aim is to drop the assumption that f should be bounded away from zero by a positive constant. By requiring f to grow away from its zeros by a polynomial order, we still obtain the appropriate optimal nondegeneracy estimate.

Let p be an integer such that $1 \le p \le n$ and $\alpha > 0$ a positive real number. For $x \in \mathbb{R}^n$ let us denote $x = (x', x'')$ where $x' = (x_1, \cdots, x_p)$ and $x'' = (x_{p+1}, \cdots, x_n)$.

Our main result is the following theorem.

Theorem 1 (Optimal Nondegeneracy)**.** *There exists a $C > 0$ (depending only on n, p and α) such that if*

$$f(x) \ge \lambda |x'|^\alpha \text{ for } x \in D \tag{1.2}$$

holds then for $x_0 \in \Omega$ and $B_r(x_0) \subset\subset D$ we have

$$\sup_{\Omega \cap \partial B_r(x_0)} u \ge u(x_0) + C\lambda r^2 (r^\alpha + |x_0'|^\alpha). \tag{1.3}$$

The proof is based on the construction of appropriate comparison functions. Homogeneous harmonic polynomials which are positive on $\{x_1 = 0\} \backslash \{0\}$ play an important role in the construction of these comparison functions.

Although the main result of this paper is the nondegeneracy result stated above, we also prove the optimal growth of the solution in Theorem 2 and the local porosity of the free boundary in Theorem 4.

This paper is structured as follows. In Section 2, we state some estimates, which will be used later. In Section 3 we prove the nondegeneracy estimate. In Section 4 we prove the growth estimate. In Section 5 we prove the local porosity of the free boundary Γ.

2. Preliminary Analysis

Let us define

$$\psi(y) = \frac{1}{(\alpha+2)(\alpha+p)} |y|^{\alpha+2} \text{ for } y \in \mathbb{R}^p.$$

Because $\alpha > 0$ we have $\psi \in C^2(\mathbb{R}^p)$. It is easy to see that $\triangle \psi = |y|^\alpha$. Functions v which satisfy $|\triangle v| \le C|y|^\alpha$ in $B_1^p(0)$ have also been studied in [2].

There exists $C > 0$ (depending on α and p) such that

$$|D^2\psi(y)| = \Big(\sum_{i,j=1}^n (\partial_{y_i y_j} \psi(y))^2 \Big)^{\frac{1}{2}} = C|y|^\alpha \text{ for } y \in \mathbb{R}^p, \tag{2.1}$$

and there exists $C > 0$ (depending on α and p) such that

$$C|y|^\alpha |\zeta|^2 \le \zeta^T D^2\psi(y) \zeta \text{ for } y, \zeta \in \mathbb{R}^p. \tag{2.2}$$

For $y_0 \in \mathbb{R}^p$, let us denote by w_{y_0} the difference between ψ and its affine tangent, i.e.

$$w_{y_0}(y) = \psi(y) - \psi(y_0) - \nabla\psi(y_0) \cdot (y - y_0) \text{ for } y \in \mathbb{R}^p. \tag{2.3}$$

We have $w_{y_0} \in C^2(\mathbb{R}^p)$ and from the convexity of ψ it follows that w_{y_0} is convex and nonnegative. Obviously we have $w_{y_0}(y_0) = 0$ and $\nabla w_{y_0}(y_0) = 0$.

We will use the function w_{y_0} to construct appropriate comparison functions in Theorems 1 and 2.

In the following Lemma we prove estimates of w_{y_0} both from above and below.

Lemma 1. *There exist $0 < C_1 < C_2$ such that for $y, y_0 \in \mathbb{R}^p$ we have*

$$C_1|y - y_0|^2 (|y_0|^\alpha + |y - y_0|^\alpha) \le w_{y_0}(y) \le C_2 |y - y_0|^2 (|y_0|^\alpha + |y - y_0|^\alpha). \tag{2.4}$$

Proof. We prove the first inequality in (2.4). The second inequality follows from (2.1) after some computations.

To obtain the first inequality in (2.4), we compute, using (2.2)

$$
\begin{aligned}
(2.5)\quad w_{y_0}(y) &= \int_0^1 (1-s)(y-y_0)^T D^2 w_{y_0}(y_0+s(y-y_0))(y-y_0)ds \\
&\geq C|y-y_0|^2 \int_0^1 (1-s)|y_0+s(y-y_0)|^\alpha ds.
\end{aligned}
$$

Now if we show that for some $C_3 > 0$

$$
(2.6)\qquad \int_0^1 (1-s)|y_0+s(y-y_0)|^\alpha ds \geq C_3\big(|y_0|^\alpha + |y-y_0|^\alpha\big) \text{ for } y, y_0 \in \mathbb{R}^p
$$

then by (2.5) the proof will be complete.

In the case $y_0 = 0$, for small enough $C_3 > 0$ the inequality (2.6) holds, so we consider the case when $y_0 \neq 0$. Let O_{y_0} be an orthonormal transformation such that $O_{y_0}e_1 = \frac{y_0}{|y_0|}$. Then by the change of variable $y = y_0 + |y_0|O_{y_0}z$ it is easy to see that (2.6) is equivalent to the inequality

$$
(2.7)\qquad \int_0^1 (1-s)|e_1+sz|^\alpha ds \geq C_3(1+|z|^\alpha) \text{ for } z \in \mathbb{R}^p.
$$

From the inequality $z_1 \geq -|z|$ it follows that

$$
|e_1+sz| \geq \big|e_1 - s|z|e_1\big| = \big|1-s|z|\big|.
$$

So denoting $r = |z|$ the inequality (2.7) follows from the inequality

$$
\int_0^1 (1-s)|1-sr|^\alpha ds \geq C_3(1+r^\alpha) \text{ for } r \geq 0
$$

and one may prove this by direct integration. □

3. Optimal Nondegeneracy

In this section we first define for $k \in \mathbb{N}\cup\{0\}$ the polynomials p_{2k} and prove some properties of these polynomials that we will use later on. For $x_0 \in \mathbb{R}^n$ we will consider the function $w_{x_0'}(x')$ in $\mathbb{R}^n$ and by adding appropriate scaled and translated polynomials p_{2k} to $w_{x_0'}$ we improve the lower bound in (2.4). In Theorem 1, using these improved lower bounds we prove the optimal nondegeneracy estimate. The optimality of our estimate is evident by the optimal growth estimate proved in Theorem 2.

Let us define for $k \in \mathbb{N}\cup\{0\}$

$$
p_{2k}(x) = \sum_{j=0}^{k} a_j x_1^{2j}|x|^{2k-2j}
$$

where $a_0 = 1$ and for $1 \leq j \leq k$, a_j are given by the recursive equation

$$
(3.1)\qquad 0 = j(2j-1)a_j + (k-j+1)(2j+2k+n-4)a_{j-1}.
$$

In the following lemma we prove some properties of the polynomials p_{2k} which we will use later.

Lemma 2. *p_{2k} is a $2k$-th order homogeneous harmonic polynomial such that for all $\beta > 0$ there exists $C > 0$ (depending only on β, k and n) such that*

$$
(3.2)\qquad \inf_{x\in\partial B_1}\big(|x_1|^\beta + Cp_{2k}(x)\big) > 0.
$$

Proof. It is clear that p_{2k} is a $2k$-th order homogeneous polynomial. To prove that p_{2k} is harmonic one computes its Laplacian and uses the fact that the coefficients satisfy the equations (3.1).

To prove (3.2) we write

$$p_{2k}(x) = |x|^{2k} + q_{2k-2}(x)x_1^2 \tag{3.3}$$

where in the case $k = 0$ we set $q_{-2} = 0$ and for $k \geq 1$

$$q_{2k-2}(x) = \sum_{j=1}^{k} a_j x_1^{2j-2}|x|^{2k-2j}.$$

Let

$$A = \sup_{x\in\partial B_1} \left|q_{2k-2}(x)\right|$$

then from (3.3) we have

$$p_{2k}(x) \geq 1 - Ax_1^2 \text{ for } x \in \partial B_1. \tag{3.4}$$

Let $\beta > 0$ then to show (3.2), by (3.4) it is enough to show that there exists a $C > 0$ such that

$$\inf_{t\in[0,1]} \big(t^\beta + C(1 - At^2)\big) > 0. \tag{3.5}$$

To prove (3.5) let $0 < \delta < 1$ to be chosen later, now we estimate

$$t^\beta + C(1 - At^2) \geq \chi_{\{0\leq t\leq\delta\}}\big(C(1 - A\delta^2)\big) + \chi_{\{\delta<t\leq 1\}}\big(\delta^\beta + C(1 - A)\big)$$

thus if we can choose $0 < \delta < 1$ and $C > 0$ such that

$$1 > A\delta^2 \text{ and } \delta^\beta > C(A - 1) \tag{3.6}$$

then (3.5) is proved.

For a fixed $C > 0$ there exists a $0 < \delta < 1$ such that (3.6) holds if and only if

$$\frac{1}{\sqrt{A}} > (C(A - 1)^+)^{\frac{1}{\beta}}. \tag{3.7}$$

For all $A \geq 0$ it is possible to choose $C > 0$ such that (3.7) holds and this completes the proof of the lemma. □

For $x_0 \in \mathbb{R}^n$ by the first inequality in (2.4) we have $w_{x_0'}(x') \geq C|x' - x_0'|^2(|x_0'|^\alpha + |x' - x_0'|^\alpha)$. In the following two lemma by adding polynomial terms to $w_{x_0'}$ we improve this inequality, such that instead of $|x' - x_0'|$ we have $|x - x_0|$.

Lemma 3. *There exist $a > 0$ and $C > 0$ such that for all x,$x_0 \in \mathbb{R}^n$*

$$w_{x_0'}(x') + a|x_0'|^\alpha p_2(x - x_0) \geq C|x_0'|^\alpha|x - x_0|^2.$$

Proof. Let $x \neq x_0$. By Lemma 1 there exists a $C_1 > 0$ such that for x,$x_0 \in \mathbb{R}^n$

$$w_{x_0'}(x') \geq C_1|x_0'|^\alpha|x' - x_0'|^2. \tag{3.8}$$

By Lemma 2 there exist $C_2, C_3 > 0$ such that

$$|x_1|^2 + C_2p_2(x) \geq C_3 \text{ for } x \in \partial B_1. \tag{3.9}$$

Now by (3.8) and (3.9) taking $a = C_1C_2$ we compute

$$\begin{aligned} w_{x_0'}(x') &+ a|x_0'|^\alpha p_2(x - x_0) \\ &\geq |x_0'|^\alpha\big(C_1|x' - x_0'|^2 + ap_2(x - x_0)\big) \\ &= |x_0'|^\alpha|x - x_0|^2\big(C_1\big|\frac{x' - x_0'}{|x - x_0|}\big|^2 + ap_2(\frac{x - x_0}{|x - x_0|})\big) \end{aligned}$$

$$\geq |x_0'|^\alpha |x-x_0|^2\big(C_1|\frac{x_1-x_{0,1}}{|x-x_0|}|^2 + ap_2(\frac{x-x_0}{|x-x_0|})\big)$$
$$= C_1|x_0'|^\alpha |x-x_0|^2\big(|\frac{x_1-x_{0,1}}{|x-x_0|}|^2 + C_2p_2(\frac{x-x_0}{|x-x_0|})\big)$$
$$\geq C_1C_3|x_0'|^\alpha|x-x_0|^2$$

which proves the lemma. □

Lemma 4. *Let $k \in \mathbb{N}$ such that $2k \geq 2+\alpha$ then there exist $b>0$ and $C>0$ such that for $r>0$, $x_0 \in \mathbb{R}^n$ and $x \in B_r(x_0)$*

$$w_{x_0'}(x') + \frac{b}{r^{2k-(2+\alpha)}}p_{2k}(x-x_0) \geq C\frac{|x-x_0|^{2k}}{r^{2k-(2+\alpha)}}. \tag{3.10}$$

Proof. Let $x \neq x_0$ and $k \in \mathbb{N}$ be such that $2k \geq 2+\alpha$. By Lemma 1 there exists $C_1 > 0$ such that for x,$x_0 \in \mathbb{R}^n$ we have

$$w_{x_0'}(x') \geq C_1|x'-x_0'|^{2+\alpha}. \tag{3.11}$$

By Lemma 2 there exist C_2,$C_3>0$ such that

$$|x_1|^{2k} + C_2p_{2k}(x) \geq C_3 \text{ for } x \in \partial B_1. \tag{3.12}$$

We have

$$\frac{r}{|x-x_0|} \geq 1 \text{ for } x \in B_r(x_0). \tag{3.13}$$

Now by (3.11), (3.12) and (3.13) taking $b = C_1C_2$ for $x_0 \in \mathbb{R}^n$ and $x \in B_r(x_0)$ we compute

$$w_{x_0'}(x') + \frac{b}{r^{2k-(2+\alpha)}}p_{2k}(x-x_0)$$
$$\geq C_1|x'-x_0'|^{2+\alpha} + \frac{b}{r^{2k-(2+\alpha)}}p_{2k}(x-x_0)$$
$$= \frac{|x-x_0|^{2k}}{r^{2k-(2+\alpha)}}\Big(C_1\frac{r^{2k-(2+\alpha)}}{|x-x_0|^{2k}}|x'-x_0'|^{2+\alpha} + \frac{b}{|x-x_0|^{2k}}p_{2k}(x-x_0)\Big)$$
$$= \frac{|x-x_0|^{2k}}{r^{2k-(2+\alpha)}}\Big(C_1\frac{r^{2k-(2+\alpha)}}{|x-x_0|^{2k-(2+\alpha)}}|\frac{x'-x_0'}{|x-x_0|}|^{2+\alpha} + bp_{2k}(\frac{x-x_0}{|x-x_0|})\Big)$$
$$\geq \frac{|x-x_0|^{2k}}{r^{2k-(2+\alpha)}}\Big(C_1|\frac{x'-x_0'}{|x-x_0|}|^{2+\alpha} + bp_{2k}(\frac{x-x_0}{|x-x_0|})\Big)$$
$$\geq \frac{|x-x_0|^{2k}}{r^{2k-(2+\alpha)}}\Big(C_1|\frac{x_1-x_{0,1}}{|x-x_0|}|^{2+\alpha} + bp_{2k}(\frac{x-x_0}{|x-x_0|})\Big)$$
$$= C_1\frac{|x-x_0|^{2k}}{r^{2k-(2+\alpha)}}\Big(|\frac{x_1-x_{0,1}}{|x-x_0|}|^{2+\alpha} + C_2p_{2k}(\frac{x-x_0}{|x-x_0|})\Big)$$
$$\geq C_1C_3\frac{|x-x_0|^{2k}}{r^{2k-(2+\alpha)}}$$

which proves the lemma. □

Proof of Theorem 1. Let x_0 and r be as in the statement of the theorem. Let $k \in \mathbb{N}$ be such that $2k \geq \alpha+2$ and $a, b > 0$ be as in Lemma 3 and 4.

We define

$$h(x) = u(x) - u(x_0) - \lambda\big(w_{x_0'}(x')$$
$$+ \frac{a}{2}|x_0'|^\alpha p_2(x-x_0) + \frac{b}{2r^{2k-(2+\alpha)}}p_{2k}(x-x_0)\big).$$

Then by (1.1) we have

$$\triangle h(x) = \triangle u(x) - \lambda\big(\triangle w_{x_0'}(x') + \frac{a}{2}|x_0'|^\alpha \triangle p_2(x-x_0) + \frac{b}{2r^{2k-(2+\alpha)}}\triangle p_{2k}(x-x_0)\big) = f - \lambda|x'|^\alpha \geq 0 \text{ in } \Omega. \tag{3.14}$$

Because $w_{x_0'}(x_0')=0$ we have

$$h(x_0) = -\lambda\big(w_{x_0'}(x_0') + \frac{a}{2}|x_0'|^\alpha p_2(0) + \frac{b}{2r^{2k-(2+\alpha)}}p_{2k}(0)\big) = 0. \tag{3.15}$$

For $x\in\Gamma$ we have $u(x)=0$, thus because of $u(x_0)>0$ and Lemma 3 and 4 we have

$$\begin{aligned} h(x) &= -u(x_0) - \lambda\big(w_{x_0'}(x') + \frac{a}{2}|x_0'|^\alpha p_2(x-x_0) + \frac{b}{2r^{2k-(2+\alpha)}}p_4(x-x_0)\big)\\ &= -u(x_0) - \lambda\Big(\frac{1}{2}\big(w_{x_0'}(x') + a|x_0'|^\alpha p_2(x-x_0)\big) + \frac{1}{2}\big(w_{x_0'}(x') + \frac{b}{r^{2k-(2+\alpha)}}p_{2k}(x-x_0)\big)\Big) < 0 \text{ on } \Gamma. \end{aligned} \tag{3.16}$$

By (3.14) we have that h is subharmonic in the domain $\Omega\cap B_r(x_0)$. Applying the maximum principle for the domain $\Omega\cap B_r(x_0)$ and the subharmonic function h we have

$$h(x_0) \leq \sup_{\partial(\Omega\cap B_r(x_0))} h(x). \tag{3.17}$$

By (3.15) and (3.17) we obtain

$$0 \leq \sup_{\partial(\Omega\cap B_r(x_0))} h(x). \tag{3.18}$$

Because

$$\partial(\Omega\cap B_r(x_0)) = (\partial\Omega\cap B_r(x_0)) \cup (\Omega\cap\partial B_r(x_0))$$

by (3.16) and (3.18) we obtain

$$0 \leq \sup_{\Omega\cap\partial B_r(x_0)} h(x). \tag{3.19}$$

By the definition of h, from (3.19) we get the inequality

$$u(x_0) + \lambda \inf_{\Omega\cap\partial B_r(x_0)}\big(w_{x_0'}(x') + \frac{a}{2}|x_0'|^\alpha p_2(x-x_0) + \frac{b}{2r^{2k-(2+\alpha)}}p_{2k}(x-x_0)\big) \leq \sup_{\Omega\cap\partial B_r(x_0)} u. \tag{3.20}$$

Now by Lemma 3 and 4 we obtain for $x\in\partial B_r(x_0)$

$$\begin{aligned} & w_{x_0'}(x') + \frac{a}{2}|x_0'|^\alpha p_2(x-x_0) + \frac{b}{2r^{2k-(2+\alpha)}}p_{2k}(x-x_0)\\ &= \frac{1}{2}\big(w_{x_0'}(x') + a|x_0'|^\alpha p_2(x-x_0)\big) + \frac{1}{2}\big(w_{x_0'}(x') + \frac{b}{r^{2k-(2+\alpha)}}p_{2k}(x-x_0)\big)\\ &\geq \frac{1}{2}C_2|x_0'|^\alpha|x-x_0|^2 + \frac{1}{2}C_3\frac{|x-x_0|^{2k}}{r^{2k-(2+\alpha)}} = \frac{1}{2}C_2|x_0'|^\alpha r^2 + \frac{1}{2}C_3 r^{2+\alpha}\\ &\geq C_4 r^2(r^\alpha + |x_0'|^\alpha). \end{aligned} \tag{3.21}$$

By (3.20) and (3.21) the theorem is proved. □

4. Optimal Growth

In the following theorem we prove the optimal growth of solutions.

Theorem 2 (Optimal Growth)**.** *There exists a $C>0$ (depending only on n, p and α) such that if for some $\Lambda>0$ we have*

$$|f(x)| \le \Lambda |x'|^\alpha \text{ for } x \in D \tag{4.1}$$

then for $B_r(x_0)\subset D$ we have

$$u(x) \le C\big(u(x_0) + \Lambda r^2 (r^\alpha + |x_0'|^\alpha)\big) \text{ for } x \in B_{\frac{r}{2}}(x_0).$$

Proof. Let us split $u = u_1 + u_2$ where u_1 is the solution to

$$\begin{cases} \triangle u_1 = \triangle u \text{ in } B_r(x_0), \\ u_1 = 0 \text{ on } \partial B_r(x_0) \end{cases}$$

and u_2 is the solution to

$$\begin{cases} \triangle u_2 = 0 \text{ in } B_r(x_0), \\ u_2 = u \text{ on } \partial B_r(x_0). \end{cases}$$

Let

$$\phi(x) = \Lambda\big(Cr^2(r^\alpha + |x_0'|^\alpha) - w_{x_0'}(x')\big)$$

with $C>0$ as in the second inequality in (2.4). Then because of $\triangle w_{x_0'}(x') = |x'|^\alpha$ and the second inequality in (2.4) we have

$$\begin{cases} -\triangle \phi = \Lambda |x'|^\alpha \text{ in } B_r(x_0), \\ \phi \ge 0 \text{ in } \overline{B}_r(x_0). \end{cases}$$

We have by (4.1)

$$-\Lambda|x'|^\alpha \le -\chi_{\{u>0\}} f \le \Lambda |x'|^\alpha$$

thus because $-\triangle u_1 = -\triangle u = -\chi_{\{u>0\}} f$, $-\triangle\phi = \Lambda|x'|^\alpha$, $u_1 = 0$ on $\partial B_r(x_0)$ and $\phi \ge 0$ on $\partial B_r(x_0)$ we have

$$\begin{cases} -\triangle(-\phi) \le -\triangle u_1 \le -\triangle \phi \text{ in } B_r(x_0), \\ -\phi \le u_1 \le \phi \text{ on } \partial B_r(x_0) \end{cases}$$

hence by the comparison principle we obtain

$$-\phi \le u_1 \le \phi \text{ in } B_r(x_0). \tag{4.2}$$

Because $-\triangle u_1 = -\triangle u$ and $u_1 = 0 \le u$ on $\partial B_r(x_0)$ we have $u_1 \le u$ in $B_r(x_0)$ and therefore

$$u_2 = u - u_1 \ge 0 \text{ in } B_r(x_0).$$

By the first inequality in (4.2) we have

$$u_2(x_0) = u(x_0) - u_1(x_0) \le u(x_0) + \phi(x_0).$$

Thus by the Harnack inequality

$$u_2(x) \le C_1 u_2(x_0) \le C_1\big(u(x_0) + \phi(x_0)\big) \text{ for } x \in B_{\frac{r}{2}}(x_0)$$

for a dimensional constant $C_1>0$.

Now because $w_{x_0'}(x_0') = 0$ and $w_{x_0'} \ge 0$ we have $\phi(x_0) = \Lambda C r^2(r^\alpha + |x_0'|^\alpha) \ge \phi(x)$ and by the second inequality in (4.2) we obtain the estimate

$$\begin{aligned} u(x) = u_1(x) + u_2(x) &\le \phi(x) + C_1(u(x_0) + \phi(x_0)) \\ &\le C_1 u(x_0) + (1+C_1)\phi(x_0) \\ &= C_1 u(x_0) + (1+C_1)\Lambda C r^2 (r^\alpha + |x_0'|^\alpha) \quad \text{for } x \in B_{\frac{r}{2}}(x_0) \end{aligned}$$

which proves the theorem. □

5. Porosity of the Free Boundary

In this section we prove that the free boundary Γ is locally porous in D. The definition of local porosity is as follows.

Definition 1. *For the sets A_1,$A_2 \subset \mathbb{R}^n$ we say that A_1 is locally porous in A_2 if for every compact set $K \subset\subset A_2$ there exists a constant $0 < \delta_K < 1$ with the property that every ball $B_r(x) \subset \mathbb{R}^n$ contains a smaller ball $B_{\delta_K r}(x_1)$ such that $B_{\delta_K r}(x_1) \subset B_r(x) \backslash (A_1 \cap K)$.*

Let us first mention some known results about the classical obstacle problem with nondegenerate force term.

The optimal growth estimate for the classical obstacle problem states that there exists a $C > 0$ (depending only on n) such that if for a constant $\Lambda > 0$, $|f| \leq \Lambda$ in D then for $B_r(x_0) \subset D$ we have

$$u(x) \leq C\big(u(x_0) + \Lambda r^2\big) \text{ for } x \in B_{\frac{r}{2}}(x_0). \tag{5.1}$$

The optimal nondegeneracy estimate for the classical obstacle problem states that there exists a $C > 0$ (depending on n) such that if for a constant $\lambda > 0$, $\lambda \leq f$ in D then for $x_0 \in \Omega$ and $B_r(x_0) \subset\subset D$ we have

$$u(x_0) + C\lambda r^2 \leq \sup_{\Omega \cap \partial B_r(x_0)} u.$$

By the continuity of u as a corollary we have that even if $x_0 \in \Gamma$ and $B_r(x_0) \subset\subset D$ then

$$C\lambda r^2 \leq \sup_{\Omega \cap \partial B_r(x_0)} u. \tag{5.2}$$

Lemma 5. *For each $a > 0$ there exists $0 < \delta_a < 1$ with the property that if $f > 0$ in B_1 and*

$$\frac{\sup_{B_1} f}{\inf_{B_1} f} \leq a$$

then for any $u \geq 0$ solution to the obstacle problem in B_1 with the force term f such that $0 \in \Gamma$, there exists $B_{\delta_a}(x_0) \subset \Omega$.

Proof. By (5.2) we have for some $x_0 \in \partial B_{\frac{1}{2}}$

$$C_1(\frac{1}{2})^2 \inf_{B_1} f \leq \sup_{\Omega \cap \partial B_{\frac{1}{2}}} u = u(x_0). \tag{5.3}$$

Let

$$0 < r \leq \frac{1}{6} \tag{5.4}$$

then for $x_1 \in B_r(x_0)$ we have

$$x_0 \in B_r(x_1) \subset B_{2r}(x_1) \subset B_{2r+|x_1-x_0|+|x_0|}(0) \subset B_{3r+\frac{1}{2}}(0) \subset B_1(0)$$

hence by (5.1) we have

$$u(x_0) \leq C_2(u(x_1) + (2r)^2 \sup_{B_1} f). \tag{5.5}$$

From (5.3) and (5.5) it follows that if r satisfies

$$r < \frac{1}{4}\sqrt{\frac{C_1}{C_2}\frac{1}{\sqrt{a}}} \tag{5.6}$$

then $B_r(x_0) \subset \Omega$.

Now if we define

$$\delta_a = \min\big(\frac{1}{6}, \frac{1}{8}\sqrt{\frac{C_1}{C_2}\frac{1}{\sqrt{a}}}\big)$$

then $r = \delta_a$ satisfies both (5.4) and (5.6). $\square$

In the following theorem we prove a general porosity result.

Theorem 3. *Let $u \geq 0$ be a solution of the obstacle problem* (1.1). *If there exists $0 < \delta_f < 1$ and $a > 0$ with the property that for all balls $B_r(x) \subset D$ there exists $B_{\delta_f r}(x_1) \subset B_r(x)$ such that*

$$\frac{\sup_{B_{\delta_f r}(x_1)} f}{\inf_{B_{\delta_f r}(x_1)} f} \leq a$$

then Γ is locally porous in D.

Proof. Let $K \subset\subset D$ then we should find a $0 < \delta_K < 1$ such that for arbitrary $B_r(x) \subset \mathbb{R}^n$ there exists $B_{\delta_K r}(x_0) \subset B_r(x) \backslash (\Gamma \cap K)$.

Now fix $K \subset\subset D$ and let $B_r(x) \subset \mathbb{R}^n$ be an arbitrary ball. We consider the two cases $r \geq \operatorname{diam}(K)$ and $r < \operatorname{diam}(K)$ separately.

Assume first that $r \geq \operatorname{diam}(K)$.

If $B_{\frac{r}{4}}(x - \frac{3}{4} r e_1) \cap (\Gamma \cap K) = \emptyset$ we set $x_0 = x - \frac{3}{4} r e_1$ and by taking $\delta_K \leq \frac{1}{4}$ we obtain $B_{\delta_K r}(x_0) \subset B_r(x) \backslash (\Gamma \cap K)$. If $B_{\frac{r}{4}}(x - \frac{3}{4} r e_1) \cap (\Gamma \cap K) \neq \emptyset$ then because $r \geq \operatorname{diam}(K)$ we have that $B_{\frac{r}{4}}(x + \frac{3}{4} r e_1) \cap (\Gamma \cap K) = \emptyset$ and similarly as in the previous case we set $x_0 = x + \frac{3}{4} r e_1$ and obtain $B_{\delta_K r}(x_0) \subset B_r(x) \backslash (\Gamma \cap K)$. This finishes the analysis of the case $r \geq \operatorname{diam}(K)$.

Assume now that $r < \operatorname{diam}(K)$.

If $B_{\frac{r}{2}}(x) \cap (\Gamma \cap K) = \emptyset$ then we set $x_0 = x$ and taking $\delta_K \leq \frac{1}{2}$ we have $B_{\delta_K r}(x_0) \subset B_{\frac{r}{2}}(x_0) = B_{\frac{r}{2}}(x) = B_{\frac{r}{2}}(x) \backslash (\Gamma \cap K) \subset B_r(x) \backslash (\Gamma \cap K)$.

If $B_{\frac{r}{2}}(x) \cap (\Gamma \cap K) \neq \emptyset$ then there exists $x_1 \in B_{\frac{r}{2}}(x) \cap (\Gamma \cap K)$.

Let us denote

$$\tilde{\delta}_K = \min\Big(\frac{1}{2}, \frac{\operatorname{dist}(K, D^c)}{\operatorname{diam}(K)}\Big).$$

Since

$$\operatorname{dist}(x_1, D^c) \geq \operatorname{dist}(K, D^c) \geq \tilde{\delta}_K \operatorname{diam}(K) > \tilde{\delta}_K r$$

we have

$$B_{\tilde{\delta}_K r}(x_1) \subset D. \tag{5.7}$$

Also we have

$$B_{\tilde{\delta}_K r}(x_1) \subset B_{\tilde{\delta}_K r + |x_1 - x|}(x) \subset B_r(x). \tag{5.8}$$

By (5.7) and (5.8) we have

$$B_{\tilde{\delta}_K r}(x_1) \subset D \cap B_r(x). \tag{5.9}$$

By the condition on f there exists $B_{\delta_f \tilde{\delta}_K r}(x_2) \subset B_{\tilde{\delta}_K r}(x_1)$ such that

$$\frac{\sup_{B_{\delta_f \tilde{\delta}_K r}(x_2)} f}{\inf_{B_{\delta_f \tilde{\delta}_K r}(x_2)} f} \leq a. \tag{5.10}$$

If $B_{\frac{1}{2}\delta_f \tilde{\delta}_K r}(x_2) \cap (\Gamma \cap K) = \emptyset$ then we set $x_0 = x_2$ and taking $\delta_K \leq \frac{1}{2}\delta_f \tilde{\delta}_K$ we have $B_{\delta_K r}(x_0) = B_{\delta_K r}(x_2) \subset B_{\frac{1}{2}\delta_f \tilde{\delta}_K r}(x_2) \subset B_r(x) \backslash (\Gamma \cap K)$.

If $B_{\frac{1}{2}\delta_f \tilde{\delta}_K r}(x_2) \cap (\Gamma \cap K) \neq \emptyset$ then there exists $x_3 \in B_{\frac{1}{2}\delta_f \tilde{\delta}_K r}(x_2) \cap (\Gamma \cap K)$ and we have

$$B_{\frac{1}{2}\delta_f \tilde{\delta}_K r}(x_3) \subset B_{\frac{1}{2}\delta_f \tilde{\delta}_K r + |x_3 - x_2|}(x_2) \subset B_{\delta_f \tilde{\delta}_K r}(x_2). \tag{5.11}$$

Now by (5.10) and (5.11) we have

$$\frac{\sup_{B_{\frac{1}{2}\delta_f \tilde{\delta}_K r}(x_3)} f}{\inf_{B_{\frac{1}{2}\delta_f \tilde{\delta}_K r}(x_3)} f} \leq \frac{\sup_{B_{\delta_f \tilde{\delta}_K r}(x_2)} f}{\inf_{B_{\delta_f \tilde{\delta}_K r}(x_2)} f} \leq a. \tag{5.12}$$

After a scaling, by Lemma 5 because of (5.12) we obtain

$$B_{\frac{1}{2}\delta_a\delta_f\tilde{\delta}_K r}(x_4) \subset B_{\frac{1}{2}\delta_f\tilde{\delta}_K r}(x_3)\backslash\Gamma. \tag{5.13}$$

So we have

$$B_{\frac{1}{2}\delta_a\delta_f\tilde{\delta}_K r}(x_4) \subset B_{\frac{1}{2}\delta_f\tilde{\delta}_K r}(x_3)\backslash\Gamma \subset B_{\delta_f\tilde{\delta}_K r}(x_2)\backslash\Gamma \subset B_{\tilde{\delta}_K r}(x_1)\backslash\Gamma \subset B_r(x)\backslash\Gamma \subset B_r(x)\backslash(\Gamma\cap K)$$

and by setting $x_0 = x_4$ and taking $\delta_K \leq \frac{1}{2}\delta_a\delta_f\tilde{\delta}_K$ the lemma is proved. □

Theorem 4 (Porosity)**.** *If for some* $0 < \lambda \leq \Lambda$ *the following inequalities hold*

$$\lambda|x'|^\alpha \leq f(x) \leq \Lambda|x'|^\alpha \text{ for } x \in D \tag{5.14}$$

then Γ *is locally porous in* D*.*

Proof. Let us check that the condition on f in Theorem 3 holds. Let $B_r(x) \subset D$ then because the set $\{x' = 0\}$ is porous with porosity constant $\frac{1}{2}$ there exists $B_{\frac{r}{2}}(x_1) \subset B_r(x)\backslash\{x' = 0\} \subset D\backslash\{x' = 0\}$. Now by (5.14) because $\frac{|x_1'|}{r} \geq \frac{1}{2}$ we have

$$\frac{\sup_{B_{\frac{1}{4}r}(x_1)} f}{\inf_{B_{\frac{1}{4}r}(x_1)} f} \leq \frac{\Lambda}{\lambda}\Big(\frac{\sup_{B_{\frac{1}{4}r}(x_1)} |x'|}{\inf_{B_{\frac{1}{4}r}(x_1)} |x'|}\Big)^\alpha \leq \frac{\Lambda}{\lambda}\Big(\frac{|x_1'| + \frac{1}{4}r}{|x_1'| - \frac{1}{4}r}\Big)^\alpha = \frac{\Lambda}{\lambda}\Big(\frac{\frac{|x_1'|}{r} + \frac{1}{4}}{\frac{|x_1'|}{r} - \frac{1}{4}}\Big)^\alpha \leq \frac{\Lambda}{\lambda}3^\alpha$$

so f satisfies the condition of Theorem 3 with $\delta_f = \frac{1}{4}$ and $a = \frac{\Lambda}{\lambda}3^\alpha$, and the theorem is proved. □

Acknowledgments: The author would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme Free Boundary Problems and Related Topics, where work on this paper was undertaken. The author is grateful to Henrik Shahgholian and Erik Lindgren for inspiring discussions.

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