Two Robin boundary value problems with opposite sign.

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Abstract

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Abstract

In this paper we discuss a boundary value and an eigenvalue problem with Robin boundary conditions of opposite sign. The eigenvalue problem arises in the study of the reaction-diffusion equation with dynamical boundary conditions. The dependence of the energy and the principle eigenvalues of the domains is investigated. The first and second domain variations are derived for nearly spherical domains. It is shown that in contrast to the classical Robin conditions the second variation changes sign and has singularities which depend on the eigenvalues of a Steklov problem. By means of the harmonic transplantation isoperimetric inequalities are derived for the principal eigenvalues in arbitrary domains.

1 Introduction

In this paper we discuss an elliptic boundary value problem with Robin boundary conditions of the opposite sign and a corresponding eigenvalue problem with the eigenvalue in the domain and on the boundary. The motivation comes from a classical model for the heat distribution in a body $\Omega \subset \mathbb{R}^3$, expressed by the heat equation

$$\partial_t T(x,t) - \Delta T(x,t) = 1 \text{ in } \Omega \times \mathbb{R}^+,$$

where ∂_t denotes partial differentiation with respect to t, Δ is the Laplace operator in $\mathbb{R}^3$ and the right-hand side describes a source of constant density 1. We suppose that the body is immersed into an ice bath and that on the boundary there is a regulating system with the following property: it the temperature drops below zero heat is carried into and otherwise out of the body, according to the law $\partial_\nu T = -\alpha T$ on $\partial\Omega$. Here ∂_ν denotes the outer normal derivative of Ω and $\alpha \in \mathbb{R}^+$ is a fixed positive number. This boundary condition is of the *Neumann-Robin* type. Observe that this flux condition is different from the classical Newton's law where α is of opposite sign.

We shall also consider the case where at the boundary a reaction takes place. Then on $\partial\Omega$, T satisfies the *dynamical boundary conditions* $\partial_\nu T = -\sigma\partial_t T$ where $\sigma < 0$.

In the first case the stationary solutions satisfy the elliptic boundary value problem

$$\Delta u + 1 = 0 \text{ in } \Omega, \quad \partial_\nu u = \alpha u \text{ on } \partial\Omega. \tag{1.1}$$

Under suitable regularity assumptions it has a unique solution provided α does not coincide with a eigenvalue $0 = \mu_1 < \mu_2 \leq \dots$ of the *Steklov problem*

$$\Delta\phi = 0 \text{ in } \Omega, \quad \partial_\nu\phi = \mu\phi \text{ on } \partial\Omega. \tag{1.2}$$

If $\alpha = \mu_i$ then the solutions are not unique. To (1.1) we associate the energy

$$\mathcal{E}(\Omega) := E(u,\Omega) \text{ where } E(\Omega, v) = \int\limits_\Omega |\nabla v|^2\,dx - 2\int\limits_\Omega v\,dx - \alpha\int\limits_{\partial\Omega} v^2\,dS.$$

It is well-known that the solution of (1.1) is a critical point of $E(v,\Omega)$ in $W^{1,2}(\Omega)$ in the sense that the Frechet derivative vanishes. However it is not a local extremum.

In this paper we study the dependence of $\mathcal{E}(\Omega)$ on the domain Ω. In contrast to the case $\alpha < 0$ the techniques based on the minimum principle $\mathcal{E}(\Omega) = \min_{W^{1,2}(\Omega)} E(v,\Omega)$ do not apply. Therefore, in the spirit of the previous investigations in [5] and [6], we compute the first and second order shape derivative and discuss the behavior under small perturbations of the domain which are volume preserving. This techniques fails if α is a Steklov eigenvalue. It turns out that the first variation vanishes if Ω is a ball. The second variation for the ball depends on $\oint_{\partial B_R}(v\cdot\nu)^2\,dS$ and on α. It is singular if $\alpha R = \mu_k$ and if the kth-Fourier coefficient of $(v\cdot\nu)$ with repect to orthonormal system of Steklov eigenfunctions does not vanish. At this point the second variation changes sign. This is in contrast to the case $\alpha < 0$ discussed in [5] where the ball is a local minimum for all $\alpha < 0$. In a recent paper Bucur and Giacomini [7] have shown that it is also a global minimum for the same choice of α. This first section is a completes the investigations in our previous paper [6] where the case $\alpha < 0$ is discussed.

The eigenvalue problem we are dealing with has its origin in the heat equation with dynamical boundary conditions. A standard method to find solutions is the separation of variables. In fact $T(x,t) = e^{-\lambda t}\phi(x)$ solves the homogeneous problem if ϕ is an eigenfunction of

$$\Delta\phi + \lambda\phi = 0 \text{ in } \Omega, \quad \partial_\nu \phi = \sigma\lambda\phi \text{ on } \partial\Omega. \tag{1.3}$$

It was shown in [4] that there exist two sequences of eigenvalues

$$\ldots \lambda_{-k} \leq \ldots \lambda_{-2} \leq \lambda_{-1} < 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \ldots,$$

such that $\lim_{n\to\infty} \lambda_{-n} = -\infty$ and $\lim_{n\to\infty} \lambda_n = \infty$. These eigenvalues can be characterized by a min-max principle, cf. [4].

In the second part of this paper we study the domain dependence of λ_1 and λ_{-1}. By means of the harmonic transplantation [8] a global upper bound for $\lambda_1(\Omega)$ and a global lower bound for λ_{-1} is derived. They are expressed in terms of the *harmonic radius* which plays an essential role. Both bounds are sharp. In [3] it was shown that for small σ the ball has for all domains of given volume the smallest λ_1. The situation is more involved if λ_{-1}. An investigation is carried out for nearly spherical domains of prescribed volume by means of the first and second domain variations.

Throughout this paper we shall assume that Ω is a bounded Lipschitz domain. This guarantees that the embedding $W^{1,2}(\Omega)$ into $L^2(\Omega)$ as well as the trace operator $\Gamma : W^{1,2}(\Omega) \to L^2(\partial\Omega)$ is compact. Under this condition both problems (1.1) and (1.3) are solvable in $W^{1,2}(\Omega)$ in the classical sense.

2 The boundary value problem

2.1 First domain variation

Consider problem (1.1) in a class of domains Ω_t which are small perturbations of Ω. We assume that

$$\overline{\Omega}_t = \left\{ y : y = x + tv(x) + \frac{t^2}{2} w(x) + o(t^2) \; : \; x \in \overline{\Omega} \right\}. \tag{2.1}$$

where v and w are vector fields such that

$$v, w : \overline{\Omega} \to \mathbb{R}^n \quad \text{are in } C^1(\overline{\Omega}).$$

The solutions of (1.1) in Ω_t will be denoted by $u(y;t)$ and the corresponding energy by $\mathcal{E}(t)$.

Example 1 *If Ω is the ball B_R of radius R then for all α there exists a unique radial solution*

$$u(r) = \frac{R^2}{2n} - \frac{R}{\alpha n} - \frac{r^2}{2n}.$$

The corresponding energy is

$$\mathcal{E}(B_R) = -\int_{B_R} u \, dx = |B_R| \left(\frac{R}{\alpha n} - \frac{R^2}{n(n+2)} \right).$$

If α is not a Steklov eigenvalue $u(y,;t)$ is continuous and continuously differentiable in t. Under this condition it was shown in [5] that the first domain variation is

$$\dot{\mathcal{E}}(0) = \int_{\partial\Omega} (v\cdot\nu)[|\nabla u|^2 - 2u - 2\alpha^2 u^2 - \alpha(n-1)u^2 H]\, dS,$$

where H is the mean curvature of $\partial\Omega$ and u is the solution of (1.1) in Ω.

Example 2 *If $\Omega = B_R$ then*

$$\dot{\mathcal{E}}(0) = \left(\frac{(n+1)R}{\alpha n^2} - \frac{R^2}{n^2}\right)\int_{\partial B_R} (v\cdot\nu)\, dS. \tag{2.2}$$

This leads to the following

Corollary 1 *Let Ω_t be a family of nearly spherical domains with prescribed volume $|\Omega_t| = |B_R|$. Then $\dot{\mathcal{E}}(0) = 0$.*

Proof The volume of Ω_t is given by $\int_{B_R} J(t)\, dx$ where $J(t)$ is the Jacobian determinant corresponding to the transformation $y : B_R \to \Omega_t$.The Jacobian matrix corresponding to this transformation is up to second order terms of the form

$$I + tD_v + \frac{t^2}{2}D_w, \text{ where } (D_v)_{ij} = \partial_j v_i \text{ and } \partial_j = \partial/\partial x_j.$$

By Jacobi's formula we have for small t

$$\begin{aligned} J(t) &:= \det\,(I + tD_v + \frac{t^2}{2}D_w) \\ &= 1 + t\,\mathrm{div}\, v + \frac{t^2}{2}\left((\mathrm{div}\, v)^2 - D_v : D_v + \mathrm{div}\ w\right) + o(t^2). \end{aligned} \tag{2.3}$$

Here we used the notation

$$D_v : D_v := \partial_i v_j \partial_j v_i.$$

Hence

$$|\Omega_t| = \int_{B_R} J(t)dx = |B_R| + t\int_{B_R} \mathrm{div}\ v dx + \frac{t^2}{2}\int_{B_R}((\mathrm{div} v)^2 - D_v : D_v + \mathrm{div} w)dx + o(t^2).$$

For the first variation we have only to require that y is volume preserving of the first order, that is

$$\int_{B_R} \mathrm{div}\, v\, dx = \int_{\partial B_R} (v\cdot\nu)\, dS = 0. \tag{2.4}$$

This together with (2.2) proves the assertion. □

A further consequence of (2.2) is the local monotonicity property

Corollary 2 *If $0 < \alpha R < n+1$ and $|\Omega_t| > |B_R|$ then $\dot{\mathcal{E}}(0) > 0$, otherwise if $\alpha R > n+1$ then $\dot{\mathcal{E}}(0) < 0$.*

Proof By our assumption we have $\int_{\partial B_R}(v\cdot\nu)\, dS > 0$ The sign of $\dot{\mathcal{E}}(0)$ depends therefore on the sign of $(n+1)\alpha R - (\alpha R)^2$. □

2.2 Second domain variation for nearly spherical domains

Corollary 1 gives rise to the following question: *is $\mathcal{E}(B_R)$ a local extremum among the family $(\Omega_t)_t$ of perturbed domains with the same volume as B_R?* The answer will be obtained from the second variation.

Let $u_t(x) := u(y(x);t)$ be the solution of $\Delta u + 1 = 0$ in Ω_t, $\partial_\nu u = \alpha u$ on $\partial\Omega_t$ transformed onto Ω. If u_t is differentiable then

$$\frac{d}{dt}u_t(x)|_{t=0} = u'(x) + v \cdot \nabla u,$$

where $u = u_0$ is the solution of (1.1) in B_R.

It was shown in [5] that formally u' solves the inhomogeneous boundary value problem

$$\Delta u' = 0 \quad \text{in } B_R \tag{2.5}$$

$$\partial_\nu u' = \alpha u' + \left(\frac{1-\alpha R}{n}\right) v \cdot \nu \qquad \text{on } \partial B_R. \tag{2.6}$$

Let us assume for the moment that such a u' exists. This is certainly the case if α does not coincide with a Steklov eigenvalue μ_i.

For the next result we consider perturbations which, in addition to the condition (2.4), satisfy the volume preservation of the second order, namely

$$\int_{B_R} ((\text{div } v)^2 - D_v : D_v + \text{div } w)\, dx = 0. \tag{2.7}$$

Set

$$Q(u') := \int_{B_R} |\nabla u'|^2\, dx - \alpha \int_{\partial B_R} u'^2\, dS.$$

Then the following formula was derived in [5].

Lemma 1 *Assume $\alpha \neq \mu_i$, (2.4) and (2.7). Put $\mathcal{S}(t) := |\partial\Omega_t|$. Then*

$$\ddot{\mathcal{E}}(0) = -2Q(u') + \frac{2R}{n^2}(1-\alpha R)\int_{\partial B_R} (v \cdot \nu)^2\, dS - \frac{R^2}{\alpha n^2}\ddot{\mathcal{S}}(0). \tag{2.8}$$

For a ball and for volume preserving perturbations the second variation of the surface is of the form

$$\ddot{\mathcal{S}}(0) = \oint_{\partial B_R} \left(|\nabla^\tau (v\cdot\nu)|^2 - \frac{(n-1)}{R^2}(v\cdot\nu)^2\right)\, dS,$$

where ∇^τ stands for the tangential gradient on ∂B_R.

2.2.1 Discussion of $\ddot{\mathcal{E}}(0)$

We write for short

$$\mathcal{F} := -2Q_g(u') + \frac{2R}{n^2}(1-\alpha R)\int_{\partial B_R}(v\cdot\nu)^2\,dS. \tag{2.9}$$

In order to estimate $\mathcal{F}$ we consider the Steklov eigenvalue problem (1.2) An elementary computation yields $\mu_1 = 0$, and $\mu_k = \frac{k-1}{R}$ (for $k \geq 2$ and counted without multiplicity). The second eigenvalue $\mu_2 = 1/R$ has multiplicity n and its eigenfunctions are $\frac{x_1}{R},\ldots,\frac{x_n}{R}$.

From now on we shall count the eigenvalues μ_i with their multiplicity, i.e. $\mu_2 = \mu_3 = \mu_{n+1} = 1/R$ and $\mu_{n+2} = 2/R$ etc. There exists a complete system of eigenfunctions $\{\phi_i\}_{i\geq 1}$ such that

$$\oint_{\partial B_R}\phi_i\phi_j\,dS = \delta_{ij}. \tag{2.10}$$

In view of the completeness we can write

$$u'(x) = \sum_{i=1}^{\infty} c_i\phi_i \quad \text{and} \quad (v\cdot\nu) = \sum_{i=1}^{\infty} b_i\phi_i.$$

Note that the first eigenfunction $\phi_1 \equiv$ constant. The condition

$$0 = \oint_{\partial B_R}(v\cdot\nu)\,dS = \oint_{\partial B_R}\phi_1(v\cdot\nu)\,dS$$

implies that $b_1 = 0$. From (2.6) we have also $c_1 = 0$. The coefficients b_i for $i \geq 2$ are determined from the boundary value problem (2.5), (2.6). In fact

$$b_i = \frac{n\,c_i\,(\mu_i-\alpha)}{1-\alpha\,R} \qquad \text{for} \quad i = 2,3,\ldots. \tag{2.11}$$

From the orthonormality conditions of the eigenfunctions it follows that

$$Q(u') = \sum_{i=2}^{\infty} c_i^2(\mu_i-\alpha).$$

Inserting this into (2.9) we get

$$\mathcal{F} = 2\sum_{2}^{\infty} c_i^2\,(\mu_i-\alpha)^2\left[\frac{R}{1-\alpha\,R} - \frac{1}{\mu_i-\alpha}\right]. \tag{2.12}$$

Discussion of $\ddot{\mathcal{S}}(0)$. Observe that

$$\mathcal{R}[\chi] = \frac{\oint_{\partial B_R}|\nabla^\tau\chi|^2\,dS}{\oint_{\partial B_R}\chi^2\,dS}$$

is the Rayleigh quotient of the Laplace- Beltrami operator on the $(n-1)$-dimensional sphere of radius R. Its eigenvalues are $k(n-2+k)/R^2$, $k \in \mathbb{N}^+$. For volume preserving perturbations of the first order $(v \cdot \nu)$ is orthogonal to the first eigenfunction $(\oint_{\partial B_R} (v \cdot \nu)\, dS = 0)$ and thus

$$\mathcal{R}[(v \cdot \nu)] \geq \frac{n-1}{R^2}.$$

Equality holds if and only if $(v \cdot \nu)$ belongs to the eigenspace spanned by $\{x_i\}_{i=1}^n$, $|x_i| = R$. This does not occur if we exclude small translations. Consequently $\ddot{\mathcal{S}}(0) > 0$.

Sign of $\ddot{\mathcal{E}}(0)$ Let us write for short

$$A := \frac{2(1-\alpha R)^2}{n^2} \text{ and } d_k := \frac{R}{1-\alpha\, R} - \frac{R}{k-1-\alpha R} = \frac{k-2}{(1-\alpha R)(k-1-\alpha R)}.$$

Clearly

$$d_k \begin{cases} > 0 & \text{if } \alpha R < 1 \text{ or } \alpha R > k-1, \\ < 0 & \text{if } 1 < \alpha R < k-1. \end{cases}$$

By (2.11)

$$\mathcal{F} = \frac{2(1-\alpha R)^2}{n^2} \sum_2^\infty b_i^2 \left[\frac{R}{1-\alpha\, R} - \frac{1}{\mu_i - \alpha} \right] = A \sum_3^\infty \tilde{b}_k^2 d_k.$$

Remark 1 *The last equality is read in the following way. Assume for some $k \geq 2$ there are m identical eigenvalues*

$$\mu_k = \mu_{k_1} = \ldots = \mu_{k_m}$$

The sum to the left thus contains the finite sum

$$\sum_{j=1}^m b_{k_j}^2 \left[\frac{R}{1-\alpha\, R} - \frac{1}{\mu_k - \alpha} \right].$$

The sum on the right side abbreviates this by setting $\tilde{b}_k^2 = \sum_{j=1}^m b_{k_j}^2$.

Let μ be the largest eigenvalue such that $\mu_\alpha := \max\{\mu_i : \mu_i < \alpha\}$. Suppose that $0 < p-1 < \alpha R < p$. Then $d_k > 0$ for $k = 1, \ldots, p$ and $d_k < 0$ for $k > p$. Then $\mathcal{F}$ can be split into a positive and negative part,

$$\mathcal{F} = \mathcal{F}^+ + \mathcal{F}^-, \text{ where } \mathcal{F}^+ = A \sum_3^p \tilde{b}_k^2 d_k \geq 0, \quad \mathcal{F}^- = A \sum_{p+1}^\infty \tilde{b}_k^2 d_k \leq 0.$$

If $0 < \alpha R < 1$ then $d_k > 0$ for all $k = 3, \ldots$. Hence

$$\mathcal{F} = \mathcal{F}^+ = A \sum_3^\infty \tilde{b}_k^2 d_k. \tag{2.13}$$

Moreover

$$\mathcal{F}^+ \leq A\, d_p \sum_3^p \tilde{b}_k^2 \leq A\, d_p \int_{\partial B_R} (v\cdot\nu)^2\, dS \tag{2.14}$$

$$\mathcal{F}^- \geq A\, d_{p+1} \sum_{p+1}^{\infty} \tilde{b}_k^2 \geq A\, d_{p+1} \int_{\partial B_R} (v\cdot\nu)^2\, dS,$$

$$\mathcal{F} \leq A \left[\frac{R}{1-\alpha R} - \frac{1}{\mu_\alpha - \alpha}\right] \oint_{\partial B_R} (v\cdot\nu)^2\, dS.$$

Recall that

$$\ddot{\mathcal{E}}(0) = \mathcal{F}^+ + \mathcal{F}^- - \frac{R^2}{\alpha n^2}\ddot{\mathcal{S}}(0).$$

Theorem 1 *Suppose that the perturbations y are different from translations and rotations. Under the assumptions of Lemma 1 and if $0 < \alpha R < \epsilon$ for ϵ sufficiently small then $\ddot{\mathcal{E}}(0) < 0$. The ball is a local maximizer for $\mathcal{E}(t)$.*

Proof By (2.13), $\mathcal{F}^+$ is bounded from above for $\alpha R < \epsilon$. Hence the last term dominates and $\ddot{\mathcal{S}}(0) < 0$. □

Sign changes occur if α is in a neighborhood of μ_α. More precisely we have

Theorem 2 *Let $\tilde{b}_p \neq 0$. Suppose that α is close to $\mu_\alpha = p-1$. Then there exists $\epsilon_0 > 0$ sufficiently small and depending only on v and w such that $\ddot{\mathcal{E}}(0) > 0$ if $p-1 < \alpha < p-1+\epsilon_0$. On the other hand if $\tilde{b}_{p+1} \neq 0$ there exists $\epsilon_1 > 0$ sufficiently small such that $\ddot{\mathcal{E}}(0) < 0$ if $p-\epsilon_1 < \alpha < p$. If $\tilde{b}_p = 0$ in the first case then $\ddot{\mathcal{E}}(0)$ can be positive or negative depending on $(v\cdot\nu)$. The same is true in the second case if $\tilde{b}_{p+1} = 0$.*

Proof The expressions containing d_p in the first case and, d_{p+1} in the second case dominate. □

In order to get a sharper upper bound for $\ddot{\mathcal{E}}(0)$ in terms of v we impose the "barycenter" condition

$$\oint_{\partial B_R} x\,(v(x)\cdot\nu(x))\, dS = 0. \tag{2.15}$$

Setting $N = (v\cdot\nu)$ we get

$$\oint_{\partial B_R} |\nabla^\tau N|^2\, dS \geq \frac{2n}{R^2} \oint_{\partial B_R} N^2\, dS.$$

Thus $\ddot{\mathcal{S}}(0) \geq \frac{n+1}{R^2} \oint_{\partial B_R} N^2\, dS \geq \frac{n+1}{R^2} \sum_3^p \tilde{b}_k^2$.

We then get the following upper bound for $\ddot{\mathcal{E}}(0)$.

$$\ddot{\mathcal{E}}(0) \leq \left\{ -\frac{n+1}{\alpha\, n^2} + 2\,\frac{R(1-\alpha\, R)}{n^2} - \frac{2(1-\alpha\, R)^2}{n^2(\mu_\alpha - \alpha)} \right\} \sum_3^p \tilde{b}_k^2. \tag{2.16}$$

Remark 2 *Observe that since* (2.15) *is assumed,* $b_i = 0$ *for* $i = 1, \dots, n$*. By* (2.11) *this implies* $c_i = 0$ *for* $i = 1, \dots, n$*. If*

$$p - 1 < \alpha R < p$$

then

$$\mu_\alpha = \frac{p-1}{R}.$$

Consequently

$$\ddot{\mathcal{E}}(0) \le -\frac{1}{\alpha n^2} \left\{ n + 1 - \frac{2 \alpha R (1 - \alpha R)(p-2)}{p - 1 - \alpha R} \right\} \sum_3^p \tilde{b}_k^2. \tag{2.17}$$

If $p = 2$ then it follows immediately that $\ddot{\mathcal{E}}(0) \le 0$. Let $p > 2$. The second term in the brackets is monotone decreasing in αR. Thus

$$\inf_{\alpha R \in (p-1, p)} \frac{2 \alpha R (1 - \alpha R)(p-2)}{p - 1 - \alpha R} = 2p(p^2 - 3p + 2) \ge 12.$$

Hence if $n \ge 12$, $\ddot{\mathcal{E}}(0) < 0$ for αR close to p.

Example 3 *Let* $\Omega_t \subset \mathbb{R}^2$ *be the ellipse whose boundary* $\partial\Omega_t$ *is given by*

$$\left\{ \frac{R\cos(\theta)}{1+t}, (1+t)R\sin(\theta) \right\},$$

where (r, θ) *are the polar coordinates in the plane. This ellipse has the same area as the circle* B_R *and can be interpreted as a perturbation described in* (2.1)*. We have* $y = x + t(-x_1, x_2) + \frac{t^2}{2}(x_1, 0) + o(t^2)$*. The eigenvalues and eigenfunctions of the Steklov eigenvalue problem* (1.3) *are*

$$\mu = \frac{m}{R} + \alpha \text{ and } \phi = r^m(a_1 \cos(m\theta) + a_2 \sin(m\theta)).$$

We have

$$(v \cdot \nu) = -R\cos(2\theta) = b_3 \phi_3,$$

and

$$\ddot{S}(0) = \oint_{\partial B_R} \left(|\nabla^\tau N|^2 - \frac{N^2}{R^2} \right) dS = 3\pi R.$$

A straightforward computation yields

$$\ddot{\mathcal{E}}(0) = \left[-\frac{3}{4\alpha} + \frac{R(1 - \alpha R)}{2(2 - \alpha R)} \right] \oint_{\partial B_R} (v \cdot \nu)^2 \, dS.$$

with $\oint_{\partial B_R} (v \cdot \nu)^2 \, dS = \pi R^3$*. From this expression it follows immediately that*

$$\ddot{\mathcal{E}}(0) \begin{cases} > 0 \text{ if } \alpha R > 2 \\ < 0 \text{ if } \alpha R < 2. \end{cases}$$

In this example there is only one coefficient $\tilde{b}_3$ *which does not vanish. In accordance with Theorem 2,* $\ddot{\mathcal{E}}(0)$ *has only one singularity at* $\alpha = \mu_3 = \frac{2}{R}$*.*

3 Eigenvalue problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. We consider the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega, \qquad \partial_\nu u = \lambda\, \sigma\, u \quad \text{in } \partial\Omega. \tag{3.1}$$

σ is a negative number. Clearly $\lambda_0 = 0$ is a eigenvalue and the corresponding eigenfunction is $u_0 = const.$. We define

$$\sigma_0(\Omega) := -\frac{|\Omega|}{|\partial\Omega|}.$$

In [4] [3] the following properties were shown.

(P1) There exists an infinite sequence of positive eigenvalues $(\lambda_n)_n$ with $\lim_{n\to\infty} \lambda_n = \infty$.

(P2) There exists an infinite sequence of negative eigenvalues $(\lambda_{-n})_n$ with $\lim_{n\to\infty} \lambda_{-n} = -\infty$.

(P3) If $\sigma < \sigma_0(\Omega)$, then λ_1 is simple and the corresponding eigenfunction u_1 is of constant sign.

(P4) If $0 > \sigma > \sigma_0(\Omega)$, then λ_{-1} is simple and the corresponding eigenfunction u_{-1} is of constant sign.

(P5) Let B_R be a ball such that $|B_R| = |\Omega|$. If $\sigma < \sigma_0(B_R)$, then $\lambda_1(\Omega) \geq \lambda_1(B_R)$.

(P6) For any domain Ω (with $|B_R| = |\Omega|$), there exist a number $\hat{\sigma} \in (\sigma_0(\Omega), 0)$ such that $\lambda_{-1}(\Omega) \geq \lambda_{-1}(B_R)$ whenever $\sigma \in (\sigma_0(\Omega), \hat{\sigma})$.

Remark 3 *For (P5) we note that the condition $\sigma < \sigma_0(B_R)$ is more restrictive than the condition $\sigma < \sigma_0(\Omega)$ if $|\Omega| = |B_R|$. This is a consequence of the isoperimetric inequality.*

From (3.1) we obtain a representation formula for λ_{-1} and λ_1. In fact, multiplying the equation for u_1 (resp. u_{-1}) with u_1 (resp. u_{-1}) and integrating over Ω we obtain

$$\lambda_i(\Omega) = \frac{\int\limits_\Omega |\nabla u_i|^2\, dx}{\int\limits_\Omega u_i^2\, dx + \sigma \int\limits_{\partial\Omega} u_i^2\, dS} \qquad i = -1, 1\ .$$

Remark 4 *Note that from* (3.1) *we deduce*

$$\int\limits_\Omega |\nabla u_i|^2\, dx = 0$$

and hence $u_i = const.$, if the denominator

$$\int\limits_\Omega u_i^2\, dx + \sigma \int\limits_{\partial\Omega} u_i^2\, dS = 0.$$

From now on λ and u will denote λ_i and u_i in either case $i = \pm 1$.

Let $(\Omega_t)_t$ be a smooth family of small perturbations of Ω as described in (2.1). In particular they are volume preserving in the sense of (2.4) and (2.7). Denote by u_t the solution to

$$\Delta u_t + \lambda(\Omega_t) u_t = 0 \quad \text{in } \Omega_t, \qquad \partial_{\nu_t} u_t = \lambda(\Omega_t)\, \sigma\, u_t \quad \text{in } \partial\Omega_t. \tag{3.2}$$

Here $\lambda(\Omega_t)$ has the representation

$$\lambda(t) := \lambda(\Omega_t) = \frac{\int\limits_{\Omega_t} |\nabla u_t|^2 \, dy}{\int\limits_{\Omega_t} u_t^2 \, dy + \sigma \int\limits_{\partial\Omega_t} u_t^2 \, dS_t} \, . \tag{3.3}$$

As in [5] we transform the integrals onto Ω for small t and differentiate $\lambda(t)$ with respect to t. Then we get

$$\dot{\lambda}(0) = \int\limits_{\partial\Omega} \left(|\nabla u|^2 - \lambda\, u^2 - \lambda^2\, \sigma^2\, u^2 - (n-1)\lambda\, \sigma\, H \right) (v \cdot \nu) \, dS \, .$$

In particular $\Omega = B_R$ implies that $\dot{\lambda}(0) = 0$. We are interested in extremality properties of the ball. Thus let $\Omega = B_R$ from now on and let $\sigma < \sigma_0(B_R)$.

We consider the following boundary value problem.

$$\Delta u' + \lambda\, u' = 0 \quad \text{in } B_R \qquad \partial_\nu u' - \lambda\, \sigma\, u' = k(R)(v \cdot \nu) \quad \text{in } \partial B_R \tag{3.4}$$

where

$$k(R) := \lambda\, u(R) \left(1 + \frac{(n-1)\, \sigma}{R} + \lambda\, \sigma^2 \right) . \tag{3.5}$$

We determine the sign of $k(R)$. In this we follow the proof of Lemma 3 in [6]. For the sake of completeness we give the details.

Lemma 2 *Let $k(R)$ be given by* (3.5) *and let $u(r)$ be the positive radial function in the case $\lambda = \lambda_1$ or $\lambda = \lambda_{-1}$. Then we have*

$$\begin{aligned} k(R) &> 0 \qquad && \textit{if} \quad \lambda = \lambda_1 \\ k(R) &< 0 \qquad && \textit{if} \quad \lambda = \lambda_{-1}. \end{aligned}$$

Proof In the radial case either eigenfunction satisfies the differentia equation

$$u_{rr} + \frac{n-1}{r}\, u_r + \lambda\, u(r) = 0 \quad \text{in } (0, R), \qquad u'(R) = \lambda\, \sigma\, u(R).$$

We set $z = \frac{u_r}{u}$ and observe that

$$\frac{dz}{dr} + z^2 + \frac{n-1}{r} z + \lambda = 0 \text{ in } (0, R).$$

At the endpoint
$$\frac{dz}{dr}(R) + \lambda^2\,\sigma^2 + \frac{(n-1)}{R}\,\lambda\,\sigma + \lambda = 0.$$
We know that $z(0) = 0$ and $z(R) = \lambda\,\sigma$. Note that

$$z_r(0) = -\lambda. \tag{3.6}$$

We distinguish two cases.

The case $\lambda = \lambda_1(B_R)$.
In that case we have (also from (3.6))

$$z(0) = 0\,, \qquad z(R) = \lambda_1\,\sigma < 0\,, \qquad z_r(0) = -\lambda_1 < 0. \tag{3.7}$$

Thus $z(r)$ decreases near 0. We determine the sign of $z_r(R)$. If $z_r(R) \geq 0$ then because of (3.7) there exists a number $\rho \in (0, R)$ such that $z_r(\rho) = 0$, $z(\rho) < 0$ and $z_{rr}(\rho) \geq 0$. From the equation we get $z_{rr}(\rho) = \frac{n-1}{\rho^2} z(\rho) < 0$ which leads to a contradiction. Consequently

$$z_r(R) = -(\lambda_1^2\,\sigma^2 + \frac{(n-1)}{R}\,\lambda_1\,\sigma + \lambda_1) < 0.$$

This implies $k(R) > 0$ in the case $\lambda = \lambda_1(B_R)$.

The case $\lambda = \lambda_{-1}(B_R)$.
In that case we have (see also (3.6))

$$z(0) = 0\,, \qquad z(R) = \lambda_{-1}\,\sigma > 0\,, \qquad z_r(0) = -\lambda_{-1} > 0. \tag{3.8}$$

Thus $z(r)$ increases near 0. We again determine the sign of $z_r(R)$. If $z_r(R) \leq 0$ then because of (3.8) there exists a number $\rho \in (0, R)$ such that $z_r(\rho) = 0$, $z(\rho) > 0$ and $z_{rr}(\rho) \leq 0$. From the equation we get $z_{rr}(\rho) = \frac{n-1}{\rho^2} z(\rho) > 0$ which is contradictory. Consequently

$$z_r(R) = -(\lambda_{-1}^2\,\sigma^2 + \frac{(n-1)}{R}\,\lambda_{-1}\,\sigma + \lambda_{-1}) > 0.$$

This also implies $k(R) < 0$ in the case $\lambda = \lambda_{-1}(B_R)$. $\square$

To (3.4) we associate the quadratic form

$$Q(u') := \int_{B_R} |\nabla u'|^2\,dx - \lambda \int_{B_R} u'^2\,dx - \lambda\,\sigma \int_{\partial B_R} u'^2\,dS.$$

Computations as in [5] lead to the following formula for $\Omega = B_R$. These are the same computations which lead to formula (2.8) in Chapter 2.

$$\ddot{\lambda}(0) = -\lambda^2\,\sigma\,u^2(R)\,\ddot{S}(0) + \lambda\,\mathcal{F}$$

where

$$\mathcal{F} = -2Q(u') - 2\,\lambda\,\sigma\,u(R)\,k(R) \int\limits_{\partial B_R} (v\cdot\nu)^2\,dS.$$

In the remaining part of this chapter we will discuss the sign of $\ddot{\lambda}(0)$.

We modify the approach in Chapter 2.2.1 and consider the following Steklov eigenvalue problem

$$\begin{aligned} \Delta\phi + \lambda\,\phi &= 0 \text{ in } B_R, \\ \partial_\nu\phi - \lambda\,\sigma\,\phi &= \mu\,\phi \text{ on } \partial B_R. \end{aligned} \tag{3.9}$$

There exists an infinite number of eigenvalues

$$\mu_1 < \mu_2 \le \mu_3 \le \dots \lim_{i\to\infty} \mu_i = \infty.$$

and a complete system of eigenfunctions $\{\phi_i\}_{i\ge1}$. See also Chapter 7 in [5]. With the same notation we have $u' = \sum_{i=1}^{\infty} c_i\,\phi_i$ for the solution of (3.4) and we get

$$\ddot{\lambda}(0) = \lambda^2\,|\sigma|\,u^2(R)\,\ddot{S}(0) + 2\sum_{i=1}^{\infty} c_i^2\,\mu_i^2 \left(\frac{\lambda^2|\sigma|\,u(R)}{k(R)} - \frac{\lambda}{\mu_i}\right). \tag{3.10}$$

Let $\lambda = \lambda_1(B_R)$. Then (3.10) is precisely the expression in [5] Chapter 7.2.2 subsection 2. where $\alpha > 0$ is now replaced by $\lambda_1(B_R)\,|\sigma| > 0$. Thus we conclude

$$\ddot{\lambda}_1(0) \ge \lambda_1^2\,|\sigma|\,u^2(R)\,\ddot{S}(0) > 0. \tag{3.11}$$

This is a local version of (P5).

The case $\lambda = \lambda_{-1}(B_R) < 0$ is more involved. In that case $\mathcal{F} < 0$ since $k(R) < 0$ by Lemma 2. It is an open problem to show that also in this case $\ddot{\lambda}_{-1}(0) > 0$ - at least for σ close to $\sigma_0(B_R) = -\frac{R}{n}$. This conjecture is motivated by (P6).

4 Harmonic transplantation

The eigenvalues λ_1 (resp. λ_{-1}) have a variational characterization for $\sigma \ne \sigma_0(\Omega)$ (see [4]). Let

$$\mathcal{K}_\Omega := \{v \in W^{1,2}(\Omega) \,:\, \int\limits_\Omega |\nabla v|^2\,dx = 1\,, \int\limits_\Omega v\,dx + \sigma \int\limits_{\partial\Omega} v\,dS = 0\}. \tag{4.1}$$

Then for $\sigma < \sigma_0(\Omega)$

$$0 \le \lambda_1(\Omega) = \frac{1}{\sup\left\{\int\limits_\Omega v^2\,dx - |\sigma| \int\limits_{\partial\Omega} v^2\,dS \,:\, v \in \mathcal{K}_\Omega\right\}}$$

has a unique minimizer (of constant sign). The same holds in the case $0 > \sigma > \sigma_0(\Omega)$ for

$$0 \geq \lambda_{-1}(\Omega) = \frac{1}{\inf\left\{\int\limits_{\Omega} v^2\,dx - |\sigma| \int\limits_{\partial\Omega} v^2\,dS \;:\; v \in \mathcal{K}_\Omega\right\}}.$$

We shortly review the method of harmonic transplantation which has been deviced by Hersch[8], (cf. also [2]). In [6] it applied to some shape optimization problems involving Robin eigenvalues. To this end we need the Green's function with Dirichlet boundary condition

$$G_\Omega(x,y) = \gamma(S(|x-y|) - H(x,y)), \tag{4.2}$$

where

$$\gamma = \begin{cases} \frac{1}{2\pi} & \text{if } n = 2 \\ \frac{1}{(n-2)|\partial B_1|} & \text{if } n > 2 \end{cases} \qquad \text{and} \qquad S(t) = \begin{cases} -\ln(t) & \text{if } n = 2 \\ t^{2-n} & \text{if } n > 2. \end{cases} \tag{4.3}$$

For fixed $y \in \Omega$ the funcion $H(\cdot, y)$ is harmonic.

Definition 1 *The harmonic radius at a point $y \in \Omega$ is given by*

$$r(y) = \begin{cases} e^{-H(y,y)} & \text{if } n = 2, \\ H(y,y)^{-\frac{1}{n-2}} & \text{if } n > 2. \end{cases}$$

The harmonic radius vanishes on the boundary $\partial\Omega$ and takes its maximum r_Ω at the harmonic center y_h. It satisfies the isoperimetric inequality [8],[2]

$$|B_{r_\Omega}| \leq |\Omega|. \tag{4.4}$$

Note that $G_{B_R}(x,0)$ is a monotone function in $r = |x|$. Consider any radial function $\phi : B_{r_\Omega} \to \mathbb{R}$ thus $\phi(x) = \phi(r)$. Then there exists a function $\omega : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(x) = \omega(G_{B_{r_\Omega}}(x,0)).$$

To $\phi(x)$ we associate the transplanted function $U : \Omega \to \mathbb{R}$ defined by $U(x) = \omega(G_\Omega(x, y_h))$. Then for any positive function $f(s)$, the following inequalities hold true

$$\int_\Omega |\nabla U|^2\,dx = \int_{B_{r_\Omega}} |\nabla\phi|^2\,dx \tag{4.5}$$

$$\int_\Omega f(U)\,dx \geq \int_{B_{r_\Omega}} f(\phi)\,dx. \tag{4.6}$$

$$\int_\Omega f(U)\,dx \leq \gamma^n \int_{B_{r_\Omega}} f(\phi)\,dx, \tag{4.7}$$

where

$$\gamma = \left(\frac{|\Omega|}{|B_{r_\Omega}|}\right)^{\frac{1}{n}}.$$

For a proof see [8] or [2] and in particular [6] for a proof of (4.7). The following observation will be useful in the sequel.

Remark 5 *Since U is constant on $\partial\Omega$ ($U = U(\partial\Omega)$) and since ϕ is radial we deduce*

$$\int\limits_{\partial\Omega} U^2\, dS = U^2(\partial\Omega)\, |\partial\Omega| = \phi^2(r_\Omega)\, |\partial B_{r_\Omega}|\, \frac{|\partial\Omega|}{|\partial B_{r_\Omega}|} = \frac{|\partial\Omega|}{|\partial B_{r_\Omega}|} \int\limits_{\partial B_{r_\Omega}} \phi^2\, dS.$$

Let u be a positive radial eigenfunction of $\lambda_1(B_R)$. Then $U \in \mathcal{K}_\Omega$ since $u \in \mathcal{K}_{B_{r_\Omega}}$ and (4.5) holds.

The case $\sigma < \sigma_0 < 0$. By the variational characterization we observe that

$$0 \leq \lambda_1(\Omega) =: \lambda_1^\sigma(\Omega) \leq \frac{1}{\int\limits_\Omega U^2\, dx - |\sigma| \int\limits_{\partial\Omega} U^2\, dS}.$$

We use (4.6) for the first integral in the denominator and Remark 5 for the second.

$$0 \leq \lambda_1^\sigma(\Omega) \leq \frac{1}{\int\limits_{B_{r_\Omega}} u^2\, dx - |\sigma|\, \frac{|\partial\Omega|}{|\partial B_{r_\Omega}|} \int\limits_{\partial B_{r_\Omega}} u^2\, dS}.$$

Set

$$\sigma' = \sigma\, \frac{|\partial\Omega|}{|\partial B_{r_\Omega}|}$$

Then we have

$$0 \leq \lambda_1^\sigma(\Omega) \leq \lambda_1^{\sigma'}(B_{r_\Omega}).$$

Alternatively we may write

$$\frac{1}{\int\limits_{B_{r_\Omega}} u^2\, dx - |\sigma|\, \frac{|\partial\Omega|}{|\partial B_{r_\Omega}|} \int\limits_{\partial B_{r_\Omega}} u^2\, dS}$$

$$= \frac{1}{\int\limits_{B_{r_\Omega}} u^2\, dx - |\sigma| \int\limits_{\partial B_{r_\Omega}} u^2\, dS} \left(1 - \frac{|\sigma| \int\limits_{\partial B_{r_\Omega}} u^2\, dS}{\int\limits_{B_{r_\Omega}} u^2\, dx - |\sigma| \int\limits_{\partial B_{r_\Omega}} u^2\, dS} \left(\frac{|\partial\Omega|}{|\partial B_{r_\Omega}|} - 1 \right) \right)^{-1}$$

Thus

$$0 \leq \lambda_1^\sigma(\Omega) \leq \lambda_1^\sigma(B_{r_\Omega}) \underbrace{\left(1 - \frac{|\sigma| \int\limits_{\partial B_{r_\Omega}} u^2\, dS}{\lambda_1^\sigma(B_{r_\Omega})} \left(\frac{|\partial\Omega|}{|\partial B_{r_\Omega}|} - 1 \right) \right)^{-1}}_{=:A(\Omega,\sigma)}. \tag{4.8}$$

Note that the multiplicative term on the right hand side is close to one if the isoperimetric defect $\frac{|\partial\Omega|}{|\partial B_{r_\Omega}|} - 1$ is small. In fact for $\Omega = B_R$ we have $r_\Omega = R$ and thus $A(B_R, \sigma) = 1$.

The case $\sigma_0 < \sigma < 0$. Again by the variational characterization we have

$$0 \geq \lambda_{-1}(\Omega) \geq \frac{1}{\int\limits_{\Omega} U^2\,dx - |\sigma| \int\limits_{\partial\Omega} U^2\,dS}.$$

We apply (4.7) to the first integral in the denominator and again Remark 5 to the second.

$$\begin{aligned} 0 \geq \lambda_{-1}(\Omega) &\geq \frac{1}{\gamma^n \int\limits_{B_{r_\Omega}} u^2\,dx - |\sigma| \frac{|\partial\Omega|}{|\partial B_{r_\Omega}|} \int\limits_{\partial\Omega} u^2\,dS} \\ &\geq \frac{1}{\gamma^n \int\limits_{B_{r_\Omega}} u^2\,dx - |\sigma|\,\gamma^n \int\limits_{\partial\Omega} u^2\,dS}. \end{aligned}$$

The last inequality holds since the isoperimetric inequality

$$\frac{|\partial\Omega|}{|\partial B_{r_\Omega}|} \geq \frac{|\Omega|}{|B_{r_\Omega}|} = \gamma^n$$

was applied. Thus

$$0 \geq \lambda_{-1}(\Omega) \geq \frac{1}{\gamma^n} \lambda_{-1}(B_{r_\Omega})$$

We may rewrite this as

$$|\Omega|\,\lambda_{-1}(\Omega) \geq |B_{r_\Omega}|\,\lambda_{-1}(B_{r_\Omega}).$$

This proves the following theorem.

Theorem 3 *Let Ω be any domain for which the trace operator $W^{1,2}(\Omega) \to L^2(\partial\Omega)$ is well defined. Let $\lambda_{\pm 1}(\Omega)$ be the first positive (negative) eigenvalue of* (3.1) *and let r_Ω be the harmonic radius of Ω. Then the following optimality result holds.*

1) *In the case $\sigma < \sigma_0(\Omega) < 0$ we have $0 \leq \lambda_1(\Omega) \leq A(\Omega,\sigma)\lambda_1(B_{r_\Omega})$, where the factor A is given in* (4.8) *and $A = 1$ for the ball.*
2) *In the case $\sigma_0(\Omega) < \sigma < 0$ we have $0 \geq |\Omega|\,\lambda_{-1}(\Omega) \geq |B_{r_\Omega}|\,\lambda_{-1}(B_{r_\Omega})$.*

Remark 6 *It is interesting to compare 1) in Theorem 3 with (P5). We get the following two sided bounds.*
If B_R is a ball of equal volume with Ω and if $\sigma < \sigma_0(B_R) < 0$ then

$$\lambda_1(B_R) \leq \lambda_1(\Omega) \leq A(\Omega,\sigma)\lambda_1(B_{r_\Omega}).$$

Equality holds for the ball.

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