COMPETITION PHENOMENA FOR DIFFERENCE EQUATIONS WITH OSCILLATORY NONLINEARITIES

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ABSTRACT. In this paper, we study the following discrete boundary value problem

$$\begin{cases} -\Delta(\Delta u(k-1)) = \lambda a(k)u(k)^p + f(u(k)) \text{ for all } k \in [1,T], \\ u(0) = u(T+1) = 0, \end{cases}$$

where $f : [0,+\infty) \to \mathbb{R}$ is a continuous function oscillating near the origin or at infinity. By using direct variational methods, we prove that, when f oscillates near the origin, the problem admits a sequence of non-negative, distinct solutions which converges to 0 if $p \geqslant 1$ and at least a finite number of solutions if $0 < p < 1$. While, when f oscillates at infinity, the converse holds true, that is, there is a sequence of non-negative, distinct solutions which converges to $+\infty$ if $0 < p \leqslant 1$ and at least a finite number of solutions if $p > 1$.

Dedicated with esteem to Professor Enzo L. Mitidieri on his 60th anniversary

1. Introduction and preliminaries results

The study of discrete boundary value problems has captured special attention in the last decade. In this context we point out the results obtained in the papers of R. Agarwal, K. Perera and D. O'Regan [1], A. Cabada, A. Iannizzotto and S. Tersian [2], P. Candito, G. Molica Bisci [3], M. Mihăilescu, V. Rădulescu, S. Tersian [4], A. Iannizzotto and V. Rădulescu [5]. In all these papers, variational methods are applied to boundary value problems on "bounded" discrete intervals (that is, sets of the type $\{0,\ldots,n\}$). Most results combine minimization and versions of the minimax principle, which usually do not require the Palais-Smale condition as the energy functional is defined on a finite-dimensional Banach space.

The studies regarding such type of problems can be placed at the interface of certain mathematical fields such as nonlinear partial differential equations and numerical analysis.

In many cases a problem in a continuous framework can be handled by using a suitable method from discrete mathematics and conversely (see L. Lovász [6]). The modeling/ simulation of certain nonlinear problems from economics, biological, neural networks, optimal control and others enforced in a natural manner the rapid development of the theory of difference equations. For instance, we may consult the monographs of W.G. Kelley and A.C. Peterson [7], V. Lakshmikantham and D. Trigiante [8].

This paper deals with the following problem

$$\begin{cases} -\Delta(\Delta u(k-1)) = \lambda a(k)u(k)^p + f(u(k)) \text{ for all } k \in [1,T], \\ u(0) = u(T+1) = 0, \end{cases} \qquad (P_\lambda)$$

where $T \geqslant 2$ is an integer, $[1,T]$ is the discrete interval $\{1,\ldots,T\}$, $a \in l^\infty$, $f : [0,+\infty) \to \mathbb{R}$ is a continuous nonlinearity, while $p > 0$ and λ are two real numbers. Moreover, the forward difference operator is defined as

$$\Delta u(k-1) = u(k) - u(k-1) \text{ for all } k \in [1,T].$$

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We would like to emphasize that problem (P_λ) is the discrete variant of the Laplacian equation given in [9], that is,

$$
\begin{cases} -\Delta u = \lambda a(x) u^p + f(u) \text{ in } \Omega, \\ u \geqslant 0,\ u \neq 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}
$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a smooth bounded domain with boundary $\partial\Omega$, $a \in L^\infty(\Omega)$, $f : [0, \infty) \to \mathbb{R}$ is a continuous function, $p > 0$ and $\lambda \in \mathbb{R}$ are some parameters. So, in [9] the authors showed that the number of solutions of the problem (1) is influenced by the competition between the power u^p and the oscillatory term f, namely, when there is an oscillatory term near the origin the equation (1) admits infinitely many distinct solutions if the power is convex, while it has a finite number of distinct solutions when the power is concave. In the case of oscillations at infinity the converse result hold true.

Moreover, problem (1) was recently extend by G. Molica Bisci, V. Rădulescu, R. Servadei in [10] to quasilinear equations of p-Laplacian type. More exactly, they studied the following problem

$$
\begin{cases} -divA(x, \nabla u) = \lambda \beta(x) u^q + f(u) \text{ in } \Omega, \\ u \geqslant 0,\ \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}
$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded domain with smooth boundry $\partial\Omega$, $q > 0$ and $\lambda \in \mathbb{R}$ are parameters, while $\beta \in L^\infty(\Omega)$ and $f : [0, \infty) \to \mathbb{R}$ is a continuous function and $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a function satisfying some general assumptions.

Motivated by the studies in [9] and [10], we focus in the present paper on the case of nonlinear difference equations. Hence, the purpose of this work is to study the number and behaviour of solutions to problem (P_λ), where f oscillates near the origin or at infinity. Usually, equations involving oscillatory nonlinearities give infinitely many distinct solutions (see, for instance, the papers [11], [12], [13]), but the presence of an additional term may alter the situation.

Here, we are interested in finding weak solutions of the problem (P_λ). For this purpose, we define the function space

$$H = \{u : [0, T+1] \to \mathbb{R} \text{ such that } u(0) = u(T+1) = 0\}.$$

Clearly, H is a T-dimensional Hilbert space (see [1]) with the inner product

$$\langle u, v \rangle = \sum_{k=1}^{T+1} \Delta u(k-1) \Delta v(k-1),\ \forall u, v \in H.$$

The associated norm is defined by

$$\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \right)^{\frac{1}{2}}.$$

We also put for every $u \in H$,

$$\|u\|_\infty = \max_{k \in [1,T]} |u(k)|. \tag{3}$$

We point out that the space H is finite-dimensional. Hence, by classical results, the norm $\| \cdot \|$ and $\| \cdot \|_\infty$ are equivalent on H.

Moreover, we denote l^∞ the set of all functions $u : [1, T] \to \mathbb{R}$ such that

$$\|u\|_\infty < +\infty,$$

where $\| \cdot \|_\infty$ is given in (3).

Definition 1. *We say that a function $u \in H$ is a weak solution for the problem (P_λ) if*

$$\sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1) + \lambda \sum_{k=1}^{T} a(k)u(k)^p v(k) - \sum_{k=1}^{T} f(u(k))v(k) = 0, \text{ for all } v \in H.$$

The paper is organized as follows. In Section 2 we will state the main results of the paper in the two different situations, when f oscillates near the origin or at infinity. In Section 3 we will consider an auxiliary problem and for it we will prove the existence of solutions by using direct minimization methods. Finally, in Section 4 we will study the problem (P_λ) in presence of an oscillation term near zero while Section 5 is devoted to the case of oscillation at infinity.

2. Main results of the paper

In this section, we state our main results, treating separately the two cases, that is, when f oscillates near the origin and at infinity, respectively.

Throughout this paper, we assume that $f : [0, +\infty) \to \mathbb{R}$ is a continuous function and we denote by F the function defined as

$$F(s) := \int_0^s f(t)dt, \text{ for any } s \in (0, +\infty).$$

• **Oscillation near the origin**.

In this framework we assume that the following conditions are satisfied

(f_1^0) $-\infty < \liminf_{s\to 0^+} \frac{F(s)}{s^2}$; $\limsup_{s\to 0^+} \frac{F(s)}{s^2} > \frac{1}{T}$;

(f_2^0) $l_0 := \liminf_{s\to 0^+} \frac{f(s)}{s} < 0$.

Remark 1. *As a consequence of assumptions (f_1^0) and (f_2^0) we have that*

$$f(0) = 0. \tag{4}$$

Indeed, arguing by contradiction, suppose that $f(0) = l \in \mathbb{R} \setminus \{0\}$. Then, by the continuity of f and (f_2^0) we would get

$$\lim_{s\to 0^+} \frac{f(s)}{s} = -\infty,$$

so that, by l'Hôspital's rule we would deduce that

$$\lim_{s\to 0^+} \frac{F(s)}{s^2} = \lim_{s\to 0^+} \frac{f(s)}{2s} = -\infty,$$

which contradicts (f_1^0). This obviously implies assertion (4).

We point out that hypotheses (f_1^0) and (f_2^0) imply an oscillatory behaviour of f near the origin. Moreover, assumption (f_1^0) allows us to deduce some information about the number of solutions for problem (P_λ), while (f_2^0) yields the existence of the solutions.

As a model for f we can take the continuous function $f_0 : [0, +\infty) \to \mathbb{R}$ such that

$$f_0(s) = \begin{cases} 0 & \text{if } s = 0, \\ s^\alpha(\gamma + \sin s^{-\beta}) & \text{if } s > 0, \end{cases} \tag{5}$$

where α, β and γ are such that $0 < \alpha < 1 < \alpha + \beta$ and $\gamma \in (0, 1)$. By direct calculations it is easy to show that the function f_0 defined above satisfies assumptions (f_1^0) and (f_2^0).

In this case our main results are the following.

Theorem 2. *Let $a \in l^\infty$, $\lambda \in \mathbb{R}$ and $p \geqslant 1$. Assume that $f \in C([0, +\infty); \mathbb{R})$ satisfies conditions (f_1^0) and (f_2^0). If either*

(i) *$p = 1$, $l_0 \in (-\infty, 0)$ and $\lambda a(k) < \lambda_0$ for all $k \in [1, T]$ and some $\lambda_0 \in (0, -l_0)$ or*
(ii) *$p = 1$, $l_0 = -\infty$ and $\lambda \in \mathbb{R}$ is arbitrary or*

(iii) $p > 1$ *and* $\lambda \in \mathbb{R}$ *is arbitrary,*

then there exists a sequence $\{u_i\}_i$ *in* H *of non-negative, distinct weak solutions of problem* (P_λ) *such that*

$$\lim_{i\to+\infty} \|u_i\| = \lim_{i\to+\infty} \|u_i\|_\infty = 0. \tag{6}$$

Theorem 3. *Let* $a \in l^\infty$, $\lambda \in \mathbb{R}$ *and* $0 < p < 1$. *Assume that* $f \in C([0,+\infty);\mathbb{R})$ *satisfies conditions* (f_1^0) *and* (f_2^0). *Then, for every* $n \in \mathbb{N}$, *there exists* $\Lambda_n > 0$ *such that problem* (P_λ) *has at least* n *distinct weak solutions* $u_{1,\lambda}, \dots, u_{n,\lambda} \in H$ *such that*

$$\|u_{i,\lambda}\| < \frac{1}{i} \text{ and } \|u_{i,\lambda}\|_\infty < \frac{1}{i}, \text{ for any } i = 1, \dots, n, \tag{7}$$

provided $\lambda \in [-\Lambda_n, \Lambda_n]$.

• **Oscillation at infinity**.

In this framework we assume that the following assumptions are fulfilled

(f_1^∞) $-\infty < \liminf\limits_{s\to+\infty} \frac{F(s)}{s^2}$; $\limsup\limits_{s\to+\infty} \frac{F(s)}{s^2} > \frac{1}{T}$;

(f_2^∞) $l_\infty := \liminf\limits_{s\to+\infty} \frac{f(s)}{s} < 0$.

Here, we point out that assumptions (f_1^∞) and (f_2^∞) imply an oscillatory behaviour of f at infinity. As in the case of the oscillations near the origin, here assumption (f_2^∞) is used in order to prove the existence of solutions for problem (P_λ), while (f_1^∞) guarantees that these solutions are infinitely many, when $0 < p \leqslant 1$, and at least a finite number, if $p > 1$.

Also, we can construct a prototype for f taking the continuous function $f_\infty : [0,+\infty) \to \mathbb{R}$ such that

$$f_\infty(s) = s^\alpha(\gamma + sins^\beta), \tag{8}$$

where α, β, γ are such that $1 < \alpha$, $|\alpha - \beta| < 1$, and $\gamma \in (0,1)$. Also in this case, direct calculations show that the function f_∞ defined in (8) satisfies assumptions (f_1^∞) and (f_2^∞).

In this setting a perfect counterpart of Theorem 2 and Theorem 3 are given as follows.

Theorem 4. *Let* $a \in l^\infty$, $\lambda \in \mathbb{R}$ *and* $0 < p \leqslant 1$. *Assume that* $f \in C([0,+\infty);\mathbb{R})$ *satisfies conditions* (f_1^∞) *and* (f_2^∞) *with* $f(0) = 0$. *If either*

(i) $p = 1$, $l_\infty \in (-\infty, 0)$ *and* $\lambda a(k) < \lambda_\infty$ *for all* $k \in [1,T]$ *and some* $\lambda_\infty \in (0, -l_\infty)$ *or*

(ii) $p = 1$, $l_\infty = -\infty$ *and* $\lambda \in \mathbb{R}$ *is arbitrary or*

(iii) $0 < p < 1$ *and* $\lambda \in \mathbb{R}$ *is arbitrary,*

then there exists a sequence $\{u_i\}_i$ *in* H *of non-negative, distinct weak solutions of problem* (P_λ) *such that*

$$\lim_{i\to+\infty} \|u_i\| = \lim_{i\to+\infty} \|u_i\|_\infty = +\infty. \tag{9}$$

Theorem 5. *Let* $a \in l^\infty$, $\lambda \in \mathbb{R}$ *and* $p > 1$. *Assume that* $f \in C([0,+\infty);\mathbb{R})$ *satisfies conditions* (f_1^∞) *and* (f_2^∞), *with* $f(0) = 0$. *Then, for every* $n \in \mathbb{N}$, *there exists* $\Lambda_n > 0$ *such that problem* (P_λ) *has at least* n *distinct weak solutions* $u_{1,\lambda}, \dots, u_{n,\lambda} \in H$ *such that*

$$\|u_{i,\lambda}\| > i - 1 \text{ and } \|u_{i,\lambda}\|_\infty > i - 1, \text{ for any } i = 1, \dots, n, \tag{10}$$

provided $\lambda \in [-\Lambda_n, \Lambda_n]$.

In all these situations, when there is an oscillation near zero or at infinity, and for any value of p, the idea is to prove the existence of solutions for problem (P_λ) using variational methods. More exactly, first of all, we will consider an auxiliary problem and, under suitable assumptions on the data, we will prove the existence of solutions for this equation studying the energy functional associated with it and proving that this functional admits a minimum, using the direct methods of the calculus of variations and then, we will apply this result to problem (P_λ) in order to get Theorems 2, 3, 4 and 5.

3. A key problem

In this section we consider the problem

$$\begin{cases} -\Delta(\Delta u(k-1)) + c(k)u(k) = g(k, u(k)), \ k \in [1, T], \\ u(0) = u(T+1) = 0 \ . \end{cases} \tag{P_g^c}$$

Here, we assume that $c : [1, T] \to \mathbb{R}$ is such that

$$c \in l^\infty; \ \min_{k\in[1,T]} c(k) > 0, \tag{11}$$

while $g : [1, T] \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions

$$g(k, 0) = 0 \text{ for every } k \in [1, T]; \tag{12}$$

$$\text{there exists } M_g > 0 \text{ such that } |g(k, s)| \leqslant M_g \text{ for every } k \in [1, T] \text{ and all } s \geqslant 0; \tag{13}$$

$$\begin{gathered} \text{there exist } \delta \text{ and } \eta, \text{ with } 0 < \delta < \eta \text{ such that} \\ g(k, s) \leqslant 0 \text{ for every } k \in [1, T] \text{ and all } s \in [\delta, \eta]. \end{gathered} \tag{14}$$

In the sequel we extend the function g by taking $g(k, s) = 0$ for every $k \in [1, T]$ and $s \leqslant 0$.

Definition 6. *By a weak solution for problem* (P_g^c) *we understand a function* $u \in H$ *such that*

$$\sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1) + \sum_{k=1}^{T} c(k)u(k)v(k) - \sum_{k=1}^{T} g(k, u(k))v(k) = 0, \ \textit{for every } v \in H.$$

Let $E_{c,g} : H \to \mathbb{R}$ be the energy functional associated to problem (P_g^c) defined by

$$E_{c,g}(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\sum_{k=1}^{T} c(k)u(k)^2 - \sum_{k=1}^{T} G(k, u(k)), \ u \in H, \tag{15}$$

where $G(k, s) := \int_0^s g(k, t)dt$ for any $s \in \mathbb{R}$ and $k \in [1, T]$.

Standard arguments assure that $E_{c,g}$ is well-defined, it belongs to $C^1(H; \mathbb{R})$ and

$$\begin{aligned} \langle E'_{c,g}(u), v\rangle &= -\sum_{k=1}^{T+1} \Delta(\Delta u(k-1))v(k) + \sum_{k=1}^{T} c(k)u(k)v(k) - \sum_{k=1}^{T} g(k, u(k))v(k) \\ &= \langle u, v\rangle + \sum_{k=1}^{T} c(k)u(k)v(k) - \sum_{k=1}^{T} g(k, u(k))v(k), \ \forall u, v \in H. \end{aligned}$$

Thus, the weak solutions of (P_g^c) coincide with the critical points of $E_{c,g}$.

Finally, we introduce the set W^η defined as follows

$$W^\eta := \{u \in H : \ \|u\|_\infty \leqslant \eta\},$$

where η is a positive parameter given in (14).

Since $g(k, 0) = 0$ for every $k \in [1, T]$ by (12), then $u \equiv 0$ is clearly a weak solution of problem (P_g^c). In the sequel, under some general assumptions, we prove the existence of a non-negative weak solution for problem (P_g^c).

Thus, the main result of this section is the following.

Theorem 7. *Assume that* $c : [1, T] \to \mathbb{R}$ *is a function verifying (11) and that* $g : [1, T] \times [0, +\infty) \to \mathbb{R}$ *is a Carathéodory function satisfying (12), (13) and (14). Then*

(a) *the functional* $E_{c,g}$ *is bounded from below on* W^η *attaining its infimum at some* $\tilde{u} \in W^\eta$;

(b) $\tilde{u}(k) \in [0, \delta]$ *for every* $k \in [1, T]$, *where* δ *is the positive parameter given in (14);*

(c) $\tilde{u}$ *is a non-negative weak solution of problem* (P_g^c).

Proof. (a) Since the norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent in the finite-dimensional space H, the set W^η is compact in H. Combining this fact with the continuity of $E_{c.g}$, we infer that $E_{c,g}\Big|_{W^\eta}$ attains its infimum at $\tilde{u} \in W^\eta$.

(b) Let δ be as in assumption (14) and let M the following set

$$M := \{k \in [1,T] : \ \tilde{u}(k) \notin [0,\delta]\}.$$

Hence, arguing by contradiction, we suppose that $M \neq \emptyset$.

Define the truncation function $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\gamma(s) := \min\{s_+, \delta\},$$

where $s_+ = \max\{s, 0\}$. Now, set $w := \gamma \circ \tilde{u}$, that is

$$w(k) = \begin{cases} \delta & \text{if } \tilde{u}(k) > \delta, \\ \tilde{u}(k) & \text{if } 0 \leqslant \tilde{u}(k) \leqslant \delta, \\ 0 & \text{if } \tilde{u}(k) < 0, \end{cases}$$

for every $k \in [1,T]$.

Since $\gamma(0) = 0$, we have $w(0) = w(T+1) = 0$, so $w \in H$. Besides, $0 \leqslant w(k) \leqslant \delta$ for every $k \in [1,T]$. By assumption (14) we know that $\delta < \eta$, and so $w \in W^\eta$.

We introduce the sets

$$M_- := \{k \in M : \ \tilde{u}(k) < 0\} \text{ and } M_+ := \{k \in M : \ \tilde{u}(k) > \delta\}.$$

Thus, $M = M_- \cup M_+$ and we have that

$$w(k) = \begin{cases} \tilde{u}(k) & \text{for all } k \in [1,T] \setminus M, \\ 0 & \text{for all } k \in M_-, \\ \delta & \text{for all } k \in M_+. \end{cases}$$

Moreover, we have

$$\begin{aligned} E_{c,g}(w) - E_{c,g}(\tilde{u}) &= \frac{1}{2}\left(\|w\|^2 - \|\tilde{u}\|^2\right) + \frac{1}{2}\sum_{k=1}^{T} c(k)[(w(k))^2 - (\tilde{u}(k))^2] \\ &\quad - \sum_{k=1}^{T}[G(k, w(k)) - G(k, \tilde{u}(k))] \\ &=: \frac{1}{2}J_1 + \frac{1}{2}J_2 - J_3. \end{aligned} \tag{16}$$

Since γ is a Lipschitz function with Lipschitz constant 1, and $w = \gamma \circ \tilde{u}$, we have

$$\begin{aligned} J_1 &= \|w\|^2 - \|\tilde{u}\|^2 = \sum_{k=1}^{T+1}[|\Delta w(k-1)|^2 - |\Delta \tilde{u}(k-1)|^2] \\ &= \sum_{k=1}^{T+1}\left[|w(k) - w(k-1)|^2 - |\tilde{u}(k) - \tilde{u}(k-1)|^2\right] \\ &\leqslant 0. \end{aligned} \tag{17}$$

Since $\min\limits_{k\in[1,T]} c(k) > 0$ by (11), one has

$$\begin{aligned} J_2 &= \sum_{k=1}^{T} c(k)[(w(k))^2 - (\tilde{u}(k))^2] = \sum_{k\in M} c(k)[(w(k))^2 - (\tilde{u}(k))^2] \\ &= -\sum_{k\in M_-} c(k)(\tilde{u}(k))^2 + \sum_{k\in M_+} c(k)[\delta^2 - (\tilde{u}(k))^2] \\ &\leqslant 0. \end{aligned} \tag{18}$$

Next, we estimate J_3. Due to the fact that $g(k,s)=0$ for all $s \leqslant 0$ and for every $k \in [1,T]$, we have

$$\sum_{k\in M_-} [G(k,w(k)) - G(k,\tilde{u}(k))] = 0. \tag{19}$$

Moreover, by the mean value theorem, for every $k \in M_+$, there exists $\theta(k) \in [\delta, \tilde{u}(k)] \subset [\delta, \eta]$ such that

$$G(k,w(k)) - G(k,\tilde{u}(k)) = G(k,\delta) - G(k,\tilde{u}(k)) = g(k,\theta(k))(\delta - \tilde{u}(k)).$$

Thus, taking into account hypothesis (14) and definition of M_+, we have

$$\sum_{k\in M_+} [G(k,w(k)) - G(k,\tilde{u}(k))] \geqslant 0. \tag{20}$$

Hence, by (19) and (20), we obtain

$$\begin{aligned} J_3 &= \sum_{k\in M} [G(k,w(k)) - G(k,\tilde{u}(k))] = \sum_{k\in M_+} [G(k,w(k)) - G(k,\tilde{u}(k))] \\ &\geqslant 0. \end{aligned} \tag{21}$$

Combining relations (17), (18), (21) with (16), we get

$$E_{c,g}(w) - E_{c,g}(\tilde{u}) \leqslant 0. \tag{22}$$

On the other hand, since $w \in W^\eta$, it is easy to see that

$$E_{c,g}(w) \geqslant E_{c,g}(\tilde{u}) = \inf_{u\in W^\eta} E_{c,g}(u).$$

By this and (22) we get that every term in $E_{c,g}(w) - E_{c,g}(\tilde{u})$ should be zero.

In particular, from J_2 and due to (11), we have

$$\sum_{k\in M_-} c(k)(\tilde{u}(k))^2 = \sum_{k\in M_+} c(k)[\delta^2 - (\tilde{u}(k))^2] = 0,$$

which imply that $\tilde{u}(k) = \begin{cases} 0 \text{ for every } k \in M_-, \\ \delta \text{ for every } k \in M_+. \end{cases}$

Due to the definition of the sets M_- and M_+, we must have $M_- = M_+ = \emptyset$, which contradicts $M_- \cup M_+ = M \neq \emptyset$.

(c) Let us fix $v \in H$ arbitrarily and let

$$\varepsilon_0 := \frac{\eta - \delta}{\|v\|_\infty + 1} > 0,$$

where δ and η are given as in (14). Moreover, let $I : [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}$ be the function defined as

$$I(\varepsilon) := E_{c,g}(\tilde{u} + \varepsilon v).$$

First of all, thanks to (b), for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ we have

$$\begin{aligned} |\tilde{u}(k) + \varepsilon v(k)| &\leqslant |\tilde{u}(k)| + \varepsilon |v(k)| \\ &\leqslant \tilde{u}(k) + \frac{\eta - \delta}{\|v\|_\infty + 1}\|v\|_\infty \\ &\leqslant \delta + \eta - \delta = \eta, \end{aligned}$$

for every $k \in [1,T]$. Thus, $\tilde{u} + \varepsilon v \in W^\eta$.

Consequently, due to (a), we have $I(\varepsilon) \geqslant I(0)$ for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, that is, 0 is an interior minimum point for I. Then, since I is differentiable at 0, it is easy to see that

$$I'(0) = 0 \text{ and } \langle E'_{c,g}(\tilde{u}), v\rangle = 0.$$

Taking into account that $v \in H$ is arbitrary and using the definition of $E_{c,g}$, we obtain that $\tilde{u}$ is a weak solution of problem (P^c_g). Moreover, due to (b), $\tilde{u}$ is non-negative in $[1,T]$.

□

We note that, Theorem 7 does not guarantee that the solution $\tilde{u}$ of problem (P_g^c) is not the trivial one. In spite of this, by Theorem 7 we will derive the existence of non-trivial solutions for the original problem (P_λ), provided the nonlinear term f is chosen appropriately.

Finally, we define the truncation function $\tau_\eta : [0,+\infty) \to \mathbb{R}$ as follows

$$\tau_\eta(s) := \min\{\eta, s\} \text{ for every } s \geqslant 0, \tag{23}$$

where η is the positive constant given in assumption (14). Note that τ_η is a continuous function in $[0,+\infty)$.

4. Oscillation near the origin

In this section we study problem (P_λ) in the case when the nonlinear term f oscillates near the origin.

In order to prove Theorem 2 and Theorem 3, we consider again the problem from the previous section, that is

$$\begin{cases} -\Delta(\Delta u(k-1)) + c(k)u(k) = g(k,u(k)), \ k \in [1,T], \\ u(0) = u(T+1) = 0 \ , \end{cases} \tag{P_g^c}$$

where $c : [1,T] \to \mathbb{R}$ fulfills (11) and $g : [1,T] \times [0,+\infty) \to \mathbb{R}$ is a Carathéodory function which satisfies the following assumptions

$$\begin{aligned} &g(k,0) = 0 \text{ for all } k \in [1,T], \text{ and} \\ &\text{there exists } \overline{s} > 0 \text{ and } M > 0 \text{ such that } \max_{s\in[0,\overline{s}]} |g(k,s)| \leqslant M \text{ for all } k \in [1,T]; \end{aligned} \tag{24}$$

$$\begin{aligned} &\text{there exist two sequences } \{\delta_i\}_i \text{ and } \{\eta_i\}_i \text{ with } 0 < \eta_{i+1} < \delta_i < \eta_i \text{ such that} \\ &\lim_{i\to+\infty} \eta_i = 0 \text{ and } g(k,s) \leqslant 0 \text{ for every } k \in [1,T] \text{ and all } s \in [\delta_i,\eta_i], \ i \in \mathbb{N}; \end{aligned} \tag{25}$$

$$\begin{aligned} &-\infty < \liminf_{s\to 0^+} \frac{G(k,s)}{s^2} \text{ and } \limsup_{s\to 0^+} \frac{G(k,s)}{s^2} > \frac{1}{T} \text{ uniformly for all } k \in [1,T], \\ &\text{where } G(k,s) = \int_0^s g(k,t)dt. \end{aligned} \tag{26}$$

In the sequel, we will prove Theorem 2. The strategy will consists in applying Theorem 7 with a suitable choice of the functions c and g.

4.1. **Proof of Theorem 2.** First of all, we show that, under suitable assumptions, problem (P_λ) has infinitely many distinct weak solutions, provided $p \geqslant 1$. We will consider separately the case when $p = 1$ and the one when $p > 1$ and in both the situations the strategy will consist in using Theorem 7.

We start by proving assertion (i). In this setting we suppose that $p = 1$ and $l_0 \in (-\infty, 0)$. Let $\lambda \in \mathbb{R}$ be such that $\lambda a(k) < \lambda_0$ for all $k \in [1,T]$ and some $0 < \lambda_0 < -l_0$.

Let us choose $\overline{\lambda}_0 \in (\lambda_0, -l_0)$ and let

$$c(k) := \overline{\lambda}_0 - \lambda a(k) \text{ and } g(k,s) := f(s) + \overline{\lambda}_0 s \text{ for all } (k,s) \in [1,T] \times [0,+\infty). \tag{27}$$

The first step consist in proving that the functions c and g given in (27) satisfy the assumptions (11), (24), (25) and (26).

First of all, note that $c \in l^\infty$ thanks to the fact that $a \in l^\infty$ and

$$\min_{k\in[1,T]} c(k) > \overline{\lambda}_0 - \lambda_0 > 0,$$

which obviously implies (11). By (4) we know that $f(0) = 0$. Thus, using the regularity of f, it is easy to see that g is a continuous function in $[1,T] \times [0,+\infty)$ and $g(k,0) = 0$ for all $k \in [1,T]$. Also, the continuity of $s \mapsto g(\cdot,s)$ and Weierstrass theorem yield (24).

Moreover, since for any $k \in [1, T]$ and $s > 0$ we have

$$\frac{G(k,s)}{s^2} = \frac{\overline{\lambda}_0}{2} + \frac{F(s)}{s^2},$$

hypothesis (f_1^0) immediately implies (26).

Next, we show that g satisfies (25). At this purpose, note that, by (f_2^0), we get that there exists a sequence $\{s_i\}_i \subset (0,1)$ converging to 0 as $i \to +\infty$ such that

$$\lim_{i\to+\infty} \frac{f(s_i)}{s_i} = l_0.$$

Since $\overline{\lambda}_0 < -l_0$ by assumption, there exists $\overline{\varepsilon} > 0$ such that $\overline{\lambda}_0 + \overline{\varepsilon} < -l_0$. By this and the above relation we get that, for i large enough, say $i \geqslant i^* \in \mathbb{N}$,

$$f(s_i) < -\overline{\lambda}_0 s_i. \tag{28}$$

Thus, we have

$$g(k, s_i) = f(s_i) + \overline{\lambda}_0 s_i < -\overline{\lambda}_0 s_i + \overline{\lambda}_0 s_i = 0.$$

Consequently, by using the continuity of f, there exists a neighborhood of s_i, say (δ_i, η_i) and we may choose two sequences $\{\delta_i\}_i$, $\{\eta_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim\limits_{i\to+\infty} \eta_i = 0$ and $g(k,s) = \overline{\lambda}_0 s + f(s) \leqslant 0$ for any $k \in [1,T]$ and all $s \in [\delta_i, \eta_i]$ and $i \geqslant i^*$.

In this way, hypothesis (25) is verified for g on every interval $[\delta_i, \eta_i]$, $i \in \mathbb{N}$.

In the sequel, since $\eta_i \to 0$ as $i \to +\infty$, by (25), without any loss of generality, we may assume that

$$0 < \delta_i < \eta_i < \overline{s}, \tag{29}$$

for i sufficiently large, where $\overline{s} > 0$ is given by (24).

For every $i \in \mathbb{N}$, let $g_i : [1,T] \times [0,+\infty) \to \mathbb{R}$ be the truncation function defined by

$$g_i(k,s) := g(k, \tau_{\eta_i}(s)) \text{ and } G_i(k,s) := \int_0^s g_i(k,t)dt, \tag{30}$$

for every $k \in [1,T]$ and $s \geqslant 0$, where τ_{η_i} is the function defined in (23) with $\eta = \eta_i$.

Let $E_i : H \to \mathbb{R}$ be the energy functional associated with problem $(P^c_{g_i})$, that is $E_i := E_{c,g_i}$, where E_{c,g_i} is the functional given in (15) with $g = g_i$.

We note that the function g_i verifies all the assumptions of Theorem 7 for $i \in \mathbb{N}$ large enough with $[\delta_i, \eta_i]$. Indeed, thanks to the regularity of g, the continuity of τ_η and the fact that $g(k,0) = 0$ for all $k \in [1,T]$, the function g_i is Carathéodory and such that $g_i(k,0) = 0$ for every $k \in [1,T]$. Moreover, by (24), (29) and (30), g_i satisfies (12) and (13). Finally, condition (14) is satisfied thanks to (25).

Hence, as a consequence of Theorem 7, for every $i \in \mathbb{N}$, there exists $u_i \in W^{\eta_i}$ such that

$$\min_{u\in W^{\eta_i}} E_i(u) = E_i(u_i); \tag{31}$$

$$u_i(k) \in [0, \delta_i] \text{ for every } k \in [1,T]; \tag{32}$$

$$u_i \text{ is a non-negative weak solution of } (P^c_{g_i}). \tag{33}$$

Using the definition of τ_η, relation (30) and the fact that

$$0 \leqslant u_i(k) \leqslant \delta_i < \eta_i \text{ for every } k \in [1,T],$$

we have

$$g_i(k, u_i(k)) = g(k, \tau_{\eta_i}(u_i(k))) = g(k, u_i(k)) \text{ for every } k \in [1,T].$$

Thus, by the above relation and (33), u_i is a non-negative weak solution not only for $(P^c_{g_i})$ but also for problem (P^c_g).

In the sequel, we prove that there are infinitely many distinct elements in the sequence $\{u_i\}_i$. In order to see this, the first step consists in proving that

$$E_i(u_i) < 0 \text{ for } i \in \mathbb{N} \text{ large enough and} \tag{34}$$

$$\lim_{i\to+\infty} E_i(u_i) = 0. \tag{35}$$

Due to (f_1^0) and (27), we have that

$$\begin{aligned}\limsup_{s\to 0^+} \frac{G(k,s)}{s^2} &= \frac{\overline{\lambda}_0}{2} + \limsup_{s\to 0^+} \frac{F(s)}{s^2} \\ &> \frac{\overline{\lambda}_0}{2} + \frac{1}{T}.\end{aligned}$$

In particular, there exists a sequence $\{\tilde{s}_i\}_i$, with

$$0 < \tilde{s}_i \leqslant \delta_i \text{ for all } i \in \mathbb{N} \text{ and} \tag{36}$$

$$G(k, \tilde{s}_i) > \left(\frac{1}{T} + \frac{\overline{\lambda}_0}{2}\right) \tilde{s}_i^2. \tag{37}$$

Now, let us fix $i \in \mathbb{N}$ sufficiently large and let us define the function $w_i \in H$ by

$$w_i(k) := \tilde{s}_i \text{ for every } k \in [1, T].$$

Then $\|w_i\|_\infty = \tilde{s}_i \leqslant \delta_i < \eta_i < 1$ by (25) and (36). Hence, $w_i \in W^{\eta_i}$. This yields that for every $k \in [1, T]$, we have

$$\begin{aligned}G_i(k, w_i(k)) &= G_i(k, \tilde{s}_i) = \int_0^{\tilde{s}_i} g_i(k,t)dt \\ &= \int_0^{\tilde{s}_i} g(k, \tau_{\eta_i}(t))dt = \int_0^{\tilde{s}_i} g(k,t)dt \\ &= G(k, \tilde{s}_i).\end{aligned} \tag{38}$$

By this and taking into account (11), (27), (37), (38), for i sufficiently large we have

$$\begin{aligned}E_i(w_i) &= \frac{1}{2}\sum_{k=1}^{T+1} |\Delta w_i(k-1)|^2 + \frac{1}{2}\sum_{k=1}^{T} c(k)(w_i(k))^2 - \sum_{k=1}^{T} G_i(k, w_i(k)) \\ &< (\tilde{s}_i)^2 + \frac{1}{2}\overline{\lambda}_0 T(\tilde{s}_i)^2 - TG(k, \tilde{s}_i) \\ &< (\tilde{s}_i)^2 + \frac{1}{2}\overline{\lambda}_0 T(\tilde{s}_i)^2 - T\left(\frac{1}{T} + \frac{\overline{\lambda}_0}{2}\right)(\tilde{s}_i)^2 \\ &= (\tilde{s}_i)^2 + \frac{1}{2}\overline{\lambda}_0 T(\tilde{s}_i)^2 - (\tilde{s}_i)^2 - T\frac{\overline{\lambda}_0}{2}(\tilde{s}_i)^2 \\ &= 0.\end{aligned}$$

Consequently, using also (31) for i sufficiently large, the above estimation and $w_i \in W^{\tilde{s}_i} \subset W^{\eta_i}$ show that

$$E_i(u_i) = \min_{u \in W^{\eta_i}} E_i(u) \leqslant E_i(w_i) < 0, \tag{39}$$

which proves in particular (34).

In the sequel, we will prove (35). For every $i \in \mathbb{N}$ sufficiently large, by using the definition of G_i, the mean value theorem, (24), (25), (29), (30) and (32), we have

$$\begin{aligned} E_i(u_i) &= \frac{1}{2}\sum_{k=1}^{T+1} |\Delta u_i(k-1)|^2 + \frac{1}{2}\sum_{k=1}^{T} c(k)(u_i(k))^2 - \sum_{k=1}^{T} G_i(k, u_i(k)) \\ &\geqslant -\sum_{k=1}^{T} G_i(k, u_i(k)) = -\sum_{k=1}^{T} G(k, u_i(k)) \\ &= -\sum_{k=1}^{T} \int_0^{u_i(k)} g(k,s)ds \\ &\geqslant -\sum_{k=1}^{T} \max_{s\in[0,\overline{s}]} |g(k,s)| u_i(k) \\ &\geqslant -\delta_i T \max_{s\in[0,\overline{s}]} |g(\cdot, s)| \\ &\geqslant -\delta_i T M. \end{aligned}$$

Since $\lim_{i\to+\infty} \delta_i = 0$, the above estimate and (39) leads to (35).

Finally, it is easy to see that relation (6) is an immediate consequence of (32) combined with $\lim_{i\to+\infty} \delta_i = 0$, and to the fact that norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent.

Thus, we get the existence of infinitely many distinct nontrivial non-negative solutions $\{u_i\}_i$ for problem (P_g^c) satisfying condition (6). Due to the choice of c and g in (27) and taking into account that $p = 1$, it is easy to see that u_i is a weak solution of problem (P_λ) and this ends the proof of assertion (i) in Theorem 2 in the case $p = 1$.

Now, let us consider assertion (ii). At this purpose, let $p = 1$, $l_0 = -\infty$ and $\lambda \in \mathbb{R}$ be arbitrary fixed. In this setting we choose $\overline{\lambda}_0 \in (\lambda_0, -l_0)$ and

$$c(k) := \overline{\lambda}_0 \text{ and } g(k,s) = f(s) + (\lambda a(k) + \overline{\lambda}_0)s \text{ for all } (k,s) \in [1,T] \times [0,+\infty).$$

This case can be dealt with in a similar way as (i), using relation

$$f(s_i) < -(|\lambda| \cdot \|a\|_\infty + \overline{\lambda}_0)s_i,$$

instead of $f(s_i) < -\overline{\lambda}_0 s_i$, for i large enough, and taking into account that for every $k \in [1,T]$ and $s \geqslant 0$ one has

$$g(k,s) = f(s) + (\lambda a(k) + \overline{\lambda}_0)s \leqslant f(s) + (|\lambda| \cdot \|a\|_\infty + \overline{\lambda}_0)s.$$

Now, let us prove assertion (iii). At this purpose, let $p > 1$ and $\lambda \in \mathbb{R}$ be arbitrary fixed. Let us also fix a number $\overline{\lambda}_0 \in (0, -l_0)$ and choose

$$c(k) := \overline{\lambda}_0 \text{ and } g(k,s) := \lambda a(k)s^p + \overline{\lambda}_0 s + f(s) \text{ for all } (k,s) \in [1,T] \times [0,+\infty). \tag{40}$$

Also in this setting our aim is to prove that c and g given in (40) satisfy the conditions (11), (24), (25) and (26).

Clearly, (11) is satisfied and also thanks to (f_1^0), (f_2^0) we have $g(k,0) = 0$ for all $k \in [1,T]$. Moreover, since $a \in l^\infty$ the continuity of $s \mapsto g(\cdot, s)$ and the Weierstrass theorem yield that (24) holds true.

Furthermore, since $p > 1$ and

$$\frac{G(k,s)}{s^2} = \lambda \frac{a(k)}{p+1} s^{p-1} + \frac{\overline{\lambda}_0}{2} + \frac{F(s)}{s^2}, \text{ for all } k \in [1,T] \text{ and } s \in (0,+\infty),$$

hypothesis (f_1^0) implies (26).

In the sequel, note that for all $k \in [1,T]$ and every $s \in [0,+\infty)$, we have

$$g(k,s) \leqslant |\lambda| \cdot \|a\|_\infty s^p + \overline{\lambda}_0 s + f(s). \tag{41}$$

As a consequence of this and of (f_2^0) we get

$$\begin{aligned} \liminf_{s\to 0^+} \frac{g(k,s)}{s} &\leqslant \liminf_{s\to 0^+}\left(|\lambda|\cdot\|a\|_\infty s^{p-1} + \overline{\lambda}_0 + \frac{f(s)}{s}\right) \\ &= \overline{\lambda}_0 + l_0 \\ &< 0, \end{aligned} \tag{42}$$

for all $k \in [1,T]$, thanks to the choice of p.

In particular, there exists a sequence $\{s_i\}_i \subset (0,1)$ converging to 0 as $i \to +\infty$ such that $g(k,s_i) < 0$ for $i \in \mathbb{N}$ large enough and for all $k \in [1,T]$. Thus, by using the continuity of $s \mapsto g(\cdot,s)$, there exist two sequences $\{\delta_i\}_i$, $\{\eta_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim\limits_{i\to+\infty} \eta_i = 0$ and $g(k,s) \leqslant 0$, for every $k \in [1,T]$ and all $s \in [\delta_i,\eta_i]$ and $i \in \mathbb{N}$ large enough.

Summarizing what proved above, we get that hypothesis (25) hold true.

Finally, an argument analogous to that used in (i) proves that problem (P_g^c) is equivalent to problem (P_λ) through the choice (40) and so, we get the existence of infinitely many distinct nontrivial solutions $\{u_i\}_i$ for problem (P_λ) satisfying (6). This concludes the proof of Theorem 2. □

4.2. **Proof of Theorem 3.** First of all, we point out that here we will proceed as in the proof of Theorem 2 and the strategy will consist in applying Theorem 7 to problem (P_g^c) with a suitable choices of c and g.

At this purpose, let $\overline{\lambda}_0 \in (0,-l_0)$, where $l_0 < 0$ is given in assumption (f_2^0) and let us choose

$$c(k) := \overline{\lambda}_0 \text{ and } g(k,s,\lambda) := \lambda a(k) s^p + \overline{\lambda}_0 s + f(s), \tag{43}$$

for all $(k,s) \in [1,T]\times[0,+\infty)$, $\lambda \in \mathbb{R}$. Note that for all $k \in [1,T]$ and every $s \in [0,+\infty)$, we have

$$g(k,s,\lambda) \leqslant |\lambda|\cdot\|a\|_\infty s^p + \overline{\lambda}_0 s + f(s).$$

Next, on account of (f_2^0), there exists a sequence $\{s_i\}_i \subset (0,1)$ converging to 0 as $i \to +\infty$ such that

$$f(s_i) < -\overline{\lambda}_0 s_i, \text{ for } i \in \mathbb{N} \text{ large enough.}$$

Consequently, we have

$$g(k,s_i,0) = \overline{\lambda}_0 s_i + f(s_i) < 0,$$

for $i \in \mathbb{N}$ large enough and for all $k \in [1,T]$. Thus, due to the continuity of $s \mapsto g(\cdot,s,\cdot)$ we get that there exist three sequences $\{\delta_i\}_i$, $\{\eta_i\}_i$, $\{\lambda_i\}_i \subset (0,1)$ such that,

$$0 < \eta_{i+1} < \delta_i < s_i < \eta_i < 1, \ \lim_{i\to+\infty} \eta_i = 0, \tag{44}$$

and for $i \in \mathbb{N}$ large enough,

$$g(k,s,\lambda) \leqslant 0, \text{ for all } k \in [1,T], \ \lambda \in [-\lambda_i,\lambda_i] \text{ and } s \in [\delta_i,\eta_i]. \tag{45}$$

For any $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i,\lambda_i]$, let $g_i : [1,T]\times[0,+\infty)\times[-\lambda_i,\lambda_i] \to \mathbb{R}$ be the function defined by

$$g_i(k,s,\lambda) := g(k,\tau_{\eta_i}(s),\lambda) \tag{46}$$

and

$$G_i(k,s,\lambda) := \int_0^s g_i(k,t,\lambda)dt,$$

for all $k \in [1,T]$ and $s \geqslant 0$.

In the sequel, let us prove that c given in (43) and g_i satisfy all the assumptions of Theorem 7. Due to relation (4), it is easy to see that g_i satisfies condition (12). Also, the assumption (11) is trivially verified.

Moreover, the regularity of g and the continuity of τ_η show that g_i is a Carathéodory function. Also, thanks to (46), (23), the continuity of $s \mapsto g(\cdot,s,\cdot)$ and the Weierstrass theorem give that g_i satisfies (13). Finally, (45) and (46) yield (14) for i large enough. Hence, g_i satisfies all the assumptions of Theorem 7 for i large.

Next, for any $i \in \mathbb{N}$, let $E_{i,\lambda} : H \to \mathbb{R}$ be the energy functional associated with the problem $(P^c_{g_i(\cdot,\cdot,\lambda)})$, that is

$$E_{i,\lambda} := E_{c,g_i(\cdot,\cdot,\lambda)}, \tag{47}$$

where $E_{c,g_i(\cdot,\cdot,\lambda)}$ is the functional given in (15) with $g = g_i(\cdot,\cdot,\lambda)$. So, Theorem 7 allows us to deduce that, for $i \in \mathbb{N}$ sufficiently large and $\lambda \in [-\lambda_i, \lambda_i]$, there exists $u_{i,\lambda} \in W^{\eta_i}$ such that

$$\min_{u \in W^{\eta_i}} E_{i,\lambda}(u) = E_{i,\lambda}(u_{i,\lambda}), \tag{48}$$

$$u_{i,\lambda}(k) \in [0, \delta_i] \text{ for all } k \in [1, T] \tag{49}$$

and

$$u_{i,\lambda} \text{ is a non-negative weak solution of } (P^c_{g_i(\cdot,\cdot,\lambda)}). \tag{50}$$

Since for i sufficiently large

$$0 \leqslant u_{i,\lambda}(k) \leqslant \delta_i < \eta_i, \tag{51}$$

for all $k \in [1, T]$ by (44) and (49), we get

$$g_i(k, u_{i,\lambda}(k), \lambda) = g(k, u_{i,\lambda}(k), \lambda).$$

Thus, using (43) we obviously have that $u_{i,\lambda}$ is a non-negative weak solution of (P_λ), provided i is large and $|\lambda| \leqslant \lambda_i$.

In the sequel, we will prove that for any $n \in \mathbb{N}$ problem (P_λ) admits at least n distinct solutions, for suitable values of λ. At this purpose, first of all note that, due to the choices of c and g_i and (51), the functional $E_{i,\lambda}$ is given by

$$\begin{aligned} E_{i,\lambda}(u) &= \frac{1}{2}\|u\|^2 - \lambda \sum_{k=1}^{T} a(k) \frac{|u(k)|^{p+1}}{p+1} - \sum_{k=1}^{T} F(u(k)) \\ &= E_{i,0}(u) - \lambda \sum_{k=1}^{T} a(k) \frac{|u(k)|^{p+1}}{p+1}, \text{ for any } u \in H. \end{aligned} \tag{52}$$

Next, for $\lambda = 0$, the function $g_i(\cdot,\cdot,\lambda) = g_i(\cdot,\cdot,0)$ verifies the hypotheses (11), (24), (25) and (26). More precisely, $g_i(\cdot,\cdot,0)$ is exactly the function appearing in (30) and $E_i := E_{i,0}$ is the energy functional associated with problem $(P^c_{g_i(\cdot,\cdot,0)})$. Thus, besides (48)-(50), the elements $u_i := u_{i,0}$ also verify

$$E_i(u_i) = \min_{u \in W^{\eta_i}} E_i(u) \leqslant E_i(w_i) < 0 \text{ for all } i \in \mathbb{N}, \tag{53}$$

where $w_i \in W^{\eta_i}$ is given in the proof of Theorem 2, see for instance (39).

In the sequel, let $\{\theta_i\}_i$ be an increasing sequence with negative terms such that $\lim\limits_{i \to +\infty} \theta_i = 0$. On account of (53), up to a subsequence, we may assume that

$$\theta_{i-1} < E_i(u_i) \leqslant E_i(w_i) < \theta_i, \tag{54}$$

for $i \geqslant i^*$, with $i^* \in \mathbb{N}$.

Now, for any $i \geqslant i^*$ let

$$\lambda'_i := \frac{(p+1)(E_i(u_i) - \theta_{i-1})}{(\|a\|_\infty + 1)T} \text{ and } \lambda''_i := \frac{(p+1)(\theta_i - E_i(w_i))}{(\|a\|_\infty + 1)T}. \tag{55}$$

Note that λ'_i and λ''_i are strictly positive, due to (54) and they are independent of λ.

Now, for any fixed $n \in \mathbb{N}$, let

$$\Lambda_n := \min\{\lambda_{i^*+1}, \dots, \lambda_{i^*+n}, \lambda'_{i^*+1}, \dots, \lambda'_{i^*+n}, \lambda''_{i^*+1}, \dots, \lambda''_{i^*+n}\}.$$

On account of (54), $\Lambda_n > 0$ and it is independent of λ. Moreover, if $|\lambda| \leqslant \Lambda_n$, then $|\lambda| \leqslant \lambda_i$ for any $i = i^* + 1, \ldots, i^* + n$. Consequently, for any $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant \Lambda_n$, we have that

$$u_{i,\lambda} \text{ is a non-negative weak solution of problem } (P_\lambda)$$

for any $i = i^* + 1, \ldots, i^* + n$.

In the sequel, we will show that these solutions are distinct. At this purpose, note that $u_{i,\lambda} \in W^{\eta_i}$ by (51) and so for any $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant \Lambda_n$ we have

$$E_i(u_i) = \min_{u \in W^{\eta_i}} E_i(u) \leqslant E_i(u_{i,\lambda}). \tag{56}$$

Thus, by (52) and (56), for any λ with $|\lambda| \leqslant \Lambda_n$ we get

$$\begin{aligned} E_{i,\lambda}(u_{i,\lambda}) &= E_i(u_{i,\lambda}) - \frac{\lambda}{p+1}\sum_{k=1}^{T} a(k)|u_{i,\lambda}(k)|^{p+1} \\ &\geqslant E_i(u_i) - \frac{|\lambda|}{p+1}\|a\|_\infty \eta_i^{p+1} T \\ &\geqslant E_i(u_i) - \frac{\Lambda_n}{p+1}\|a\|_\infty T \\ &\geqslant E_i(u_i) - \frac{\lambda_i'}{p+1}\|a\|_\infty T \\ &> \theta_{i-1}, \end{aligned} \tag{57}$$

for any $i = i^* + 1, \ldots, i^* + n$, due to (44), (51), the choice of Λ_n and the definition of λ_i'.

On the other hand, by (52), (53) and using the fact that $\|w_i\|_\infty = \tilde{s}_i \leqslant \delta_i < \eta_i < 1$ (see the proof of Theorem 2), for any λ with $|\lambda| \leqslant \Lambda_n$ we deduce that

$$\begin{aligned} E_{i,\lambda}(u_{i,\lambda}) &= \min_{u \in W^{\eta_i}} E_{i,\lambda}(u) \\ &\leqslant E_{i,\lambda}(w_i) \\ &= E_i(w_i) - \frac{\lambda}{p+1}\sum_{k=1}^{T} a(k)|w_i(k)|^{p+1} \\ &\leqslant E_i(w_i) + \frac{|\lambda|}{p+1}\|a\|_\infty T \\ &\leqslant E_i(w_i) + \frac{\Lambda_n}{p+1}\|a\|_\infty T \\ &\leqslant E_i(w_i) + \frac{\lambda_i''}{p+1}\|a\|_\infty T \\ &< \theta_i, \end{aligned} \tag{58}$$

for any $i = i^* + 1, \ldots, i^* + n$, again thanks to the choice of Λ_n and the definition of λ_i''.

In conclusion, by (57), (58) and the properties of $\{\theta_i\}_i$, we deduce that for every $i = i^* + 1, \ldots, i^* + n$ and $\lambda \in [-\Lambda_n, \Lambda_n]$, we have

$$\theta_{i-1} < E_{i,\lambda}(u_{i,\lambda}) < \theta_i < 0, \tag{59}$$

which yields that

$$E_{1,\lambda}(u_{1,\lambda}) < \ldots < E_{n,\lambda}(u_{n,\lambda}) < 0.$$

But $u_{i,\lambda} \in W^{\eta_1}$ for every $i = i^* + 1, \ldots, i^* + n$, so $E_{i,\lambda}(u_{i,\lambda}) = E_{1,\lambda}(u_{i,\lambda})$, see relation (46). Therefore, from above, we obtain that for every $\lambda \in [-\Lambda_n, \Lambda_n]$,

$$E_{1,\lambda}(u_{1,\lambda}) < \ldots < E_{1,\lambda}(u_{n,\lambda}) < 0 = E_{1,\lambda}(0).$$

These inequalities show that the elements $u_{1,\lambda}, \ldots, u_{n,\lambda}$ are all distinct and non-trivial, provided $\lambda \in [-\Lambda_n, \Lambda_n]$.

Finally, it remains to prove conclusion (7). For this, by (44), (51), (52), (59) and the continuity of f we have that

$$\begin{aligned}\frac{1}{2}\|u_{i,\lambda}\|^2 &= E_{i,\lambda}(u_{i,\lambda}) + \frac{\lambda}{p+1}\sum_{k=1}^{T} a(k)|u_{i,\lambda}(k)|^{p+1} + \sum_{k=1}^{T} F(u_{i,\lambda}(k)) \\ &< \theta_i + \frac{|\lambda|}{p+1}\|a\|_\infty \delta_i^{p+1} T + \sum_{k=1}^{T}\int_0^{\delta_i} |f(s)|ds \\ &< \frac{\Lambda_n}{p+1}\|a\|_\infty \delta_i T + T\max_{s\in[0,1]}|f(s)|\delta_i,\end{aligned}$$

for any $i = i^*+1,\dots,i^*+n$ and $|\lambda| \leqslant \Lambda_n$. Hence, we obtain

$$\|u_{i,\lambda}\| \leqslant \tilde{c}\delta_i^{1/2},$$

where $\tilde{c} = 2^{-1}\left(\frac{\Lambda_n}{p+1}\|a\|_\infty T + T\max_{s\in[0,1]}|f(s)|\right) > 0$.

Since $\delta_i \to 0$ as $i \to +\infty$, without loss of generality, we may assume that

$$\delta_i \leqslant \min\{\tilde{c}^{-2}, 1\}\frac{1}{i^2}, \tag{60}$$

which gives that

$$\|u_{i,\lambda}\| \leqslant \frac{1}{i},$$

for any $i = i^*+1,\dots,i^*+n$, provided $|\lambda| \leqslant \Lambda_n$.

In conclusion, by (51) and (60) we obtain that

$$\|u_{i,\lambda}\|_\infty \leqslant \frac{1}{i^2} < \frac{1}{i},$$

for any $i = i^*+1,\dots,i^*+n$, with $|\lambda| \leqslant \Lambda_n$.

This concludes the proof of Theorem 3. □

5. Oscillation at infinity

This section is devoted to the study of problem (P_λ) in the case when f oscillates at infinity.

In order to prove Theorem 4 and Theorem 5, we will proceed in a similar way as in the previous section. However, for completeness, we give all the details.

We consider again the problem (P_g^c), where the Carathéodory function $g : [1,T] \times [0,+\infty) \to \mathbb{R}$ fulfills the following assumptions

$$\begin{aligned}&g(k,0) = 0 \text{ for all } k \in [1,T], \text{ and} \\ &\text{for any } s \geqslant 0, \text{ there exists } M > 0 \text{ such that } \max_{t\in[0,s]}|g(k,t)| \leqslant M \text{ for all } k \in [1,T];\end{aligned} \tag{61}$$

$$\begin{aligned}&\text{there exist two sequences } \{\delta_i\}_i \text{ and } \{\eta_i\}_i \text{ with } 0 < \delta_i < \eta_i < \delta_{i+1} \text{ such that} \\ &\lim_{i\to+\infty}\delta_i = +\infty \text{ and } g(k,s) \leqslant 0 \text{ for every } k \in [1,T] \text{ and for all } s \in [\delta_i,\eta_i],\ i \in \mathbb{N};\end{aligned} \tag{62}$$

$$\begin{aligned}&-\infty < \liminf_{s\to+\infty}\frac{G(k,s)}{s^2} \text{ and } \limsup_{s\to+\infty}\frac{G(k,s)}{s^2} > \frac{1}{T} \text{ uniformly for all } k \in [1,T], \\ &\text{where } G(k,s) = \int_0^s g(k,t)dt.\end{aligned} \tag{63}$$

In the sequel, we will prove Theorem 4. Here, the strategy is similar to that of Theorem 2 and the idea consists in applying Theorem 7 to problem (P_g^c) with a suitable choice for the functions c and g appearing in the equation.

5.1. **Proof of Theorem 4.** Let us start proving assertion (i). In this case when $p = 1$ and $l\infty \in (-\infty, 0)$, we fix $\lambda \in \mathbb{R}$ such that $\lambda a(k) < \lambda_\infty$ for all $k \in [1, T]$ and some $0 < \lambda_\infty < -l_\infty$.

Let us choose $\overline{\lambda}_\infty \in (\lambda_\infty, -l_\infty)$ and let

$$c(k) := \overline{\lambda}_\infty - \lambda a(k) \text{ and } g(k, s) := f(s) + \overline{\lambda}_\infty s \text{ for all } (k, s) \in [1, T] \times [0, +\infty). \tag{64}$$

Firstly, we show that the functions c and g given in (64) satisfy the assumptions (11), (61), (62) and (63).

It is clear that $\min_{k\in[1,T]} c(k) > \overline{\lambda}_\infty - \lambda_\infty > 0$ and $c \in l^\infty$ thanks to the fact that $a \in l^\infty$, so (11) is satisfied.

Since $f(0) = 0$ by assumption and using the regularity of f, it is easy to see that g is a continuous function in $[1, T] \times [0, +\infty)$ and $g(k, 0) = 0$ for all $k \in [1, T]$. Also, the continuity of $s \mapsto g(\cdot, s)$ and the Weierstrass theorem yield (61).

Note that

$$\frac{G(k, s)}{s^2} = \frac{\overline{\lambda}_\infty}{2} + \frac{F(s)}{s^2}, \text{ for any } k \in [1, T] \text{ and } s > 0.$$

Thus, hypothesis (f_1^∞) implies (63).

In the sequel, since $l_\infty < -\overline{\lambda}_\infty$ and using (f_2^∞), there exists a sequence $\{s_i\}_i \subset (0, +\infty)$ converging to $+\infty$ as $i \to +\infty$ such that

$$f(s_i) < -\overline{\lambda}_\infty s_i \text{ for all } i \in \mathbb{N} \text{ large enough.}$$

Thus, we have

$$g(k, s_i) = f(s_i) + \overline{\lambda}_\infty s_i < 0.$$

Consequently, by using the continuity of f, we may fix two sequences $\{\delta_i\}_i$, $\{\eta_i\}_i \subset (0, +\infty)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}$, $\lim_{i\to+\infty} \delta_i = +\infty$ and $g(k, s) = \overline{\lambda}_\infty s + f(s) \leqslant 0$ for any $k \in [1, T]$ and all $s \in [\delta_i, \eta_i]$ and $i \geqslant i^*$, $i^* \in \mathbb{N}$.

Therefore, hypothesis (62) is also fulfilled for g on every interval $[\delta_i, \eta_i]$, $i \in \mathbb{N}$.

For any $i \in \mathbb{N}$, we consider again the truncation function $g_i : [1, T] \times [0, +\infty) \to \mathbb{R}$ by

$$g_i(k, s) := g(k, \tau_{\eta_i(s)}) \text{ and } G_i(k, s) := \int_0^s h_i(k, t)dt, \tag{65}$$

for every $k \in [1, T]$ and $s \geqslant 0$, where τ_{η_i} is the function defined in (23) with $\eta = \eta_i$.

Let $E_i : H \to \mathbb{R}$ be the energy functional associated with problem $(P^c_{g_i})$, that is $E_i := E_{c,g_i}$, where E_{c,g_i} is the functional given in (15) with $g = g_i$.

Taking into account hypotheses (61) and (62), it is easily seen that the function g_i fulfills all the assumptions of Theorem 7 for any $i \in \mathbb{N}$. Thus, for every $i \in \mathbb{N}$, there exists $u_i \in W^{\eta_i}$ such that

$$\min_{u\in W^{\eta_i}} E_i(u) = E_i(u_i); \tag{66}$$

$$u_i(k) \in [0, \delta_i] \text{ for every } k \in [1, T]; \tag{67}$$

$$u_i \text{ is a non-negative weak solution of } (P^c_{g_i}). \tag{68}$$

Arguing as in the proof of Theorem 2 and taking into account the definition of g_i, (62) and (67), it is easily seen that

$$g_i(k, u_i(k)) = g(k, \tau_{\eta_i}(u_i(k))) = g(k, u_i(k)) \text{ for every } k \in [1, T].$$

Thus, by the above relation and (68), u_i is also a non-negative weak solution for the problem (P^c_g).

In the sequel, we need to show that there are infinitely many distinct elements in the sequence $\{u_i\}_i$. To this end, first of all we claim that, up to a subsequence,

$$\lim_{i\to+\infty} E_i(u_i) = -\infty. \tag{69}$$

Indeed, due to (f_1^∞) and (64), we have that

$$\begin{aligned}\limsup_{s\to+\infty}\frac{G(k,s)}{s^2} &= \frac{\overline{\lambda}_\infty}{2}+\limsup_{s\to+\infty}\frac{F(s)}{s^2}\\ &> \frac{\overline{\lambda}_\infty}{2}+\frac{1}{T}.\end{aligned}$$

In particular, for a small $\varepsilon_\infty>0$, there exists a sequence $\{\tilde{s}_i\}_i$, tending to $+\infty$ such that

$$G(k,\tilde{s}_i)>\left(\frac{1}{T}+\frac{\overline{\lambda}_\infty}{2}+\varepsilon_\infty\right)\tilde{s}_i^2. \tag{70}$$

Since $\delta_i\nearrow+\infty$ by (62), we can choose a subsequence of $\{\delta_i\}_i$ still denoted by $\{\delta_i\}_i$ such that

$$0<\tilde{s}_i\leqslant\delta_i \text{ for all } i\in\mathbb{N}. \tag{71}$$

Let $i\in\mathbb{N}$ be fixed and let us define the function $w_i\in H$ by

$$w_i(k)=\tilde{s}_i \text{ for every } k\in[1,T].$$

Then $\|w_i\|_\infty=\tilde{s}_i\leqslant\delta_i<\eta_i$ by (62) and (71). Hence, $w_i\in W^{\eta_i}$. This yields that for every $k\in[1,T]$, we have

$$\begin{aligned}G_i(k,w_i(k)) &= G_i(k,\tilde{s}_i)=\int_0^{\tilde{s}_i}g_i(k,t)dt\\ &= \int_0^{\tilde{s}_i}g(k,\tau_{\eta_i}(t))dt=\int_0^{\tilde{s}_i}g(k,t)dt\\ &= G(k,\tilde{s}_i).\end{aligned} \tag{72}$$

Then, by using (11), (64), (70) and (72), for i sufficiently large we have

$$\begin{aligned}E_i(w_i) &= \frac{1}{2}\sum_{k=1}^{T+1}|\Delta w_i(k-1)|^2+\frac{1}{2}\sum_{k=1}^{T}c(k)(w_i(k))^2-\sum_{k=1}^{T}G_i(k,w_i(k))\\ &< (\tilde{s}_i)^2+\frac{1}{2}\overline{\lambda}_\infty T(\tilde{s}_i)^2-TG(k,\tilde{s}_i)\\ &< (\tilde{s}_i)^2+\frac{1}{2}\overline{\lambda}_\infty T(\tilde{s}_i)^2-T\left(\frac{1}{T}+\frac{\overline{\lambda}_\infty}{2}+\varepsilon_\infty\right)(\tilde{s}_i)^2\\ &= -T\varepsilon_\infty(\tilde{s}_i)^2.\end{aligned}$$

By construction, we know that $w_i\in W^{\tilde{s}_i}\subset W^{\eta_i}$.

Consequently, by the above relations and (66), we have

$$E_i(u_i)=\min_{u\in W^{\eta_i}}E_i(u)\leqslant E_i(w_i)<-T\varepsilon_\infty(\tilde{s}_i)^2 \text{ for all } i\in\mathbb{N}. \tag{73}$$

Since $\lim_{i\to+\infty}\tilde{s}_i=+\infty$, by relation (73) it easily follows claim (69).

As a consequence of (69) we get that the sequence $\{u_i\}_i$ has infinitely many distinct elements (and, in particular, $u_i\neq0$ in $[1,T]$, being $E_i(0)=0$). Indeed, let us assume that in the sequence $\{u_i\}_i$ there is only a finite number of elements, say $\{u_1,\dots,u_n\}$ for some $n\in\mathbb{N}$. Consequently, due to (65), the sequence $\{E_i(u_i)\}_i$ reduces to at most the finite set $\{E_1(u_1),\dots,E_n(u_n)\}$ which contradicts relation (69). Hence problem (P_g^c) admits infinitely many distinct weak solutions.

It remains to prove (9). Since the norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent, it is enough to prove that $\lim_{i\to+\infty}\|u_i\|_\infty=+\infty$. Arguing by contradiction, we assume that for a subsequence of $\{u_i\}_i$, still denoted by $\{u_i\}_i$, there exists a constant $C > 0$ such that

$$\|u_i\|_\infty \leqslant C \text{ for all } i \in \mathbb{N}.$$

Therefore, we have

$$\begin{aligned}
E_i(u_i) &= \frac{1}{2}\sum_{k=1}^{T+1} |\Delta u_i(k-1)|^2 + \frac{1}{2}\sum_{k=1}^{T} c(k)(u_i(k))^2 - \sum_{k=1}^{T} G_i(k, u_i(k)) \\
&\geqslant -\sum_{k=1}^{T} G_i(k, u_i(k)) = -\sum_{k=1}^{T} G(k, u_i(k)) \\
&= -\sum_{k=1}^{T} \int_0^{u_i(k)} g(k,s)ds \\
&\geqslant -\sum_{k=1}^{T} \max_{s\in[0,C]} |g(k,s)| u_i(k) \\
&\geqslant -\delta_i T \max_{s\in[0,C]} |g(\cdot, s)|.
\end{aligned}$$

Since $\lim\limits_{i\to+\infty} \delta_i = +\infty$, by (62) the above inequality contradicts relation (69).

Thus, we get the existence of infinitely many distinct nontrivial non-negative solutions $\{u_i\}_i$ for problem (P_g^c) satisfying condition (9).

Due to the choice of c and g in (64) and taking into account that $p = 1$, it is easy to see that (P_g^c) is equivalent to problem (P_λ). So, u_i is a weak solution of problem (P_λ) which concludes the proof of assertion (i).

Now, let us consider assertion (ii). In this case when $p = 1$ and $l_\infty = -\infty$, we take $\lambda \in \mathbb{R}$ be arbitrary fixed, $\overline{\lambda}_\infty \in (0, -l_\infty)$ and

(74) $\quad c(k) := \overline{\lambda}_\infty$ and $g(k,s) = f(s) + (\lambda a(k) + \overline{\lambda}_\infty)s$ for all $(k,s) \in [1,T] \times [0, +\infty)$.

In this setting the arguments are the same of the ones used in the previous case.

Now, let us prove assertion (iii). In this case when $0 < p < 1$, let $\lambda \in \mathbb{R}$ be arbitrary fixed and we choose $\overline{\lambda}_\infty \in (0, -l_\infty)$ and

(75) $\quad c(k) := \overline{\lambda}_\infty$ and $g(k,s) := \lambda a(k)s^p + \overline{\lambda}_\infty s + f(s)$ for all $(k,s) \in [1,T] \times [0, +\infty)$.

Also in this setting our aim is to prove that c and g given in (75) satisfy the conditions (11), (61), (62) and (63).

Hypothesis (11) is clearly satisfied. By assumption we know that $f(0) = 0$ and thus $g(k, 0) = 0$ for all $k \in [1,T]$. Due to the fact that $a \in l^\infty$, the continuity of $s \mapsto g(\cdot, s)$ and the Weierstrass theorem yield that (61) hold too.

Furthermore, since $p < 1$ and

$$\frac{G(k,s)}{s^2} = \lambda \frac{a(k)}{p+1} s^{p-1} + \frac{\overline{\lambda}_\infty}{2} + \frac{F(s)}{s^2}, \text{ for all } k \in [1,T] \text{ and } s \in (0, +\infty),$$

hypothesis (f_1^∞) implies (63).

Next, for all $k \in [1,T]$ and every $s \in [0, +\infty)$, we have

$$g(k,s) \leqslant |\lambda| \cdot \|a\|_\infty s^p + \overline{\lambda}_\infty s + f(s). \tag{76}$$

Due to (f_2^∞) and (76) we have

$$\begin{aligned}
\liminf_{s\to+\infty} \frac{h(k,s)}{s} &\leqslant \liminf_{s\to+\infty} \left(|\lambda| \cdot \|a\|_\infty s^{p-1} + \overline{\lambda}_\infty + \frac{f(s)}{s} \right) \\
&= \overline{\lambda}_\infty + l_\infty \\
&< 0,
\end{aligned} \tag{77}$$

for all $k \in [1, T]$, thanks to the choice of p, i.e. $p < 1$.

Therefore, we can fix a sequence $\{s_i\}_i \subset (0, +\infty)$ converging to $+\infty$ as $i \to +\infty$ such that $g(k, s_i) < 0$ for all $i \in \mathbb{N}$ large enough and for every $k \in [1, T]$. Thus, by using the continuity of $s \mapsto h(\cdot, s)$, there exist two sequences $\{\delta_i\}_i$, $\{\eta_i\}_i \subset (0, +\infty)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}$, $\lim_{i\to+\infty} \delta_i = +\infty$ and $g(k, s) \leqslant 0$, for every $k \in [1, T]$ and all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$ large enough. Thus, hypothesis (62) hold true.

Finally, arguing as in the proof of assertion (i) we observe that problem (P_g^c) is equivalent to problem (P_λ) through the choice (75). This ends the proof of Theorem 4. $\square$

5.2. **Proof of Theorem 5.** Here, we point out that we will use an argument analogous to that used in Theorem 3, so the strategy will consist in applying Theorem 7 to problem (P_g^c) with a suitable choices of c and g.

At this purpose, let $\overline{\lambda}_\infty \in (0, -l_\infty)$, where $l_\infty < 0$ is given in assumption (f_2^∞) and let us choose

$$c(k) := \overline{\lambda}_\infty \text{ and } g(k, s, \lambda) := \lambda a(k) s^p + \overline{\lambda}_\infty s + f(s), \tag{78}$$

for all $(k, s) \in [1, T] \times [0, +\infty)$, $\lambda \in \mathbb{R}$. Note that for all $k \in [1, T]$ and every $s \in [0, +\infty)$, we have

$$g(k, s, \lambda) \leqslant |\lambda| \cdot \|a\|_\infty s^p + \overline{\lambda}_\infty s + f(s).$$

In the sequel, since $l_\infty < -\lambda_\infty$ and using (f_2^∞), there exists a sequence $\{s_i\}_i \subset (0, +\infty)$ converging to $+\infty$ as $i \to +\infty$ such that

$$f(s_i) < -\overline{\lambda}_\infty s_i, \text{ for } i \in \mathbb{N} \text{ large enough.}$$

Thus, we have

$$g(k, s_i, 0) = \overline{\lambda}_\infty s_i + f(s_i) < 0,$$

for $i \in \mathbb{N}$ large enough and for all $k \in [1, T]$. Due to the continuity of $s \mapsto g(\cdot, s, \cdot)$ we can fix three sequences $\{\delta_i\}_i$, $\{\eta_i\}_i \subset (0, +\infty)$, $\{\lambda_i\}_i \subset (0, 1)$ such that

$$0 < \delta_i < s_i < \eta_i < \delta_{i+1}, \lim_{i\to+\infty} \delta_i = +\infty, \tag{79}$$

and for $i \in \mathbb{N}$ large enough,

$$g(k, s, \lambda) \leqslant 0, \text{ for all } k \in [1, T], \ \lambda \in [-\lambda_i, \lambda_i] \text{ and } s \in [\delta_i, \eta_i]. \tag{80}$$

Without any loss of generality, we may assume that

$$\delta_i \geqslant i, \ i \in \mathbb{N}. \tag{81}$$

For any $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$, let $g_i : [1, T] \times [0, +\infty) \times [-\lambda_i, \lambda_i] \to \mathbb{R}$ be the function defined by

$$g_i(k, s, \lambda) := g(k, \tau_{\eta_i}(s), \lambda) \tag{82}$$

and

$$G_i(k, s, \lambda) := \int_0^s g_i(k, t, \lambda) dt,$$

for all $k \in [1, T]$ and $s \geqslant 0$.

Let $E_{i,\lambda} : H \to \mathbb{R}$ be the energy functional associated with the problem $(P^c_{g_i(\cdot,\cdot,\lambda)})$, which is the same as in the proof of Theorem 3 (see relation (47)).

Note that for every $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$ the functions c given in (78) and g_i fulfills all the hypotheses of Theorem 7, the arguments being the same as in the proof of Theorem 3.

Consequently, applying Theorem 7 we get that, for $i \in \mathbb{N}$ sufficiently large and $\lambda \in [-\lambda_i, \lambda_i]$, there exists $u_{i,\lambda} \in W^{\eta_i}$ such that

$$\min_{u \in W^{\eta_i}} E_{i,\lambda}(u) = E_{i,\lambda}(u_{i,\lambda}), \tag{83}$$

$$u_{i,\lambda}(k) \in [0, \delta_i] \text{ for all } k \in [1, T] \tag{84}$$

and

$$u_{i,\lambda} \text{ is a non-negative weak solution of } (P^c_{g_i(\cdot,\cdot,\lambda)}). \tag{85}$$

Now, by (79) and (84) for i sufficiently large and for all $k \in [1, T]$, we have

$$0 \leqslant u_{i,\lambda}(k) \leqslant \delta_i < \eta_i, \tag{86}$$

and thus

$$g_i(k, u_{i,\lambda}(k), \lambda) = g(k, u_{i,\lambda}(k), \lambda). \tag{87}$$

On account of the definition of the functions g_i and c, and relations (85) and (87), $u_{i,\lambda}$ is also a non-negative weak solution of problem (P_λ), provided i is large and $|\lambda| \leqslant \lambda_i$.

It remains to prove that for any $n \in \mathbb{N}$ problem (P_λ) admits at least n distinct solutions, for suitable values of λ. At this purpose, thanks to the choices of c and g_i and (86), the functional $E_{i,\lambda}$ is given by

$$\begin{aligned} E_{i,\lambda}(u) &= \frac{1}{2}\|u\|^2 - \lambda \sum_{k=1}^{T} a(k) \frac{|u(k)|^{p+1}}{p+1} - \sum_{k=1}^{T} F(u(k)) \\ &= E_{i,0}(u) - \lambda \sum_{k=1}^{T} a(k) \frac{|u(k)|^{p+1}}{p+1}, \text{ for any } u \in H. \end{aligned} \tag{88}$$

Note that, for $\lambda = 0$, the function $g_i(\cdot, \cdot, \lambda) = g_i(\cdot, \cdot, 0)$ verifies the hypotheses (11), (61), (62) and (63). In fact, $g_i(\cdot, \cdot, 0)$ is the function appearing in (65) and $E_i := E_{i,0}$ is the energy functional associated with the problem $(P^c_{g_i(\cdot,\cdot,0)})$. Denoting $u_i := u_{i,0}$, up to a subsequence we also have

$$E_i(u_i) = \min_{u \in W^{\eta_i}} E_i(u) \leqslant E_i(w_i) \text{ for all } i \in \mathbb{N}, \tag{89}$$

$$\lim_{i \to +\infty} E_i(u_i) = -\infty, \tag{90}$$

where $w_i \in W^{\eta_i}$ appear in the proof of Theorem 4, see relations (69) and (73), respectively.

Let us fix a sequence $\{\theta_i\}_i$ with negative terms such that $\lim\limits_{i \to +\infty} \theta_i = -\infty$. Due to (89) and (90), up to a subsequence, we may assume that

$$\theta_i < E_i(u_i) \leqslant E_i(w_i) < \theta_{i-1}, \tag{91}$$

for $i \geqslant i^*$, with $i^* \in \mathbb{N}$.

For any $i \geqslant i^*$ let

$$\lambda_i' := \frac{(p+1)(E_i(u_i) - \theta_i)}{(\|a\|_\infty + 1) T \delta_i^{p+1}} \text{ and } \lambda_i'' := \frac{(p+1)(\theta_{i-1} - E_i(w_i))}{(\|a\|_\infty + 1) T \delta_i^{p+1}}. \tag{92}$$

Note that λ_i' and λ_i'' are strictly positive, due to (91) and they are independent of λ.

Now, fix $n \in \mathbb{N}$ and let

$$\Lambda_n := \min\{\lambda_{i^*+1}, \ldots, \lambda_{i^*+n}, \lambda'_{i^*+1}, \ldots, \lambda'_{i^*+n}, \lambda''_{i^*+1}, \ldots, \lambda''_{i^*+n}\}.$$

On account of (91), $\Lambda_n > 0$ and it is independent of λ. Moreover, if $|\lambda| \leqslant \Lambda_n$, then $|\lambda| \leqslant \lambda_i$ for any $i = i^* + 1, \ldots, i^* + n$. Thus, for any $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant \Lambda_n$, we have that

$$u_{i,\lambda} \text{ is a non-negative weak solution of problem } (P_\lambda)$$

for any $i = i^* + 1, \ldots, i^* + n$.

Next, we will show that these solutions are distinct. At this purpose, note that $u_{i,\lambda} \in W^{\eta_i}$ by (86) and so for any $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant \Lambda_n$ we have

$$E_i(u_i) = \min_{u \in W^{\eta_i}} E_i(u) \leqslant E_i(u_{i,\lambda}). \tag{93}$$

Thus, by (88) and (93), for any λ with $|\lambda| \leqslant \Lambda_n$ we have

$$
\begin{aligned}
E_{i,\lambda}(u_{i,\lambda}) &= E_i(u_{i,\lambda}) - \frac{\lambda}{p+1}\sum_{k=1}^{T} a(k)|u_{i,\lambda}(k)|^{p+1} \\
&\geqslant E_i(u_i) - \frac{|\lambda|}{p+1}\|a\|_\infty \delta_i^{p+1} T \\
&\geqslant E_i(u_i) - \frac{\Lambda_n}{p+1}\|a\|_\infty \delta_i^{p+1} T \\
&\geqslant E_i(u_i) - \frac{\lambda_i'}{p+1}\|a\|_\infty \delta_i^{p+1} T \\
&> \theta_i,
\end{aligned} \tag{94}
$$

for any $i = i^* + 1, \ldots, i^* + n$, thanks to (86), the choice of Λ_n and the definition of λ_i'.

On the other hand, by (88), (89) and using the fact that $\|w_i\|_\infty = \tilde{s}_i \leqslant \delta_i$ (see the proof of Theorem 4) and the definition of λ_i'', for any λ with $|\lambda| \leqslant \Lambda_n$ and for any $i = i^*+1, \ldots, i^*+n$ we deduce that

$$
\begin{aligned}
E_{i,\lambda}(u_{i,\lambda}) &= \min_{u \in W^{\eta_i}} E_{i,\lambda}(u) \\
&\leqslant E_{i,\lambda}(w_i) \\
&= E_i(w_i) - \frac{\lambda}{p+1}\sum_{k=1}^{T} a(k)|w_i(k)|^{p+1} \\
&\leqslant E_i(w_i) + \frac{|\lambda|}{p+1}\|a\|_\infty \delta_i^{p+1} T \\
&\leqslant E_i(w_i) + \frac{\Lambda_n}{p+1}\|a\|_\infty \delta_i^{p+1} T \\
&\leqslant E_i(w_i) + \frac{\lambda_i''}{p+1}\|a\|_\infty \delta_i^{p+1} T \\
&< \theta_{i-1}.
\end{aligned} \tag{95}
$$

Consequently, for every $i = i^* + 1, \ldots, i^* + n$ and $\lambda \in [-\Lambda_n, \Lambda_n]$, by (94) and (95) and the properties of $\{\theta_i\}_i$, we have

$$\theta_i < E_{i,\lambda}(u_{i,\lambda}) < \theta_{i-1} < 0, \tag{96}$$

and therefore

$$E_{n,\lambda}(u_{n,\lambda}) < \ldots < E_{1,\lambda}(u_{1,\lambda}) < 0. \tag{97}$$

Note that $u_{i,\lambda} \in W^{\eta_n}$ for every $i = i^* + 1, \ldots, i^* + n$, so $E_{i,\lambda}(u_{i,\lambda}) = E_{n,\lambda}(u_{i,\lambda})$, see relation (82). From above, for every $\lambda \in [-\Lambda_n, \Lambda_n]$, we have

$$E_{n,\lambda}(u_{n,\lambda}) < \ldots < E_{n,\lambda}(u_{1,\lambda}) < 0 = E_{n,\lambda}(0).$$

In particular, the solutions $u_{1,\lambda}, \ldots, u_{n,\lambda}$ are all distinct and non-trivial, whenever $\lambda \in [-\Lambda_n, \Lambda_n]$.

Finally, it remains to prove conclusion (10). For this, we assume that $n \geqslant 2$ and fix $\lambda \in [-\Lambda_n, \Lambda_n]$. We prove that

$$\|u_{i,\lambda}\|_\infty > \delta_{i-1} \text{ for all } i \in \{2, \ldots, n\}. \tag{98}$$

Let us assume that there exists an element $i_0 \in \{2, \ldots, n\}$ such that $\|u_{i_0,\lambda}\|_\infty \leqslant \delta_{i_0-1}$. Since $\delta_{i_0-1} < \eta_{i_0-1}$, then $u_{i_0,\lambda} \in W^{\eta_{i_0-1}}$. Thus, on account of (83) and (82), we have

$$E_{i_0-1,\lambda}(u_{i_0-1,\lambda}) = \min_{u \in W^{\eta_{i_0-1}}} E_{i_0-1,\lambda} \leqslant E_{i_0-1,\lambda}(u_{i_0,\lambda}) = E_{i_0,\lambda}(u_{i_0,\lambda}),$$

which contradicts (97). Therefore, (98) hold true.

Thus, from (81) we have $\|u_{i,\lambda}\|_\infty > i - 1$ for all $i \in \{1, \ldots, n\}$. This ends the proof of Theorem 5. $\square$

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