FIELD THEORY FOR MULTI-PARTICLE SYSTEM

TIAN MA AND SHOUHONG WANG

Abstract. The main objectives of this article are 1) to introduces some basic postulates for quantum multi-particle systems, and 2) to develop a universal field theory for interacting multi-particle systems coupling both particle fields and interacting fields. By carefully examining the nature of interactions between multi-particles, we conclude that multi-particle systems must obey both the gauge symmetry and the principle of interaction dynamics (PID). Hence a few basic postulates for multi-particle systems are introduced based on gauge invariance and PID, leading to a field theory for interacting multi-particle systems. A direct consequence of the field theory is the derivation of general atomic spectrum equations. Another important application of this field theory is a unified field model coupling matter fields, with the energy-momentum tensor $T_{\mu\nu}$, *geometrized* as hoped by Einstein and Nambu.

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1. Introduction

Classical quantum dynamic equations describe single or a few particle systems. The existing model for a multi-particle system is non-relativistic and is based on

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prescribing the interaction between particles using such potentials as the Coulomb potential. As far as we know, there is still no good model for a multi-particle system, which takes also into consideration the dynamic interactions between particles.

The main objectives of this article are 1) to introduces some basic postulates for quantum multi-particle systems, and 2) to develop a universal interactive field theory for multi-particle systems coupling both particle fields and interaction fields. Hereafter we describe some main ingredients of this study.

1. The main obstacle for establishing a field theory for an interacting multi-particle system is the lack of basic principles to describe the dynamic interactions of the particles. To seek the needed principles, we proceed with two observations.

The first observation is that one natural outcome of the field theory of four interactions developed recently by the authors is that the coupling constants for the $U(1) \times SU(2) \times SU(3)$ gauge theory play the role of the three charges e, g_w and g_s for electromagnetism, the weak and the strong interaction.

Now we consider an N-particle system with each particle carrying an interaction charge g. Let this be a fermionic system, and the Dirac spinors be given by

$$\Psi = (\psi_1, \cdots, \psi_N)^T,$$

which obeys the Dirac equations:

$$i\gamma^\mu D_\mu \Psi + M\Psi = 0, \tag{1.1}$$

where M is the mass matrix, and

$$D_\mu \Psi = \partial_\mu \Psi + igG\Psi, \tag{1.2}$$

where $G = (G^{ij}_\mu)$ is an Hermitian matrix, representing the interacting potentials between the N-particles generated by the interaction charge g.

Now let

$$\{\tau_0, \ \tau_1, \ \cdots, \ \tau_K \mid K = N^2 - 1\}$$

be a basis of the set of all Hermitian matrices, where $\tau_0 = I$ is the identity, and τ_a $(1 \leq a \leq N^2 - 1)$ are the traceless Hermitian matrices. Then the Hermitian matrix $G = (G^{ij}_\mu)$ and the differential operator D_μ in (1.1) can be expressed as

$$\begin{aligned} G &= G^0_\mu I + G^a_\mu \tau_a, \\ D_\mu &= \partial_\mu + igG^0_\mu + igG^a_\mu \tau_a. \end{aligned}$$

Consequently the Dirac equations (1.1) are rewritten as

$$i\gamma^\mu [\partial_\mu + igG^0_\mu + igG^a_\mu \tau_a]\Psi + M\Psi = 0. \tag{1.3}$$

The second observation is that the energy contributions of the N particles are indistinguishable, which implies that the $SU(N)$ gauge invariance. Hence (1.3) are exactly the Dirac equations in the form of $SU(N)$ gauge fields $\{G^a_\mu \mid 1 \leq a \leq N^2-1\}$ with a given external interaction field G^0_μ.

With these two observations, it is natural for us to postulate that (see Postulate 3.7)

> *the Lagrangian action for an N-particle system satisfy the $SU(N)$ gauge invariance.*

Also, it is then natural to postulate that (see Postulate 3.8)

> gG^a_μ *represent the interaction potentials between the particles.*

2. In an $SU(N)$ gauge theory, the gauge fields depend on the specific representation generators $\{\tau_1, \cdots, \tau_K\}$. The principle of representation invariance (PRI) introduced in [5] amounts to saying that an $SU(N)$ gauge theory should be invariant under the $SU(N)$ representation transformation

$$\widetilde{\tau}_a = x_a^b \tau_b \qquad \text{for } X = (x_a^b) \text{ being a nondegenerate complex matrix.} \tag{1.4}$$

One important consequence of PRI is that there exists a constant $SU(N)$ tensor

$$\alpha_a^N = (\alpha_1^N, \cdots, \alpha_N^N),$$

such that the contraction field using PRI

$$G_\mu = \alpha_a^N G^a \tag{1.5}$$

is independent of the $SU(N)$ representation τ_a, and is the interaction field which can be experimentally observed. This observation leads us to postulate that (see Postulate 3.9)

> *for an N-particle system, only the interaction field G_μ in (1.5) can be measured, and is the interaction field under which this system interacts with other external systems.*

3. Multi-particle systems are layered, and with the aforementioned postulates and basic symmetry principles, we are able to determine in a unique fashion field equations for different multi-particle systems.

For example, given an N-particle system consisting of N fermions with given charge g, the $SU(N)$ gauge symmetry dictates uniquely the Lagrangian density, given in two parts: 1) the sector of $SU(N)$ gauge fields $\mathcal{L}_G$ and 2) the Dirac sector of particle fields $\mathcal{L}_D$ (see (5.4)):

$$\begin{aligned} &\mathcal{L}_G = -\frac{1}{4\hbar c}\mathcal{G}_{ab} g^{\mu\alpha} g^{\nu\beta} G^a_{\nu\mu} G^b_{\alpha\beta}, \\ &\mathcal{L}_D = \bar{\Psi}\left[i\gamma^\mu\left(\partial_\mu + \frac{ig}{\hbar c}G^0_\mu + \frac{ig}{\hbar c}G^a_\mu \tau_a\right) - \frac{c}{\hbar}M\right]\Psi, \end{aligned} \tag{1.6}$$

where

$$\begin{aligned} &\mathcal{G}_{ab} = \frac{1}{2}\mathrm{Tr}(\tau_a \tau_b^\dagger), \\ &G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + \frac{g}{\hbar c}\lambda^a_{bc} G^b_\mu G^c_\nu. \end{aligned}$$

The combined action is invariant under 1) the $SU(N)$ gauge transformation, 2) the representation generator transformation (1.4), and 3) the Lorentz transformation.

The field equations are then derived by using the principle of interaction dynamics (PID) for the interaction fields G^a_μ, and by using the principle of Lagrangian Dynamics (PLD) for the Dirac spinor fields:

$$\begin{aligned} &\mathcal{G}_{ab}\left[\partial^\nu G^b_{\nu\mu} - \frac{g}{\hbar c}\lambda^b_{cd} g^{\alpha\beta} G^c_{\alpha\mu} G^d_\beta\right] - g\bar{\Psi}\gamma_\mu \tau_a \Psi \\ &\quad = \left[\partial_\mu - \frac{k^2}{4}x_\mu + \frac{g\alpha}{\hbar c}G_\mu + \frac{g\beta}{\hbar c}G^0_\mu\right]\phi_a \qquad \text{for } 1 \le a \le N^2 - 1, \end{aligned} \tag{1.7}$$

$$i\gamma^\mu\left[\partial_\mu + \frac{ig}{\hbar c}G^0_\mu + \frac{ig}{\hbar c}G^a_\mu \tau_a\right]\Psi - \frac{c}{\hbar}M\Psi = 0, \tag{1.8}$$

where G^0_μ is the interaction field of external systems, α and β are constants, taking values 0 and ±1 determined by the underlying physical situations. Here PID was first postulated in [4] by the authors, evidenced by the existence of dark matter and dark energy.

Now the interaction between the N particles is clearly described by the gauge fields G^a_μ and the dual fields ϕ_a based on PID. In a nutshell, the field equations for an N fermionic particle system are completely determined by the gauge invariance combined with PID, stated as basic postulates earlier.

We also note that the Lagrangian action obeys the gauge invariance, but the field equations spontaneously break the gauge symmetry, due essentially to the fields on the right-hand sides of the field equations.

4. Also, we establish a unified field model coupling matter fields, which matches the vision of Einstein and Nambu, as stated in Nambu's Nobel lecture [10] (see Section 6.2). Basically, one needs to geometrize the energy-momentum tensor $T_{\mu\nu}$ appearing in the Einstein field equations. For example, for multi-particle system under gravity and electromagnetism, using the basic postulates as outlined above, a unified field model can be naturally derived so that the energy-momentum tensor $T_{\mu\nu}$ is derived from first principles and is geometrized; see Section 6.2 for details.

The paper is organized as follows. Section 2 examines the classical multi-particle systems and Section 4 recalls PID and PRI. Section 3 introduces new basic postulates for multi-particle systems, leading to field equations for multi-particle Sections in Section 5. Section 6 derives unified field model coupling matter fields and Section 7 gives atomic spectrum equations.

2. Classical Theory of Multi-Particle Systems

We start with the known model of multi-particle systems. Consider an N-particle system with particles

$$A_1, \cdots, A_N. \tag{2.1}$$

Let $x_k = (x^1_k, x^2_k, x^3_k) \in \mathbb{R}^3$ be the coordinate of A_k, and

$$\psi = \psi(t, x_1, \cdots, x_k) \tag{2.2}$$

be the wave function describing the N-particle system (2.1). Then, the classical theory for (2.1) is provided by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\sum_{k=1}^N \frac{\hbar^2}{2m_k} \Delta_k \psi + \sum_{j \neq k} V(x_j, x_k)\psi, \tag{2.3}$$

where $V(x_j, x_k)$ is the potential energy of interactions between A_j and A_k, m_k is the mass of A_k, and

$$\Delta_k = \frac{\partial^2}{(\partial x^1_k)^2} + \frac{\partial^2}{(\partial x^2_k)^2} + \frac{\partial^2}{(\partial x^3_k)^2}.$$

The wave function ψ satisfies the normalization condition

$$\int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} |\psi|^2 dx_1 \cdots dx_N = 1.$$

Namely, the physically

$$|\psi(t, x_1, \cdots, x_N)|^2$$

represents the probability density of $A_1, \cdots, A_N$ appearing at $x_1, \cdots, x_N$ at time t.

It is clear that the Schrödinger equation (2.3) for an N-particle system is only an approximate model:

- It is non-relativistic model;
- The model does not involve the vector potentials $\vec{A}$ of the interactions between particles.
- By using coordinate x_k to represent the particle A_k amounts essentially to saying that the wave function ψ satisfying (2.3) can only describe the statistic properties of the system (2.1), and contains no information for each individual particle A_k ($1 \leq k \leq N$).
- The model is decoupled with interaction fields, which are given functions appearing in the interacting multi-particle model.

In fact, the most remarkable characteristic of interacting multi-particle systems is that both particle fields and interaction fields are closely related. Therefore, a complete field model of multi-particle systems have to couple both the particle field equations and the interaction field equations. In particular, a precise unified field theory should be based on the field model of the multi-particle system coupled with the four fundamental interactions.

3. Basic Postulates for Multi-Particle Quantum Physics

3.1. **Basic postulates of quantum mechanics.** For completeness, we first recall the basic postulates of quantum physics.

Postulate 3.1. *A quantum system consists of some micro-particles, which are described by a set of complex value functions $\psi = (\psi_1, \cdots, \psi_N)^T$, called wave functions. In other words, each quantum system is identified by a set of wave functions ψ:*

$$\text{a quantum system} = \psi, \tag{3.1}$$

which contain all quantum information of this system.

Postulate 3.2. *For a single particle system described by a wave function ψ, its modular square*

$$|\psi(x,t)|^2$$

represents the probability density of the particle being observed at point $x \in \mathbb{R}^3$ and at time t. Hence, ψ satisfies that

$$\int_{\mathbb{R}^3} |\psi|^2 dx = 1.$$

Postulate 3.3. *Each observable physical quantity L corresponds to an Hermitian operator $\hat{L}$, and the values of the physical quantity L are given by eigenvalues λ of $\hat{L}$:*

$$\hat{L}\psi_\lambda = \lambda\psi_\lambda,$$

and the eigenfunction ψ_λ is the state function in which the physical quantity L takes value λ. In particular, the Hermitian operators corresponding to position x, momentum p and energy E are given by

$$\begin{aligned} &\text{position operator:} && \hat{x}\psi = x\psi, \\ &\text{momentum operator:} && \hat{p}\psi = -i\hbar\nabla\psi, \\ &\text{energy operator:} && \hat{E}\psi = i\hbar\frac{\partial\psi}{\partial t}. \end{aligned} \tag{3.2}$$

Postulate 3.4. *For a quantum system ψ and a physical Hermitian operator $\hat{L}, \psi$ can be expanded as*

$$\psi = \sum \alpha_k \psi_k + \int \alpha_\lambda \psi_\lambda d\lambda, \tag{3.3}$$

where ψ_k and ψ_λ are the eigenfunctions of $\hat{L}$ corresponding to discrete and continuous eigenvalues respectively. In (3.3) for the coefficients α_k and α_λ, their modular square $|\alpha_k|^2$ and $|\alpha_\lambda|^2$ represent the probability of the system ψ in the states ψ_k and ψ_λ. In addition, the following integral, denoted by

$$\langle\psi|\hat{L}|\psi\rangle = \int \psi^\dagger(\hat{L}\psi)dx, \tag{3.4}$$

represents the average value of physical quantity $\hat{L}$ of system ψ.

Postulate 3.5. *For a quantum system with observable physical quantities $l_1, \cdots, l_N$, if they satisfy a relation*

$$R(l_1, \cdots, l_N) = 0,$$

then the quantum system ψ (see (3.1)) satisfies the equation

$$R(\hat{L}_1, \cdots, \hat{L}_N)\psi = 0,$$

where $\hat{L}_k$ are the Hermitian operators corresponding to l_k $(1 \leq k \leq N)$, provided that $R(\hat{L}_1, \cdots, \hat{L}_N)$ is a Hermitian.

We remark that in addition to the three basic Hermitian operators given by (3.2), the other Hermitian operators often used in quantum physics are as follows:

$$\begin{aligned} &\text{angular momentum:} && \hat{L} = \hat{x} \times \hat{p} = -i\hbar\vec{r} \times \nabla, \\ &\text{spin operator:} && \hat{S} = s\hbar\vec{\sigma}, \\ &\text{scalar momentum:} && \hbar p_0 = i\hbar(\vec{\sigma} \cdot \nabla) \text{ (massless fermion)}, \\ &\text{scalar momentum:} && \hat{p}_1 = -i\hbar(\vec{\alpha} \cdot \nabla) \text{ (massive fermion)}, \\ &\text{Hamiltonian energy :} && \hat{H} = \hat{K} + \hat{V} + \hat{M}, \end{aligned} \tag{3.5}$$

where s is the spin, $\vec{\sigma}$ and $\vec{\alpha}$ are the Pauli and Dirac matrix vectors, and $\hat{K}, \hat{V}, \hat{M}$ are the kinetic energy, potential energy, and mass operators.

3.2. **Basic postulates for multi-particle quantum systems.** As mentioned earlier, the dynamic models for multi-particle quantum systems have to couple both particle and interaction fields. Therefore there should be some added quantum rules for the systems. In the following we propose the basic postulates for N-particle quantum systems.

First of all, the physical systems have to satisfy a few fundamental physical principles introduced below.

Postulate 3.6. *Any N-particle quantum system has to obey the physical fundamental principles such as:*

$$\begin{aligned}&\textit{Einstein General Relativity,}\\&\textit{Lorentz Invariance,}\\&\textit{Gauge Invariance,}\\&\textit{Gauge Representation Invariance (PRI),}\\&\textit{Principle of Lagrange Dynamics (PLD),}\\&\textit{Principle of Interaction Dynamics (PID),}\end{aligned}\tag{3.6}$$

where the gauge invariance means the invariance of the Lagrangian action under corresponding gauge transformations.

We note that in general multi-particle systems are layered, and may consist of numerous sub-systems. In particular, we know that the weak and strong interactions are also layered. Hence, here we consider the same level systems, i.e. the systems which consist of identical particles or sub-systems possessing the same level of interactions.

For multi-particle systems with N same level subsystems A_k $(1 \leq k \leq N)$, the energy contributions of A_k are indistinguishable. Hence, the Lagrangian actions for the N-particle systems satisfy $SU(N)$ gauge invariance. Thus we propose the following basic postulate:

Postulate 3.7. *An N-particle system obeys the $SU(N)$ gauge invariance, i.e. the Lagrangian action of this system is invariant under the $SU(N)$ gauge transformation*

$$\begin{pmatrix}\widetilde{\psi}_1\\ \vdots\\ \widetilde{\psi}_N\end{pmatrix} = \Omega \begin{pmatrix}\psi_1\\ \vdots\\ \psi_N\end{pmatrix}, \qquad \Omega \in SU(N),\tag{3.7}$$

where $\psi_1, \cdots, \psi_N$ are the wave functions of the N particles.

We now need to explain the physical significance of the $SU(N)$ gauge fields induced by Postulate 3.7.

Let each particle of the N-particle system carry an interaction charge g (for example a weak charge $g = g_w$). Then, there are interactions present between the N particles. By the $SU(N)$ gauge theory, the gauge invariant 4-dimensional energy-momentum operator is given by

$$D_\mu = \partial_\mu + igG^a_\mu \tau_a \qquad \text{for } 1 \leq a \leq N^2 - 1,\tag{3.8}$$

and the interaction energy generated by the N particles is

$$E = \begin{cases} \bar{\Psi}(i\gamma^\mu D_\mu \Psi) & \text{for fermions,}\\ |D_\mu \Psi|^2 & \text{for bosons,}\end{cases}\tag{3.9}$$

where $\Psi = (\psi_1, \cdots, \psi_N)^T$, and D_μ is as in (3.8). From (3.8) and (3.9) we obtain the physical explanation to the $SU(N)$ gauge fields G^a_μ, stated in the following postulate:

Postulate 3.8. *For an N-particle system with each particle carrying an interaction charge g, the N particles induce dynamic interactions between them, and the $SU(N)$ gauge fields*

$$gG^a_\mu \qquad \text{for } 1 \leq a \leq N^2-1 \tag{3.10}$$

stand for the interaction potentials between the N particles.

The N particles induce dynamic interactions between them in terms of the $SU(N)$ gauge fields (3.10). These interaction fields cannot be measured experimentally because they depend on the choice of generator representation τ_a of $SU(N)$. By PRI given in the next section, there is a constant $SU(N)$ tensor

$$\alpha^N_a = (\alpha^N_1, \cdots, \alpha^N_N), \tag{3.11}$$

such that the contraction field using PRI

$$G_\mu = \alpha^N_a G^a \tag{3.12}$$

is independent of the $SU(N)$ representation τ_a. The field (3.12) is the interaction field which can be experimentally observed. Thus we propose the following basic postulate.

Postulate 3.9. *For an N-particle system, only the interaction field given by (3.12) can be measured, and is the interaction field under which this system interacts with other external systems.*

Remark 3.1. *Postulates 3.6-3.9, together with the Postulates 3.1-3.5, form a complete foundation for quantum physics. In fact, without Postulates 3.6-3.9, we cannot establish the quantum physics of multi-particle systems.*

The main motivation to introduce Postulates 3.7 and 3.8 are as follows. Consider an N-particle system with each particle carrying an interaction charge g. Let this be a fermionic system, and the Dirac spinors be given by

$$\Psi = (\psi_1, \cdots, \psi_N)^T.$$

By Postulates 3.3 and 3.5, the Dirac equations for this system can be expressed in the general form

$$i\gamma^\mu D_\mu \Psi + M\Psi = 0, \tag{3.13}$$

where M is the mass matrix, and

$$D_\mu \Psi = \partial_\mu \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} + ig \begin{pmatrix} G^{11}_\mu & \cdots & G^{1N}_\mu \\ \vdots & & \vdots \\ G^{N1}_\mu & \cdots & G^{NN}_\mu \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}, \tag{3.14}$$

where $G = (G^{ij}_\mu)$ is an Hermitian matrix, representing the interaction potentials between the N particles generated by the interaction charge g.

Notice that the space consisting of all Hermitian matrices

$$H(N) = \{G \mid G \ \text{ is an N-th order Hermitian matrix}\}$$

is an N^2-dimensional linear space with basis

$$\tau_0, \ \tau_1, \ \cdots, \ \tau_K \qquad \text{with } K = N^2-1, \tag{3.15}$$

where $\tau_0 = I$ is the identity, and τ_a $(1 \leq a \leq N^2 - 1)$ are the traceless Hermitian matrices. Hence, the Hermitian matrix $G = (G^{ij}_\mu) \in H(N)$ in (3.14) can be expressed as

$$G = G^0_\mu I + G^a_\mu \tau_a \qquad \text{with } \ \tau_a \ \text{ as in (3.15).}$$

Thus, the differential operator in (3.14) is in the form

$$D_\mu = \partial_\mu + igG^0_\mu + igG^a_\mu \tau_a. \tag{3.16}$$

The equations (3.13) with (3.16) are just the Dirac equations in the form of $SU(N)$ gauge fields $\{G^a_\mu \mid 1 \leq a \leq N^2 - 1\}$ with a given external interaction field G^0_μ. Thus, based on Postulate 3.6, the gauge invariance of an N-particle system and the expressions (3.13) and (3.16) of the N fermionic particle field equations dictate Postulates 3.7 and 3.8.

The derivation here indicates that Postulates 3.7 and 3.8 can be considered as the consequence of 1) the gauge invariance stated in Postulate 3.6, and 2) the existence of interactions between particles as stated in (3.14), which can be considered as an axiom.

4. Two Fundamental Principles

4.1. **Principle of Interaction Dynamics (PID).** The main objective in this section is to recall a fundamental principle of physics, the principle of interaction dynamics (PID), first introduced in [4] by the authors. Intuitively, PID takes the variation of the action functional under energy-momentum conservation constraint. As demonstrated in [6, 4], there are strong physical evidence and motivations for the validity of PID, including

(1) the discovery of dark matter and dark energy,
(2) the non-existence of solutions for the classical Einstein gravitational field equations in general cases,
(3) the principle of spontaneous gauge-symmetry breaking, and
(4) the theory of Ginzburg-Landau superconductivity.

Let $(M, g_{\mu\nu})$ be the 4-dimensional space-time Riemannian manifold with $\{g_{\mu\nu}\}$ the Minkowski type Riemannian metric. For an (r, s)-tensor u we define the A-gradient and A-divergence operators ∇_A and div_A as

$$\begin{aligned} \nabla_A u &= \nabla u + u \otimes A, \\ \text{div}_A u &= \text{ div } u - A \cdot u, \end{aligned}$$

where A is a vector field and here stands for a gauge field, ∇ and div are the usual gradient and divergent covariant differential operators. Let $F = F(u)$ be a functional of a tensor field u. A tensor u_0 is called an extremum point of F with the div_A-free constraint, if u_0 satisfies the equation

$$\frac{d}{d\lambda}\Big|_{\lambda=0} F(u_0 + \lambda X) = \int_M \delta F(u_0) \cdot X \sqrt{-g} dx = 0 \qquad \forall X \text{ with } \ \text{div}_A X = 0. \tag{4.1}$$

Principle 4.1 (Principle of Interaction Dynamics). (1) *For all physical interactions there are Lagrangian actions*

$$L(g, A, \psi) = \int_M \mathcal{L}(g_{\mu\nu}, A, \psi) \sqrt{-g} dx, \tag{4.2}$$

where $g = \{g_{\mu\nu}\}$ is the Riemannian metric representing the gravitational potential, A is a set of vector fields representing the gauge potentials, and ψ are the wave functions of particles.

(2) *The actions (4.2) satisfy the invariance of general relativity, Lorentz invariance, gauge invariance and the gauge representation invariance.*

(3) *The states (g, A, ψ) are the extremum points of (4.2) with the div_A-free constraint (4.1).*

Based on PID and the Orthogonal Decomposition Theorems in [4], the field equations with respect to the action (4.2) are given in the form

$$\frac{\delta}{\delta g_{\mu\nu}} L(g, A, \psi) = (\nabla_\mu + \alpha_b A^b_\mu)\Phi_\nu, \tag{4.3}$$

$$\frac{\delta}{\delta A^a_\mu} L(g, A, \psi) = (\nabla_\mu + \beta^a_b A^b_\mu)\varphi^a, \tag{4.4}$$

$$\frac{\delta}{\delta \psi} L(g, A, \psi) = 0, \tag{4.5}$$

where $A^a_\mu = (A^a_0, A^a_1, A^a_2, A^a_3)$ are the gauge vector fields for the electromagnetic, the weak and the strong interactions, $\Phi_\nu = (\Phi_0, \Phi_1, \Phi_2, \Phi_3)$ in (4.3) is a vector field induced by gravitational interaction, φ^a are the scalar fields generated from the gauge fields A^a_μ, and α_b, β^a_b are coupling parameters.

PID is based on variations with div_A-free constraint defined by (4.1). Physically, the conditions

$$\text{div}_A X = 0 \qquad \text{in (4.1)}$$

stand for the energy-momentum conservation constraints.

4.2. **Principle of representation invariance (PRI).** We now recall the principle of representation invariance (PRI) first postulated in [5]. We proceed with the $SU(N)$ representation. In a neighborhood $U \subset SU(N)$ of the unit matrix, a matrix $\Omega \in U$ can be written as

$$\Omega = e^{i\theta^a \tau_a},$$

where

$$\tau_a = \{\tau_1, \cdots, \tau_K\} \subset T_e SU(N), \qquad K = N^2 - 1, \tag{4.6}$$

is a basis of generators of the tangent space $T_e SU(N)$. An $SU(N)$ representation transformation is a linear transformation of the basis in (4.6) as

$$\widetilde{\tau}_a = x^b_a \tau_b, \tag{4.7}$$

where $X = (x^b_a)$ is a nondegenerate complex matrix.

Mathematical logic dictates that a physically sound gauge theory should be invariant under the $SU(N)$ representation transformation (4.7). Consequently, the following principle of representation invariance (PRI) must be universally valid and was first postulated in [5].

Principle 4.2 (Principle of Representation Invariance)**.** *All $SU(N)$ gauge theories are invariant under the transformation (4.7). Namely, the actions of the gauge fields are invariant and the corresponding gauge field equations as given by (4.3)-(4.5) are covariant under the transformation (4.7).*

Direct consequences of PRI include the following; see also [5] for details:

- The physical quantities such as θ^a, A^a_μ, and λ^c_{ab} are $SU(N)$-tensors under the generator transformation (4.7).
- The tensor

$$\mathcal{G}_{ab} = \frac{1}{4N}\lambda^c_{ad}\lambda^d_{cb} = \frac{1}{2}\mathrm{Tr}(\tau_a\tau_b^\dagger) \tag{4.8}$$

  is a symmetric positive definite 2nd-order covariant $SU(N)$-tensor, which can be regarded as a Riemannian metric on $SU(N)$.
- The representation invariant action is

$$L = \int_M -\frac{1}{4}\mathcal{G}_{ab}g^{\mu\alpha}g^{\nu\beta}F^a_{\mu\nu}F^b_{\alpha\beta} + \bar{\Psi}\left[i\gamma^\mu(\partial_\mu + igA^a_\mu\tau_a) - m\right]\Psi,$$

  and the representation invariant gauge field equations are

$$\begin{aligned}&\mathcal{G}_{ab}\left[\partial^\nu F^b_{\nu\mu} - g\lambda^b_{cd}g^{\alpha\beta}F^c_{\alpha\mu}A^d_\beta\right] - g\bar{\Psi}\gamma_\mu\tau_a\Psi = (\partial_\mu + \alpha_b A^b_\mu)\phi_a,\\ &(i\gamma^\mu D_\mu - m)\Psi = 0.\end{aligned}$$

As we indicated in [5], the field models based on PID appear to be the only model which obeys PRI. In particular, both the standard model and the electroweak theory violate PRI, and consequently they are approximate models of the fundamental interactions of Nature.

5. Field Equations of Multi-Particle Systems

Based on the basic axioms given by Postulates 3.6-3.9, we can establish field equations for various levels of N-particle systems. We proceed in several different cases.

Fermionic systems

Consider N fermions with interaction charge g, the wave functions (Dirac spinors) are given by

$$\Psi = (\psi_1, \cdots, \psi_N)^T, \qquad \psi_k = (\psi^1_k, \psi^2_k, \psi^3_k, \psi^4_k)^T \quad \text{for } 1 \le k \le N, \tag{5.1}$$

with the mass matrix

$$M = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_N \end{pmatrix}. \tag{5.2}$$

By Postulates 3.6 and 3.7, the Lagrangian action for the N-particle system (5.1)-(5.2) must be in the form

$$L = \int(\mathcal{L}_G + \mathcal{L}_D)dx, \tag{5.3}$$

where $\mathcal{L}_G$ is the sector of the $SU(N)$ gauge fields, and $\mathcal{L}_D$ is the Dirac sector of particle fields:

$$\begin{aligned}\mathcal{L}_G &= -\frac{1}{4\hbar c}\mathcal{G}_{ab}g^{\mu\alpha}g^{\nu\beta}G^a_{\nu\mu}G^b_{\alpha\beta},\\ \mathcal{L}_D &= \bar{\Psi}\left[i\gamma^\mu\left(\partial_\mu + \frac{ig}{\hbar c}G^0_\mu + \frac{ig}{\hbar c}G^a_\mu\tau_a\right) - \frac{c}{\hbar}M\right]\Psi,\end{aligned} \tag{5.4}$$

where G^a_μ $(1 \leq a \leq N^2-1)$ are the $SU(N)$ gauge fields representing the interactions between the N particles, τ_a $(1 \leq a \leq N^2-1)$ are the generators of $SU(N)$, and

$$\begin{aligned} \mathcal{G}_{ab} &= \frac{1}{2}\mathrm{Tr}(\tau_a\tau_b^\dagger), \\ G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + \frac{g}{\hbar c}\lambda^a_{bc}G^b_\mu G^c_\nu. \end{aligned}$$

According to PID and PLD, for the action (5.3) the field equations are given by

$$\begin{aligned} &\frac{\delta L}{\delta G^a_\mu} = D_\mu\phi_a && \text{by PID,} \\ &\frac{\delta L}{\delta \Psi} = 0 && \text{by PLD,} \end{aligned} \tag{5.5}$$

where D_μ is the PID gradient operator given by

$$D_\mu = \frac{1}{\hbar c}\left(\partial_\mu - \frac{1}{4}k^2 x_\mu + \frac{g\alpha}{\hbar c}G_\mu + \frac{g\beta}{\hbar c}G^0_\mu\right),$$

G_μ is as in (3.12), α and k are parameters, k^{-1} stands for the range of attracting force of the interaction, and $\left(\frac{g\alpha}{\hbar c}\right)^{-1}$ is the range of the repelling force.

Thus, by (5.4) and (5.5) we derive the field equations of the N-particle system (5.1)-(5.2) as follows

$$\begin{aligned} &\mathcal{G}_{ab}\left[\partial^\nu G^b_{\nu\mu} - \frac{g}{\hbar c}\lambda^b_{cd}g^{\alpha\beta}G^c_{\alpha\mu}G^d_\beta\right] - g\bar{\Psi}\gamma_\mu\tau_a\Psi \\ &\qquad = \left[\partial_\mu - \frac{1}{4}k^2x_\mu + \frac{g\alpha}{\hbar c}G_\mu + \frac{g\beta}{\hbar c}G^0_\mu\right]\phi_a \qquad \text{for } 1 \leq a \leq N^2-1, \end{aligned} \tag{5.6}$$

$$i\gamma^\mu\left[\partial_\mu + \frac{ig}{\hbar c}G^0_\mu + \frac{ig}{\hbar c}G^a_\mu\tau_a\right]\begin{pmatrix}\psi_1\\ \vdots \\ \psi_N\end{pmatrix} - \frac{c}{\hbar}M\begin{pmatrix}\psi_1\\ \vdots \\ \psi_N\end{pmatrix} = 0, \tag{5.7}$$

where $\gamma_\mu = g_{\mu\nu}\gamma^\nu$, and G^0_μ is the interaction field of external systems. It is by this field G^0_μ that we can couple external sub-systems to the model (5.6)-(5.7).

Remark 5.1. *In the field equations of multi-particle systems there is a gauge fixing problem. In fact, we know that the action (5.3)-(5.4) is invariant under the gauge transformation*

$$\begin{aligned} &\widetilde{\Psi} = e^{i\theta^a\tau_a}\Psi, \\ &\widetilde{G}^a_\mu\tau_a = G^a_\mu e^{i\theta^b\tau_b}\tau_a e^{-i\theta^b\tau_b} + \frac{i}{g}\partial_\mu e^{i\theta^b\tau_b}e^{-i\theta^b\tau_b}. \end{aligned} \tag{5.8}$$

Hence if (Ψ, G^a_μ) *is a solution of*

$$\delta L = 0, \tag{5.9}$$

then $(\widetilde{\Psi}, \widetilde{G}^a_\mu)$ *is a solution of (5.9) as well. In (5.8) we see that* $\widetilde{G}^a_\mu$ *have* N^2-1 *free functions*

$$\theta^a(x) \qquad \text{with } 1 \leq a \leq N^2-1. \tag{5.10}$$

In order to eliminate the N^2-1 *freedom of (5.10), we have to supplement* N^2-1 *gauge fixing equations for the equation (5.9). Now, as we replace the PLD equation (5.9). By the PID equations (5.5), (5.8) breaks the gauge invariance. Therefore the* N^2-1 *freedom of (5.10) is eliminated. However, in the PID equations (5.5)* *there are additional* N^2-1 *new unknown functions* ϕ_a $(1 \le a \le N^2-1)$. *Hence, the gauge fixing problem still holds true. There are two possible ways to solve this problem:*

(1) *there might exist some unknown fundamental principles, which can provide the all or some of the* N^2-1 *gauge fixing equations; and*
(2) *there might be no general physical principles to determine the gauge fixing equations, and these equations will be determined by underlying physical system.*

Bosonic systems

Consider N bosons with charge g, the Klein-Gordon fields are

$$\Phi = (\varphi_1, \cdots, \varphi_N)^T,$$

and the mass matrix is given by (5.2). The action is

$$L = \int (\mathcal{L}_G + \mathcal{L}_{KG}) dx \tag{5.11}$$

where $\mathcal{L}_G$ is as given by (5.4), and $\mathcal{L}_{KG}$ is the Klein-Gordon sector given by

$$\begin{aligned} \mathcal{L}_{KG} &= \frac{1}{2}|D_\mu \Phi|^2 + \frac{1}{2}\left(\frac{c}{\hbar}\right)^2 |M\Phi|^2 \\ D_\mu &= \partial_\mu + \frac{ig}{\hbar c} G^0_\mu + \frac{ig}{\hbar c} G^a_\mu \tau_a. \end{aligned}$$

Then, the PID equations of (5.11) are as follows

$$\begin{aligned} \mathcal{G}_{ab}\left[\partial^\nu G^b_{\nu\mu} - \frac{g}{\hbar c}\lambda^b_{cd} g^{\alpha\beta} G^c_{\alpha\mu} G^d_\beta\right] &+ \frac{ig}{2}\left[(D_\mu\Phi)^\dagger(\tau_a\Phi) - (\tau_a\Phi)^\dagger(D_\mu\Phi)\right] \\ &= \left[\partial_\mu - \frac{1}{4}k^2 x_\mu + \frac{g}{\hbar c}\alpha G_\mu + \frac{g}{\hbar c}\beta G^0_\mu\right]\phi_a \qquad \text{for } 1 \le a \le N^2-1, \end{aligned} \tag{5.12}$$

$$D^\mu D_\mu \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix} - \left(\frac{c}{\hbar}\right)^2 M^2 \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix} = 0. \tag{5.13}$$

Mixed systems

Consider a maxed system consisting of N_1 fermions with n_1 charges g and N_2 bosons with n_2 charges g, and the fields are

Dirac fields:	$\Psi = (\psi_1, \cdots, \psi_{N_1})^T$,
Klein-Gordon fields:	$\Phi = (\varphi_1, \cdots, \varphi_{N_2})^T$.

The interaction fields of this system are $SU(N_1) \times SU(N_2)$ gauge fields, $SU(N_1)$ gauge fields are for fermions, and $SU(N_2)$ for bosons:

$\{G^a_\mu \mid 1 \le a \le N_1^2 - 1\}$	for Dirac fields Ψ,
$\{\widetilde{G}^k_\mu \mid 1 \le k \le N_2^2 - 1\}$	for Klein-Gordon fields Φ.

The action is given by

$$L = \int \left[\mathcal{L}^1_G + \mathcal{L}^2_G + \mathcal{L}_D + \mathcal{L}_{KG}\right] dx, \tag{5.14}$$

where $\mathcal{L}_G^1$ and $\mathcal{L}_G^2$ are the sectors of $SU(N_1)$ and $SU(N_2)$ gauge fields as given in (5.4) with $N = N_1$ and $N = N_2$ respectively.

Define the two total gauge fields of $SU(N_1)$ and $SU(N_2)$, as defined by (3.11)-(3.12):

$$\begin{aligned} G_\mu &= \alpha_a^{N_1} G_\mu^a \qquad \text{for } 1 \leq a \leq N_1^2 - 1, \\ \widetilde{G}_\mu &= \alpha_k^{N_2} \widetilde{G}_\mu^k \qquad \text{for } 1 \leq k \leq N_2^2 - 1. \end{aligned} \tag{5.15}$$

Namely, $\mathcal{L}_D$ and $\mathcal{L}_{KG}$ are given by

$$\begin{aligned} \mathcal{L}_D &= \bar{\Psi} \left[i\gamma^\mu \left(\partial_\mu + \frac{in_1 g}{\hbar c} G_\mu^0 + \frac{in_1 g}{\hbar c} G_\mu^a \tau_a^1 \right) - \frac{c}{\hbar} M_1 \right] \Psi, \\ \mathcal{L}_{KG} &= \frac{1}{2} \left| \left(\partial_\mu + \frac{in_2 g}{\hbar c} G_\mu^0 + \frac{in_2 g}{\hbar c} \widetilde{G}_\mu^k \tau_k^2 \right) \Phi \right|^2 + \frac{1}{2} \left(\frac{c}{\hbar} \right)^2 |M_2 \Phi|^2, \end{aligned}$$

where G_μ^0 is the external field.

Thus, we derive the field equations for mixed multi-particle systems expressed in the following form

$$\begin{aligned} & \mathcal{G}_{ab}^1 \left[\partial^\nu G_{\nu\mu}^b - \frac{n_1 g}{\hbar c} \lambda_{1cd}^b g^{\alpha\beta} G_{\alpha\mu}^c G_\beta^d \right] - n_1 g \bar{\Psi} \gamma_\mu \tau_a^1 \Psi \\ & \quad = \left[\partial_\mu - \frac{1}{4} k_1^2 x_\mu + \frac{n_1 g}{\hbar c} \alpha_1 G_\mu + \frac{n_2 g}{\hbar c} \alpha_2 \widetilde{G}_\mu \right] \phi_a \qquad \text{for } 1 \leq a \leq N^2 - 1, \end{aligned} \tag{5.16}$$

$$\begin{aligned} & \mathcal{G}_{kl}^2 \left[\partial^\nu \widetilde{G}_{\nu\mu}^l - \frac{n_2 g}{\hbar c} \lambda_{2ij}^l g^{\alpha\beta} \widetilde{G}_{\alpha\mu}^i \widetilde{G}_\beta^j \right] + \frac{in_1 g}{2} \left[(D_\mu \Phi)^\dagger (\tau_k^2 \Phi) - (\tau_k^2 \Phi)^\dagger (D_\mu \Phi) \right] \\ & \quad = \left[\partial_\mu - \frac{1}{4} k_2^2 x_\mu + \frac{n_1 g}{\hbar c} \beta_1 G_\mu + \frac{n_2 g}{\hbar c} \beta_2 \widetilde{G}_\mu \right] \widetilde{\phi}_k \qquad \text{for } 1 \leq k \leq N_2^2 - 1, \end{aligned} \tag{5.17}$$

$$i\gamma^\mu \left(\partial_\mu + \frac{in_1 g}{\hbar c} G_\mu^0 + \frac{in_1 g}{\hbar c} G_\mu^a \tau_a^1 \right) \Psi - \frac{c}{\hbar} M_1 \Psi = 0, \tag{5.18}$$

$$g^{\mu\nu} D_\mu D_\nu \Phi - \left(\frac{c}{\hbar} \right)^2 M_2^2 \Phi = 0, \tag{5.19}$$

where G_μ and $\widetilde{G}_\mu$ are as in (5.15), and D_μ is defined by

$$D_\mu = \partial_\mu + \frac{in_2 g}{\hbar c} G_\mu^0 + \frac{in_1 g}{\hbar c} \widetilde{G}_\mu^k \tau_k^2.$$

We remark here that the coupling interaction between fermions and bosons is directly represented on the right hand side of gauge field equations (5.16) and (5.17), due to the presence of the dual interaction fields based on PID. Namely, the interactions between particles in an N-particle system are achieved through both the interaction gauge fields and the corresponding dual fields. This fact again validates the importance of PID.

Another remark is that the Lagrangian action (5.14) obeys gauge invariance, but the field equations (5.16) and (5.17) spontaneously break the gauge symmetry, due essentially to the fields G_μ and $\widetilde{G}_\mu$ on the right-sides of the field equations.

Layered systems

Let a system be layered consisting of two levels: 1) level A consists of K sub-systems $A_1, \cdots, A_K$, and 2) level B is level inside of each sub-system A_j, which consists of N particles $B_1^j, \cdots, B_N^j$:

$$\begin{aligned} &\text{at level } A: \quad A = \{A_1, \cdots, A_K\}, \\ &\text{at level } B: \quad A_j = \{B_1^j, \cdots, B_N^j\} \qquad \text{for } 1 \leq j \leq K. \end{aligned} \tag{5.20}$$

Each particle B_i^j carries n charges g.

Let the particle field functions be

$$\begin{aligned} &\text{at level } A: && \Psi_A = (\psi_{A_1}, \cdots, \psi_{A_K}), \\ &\text{at level } B: && \Psi_{B_j} = (\psi_{B_{j_1}}, \cdots, \psi_{B_{j_N}}) \qquad \text{for } 1 \leq j \leq K. \end{aligned}$$

The interaction is the $SU(K) \times SU(N)$ gauge fields:

$$\begin{aligned} &\text{at level } A: && SU(K) \ \text{ gauge fields } A_\mu^a && 1 \leq a \leq K^2 - 1, \\ &\text{at level } B: && SU(N) \ \text{ gauge fields } (B_j)_\mu^k && 1 \leq k \leq N^2 - 1. \end{aligned}$$

Without loss of generality, we assume A and B are the fermion systems. Thus the action of this layered system is

$$L = \int \Big[\mathcal{L}_{AG} + \sum_{j=1}^{K} \mathcal{L}_{B_jG} + \mathcal{L}_{AD} + \sum_{j=1}^{K} \mathcal{L}_{B_jD}\Big] dx, \tag{5.21}$$

where

$$\begin{aligned} \mathcal{L}_{AG} = &\ \text{the sector of } SU(K) \ \text{ gauge fields}, \\ \mathcal{L}_{AD} = &\ \bar{\Psi}_A \Big[i\gamma^\mu \Big(\partial_\mu + \frac{inN}{\hbar c} g G_\mu^0 + \frac{inN}{\hbar c} g A_\mu^a \tau_a^K\Big) - \frac{c}{\hbar} M_A\Big] \Psi_A, \\ \mathcal{L}_{B_jG} = &\ \text{the j-th the sector of } SU(N) \text{ gauge fields}, \\ \mathcal{L}_{B_jD} = &\ \bar{\Psi}_{B_j} \Big[i\gamma^\mu \Big(\partial_\mu + \frac{ing}{\hbar c} G_\mu^0 + \frac{ing}{\hbar c} (B_j)_\mu^k \tau_k^N\Big) - \frac{c}{\hbar} M_{B_j}\Big] \Psi_{B_j}, \end{aligned} \tag{5.22}$$

where G_μ^0 is the external field. The corresponding PID field equations of the layered multi-particle system (5.20) follow from (5.21) and (5.22), and here we omit the details.

Remark 5.2. *Postulate 3.9 is essentially another expression of PRI, which is very crucial to couple all sub-systems together to form a complete set of field equations for a given multi-particle system. In particular, this approach is natural and unique to derive models for multi-particle systems, satisfying all fundamental principles of (3.6) and the gauge symmetry breaking principle (Principle 6.3). It is also a unique way to establish a unified field theory coupling the gravity and other interactions in various levels of multi-particle systems. In the next section we discuss this topic.*

6. Unified Field Model Coupling Matter Fields

6.1. **General Principles of the Unified Field Theory.** We recall and examine some basic ingredients of the unified field theory developed recently by the authors, based on PID and PRI recapitulated earlier. This theory focuses on 1) the interaction field particles, and 2) the interaction potentials.

Symmetry of fundamental interactions

One crucial component of this theory is that laws of the fundamental interactions are dictated by the following symmetries:

$$\begin{array}{ll} \text{gravity:} & \text{general relativity,} \\ \text{electromagnetism:} & \text{gauge invariance,} \\ \text{weak interaction:} & \text{gauge invariance,} \\ \text{strong interaction:} & \text{gauge invariance,} \end{array} \tag{6.1}$$

Also, the last three interactions in (6.1) obey the Lorentz invariance and PRI. As a natural outcome, the three charges e, g_w, g_s are the coupling constants of the corresponding gauge fields.

Following the simplicity principle of laws of Nature, the three basic symmetries—the Einstein general relativity, the Lorentz invariance and the gauge invariance—uniquely determine the interaction fields and their Lagrangian actions for the four interactions.

Mechanism of fundamental interactions

Albert Einstein was the first physicist who postulated that the gravitational force is caused by the space-time curvature. However, Yukawa's viewpoint, entirely different from Einstein's, is that the other three fundamental forces take place through exchanging intermediate bosons such as photons for the electromagnetic interaction, $\mathrm{W}^{\pm}$ and Z intermediate vector bosons for the weak interaction, and gluons for the strong interaction.

In the same spirit as the Einstein's principle of equivalence of gravitational force, it is natural for us to postulate an alternate mechanism for all four interactions. The rigorous mathematical foundation of this mechanism is developed in [4].

Geometric Interaction Mechanism 6.1. *The gravitational force is the curved effect of the time-space, and the electromagnetic, weak, strong interactions are the twisted effects of the underlying complex vector bundles* $M \otimes_p \mathbb{C}^n$.

As mentioned earlier, traditionally one adopts Yukawa's viewpoint that forces of the interactions of Nature are caused by exchanging the field mediators.

Yukawa Interaction Mechanism 6.2. *The four fundamental interactions of Nature are mediated by exchanging interaction field particles, called the mediators. The gravitational force is mediated by the graviton, the electromagnetic force is mediated by the photon, the strong interaction is mediated by the gluons, and the weak interaction is mediated by the intermediate vector bosons* $W^{\pm}$ *and* Z.

It is the Yukawa mechanism that leads to the $SU(2)$ and $SU(3)$ gauge theories for the weak and the strong interactions. In fact, the three mediators $W^{\pm}$ and Z for the weak interaction are regarded as the $SU(2)$ gauge fields W^a_μ $(1 \leq a \leq 3)$, and the eight gluons for the strong interaction are considered as the $SU(3)$ gauge fields S^k_μ $(1 \leq k \leq 8)$. Of course, the three color quantum numbers for the quarks are an important fact to choose $SU(3)$ gauge theory to describe the strong interaction.

The two interaction mechanisms lead to two entirely different directions to develop the unified field theory. The need for quantization for all current theories for the four interactions are based on the Yukawa Interaction Mechanism. The new unified field theory is based on the Geometric Mechanism, focusing directly on the four interaction forces, and does not involve a quantization process.

A radical difference for the two direction mechanisms is that the Yukawa Mechanism is oriented toward to computing the transition probability for the particle decays and scatterings, and the Geometric Interaction Mechanism is oriented toward to fundamental laws, such as interaction potentials, of the four interactions.

Gauge symmetry breaking

In physics, symmetries are displayed in two levels in the laws of Nature:

(6.2) the invariance of Lagrangian actions L,

(6.3) the covariance of variation equations of L.

The implication of the following three symmetries:

$$\begin{aligned}&\text{Einstein General Relativity},\\&\text{Lorentz Invariance},\\&\text{Gauge Representation Invariance (PRI)},\end{aligned}\tag{6.4}$$

is the universality of physical laws, i.e. the validity of laws of Nature is independent of the coordinate systems expressing them. Consequently, the symmetries in (6.4) cannot be broken at both levels of (6.2) and (6.3).

However, the physical implication of the gauge symmetry is different at the two levels (6.2) and (6.3):

1. The gauge invariance of the Lagrangian action, (6.2), amounts to saying that the energy contributions of particles in a physical system are indistinguishable.
2. The gauge invariance of the variation equations, (6.3), means that the particles involved in the interaction are indistinguishable.

It is clear that the first aspect (1) above is universally true, while the second aspect (2) is not universally true. In other words, the Lagrange actions obey the gauge invariance, but the corresponding variation equations break the gauge symmetry. This suggests us to postulate the following principle of gauge symmetry breaking for interactions described by a gauge theory.

Principle 6.3 (Gauge Symmetry Breaking). *The gauge symmetry holds true only for the Lagrangian actions for the electromagnetic, week and strong interactions, and it will be violated in the field equations of these interactions.*

The principle of gauge symmetry breaking can be regarded as part of the spontaneous symmetry breaking, which is a phenomenon appearing in various physical fields. In 2008, the Nobel Prize in Physics was awarded to Y. Nambu for the discovery of the mechanism of spontaneous symmetry breaking in subatomic physics. In 2013, F. Englert and P. Higgs were awarded the Nobel Prize for the theoretical discovery of a mechanism that contributes to our understanding of the origin of mass of subatomic particles.

Although the phenomenon was discovered in superconductivity by Ginzburg-Landau in 1951, the mechanism of spontaneous symmetry breaking in particle physics was first proposed by Y. Nambu in 1960; see [9, 11, 12]. The Higgs mechanism, discovered in [3, 1, 2], is a special case of the Nambo-Jona-Lasinio spontaneous symmetry breaking, leading to the mass generation of sub-atomic particles.

PID provides a new mechanism for gauge symmetry breaking and mass generation. The difference between both the PID and the Higgs mechanisms is that the first one is a natural sequence of the first principle, and the second is to add artificially a Higgs field in the Lagrangian action. Also, the PID mechanism obeys PRI, and the Higgs mechanism violates PRI.

6.2. **Unified Field Model Coupling Matter Fields.** The unified field theory introduced in [4, 5] considers two aspects: 1) the interaction field particles, and 2) the interaction potentials. Hence, it restricted the unified field model to be the theory based on

$$\text{Einstein relativity} + U(1) \times SU(2) \times SU(3) \text{ symmetry.} \tag{6.5}$$

However, if we consider the interaction potentials between the particles of N-particle systems, then the unified field theory has to be based on the following symmetries instead of (6.5):

$$\text{Einstein relativity} + SU(N_1) \times \cdots \times SU(N_K) \text{ symmetry,} \tag{6.6}$$

where $N_1, \cdots, N_K$ are the particle numbers of various sub-systems and layered systems.

The two types of unified field models based on (6.5) and (6.6) are mutually complementary. They have different roles in revealing the essences of interactions and particle dynamic behaviors.

In this subsection, we shall establish the unified field model of multi-particle systems based on (6.6), which matches the vision of Einstein and Nambu. In his Nobel lecture [10], Nambu stated that

> *Einstein used to express dissatisfaction with his famous equation of gravity*
>
> $$G_{\mu\nu} = 8\pi T_{\mu\nu}$$
>
> *His point was that, from an aesthetic point of view, the left hand side of the equation which describes the gravitational field is based on a beautiful geometrical principle, whereas the right hand side, which describes everything else, . . . looks arbitrary and ugly.*
>
> *... [today] Since gauge fields are based on a beautiful geometrical principle, one may shift them to the left hand side of Einsteins equation. What is left on the right are the matter fields which act as the source for the gauge fields ... Can one geometrize the matter fields and shift everything to the left?*

The gravity will be considered only in systems possessing huge amounts of particles, which we call gravitational systems. Many gravitational systems have very complicated structures. But they are composites of some simple systems. Here we only discuss two cases.

Systems with gravity and electromagnetism

Consider the system consisting of N_1 fermions with n_1 electric charges $n_1 e$ and N_2 bosons with n_2 charges $n_2 e$:

$$\Psi = (\psi_1, \cdots, \psi_{N_1}) \qquad \text{for fermions,}$$
$$\Phi = (\varphi_1, \cdots, \varphi_{N_2}) \qquad \text{for bosons.}$$

The action is given by

$$L = \int \left[\frac{c^4}{8\pi G} R + \mathcal{L}_A^{N_1} + \mathcal{L}_A^{N_2} + \hbar c \mathcal{L}_D + \hbar c \mathcal{L}_{KG} \right] \sqrt{-g} dx \tag{6.7}$$

where R is the scalar curvature, G is the gravitational constant, $g = \det(g_{\mu\nu})$, $\mathcal{L}_A^{N_1}$ and $\mathcal{L}_A^{N_2}$ are the sectors of $SU(N_1)$ and $SU(N_2)$ gauge fields for the electromagnetic interaction

$$
\begin{aligned}
\mathcal{L}_A^{N_1} &= -\frac{1}{4}\mathcal{G}_{ab}g^{\mu\alpha}g^{\nu\beta}A^a_{\mu\nu}A^b_{\alpha\beta} && 1 \leq a, b \leq N_1^2 - 1,\\
\mathcal{L}_A^{N_2} &= -\frac{1}{4}\widetilde{\mathcal{G}}_{kl}g^{\mu\alpha}g^{\nu\beta}\widetilde{A}^k_{\mu\nu}\widetilde{A}^l_{\alpha\beta} && 1 \leq k, l \leq N_2^2 - 1,\\
A^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \frac{n_1 e}{\hbar c}\lambda^a_{bc}A^b_\mu A^c_\nu && n_1 \in \mathbb{Z},\\
\widetilde{A}^k_{\mu\nu} &= \partial_\mu \widetilde{A}^k_\nu - \partial_\nu \widetilde{A}^k_\mu + \frac{n_2 e}{\hbar c}\widetilde{\lambda}^k_{ij}\widetilde{A}^i_\mu \widetilde{A}^j_\nu && n_2 \in \mathbb{Z},
\end{aligned}
\tag{6.8}
$$

and $\mathcal{L}_D, \mathcal{L}_{KG}$ are the Dirac and Klein-Gordon sectors:

$$
\begin{aligned}
\mathcal{L}_D &= \bar{\Psi}\left[i\gamma^\mu\left(\partial_\mu + \frac{in_1 e}{\hbar c}A^0_\mu + \frac{in_1 e}{\hbar c}A^a_\mu \tau_a\right) - \frac{c}{\hbar}M_1\right]\Psi,\\
\mathcal{L}_{KG} &= \frac{1}{2}g^{\mu\nu}(D_\mu\Phi)^\dagger(D_\nu\Phi) + \frac{1}{2}\left(\frac{c}{\hbar}\right)^2|M_2\Phi|^2,\\
D_\mu &= \nabla_\mu + \frac{in_2 e}{\hbar c}A^0_\mu + \frac{in_2 e}{\hbar c}\widetilde{A}^k_\mu\widetilde{\tau}_k,
\end{aligned}
\tag{6.9}
$$

where M_1 and M_2 are the masses, ∇_μ is the covariant derivative, and A^0_μ is the external electromagnetic field.

Based on PID and PLD, the field equations of (6.7) are given by

$$
\begin{aligned}
&\frac{\delta}{\delta g_{\mu\nu}}L = \frac{c^4}{8\pi G}D^G_\mu \phi^g_\nu, && \text{(PID)}\\
&\frac{\delta}{\delta A^a_\mu}L = D^A_\mu \phi_a, && \text{(PID)}\\
&\frac{\delta}{\delta \widetilde{A}^k_\mu}L = D^{\widetilde{A}}_\mu \widetilde{\phi}_k, && \text{(PID)}\\
&\frac{\delta}{\delta \Psi}L = 0, && \text{(PLD)}\\
&\frac{\delta}{\delta \Phi}L = 0, && \text{(PLD)}
\end{aligned}
\tag{6.10}
$$

where

$$
\begin{aligned}
D^G_\mu &= \nabla_\mu + \frac{n_1 e}{\hbar c}A_\mu + \frac{n_2 e}{\hbar c}\widetilde{A}_\mu,\\
D^A_\mu &= \partial_\mu - \frac{1}{4}k_1^2 x_\mu + \frac{n_1 e}{\hbar c}\alpha A_\mu + \frac{n_2 e}{\hbar c}\widetilde{\alpha}\widetilde{A}_\mu,\\
D^{\widetilde{A}}_\mu &= \partial_\mu - \frac{1}{4}k_2^2 x_\mu + \frac{n_1 e}{\hbar c}\beta A_\mu + \frac{n_2 e}{\hbar c}\widetilde{\beta}\widetilde{A}_\mu.
\end{aligned}
\tag{6.11}
$$

Here $A_\mu = \alpha^{N_1}_a A^a_\mu$ and $\widetilde{A}_\mu = \alpha^{N_2}_k \widetilde{A}^k_\mu$ are the total electromagnetic fields generated by the fermion system and the boson system.

By (6.7)-(6.9), the equations (6.10)-(6.11) are written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} + \left(\nabla_\mu + \frac{n_1 e}{\hbar c} A_\mu + \frac{n_2 e}{\hbar c} \widetilde{A}_\mu\right) \phi^g_\nu, \tag{6.12}$$

$$\begin{aligned} &\mathcal{G}_{ab}\left[\partial^\nu A^b_{\nu\mu} - \frac{n_1 e}{\hbar c} \lambda^b_{cd} g^{\alpha\beta} A^c_{\alpha\mu} A^d_\beta\right] - n_1 e \bar{\Psi} \gamma_\mu \tau_a \Psi \\ &= \left[\partial_\mu - \frac{1}{4} k_1^2 x_\mu + \frac{n_1 e}{\hbar c} \alpha A_\mu + \frac{n_2 e}{\hbar c} \widetilde{\alpha} \widetilde{A}_\mu\right] \phi_a, \end{aligned} \tag{6.13}$$

$$\begin{aligned} &\widetilde{\mathcal{G}}_{kl}\left[\partial^\nu \widetilde{A}^l_{\nu\mu} - \frac{n_2 e}{\hbar c} \widetilde{\lambda}^l_{ij} g^{\alpha\beta} \widetilde{A}^i_{\alpha\mu} \widetilde{A}^j_\beta\right] + \frac{i}{2} n_2 e \left[(D_\mu \Phi)^\dagger (\widetilde{\tau}_k \Phi) - (\widetilde{\tau}_k \Phi)^\dagger (D_\mu \Phi)\right], \\ &= \left[\partial_\mu - \frac{1}{4} k_2^2 x_\mu + \frac{n_1 e}{\hbar c} \beta A_\mu + \frac{n_2 e}{\hbar c} \widetilde{\beta} \widetilde{A}_\mu\right] \widetilde{\phi}_k, \end{aligned} \tag{6.14}$$

$$i\gamma^\mu \left[\partial_\mu + \frac{i n_1 e}{\hbar c} A^0_\mu + \frac{i n_1 e}{\hbar c} A^a_\mu \tau_a\right] \Psi - \frac{c}{\hbar} M_1 \Psi = 0, \tag{6.15}$$

$$g^{\mu\nu} D_\mu D_\nu \Phi - \left(\frac{c}{\hbar}\right)^2 M_2^2 \Phi = 0, \tag{6.16}$$

where the energy-momentum tensor $T_{\mu\nu}$ in (6.12) is

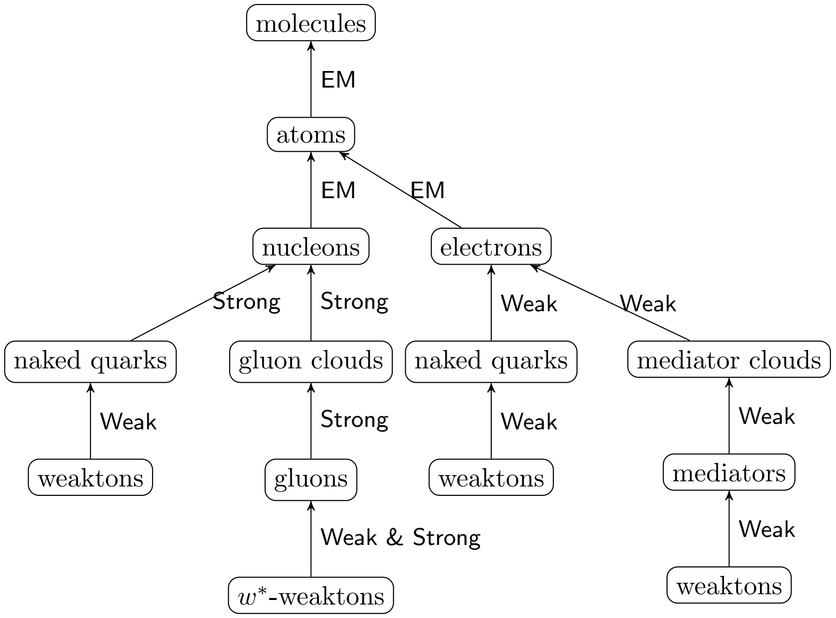
$$\begin{aligned} T_{\mu\nu} = & -\frac{1}{2} g_{\mu\nu} (\mathcal{L}_A^{N_1} + \mathcal{L}_A^{N_2} + \hbar c \mathcal{L}_D + \hbar c \mathcal{L}_{KG}) + \frac{1}{2} (D_\mu \Phi)^\dagger (D_\nu \Phi) \\ & -\frac{1}{4} \mathcal{G}_{ab} g^{\alpha\beta} A^a_{\mu\alpha} A^b_{\nu\beta} - \frac{1}{4} \widetilde{\mathcal{G}}_{kl} g^{\alpha\beta} \widetilde{A}^k_{\mu\alpha} \widetilde{A}^l_{\nu\beta}. \end{aligned} \tag{6.17}$$

The energy-momentum tensor $T_{\mu\nu}$ contains the masses M_1, M_2, the kinetic energy and electromagnetic energy.

It is clear that both sides of the field equations (6.12)-(6.16) are all generated by the fundamental principles. It is the view presented by Einstein and Nambu and shared by many physicists that the Nature obeys simple beautiful laws based on a few first physical principles. In other words, the energy-momentum tensor $T_{\mu\nu}$ is now derived from first principles and is geometrized as Einstein and Nambu hoped.

Systems with four interactions

The above systems with gravity and electromagnetism in general describe the bodies in lower energy density. For the systems in higher energy density, we have to also consider the weak and strong interactions. The interactions are layered as shown below, which were derived in [8, 7]:



The layered systems and sub-systems above determine the action of the system with four interactions as follows:

$$L = \int \frac{c^4}{8\pi G} R\sqrt{-g}dx + \text{ actions of all levels,} \tag{6.18}$$

and the action of each layered level is as given by the manner as used in (5.21)-(5.22).

Hence, the unified field model of a multi-particle system is completely determined by the layered structure of this system, as given by (6.18). It is very natural that a rationale unified field theory must couple the matter fields and interaction fields together.

Remark 6.1. *Once again we emphasize that, using PRI contractions as given by (3.12) and proper gauge fixing equations, from the unified field model (6.18) coupling matter fields for multi-particle system, we can easily deduce that the total electromagnetic field* A_μ *obtained from (6.18) satisfies the* $U(1)$ *electromagnetic gauge field equations, and derive the weak and strong interaction potentials as given in* [4, 5].

7. Atomic Spectrum

Classical quantum mechanics is essentially a subject to deal with single particle systems. Hence, the hydrogen spectrum theory was perfect under the framework of the Dirac equations. But, for general atoms the spectrum theory was defective due to lack of precise field models of multi-particle systems.

In this subsection, we shall apply the field model of multi-particle systems to establish the spectrum equations for general atoms.

Classical theory of atomic shell structure

We recall, see among others [13], that an atom with atomic number Z has energy spectrum

$$E_n = -\frac{Z^2}{n^2}\frac{me^4}{2\hbar^2}, \qquad n = 1, 2, \cdots . \tag{7.1}$$

If we ignore the interactions between electrons, the orbital electrons of this atom have the idealized discrete energies (7.1). The integers n in (7.1) are known as principal quantum number, which characterizes the electron energy levels and orbital shell order:

$$\begin{array}{llllllll} n: & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \text{shell symbol:} & K & L & M & N & O & P & Q. \end{array} \tag{7.2}$$

Each orbital electron is in some shell of (7.2) and possesses the following four quantum numbers:

(1) principle quantum number $n = 1, 2, \cdots$,
(2) orbital quantum number $l = 0, 1, 2, \cdots, (n-1)$,
(3) magnetic quantum number $m = 0, \pm 1, \cdots, \pm l$,
(4) spin quantum number $J = \pm\frac{1}{2}$.

For each given shell n, there are sub-shells characterized by orbital quantum number l, whose symbols are:

$$\begin{array}{lllllll} l: & 0 & 1 & 2 & 3 & 4 & \cdots \\ \text{sub-shell:} & s & p & d & f & g & \cdots . \end{array} \tag{7.3}$$

By the Pauli exclusion principle, at a give sub-shell nl, there are at most the following electron numbers

$$N_{nl} = N_l = 2(2l+1) \qquad \text{for } 0 \leq l \leq n-1.$$

Namely, for the sub-shells $s(l=0), p(l=1), d(l=2), f(l=3), g(l=4)$, their maximal electron numbers are

$$N_s = 2, \ N_p = 6, \ N_d = 10, \ N_f = 14, \ N_g = 18.$$

Thus, on the n-th shell, the maximal electron number is

$$N_n = \sum_{l=0}^{n-1} N_l = 2n^2. \tag{7.4}$$

Atomic field equations

Based on the atomic shell structure, the electron system of an atom consists of shell systems as (7.2), which we denote by

$$S_n = \text{ the } n\text{-th shell system for } n = 1, 2, \cdots . \tag{7.5}$$

Each shell system S_n has n sub-shell systems as in (7.3), denoted by

$$S_{nl} = \text{ the } l\text{-th sub-shell system of } S_n \qquad \text{for } 0 \leq l \leq n-1. \tag{7.6}$$

Thus, we have two kinds of classifications (7.5) and (7.6) of sub-systems for atomic orbital electrons, which lead to two different sets of field equations.

A. Field equation of system S_n. If we ignore the orbit-orbit interactions, then we take (7.5) as an N-particle system. Let S_n have K_n electrons:

$$S_n: \ \Psi_n = (\psi_n^1, \cdots, \psi_n^{K_n}), \qquad K_n \leq N_n, \ 1 \leq n \leq N, \tag{7.7}$$

where N_n is as in (7.4). Hence, the model of (7.7) is reduced to the $SU(K_1) \times \cdots \times SU(K_N)$ gauge fields of fermions. Referring to the single fermion system (5.1)-(5.7), the action of (7.7) is

$$L = \int \sum_{n=1}^{N} (\mathcal{L}_{SU(K_n)} + \mathcal{L}_D^n) dx, \tag{7.8}$$

where

$$\begin{aligned}
&\mathcal{L}_{SU(K_n)} = -\frac{1}{4\hbar c} g^{\mu\alpha} g^{\nu\beta} A^{a_n}_{\mu\nu} A^{a_n}_{\alpha\beta} && 1 \leq a_n \leq K_n, \\
&\mathcal{L}_D^n = \bar{\Psi}_n (i\gamma^\mu D_\mu - \frac{m_e c}{\hbar}) \Psi_n && 1 \leq n \leq N, \\
&A^{a_n}_{\mu\nu} = \partial_\mu A^{a_n}_\nu - \partial_\nu A^{a_n}_\mu - \frac{e}{\hbar c} \lambda^{a_n}_{b_n c_n} A^{b_n}_\mu A^{c_n}_\nu, \\
&D_\mu \Psi_n = (\partial_\mu - \frac{ie}{\hbar c} A^0_\mu - \frac{ie}{\hbar c} A^{a_n}_\mu \tau_{a_n}) \Psi_n,
\end{aligned} \tag{7.9}$$

where $A^{a_n}_\mu$ are the $SU(K_n)$ gauge fields representing the electromagnetic (EM) potential of the electrons in S_n, $\lambda^{a_n}_{b_n c_n}$ are the structure constants of $SU(K_n)$ such that $\mathcal{G}_{a_n b_n} = \frac{1}{2} \text{tr}(\tau_{a_n} \tau^\dagger_{b_n}) = \delta_{a_n b_n}$, A^0_μ is the EM potential generated by the nuclear, $g = -e$ $(e > 0)$ is the charge of an electron, and m_e is the electron mass.

The PID gradient operators for $SU(K_1) \times \cdots \times SU(K_N)$ in (5.5) are given by

$$D^n_\mu = \frac{1}{\hbar c} \left[\partial_\mu + \frac{e}{\hbar c} \sum_{k \neq n} A^{(k)}_\mu \right] \qquad \text{for } 1 \leq n \leq N, \tag{7.10}$$

where $A^{(k)}_\mu = \alpha^{K_k}_{a_K} A^{a_k}_\mu$ is the total EM potential of S_k shell as defined in (3.12).

Then by (7.8)-(7.10), the field equations of (7.7) can be written in the following form

$$\begin{aligned}
&\partial^\nu A^{a_n}_{\nu\mu} + \frac{e}{\hbar c} \lambda^{a_n}_{b_n c_n} g^{\alpha\beta} A^{b_n}_{\alpha\mu} A^{c_n}_\beta + e \bar{\Psi} \gamma_\mu \tau^{a_n} \Psi_n \\
&\quad = \left[\partial_\mu + \frac{e}{\hbar c} \sum_{k \neq n} A^{(k)}_\mu \right] \phi^{a_n} \qquad \text{for } 1 \leq a_n \leq K_n^2 - 1,\ 1 \leq n \leq N,
\end{aligned} \tag{7.11}$$

$$i\gamma^\mu \left[\partial_\mu - \frac{ie}{\hbar c} A^0_\mu - \frac{ie}{\hbar c} A^{a_n}_\mu \tau_{a_n} \right] \Psi_n - \frac{m_e c}{\hbar} \Psi_n = 0. \tag{7.12}$$

B. Field equation of system S_{nl}. The precise model of atomic spectrum should take (7.6) as an N-particle system. Also, $S_n = \sum\limits_{l=0}^{n-1} S_{nl}$ is again divided into n sub-systems

$$S_n : \ S_{n0}, \cdots, S_{nn-1}.$$

Hence, the system S_{nl} has more sub-systems than S_n, i.e. if S_n has N sub-systems, then S_{nl} has $\frac{1}{2} N(N+1)$ sub-systems.

Let S_{nl} have K_{nl} electrons with wave functions:

$$S_{nl} : \ \Psi_{nl} = (\psi^1_{nl}, \cdots, \psi^{K_{nl}}_{nl}), \qquad 1 \leq n \leq N,\ 0 \leq l \leq n-1, \tag{7.13}$$

and $K_{nl} \leq 2(2l+1)$. Then the action of (7.13) takes as

$$L = \int \sum_{l=0}^{n-1} \sum_{n=1}^{N} (\mathcal{L}_{SU(K_{nl})} + \mathcal{L}_D^{nl}) dx, \tag{7.14}$$

where $\mathcal{L}_{SU(K_{nl})}$ and $\mathcal{L}_D^{nl}$ are similar to that of (7.9). Thus, the field equation of the system (7.13) is determined by (7.14).

Remark 7.1. *The reason why atomic spectrum can be divided into two systems (7.7) and (7.13) to be considered is that in the system (7.13) the electrons in each S_{nl} have the same energy, and in (7.7) the electons in each S_n have the same energy if we ignore the interaction energy between different l-orbital electrons of S_{nl}. Hence, the system of S_{nl} is precise and the system of S_n is approximative.*

Atomic spectrum equations

For simplicity, we only consider the system S_n, and for S_{nl} the case is similar. Since the electrons in each S_n have the same energy λ_n, the wave functions in (7.7) can take as

$$\psi_n^j = \varphi_n^j(x) e^{-i\lambda_n t/\hbar} \qquad \text{for } 1 \leq j \leq K_n. \tag{7.15}$$

It is known that the EM fields A_μ^a in atomic shells are independent of time t, i.e. $\partial_t A_\mu^a = 0$. Therefore, inserting (7.15) into (7.11) and (7.12) we derive the spectrum equation in the form

$$\begin{aligned}
\lambda_n \Phi_n = {} & ic\hbar(\vec{\alpha} \cdot D)\Phi_n - eV\Phi_n \\
& + m_e c^2 \alpha_0 \Phi_n + eA_0^{a_n} \tau_{a_n} \Phi_n \qquad \text{for } 1 \leq n \leq N,
\end{aligned} \tag{7.16}$$

$$\begin{aligned}
& \Delta A_0^{a_n} - \frac{e}{\hbar c} \lambda_{b_n c_n}^{a_n} \vec{A}^{b_n} \cdot (\nabla A_0^{c_n} + \frac{e}{\hbar c} \lambda_{d_n f_n}^{c_n} A^{d_n} \vec{A}^{f_n}) - e\Phi_n^\dagger \tau_{a_n} \Phi_n \\
& = \frac{e}{\hbar c} \sum_{k \neq n} A_0^{(k)} \phi^{a_n},
\end{aligned} \tag{7.17}$$

$$\begin{aligned}
& \Delta \vec{A}^{a_n} - \nabla(\text{ div } \vec{A}^{a_n}) + \frac{e}{\hbar c} \lambda_{b_n c_n}^{a_n} g^{\alpha\alpha} \vec{A}_\alpha^{b_n} A_\alpha^{c_n} + e\bar{\Phi}_n \vec{\gamma} \tau_{a_n} \Phi_n \\
& = (\nabla + \frac{e}{\hbar c} \sum_{k \neq n} \vec{A}^{(k)}) \phi^{a_n},
\end{aligned} \tag{7.18}$$

where $\Phi_n = (\varphi_n^1, \cdots, \varphi_n^{K_n})^T$, $A_\mu^{a_n} = (A_0^{a_n}, \vec{A}^{a_n})$, $V = ze/r$ is the Coulomb potential of the nuclear, $\vec{A} = (A_1, A_2, A_3)$ is the magnetic potential of the nuclear, and

$$\begin{aligned}
D\Phi_n &= (\nabla - \frac{ie}{\hbar c} \vec{A} - \frac{ie}{\hbar c} \vec{A}^{a_n} \tau_{a_n}) \Phi_n, \\
\vec{A}_\alpha^{b_n} &= \partial_\alpha \vec{A}^{b_n} - \nabla A_\alpha^{b_n} - \frac{e}{\hbar c} \lambda_{c_n d_n}^{b_n} A_\alpha^{c_n} \vec{A}^{d_n}.
\end{aligned}$$

The equations (7.16)-(7.18) need to be complemented with some gauge fixing equations; see Remark 5.1.

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(TM) Department of Mathematics, Sichuan University, Chengdu, P. R. China

(SW) Department of Mathematics, Indiana University, Bloomington, IN 47405
E-mail address: `showang@indiana.edu, http://www.indiana.edu/ fluid`