Quantum control via Wigner measures and Wigner functions

Tudor S. Ratiu[1] and Oleg G. Smolyanov[2]

We develop an approach for analyzing open quantum systems which can be used to investigate quantum control problems, based on the use of both the Wigner functions and the so-called Wigner measures. We also propose an axiomatic definition of coherent quantum feedback control (see [1] and the collection of articles in [2]). While the results relating to the Wigner functions and measures are quite technical, the latter topic is more conceptual.

The main advantage of using the Wigner functions and measures is the fact that their domains are the phase spaces, and hence the transition from the Wigner measure or the Wigner function of the composition of two subsystems to the Wigner measure or function of any of the subsystems, is quite similar to the transition from the usual probability on the product of two phase spaces to the probability on any of these spaces; the latter probability is just the projection of the probability on the Cartesian product.

Actually, if the dimension of the phase space is finite, we can consider only the Wigner function, because it is the density of the Wigner measure with respect to the Liouville measure on phase space.

[1]Section de Mathématiques and Bernoulli Center, École Polytechnique Fédérale de Lausanne. CH–1015 Lausanne. Switzerland. tudor.ratiu@epfl.ch

[2]Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, Moscow, Russia. smolyanov@yandex.ru

However, if the phase space is infinite dimensional, then there does not exist a Lebesgue type Liouville measure (on the σ-algebra of Borel subsets), which means that the measure is translation invariant, σ-additive, σ-finite, and locally finite (each point has a neighborhood of finite measure)[1]. Then we can, on one hand, use the Wigner measure itself. On the other hand, we can employ, instead of the Lebesgue measure, a "good enough" measure, e.g., a Gaussian measure like in "white noise analysis", again substituting the Wigner measure by a proper Wigner function. Below, we consider in parallel, both the Wigner measure and the Wigner function.

The paper is organized as follows. In the first section, which is of independent interest, we consider some properties of the Wigner measures and functions, describing the state of a quantum system. Some of these properties are known but few can be found in the literature. In the second section, we present some equations, which can be called the *Liouville-Moyal equations* (cf. [5]), that describe the evolution of the Wigner measure and function. It is worth pointing out that the Wigner measure is a signed cylindrical measure and it would be interesting to get estimates of its variation and to find conditions for its σ-additivity; we will not address these issues in the current paper.

In the next section, we discuss how one can describe the evolution of the Wigner measures and functions of an open quantum system starting from the evolution of these objects related to the larger (closed) system. In this section, we also consider a couple of models of quantum control. In particular, we formulate an axiomatic definition of coherent quantum feedback that, to our knowledge, is not present in the literature.

We also consider a general model, which can be specified to quantum control with or

[1]This statement is a particular case of a theorem of A. Weil.

without feedback, depending on the inner design of the model (i.e., on the Hamiltonians describing both dynamics of the subsystems, alone and in interaction). Also, we compare two versions of (coherent) open loop quantum control.

1 Wigner measures and functions

In this section we give four (equivalent) definitions of the Wigner measure and function; in particular, we recall some definitions from [5]. Some of these definitions appear in the form of propositions. Let $E := Q \times P$ be the phase space of a Hamiltonian system; here Q and P are real locally convex spaces (LCS), $P = Q^*$, $Q = P^*$, and hence $E^* = P \times Q$. Actually, below we assume that all these spaces are Hilbert and identify canonically Q with Q^* and P with P^*. Denote by $\langle \cdot, \cdot \rangle : P \times Q \to \mathbb{R}$ the duality pairing. The linear map $J : E \ni (q, p) \mapsto (p, q) \in E^*$ is an isomorphism. Below, we usually identify the elements $h \in E$ and $Jh \in E^*$. In particular, for each $h \in E$, we denote by $\widehat{h}$ the pseudodifferential operator in $\mathcal{L}_2(Q, \mu)$ whose symbol is $Jh \in E^*$. Here, μ is the P-cylindrical Gaussian measure on Q whose Fourier transform $\Phi_\mu : P \to \mathbb{R}$ is defined by

$$\Phi_\mu(p) := \exp\left(-\frac{1}{2}\langle p, B_\mu p\rangle\right),$$

where $B_\mu : P \to Q$ is a continuous linear operator such that $\langle p, B_\mu p\rangle > 0$ for $p \neq 0$. Similarly, we denote by ν the Q-cylindrical Gaussian measure on P whose Fourier transform $\Phi_\nu : Q \to \mathbb{R}$ is defined by

$$\Phi_\nu(q) := \exp\left(-\frac{1}{2}\left\langle B_\mu^* q, q\right\rangle\right).$$

If Q, and hence P, are Hilbert spaces, then $B_\mu^* = B_\mu$ and $B_\mu > 0$; moreover, μ and ν are σ-additive if and only if B_μ is a trace class operator. If $\dim Q < \infty$, then, instead of the

Gaussian measures μ and ν we can use the Lebesgue measures.

The Weyl operator $\mathcal{W}(h)$ generated by $h \in E$ is defined by $W(h) := e^{-\mathrm{i}\widehat{h}}$; recall that we identify h and Jh, so $\widehat{h}$ is the pseudodifferential operator acting on $\mathcal{L}_2(Q, \mu)$ having symbol h (the definition of a pseudodifferential operator on $\mathcal{L}_2(Q, \mu)$ can be found in [5]). It is worth noticing that, in this special case, $e^{-\mathrm{i}\widehat{h}} = \widehat{e^{\mathrm{i}h}}$, i.e., the exponential of the pseudodifferential operator coincides with the pseudodifferential operator whose symbol is the exponential of the symbol h. In general, this is not the case.

The Weyl function corresponding to the density operator T (a trace class positive operator on $\mathcal{L}_2(Q, \mu)$ of trace one) is the function $\mathcal{W}_T : E \to \mathbb{R}$ defined by $\mathcal{W}_T(h) := \mathrm{tr}\,(T\mathcal{W}(h))$ (see [4]).

Definition 1 (see [5]) *The Wigner measure corresponding to the density operator T is the E^*-cylindrical measure W_T on E defined by*

$$\int_{Q\times P} e^{\mathrm{i}(\langle p_1, q_2\rangle + \langle p_2, q_1\rangle)} W_T(dq_1, dp_1) = \mathcal{W}_T(h)(q_2, p_2).$$

This means that W_T is the (inverse) Fourier transform of the function $\mathcal{W}_T(h)$, hence the following identity holds:

$$W_T(dq, dp) = \int_Q \int_P \mathcal{W}_T(h)(q_2, p_2) F_{E\times E}(dq_2, dp_2, dq, dp),$$

where $F_{E\times E}$ is the Hamiltonian Feynman pseudo measure on $E \times E$.

Any Feynman pseudomeasure $F_{\mathcal{K}}$ on a Hilbert space $\mathcal{K}$ is a distribution (like in Sobolev-Schwartz theory) on $\mathcal{K}$, i.e., a linear functional (continuous in the appropriate sense) on a space of functions on $\mathcal{K}$. As in the case of usual measures, it is convenient to define $F_{\mathcal{K}}$ by its Fourier transform $\widetilde{F_{\mathcal{K}}} : \mathcal{K} \ni z \mapsto F_{\mathcal{K}}(\varphi_z) \in \mathbb{C}$, where $\varphi_z : \mathcal{K} \to \mathbb{C}$ is given by $\varphi_z(x) := e^{\mathrm{i}\langle z, x\rangle}$.

If $\mathcal{K} = E = Q \times P$ and $\widetilde{F}_{\mathcal{K}}(q,p) = e^{\mathrm{i}\langle q,p\rangle}$, then $F_{\mathcal{K}}$ is called the Hamiltonian Feynman pseudomeasure; it can be used, instead of the exponential, in defining the Fourier transform on infinite dimensional spaces, which maps functions to measures. The structure of the Hilbert space is actually not important; the Feynman pseudomeasure can be defined on any locally convex space and the Hamiltonian Feynman measure on any symplectic space (some information about all of this can be found in [3], [9], [11]).

Proposition 2 ([5]) *If G is the Weyl symbol of a bounded pseudodifferential operator acting on $\mathcal{L}_2(Q,\mu)$, then*

$$\int_P \int_Q G(q,p) W_T(dq,dp) = \operatorname{tr}\left(T\widehat{G}\right)$$

This proposition can also be used as a definition (cf. [4, Definition 3], where $\dim Q = \dim P < \infty$ and thus only Wigner functions, but not Wigner measures, are considered). The density of W_T with respect to μ (if it exists) is called the μ-Wigner function and it is denoted by Φ_T; the μ_L-Wigner function is just the standard Wigner function.

Corollary 3 *If the assumptions of Proposition 2 hold, then*

$$\int_P \int_Q G(q,p)\Phi_T(q,p)\mu \otimes \nu(dq,dp) = \operatorname{tr}(T\widehat{G}).$$

In [4], two other definitions of the Wigner function have also been considered (but only for finite dimensional Q and P). One of them has mainly a conceptual character, whereas the other, going back to Wigner himself, can be used to develop the equation describing the evolution of the Wigner measure. In this paper, we give the general definition if both Q and P are infinite dimensional. In the definition below, similar to Definition 1, it is assumed that both Q and P are Hilbert spaces.

Definition 4 *The μ-Wigner function Φ_T on E is defined by*

$$\Phi_T(q,p) := e^{\frac{1}{2}\left(\left\langle p_1, B_\mu^{-1} p_1\right\rangle + \left\langle q_1, B_\mu^{-1} q_1\right\rangle\right)}$$
$$\int_{Q\times P} e^{-\mathrm{i}(\langle p_1,q_2\rangle + \langle p_2,q_1\rangle)} \mathcal{W}_T(h)(q_2,p_2) e^{\frac{1}{2}\left(\left\langle p_2, B_\mu^{-1} p_2\right\rangle + \left\langle q_2, B_\mu^{-1} q_2\right\rangle\right)} (\mu\otimes\nu)(dq_2,dp_2)$$

Here, the function

$$(q,p) \mapsto e^{-\frac{1}{2}\left(\left\langle p, B_\mu^{-1} p\right\rangle + \left\langle q, B_\mu^{-1} q\right\rangle\right)}$$

is a generalized density of the Gaussian measure $\mu\otimes\nu$ (see [8] and references therein). There is a heuristic algorithm to develop this and similar formulae. This algorithm can be described as follows. First, we write, for the case when $\dim Q < \infty$, some formulae using the standard Gaussian density with respect to the Lebesgue measure μ_L and then we pass from the space $\mathcal{L}_2(Q,\mu_L)$ to $\mathcal{L}_2(Q,\mu)$. After that, we substitute the Gaussian density with respect to μ_L by the generalized Gaussian density. To do this, it is necessary to recall that the generalized Gaussian density is defined only up to real multiples which means that only the finite dimensional formulae which are invariant with respect to real multiples can be generalized to infinite dimensional spaces¿

Next, we formulate some propositions which, actually, are equivalent definitions of the Wigner measure and function and are similar to the definitions mentioned above of the Wigner function given in [4].

Proposition 5 *For any density operator T acting on $\mathcal{L}_2(Q,\mu)$ and any $\varphi\in\mathcal{L}_2(Q,\mu)$, the following identities hold:*

$$(T\varphi)(q) = e^{\frac{1}{4}\left\langle B_\mu^{-1} q,q\right\rangle} \int_P\int_Q e^{-\mathrm{i}\langle p,q_1-q\rangle}\varphi(q_1) e^{-\frac{1}{4}\left\langle B_\mu^{-1} q_1,q_1\right\rangle} W_T\left(\frac{dq_1+q}{2},dp\right),$$
$$(T\varphi)(q) = e^{\frac{1}{4}\left\langle B_\mu^{-1} q,q\right\rangle} \int_P\int_Q e^{-\mathrm{i}\langle p,q_1-q\rangle}\varphi(q_1) e^{\frac{1}{4}\left\langle B_\mu^{-1} q_1,q_1\right\rangle} \Phi_T\left(\frac{q_1+q}{2},p\right) e^{\frac{1}{2}\left\langle B_\mu^{-1} p,p\right\rangle}(\mu\otimes\nu)(dq,dp).$$

The notation in the first formula means that $q \mapsto W_T\left(\frac{dq_1+q}{2}, dp\right)$ is a function, whereas $(dq_1, dp) \mapsto W_T\left(\frac{dq_1+q}{2}, dp\right)$ is a measure. The function $q \mapsto e^{-\frac{1}{2}\left\langle B_\mu^{-1}q,q\right\rangle}$ is the generalized density of the Gaussian measure μ and the function $p \mapsto e^{-\frac{1}{2}\left\langle B_\mu^{-1}p,p\right\rangle}$ is the generalized density of ν.

Let ρ_T^1 be the integral kernel of the density operator T acting on $\mathcal{L}_2(Q,\mu)$, defined by the identity

$$(T\varphi)(q) = e^{\frac{1}{4}\left\langle B_\mu^{-1}q,q\right\rangle} \int_Q e^{\frac{1}{4}\left\langle B_\mu^{-1}q_1,q_1\right\rangle} \varphi(q_1)\rho_T^1(q,q_1)\mu(dq_1)$$

for any $\varphi \in \mathcal{L}_2(Q,\mu)$. We have the following result.

Proposition 6 *For any $\varphi \in \mathcal{L}_2(Q,\mu)$, the following identity holds*

$$\Phi_T(q,p) = e^{\frac{1}{2}(\langle B_\mu q,q\rangle + \langle B_\mu p,p\rangle)} \int_Q \rho_T^1\left(q - \frac{1}{2}r, q + \frac{1}{2}r\right) e^{\mathrm{i}\langle r,p\rangle} e^{\frac{1}{2}\left\langle B_\mu^{-1}r,r\right\rangle} \mu(dr)$$

Let ρ_T^2 be the integral kernel of the density operator in $\mathcal{L}_2(Q,\mu)$ defined by the identity

$$(T\varphi)(q) = e^{\frac{1}{4}\left\langle B_\mu^{-1}q,q\right\rangle} \int_Q \varphi(q_1) e^{-\frac{1}{4}\left\langle B_\mu^{-1}q_1,q_1\right\rangle} \rho_T^2(q,dq_1)$$

for any $\varphi \in \mathcal{L}_2(Q,\mu)$. Thus, ρ_T^2 is a function with respect to the first argument and a measure with respect to the second argument.

From Proposition 2, it follows that

$$\rho_T^2(q,dq_1) = \int_P e^{-\mathrm{i}\langle p,q_1-q\rangle} W_T\left(\frac{dq_1+q}{2}, dp\right).$$

Then the change of variables formula (in which $s-r=q$, $s+r=q_1$) implies

$$\rho_T^2(s-r, ds+r) = \int_P e^{-\mathrm{i}\langle p,2r\rangle} W_T(ds,dp)$$

or

$$\rho_T^2\left(q - \frac{r}{2}, dq + \frac{r}{2}\right) = \int_P e^{-\langle p,r\rangle} W_T(dq,dp)$$

and hence the "measure" $dp \mapsto W_T(dq, dp)$ is the inverse Fourier transform of the function $r \mapsto \rho_T^2\left(q - \frac{r}{2}, dq + \frac{r}{2}\right)$. Therefore, we get the following result.

Proposition 7 *Let F_E be the Hamiltonian Feynman pseudomeasure on $E := Q \times P$. Then*

$$W_T(dq, dp) = \int_Q \rho_T^2\left(q - \frac{r}{2}, dq + \frac{r}{2}\right) F_E(dr, dp);$$

here, to integrate with respect to the "measure" $dq \mapsto W_T(dq, dp)$ one needs to use the so-called Kolmogorov integral (see [7][2]).

2 The evolution of Wigner functions and measures

We keep the assumptions and notations of the preceding section. For any $t \in \mathbb{R}$, let $W_T(t)$ be the Wigner measure which describes the state of the quantum system at the moment t (so, in this section, $W_T(\cdot)$ denotes a function of a real variable whose values are Wigner measures, whereas in the preceding section, W_T denoted a Wigner measure). Then $W_T(\cdot)$ satisfies the following equation [5]:

$$\dot{W}_T(t) = 2\sin\left(\frac{1}{2}\mathcal{L}_{\mathcal{H}}^*(W_T(t))\right), \tag{1}$$

where, for any $a \in \mathbb{R}$, $\sin(a\mathcal{L}_{\mathcal{H}}^*)$ is the linear operator acting on the space $\mathcal{H}$ of E^*-cylindrical measures on E which is adjoint to the operator $\sin(a\mathcal{L}_{\mathcal{H}})$ acting on the space of functions on E, defined by

$$\sin(a\mathcal{L}_{\mathcal{H}}) := \sum_{n=1}^{\infty} \frac{a^{2n-1}}{(2n-1)!}\mathcal{L}_{\mathcal{H}}^{(2n-1)}.$$

[2] The Kolmogorov integral is just the trace in a tensor product of a space of functions on Q and a space of measures on Q; ρ_T^2 is an element of such a space

Here, $\mathcal{L}_{\mathcal{H}}^{(n)}$ is defined in the following way: for any function $\Psi : E \to \mathbb{R}$ and any $n \in \mathbb{N}$,

$$\mathcal{L}_{\mathcal{H}}^{(n)}\Psi(x) := \{\Psi, \mathcal{H}\}^{(n)}(x), \quad x \in E,$$

where

$$\{\Psi, \mathcal{H}\}^{(n)}(x) := \Psi^{(n)}(x) I^{\otimes n} \mathcal{H}^{(n)}(x),$$

$\Psi^{(n)}$, $\mathcal{H}^{(n)}$ denote the derivatives of order n of the functions Ψ and $\mathcal{H}$, respectively, and $I^{\otimes n}$ is the nth tensor power of the operator I which defines the symplectic structure on the phase space E ([5]).

The identity (1) leads to the equation describing the evolution of the μ-Wigner function. To do this, it is enough to recall that for any function $\Phi : E \to \mathbb{R}$, the nth order derivative of the product $\Phi^n\mu$ can be calculated using the Leibniz rule and that derivatives of the Gaussian measure μ can be calculated as follows. For any $h, h_1, h_2, \ldots \in B_\mu^{\frac{1}{2}}Q$, we have

$$\mu' h = -\left\langle B_\mu^{-1}h, \cdot\right\rangle \mu$$
$$\mu'' h_1 h_2 = -\left\langle B_\mu^{-1}h_1, h_2\right\rangle \mu + \left\langle B_\mu^{-1}h_1, \cdot\right\rangle \left\langle B_\mu^{-1}h_2, \cdot\right\rangle \mu, \qquad \text{etc.}$$

These expressions are some sort of Wick formulae. Here, the symbols $\left\langle B_\mu^{-1}h, \cdot\right\rangle$ and $\left\langle B_\mu^{-1}h_j, \cdot\right\rangle$ mean the unique μ-almost everywhere defined measurable functions on Q having the following properties (see [10]):

(1) if $x \in B_\mu^{\frac{1}{2}}Q$, then $\left\langle B_\mu^{-1}h, x\right\rangle = \left\langle B_\mu^{-\frac{1}{2}}h, B_\mu^{-\frac{1}{2}}x\right\rangle$;

(ii) the domains of each of these functions are linear measurable subspaces of Q of measure 1;

(iii) these functions are linear on their domains.

For any $a > 0$, the operator $\sin(aL^*_{\mathcal{H}})$, acting on functions on E, is defined by

$$\sin(aL^*_{\mathcal{H}})\,\varphi(\mu\otimes\nu) := (\sin a\mathcal{L}^*_{\mathcal{H}})\,(\varphi\mu\otimes\nu).$$

For any $t \in \mathbb{R}$, let $\Phi_T(t)$ be the μ-Wigner function which describes the state of the quantum system at the moment t. Then, the following holds.

Theorem 8 *The μ-Wigner function valued map $\Phi_T(\cdot)$ satisfies the following equation*

$$\dot{\Phi}_T(t) = 2\sin\left(\frac{1}{2}L^*_{\mathcal{H}}\left(\Phi_T(t)\right)\right).$$

3 Reduced evolution of the Wigner measure

To get the Wigner measure and function of a subsystem of a quantum system, it is necessary to use Propositions 6 and 7. In fact, if ρ^1_T and ρ^2_T are the integral kernels in the sense of the above definitions of the operator T of the quantum system which is the quantum version of the classical Hamiltonian system with phase space $E_1 \times E_2$, $E_1 = Q_1 \times P_1$, $E_2 = Q_2 \times P_2$, then the corresponding (reduced) density operators T_i acting on $\mathcal{L}_2(Q_i, \mu_i)$, $i = 1, 2$ (here and below we use the natural generalizations of the above notations and assumptions), are given by

$$\rho^1_{T_1}(q^1_1, q^1_2) = \int_{Q_2} \rho^1_T(q^1_1, q^1_2, q^1, q^2)e^{\frac{1}{2}\left\langle B_{\mu_1\otimes\mu_2}(q^1,q^2),(q^1,q^2)\right\rangle}\mu_2(dq_2)$$

$$\rho^2_T(q^1, dq^1_2) = \int_{Q_2} \rho^2_T(q^1, dq^1_2, q^2, dq^2)$$

where the latter integral is again the Kolmogorov integral. Hence, due to Propositions 6 and 7, the following statement holds.

Theorem 9 *Let W_T and Φ_T be the Wigner measure and function of the quantum system whose Hilbert space is $\mathcal{L}_2(Q_1 \times Q_2, \mu_1 \otimes \mu_2)$. Then the Wigner measure W_T and function Φ_T of the subsystem of this system with Hilbert space $\mathcal{L}_2(Q_1, \mu_1)$ are given by*

$$W_{T_1}(dq_1, dp_1) = \int_{Q_2 \times P_2} W_T(dq_1, dp_1, dq_2, dp_2),$$

$$\Phi_T(q_1, p_1) =$$

$$e^{\frac{1}{2}\left(\left\langle B_{\mu_1}^{-1} q_1, q_1\right\rangle + \left\langle B_{\mu_1}^{-1} p_1, p_1\right\rangle\right)} \int_{Q_2 \times P_2} e^{\frac{1}{2}\left(\left\langle B_{\mu_2}^{-1} q_2, q_2\right\rangle + \left\langle B_{\mu_2}^{-1} p_2, p_2\right\rangle\right)} \Phi_T(q_1, p_1, q_2, p_2)(\mu_2 \otimes \nu_2)(dq_2, dp_2).$$

Now we will consider the models mentioned in the introduction. We use below the following notation: if $\mathcal{T}$ is a Hilbert space, then $\mathcal{L}^s(\mathcal{T})$ denotes the collection of all selfadjoint operators on $\mathcal{T}$.

Let $\mathscr{P}$, $\mathscr{P}_1$, $\mathscr{P}_2$, $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$, be Hilbert spaces. We think of $\mathscr{P}$, as the Hilbert space of a quantum system under control (usually called a plant) and $\mathscr{C}$ as the Hilbert space of a quantum controller; $\mathscr{P}_j$, and $\mathscr{C}_j$, $j = 1.2$, correspond to parts of the of the plant and of the controller, respectively. Let $\mathscr{H} := \mathscr{P} \otimes \mathscr{C}$ be the Hilbert space of the composed quantum system and $\widehat{\mathscr{H}_{\mathscr{P}}} \in \mathcal{L}^*(\mathscr{P})$, $\widehat{\mathscr{H}_{\mathscr{C}}} \in \mathcal{L}^*(\mathscr{C})$, $\widehat{\mathscr{K}_{\mathscr{P}_1 \otimes \mathscr{C}_1}} \in \mathcal{L}^*(\mathscr{P}_1 \otimes \mathscr{C}_1)$, $\widehat{\mathscr{K}_{\mathscr{P}_2 \otimes \mathscr{C}_2}} \in \mathcal{L}^*(\mathscr{P}_e \otimes \mathscr{C}_2)$. Define $\widehat{\mathscr{H}_{\text{feedback}}} := \widehat{\mathscr{H}_{\mathscr{P}}} \otimes \mathscr{I}d_{\mathscr{C}} + \mathscr{I}d_{\mathscr{P}} \otimes \widehat{\mathscr{H}_{\mathscr{C}}} + \widehat{\mathscr{K}_{\mathscr{P}_1 \otimes \mathscr{C}_1}} \otimes \mathscr{I}d_{\mathscr{P}_2 \otimes \mathscr{C}_2} + \mathscr{I}d_{\mathscr{P}_1 \otimes \mathscr{C}_1} \otimes \widehat{\mathscr{K}_{\mathscr{P}_2 \otimes \mathscr{C}_2}} \in \mathcal{L}^s(\mathcal{H})$, where $\mathscr{I}d_{\mathscr{P}} \in \mathcal{L}^s(\mathcal{P})$, $\mathscr{I}d_{\mathscr{C}} \in \mathcal{L}^s(\mathcal{C})$, $\mathscr{I}d_{\mathscr{P}_1 \otimes \mathscr{C}_1} \in \mathcal{L}^s(\mathscr{P}_1 \otimes \mathscr{C}_1)$, $\mathscr{I}d_{\mathscr{P}_2 \otimes \mathscr{C}_2} \in \mathcal{L}^s(\mathscr{P}_2 \otimes \mathscr{C}_2)$, are the identity operators in the corresponding spaces. The first term in $\widehat{\mathscr{H}_{\text{feedback}}}$ describes the evolution of the system under control alone, the second term describes the evolution of the quantum controller alone, and the last two terms describe the (coherent) quantum feedback. It is worth noting that the definition of $\widehat{\mathscr{H}_{\text{feedback}}}$ is symmetric with respect to the plant, the controller, and the feedback.

The more general Hamiltonian $\widehat{\mathscr{H}} := \widehat{\mathscr{H}_{\mathscr{P}}} \otimes \mathscr{I}\mathrm{d}_{\mathscr{C}} + \mathscr{I}\mathrm{d}_{\mathscr{P}} \otimes \widehat{\mathscr{H}_{\mathscr{C}}} + \widehat{\mathscr{K}}$, where $\widehat{\mathscr{K}} \in \mathcal{L}^s(\mathcal{P} \otimes \mathscr{C})$ (cf. [6]) can be used to describe both open loop (coherent) quantum control and coherent quantum control with feedback. It is clear that the former model of quantum feedback is a particular case of this one (when $\widehat{\mathscr{K}} = \widehat{\mathscr{K}_{\mathscr{P}_1 \otimes \mathscr{C}_1}} \otimes \mathscr{I}\mathrm{d}_{\mathscr{P}_2 \otimes \mathscr{C}_2} + \mathscr{I}\mathrm{d}_{\mathscr{P}_1 \otimes \mathscr{C}_1} \otimes \widehat{\mathscr{K}_{\mathscr{P}_2 \otimes \mathscr{C}_2}}$).

On the other hand, if $\widehat{\mathscr{K}} := \widehat{\mathscr{K}_1} \otimes \mathscr{I}\mathrm{d}_{\mathscr{P}_2 \otimes \mathscr{C}_2}$, we get an open loop quantum control system. Here we have not assumed that the quantum system is obtained by a quantization procedure of a classical Hamiltonian system. However, if this were the case, then we would take, with natural notations, $\mathcal{P}_j = \mathcal{L}_2(Q_{\mathscr{P}_j}, \mu_j)$, $j = 1, 2$, $\mathscr{P} = \mathcal{L}_2(Q_{\mathscr{P}_1} \otimes Q_{\mathscr{P}_2}, \mu_1 \otimes \mu_2)$.

Then, one can describe the evolution of the Wigner function or measure of the whole system (with Hilbert space $\mathcal{L}_2(Q_{\mathscr{P}_1} \otimes Q_{\mathscr{P}_2}, \mu_1 \otimes \mu_2)$) using the equations of Theorem 9 and then we can describe the reduced dynamics of the system with Hilbert space $\mathcal{L}_2(Q_{\mathscr{P}_1}, \mu_1)$ using the formulae above for the reduced Wigner function and measure.

After that, our task is to find Hamiltonians $\mathscr{K}_1$ and $\mathscr{K}_2$ (respectively $\mathscr{K}$) to realize some prescribed dynamics of the first system.

Remark 10 This task is similar to the simpler one to choose the time dependent Hamiltonian function $\mathscr{K}_1(\cdot)$ on $Q_{\mathscr{P}_1}$ to realize a prescribed dynamics on $\mathcal{L}_2(Q_{\mathscr{P}_1}, \mu_1)$, assuming that $\widehat{\mathscr{H}} = \widehat{\mathscr{H}_1} + \widehat{\mathscr{K}_1(t)}$, where $\widehat{\mathscr{H}_1} \in \mathcal{L}^s(\mathscr{P})$, $\widehat{\mathscr{K}(t)} \in \mathcal{L}^s(\mathscr{P})$. We expect that the latter model can be obtained as a limit of a family, depending on a parameter, of models of the first type.

Remark 11 Along the same lines, we can extend our model by including an additional quantum system that, coupled to the plant, produces some perturbations. However, we can also assume that this source of perturbations is already part of the plant.

Remark 12 The approach presented in the first two sections of this paper can be applied directly only to quantum systems which can be obtained by Schrödinger quantization of classical Hamiltonian systems.

To consider the more general case which includes, e.g., some spin systems, we need to extend our approach using the methods of superanalysis. We expect that all our results can be generalized to this case.

Remark 13 In a similar way, one can define feedback for classical Hamiltonian systems.

Remark 14 In our quantum model with feedback, we can also separate the inner dynamics of the plant, of the controller, and of the corresponding coupling. Then

$$\begin{aligned}\widehat{\mathscr{H}} &= \left(\widehat{\mathscr{H}_{\mathscr{P}_1}} \otimes \mathscr{I}\mathrm{d}_{\mathscr{P}_2} + \mathscr{I}\mathrm{d}_{\mathscr{P}_1} \otimes \widehat{\mathscr{H}_{\mathscr{P}_2}}\right) \otimes \mathscr{I}\mathrm{d}_{\mathscr{C}} \\ &+ \mathscr{I}\mathrm{d}_{\mathscr{P}} \otimes \left(\widehat{\mathscr{H}_{\mathscr{C}_1}} \otimes \mathscr{I}\mathrm{d}_{\mathscr{C}_2} + \mathscr{I}\mathrm{d}_{\mathscr{C}_1} \otimes \widehat{\mathscr{H}_{\mathscr{C}_2}}\right) \\ &+ \widehat{\mathscr{K}_{\mathscr{P}_1\otimes\mathscr{P}_2}} \otimes \mathscr{I}\mathrm{d}_{\mathscr{C}_1\otimes\mathscr{C}_2} + \mathscr{I}\mathrm{d}_{\mathscr{P}_1\otimes\mathscr{P}_2} \otimes \widehat{\mathscr{K}_{\mathscr{C}_1\otimes\mathscr{C}_2}} \\ &+ \widehat{\mathscr{K}_{\mathscr{P}_1\otimes\mathscr{C}_1}} \otimes \mathscr{I}\mathrm{d}_{\mathscr{P}_2\otimes\mathscr{C}_2} + \widehat{\mathscr{K}_{\mathscr{P}_2\otimes\mathscr{C}_2}} \otimes \mathscr{I}\mathrm{d}_{\mathscr{P}_1\otimes\mathscr{C}_1}.\end{aligned}$$

Here, the plant, the controller, and the two systems responsible for feedback again look similar.

Acknowldegment. We thank John Gough for useful discussions and remarks. Both authors were partially supported by the government grant of the Russian Federation for support of research projects implemented by leading scientists, Lomonosov Moscow State University, under the agreement No. 11.G34.31.0054. TSR was partially supported by NCCR SwissMAP and grant 200021-140238, both of the Swiss National Science Foundation. OGS was partially supported by the Russian Fund for Fundamental Research.

References

1 Gough, J. and James, M. R., Quantum feedback networks: Hamiltonian formulation, *Comm. Math. Phys.*, **287**(3) (2009), 1109–1132.

2 Gough, J., James, M. R., Mabuchi, H., Mølmer, K., and Walmsley, I., eds., *Principles and Applications of Quantum Control Engineering*, Kavli Royal Society International Centre, Chicheley Hall, 12–13 December 2011, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, **370**(1979) (2012).

3 Gough, J., Ratiu, T. S., and Smolyanov, O. G., Feynman, Wigner, and Hamiltonian structures describing the dynamics of open quantum systems (Russian), *Dokl. Akad. Nauk*, **454**(4) (2014), 379–382; English translation *Dokl. Math.*, **89**(1) (2014), 68–71.

4 Kozlov, V. V. and Smolyanov, O. G., The Wigner function and diffusion in a collisionless medium consisting of quantum particles (Russian), *Teor. Veroyatn. Primen.*, **51**(1) (2006), 109–125; translation in *Theory Probab. Appl.*, **51**(1) (2007), 1–13.

5 Kozlov, V. V. and Smolyanov, O. G., Wigner measures on infinite dimensional spaces and Bogolyubov equations for quantum systems (Russian), *Dokl. Akad. Nauk*, **439**(5) (2011), 600–609; English translation in *Dokl. Math.*, **84**(1) (2011), 571–575.

6 Lloyd, S., Coherent quantum feedback, *Phys. Rev. A (3)*, **62**(2) (2000), 022108.

7 Loève, M., *Probability Theory*, Vol I, fourth edition, Graduate Texts in Mathematics, **45**, Springer-Verlag, New York-Heidelberg, 1977.

8 Montaldi, J. and Smolyanov, O. G., Transformations of measures via their generalized densities, *Russian Journ. Math. Phys.*, **21**(3) (2014), 1–10.

9 Ratiu T., Smolyanov O.G. Hamiltonian and Feynman aspects of secondary quantization (Russian), *Dokl. Akad. Nauk*, **450**(2) (2013), 150–153; English translation *Dokl. Math.*, **87**(3) (2013), 289–292.

10 Smolyanov, O. G., On measurable and multilinear and power functionals on some linear spaces with measure, *Dokl. Akad. Nauk*, **170**(3) (1966), 526–529.

11 Smolyanov, O. G., Tokarev, A. G., and Truman, A., Hamiltonian Feynman path integrals via the Chernoff formula, *J. Math. Phys.*, **43**(10) (2002), 5161–5171.