OBSTACLE PROBLEM WITH A DEGENERATE FORCE TERM

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Abstract. In this paper we study the regularity of the free boundary at its intersection with the line $\{x_1 = 0\}$ in the obstacle problem

$$\triangle u = |x_1|\chi_{\{u>0\}} \text{ in } D,$$

where $D \subset \mathbb{R}^2$ is a bounded domain such that $D \cap \{x_1 = 0\} \neq \emptyset$.

We obtain a uniform $C^{1,1}$ bound on cubic blowups, we find all homogenous global solutions, we prove the uniqueness of the blowup limit in all cases, we prove the convergence of the free boundary to the free boundary of the blowup limit, at the points with lowest Weiss balanced energy we prove the convergence of the normal of the free boundary to the normal of the free boundary of the blowup limit and that locally the free boundary is a graph and finally for a particular case we prove that the free boundary is not $C^{1,\alpha}$ regular near to a degenerate point for any $0 < \alpha < 1$.

1. Introduction

Let $D \subset \mathbb{R}^2$ be a bounded domain such that $D \cap \{x_1 = 0\} \neq \emptyset$. Let $g \in H^1(D)$ such that $g \geq 0$ on ∂D. Let $u \in H^1(D)$ be the unique minimiser of the functional

$$\int_D \big(|\nabla u|^2 + 2|x_1|u\big)dx \tag{1.1}$$

in the admissible set of functions

$$\big\{u \geq 0 \text{ a.e. in } D \text{ and } u = g \text{ on } \partial D\big\}.$$

For the existence and uniqueness of the minimiser u one may refer to [5].

It is known (cf. [6]) that $u \in C^{1,1}_{loc}(D)$ and

$$\triangle u = |x_1|\chi_{\{u>0\}} \text{ in } D \tag{1.2}$$

in the sense of distributions.

Let us denote by Ω the noncoincidence set and by Γ the free boundary, i.e.

$$\Omega = \big\{x \in D \bigm| u(x) > 0\big\} \qquad \text{and} \qquad \Gamma = D \cap \partial\Omega.$$

Let us consider two examples. Set $D = (-1, 1)^2$. For the first example we take $g(x) = \frac{1}{16}(x_1 + x_2)^+$ and for the second example we take $g(x) = x_1^+(c - |x_2|)^+$ where $c \approx 0.42559$. The noncoincidence set and the free boundary are depicted in figure 1 for both examples.

In [2], in the case of nondegenerate obstacle problem, i.e. when instead of $|x_1|$ we have f satisfying $f \geq c$ in D for some $c > 0$, the Lipschitz and C^1 regularity of the free boundary was proved for the first time. A good reference for nondegenerate obstacle problems is [3] and a good reference for obstacle type problems is [6].

In [13] for a class of degenerate obstacle problems the optimal nondegeneracy of the solution is obtained. The proof of the optimal nondegeneracy is based on specially constructed comparison functions using harmonic polynomials. In this paper the nondegeneracy result in [13] will be used numerous times. Also the

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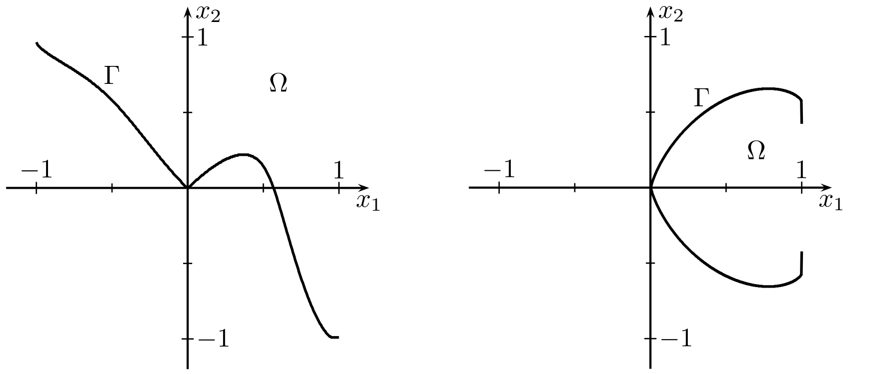

(A) $g(x) = \frac{1}{16}(x_1 + x_2)^+$ (B) $g(x) = x_1^+(c - |x_2|)^+$ with $c \approx 0.42559$

FIGURE 1. Ω and Γ in the examples.

special harmonic polynomials used in the proof of the nondegeneracy result will be used to prove one of the directional monotonicity results in this paper.

Our approach to prove the regularity of the free boundaries is based on some directional monotonicity properties satisfied by the solutions. This method is based on the proof of C^1 regularity in [6] and is closely related to the work [1].

We use Hopf's Lemma to prove the irregularity of the free boundary in a particular case which corresponds to the free boundary near to the origin in the example depicted in figure 1b.

Studying obstacle problems with a degenerate force term reveals rather unexpected behaviour of the solution. Such as the fact that the free boundary usually is forming a certain angle at its intersections with the line $\{x_1 = 0\}$ where the force term is degenerate.

In the problem of the free boundary near contact points with the fixed boundary, cf. [7], where the solution satisfies a homogenous Dirichlet boundary condition, a similar strong influence of the data of the problem on the structure of the free boundary has been observed.

In [8, 9, 10] the authors have studied 2-dimensional or axisymmetric 3-dimensional inviscid incompressible fluids acted on by gravity and with a free surface. These problems are in the class of Bernoulli free boundary problems. But the degeneracies in the force terms give rise to similar situations as encountered in this paper and has been a motivation for considering the problem in this paper.

This paper is structured as follows. In Section 2, the main results of this paper are presented. In Section 3, we prove uniform $C^{1,1}$ bounds on cubic blowups. In Section 4, using the Weiss balanced energy we prove the homogeneity of the blowup limits. In Section 5, we classify all possible homogeneous global solutions. In Section 6, using a lower bound for homogenous global solutions and the optimal nondegeneracy result in [13] we prove closeness of the free boundary to the free boundary of a homogenous global solution. In Section 7, we prove the uniqueness of blowup limits at degenerate points. In Section 8, we prove the uniqueness of the blowup limits at the points with lowest Weiss balanced energy, i.e. regular points. In Section 9, we prove the convergence of the free boundary to the free boundary of the blowup limit. In Section 10, we prove the convergence of the normal of the free boundary to the normal of the free boundary of the blowup limit at regular points. In Section 11, we prove that in a neighbourhood of a regular point the free boundary might be given as a graph. In Section 12, we prove that under some assumptions the free boundary near to a degenerate point is either flat or not $C^{1,\alpha}$

for any $0 < \alpha < 1$. In Section 13, we conclude this paper with a discussion about further directions of research on obstacle problems with degenerate forces.

2. Main Results

Let us define a cubic blowup of u as follows

Definition 1. *Let $B_{r_0} \subset D$, then we define for $0 < r < r_0$*

$$u_r(x) = \frac{u(rx)}{r^3} \text{ for } x \in B_1$$

and call u_r the (cubic) blowup of u at 0.

In the following theorem we prove that for $r > 0$ the family u_r is uniformly bounded in $C^{1,1}(B_1)$.

Theorem 1 (Uniform $C^{1,1}$ Bounds on Blowups)**.** *There exists a $C > 0$ such that if u is a solution in D, $r_0 > 0$, $B_{r_0} \subset D$ and $0 \in \Gamma$ then we have the estimate*

$$\|u_r\|_{C^{1,1}(B_1)} \leq C \tag{2.1}$$

for $0 < r < \frac{1}{6} r_0$.

The proof of this theorem is based on the optimal growth result proved in [13].

From the uniform bound (2.1) it follows that for any sequence r_j there exists a subsequence r_{j_k} and $v \in C^{1,1}(B_1)$ such that $u_{j_k} \to v$ in $C^1(B_1)$.

Let us consider for $u \in H^1(B_r)$ the Weiss balanced energy

$$W(r,u) = \frac{1}{r^6} \int_{B_r} \big(|\nabla u|^2 + 2|x_1|u\big)dx - \frac{3}{r^7} \int_{\partial B_r} u^2 s(dx). \tag{2.2}$$

The Weiss balanced energy has been introduced to study the free boundary in the nondegenerate obstacle problem in [11, 12]. The energy in (2.2) has been adapted to the first order homogeneity of the force term $|x_1|$. For the Weiss balanced energy for different homogeneities one may refer to [6].

As we will see, for u a solution in D with $0 \in D$, by a monotonicity result for the Weiss balanced energy the right limit $W(+0,u)$ exists but might be $-\infty$. If $0 \in \Gamma$ then $W(+0,u) > -\infty$.

Definition 2. *Let u be a solution in D, $0 \in D$ and $0 \in \Gamma$. Then we call $v \in C^{1,1}(B_1)$ a blowup limit if there exists $r_j \to 0$ such that $u_{r_j} \to v$ in $C^1(B_1)$.*

Using Weiss balanced energy, if v is a blowup limit at 0 then v is a third order homogenous global solution and $W(+0,u) = W(1,v)$.

So we are lead to find all the solutions of the obstacle problem

$$\begin{cases} \triangle u = |x_1|\chi_{u>0} \text{ in } \mathbb{R}^2, \\ u \text{ third order homogeneous.} \end{cases} \tag{2.3}$$

Clearly $u = 0$ is a trivial solution of (2.3).

Let us define

$$u_{hs}(x) = \frac{1}{6}(x_1^+)^3 \quad\text{and}\quad u_w(x) = (\frac{1}{6}|x_1|^3 + \frac{1}{12}x_2^3 - \frac{1}{4}x_1^2 x_2)\chi_{x_2 > |x_1|}. \tag{2.4}$$

Theorem 2 (Classification of Homogenous Global Solutions)**.** *There exists only the following nontrivial solutions of (2.3), u_w, $u_w(x_1, -x_2)$, $u_w + u_w(x_1, -x_2)$, u_{hs}, $u_{hs}(-x_1, x_2)$ and $u_{hs} + u_{hs}(-x_1, x_2)$.*

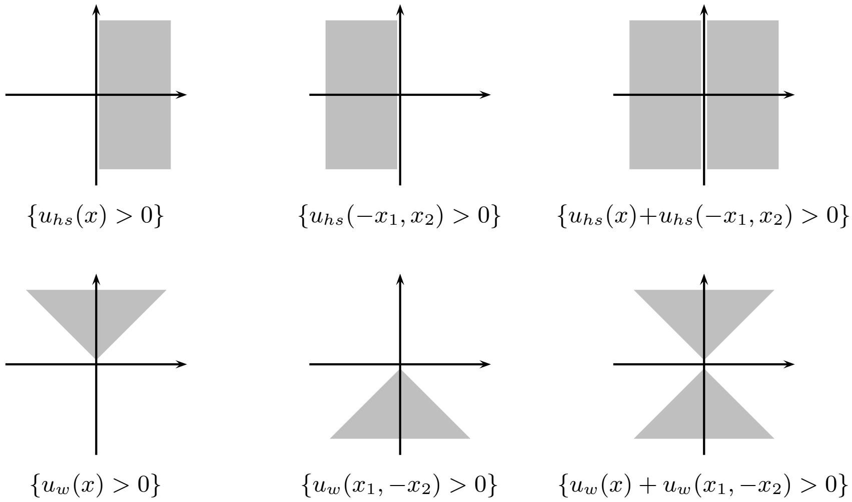

FIGURE 2. The only possible noncoincidence sets of nontrivial homogenous global solutions.

To prove Theorem 2 we first find all the solutions of the corresponding no-sign obstacle problem and then among these solutions we find the nonnegative ones.

All possible noncoincidence sets of nontrivial homogenous global solutions, i.e. the noncoincidence sets of the nontrivial solutions of (2.3), are depicted in figure 2.

It is easy to see that $W(1, u_w) = W(1, u_w(x_1, -x_2))$, $W(1, u_w + u_w(x_1, -x_2)) = 2W(1, u_w)$, $W(1, u_{hs}) = W(1, u_{hs}(-x_1, x_2))$, $W(1, u_{hs}+u_{hs}(-x_1, x_2)) = 2W(1, u_{hs})$ and by direct computation we see that $0 < W(1, u_w)$ and

$$2W(1, u_w) < W(1, u_{hs}).$$

So we have the following four possible energy levels together with the order between them

$$W(1, u_w) < 2W(1, u_w) < W(1, u_{hs}) < 2W(1, u_{hs}).$$

Let us define for $y \in \Gamma \cap \{x_1 = 0\}$ and $r > 0$

$$W(r, y, u) = W(r, u(\cdot + y)). \tag{2.5}$$

Based on four possible values of $W(+0, x, u)$ (the value 0 is excluded by the nondegeneracy) for $x \in \Gamma \cap \{x_1 = 0\}$ the points of $\Gamma \cap \{x_1 = 0\}$ get classified in four types.

Definition 3. *We call $y \in \Gamma \cap \{x_1 = 0\}$ a degenerate free boundary point if there exists $r_j \to 0$ such that $u_{r_j}(x + y) \to u_{hs}(x)$ or $u_{r_j}(x + y) \to u_{hs}(-x_1, x_2)(x)$ in $C^1(B_1)$.*

In the example depicted in figure 1b the origin is a degenerate free boundary point with u_{hs} as a blowup limit.

By our uniform bounds on the blowups it follows that 0 is degenerate if and only if $W(+0, u) = W(1, u_{hs})$.

Theorem 3 (Uniqueness of Degenerate Blowup Limits)**.** *If u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a degenerate free boundary point then the blowup limit is unique.*

The proof of this theorem is not based on directional monotonicity results. The proof is based on the observation that at the degenerate points the free boundary converges tangentially to the line $\{x_1 = 0\}$. By this observation and the nondegeneracy result proved in [13] we are able to prove the uniqueness of the blowup limit.

Definition 4. *We call $y \in \Gamma \cap \{x_1 = 0\}$ a regular free boundary point if there exists $r_j \to 0$ such that $u_{r_j}(x+y) \to u_w(x)$ or $u_{r_j}(x+y) \to u_w(x_1, -x_2)(x)$ in $C^1(B_1)$.*

In the example depicted in figure 1a a point close to the origin is a regular free boundary point with u_w as a blowup limit.

By our uniform bounds on the blowups it follows that 0 is regular if and only if $W(+0, u) = W(1, u_w)$, i.e. it has the lowest Weiss balanced energy.

Theorem 4 (Uniqueness of Regular Blowup Limits)**.** *If u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular free boundary point then the blowup limit is unique.*

The proof of Theorem 4 is based on a directional monotonicity result which shows that if 0 is a regular free boundary point with u_w as a blowup limit, then in a neighbourhood of 0 the function u is nondecreasing in the direction e_2.

In the cases when $W(+0, x, u) \in \{2W(1, u_w), 2W(1, u_{hs})\}$ there exists only one possible blowup limit and thus the limit is unique. So by Theorems 3 and 4 and this observation we have that always the blowup limit is unique.

Let us define for $\delta > 0$ and $k = 0, 1$

$$\sigma_k(\delta) = \sup_{0<r\le\delta} \|u_r - u_0\|_{C^k(B_1)} \tag{2.6}$$

where u_0 is the unique blowup limit.

Theorem 5 (Convergence of the Free Boundary)**.** *There exists $C_1 > 0$ and $C_2 > 0$ such that if u is a solution in D, $0 \in D$ and $0 \in \Gamma$ then for $x \in \Gamma$ and close enough to 0 if $W(+0, u) \in \{W(1, u_w), 2W(1, u_w)\}$ then we have*

$$d(x, \Gamma_{u_0}) \le C_1(\sigma_0(C_2|x|))^{\frac{1}{2}}|x| \tag{2.7}$$

where Γ_{u_0} is the free boundary of the unique blowup limit and if $W(+0, u) \in \{W(1, u_{hs}), 2W(1, u_{hs})\}$ then

$$|x_1| \le C_1(\sigma_0(C_2|x|))^{\frac{1}{3}}|x|. \tag{2.8}$$

The proof of this theorem is based on a lower bound for the nontrivial homogenous global solutions and the nondegeneracy result proved in [13].

From Theorem 5, in particular, it follows that all points of $\Gamma \cap \{x_1 = 0\} \cap \{W(+0, x, u) \in \{W(1, u_w), 2W(1, u_w)\}\}$ are isolated points of $\Gamma \cap \{x_1 = 0\}$ (in the topology of $\{x_1 = 0\}$).

Theorem 6 (Convergence of Normals and the Free Boundary as a Graph at Regular Points)**.** *There exists $C_1 > 0$ and $C_2 > 0$ such that if u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular free boundary point with blowup limit u_w then there exists $\epsilon > 0$ and*

$$\gamma \in C(-\frac{\epsilon}{4}, \frac{\epsilon}{4}) \cap C^1\big((-\frac{\epsilon}{4}, \frac{\epsilon}{4})\backslash\{0\}\big)$$

such that

$$\Gamma \cap \big\{|x_1| < \frac{\epsilon}{4}\big\} \cap B_\epsilon = \Big\{(x_1, \gamma(x_1)) \;\Big|\; x_1 \in (-\frac{\epsilon}{4}, \frac{\epsilon}{4})\Big\},$$

$$\big|\gamma(x_1) - |x_1|\big| \le C_1(\sigma_0(C_2|x_1|))^{\frac{1}{2}}|x_1| \text{ for } x_1 \in (-\frac{\epsilon}{4}, \frac{\epsilon}{4})$$

and

$$\big|\gamma'(x_1) - \frac{x_1}{|x_1|}\big| \le C_1(\sigma_1(C_2|x_1|))^{\frac{1}{2}} \text{ for } x_1 \in (-\frac{\epsilon}{4}, \frac{\epsilon}{4})\backslash\{0\}.$$

The proof of this theorem is mainly based on a directional monotonicity result proved in Lemma 25. There we prove that $\partial_\nu u \geq 0$ in $B_r(x)$ for $x \in \Gamma \cap \{x_1 > 0\} \cap \partial B_{\frac{1}{4}}$ if for a given $\nu \in \partial B_1$ with $\nu \cdot \nu_w > 0$, r is small enough and u is close enough to u_w in $C^1(B_1)$. The vector ν_w is the normal to the free boundary of u_w in the half plain $\{x_1 > 0\}$, pointing into the noncoincidence set of u_w. This directional monotonicity result establishes the convergence of the normal of the free boundary to the normal of the free boundary of the blowup limit.

As we will see, from Theorem 6 it follows that in the case when 0 is a regular point but with $u_w(x_1, -x_2)$ as blowup limit and in the case when $W(+0, u) = 2W(1, u_w)$ the free boundary is respectively a graph or a union of two graphs.

In the following theorem, in particular cases we show that the free boundary near to a degenerate point is not $C^{1,\alpha}$ smooth.

Theorem 7 (An Irregularity Result at Degenerate Points)**.** *If u is a solution in D, $0 \in D$, there exists $\delta > 0$ such that $B_\delta \subset D$, $\partial_{x_2} u \leq 0$ in $B_\delta \cap \{x_1 > 0, x_2 > 0\}$, there exists $\rho \in C([0, \frac{1}{2}\delta)) \cap C^1([0, \frac{1}{2}\delta))$ such that $\rho(0) = \rho'(+0) = 0$, $\rho \geq 0$ in $(0, \frac{1}{2}\delta)$, ρ is convex and*

$$\Omega \cap B_\delta \cap \{x_1 > 0, 0 < x_2 < \frac{1}{2}\delta\} = B_\delta \cap \{0 < x_2 < \frac{1}{2}\delta,\ \rho(x_2) < x_1\}$$

then either $\rho = 0$ and $u = u_{hs}$ in $\Omega \cap B_\delta \cap \{x_1 > 0, 0 < x_2 < \frac{1}{2}\delta\}$ or the free boundary part $\Gamma \cap \{x_1 > 0\}$ is not $C^{1,\alpha}$ regular at 0 for any $0 < \alpha < 1$.

Let us notice that the conditions in this theorem correspond to the example depicted in figure 1b.

The proof of this theorem relies on considering the nonnegative function $v = -\partial_{x_2} u$ and using the quantitative Hopf Lemma (cf. [4]).

3. Uniform Bounds on Blowups

The following theorem is a special case of the optimal growth theorem in [13].

Theorem 8. *There exists a $C > 0$ such that if $B_r(y) \subset D$ then we have*

$$u(x) \leq C\big(u(y) + r^2(r + |y_1|)\big) \text{ for } x \in B_{\frac{r}{2}}(y).$$

Based on this optimal growth estimate in the following theorem we prove an estimate on the growth of the solution near the free boundary.

Lemma 1. *There exists a $C > 0$ such that if u is a solution in D, $y \in \Omega$, $d = d(y, \Gamma)$ and $B_{5d}(y) \subset D$ then*

$$u(x) \leq Cd^2(d + |y_1|) \text{ for } x \in B_d(y). \tag{3.1}$$

Proof. Let $z \in \Gamma$ such that $d = |y - z|$. We have for $r = 4d$

$$B_r(z) = B_{4d}(z) \subset B_{4d+|y-z|}(y) = B_{5d}(y) \subset D.$$

By Theorem 8 we have that because $z \in \Gamma$ and $B_r(z) \subset D$

$$u(x) \leq C_1 r^2(r + |z_1|) \text{ for } x \in B_{\frac{r}{2}}(z). \tag{3.2}$$

We have

$$B_d(y) \subset B_{d+|y-z|}(z) = B_{2d}(z) = B_{\frac{r}{2}}(z). \tag{3.3}$$

By (3.2) and (3.3) we obtain

$$\begin{aligned} u(x) \leq C_1 r^2(r + |z_1|) = C_1(4d)^2(4d + |z_1|) &\leq C_2 d^2(d + |z_1|) \\ &\leq C_2 d^2\big(d + |z_1 - y_1| + |y_1|\big) \\ &\leq C_2 d^2\big(2d + |y_1|\big) \leq C_3 d^2(d + |y_1|) \text{ for } x \in B_d(y) \end{aligned}$$

which proves the lemma. ∎

Let us define

$$\psi(t) = \frac{1}{6}|t|^3 \text{ for } t \in \mathbb{R}. \tag{3.4}$$

and for $t_0 \in \mathbb{R}$

$$w_{t_0}(t) = \psi(t) - \psi(t_0) - \psi'(t_0)(t - t_0) \text{ for } t \in \mathbb{R}.$$

Then there exists $C > 0$ such that for $t, t_0 \in \mathbb{R}$ we have

$$w_{t_0}(t) \le C|t - t_0|^2\big(|t_0| + |t - t_0|\big). \tag{3.5}$$

Proof of Theorem 1. We have

$$\|u_r\|_{L^\infty(B_1)} = \frac{1}{r^3}\|u\|_{L^\infty(B_r)},\ \|\nabla u_r\|_{L^\infty(B_1)} = \frac{1}{r^2}\|\nabla u\|_{L^\infty(B_r)}$$

and

$$[\nabla u_r]_{C^{0,1}(B_1)} = \frac{1}{r}[\nabla u]_{C^{0,1}(B_r)}.$$

So if we prove that for some $C > 0$ we have

$$\|u\|_{L^\infty(B_r)} \le Cr^3, \tag{3.6}$$

$$\|\nabla u\|_{L^\infty(B_r)} \le Cr^2 \tag{3.7}$$

and

$$[\nabla u]_{C^{0,1}(B_r)} \le Cr \tag{3.8}$$

then the lemma is proved.

There exists $C > 0$ such that for v a harmonic function in B_1 we have

$$|\nabla v(0)| \le C\|v\|_{L^\infty(B_1)} \text{ and } [\nabla v]_{C^{0,1}(B_{\frac{1}{2}})} \le C\|v\|_{L^\infty(B_1)}.$$

By scaling we obtain that for v harmonic in B_η we have

$$|\nabla v(0)| \le \frac{C}{\eta}\|v\|_{L^\infty(B_\eta)} \tag{3.9}$$

and

$$[\nabla v]_{C^{0,1}(B_{\frac{\eta}{2}})} \le \frac{C}{\eta^2}\|v\|_{L^\infty(B_\eta)}. \tag{3.10}$$

For $x \in \Omega$ let $d = d(x, \Gamma)$ then we have

$$B_{5d}(x) \subset B_{5d+|x|} \subset B_{5|x|+|x|} = B_{6|x|}$$

so if $x \in B_{\frac{1}{6}r_0}$ then $B_{5d}(x) \subset D$.

Now by Lemma 1 we obtain that for $x \in B_{\frac{1}{6}r_0}$ we have

$$\|u\|_{L^\infty(B_d(x))} \le Cd^2(d + |x_1|). \tag{3.11}$$

Let $0 < r < \frac{1}{6}r_0$.

To prove (3.6) we compute for $x \in B_r$

$$|u(x)| \le \|u\|_{L^\infty(B_d(x))} \le Cd^2(d + |x_1|) \le C|x|^2(|x| + |x_1|) = 2C|x|^3 \le 2Cr^3.$$

To prove (3.7) using $w'_{x_1}(x_1) = 0$, (3.9), (3.11) and (3.5) we compute for $x \in B_r$

$$\begin{aligned} |\nabla u(x)| = |\nabla(u - w_{x_1})(x)| &\le \frac{C_1}{d}\|u - w_{x_1}\|_{L^\infty(B_d(x))} \\ &\le \frac{C_1}{d}\|u\|_{L^\infty(B_d(x))} + \frac{C_1}{d}\|w_{x_1}\|_{L^\infty(B_d(x))} \\ &\le C_2 d(d + |x_1|) + C_3 d(d + |x_1|) = C_4 d(d + |x_1|). \end{aligned} \tag{3.12}$$

From (3.12) it follows that

$$|\nabla u(x)| \le 2C_4|x|^2 \le 2C_4 r^2. \tag{3.13}$$

It remains to prove (3.8). We should show that

$$|\nabla u(x)-\nabla u(y)|\le Cr|x-y|,\ \forall x,y\in B_r.$$

Fix $x,y\in B_r$. In the case $B_{|x-y|}(\frac{x+y}{2})\subset\Omega$ let us denote $z=\frac{x+y}{2}$. We have $d=d(z,\Gamma)\ge|x-y|$.

By (3.10) and (3.11) we compute

$$\begin{aligned}\frac{|\nabla u(x)-\nabla u(y)|}{|x-y|}&\le[\nabla u]_{C^{0,1}(B_{\frac{|x-y|}{2}}(z))}\le[\nabla u]_{C^{0,1}(B_{\frac{d}{2}}(z))}\\&\le[\nabla(u-w_{z_1})]_{C^{0,1}(B_{\frac{d}{2}}(z))}+[w'_{z_1}]_{C^{0,1}(B_{\frac{d}{2}}(z))}\\&\le\frac{C_1}{d^2}\|u-w_{z_1}\|_{L^\infty(B_d(z))}+[w_{z_1}]_{C^2(B_{\frac{d}{2}}(z))}\\&\le\frac{C_1}{d^2}\|u\|_{L^\infty(B_d(z))}+\frac{C_1}{d^2}\|w_{z_1}\|_{L^\infty(B_d(z))}+[\psi]_{C^2(B_{\frac{d}{2}}(z))}\\&\le\frac{C_1}{d^2}C_2d^2(d+|z_1|)+\frac{C_1}{d^2}C_3d^2(d+|z_1|)+C_4(d+|z_1|)\\&=C_5(d+|z_1|)\le 2C_5r.\end{aligned}$$

In the case $B_{|x-y|}(\frac{x+y}{2})\cap\Omega^c\neq\emptyset$ by (3.12) we compute

$$\begin{aligned}|\nabla u(x)-\nabla u(y)|&\le|\nabla u(x)|+|\nabla u(y)|\\&\le Cd(x,\Gamma)\big(d(x,\Gamma)+|x_1|\big)+Cd(y,\Gamma)\big(d(y,\Gamma)+|y_1|\big)\\&\le\frac{3}{2}C|x-y|\big(d(x,\Gamma)+|x_1|\big)+\frac{3}{2}C|x-y|\big(d(y,\Gamma)+|y_1|\big)\\&\le C_1r|x-y|\end{aligned}$$

and this finishes the proof of the theorem. □

4. Homogeneity of Blowup Limits

Most of the results in this section are well known, one may refer to [6, 11, 12]. But for the sake of completeness we include the proofs.

The Weiss balanced energy $W(r,u)$ is defined in (2.2).

Lemma 2. *For $r,s>0$ and $u\in H^1(B_{rs})$ we have $W(rs,u)=W(s,u_r)$.*

For $u\in H^1(B_{r_0})$, $W(r,u)$ as a function of $0<r<r_0$ is locally bounded and absolutely continuous.

For u solution in B_{r_0} and $0<r<r_0$ we have

$$\frac{d}{dr}W(r,u)=2r\int_{\partial B_1}(\partial_r u_r)^2 s(dx).\tag{4.1}$$

For u a third order homogenous solution in B_1 we have

$$W(1,u)=\int_{B_1}|x_1|u\,dx.\tag{4.2}$$

Proof. Let $r,s>0$ and $u\in H^1(B_{rs})$. We compute

$$\begin{aligned}W(rs,u)&=\frac{1}{(rs)^6}\int_{B_{rs}}\big(|\nabla u|^2+2|x_1|u\big)dx-\frac{3}{(rs)^7}\int_{\partial B_{rs}}u^2s(dx)\\&=\frac{1}{s^6}\frac{1}{r^4}\int_{B_s}\big(|\nabla u(rx)|^2+2r|x_1|u(rx)\big)dx-\frac{3}{s^7}\frac{1}{r^6}\int_{\partial B_s}u^2(rx)s(dx)\\&=\frac{1}{s^6}\int_{B_s}\big(|\nabla u_r(x)|^2+2|x_1|u_r\big)dx-\frac{3}{s^7}\int_{\partial B_s}u_r^2s(dx)=W(s,u_r)\end{aligned}$$

and this proves the first claim.

Let $u \in H^1(B_{r_0})$ then for $0 < r < r_0$ by direct computation using polar coordinates we have

$$\int_{\partial B_r} u^2 s(dx) = -2r \int_{B_{r_0} \setminus B_r} \frac{1}{|x|^2} u(x) \nabla u(x) \cdot x dx + \frac{r}{r_0} \int_{\partial B_{r_0}} u^2(x) s(dx). \tag{4.3}$$

The equation (4.3) together with the fact that for $f \in L^1_{loc}(\mathbb{R}^2)$, $\int_{B_r} f dx$ as a function of r is bounded and absolutely continuous proves the second claim.

Let u be a solution in B_{r_0} then we have (cf. [6]) $u \in C^{1,1}_{loc}(B_{r_0})$. Let $0 < r < r_0$ then we compute

$$\begin{aligned}
\frac{1}{2}\frac{d}{dr} W(r,u) &= \frac{1}{2}\frac{d}{dr} W(1,u_r) \\
&= \frac{1}{2}\Big(\int_{B_1} \big(2\nabla u_r(x) \cdot \nabla \partial_r u_r(x) + 2|x_1| \partial_r u_r\big) dx - 6 \int_{\partial B_1} u_r \partial_r u_r s(dx) \Big) \\
&= \int_{B_1} \big(\nabla u_r(x) \cdot \nabla \partial_r u_r(x) + |x_1| \partial_r u_r\big) dx - 3 \int_{\partial B_1} u_r \partial_r u_r s(dx) \\
&= \int_{B_1} \big(-\triangle u_r(x) \partial_r u_r(x) + |x_1| \partial_r u_r\big) dx \\
&\quad + \int_{\partial B_1} \partial_\nu u_r(x) \partial_r u_r(x) s(dx) - 3 \int_{\partial B_1} u_r \partial_r u_r s(dx) \\
&= \int_{\partial B_1} \big(\partial_\nu u_r(x) - 3u_r\big) \partial_r u_r s(dx).
\end{aligned}$$

It is easy to see that on ∂B_1 we have

$$\partial_\nu u_r(x) - 3u_r = r \partial_r u_r$$

and this proves the third claim.

Let u be a solution in B_1. We compute

$$\begin{aligned}
W(1,u) &= \int_{B_1} \big(|\nabla u(x)|^2 + 2|x_1| u\big) dx - 3 \int_{\partial B_1} u^2 s(dx) \\
&= \int_{B_1} (-\triangle u(x)) u(x) dx + \int_{\partial B_1} \partial_\nu u(x) u(x) s(dx) + \int_{B_1} 2|x_1| u dx - 3 \int_{\partial B_1} u^2 s(dx) \\
&= \int_{\partial B_1} \partial_\nu u(x) u(x) s(dx) + \int_{B_1} |x_1| u dx - 3 \int_{\partial B_1} u^2 s(dx) \\
&= \int_{B_1} |x_1| u dx + \int_{\partial B_1} \big(\partial_\nu u - 3u\big) u s(dx).
\end{aligned}$$

For a third order homogenous function we have $\partial_\nu u = 3u$ thus the last integral is null and this proves the last claim. □

If u is a solution in B_{r_0} for some $r_0 > 0$ then by (4.1), $W(r,u)$ is nondecreasing in $0 < r < r_0$, thus the limit $\lim_{r \to 0, r > 0} W(r,u) = W(+0,u)$ exists but might be $-\infty$. If $0 \in \Gamma$ then by Theorem 1 we have $\|u_r\|_{L^\infty(B_1)} \leq C$ for small enough $0 < r$ and from this it follows that

$$-\frac{1}{r^7} \int_{\partial B_r} u^2 s(dx) = -\int_{\partial B_1} u_r^2 s(dx) \geq -C_1,$$

thus $W(r,u) \geq -3C_1$ and $W(+0,u) \geq -3C_1 > -\infty$.

For $y \in \Gamma \cap \{x_1 = 0\}$ and $r > 0$, $W(r,y,u)$ is defined in (2.5).

Lemma 3. *$W(+0,x,u)$ is an upper semicontinuous function of $x \in \Gamma \cap \{x_1 = 0\}$.*

Proof. For $x \in \Gamma \cap \{x_1 = 0\}$ by the monotonicity of $W(r, x, u)$ as a function of $r > 0$ and its continuity as a function of x it follows that $W(+0, x, u) = \lim_{r\to 0, r>0} W(r, x, u)$ is upper semicontinuous in $\Gamma \cap \{x_1 = 0\}$. □

Assume v is a third order homogenous function in B_1, i.e. $v(0) = 0$ and $v(x) = v(\frac{x}{2|x|})(2|x|)^3$ for all $x \in B_1\backslash\{0\}$. Then we might extend v as a third order homogenous function in $\mathbb{R}^2$ as follows, $v(x) = v(\frac{x}{2|x|})(2|x|)^3$ for all $x \in B_1^c$. Let us note that the term on the right hand side is well defined because for $x \in B_1^c$ we have $\frac{x}{2|x|} \in B_1$. From this definition of extension it follows that $v(rx) = r^3 v(x)$ for all $x \in \mathbb{R}^2$ and $r \geq 0$.

The following theorem is a special case of the main theorem in [13].

Theorem 9. *There exists a $C > 0$ such that if u is a solution in D, $y \in \Omega$ and $B_r(y) \subset\subset D$ then we have*

$$\sup_{\Omega\cap\partial B_r(y)} u \geq u(y) + Cr^2(r + |y_1|). \tag{4.4}$$

A blowup limit is defined in Definition 2.

Lemma 4. *Let v be a blowup limit. Then v is a third order nontrivial homogenous solution in B_1, the third order homogenous extension of v in $\mathbb{R}^2$ is a global solution and $W(+0, u) = W(r, v)$ for $r > 0$.*

Proof. Assume $v \in C^{1,1}(B_1)$ is a blowup limit and $u_{r_j} \to v$ in $C^1(B_1)$.

From $u_{r_j} \geq 0$ in B_1 it follows that $v \geq 0$ in B_1. By the convergence $u_{r_j} \to v$ in $C^1(B_1)$ it follows that $\triangle u_{r_j} \to \triangle v$ in $H^{-1}(B_1)$ and in particular as distributions. Also $\chi_{u_{r_j}>0} \to \chi_{v>0}$ in $L^1(B_1)$ and thus $|x_1|\chi_{u_{r_j}>0} \to |x_1|\chi_{v>0}$ as distributions. Now the equation (1.2) holds for u_{r_j} in B_1, passing to the limit as $j \to \infty$ we obtain that v satisfies (1.2) in B_1. This together with $v \geq 0$ in B_1 proves that v is a solution to the obstacle problem in B_1.

For $0 < s < 1$ we compute

$$W(+0, u) = \lim_{j\to\infty} W(sr_j, u) = \lim_{j\to\infty} W(s, u_{r_j}) = W(s, v). \tag{4.5}$$

Thus $W(s, v)$ is independent of $0 < s < 1$.

Now by (4.1) we obtain that for $0 < s < 1$

$$0 = \frac{d}{ds} W(s, v) = 2s \int_{\partial B_1} (\partial_s v_s)^2 s(dx).$$

From here it follows that $\nabla v \cdot x - 3v = 0$ in B_1 and hence v is third order homogenous in B_1.

Now let us prove that v is not 0 in B_1, i.e. v is nontrivial.

Let $\delta > 0$ and $B_\delta \subset D$. Let $0 < r < \delta$ and $y \in B_{\frac{1}{2}r} \cap \Omega$ then we have

$$B_{\frac{1}{4}r}(y) \subset B_{\frac{1}{4}r+|y|} \subset B_{\frac{1}{4}r+\frac{1}{2}r} = B_{\frac{3}{4}r} \subset\subset D$$

thus by Theorem 9 we have

$$\sup_{\partial B_{\frac{1}{4}r}(y)} u \geq u(y) + C(\frac{1}{4}r)^3.$$

We compute

$$\partial B_{\frac{1}{4}r}(y) \subset B_{\frac{1}{2}r}(y) \subset B_{\frac{1}{2}r+|y|} \subset B_r$$

so we have

$$\sup_{B_r} u \geq \sup_{\partial B_{\frac{1}{4}r}(y)} u \geq u(y) + C(\frac{1}{4}r)^3 \geq \frac{1}{4^3} Cr^3$$

and thus

$$\sup_{B_1} u_r \geq \frac{1}{4^3} C.$$

From this inequality taking $r = r_j \to 0$ we obtain that v is not identically 0 in B_1.

Let us again denote by v the extension of v in $\mathbb{R}^2$. Then it is easy to see that because v is a solution in B_1 and $v(rx) = r^3 v(x)$ for $x \in \mathbb{R}^2$ and $r \geq 0$, v is a solution in $\mathbb{R}^2$, i.e. a global solution.

By third order homogeneity of v we have $W(r, v) = W(\frac{1}{2}, v)$ for $r > 0$ and this together with (4.5) proves the last claim of the lemma. □

5. Homogeneous Global Solutions

In this section we classify all the possible solutions of the problem (2.3). The solutions of (2.3) form the subset of nonnegative solutions of the following no-sign obstacle problem (cf. [6] for more on no-sign obstacle problems)

$$\begin{cases} \triangle u = |x_1| \chi_{\Omega(u)} \text{ in } \mathbb{R}^2, \\ \Omega(u) = \{u = |\nabla u| = 0\}^c, \\ u \text{ third order homogeneous.} \end{cases} \tag{5.1}$$

We first classify the nontrivial solutions of (5.1) and then find the subset of nonnegative and nontrivial solutions of (5.1) and thus obtain the classification of the nontrivial solutions of the problem (2.3).

In the rest of this section we always assume that $u \neq 0$ in $\mathbb{R}^2$, i.e. we discuss only the nontrivial solutions, so $\Omega \neq \emptyset$.

In both problems, by homogeneity, the set Ω is an open cone in $\mathbb{R}^2 \backslash \{0\}$, i.e. for $x \in \Omega$ and $r > 0$ we have $rx \in \Omega$.

Either Ω is equal to $\mathbb{R}^2 \backslash \{0\}$ or it is at most a countable union of disjoint connected open cones in $\mathbb{R}^2 \backslash \{0\}$.

To classify the solutions in both problems we first establish if there exists a solution with $\Omega = \mathbb{R}^2 \backslash \{0\}$. Then we find all the connected cones Ω not equal to $\mathbb{R}^2 \backslash \{0\}$ for which there exists a corresponding solution.

Lemma 5. *If in a connected open cone $\Omega \subset \mathbb{R}^2$, u is a third order homogenous function such that $\triangle u = |x_1|$ then there exists $a \in \mathbb{C}$ such that*

$$u(e^{i\theta}) - \frac{i}{3} \partial_\theta u(e^{i\theta}) = \frac{1}{6} |\cos(\theta)| \cos(\theta) e^{i\theta} + \bar{a} e^{3i\theta} \tag{5.2}$$

for all $e^{i\theta} \in \Omega$ (in the rest of this section we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$).

Proof. Let us denote $v(x) = u(x) - \psi(x_1)$, ψ is defined in (3.4), then v is a third order homogenous harmonic function in the connected open cone $\Omega \subset \mathbb{R}^2$. Thus there exists $a \in \mathbb{C}$ such that

$$v(x) = \Re\big(\bar{a}(x_1 + i x_2)^3\big) \text{ for all } x \in \Omega.$$

So we have

$$u(e^{i\theta}) = \frac{1}{6} |\cos(\theta)|^3 + \Re(\bar{a} e^{3i\theta}) \tag{5.3}$$

for all $e^{i\theta} \in \Omega$.

Differentiating (5.3) with respect to θ we obtain the desired equation. □

Let us denote by $U(\theta)$ the expression on the left hand side of (5.2), i.e.

$$U(\theta) = \frac{1}{6} |\cos(\theta)| \cos(\theta) e^{i\theta} + \bar{a} e^{3i\theta}. \tag{5.4}$$

By the homogeneity of u it follows that

$$\{x \in \bar{\Omega} \mid u(x) = |\nabla u(x)| = 0\} = \{re^{i\theta} \in \bar{\Omega} \mid U(\theta) = 0,\, r > 0\}.$$

If $\Omega = \mathbb{R}^2 \backslash \{0\}$ then for u to be a solution to (5.1), U should be a periodic function with period 2π such that $U(\theta) \neq 0$ for all $\theta \in \mathbb{R}$ and if in addition u is a solution to (2.3) then we should have $\Re U(\theta) > 0$ for all $\theta \in \mathbb{R}$.

In the case Ω is an open connected cone not equal to $\mathbb{R}^2 \backslash \{0\}$ then there exists $\theta_1, \theta_2 \in \mathbb{R}$ such that $\theta_1 < \theta_2 \leq \theta_1 + 2\pi$ and $\Omega = \{re^{i\theta} \mid r > 0,\, \theta_1 < \theta < \theta_2\}$. In this case if u is a solution to (5.1) with $\Omega = \Omega(u)$, then U should satisfy $U(\theta_1) = U(\theta_2) = 0$, $U(\theta) \neq 0$ for $\theta_1 < \theta < \theta_2$. And if in addition u is a solution to (2.3) then we should have $\Re U(\theta) > 0$ for $\theta_1 < \theta < \theta_2$.

Let us define

$$V(\theta) = |\cos(\theta)| \cos(\theta) e^{2i\theta}. \tag{5.5}$$

It follows that

$$6e^{3i\theta}\bar{U}(\theta) = V(\theta) + 6a. \tag{5.6}$$

Lemma 6. *u is a solution of (5.1) with $\Omega = \mathbb{R}^2 \backslash \{0\}$ if and only if $-6a \notin V(\mathbb{R})$.*

Proof. u is a solution of (5.1) with $\Omega = \mathbb{R}^2 \backslash \{0\}$ if and only if U is 2π periodic and $U(\theta) \neq 0$ for all $\theta \in \mathbb{R}$.

From (5.2) it follows that U is 2π periodic and by (5.6) it is clear that $U(\theta) \neq 0$ for all $\theta \in \mathbb{R}$ if and only if $-6a \notin V(\mathbb{R})$. □

From the definition of V in (5.5) it is clear that $B_1^c \subset (V(\mathbb{R}))^c$, thus by Lemma 6 it follows that there are many solutions of (5.1) with $\Omega = \mathbb{R}^2 \backslash \{0\}$.

Let us notice that for a connected cone specified by θ_1 and θ_2 the solution with such a cone is unique. This follows from the fact that because $U(\theta_1) = 0$ then by (5.4), a is uniquely obtained and by this value of a the solution u is uniquely given by (5.3). Based on this observation, in the following we do not distinguish between a connected cone and the corresponding solution.

Lemma 7. *u is a solution of (5.1) with a connected open cone $\Omega \neq \mathbb{R}^2 \backslash \{0\}$ if and only if one of the following cases hold*

(i) $\theta_1 \notin \mathbb{Z}\pi + \{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$ *and* $\theta_2 = \theta_1 + 2\pi$,
(ii) $\theta_1 \in \mathbb{Z}\pi + \frac{\pi}{2}$ *and* $\theta_2 = \theta_1 + \pi$,
(iii) $\theta_1 \in \mathbb{Z}\pi + \frac{\pi}{4}$ *and* $\theta_2 = \theta_1 + \frac{\pi}{2}$,
(iv) $\theta_1 \in \mathbb{Z}\pi + \frac{3\pi}{4}$ *and* $\theta_2 = \theta_1 + \frac{3}{2}\pi$.

Proof. Let us remember that we should have $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_1 < \theta_2 \leq \theta_1 + 2\pi$, $U(\theta_1) = U(\theta_2) = 0$, $U(\theta) \neq 0$ for $\theta_1 < \theta < \theta_2$. Although it is possible to find all such θ_1 and θ_2 by algebraic computations, but for ease of presentation we resort to geometric arguments.

By (5.6), $U(\theta) = 0$ if and only if $-6a = V(\theta)$ hence we should have $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_1 < \theta_2 \leq \theta_1 + 2\pi$, $V(\theta_1) = V(\theta_2)$, $V(\theta) \neq V(\theta_1)$ for $\theta_1 < \theta < \theta_2$. Thus we should find smallest closed loops in the range graph of V. The range graph of V, i.e. the set $V(\mathbb{R})$ is depicted in the figure 3.

Then we have the following four cases

(i) $-6a = V(\theta_1) \in V(\mathbb{R}) \backslash \{0, \pm\frac{i}{2}\}$ with $\theta_1 \in \mathbb{R} \backslash (\mathbb{Z}\pi + \{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\})$ and the smallest loop is when $\theta_2 = \theta_1 + 2\pi$,
(ii) $-6a = V(\theta_1) = 0$ with $\theta_1 \in \mathbb{Z}\pi + \frac{\pi}{2}$ and the smallest loop is when $\theta_2 = \theta_1 + \pi$,
(iii) $-6a = V(\theta_1) \in \{\pm\frac{i}{2}\}$ with $\theta_1 \in \mathbb{Z}\pi + \frac{\pi}{4}$ and the smallest loop is when $\theta_2 = \theta_1 + \frac{\pi}{2}$,
(iv) $-6a = V(\theta_1) \in \{\pm\frac{i}{2}\}$ with $\theta_1 \in \mathbb{Z}\pi + \frac{3\pi}{4}$ and the smallest loop is when $\theta_2 = \theta_1 + \frac{3}{2}\pi$. □

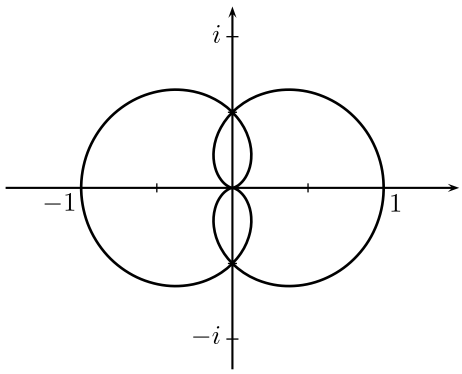

FIGURE 3. $V(\mathbb{R})$.

There is some redundancy in the solutions specified in the previous lemma. In the following lemma we prove that if for two solutions the corresponding connected cones are rotations of each other by a multiple of π then the corresponding solutions are also rotated by the same angle.

Lemma 8. *Let $a, a' \in \mathbb{C}$ and U, U' be the corresponding functions. If $n \in \mathbb{Z}$ and $\theta_0 \in \mathbb{R}$ such that $U'(\theta_0 + n\pi) = U(\theta_0)$ then $U'(\theta + n\pi) = U(\theta)$ for all $\theta \in \mathbb{R}$.*

Proof. For any $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$ we have

$$\begin{aligned} U'(\theta + n\pi) &= \overline{a'} e^{3i(\theta+n\pi)} + \frac{1}{6}|\cos(\theta + n\pi)|\cos(\theta + n\pi)e^{i(\theta+n\pi)} \\ &= (-1)^n \overline{a'} e^{3i\theta} + \frac{1}{6}|\cos(\theta)|\cos(\theta)e^{i\theta} = ((-1)^n \overline{a'} - \overline{a})e^{3i\theta} + U_0(\theta) \end{aligned}$$

from which the lemma follows because if $U'(\theta_0 + n\pi) = U(\theta_0)$ for some θ_0 then $(-1)^n\overline{a'} - \overline{a} = 0$ from which in turn it follows that $U'(\theta + n\pi) = U(\theta)$ for all θ. □

Corollary 1. *Let u and u' be solutions of* (5.1) *respectively with $\Omega(u) = \{re^{i\theta} \mid \theta_1 < \theta < \theta_2,\ r > 0\}$ and $\Omega(u') = \{re^{i\theta} \mid \theta_1' < \theta < \theta_2',\ r > 0\}$ where $\theta_1 < \theta_2 \leq \theta_1 + 2\pi$ and $\theta_1' < \theta_2' \leq \theta_1' + 2\pi$. If there exists $n \in \mathbb{Z}$ such that $\theta_1' = \theta_1 + n\pi$ and $\theta_2' = \theta_2 + n\pi$ then $u'(e^{i(\theta+n\pi)}) = u(e^{i\theta})$ for $\theta_1 < \theta < \theta_2$.*

Proof. Let $U(\theta)$ correspond to $u(x)$ and $U'(\theta)$ to $u'(x)$. Then $U(\theta_1) = 0$ and $U'(\theta_1') = 0$. Thus $U(\theta_1) = U'(\theta_1') = U'(\theta_1 + n\pi)$. Now by Lemma 8 the corollary is proved. □

By this corollary we are able to remove some of the redundancies in Lemma 7 as stated in the following corollary.

Corollary 2. *u is a solution of* (5.1) *with a connected open cone $\Omega \neq \mathbb{R}^2 \backslash \{0\}$ if and only if one of the following cases hold*

(i) $\theta_1 \in [0, 2\pi)\backslash\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}\}$ *and* $\theta_2 = \theta_1 + 2\pi$*, the solutions corresponding to* $\theta_1 \in [\pi, 2\pi)\backslash\{\frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}\}$ *are respectively equal to the solutions corresponding to* $\theta_1 \in [0, \pi)\backslash\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$ *rotated by* π*,*

(ii) $\theta_1 \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ *and* $\theta_2 = \theta_1 + \pi$*, the solution corresponding to* $\theta_1 = \frac{3\pi}{4}$ *is equal to the solution corresponding to* $\theta_1 = \frac{\pi}{2}$ *rotated by* π*,*

(iii) $\theta_1 \in \{\frac{\pi}{4}, \frac{5\pi}{4}\}$ *and* $\theta_2 = \theta_1 + \frac{\pi}{2}$*, the solution corresponding to* $\theta_1 = \frac{5\pi}{4}$ *is equal to the solution corresponding to* $\theta_1 = \frac{\pi}{4}$ *rotated by* π*,*

(iv) $\theta_1 \in \{\frac{3\pi}{4}, \frac{7\pi}{4}\}$ *and* $\theta_2 = \theta_1 + \frac{3}{2}\pi$*, the solution corresponding to* $\theta_1 = \frac{7\pi}{4}$ *is equal to the solution corresponding to* $\theta_1 = \frac{3\pi}{4}$ *rotated by* π*.*

By Lemma 6 we have obtained the solutions of (5.1) with $\Omega = \mathbb{R}^2\backslash\{0\}$ and by Corollary 2 we have obtained all the solutions of (5.1) with a connected open cone $\Omega \neq \mathbb{R}^2\backslash\{0\}$. Now we turn to finding the nonnegative solutions among these solutions.

To check the nonnegativity of a solution u, in the following lemma we write $u(e^{i\theta})$ in a closed form.

Lemma 9. *Let $\theta_1 < \theta_2 \leq \theta_1 + 2\pi$ and u be a solution to* (5.1) *in the cone corresponding to θ_1 and θ_2. Then we have*

$$6u(e^{i\theta}) = |\cos(\theta)|^3 - |\cos(\theta_1)|\cos(\theta_1)\cos(3\theta - 2\theta_1). \tag{5.7}$$

Proof. Because $U(\theta_1) = 0$, by (5.6) we have $6\bar{a} = -\bar{V}(\theta_1)$.

Now by (5.3) we compute

$$\begin{aligned} 6u(e^{i\theta}) &= |\cos(\theta)|^3 + \Re\big(6\bar{a}e^{3i\theta}\big) = |\cos(\theta)|^3 - \Re\big(\bar{V}(\theta_1)e^{3i\theta}\big) \\ &= |\cos(\theta)|^3 - \Re\big(|\cos(\theta_1)|\cos(\theta_1)e^{-2i\theta_1}e^{3i\theta}\big) \\ &= |\cos(\theta)|^3 - \Re\big(|\cos(\theta_1)|\cos(\theta_1)e^{(3\theta-2\theta_1)i}\big) \\ &= |\cos(\theta)|^3 - |\cos(\theta_1)|\cos(\theta_1)\Re\big(e^{(3\theta-2\theta_1)i}\big) \\ &= |\cos(\theta)|^3 - |\cos(\theta_1)|\cos(\theta_1)\cos(3\theta - 2\theta_1). \quad \square \end{aligned}$$

Lemma 10. *There exists no solution to the problem* (2.3) *with $\Omega = \{u > 0\} = \mathbb{R}^2\backslash\{0\}$.*

Proof. On the line segments $\{x_1 = 0\}\backslash\{0\}$, i.e. for $\theta = \pm\frac{\pi}{2}$ we have

$$\begin{aligned} 6u(e^{\pm i\frac{\pi}{2}}) &= |\cos(\pm\frac{\pi}{2})|^3 - |\cos(\theta_1)|\cos(\theta_1)\cos(\pm\frac{3\pi}{2} - 2\theta_1) \\ &= -|\cos(\theta_1)|\cos(\theta_1)\cos(\pm\frac{3\pi}{2} - 2\theta_1) = \pm|\cos(\theta_1)|\cos(\theta_1)\sin(2\theta_1) \end{aligned} \tag{5.8}$$

If $|\cos(\theta_1)|\cos(\theta_1)\sin(2\theta_1) = 0$ then $u(e^{\pm i\frac{\pi}{2}}) = 0$ which is in contradiction with $\Omega = \{u > 0\} = \mathbb{R}^2\backslash\{0\}$. If $|\cos(\theta_1)|\cos(\theta_1)\sin(2\theta_1) \neq 0$ then we can choose $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$ and obtain $u(e^{i\theta}) < 0$ which is again in contradiction with $\Omega = \{u > 0\} = \mathbb{R}^2\backslash\{0\}$. $\square$

Lemma 11. *u is a solution of* (2.3) *with a connected open cone $\Omega \neq \mathbb{R}^2\backslash\{0\}$ if and only if one of the following cases hold*

(i) $\theta_1 \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ and $\theta_2 = \theta_1 + \pi$, solution corresponding to $\theta_1 = \frac{3\pi}{2}$ is equal to the solution corresponding to $\theta_1 = \frac{\pi}{2}$ rotated by π,

(ii) $\theta_1 \in \{\frac{\pi}{4}, \frac{5\pi}{4}\}$ and $\theta_2 = \theta_1 + \frac{\pi}{2}$, solution corresponding to $\theta_1 = \frac{5\pi}{4}$ is equal to the solution corresponding to $\theta_1 = \frac{\pi}{4}$ rotated by π.

Proof. We first show that the solutions given in parts (i) and (iv) of Corollary 2 are not nonnegative and then we show that the solutions given in parts (ii) and (iii) are nonnegative.

To prove the not nonnegativity of solutions given in part (i) of Corollary 2 we need only to consider $\theta_1 \in [0, \pi)\backslash\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$ with $\theta_2 = \theta_1 + 2\pi$ and to prove the not nonnegativity of solutions given in part (iv) we need only to consider $\theta_1 = \frac{3\pi}{4}$ with $\theta_2 = \theta_1 + \frac{3\pi}{2}$.

For all these cases let us consider $\theta = \frac{3\pi}{2}$ then $\theta_1 < \theta < \theta_2$ and by a similar computation as in (5.8) we obtain that

$$6u(e^{i\frac{3\pi}{2}}) = -|\cos(\theta_1)|\cos(\theta_1)\sin(2\theta_1).$$

Because for $\theta_1 \in [0, \pi)$ we have

$$|\cos(\theta_1)|\cos(\theta_1)\sin(2\theta_1) = 2|\cos(\theta_1)|\cos^2(\theta_1)\sin(\theta_1) \geq 0$$

this proves that the respective solutions take a nonpositive value at $\theta = \frac{3\pi}{2}$. If $u(e^{i\frac{3\pi}{2}}) < 0$ then u is not nonnegative. If $u(e^{i\frac{3\pi}{2}}) = 0$ and u was nonnegative then we would have $\partial_\theta u(e^{i\frac{3\pi}{2}}) = 0$ which is in contradiction with the connectedness of Ω.

To prove that the solutions given in part (ii) of Corollary 2 are solutions of (2.3), we need only to consider the case when $\theta_1 = \frac{\pi}{2}$ with $\theta_2 = \theta_1 + \pi$. We compute

$$6u(e^{i\theta}) = |\cos(\theta)|^3 - |\cos(\frac{\pi}{2})|\cos(\frac{\pi}{2})\cos(3\theta - 2(\frac{\pi}{2})) = |\cos(\theta)|^3 \tag{5.9}$$

and because $|\cos(\theta)| > 0$ for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ we obtain that u is a solution of (2.3).

To prove that the solutions given in part (iii) of Corollary 2 are solutions of (2.3), we need only to consider the case when $\theta_1 = \frac{\pi}{4}$ with $\theta_2 = \theta_1 + \frac{\pi}{2}$. We compute

$$\begin{aligned} 6u(e^{i\theta}) &= |\cos(\theta)|^3 - |\cos(\frac{\pi}{4})|\cos(\frac{\pi}{4})\cos(3\theta - \frac{\pi}{2}) \\ &= |\cos(\theta)|^3 - \frac{1}{2}\cos(3\theta - \frac{\pi}{2}) = |\cos(\theta)|^3 - \frac{1}{2}\sin(3\theta). \end{aligned} \tag{5.10}$$

Let $\theta = \frac{\pi}{2} + \gamma$ for $-\frac{\pi}{4} < \gamma < \frac{\pi}{4}$ then

$$6u(e^{i(\frac{\pi}{2}+\gamma)}) = |\cos(\frac{\pi}{2} + \gamma)|^3 - \frac{1}{2}\sin(3(\frac{\pi}{2} + \gamma)) = |\sin(\gamma)|^3 + \frac{1}{2}\cos(3\gamma).$$

It follows that $6u(e^{i(\frac{\pi}{2}+\gamma)}) = 6u(e^{i(\frac{\pi}{2}-\gamma)})$ so we need only to consider $0 \le \gamma < \frac{\pi}{4}$. For $0 \le \gamma < \frac{\pi}{4}$ we have $\sin(\gamma) \ge 0$ thus

$$6u(e^{i(\frac{\pi}{2}+\gamma)}) = \sin^3(\gamma) + \frac{1}{2}\cos(3\gamma) = \frac{1}{2}\cos^3(\gamma)(\tan(\gamma) - 1)^2(2\tan(\gamma) + 1) > 0$$

therefore we obtain that u is a solution of (2.3). □

Lemma 12. *In the original variable $x \in \mathbb{R}^2$ the only solutions of* (2.3) *with a connected open cone $\Omega \neq \mathbb{R}^2\backslash\{0\}$ are the following four solutions together with their noncoincidence cone Ω and their free boundary Γ*

$$u(x) = u_{hs}(x),\ \Omega = \{x_1 > 0\},\ \Gamma = \{x_1 = 0\},$$

$$u(x) = u_{hs}(-x_1, x_2),\ \Omega = \{x_1 < 0\},\ \Gamma = \{x_1 = 0\},$$

$$u(x) = u_w(x),\ \Omega = \{x_2 > |x_1|\},\ \Gamma = \{x_2 = |x_1|\},$$

and

$$u(x) = u_w(x_1, -x_2),\ \Omega = \{x_2 < -|x_1|\},\ \Gamma = \{x_2 = -|x_1|\}.$$

Proof. We compute the solutions given in the Lemma 11 in the original variable.

For solutions given in part (i) of Lemma 11, we only consider the case when $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \theta_1 + \pi$. We have

$$\Big\{x = re^{i\theta} \ \Big|\ r > 0,\ \frac{\pi}{2} < \theta < \frac{3\pi}{2}\Big\} = \big\{x_1 < 0\big\}.$$

Now for $x = re^{i\theta} \in \{x_1 < 0\}$ using the computation in (5.9) we compute

$$6u(x) = 6u(re^{i\theta}) = 6r^3u(e^{i\theta}) = r^3|\cos(\theta)|^3 = r^3|\frac{x_1}{r}|^3 = |x_1|^3 = (x_1^-)^3.$$

For solutions given in part (ii) of Lemma 11 we only consider the case when $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \theta_1 + \frac{\pi}{2}$. We have

$$\Big\{x = re^{i\theta} \ \Big|\ r > 0,\ \frac{\pi}{4} < \theta < \frac{3\pi}{4}\Big\} = \big\{x_2 > |x_1|\big\}.$$

Now for $x = re^{i\theta} \in \{x_2 > |x_1|\}$ using the computation in (5.10) we compute

$$\begin{aligned}6u(x) = 6u(re^{i\theta}) = 6r^3u(e^{i\theta}) &= r^3\big(|\cos(\theta)|^3 - \frac{1}{2}\sin(3\theta)\big)\\ &= r^3\big(|\cos(\theta)|^3 - \frac{1}{2}\big(3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)\big)\big)\\ &= r^3\big(|\frac{x_1}{r}|^3 - \frac{1}{2}\big(3(\frac{x_1}{r})^2\frac{x_2}{r} - (\frac{x_2}{r})^3\big)\big) = |x_1|^3 - \frac{1}{2}\big(3x_1^2x_2 - x_2^3\big). \quad \square\end{aligned}$$

Proof of Theorem 2. By Lemma 10 there exists no solution to the problem (2.3) with $\Omega = \{u > 0\} = \mathbb{R}^2\backslash\{0\}$.

So we are left only with solutions whose noncoincidence open cone Ω is a countable union of disjoint connected open cones. But considering the only possible connected open cones as noncoincidence sets enumerated in Lemma 12, we come to the conclusion that except the solutions with connected cones, there exists two additional solutions $u_w + u_w(x_1, -x_2)$ and $u_{hs} + u_{hs}(-x_1, x_2)$, each a combination of two solutions with connected open cones. $\square$

Lemma 13. *We have*

$$W(1, u_{hs}) = \frac{\pi}{96} \quad \text{and} \quad W(1, u_w) = \frac{1}{192}(\pi - \frac{8}{3}).$$

Proof. For any solution of (2.3) with connected open cone we have using (4.2)

$$\begin{aligned}W(1, u) = \int_{B_1}|x_1|u dx &= \int_0^1\int_{\partial B_r}|x_1|us(dx)dr = \int_0^1\int_{\partial B_1}|ry_1|u(ry)rs(dy)dr\\ &= \int_0^1 r^5 dr\int_{\partial B_1}|y_1|u(y)s(dy) = \frac{1}{6}\int_{\theta_1}^{\theta_2}|\cos(\theta)|u(e^{i\theta})d\theta.\end{aligned}$$

For the half space solution u_{hs}, we compute using the computation (5.9)

$$W(1, u_{hs}) = \frac{1}{36}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}|\cos(\theta)|^4 d\theta = \frac{1}{18}\int_0^{\frac{\pi}{2}}\cos^4(\theta)d\theta = \frac{\pi}{96}.$$

For the wedge solution u_w we compute using the computation (5.10)

$$\begin{aligned}W(1, u_w) &= \frac{1}{36}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\big(|\cos(\theta)|^4 - \frac{1}{2}|\cos(\theta)|\sin(3\theta)\big)d\theta\\ &= \frac{1}{18}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^4(\theta)d\theta - \frac{1}{36}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos(\theta)\sin(3\theta)d\theta = \frac{1}{192}\big(\pi - \frac{8}{3}\big). \quad \square\end{aligned}$$

Corollary 3. *We have*

$$\begin{aligned}&0 < W(1, u_w) = W(1, u_w(x_1, -x_2)) < W(1, u_w + u_w(x_1, -x_2)) = 2W(1, u_w)\\ &< W(1, u_{hs}) = W(1, u_{hs}(-x_1, x_2)) < W(1, u_{hs} + u_{hs}(-x_1, x_2)) = 2W(1, u_{hs}).\end{aligned}$$

Proof. The only inequality that is not clear is the inequality $2W(1, u_w) < W(1, u_{hs})$. But this is verified by the explicit values computed in the previous lemma. $\square$

Corollary 4. *The set $\Gamma \cap \{x_1 = 0\}$ might be decomposed in four disjoint sets according to four possible values of the Weiss balanced energy. The closure of the set of points with a given energy w is a subset of the set of points with energy larger than or equal to w.*

Proof. Let $y \in \Gamma \cap \{x_1 = 0\}$ then by the translation $u(x + y)$ we might assume that $y = 0$. Let $0 < \delta$ be such that $B_\delta \subset D$. Let us consider the family u_r for $0 < r < \frac{1}{6}\delta$. By Theorem 1 this family is uniformly bounded in $C^{1,1}(B_1)$. Thus there exists $r_j \to 0$ and $v \in C^{1,1}(B_1)$ such that $u_{r_j} \to v$ in $C^1(B_1)$. By Lemma 4, v

is a nontrivial homogenous global solutions and $W(+0,u) = W(1,v)$. The possible values of $W(1,v)$ are only of the four values given in the previous corollary and this shows that the free boundary points $\Gamma \cap \{x_1 = 0\}$ divide into four disjoint sets depending on the Weiss balanced energy of the blowups at that point.

The last claim follows from the upper semicontinuity of $W(+0,x,u)$ stated in Lemma 3. □

For example from Corollary 4 it follows that the set $\Gamma\cap\{x_1 = 0\}\cap\{W(+0,x,u) = 2W(1,u_{hs})\}$ is closed. Actually at the end of section 6 we will show that all points of $\Gamma \cap \{x_1 = 0\} \cap \{W(+0,x,u) \in \{W(1,u_w), 2W(1,u_w)\}\}$ are isolated points of $\Gamma \cap \{x_1 = 0\}$.

In the following lemma we obtain a lower bound for the homogenous global solutions which will be used in Lemma 16.

Lemma 14. *There exists a $C > 0$ such that for all homogenous global solutions u we have*

$$u(x) \geq Cd^2(x,\{u = 0\})\big(d(x,\{u = 0\}) + |x_1|\big) \text{ for } x \in \mathbb{R}^2. \tag{5.11}$$

Proof. It is easy to see that we need to prove (5.11) for the cases when $u = u_w$ or $u = u_{hs}$.

In the case $u = u_{hs}$ for $x_1 \leq 0$ both sides of the inequality (5.11) are 0. For $x_1 > 0$ we have $d(x,\{u_{hs} = 0\}) = x_1$ hence

$$\begin{aligned} u_{hs}(x) = \frac{1}{6}x_1^3 = \frac{1}{6}d^2(x,\{u_{hs} = 0\})\Big(\frac{d(x,\{u_{hs} = 0\})}{2} + \frac{x_1}{2}\Big) \\ = \frac{1}{12}d^2(x,\{u_{hs} = 0\})\big(d(x,\{u_{hs} = 0\}) + x_1\big) \end{aligned}$$

and this proves (5.11) for $u = u_{hs}$.

In the case $u = u_w$ for $x_2 < |x_1|$ both sides of the inequality are 0. Also by the symmetry $u_w(x_1,x_2) = u_w(-x_1,x_2)$ we need only to consider the case $x_2 > x_1 > 0$.

For $x_2 > x_1 > 0$ it is easy to see that $d(x,\{u_w = 0\}) = \frac{x_2 - x_1}{\sqrt{2}}$ thus for $x_2 > x_1 > 0$ we compute

$$\begin{aligned} u_w(x) &= \frac{1}{6}x_1^3 + \frac{1}{12}x_2^3 - \frac{1}{4}x_1^2x_2 = \frac{1}{12}(x_2 - x_1)^2(2x_1 + x_2) \\ &= \frac{1}{12}\big(\sqrt{2}d(x,\{u_w = 0\})\big)^2\big(3x_1 + \sqrt{2}d(x,\{u_w = 0\})\big) \\ &\geq \frac{\sqrt{2}}{6}d^2(x,\{u_w = 0\})\big(d(x,\{u_w = 0\}) + x_1\big) \end{aligned}$$

and this proves the desired inequality. □

In the following lemma we prove directional monotonicity type inequalities which will be used in Lemmas 22 and 25.

Lemma 15. *There exists a $C > 0$ such that*

(i) $a\partial_{x_2}u_w - u_w \geq 0$ *in* $B_1 \cap \{x_2 > |x_1|\}$ *if* $a \geq C$,

(ii) $a\partial_\nu u_w - u_w \geq 0$ *in* $B_1 \cap \{(1+\epsilon)x_1 > x_2 > x_1 > 0\}$ *if* $\nu = e^{i(\frac{3\pi}{4}+\gamma)}$, $0 < \epsilon$, $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$ *and* $C(\frac{1}{a} + 1)\epsilon \leq \cos(\gamma)$.

Proof. For $x_2 > x_1 > 0$ we have

$$u_w(x) = \frac{1}{6}x_1^3 + \frac{1}{12}x_2^3 - \frac{1}{4}x_1^2x_2 = \frac{1}{12}(x_2 - x_1)^2(2x_1 + x_2),$$

$$\partial_{x_1}u_w(x) = \frac{1}{2}x_1^2 - \frac{1}{2}x_1x_2 = -\frac{1}{2}(x_2 - x_1)x_1$$

and

$$\partial_{x_2} u_w(x) = \frac{1}{4}x_2^2 - \frac{1}{4}x_1^2 = \frac{1}{4}(x_2 - x_1)(x_1 + x_2).$$

Thus we may compute for $x_2 > x_1 > 0$

$$\begin{aligned} a\partial_\nu u_w(x) - u_w(x) &= a\Big(\nu_1\big(-\frac{1}{2}(x_2 - x_1)x_1\big) + \nu_2\big(\frac{1}{4}(x_2 - x_1)(x_1 + x_2)\big)\Big) \\ &\qquad - \frac{1}{12}(x_2 - x_1)^2(2x_1 + x_2) \\ &= \frac{1}{2}(x_2 - x_1)\Big(a\big(-\nu_1 x_1 + \nu_2(\frac{1}{2}(x_1 + x_2))\big) - \frac{1}{6}(x_2 - x_1)(2x_1 + x_2)\Big). \end{aligned} \tag{5.12}$$

Thus to have $a\partial_\nu u_w(x) - u_w(x) \geq 0$ for $x \in \mathbb{R}^2$ satisfying $x_2 > x_1 > 0$ we should have

$$a\big(-\nu_1 x_1 + \nu_2(\frac{1}{2}(x_1 + x_2))\big) \geq \frac{1}{6}(x_2 - x_1)(2x_1 + x_2)$$

and rearranging this further, we get the equivalent inequality

$$\nu_2 - \nu_1 \geq \frac{1}{2x_1}(x_2 - x_1)\big(\frac{1}{3a}(2x_1 + x_2) - \nu_2\big).$$

Now for $x \in B_1$ we have the bounds $x_1 < 1$, $x_2 < 1$. And if $0 < x_1 < x_2$ then $x_2 - x_1 > 0$. So it is sufficient to have the inequality

$$\nu_2 - \nu_1 \geq \frac{1}{2x_1}(x_2 - x_1)(\frac{1}{a} - \nu_2). \tag{5.13}$$

For the first part we have $\nu = e_2$, so it is sufficient to have the following inequality

$$1 \geq \frac{1}{2x_1}(x_2 - x_1)(\frac{1}{a} - 1).$$

If $a \geq 1$ then $\frac{1}{a} - 1 \leq 0$ hence because also $x_2 - x_1 > 0$ and $x_1 > 0$ the inequality above holds for $x \in B_1 \cap \{x_2 > x_1 > 0\}$. By the symmetry equation $u_w(x_1, x_2) = u_w(-x_1, x_2)$, the desired inequality holds also for $x \in B_1 \cap \{x_2 > -x_1,\, x_1 < 0\}$ and this finishes the proof of the first part with any $C \geq 1$.

For the second part by $0 < x_1 < x_2 < (1+\epsilon)x_1$ we have $0 < \frac{1}{x_1}(x_2 - x_1) < \epsilon$. Thus if $\frac{1}{a} - \nu_2 > 0$ then we should have

$$\nu_2 - \nu_1 \geq \frac{\epsilon}{2}\Big(\frac{1}{a} - \nu_2\Big)$$

and if $\frac{1}{a} - \nu_2 \leq 0$ then we should have $\nu_2 - \nu_1 \geq 0$. Because $\nu_2 \geq -1$ for both cases it is sufficient to have

$$\nu_2 - \nu_1 \geq \frac{\epsilon}{2}\Big(\frac{1}{a} + 1\Big). \tag{5.14}$$

We compute

$$\nu_2 - \nu_1 = \sin(\frac{3\pi}{4} + \gamma) - \cos(\frac{3\pi}{4} + \gamma) = \sqrt{2}\cos(\gamma). \tag{5.15}$$

From (5.14) and (5.15) it follows that it is sufficient to have

$$\cos(\gamma) \geq \frac{\sqrt{2}}{4}\Big(\frac{1}{a} + 1\Big)\epsilon$$

and taking $C \geq \frac{\sqrt{2}}{4}$ the second part is also proved. □

6. Closeness of the Free Boundary to the Free Boundary of a Homogenous Global Solution

In the following lemma, roughly speaking, we prove two inclusions. First, if u is close to u_0 a nontrivial homogenous global solution then for x far from $\{u_0 = 0\}$ we have $u(x) > 0$. Second, if u is close to a solution u_0 and for x far from $\{u_0 > 0\}$ we have $x \in \{u = 0\}^\circ$.

Lemma 16. *There exists $C > 0$ such that if u_0 is a nontrivial homogenous global solution and u is a solution in B_1, then we have*

$$\Big\{x \in B_1 \ \Big| \ Cd^2(x, \{u_0 = 0\})\big(d(x, \{u_0 = 0\}) + |x_1|\big) > \|u - u_0\|_{L^\infty(B_1)}\Big\} \subset \{u > 0\} \tag{6.1}$$

here $\{u_0 = 0\} = \{x \in \mathbb{R}^2 \mid u_0(x) = 0\}$ and $\{u > 0\} = \{x \in B_1 \mid u(x) > 0\}$.

If u_0 and u are solutions in B_1 and

$$\|u - u_0\|_{L^\infty(B_1)} < C$$

then

$$\Big\{x \in B_{\frac{1}{2}} \ \Big| \ Cd^2(x, \{u_0 > 0\})\big(d(x, \{u_0 > 0\}) + |x_1|\big) > \|u - u_0\|_{L^\infty(B_1)}\Big\} \subset \{u = 0\}^\circ \tag{6.2}$$

here $\{u_0 = 0\} = \{x \in B_1 \mid u_0(x) = 0\}$ and $\{u = 0\} = \{x \in B_1 \mid u(x) = 0\}$.

Proof. Assume u_0 is a nontrivial homogenous global solution and u is a solution in B_1. Using Lemma 14 for $x \in B_1$ we compute

$$\begin{aligned} u(x) = u_0(x) + u(x) - u_0(x) &\geq u_0(x) - \|u - u_0\|_{L^\infty(B_1)} \\ &\geq C_1 d^2(x, \{u_0 = 0\})\big(d(x, \{u_0 = 0\}) + |x_1|\big) - \|u - u_0\|_{L^\infty(B_1)} \end{aligned}$$

here C_1 is the constant in Lemma 14. So if

$$\|u - u_0\|_{L^\infty(B_1)} < \frac{1}{2} C_1 d^2(x, \{u_0 = 0\})\big(d(x, \{u_0 = 0\}) + |x_1|\big)$$

then

$$u(x) > \frac{1}{2} C_1 d^2(x, \{u_0 = 0\})\big(d(x, \{u_0 = 0\}) + |x_1|\big)$$

and this proves (6.1) with $0 < C \leq \frac{1}{2} C_1$.

Assume u_0 and u are solutions in B_1. By Theorem 9 there exists $C_2 > 0$ such that if $y \in B_1$, $u(y) > 0$ and $B_r(y) \subset\subset B_1$ then we have

$$\sup_{\{u>0\} \cap \partial B_r(y)} u \geq u(y) + C_2 r^2 (r + |y_1|).$$

Thus if $y \in B_1$, $u(y) > 0$, $B_r(y) \subset\subset \{u_0 = 0\} \cap B_1$ and $C_2 r^2(r + |y_1|) > \|u - u_0\|_{L^\infty(B_1)}$ then we have

$$\begin{aligned} 0 = \sup_{\{u>0\} \cap \partial B_r(y)} u_0 = \sup_{\{u>0\} \cap \partial B_r(y)} (u - (u - u_0)) \\ \geq \sup_{\{u>0\} \cap \partial B_r(y)} u - \|u - u_0\|_{L^\infty(B_1)} \\ \geq u(y) + C_2 r^2 (r + |y_1|) - \|u - u_0\|_{L^\infty(B_1)} \\ \geq C_2 r^2 (r + |y_1|) - \|u - u_0\|_{L^\infty(B_1)} \end{aligned}$$

a contradiction. Thus if $y \in B_1$, $B_r(y) \subset\subset \{u_0 = 0\} \cap B_1$ and $C_2 r^2(r + |y_1|) > \|u - u_0\|_{L^\infty(B_1)}$ then $u(y) = 0$.

For $y \in (\{u_0 = 0\} \cap B_1)^\circ$ setting $r = \frac{1}{2} d\big(y, (\{u_0 = 0\} \cap B_1)^c\big)$ it follows that if

$$\frac{C_2}{4} d^2(y, (\{u_0 = 0\} \cap B_1)^c)\big(\frac{1}{2} d(y, (\{u_0 = 0\} \cap B_1)^c) + |y_1|\big) > \|u - u_0\|_{L^\infty(B_1)}$$

then $u(y) = 0$. This proves that

$$\Big\{x \in B_1 \;\Big|\; \frac{C_2}{8} d^2(x, (\{u_0 = 0\} \cap B_1)^c)\big(d(x, (\{u_0 = 0\} \cap B_1)^c) + |x_1|\big) > \|u - u_0\|_{L^\infty(B_1)}\Big\} \subset \big\{u = 0\big\}.$$

By the continuity of $d(x, (\{u_0 = 0\} \cap B_1)^c)$ as a function of x it follows that

$$\Big\{x \in B_1 \;\Big|\; \frac{C_2}{8} d^2(x, (\{u_0 = 0\} \cap B_1)^c)\big(d(x, (\{u_0 = 0\} \cap B_1)^c) + |x_1|\big) > \|u - u_0\|_{L^\infty(B_1)}\Big\} \subset \big\{u = 0\big\}^\circ. \tag{6.3}$$

Let $x \in B_{\frac{1}{2}}$ then we compute

$$\begin{aligned} d(x, (\{u_0 = 0\} \cap B_1)^c) &= d(x, \{u_0 > 0\} \cup B_1^c) \\ &= \min\big(d(x, \{u_0 > 0\}), d(x, B_1^c)\big) \geq \min\big(d(x, \{u_0 > 0\}), \frac{1}{2}\big) \end{aligned}$$

so we have

$$\begin{aligned} & d^2(x, (\{u_0 = 0\} \cap B_1)^c)\big(d(x, (\{u_0 = 0\} \cap B_1)^c) + |x_1|\big) \\ &= \min\Big(d^2(x, \{u_0 > 0\})\big(d(x, \{u_0 > 0\}) + |x_1|\big), (\frac{1}{2})^2(\frac{1}{2} + |x_1|)\Big) \\ &\geq \min\Big(d^2(x, \{u_0 > 0\})\big(d(x, \{u_0 > 0\}) + |x_1|\big), \frac{1}{8}\Big). \end{aligned} \tag{6.4}$$

So by (6.3) and (6.4), if

$$\|u - u_0\|_{L^\infty(B_1)} < \frac{C_2}{64}$$

then

$$\Big\{x \in B_{\frac{1}{2}} \;\Big|\; \frac{C_2}{8} d^2(x, \{u_0 > 0\})\big(d(x, \{u_0 > 0\}) + |x_1|\big) > \|u - u_0\|_{L^\infty(B_1)}\Big\} \subset \big\{u = 0\big\}^\circ \tag{6.5}$$

and by choosing $0 < C \leq \frac{C_2}{64}$ this finishes the proof the lemma. □

By the inclusions proved in the previous lemma, in the following lemma we show that for u a solution and u_0 a nontrivial homogenous global solution, if u is close enough to u_0 then the free boundary of u is in a quantitatively specified neighbourhood of the free boundary of u_0.

Lemma 17. *There exists $C > 0$ such that if u is a solution in B_1 and u_0 is a nontrivial homogenous global solution then if*

$$\|u - u_0\|_{L^\infty(B_1)} < C \tag{6.6}$$

we have

$$\Gamma \cap B_{\frac{1}{2}} \subset \Big\{C d^2(x, \Gamma_{u_0})\big(d(x, \Gamma_{u_0}) + |x_1|\big) \leq \|u - u_0\|_{L^\infty(B_1)}\Big\}.$$

Proof. If $u = u_0$ in B_1 then the claim is obvious, so we assume that $u_0 \neq u$ in B_1.

Assume there exists $x \in \Gamma \cap B_{\frac{1}{2}}$ such that

$$Cd^2(x, \Gamma_{u_0})\big(d(x, \Gamma_{u_0}) + |x_1|\big) > \|u - u_0\|_{L^\infty(B_1)}$$

here $C > 0$ is as in Lemma 16.

Then because

$$d(x, \Gamma_{u_0}) = \max\big(d(x, \{u_0 = 0\}), d(x, \{u_0 > 0\})\big)$$

we should have either

$$Cd^2(x, \{u_0 = 0\})\big(d(x, \{u_0 = 0\}) + |x_1|\big) > \|u - u_0\|_{L^\infty(B_1)} \tag{6.7}$$

or

$$Cd^2(x, \{u_0 > 0\})\big(d(x, \{u_0 > 0\}) + |x_1|\big) > \|u - u_0\|_{L^\infty(B_1)}. \tag{6.8}$$

In the case when (6.7) holds then by (6.1) we obtain that $u(x) > 0$ which is in contradiction with $x \in \Gamma$.

In the case when (6.8) holds then because also (6.6) holds by (6.1) we obtain that $x \in \{u = 0\}^\circ$ which is in contradiction with $x \in \Gamma$ and this finishes the proof of the lemma. □

Lemma 18. *There exists $C > 0$ and $C_1 > 0$ such that if u_0 is a nontrivial homogenous global solution, u is a solution in D, $0 \in D$ and $0 \in \Gamma$ then for $x \in \Gamma$ such that $B_{4|x|} \subset D$ and*

$$\|u_{4|x|} - u_0\|_{L^\infty(B_1)} < C$$

we have

$$C_1 d^2(x, \Gamma_{u_0})\big(d(x, \Gamma_{u_0}) + |x_1|\big) \leq |x|^3 \|u_{4|x|} - u_0\|_{L^\infty(B_1)}.$$

Proof. Let C be as in Lemma 16.

Let $r > 0$ and assume

$$\|u_r - u_0\|_{L^\infty(B_1)} < C$$

then by Lemma 17 we have

$$\Gamma_{u_r} \cap B_{\frac{1}{2}} \subset \Big\{Cd^2(y, \Gamma_{u_0})\big(d(y, \Gamma_{u_0}) + |y_1|\big) \leq \|u_r - u_0\|_{L^\infty(B_1)}\Big\}.$$

Then because Γ_{u_0} is a cone and $\Gamma_u \cap B_{\frac{r}{2}} = r(\Gamma_{u_r} \cap B_{\frac{1}{2}})$ we obtain

$$\begin{aligned}
\Gamma_u \cap B_{\frac{r}{2}} &\subset \Big\{ry \in B_{\frac{r}{2}} \;\Big|\; Cd^2(y, \Gamma_{u_0})\big(d(y, \Gamma_{u_0}) + |y_1|\big) \leq \|u_r - u_0\|_{L^\infty(B_1)}\Big\} \\
&= \Big\{x \in B_{\frac{r}{2}} \;\Big|\; Cd^2(\frac{x}{r}, \Gamma_{u_0})\big(d(\frac{x}{r}, \Gamma_{u_0}) + |\frac{x_1}{r}|\big) \leq \|u_r - u_0\|_{L^\infty(B_1)}\Big\} \\
&= \Big\{x \in B_{\frac{r}{2}} \;\Big|\; Cd^2(x, \Gamma_{u_0})\big(d(x, \Gamma_{u_0}) + |x_1|\big) \leq r^3\|u_r - u_0\|_{L^\infty(B_1)}\Big\}.
\end{aligned}$$

For those $x \in \Gamma_u$ such that $B_{4|x|} \subset D$ we may consider $r = 4|x|$.

So if

$$\|u_{4|x|} - u_0\|_{L^\infty(B_1)} < C$$

then because $x \in \Gamma_u \cap B_{2|x|}$ we have

$$Cd^2(x, \Gamma_{u_0})\big(d(x, \Gamma_{u_0}) + |x_1|\big) \leq 4^3|x|^3\|u_{4|x|} - u_0\|_{L^\infty(B_1)}.$$ □

7. Uniqueness of Degenerate Blowup Limits

Lemma 19. *If u is a solution in D, $0 \in D$, $0 \in \Gamma$ and $W(+0,u) = W(1,u_{hs})$ then there exists $\epsilon > 0$ such that*

$$\Gamma \cap B_\epsilon \cap \{x_1 > |x_2|\} = \emptyset.$$

Proof. Let us denote $\tilde{u}_{hs}(x) = u_{hs}(-x_1, x_2)$.

Assume $x \in \Gamma$, $B_{4|x|} \subset D$ and

$$\min\big(\|u_{4|x|} - u_{hs}\|_{L^\infty(B_1)}, \|u_{4|x|} - \tilde{u}_{hs}\|_{L^\infty(B_1)}\big) < C$$

where C is as in Lemma 18. Then by Lemma 18 it follows that

$$C_1 \min\Big(d^2(x,\Gamma_{u_{hs}})\big(d(x,\Gamma_{u_{hs}}) + |x_1|\big), d^2(x,\Gamma_{\tilde{u}_{hs}})\big(d(x,\Gamma_{\tilde{u}_{hs}}) + |x_1|\big)\Big) \\ \leq |x|^3 \min\big(\|u_{4|x|} - u_{hs}\|_{L^\infty(B_1)}, \|u_{4|x|} - \tilde{u}_{hs}\|_{L^\infty(B_1)}\big).$$

We have $\Gamma_{u_{hs}} = \Gamma_{\tilde{u}_{hs}} = \{x_1 = 0\}$ thus we have $d(x,\Gamma_{u_{hs}}) = d(x,\Gamma_{\tilde{u}_{hs}}) = |x_1|$ and

$$2C_1|x_1|^3 \leq |x|^3 \min\big(\|u_{4|x|} - u_{hs}\|_{L^\infty(B_1)}, \|u_{4|x|} - \tilde{u}_{hs}\|_{L^\infty(B_1)}\big). \tag{7.1}$$

From Theorem 1 it follows that

$$\min\big(\|u_r - u_{hs}\|_{L^\infty(B_1)}, \|u_r - \tilde{u}_{hs}\|_{L^\infty(B_1)}\big) \to 0 \text{ as } r \to 0. \tag{7.2}$$

From (7.1) and (7.2) it follows that for small enough $x \in \Gamma$ we have

$$|x_1| \leq \frac{\sqrt{2}}{2}|x|. \tag{7.3}$$

Because $\{x_1 > |x_2|\} = \{x_1 > \frac{\sqrt{2}}{2}|x|\}$, (7.3) proves the lemma. □

Proof of Theorem 3. By Lemma 19 there exists $\epsilon > 0$ such that

$$\Gamma \cap B_\epsilon \cap \{x_1 > |x_2|\} = \emptyset. \tag{7.4}$$

For short notation let us denote $A = B_\epsilon \cap \{x_1 > |x_2|\}$.

We claim that

$$\text{either } A \subset \{u = 0\} \text{ or } A \subset \{u > 0\}. \tag{7.5}$$

To prove this claim let us assume that $A \not\subset \{u = 0\}$ and $A \not\subset \{u > 0\}$. Then there exists $y, z \in A$ such that $u(y) > 0$ and $u(z) = 0$. By the C^1 regularity of u it follows that there exists a point x on the line segment connecting y and z such that $x \in \Gamma$. From the convexity of A it follows that $x \in A$ which contradicts with $x \in \Gamma$ and (7.4). This contradiction proves (7.5).

Let us consider the case when $A \subset \{u = 0\}$. Then for $0 < r < 2\epsilon$ we have

$$u_r(\frac{1}{2}e_1) = \frac{1}{r^3}u(\frac{r}{2}e_1) = 0$$

and this proves that in the case $A \subset \{u = 0\}$ we have $u_r \to u_{hs}(-x_1, x_2)$ as $r \to 0$.

Now let us consider the case when $A \subset \{u > 0\}$.

Let $0 < r < 2\epsilon$ then because $\frac{1}{2}re_1 \in \{u > 0\}$ by Theorem 9 we have

$$\sup_{\partial B_{\frac{r}{4}}(\frac{1}{2}re_1)} u \geq C(\frac{r}{4})^3$$

which might be written as

$$\sup_{\partial B_{\frac{1}{4}}(\frac{1}{2}e_1)} u_r \geq C(\frac{1}{4})^3$$

and this proves that in the case $A \subset \{u > 0\}$ we have $u_r \to u_{hs}$ as $r \to 0$. □

8. Uniqueness of the Regular Blowup Limits

Although it is possible to prove the uniqueness of the blowup limit at regular points with the same technique as we proved the uniqueness of the blowup limit at degenerate points, in this section we first prove a directional monotonicity result which establishes that near to regular points the solution is either monotonically nondecreasing or nonincreasing in the direction e_2 and then from this directional monotonicity result the uniqueness of the blowup limit follows. We will use the obtained directional monotonicity result again in Section 11.

Let us define

$$p_4(x) = x_1^4 - 6x_1^2x_2^2 + x_2^4 = |x|^4 - 8x_1^2|x|^2 + 8x_1^4.$$

The following lemma is proved in [13].

Lemma 20. *There exist $b > 0$ and $C > 0$ such that for $r > 0$, $y \in \mathbb{R}^2$ and $x \in B_r(y)$*

$$w_{y_1}(x_1) + \frac{b}{r}p_4(x-y) \geq C\frac{|x-y|^4}{r}. \tag{8.1}$$

Lemma 21. *There exists a $C > 0$ such that if u is a solution in D, $y \in \Omega$, $B_r(y) \subset\subset D$ and $u(y) - a\partial_{x_2}u(y) > 0$ then we have*

$$\sup_{\Omega\cap\partial B_r(y)} (u - a\partial_{x_2}u) \geq u(y) - a\partial_{x_2}u(y) + Cr^3. \tag{8.2}$$

Proof. Let y and r be as in the statement of the theorem. Let $b > 0$ be as in Lemma 20.

Let us define for $x \in D$

$$h(x) = u(x) - a\partial_{x_2}u(x) - (u(y) - a\partial_{x_2}u(y)) - \big(w_{y_1}(x_1) + \frac{b}{r}p_4(x-y)\big).$$

We compute

$$\begin{aligned}\triangle h(x) = \triangle u(x) - a\partial_{x_2}\triangle u(x) - \big(\triangle w_{y_1}(x_1) + \frac{b}{r}\triangle p_4(x-y)\big)\\ = |x_1| - |x_1| = 0 \text{ in } \Omega.\end{aligned} \tag{8.3}$$

Because $w_{y_1}(y_1) = 0$ we have

$$h(y) = -\big(w_{y_1}(y_1) + \frac{b}{r}p_4(0)\big) = 0. \tag{8.4}$$

For $x \in \Gamma$ we have $u(x) - a\partial_{x_2}u(x) = 0$, thus if $u(y) - a\partial_{x_2}u(y) > 0$ then by Lemma 20 we have

$$h(x) = -(u(y) - a\partial_{x_2}u(y)) - \big(w_{y_1}(x_1) + \frac{b}{r}p_4(x-y)\big) < 0 \text{ on } \Gamma. \tag{8.5}$$

By (8.3) we have that h is harmonic in the domain $\Omega \cap B_r(y)$. Applying the maximum principle for the domain $\Omega\cap B_r(y)$ and the harmonic function h we have

$$h(y) \leq \sup_{\partial(\Omega\cap B_r(y))} h. \tag{8.6}$$

By (8.4) and (8.6) we obtain

$$0 \leq \sup_{\partial(\Omega\cap B_r(y))} h. \tag{8.7}$$

Because

$$\partial(\Omega \cap B_r(y)) = (\partial\Omega \cap B_r(y)) \cup (\Omega \cap \partial B_r(y))$$

by (8.5) and (8.7) we obtain

$$0 \leq \sup_{\Omega\cap\partial B_r(y)} h. \tag{8.8}$$

By the definition of h, from (8.8) we get the inequality

$$u(y) - a\partial_{x_2}u(y) + \inf_{\Omega\cap\partial B_r(y)}\big(w_{y_1}(x_1) + \frac{b}{r}p_4(x-y)\big) \leq \sup_{\Omega\cap\partial B_r(y)}(u - a\partial_{x_2}u). \tag{8.9}$$

Now by Lemma 20 we obtain for $x \in \partial B_r(y)$

$$w_{y_1}(x_1) + \frac{b}{r}p_4(x-y) \geq C\frac{|x-y|^4}{r} = Cr^3. \tag{8.10}$$

By (8.9) and (8.10) the lemma is proved. □

In the following lemma we proof a directional monotonicity result which eventually will show that for solutions with u_w as a blowup limit, the solution is monotonically nondecreasing in the direction e_2 in a neighbourhood of 0.

Lemma 22. *There exist $\delta > 0$ and $a > 0$ such that if u is a solution in B_1 and $\|u - u_w\|_{C^1(B_1)} \leq \delta$ then $a\partial_{x_2}u - u \geq 0$ in $B_{\frac{1}{2}}$.*

Proof. By Lemma 21 there exist $C > 0$ and $a > 0$ such that if there exists $y \in \Omega \cap B_{\frac{1}{2}}$, $u(y) - a\partial_{x_2}u(y) > 0$ then

$$\sup_{\Omega\cap\partial B_{\frac{1}{2}}(y)}(u - a\partial_{x_2}u) \geq u(y) - a\partial_{x_2}u(y) + \frac{1}{8}C. \tag{8.11}$$

By Lemma 15 there exists $C_1 > 0$ such that if $a \geq C_1$ then $a\partial_{x_2}u_w - u_w \geq 0$. Thus for $a \geq C_1$ we estimate

$$\begin{aligned} &\sup_{\Omega\cap\partial B_{\frac{1}{2}}(y)}(u - a\partial_{x_2}u) \\ &\leq \sup_{\Omega\cap\partial B_{\frac{1}{2}}(y)}(u_w - a\partial_{x_2}u_w) + \sup_{\Omega\cap\partial B_{\frac{1}{2}}(y)}\big((u-u_w) - a\partial_{x_2}(u-u_w)\big) \\ &\leq \sup_{\Omega\cap\partial B_{\frac{1}{2}}(y)}(u-u_w) + \sup_{\Omega\cap\partial B_{\frac{1}{2}}(y)}\big(-a\partial_{x_2}(u-u_w)\big) \\ &\leq \|u-u_w\|_{C(B_1)} + a\|\partial_{x_2}u - \partial_{x_2}u_w\|_{C(B_1)} \\ &\leq C_2\max(1,a)\|u-u_w\|_{C^1(B_1)}. \end{aligned} \tag{8.12}$$

From (8.11) and (8.12) it follows that

$$\frac{1}{8}C < C_2\max(1,a)\|u-u_w\|_{C^1(B_1)}$$

thus if $\|u-u_w\|_{C^1(B_1)} \leq \frac{C}{8C_2\max(1,a)} = \delta$ we are in contradiction and hence $a\partial_{x_2}u - u \geq 0$ in $B_{\frac{1}{2}}$. □

A regular free boundary point is defined in Definition 4.

Lemma 23. *If u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular free boundary point with blowup limit u_w, then there exists $C > 0$ and $\epsilon > 0$ such that*

$$C\partial_{x_2}u - u \geq 0 \text{ in } B_\epsilon. \tag{8.13}$$

Proof. By assumption there exists $r_j \to 0$ such that $u_{r_j} \to u_w$ in $C^1(B_1)$.

Hence by Lemma 22 there exists $a > 0$ such that for large enough j we have $a\partial_{x_2}u_{r_j} - u_{r_j} \geq 0$ in $B_{\frac{1}{2}}$. It is easy to see that this is equivalent to $ar_j\partial_{x_2}u - u \geq 0$ in $B_{\frac{r_j}{2}}$ and this proves the lemma. □

Proof of Theorem 4. Let us assume that a blowup limit is u_w. By the previous lemma there exists a $C > 0$ and $\epsilon > 0$ such that (8.13) holds.

Assume that v is another blowup limit, i.e. there exists $r_j \to 0$ and $u_{r_j} \to v$ in $C^1(B_1)$.

It follows from (8.13) that

$$C\partial_{x_2} u_{r_j}(x) - r_j u_{r_j}(x) \geq 0 \text{ for } x \in B_{\frac{\epsilon}{r_j}}$$

passing to the limit as $r_j \to 0$ we obtain that

$$\partial_{x_2} v \geq 0 \text{ in } B_1. \tag{8.14}$$

We have that $W(1, u_{r_j}) \to W(+0, u) = W(1, u_w)$. Now by Corollary 3 it follows that either $v = u_w$ or $v = u_w(x_1, -x_2)$.

But it is easy to check that $v = u_w(x_1, -x_2)$ does not satisfy the inequality (8.14). Thus $v = u_w$ and this completes the proof of the theorem. □

9. Convergence of the Free Boundary to the Free Boundary of the Blowup Limit

By the fact that there exists only a single possible blowup limit in the cases when $W(+0, u)$ is equal to $2W(1, u_w)$ or $2W(1, u_{hs})$, Theorem 4 in the case $W(+0, u) = W(1, u_w)$ and Theorem 3 in the case $W(+0, u) = W(1, u_{hs})$ we have that always u_r converges to a unique blowup limit. Thus $\sigma_0(r) \to 0$ as $r \to 0$.

Proof of Theorem 5. Let us consider the case $W(+0, u) = W(1, u_w)$ with the blowup limit u_w. Then for $x \in \{x_1 > 0, x_2 > -x_1\}$ we have $d(x, \Gamma_{u_w}) = \frac{\sqrt{2}}{2}|x_2 - x_1|$ and for $x \in \{x_1 > 0, x_2 \leq -x_1\}$, $d(x, \Gamma_{u_w}) = |x| \geq \frac{\sqrt{2}}{2}|x_2 - x_1|$. Thus we compute for $x_1 > 0$

$$d(x, \Gamma_{u_w}) + |x_1| \geq \frac{\sqrt{2}}{2}|x_2 - x_1| + |x_1| \geq C_1|x|.$$

By symmetry we obtain the same inequality for $x_1 < 0$.

Now by Lemma 18 we obtain the inequality (2.7). For the rest of cases when $W(+0, u) \in \{W(1, u_w), 2W(1, u_w)\}$ we can compute similarly.

In the cases when $W(+0, u) \in \{W(1, u_{hs}), 2W(1, u_{hs})\}$ we have $\Gamma_{u_0} = \{x_1 = 0\}$, $d(x, \Gamma_{u_0}) = |x_1|$ and (2.8) follows immediately from Lemma 18. □

Corollary 5. *Let u be a solution in D then the points of $\Gamma \cap \{x_1 = 0\} \cap \{W(+0, x, u) \in \{W(1, u_w), 2W(1, u_w)\}\}$ are isolated points of $\Gamma \cap \{x_1 = 0\}$ (in the topology of $\{x_1 = 0\}$).*

Proof. Assume $W(+, u) \in \{W(1, u_w), 2W(1, u_w)\}$ then by (2.7) the free boundary should converge to the free boundary of the blowup limit tangentially. But this is not the case if 0 is not an isolated point of $\Gamma \cap \{x_1 = 0\}$. □

10. Convergence of the Normal of the Free Boundary to the Normal of the Free Boundary of the Blowup Limit at Regular Points

In the following lemma we prove a nondegeneracy type result for $u - a\partial_\nu u$ far from the degeneracy line $\{x_1 = 0\}$.

Lemma 24. *If u is a solution in D, $y \in \Omega$, $B_r(y) \subset\subset D \cap \{x_1 \geq \frac{1}{16}\}$ and $u(y) - \frac{1}{32}\partial_\nu u(y) > 0$ then we have*

$$\frac{r^2}{128} \leq \sup_{\Omega \cap \partial B_r(y)} (u - a\partial_\nu u).$$

Proof. Let y and r be as in the statement of the theorem.

We define for $a > 0$ and $c > 0$

$$h(x) = u(x) - a\partial_\nu u(x) - (u(y) - a\partial_\nu u(y)) - c|x-y|^2.$$

We compute

$$\triangle h(x) = |x_1| - a\nu_1 \frac{x_1}{|x_1|} - 4c \geq \frac{1}{16} - a - 4c \text{ in } \Omega \cap \{x_1 \geq \frac{1}{16}\}$$

so if we choose $a = \frac{1}{32}$ and $c = \frac{1}{128}$ then we have

$$\triangle h \geq 0 \text{ in } \Omega \cap \{x_1 \geq \frac{1}{16}\}. \tag{10.1}$$

Also we have

$$h(y) = 0. \tag{10.2}$$

For $x \in \Gamma$ we have $u(x) - \frac{1}{32}\partial_\nu u(x) = 0$, thus if $u(y) - \frac{1}{32}\partial_\nu u(y) > 0$ then we have

$$h(x) = -(u(y) - \frac{1}{32}\partial_\nu u(y)) - \frac{|x-y|^2}{128} < 0 \text{ on } \Gamma. \tag{10.3}$$

Because $B_r(y) \subset \{x_1 \geq \frac{1}{16}\}$ by (10.1) we have that h is subharmonic in the domain $\Omega \cap B_r(y)$. Applying the maximum principle for the domain $\Omega \cap B_r(y)$ and the subharmonic function h we have

$$h(y) \leq \sup_{\partial(\Omega \cap B_r(y))} h. \tag{10.4}$$

By (10.2) and (10.4) we obtain

$$0 \leq \sup_{\partial(\Omega \cap B_r(y))} h. \tag{10.5}$$

Because

$$\partial(\Omega \cap B_r(y)) = (\partial\Omega \cap B_r(y)) \cup (\Omega \cap \partial B_r(y))$$

by (10.3) and (10.5) we obtain

$$0 \leq \sup_{\Omega \cap \partial B_r(y)} h. \tag{10.6}$$

By the definition of h, from (10.6) we get the inequality

$$u(y) - \frac{1}{32}\partial_\nu u(y) + \frac{r^2}{128} \leq \sup_{\Omega \cap \partial B_r(y)} (u - \frac{1}{32}\partial_\nu u) \tag{10.7}$$

and this proves the lemma. □

Let ν_w be the normal to $\Gamma_{u_w} \cap \{x_1 > 0\}$ pointing into $\{u_w > 0\}$, i.e.

$$\nu_w = \frac{(-1,1)}{\sqrt{2}}.$$

In the following lemma we prove a crucial directional monotonicity result which will be used in the proof of the convergence of normals.

Lemma 25. *There exists $C > 0$ such that if u is a solution in B_1, $x_u \in \Gamma_u \cap \partial B_{\frac{1}{4}} \cap \{x_1 > 0\}$, $\nu \in \partial B_1$, $r > 0$ such that*

$$\|u - u_w\|_{C^1(B_1)}^{\frac{1}{2}} + r \leq C\nu \cdot \nu_w$$

then

$$\frac{1}{32}\partial_\nu u - u \geq 0 \text{ in } \Omega \cap B_r(x_u).$$

Proof. We have

$$\{x_{u_w}\} = \Gamma_{u_w} \cap \partial B_{\frac{1}{4}} \cap \{x_1 > 0\} \text{ where } x_{u_w} = \frac{\sqrt{2}}{8}(1,1).$$

Step 1. In this step we show that there exists $C_1 > 0$ such that

$$|x_u - x_{u_w}| \le C_1 \|u - u_w\|_{L^\infty(B_1)}^{\frac{1}{2}}. \tag{10.8}$$

By Lemma 17 there exists $C > 0$ such that if $\|u - u_w\|_{L^\infty(B_1)} < C$ then

$$\Gamma_u \cap B_{\frac{1}{2}} \subset \Big\{C(d(x,\Gamma_{u_w}))^2\big(d(x,\Gamma_{u_w}) + |x_1|\big) \le \|u - u_w\|_{L^\infty(B_1)}\Big\}. \tag{10.9}$$

We have $x_u \in \Gamma_u \cap \partial B_{\frac{1}{4}} \cap \{x_1 > 0\}$ thus by (10.9)

$$C(d(x_u,\Gamma_{u_w}))^2\big(d(x_u,\Gamma_{u_w}) + |x_{u,1}|\big) \le \|u - u_w\|_{L^\infty(B_1)}. \tag{10.10}$$

From (10.10) it follows that there exists $C_1 > 0$ such that

$$|x_u - x_{u_w}| \le C_1 \|u - u_w\|_{L^\infty(B_1)}^{\frac{1}{2}}. \tag{10.11}$$

Step 2. In this step we show that there exists $\delta > 0$ such that if

$$\|u - u_w\|_{L^\infty(B_1)} < \delta \text{ and } 0 < r < \frac{1}{48}$$

then for $x \in \Omega \cap B_{\frac{1}{48}}(x_u)$ if $u(x) - \frac{1}{32}\partial_\nu u(x) > 0$ we have

$$\frac{r^2}{128} \le \sup_{\Omega\cap\partial B_r(x)} (u - \frac{1}{32}\partial_\nu u).$$

By step 1 if

$$C_1 \|u - u_w\|_{L^\infty(B_1)}^{\frac{1}{2}} < \frac{1}{48}$$

then $|x_u - x_{u_w}| < \frac{1}{48}$. Thus $x_{u,1} > x_{u_w,1} - \frac{1}{48}$ and

$$B_{\frac{1}{48}}(x_u) \subset \{x_1 > x_{u_w,1} - \frac{1}{48} - \frac{1}{48}\} = \{x_1 > x_{u_w,1} - \frac{1}{24}\}$$

and for $x \in B_{\frac{1}{48}}(x_u)$ we have

$$\begin{aligned} B_{\frac{1}{48}}(x) \subset \{x_1 > x_{u_w,1} - \frac{1}{24} - \frac{1}{48}\} &= \{x_1 > x_{u_w,1} - \frac{1}{16}\} \\ &= \{x_1 > \frac{\sqrt{2}}{8} - \frac{1}{16}\} \subset \{x_1 > \frac{1}{8} - \frac{1}{16}\} = \{x_1 > \frac{1}{16}\}. \end{aligned}$$

Now by Lemma 24 if

$$0 < r < \frac{1}{48},$$

$x \in \Omega \cap B_{\frac{1}{48}}(x_u)$ and $u(x) - \frac{1}{32}\partial_\nu u(x) > 0$ then we have

$$\frac{r^2}{128} \le \sup_{\Omega\cap\partial B_r(x)} (u - \frac{1}{32}\partial_\nu u).$$

Step 3. In this step we show that there exists $C_2 > 0$ such that $\frac{1}{32}\partial_\nu u_w - u_w \ge 0$ in $B_\eta(x_{u_w})$ if $0 < \eta < \frac{1}{16}$, $\nu \in \partial B_1$ and $C_2\eta \le \nu \cdot \nu_w$.

Assume $x \in B_\eta(x_{u_w})$ with $0 < \eta < \frac{1}{16}$. Then

$$x_1 > x_{u_w,1} - \eta > x_{u_w,1} - \frac{1}{16} = \frac{\sqrt{2}}{8} - \frac{1}{16} > \frac{1}{8} - \frac{1}{16} = \frac{1}{16}$$

and

$$\frac{x_2}{x_1} = 1 + \frac{x_2 - x_1}{x_1} \leq 1 + \frac{|x_2 - x_1|}{x_1}$$
$$= 1 + 16|x_2 - x_1| = 1 + 16\sqrt{2} d(x, \{x_2 = x_1\})$$
$$\leq 1 + 16\sqrt{2}|x - x_{u_w}| \leq 1 + 16\sqrt{2}\eta$$

hence by Lemma 15 we have $\frac{1}{32}\partial_\nu u_w(x) - u_w(x) \geq 0$ if $\nu \in \partial B_1$ and

$$C(\frac{1}{\frac{1}{32}} + 1)(16\sqrt{2}\eta) \leq \nu \cdot \nu_w$$

with $C > 0$ as in Lemma 15.

Step 4. In this step we show that there exists $\delta_1 > 0$ and $C_3 > 0$ such that if

$$\|u - u_w\|_{L^\infty(B_1)} < \delta_1,\ 0 < r < \frac{1}{48},\ 0 < r_1 < \frac{1}{48}, \tag{10.12}$$

$$\nu \in \partial B_1,\ C_2(r + r_1 + C_1\|u - u_w\|^{\frac{1}{2}}_{L^\infty(B_1)}) \leq \nu \cdot \nu_w \tag{10.13}$$

and

$$C_3\|u - u_w\|^{\frac{1}{2}}_{C^1(B_1)} < r \tag{10.14}$$

then

$$u - \frac{1}{32}\partial_\nu u \leq 0 \text{ in } \Omega \cap B_{r_1}(x_u). \tag{10.15}$$

By step 1 there exists $0 < \delta_1 < \delta$ such that if

$$\|u - u_w\|_{L^\infty(B_1)} < \delta_1 \tag{10.16}$$

then

$$|x_u - x_{u_w}| < \frac{1}{48}. \tag{10.17}$$

Let

$$0 < r < \frac{1}{48} \text{ and } 0 < r_1 < \frac{1}{48}. \tag{10.18}$$

Assume now that both (10.16) and (10.18) hold.

We define

$$\eta = r + r_1 + |x_u - x_{u_w}|$$

then by (10.17) and (10.18) we have

$$0 < \eta < \frac{1}{16}. \tag{10.19}$$

By step 2 for $x \in \Omega \cap B_{r_1}(x_u)$ if $u(x) - \frac{1}{32}\partial_\nu u(x) > 0$ then

$$\frac{r^2}{128} \leq \sup_{\Omega \cap \partial B_r(x)} (u - \frac{1}{32}\partial_\nu u).$$

By (10.19) and step 3 we have $\frac{1}{32}\partial_\nu u_w - u_w \geq 0$ in $B_r(x) \subset B_\eta(x_{u_w})$ if

$$\nu \in \partial B_1 \text{ and } C_2\eta \leq \nu \cdot \nu_w. \tag{10.20}$$

Assume now that (10.20) holds.

We compute

$$\sup_{\Omega\cap\partial B_r(x)}(u-\frac{1}{32}\partial_\nu u)$$
$$\leq \sup_{\Omega\cap\partial B_r(x)}(u_w-\frac{1}{32}\partial_\nu u_w)+\sup_{\Omega\cap\partial B_r(x)}\big(u-\frac{1}{32}\partial_\nu u-(u_w-\frac{1}{32}\partial_\nu u_w)\big)$$
$$\leq C_4\|u-u_w\|_{C^1(B_1)}.$$

Therefore if

$$\frac{r^2}{128}>C_4\|u-u_w\|_{C^1(B_1)}$$

then

$$u-\frac{1}{32}\partial_\nu u\leq 0 \text{ in } \Omega\cap B_{r_1}(x_u).$$

Step 5. In this step we finish the proof of the lemma.
Choosing

$$r=2C_3\|u-u_w\|^{\frac{1}{2}}_{C^1(B_1)}$$

then (10.14) holds. Noticing that $\nu\cdot\nu_w\leq 1$ we obtain that by choosing $C>0$ small enough, if

$$\nu\in\partial B_1,\ \|u-u_w\|^{\frac{1}{2}}_{C^1(B_1)}+r_1\leq C\nu\cdot\nu_w$$

holds then (10.12) and (10.13) hold and thus by step 4, (10.15) holds and this proves the Lemma. □

For $0\leq\delta<1$ let us define the open cone

$$C_\delta=\Big\{x\in\mathbb{R}^2\ \Big|\ x\cdot\nu_w>\delta|x|\Big\}.$$

Corollary 6. *There exists $C>0$ such that if u is a solution in B_1, $x\in\Gamma\cap\partial B_{\frac{1}{4}}\cap\{x_1>0\}$, $0<\delta<1$, $r>0$ such that*

$$\|u-u_w\|^{\frac{1}{2}}_{C^1(B_1)}+r\leq C\delta$$

with $C>0$ as in Lemma 25, then

$$B_r(x)\cap\big(x+C_\delta\big)\subset\{u>0\} \text{ and } B_r(x)\cap\big(x-C_\delta\big)\subset\{u=0\}. \tag{10.21}$$

Proof. By Lemma 25 and the definition of C_δ we have that for all $\nu\in C_\delta$

$$\partial_\nu u\geq 0 \text{ in } B_r(x_u). \tag{10.22}$$

From (10.27) because $u\geq 0$

$$\text{if } z\in B_r(x) \text{ and } u(z)=0 \text{ then } B_r(x)\cap(z-C_\delta)\subset\{u=0\}. \tag{10.23}$$

In particular because $u(x)=0$ we have

$$B_r(x)\cap(x-C_\delta)\subset\{u=0\}.$$

Now assume there exists $y\in B_r(x)\cap(x+C_\delta)$ such that $u(y)=0$. By (10.23) we have that $u=0$ in $B_r(x)\cap(y-C_\delta)$. From $y\in x+C_\delta$ it follows that $x\in y-C_\delta$, thus x is in the interior of $B_r(x)\cap(y-C_\delta)$ where we have shown that $u=0$ and this contradicts with $x\in\Gamma$. □

It is easy to see that for the cone C'_δ conjugate to the cone C_δ we have

$$C'_\delta=\Big\{x\in\mathbb{R}^2\ \Big|\ x\cdot y\geq 0,\ \forall y\in C_\delta\Big\}=\overline{C_{\sqrt{1-\delta^2}}}. \tag{10.24}$$

Theorem 10. *There exists $C_1 > 0$ such that if u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular point with blowup limit u_w then there exists $\epsilon > 0$ such that all points of $\Gamma \cap \{x_1 > 0\} \cap B_\epsilon$ are usual (for $x_1 > 0$ the force term is nondegenerate) regular free boundary points and*

$$|n(x) - \nu_w| \leq C_1 \|u_{4|x|} - u_w\|^{\frac{1}{2}}_{C^1(B_1)} \tag{10.25}$$

for $x \in \Gamma \cap \{x_1 > 0\} \cap B_\epsilon$ where $n(x)$ is the normal to Γ at x, pointing into Ω.

Proof. If there exists $r > 0$ such that $u = u_w$ in B_r then the claim of the theorem holds trivially. So we might assume that for all $r > 0$ we have $u \neq u_w$ in B_r.

Let $x \in \Gamma \cap \{x_1 > 0\} \cap B_1$. By the uniqueness of the blowup limit and Theorem 1 we have that $u_{4|x|} \to u_w$ in $C^1(B_1)$ as $x \to 0$. Thus there exists $\epsilon > 0$ such that for $|x| < \epsilon$ we have

$$\|u_{4|x|} - u_w\|_{C^1(B_1)} < (\frac{C}{2})^2 \tag{10.26}$$

with $C > 0$ as in Lemma 25.

Let $y = \frac{1}{4}\frac{x}{|x|}$. Then $y \in \Gamma_{u_{4|x|}} \cap \partial B_{\frac{1}{4}} \cap \{x_1 > 0\}$. By (10.26) if we choose

$$\delta = \frac{2}{C} \|u_{4|x|} - u_w\|^{\frac{1}{2}}_{C^1(B_1)} \tag{10.27}$$

then $0 < \delta < 1$.

Also let us set

$$r = \|u_{4|x|} - u_w\|^{\frac{1}{2}}_{C^1(B_1)}. \tag{10.28}$$

Then by (10.27) and (10.28) we have

$$\|u_{4|x|} - u_w\|^{\frac{1}{2}}_{C^1(B_1)} + r = C\delta \tag{10.29}$$

and consequently by Corollary 6 we have

$$B_r(y) \cap (y + C_\delta) \subset \{u_{4|x|} > 0\} \text{ and } B_r(y) \cap (y - C_\delta) \subset \{u_{4|x|} = 0\}. \tag{10.30}$$

From (10.30) it follows that

$$B_{4|x|r}(x) \cap (x + C_\delta) \subset \{u > 0\} \text{ and } B_{4|x|r}(x) \cap (x - C_\delta) \subset \{u = 0\}. \tag{10.31}$$

Now if x is a singular free boundary point then the blowup limit is a nonzero homogenous quadratic polynomial. But by (10.31) this polynomial should be equal to 0 in $-C_\delta$ which brings us to contradiction. Thus all points of $\Gamma \cap \{x_1 > 0\} \cap B_\epsilon$ are regular points.

Now assume $|x| < \epsilon$ then because x is a regular point, Γ has a normal at this point. Let $n(x)$ be the normal to Γ pointing into Ω. From (10.31) it follows that $n(x) \in C'_\delta$. Now by (10.24) we have

$$n(x) \in \overline{C_{\sqrt{1-\delta^2}}}$$

so

$$n(x) \cdot \nu_w \geq \sqrt{1 - \delta^2}.$$

We compute

$$|n(x) - \nu_w|^2 = 2 - 2n(x) \cdot \nu_w \leq 2 - 2\sqrt{1 - \delta^2} = \frac{2\delta^2}{1 + \sqrt{1 - \delta^2}} \leq 2\delta^2 \tag{10.32}$$

and from (10.27) and (10.32), (10.25) follows. □

11. Free Boundary as a Graph Near Regular Points

The following two lemmas will be used in Lemma 28.

Lemma 26. *If u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular free boundary point with blowup limit u_w then there exists an $\epsilon > 0$ such that $u(0,t) > 0$ for $0 < t < \epsilon$ and $(0,t) \in \{u = 0\}^\circ$ for $-\epsilon < t < 0$.*

Proof. Let $x = (0,t) \in B_\epsilon$, $0 < t < \epsilon$ then we compute

$$d^2(\frac{x}{2|x|}, \{u_w = 0\})\big(d(\frac{x}{2|x|}, \{u_w = 0\}) + |\frac{x_1}{2|x|}|\big) = d^3(\frac{x}{2|x|}, \{u_w = 0\}) = d^3(\frac{1}{2}e_2, \{u_w = 0\}) = (\frac{\sqrt{2}}{4})^3. \tag{11.1}$$

For small enough ϵ if $|x| < \epsilon$ then

$$\|u_{2|x|} - u_w\|_{L^\infty(B_1)} < C(\frac{\sqrt{2}}{4})^3 \tag{11.2}$$

with C as in Lemma 16. Thus by (11.1), (11.2) and (6.1) we have $u_{2|x|}(\frac{x}{2|x|}) > 0$ so $u(x) > 0$.

Let $x = (0,t) \in B_\epsilon$, $-\epsilon < t < 0$ then we compute

$$d^2(\frac{x}{4|x|}, \{u_w > 0\})\big(d(\frac{x}{4|x|}, \{u_w > 0\}) + |\frac{x_1}{4|x|}|\big) = d^3(\frac{x}{4|x|}, \{u_w > 0\}) = d^3(-\frac{1}{4}e_2, \{u_w > 0\}) = \frac{1}{4^3}. \tag{11.3}$$

For small enough ϵ if $|x| < \epsilon$ then

$$\|u_{4|x|} - u_w\|_{L^\infty(B_1)} < \frac{1}{4^3}C. \tag{11.4}$$

Thus by (11.3), (11.4) and (6.2) we have $\frac{x}{4|x|} \in \{u_{4|x|} = 0\}^\circ$, so $x \in \{u = 0\}^\circ$. $\square$

Lemma 27. *If u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular free boundary point with blowup limit u_w then there exists an $\epsilon > 0$ such that for every $0 < x_1 < \frac{\epsilon}{4}$ there exists a unique x_2 such that $x = (x_1, x_2) \in \Gamma \cap B_\epsilon$ and for $(x_1, t) \in B_\epsilon$ we have $u(x_1, t) > 0$ if $t > x_2$ and $(x_1, t) \in \{u = 0\}^\circ$ if $t < x_2$.*

Proof. First we show that there exists $\epsilon > 0$ such that for all $0 < x_1 < \frac{\epsilon}{4}$ there exists x_2 such that $(x_1, x_2) \in \Gamma \cap B_\epsilon$.

Let $\epsilon > 0$ to be chosen later. Let $0 < x_1 < \frac{\epsilon}{4}$ then we compute

$$|(x_1, \frac{3}{4}\epsilon)|^2 < (\frac{\epsilon}{4})^2 + (\frac{3}{4}\epsilon)^2 = \frac{10}{16}\epsilon^2 < \epsilon^2$$

thus $(x_1, \frac{3}{4}\epsilon) \in B_\epsilon$. We compute

$$d\big((\frac{x_1}{\epsilon}, \frac{3}{4}), \{u_w = 0\}\big) = \frac{\sqrt{2}}{2}(\frac{3}{4} - \frac{x_1}{\epsilon}) \geq \frac{\sqrt{2}}{2}(\frac{3}{4} - \frac{1}{4}) = \frac{\sqrt{2}}{4}$$

and

$$d^2\big((\frac{x_1}{\epsilon}, \frac{3}{4}), \{u_w = 0\}\big)\big(d((\frac{x_1}{\epsilon}, \frac{3}{4}), \{u_w = 0\}) + |\frac{x_1}{\epsilon}|\big) \geq d^3((\frac{x_1}{\epsilon}, \frac{3}{4}), \{u_w = 0\}) \geq (\frac{\sqrt{2}}{4})^3.$$

Thus if ϵ is small enough such that

$$\|u_\epsilon - u_w\|_{L^\infty(B_1)} < C(\frac{\sqrt{2}}{4})^3$$

with C as in Lemma 16, then by (6.1) we obtain that

$$u_\epsilon(\frac{x_1}{\epsilon}, \frac{3}{4}) > 0$$

and therefore

$$u(x_1, \frac{3}{4}\epsilon) > 0. \tag{11.5}$$

Let $0 < x_1 < \frac{\epsilon}{4}$ then we compute

$$|(x_1, -\frac{\epsilon}{4})|^2 < (\frac{\epsilon}{4})^2 + (\frac{\epsilon}{4})^2 = (\frac{\sqrt{2}}{4}\epsilon)^2 < (\frac{1}{2}\epsilon)^2$$

thus $(x_1, -\frac{\epsilon}{4}) \in B_{\frac{1}{2}\epsilon} \subset B_1$.

We compute

$$d\big((\frac{x_1}{\epsilon}, -\frac{1}{4}), \{u_w > 0\}\big) \geq \frac{1}{4}$$

and

$$d^2\big((\frac{x_1}{\epsilon}, -\frac{1}{4}), \{u_w > 0\}\big)\Big(d\big((\frac{x_1}{\epsilon}, -\frac{1}{4}), \{u_w > 0\}\big) + |\frac{x_1}{\epsilon}|\Big) \geq \frac{1}{4^3}.$$

Thus if ϵ is small enough such that

$$\|u_\epsilon - u_w\|_{L^\infty(B_1)} < \frac{1}{4^3}C$$

then by (6.2) we obtain that

$$(\frac{x_1}{\epsilon}, -\frac{1}{4}) \in \{u_\epsilon = 0\}^\circ$$

and therefore

$$(x_1, -\frac{1}{4}\epsilon) \in \{u = 0\}^\circ. \tag{11.6}$$

From (11.5), (11.6) and the continuity of u it follows that there exists $-\frac{\epsilon}{4} < x_2 < \frac{3}{4}\epsilon$ such that $(x_1, x_2) \in \Gamma$. This finishes the proof of the existence of x_2.

By Lemma 23 if $\epsilon > 0$ is small enough then

$$\partial_{x_2} u \geq 0 \text{ in } B_\epsilon. \tag{11.7}$$

Assume there exists $t > x_2$, $(t, x_2) \in B_\epsilon$ such that $u(x_1, t) = 0$. Then from (11.7) it follows that

$$u(x_1, s) = 0 \text{ for all } x_2 < s < t. \tag{11.8}$$

From Theorem 10 it follows that for small enough ϵ, because $|x| < \epsilon$, we have

$$|n(x_1, x_2) - \nu_w| < \frac{\sqrt{2}}{2}.$$

Therefore

$$\begin{aligned} n(x_1, x_2) \cdot e_2 &= \nu_w \cdot e_2 + (n(x_1, x_2) - \nu_w) \cdot e_2 \\ &\geq \nu_w \cdot e_2 - |n(x_1, x_2) - \nu_w| = \frac{\sqrt{2}}{2} - |n(x_1, x_2) - n_w| > 0 \end{aligned}$$

and this is in contradiction with (11.8) which proves that $u(x_1, t) > 0$ for $t > x_2$ and $(x_1, t) \in B_\epsilon$.

If $t < x_2$, $(x_1, t) \in B_\epsilon$ then from (11.7) it follows that $u(x_1, t) = 0$.

Now if $(x_1, t) \in \Gamma$ then switching the place of x_2 and t, and arguing as above we come to contradiction, hence $(x_1, t) \notin \Gamma$. □

In the following lemma we prove that near to regular points the free boundary is a graph.

Lemma 28. *If u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular free boundary point with blowup limit u_w then there exists an $\epsilon > 0$ and $\gamma \in C([0, \frac{\epsilon}{4}))$ such that $\gamma(0) = 0$, for $0 < x_1 < \frac{\epsilon}{4}$ we have $(x_1, \gamma(x_1)) \in B_\epsilon$ and*

$$\{u = 0\} \cap B_\epsilon \cap \{0 \leq x_1 < \frac{\epsilon}{4}\} = \Big\{x \in B_\epsilon \;\Big|\; 0 \leq x_1 < \frac{\epsilon}{4},\, x_2 \leq \gamma(x_1)\Big\}. \tag{11.9}$$

Proof. By Lemma 27 there exists an $\epsilon > 0$ such that for every $0 < x_1 < \frac{\epsilon}{4}$ there exists a unique x_2 such that $x = (x_1, x_2) \in \Gamma \cap B_\epsilon$, let us define $\gamma(x_1) = x_2$. Let us also define $\gamma(0) = 0$.

Then by Lemmas 26 and 27, we have (11.9).

Now let us show that γ is continuous. Assume there exists $0 \leq y < \frac{\epsilon}{4}$ such that γ is discontinuous at y. Then there exists $x_j \to y$ such that $\gamma(x_j) \to z$ and either $z > \gamma(y)$ or $z < \gamma(y)$.

In the case $z > \gamma(y)$ we have $u(y, z) > 0$ which is in contradiction with $u(x_j, \gamma(x_j)) = 0$ and the continuity of u.

In the case $z < \gamma(y)$ we have $(y, z) \in \{u = 0\}^\circ$ which is in contradiction with $(x_j, \gamma(x_j)) \in \Gamma$. □

The functions σ_0 and σ_1 are defined in (2.6).

In the following lemma we formulate the convergence of the free boundary in terms of the function γ.

Lemma 29. *There exists $C_1 > 0$ and $C_2 > 0$ such that if u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular free boundary point with blowup limit u_w then with $\epsilon > 0$ and γ as in Lemma 28 we have*

$$|\gamma(x_1) - x_1| \leq C_1(\sigma_0(C_2|x_1|))^{\frac{1}{2}}|x_1| \text{ for } 0 < x_1 < \frac{\epsilon}{4}$$

where σ_0 is defined in (2.6).

Proof. By Theorem 5 we have

$$d(x, \Gamma_{u_w}) \leq C_1(\sigma_0(C_2|x|))^{\frac{1}{2}}|x|.$$

For $x_1 > 0$ we estimate

$$d(x, \Gamma_{u_w}) \geq \frac{\sqrt{2}}{2}|x_2 - x_1|$$

thus

$$\begin{aligned} |\gamma(x_1) - x_1| &\leq C_3(\sigma_0(C_2|x|))^{\frac{1}{2}}|x| \leq C_4(\sigma_0(C_2|x|))^{\frac{1}{2}}(|\gamma(x_1)| + |x_1|) \\ &\leq C_4(\sigma_0(C_2|x|))^{\frac{1}{2}}(|\gamma(x_1) - x_1| + 2|x_1|) \end{aligned} \tag{11.10}$$

By the continuity of γ at 0 we have that $\gamma(x_1) \to \gamma(0) = 0$ as $x_1 \to 0$. Hence $|x| \leq C_5(|\gamma(x_1)| + |x_1|) \to 0$ as $x_1 \to 0$. From this convergence we obtain $\sigma_0(C_2|x|) \to 0$ as $x_1 \to 0$.

Thus from (11.10) it follows that

$$|\gamma(x_1) - x_1| \leq C_6(\sigma_0(C_2|x|))^{\frac{1}{2}}|x_1|. \tag{11.11}$$

In turn from (11.11) it follows that

$$\begin{aligned} |x| &\leq C_5\big(|\gamma(x_1)| + |x_1|\big) \leq C_5\big(|\gamma(x_1) - x_1| + 2|x_1|\big) \\ &\leq C_5\big(C_6(\sigma_0(C_2|x|))^{\frac{1}{2}}|x_1| + 2|x_1|\big) = C_5\big(C_6(\sigma_0(C_2|x|))^{\frac{1}{2}} + 2\big)|x_1| \leq C_7|x_1|. \end{aligned} \tag{11.12}$$

Now by (11.11) and (11.12) the lemma is proved. □

In the following lemma we formulate the convergence of the normals in terms of the function γ.

Lemma 30. *There exists $C_1 > 0$ and $C_2 > 0$ such that if u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular free boundary point with blowup limit u_w and $\epsilon > 0$ and γ as in Lemma 28, then we have $\gamma \in C^1(0, \frac{\epsilon}{4})$ and*

$$|\gamma'(x_1) - 1| \leq C_1(\sigma_1(C_2|x_1|))^{\frac{1}{2}}$$

where σ_1 is defined in (2.6).

Proof. By Theorem 10 for small enough $\epsilon > 0$ all points of $\Gamma \cap \{x_1 > 0\} \cap B_\epsilon$ are usual regular points. Let $0 < x_1 < \frac{\epsilon}{4}$. Hence (cf. [6]) Γ is a C^1 curve in a neighbourhood of $(x_1, \gamma(x_1))$. From (10.25) it follows that for small enough ϵ and $|x| < \epsilon$ we have $n(x) \notin \{-e_1, e_1\}$. It follows that $\gamma'(x_1)$ exists and

$$n(x) = \frac{(-\gamma'(x_1), 1)}{\sqrt{1 + (\gamma'(x_1))^2}}.$$

From here it follows that there exists $C > 0$ such that for $n(x)$ close enough to ν_w we have

$$|\gamma'(x_1) - 1| \leq C|n(x) - \nu_w|. \tag{11.13}$$

Now by (10.25) and (11.13) we obtain

$$|\gamma'(x_1) - 1| \leq C_2\|u_{4|x|} - u_w\|^{\frac{1}{2}}_{C^1(B_1)}. \tag{11.14}$$

By (11.12) together with the definition of σ_1 and (11.14) the lemma is proved. □

Proof of Theorem 6. Follows from Lemmas 28, 29, 30 and the symmetry of the problem with respect to the line $\{x_1 = 0\}$. □

In the case when 0 is a regular point but with $u_w(x_1, -x_2)$ as the blowup limit, we consider the even reflection $\tilde{u}(x_1, x_2) = u(x_1, -x_2)$, apply Theorem 6 to $\tilde{u}$ and obtain that the free boundary of u is a graph with properties as in Theorem 6 but reflected with respect to the line $\{x_2 = 0\}$.

By the following two lemmas we prove that if $W(+0, u) = 2W(1, u_w)$ then u might be decomposed into the sum of two functions each having 0 as a regular point.

Lemma 31. *If u is a solution in D, $0 \in D$, $0 \in \Gamma$ and $W(+0, u) = 2W(1, u_w)$ then there exists an $\epsilon > 0$ such that $u(x_1, 0) = 0$ for $|x_1| < \epsilon$.*

Proof. Let $u_0 = u_w + u_w(x_1, -x_2)$. We have

$$d\big(\pm\frac{1}{4}e_1, \{u_0 > 0\}\big) = \frac{\sqrt{2}}{8}.$$

We compute

$$d^2\big(\pm\frac{1}{4}e_1, \{u_0 > 0\}\big)\big(d\big(\pm\frac{1}{4}e_1, \{u_0 > 0\}\big) + \frac{1}{4}\big) = (\frac{\sqrt{2}}{8})^2(\frac{\sqrt{2}}{8} + \frac{1}{4}).$$

Now if $|x_1| > 0$ is small enough such that

$$\|u_{4|x_1|} - u_0\|_{L^\infty(B_1)} < C(\frac{\sqrt{2}}{8})^2(\frac{\sqrt{2}}{8} + \frac{1}{4})$$

with C as in Lemma 16 then by (6.2) we have $u_{4|x_1|}(\pm\frac{1}{4}e_1) = 0$. Thus $u(x_1, 0) = 0$. □

Lemma 32. *If u is a solution in D, $0 \in D$, $0 \in \Gamma$ and $W(+0, u) = 2W(1, u_w)$ then there exists an $\epsilon > 0$ such that $u_+ = \chi_{x_2 > 0}u$ and $u_- = \chi_{x_2 < 0}u$ are solutions in B_ϵ. We have $W(+0, u_\pm) = W(1, u_w)$, the blowup limit of u_+ is u_w and the blowup limit of u_- is $u_w(x_1, -x_2)$.*

Proof. By Lemma 31 there exists an $\epsilon > 0$ such that $u(x_1, 0) = 0$ for $|x_1| < \epsilon$.

Because $u \geq 0$, $u \in C^1_{loc}(D)$ and $u(x_1, 0) = 0$ for $|x_1| < \epsilon$ it follows that $\nabla u(x_1, 0) = 0$ for $|x_1| < \epsilon$.

From this it follows that u_+ and u_- are solutions in B_ϵ. We have $u_r(x) \to u_w + u_w(x_1, -x_2)$ in $C^1(B_1)$ as $r \to 0$. Thus $\chi_{x_2>0} u_r \to u_w$ in $C^1(B_1)$ and $u_{+,r}(x) = r^{-3}\chi_{rx_2>0} u(rx) = \chi_{x_2>0} u_r(x)$ hence $u_{+,r}(x) \to u_w$ in $C^1(B_1)$ and

$$W(+0, u_+) = \lim_{r\to+0} W(r, u_+) = \lim_{r\to+0} W(1, u_{+,r}) = W(1, u_w).$$

Similarly we argue for u_-. □

In the case $W(+0, u) = 2W(1, u_w)$ by Lemma 32 and Theorem 6 it follows that the free boundary near to 0 is the union of two graphs, one graph is as in Theorem 6 and the other a graph with properties as in Theorem 6 but reflected with respect to the line $\{x_2 = 0\}$.

12. An Irregularity Result for the Free Boundary Near Degenerate Points

Lemma 33. *If u is a solution in D, $0 \in D$, there exists $\delta > 0$ such that $B_\delta \subset D$, $\partial_{x_2} u \leq 0$ in $B_\delta \cap \{x_1 > 0, x_2 > 0\}$, $\Gamma \cap B_\delta \cap \{x_1 = 0, x_2 > 0\} \neq \emptyset$ and $B_\delta \cap \{x_1 > 0, x_2 > 0\} \subset \Omega$ then $u = u_{hs}$ in $B_\delta \cap \{x_1 > 0, x_2 > 0\}$.*

Proof. For short notation let us denote $v = -\partial_{x_2} u$. We have that v is harmonic in Ω and $v \geq 0$ in $B_\delta \cap \{x_1 > 0, x_2 > 0\}$.

Assume $y \in \Gamma \cap B_\delta \cap \{x_1 = 0, x_2 > 0\}$, then by the optimal growth Theorem 8 we have $\partial_{x_1} v(y) = 0$. For small enough $r > 0$ we have $B_r(re_1 + y) \subset \Omega$. Now because v is nonnegative and harmonic in $B_r(re_1 + y)$ and $\partial_{x_1} v(y) = 0$ by Hopf's Lemma we conclude that $v = 0$ in $B_r(re_1 + x)$. Because v is harmonic in Ω we obtain that $v = 0$ in $B_\delta \cap \{x_1 > 0, x_2 > 0\}$. Hence $u = u(x_1)$ in $B_\delta \cap \{x_1 > 0, x_2 > 0\}$. By this and the assumption $\Gamma \cap B_\delta \cap \{x_1 = 0, x_2 > 0\} \neq \emptyset$ the claim follows. □

Lemma 34. *If u is a solution in D, $0 \in D$, there exists $\delta > 0$ such that $B_\delta \subset D$, $\partial_{x_2} u \leq 0$ in $B_\delta \cap \{x_1 > 0, x_2 > 0\}$, there exists $\rho \in C([0, \frac{1}{2}\delta)) \cap C^1([0, \frac{1}{2}\delta))$ such that $\rho(0) = \rho'(+0) = 0$, $\rho > 0$ in $(0, \frac{1}{2}\delta)$, ρ is convex and*

$$\Omega \cap B_\delta \cap \{x_1 > 0, 0 < x_2 < \frac{1}{2}\delta\} = B_\delta \cap \{0 < x_2 < \frac{1}{2}\delta,\ \rho(x_2) < x_1\} \tag{12.1}$$

then for every $q > 1$ there exist $C > 0$ and $t_0 > 0$ such that

$$\rho(t) \geq Ct^q \text{ and } \rho'(t) \geq Ct^{q-1} \text{ for } 0 < t < t_0. \tag{12.2}$$

Proof. Again for short notation let us denote $v = -\partial_{x_2} u$. The proof is divided in multiple steps.

Step 1. In this step we show that $v > 0$ in $B_\delta \cap \{0 < x_2 < \frac{1}{2}\delta,\ \rho(x_2) < x_1\}$.

If there would exist $x \in B_\delta \cap \{0 < x_2 < \frac{1}{2}\delta,\ \rho(x_2) < x_1\}$ such that $v(x) = 0$ then because v is harmonic and nonnegative in $B_\delta \cap \{0 < x_2 < \frac{1}{2}\delta,\ \rho(x_2) < x_1\}$ it follows that $v = 0$ in $B_\delta \cap \{0 < x_2 < \frac{1}{2}\delta,\ \rho(x_2) < x_1\}$, but then because $u(\rho(t), t) = 0$ for $0 < t < \frac{1}{2}\delta$ we come to contradiction with (12.1).

Step 2. In this step we show that for each $q > 1$ and $\eta > (\tan(\frac{\pi}{2q}))^{-1}$ there exist $C_1 > 0$ (depends on u) and $t_1 > 0$ such that

$$v(x_t) \geq C_1 t^{2q} \text{ for } 0 < t < t_1 \tag{12.3}$$

where

$$x_t = (\eta t, t) \in \Omega.$$

Let $q > 1$ and

$$\alpha_q = \frac{\pi}{2q}.$$

Because $\rho'(+0) = 0$ there exists $t_q > 0$ such that $\rho(t) < \frac{t}{\tan(\alpha_q)}$ for $0 < t < t_q$. Let us denote

$$r_q = \frac{t_q}{\tan(\alpha_q)}.$$

If follows that

$$\Omega_q = \Big\{ x = re^{i\theta} \ \Big| \ 0 < r < r_q,\ 0 < \theta < \alpha_q \Big\} \subset \Omega.$$

Let us define the function

$$v_q(x) = r^{2q} \sin(2q\theta) \text{ for } x = re^{i\theta} \in \Omega_q.$$

We have

$$\partial(\frac{1}{2}\Omega_q) = S_q \cup A_q$$

where

$$S_q = \Big\{ x = re^{i\theta} \ \Big| \ 0 \le r < \frac{1}{2} r_q,\ \theta \in \{0, \alpha_q\} \Big\}$$

and

$$A_q = \Big\{ x = re^{i\theta} \ \Big| \ r = \frac{1}{2} r_q,\ 0 \le \theta \le \alpha_q \Big\}.$$

Let $a = \frac{1}{2} r_q e_1$ and $b = \frac{1}{2} r_q e^{i\alpha_q}$ be the end points of the arc A_q. We have $b \in \Omega$ hence $v(b) > 0$. Either $v(a) > 0$ or $v(a) = 0$ and by Hopf's Lemma we have $\partial_{x_2} v(a) > 0$. Also we have $v > 0$ on $A_q \backslash \{a, b\}$.

Thus there exists $\epsilon > 0$ such that

$$\epsilon v_q \le v \text{ on } A_q. \tag{12.4}$$

We have $v_q = 0$ and $v \ge 0$ on S_q thus

$$\epsilon v_q \le v \text{ on } S_q. \tag{12.5}$$

Putting (12.4) and (12.5) together we have

$$\epsilon v_q \le v \text{ on } \partial(\frac{1}{2}\Omega_q).$$

Now by the maximum principle we obtain that

$$\epsilon v_q \le v \text{ in } \frac{1}{2}\Omega_q. \tag{12.6}$$

We compute $|x_t| = \sqrt{1+\eta^2}\, t$ so for

$$0 < t < \frac{1}{2} \frac{r_q}{\sqrt{1+\eta^2}}$$

we have $|x_t| < \frac{1}{2} r_q$, also we compute

$$\frac{x_{t,2}}{x_{t,1}} = \frac{1}{\eta} < \tan(\alpha_q)$$

thus we have

$$x_t \in \frac{1}{2}\Omega_q \text{ for } 0 < t < \frac{1}{2} \frac{r_q}{\sqrt{1+\eta^2}}. \tag{12.7}$$

Now by (12.6) and (12.7) we have

$$v(x_t) \ge \epsilon v_q(x_t) = \epsilon |x_t|^{2q} \sin(2q \arctan(\frac{1}{\eta})) = C_1 t^{2q} \text{ for } 0 < t < \frac{1}{2} \frac{r_q}{\sqrt{1+\eta^2}}$$

where

$$C_1 = \epsilon (1+\eta^2)^q \sin(2q \arctan(\frac{1}{\eta})) > 0.$$

Step 3. In this step we show that there exists $C_2 > 0$ (independent of u) and $t_2 > 0$ such that if

$$0 < t < t_2 \text{ and } \eta < 1$$

then there exists $y_t = (\rho(y_{t,2}), y_{t,2}) \in \Gamma$ with $0 < y_{t,2} < t_q$ such that

$$d_t = |y_t - x_t| = d(\Gamma, x_t)$$

and

$$\partial_{n(y_t)} v(y_t) \geq \frac{C_2}{d_t} v(x_t). \tag{12.8}$$

Here $n(y)$ is the normal to Γ at y, pointing into Ω.

Let

$$\Pi_q = \Big\{0 < x_1 < r_q,\ 0 < x_2 < t_q\Big\}$$

then we have

$$\Gamma_q = \Gamma \cap \Pi_q = \Big\{(\rho(t), t) \ \Big|\ 0 < t < t_q\Big\}.$$

One may see that

$$d(x_t, \partial\Pi_q) = \min\big\{\eta t, r_q - \eta t, t, t_q - t\big\} = \eta t \tag{12.9}$$

if

$$t < \min(\frac{r_q}{2\eta}, \frac{t_q}{1+\eta}) \text{ and } \eta < 1.$$

Because $\eta > (\tan(\alpha_q))^{-1}$ and $0 < t < t_q$ we have that $\rho(t) < \frac{t}{\tan(\alpha_q)} < \eta t$. Also we have $\rho(t) > 0$ thus

$$d(x_t, (\rho(t), t)) = \eta t - \rho(t) < \eta t.$$

Now because $(\rho(t), t) \in \Gamma_q$ we have

$$d(x_t, \Gamma) < \eta t. \tag{12.10}$$

By (12.9) and (12.10) there exists $y_t \in \Gamma_q$ such that

$$d_t = |y_t - x_t| = d(\Gamma, x_t). \tag{12.11}$$

Because

$$d(x_t, \partial\Pi_q) = \eta t > d(\Gamma, x_t) = d_t$$

we have

$$B_{d_t}(x_t) \subset \Pi_q \subset \Omega.$$

Because $y_t \in \partial B_{d_t}(x_t)$ by the quantitative Hopf Lemma (cf. [4]) there exists $C_2 > 0$ (independent of u and t) such that (12.8) holds.

Step 4. In this step we show that

$$\partial_{n(y)} v(y) = -n_2(y) y_1 \text{ for } y \in \Gamma_q. \tag{12.12}$$

By the equation $\triangle u = |x_1|\chi_{u>0}$ and the smoothness of the free boundary Γ_q, i.e. smoothness of ρ, it follows that in a neighbourhood of $y \in \Gamma_q$ we have

$$\triangle v = -n_2 |x_1| \mathcal{H}^1 \llcorner \Gamma. \tag{12.13}$$

From (12.1) and (12.13) the equation (12.12) follows.

Step 5. In this step we show that for $0 < t < t_2$ we have

$$y_{t,2} < (1+\eta)t. \tag{12.14}$$

We have

$$n(y) = \frac{(1, -\rho'(y_2))}{\sqrt{1 + (\rho'(y_2))^2}} \text{ for } y \in \Gamma_q \tag{12.15}$$

and

$$y_t = x_t - d_t n(y_t).$$

Thus

$$y_{t,2} = t + d_t \frac{\rho'(y_{t,2})}{\sqrt{1 + (\rho'(y_{t,2}))^2}}$$

and

$$y_{t,2} \le t + d_t < t + \eta t = (1+\eta)t.$$

Step 6. In this step we show that there exists $C_3 > 0$ and $t_3 > 0$ such that

$$\rho(y_{t,2})\rho'(y_{t,2}) \ge C_3 t^{2q-1} \text{ for } 0 < t < t_3. \tag{12.16}$$

Set $t_3 = \min(t_1, t_2)$. From (12.3), (12.8) and (12.12) it follows that

$$-n_2(y_t)y_{t,1} = \partial_{n(y_t)} v(y_t) \ge \frac{C_2}{d_t} v(x_t) \ge \frac{C_2}{d_t} C_1 t^{2q} \text{ for } 0 < t < t_3. \tag{12.17}$$

From (12.17), (12.15), (12.10) and (12.11) we get

$$\begin{aligned}\rho(y_{t,2})\rho'(y_{t,2}) = \rho'(y_{t,2})y_{t,1} &\ge \frac{\rho'(y_{t,2})}{\sqrt{1+(\rho'(y_{t,2}))^2}} y_{t,1} \\ &= -n_2(y_t)y_{t,1} \ge \frac{C_2}{d_t} C_1 t^{2q} \ge \frac{1}{\eta} C_1 C_2 t^{2q-1} = C_3 t^{2q-1}.\end{aligned}$$

Step 7. In this step using the convexity of ρ we finish the proof of the lemma.

By the convexity of ρ, the function $\rho\rho'$ is nondecreasing hence by (12.14) and (12.16) we have

$$\rho((1+\eta)t)\rho'((1+\eta)t) \ge \rho(y_{t,2})\rho'(y_{t,2}) \ge C_3 t^{2q-1} \text{ for } 0 < t < t_3.$$

Letting $\tau = (1+\eta)t$ we have that

$$\rho(\tau)\rho'(\tau) \ge C_3 (\frac{\tau}{1+\eta})^{2q-1} = C_4 \tau^{2q-1} \text{ for } 0 < \tau < (1+\eta)t_3 = \tau_0.$$

If follows that

$$(\rho^2)'(\tau) \ge 2C_4 \tau^{2q-1} \text{ for } 0 < \tau < \tau_0$$

and by an integration we obtain

$$\rho(\tau) \ge C_5 \tau^q \text{ for } 0 < \tau < \tau_0.$$

From the convexity of ρ it follows that $\tau\rho'(\tau) \ge \rho(\tau)$ hence

$$\rho'(\tau) \ge C_5 \tau^{q-1} \text{ for } 0 < \tau < \tau_0$$

and this completes the proof of the lemma. □

Proof of Theorem 7. By Lemmas 33 and 34 we have that either $\rho = 0$ in $(0, \frac{1}{2}\delta)$ and $u = u_{hs}$ in $\Omega \cap B_\delta \cap \{x_1 > 0, x_2 > 0\}$ or for all $q > 1$ there exists $C > 0$ and $t_0 > 0$ such that (12.2) holds.

In the latter case if Γ is $C^{1,\alpha}$ regular for some $0 < \alpha < 1$ at the origin then there exists $C > 0$ and $\delta_1 > 0$ such that

$$|\rho'(x_2) - \rho'(+0)| \le C|x_2|^\alpha \text{ for } 0 < x_2 < \delta_1.$$

But because $\rho'(+0) = 0$ and $\rho'(x_2) \ge 0$ we should have

$$\rho'(x_2) \le C x_2^\alpha \text{ for } 0 < x_2 < \delta_1.$$

This contradicts with (12.2) if we take $1 < q < 1 + \alpha$. □

13. Further Directions

The problem considered in this paper might be thought of as a prototype of free boundary problems, specially the obstacle problem, with a degenerate force term. There are many open questions in these problems and we are working to complete some works on these questions.

Some further directions are as follows.

1) Higher dimension. It is interesting to consider the same problem in higher dimensions with possibly different dimensions for the set where the force term vanishes. In [13] the key nondegeneracy result is proved for such higher dimensional problems when the set where the force term vanishes is a linear subspace.

2) More general force terms. Partial results show that when the force term is of the form $|x_1|^\alpha$ for $\alpha > 0$ then the number of homogenous global solutions and together with it the possible Weiss balanced energy levels grows linearly with $\alpha > 0$. Again in [13] the key nondegeneracy result is proved for such general force terms. Many results in this paper could be written for such more general forces, but to have a reasonable bound on the size of the paper we have opted to consider the case $\alpha = 1$ only.

3) Degenerate free boundary points and points where $W(+0, x, u) = 2W(1, u_{hs})$. We know that at these points the blowup limit is unique and the free boundary converges tangentially to the line $\{x_1 = 0\}$ and we know some topological structure of the set of these points based on the upper semicontinuity of the Weiss balanced energy. Also in a particular case we have proved an irregularity result for the free boundary at such points. It is interesting to study the structure of the free boundary near to such points in more details.

4) Uniform results. For the nondegenerate obstacle problems there are many results which hold uniformly for a class of problems, cf. [6]. But in this paper we have only considered a single solution alone.

5) Parabolic problem. The problem considered in this paper has a parabolic analogue. It is interesting to know the exact influence of the degeneracy of the force term in the parabolic problems.

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