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# Petrov type I spacetime and dual relativistic fluids

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The Petrov type I condition for the solutions of vacuum Einstein equations in both of the nonrelativistic and relativistic hydrodynamic expansions is checked. We show that it holds up to the third order of the nonrelativistic hydrodynamic expansion parameter, but it is violated at the fourth order even if we choose a general frame. On the other hand, it is found that the condition holds at least up to the second order of the derivative expansion parameter. Turn the logic around, through imposing the Petrov type I condition and Hamiltonian constraint on a finite cutoff surface, we show that the stress tensor of the relativistic fluid can be recovered with correct first order and second order transport coefficients dual to the solutions of vacuum Einstein equations.

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## I. INTRODUCTION

The holographic duality between gravity and one lower dimensional fluid has attracted much attention over the past years. There exist two kinds of prescriptions for the dual fluid. One is the membrane paradigm which describes a fluid living on the stretched horizon of a black hole [1–5], and the other is the AdS/fluid duality which describes a certain conformal fluid living on the anti–de Sitter (AdS) boundary [6–11]. It is expected that there exists some connection between the two descriptions [12–14]. This motivates the authors in [15] to consider the gravitational fluctuations confined inside a finite cutoff surface outside a horizon, and in this case the dual fluid lives on this hypersurface. The Dirichlet condition on the cutoff surface and the regularity on the horizon are imposed. This procedure has also been generalized to the asymptotically flat [16,17] and de Sitter [18] spacetimes.

The authors of [16] have shown that for every solution of the incompressible Navier-Stokes equations in $p+1$ dimensions, there exists a unique corresponding solution of vacuum Einstein equations in $p+2$ dimensions. On the cutoff surface, the extrinsic curvature is given by the stress tensor of the Navier-Stokes fluid. A systematical method to reconstruct the solution of vacuum Einstein gravity to an arbitrary order has been presented in both of the nonrelativistic and relativistic hydrodynamic expansions [19–22]. It is interesting to note that, instead of imposing the regularity condition on the horizon, imposing the Petrov type I condition on a hypersurface in near-horizon limit is alternatively introduced in [23]. The Petrov type I condition just gives $p(p+1)/2$ constraints on the extrinsic curvature (or say, the Brown-York stress tensor $T_{ab}$ of the dual fluid), which leads to $p+1$ independent variables. These variables are exactly the degrees of freedom of a fluid in $p+1$ dimensions. They have shown that combining the Petrov type I condition with Hamiltonian and momentum constraints can lead to the incompressible Navier-Stokes equation for the dual fluid on the cutoff surface in the near-horizon limit. Some further generalizations and discussions can be seen in [24–31].

Notice that if one considers the mathematically equivalent solution of vacuum Einstein equations in the nonrelativistic hydrodynamic expansion with parameter $\epsilon$, the Petrov type I condition holds up to order of $\epsilon^2$. An interesting question is whether the solution of vacuum Einstein equations satisfies the Petrov type I condition to higher orders. It is found in [32] that the condition holds up to order $\epsilon^3$ and is broken at order $\epsilon^4$. However, those violated terms contain only the third order terms of the derivative expansion parameter $\partial$ if an improved frame is taken. This motivates us to check the Petrov type I condition for the solution of vacuum Einstein equations in the relativistic hydrodynamic expansion. It turns out that the condition indeed holds up to the second order of the derivative expansion parameter $\partial$, by using the vacuum solution available to this order in [20].

## II. PETROV TYPE I SPACETIME IN THE NONRELATIVISTIC HYDRODYNAMIC EXPANSION

Let us start with the $p+2$ dimensional Rindler metric

$$ds^2 = g^{(r)}_{\mu\nu}dx^\mu dx^\nu = -r d\tau^2 + 2d\tau dr + dx_i dx^i, \qquad (1)$$

where $x^\mu = (r, \tau, x^i)$, and $i = 1, 2, \ldots, p$. A spacetime is at least Petrov type I if for some choice of frame, $C_{(\ell)i(\ell)j} \equiv \ell^\mu \mathbf{m}_i{}^\alpha \ell^\nu \mathbf{m}_j{}^\beta C_{\mu\alpha\nu\beta} = 0$ at each point [33,34]. Here $\ell$, $\mathbf{k}$, $\mathbf{m}_i$ are the $p+2$ Newman-Penrose-like vector fields which obey $\ell_\mu \ell^\mu = \mathbf{k}_\mu \mathbf{k}^\mu = 0$, $\ell_\mu \mathbf{k}^\mu = 1$,

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$g_{\mu\nu}\mathbf{m}_i{}^\mu\mathbf{m}_j{}^\mu = \delta_{ij}$ and all other products vanish. One can show that the whole Rindler spacetime (1) is Petrov type I with the frame chosen as [23]

$$\mathbf{m}_i = \partial_i, \qquad \sqrt{2}\boldsymbol{\ell} = \partial_0 - \mathbf{n}, \qquad \sqrt{2}\mathbf{k} = -\partial_0 - \mathbf{n}, \quad (2)$$

where $\partial_0 = \partial_\tau/\sqrt{r}$ and $\mathbf{n} = \sqrt{r}\partial_r + \partial_0$.

On a timelike hypersurface $\Sigma_c$ at $r = r_c$ with a flat induced metric $\gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b = -r_c\mathrm{d}\tau^2 + \mathrm{d}x_i\mathrm{d}x^i$, one can define the $p+1$ velocity $u^a = \gamma_v(1, v^i)$, where $\gamma_v$ is fixed through $\gamma_{ab}u^a u^b = -1$. Introducing the other parameter $P$ and regarding $v^i$ and $P$ as slowly varying functions of $x^a = (\tau, x^i)$, one can consider the perturbations of the metric (1) in nonrelativistic hydrodynamic limit [11,16] that $v_i \sim \partial_i \sim \epsilon$, $P \sim \partial_\tau \sim \epsilon^2$. The solution of vacuum Einstein equations to an arbitrary order of $\epsilon$ can be constructed through keeping the induced metric flat and demanding the regularity on the horizon [19].

In order to check whether the solution to higher orders in [19] is Petrov type I or not, we consider a frame by adding higher order corrections to the zeroth order frame (2) as

$$\begin{aligned}
\sqrt{2}\boldsymbol{\ell} &= \partial_0 - \mathbf{n}' + \boldsymbol{\ell}_{(\epsilon)} + \boldsymbol{\ell}_{(\epsilon^2)} + O(\epsilon^3),\\
\sqrt{2}\mathbf{k} &= -\partial_0 - \mathbf{n}' + \mathbf{k}_{(\epsilon)} + \mathbf{k}_{(\epsilon^2)} + O(\epsilon^3),\\
\mathbf{m}_1 &= \mathbf{m}'_1 + \mathbf{m}_{1(\epsilon)} + \mathbf{m}_{1(\epsilon^2)} + O(\epsilon^3),\\
\mathbf{m}_{i'} &= \partial_{i'} + \mathbf{m}_{i'(\epsilon)} + \mathbf{m}_{i'(\epsilon^2)} + O(\epsilon^3),
\end{aligned} \quad (3)$$

where $i', j' = 2, \ldots, p$, and the two zeroth order normalized spatial vectors are $\mathbf{n}' = (\sin\theta)\mathbf{n} - (\cos\theta)\mathbf{m}_1$, $\mathbf{m}'_1 = (\cos\theta)\mathbf{n} + (\sin\theta)\mathbf{m}_1$. As there exists the rotational symmetry among the $\mathbf{m}_i$ vectors, this choice does not lose any generality. Putting them and the Wely tensors of the spacetime with higher order corrections [19] into $C_{(\boldsymbol{\ell})i(\boldsymbol{\ell})j}$, we find that up to $\epsilon^2$,

$$\begin{aligned}
4C_{(\boldsymbol{\ell})1(\boldsymbol{\ell})1} &= r^{-1}(\sin\theta - 1)^2\partial_1 v_1,\\
4C_{(\boldsymbol{\ell})1(\boldsymbol{\ell})i'} &= [r^{-1}(\sin\theta - 1)^2 - 3r_c^{-1}(\sin^2\theta - 1)]\partial_{[1}v_{i']},\\
4C_{(\boldsymbol{\ell})i'(\boldsymbol{\ell})j'} &= r^{-1}(\sin\theta - 1)^2\partial_{(i'}v_{j')}.
\end{aligned} \quad (4)$$

If demanding $C_{(\boldsymbol{\ell})i(\boldsymbol{\ell})j}$ vanishes at this order, $\sin\theta = 1$ is the only consistent solution, which just gives the frame at the zeroth order (2). Taking into account of this, the relevant possible choice of the first order corrections in (3) is $\boldsymbol{\ell}^\tau_{(\epsilon)} = 0$, $\boldsymbol{\ell}^i_{(\epsilon)} = \lambda_{\boldsymbol{\ell}}\sqrt{r}v^i$, $\mathbf{m}_{i(\epsilon)}{}^\tau = \lambda_m v_i$, where $\lambda_m$ and $\lambda_{\boldsymbol{\ell}}$ are arbitrary functions of $r$ and $r_c$. On the other hand, the orthogonal normalization condition of the vectors up to the first order of $\epsilon$ gives constraints that $\mathbf{m}_{i(\epsilon)}{}^j = 0$ and $\mathbf{m}_{i(\epsilon)}{}^\tau - v_i/r_c = \delta_{ij}\boldsymbol{\ell}^i_{(\epsilon)}$. Putting them together we find that the nonvanishing terms in $C_{(\boldsymbol{\ell})i(\boldsymbol{\ell})j}$ first appear at order $\epsilon^4$,

$$\begin{aligned}
4C_{(\boldsymbol{\ell})i(\boldsymbol{\ell})j} = {}& \lambda_{\boldsymbol{\ell}} r_c^{-1} r[6\lambda_{\boldsymbol{\ell}} v^k\omega_{k(i}v_{j)} + 2v_{(i}\partial^2 v_{j)} - 4v^k\partial_{(i}\omega_{j)k}]\\
& + r_c^{-1} r\partial^2\partial_{(i}v_{j)} + O(\epsilon^5).
\end{aligned} \quad (5)$$

As all these terms in (5) are independent and only one free parameter $\lambda_{\boldsymbol{\ell}}$ is left, it is impossible to make $C_{(\boldsymbol{\ell})i(\boldsymbol{\ell})j}$ in (5) vanish at $\epsilon^4$ for any choice of $\lambda_{\boldsymbol{\ell}}$. We may need to consider the possible higher order corrections to the velocity and pressure like $v_i \to v_i + \delta v_{i(\epsilon^3)}$, $P \to P + \delta P_{(\epsilon^4)}$, but these corrections can be absorbed into the arbitrary functions $F_i^{(\epsilon^3)}$ and $F_\tau^{(\epsilon^4)}$ in the metric [19], which do not make any contribution to $C_{(\boldsymbol{\ell})i(\boldsymbol{\ell})j}$ up to $\epsilon^4$.

Notice that by setting $\lambda_{\boldsymbol{\ell}} = -r^{-1}$ and taking $r \to r_c$, one can recover the results in [32] that Petrov type I condition is broken at $\epsilon^4$, unless some additional physical conditions, such as the irrotational condition, are added. In particular, if setting $\lambda_{\boldsymbol{\ell}} = 0$ in (5), only the term $\partial^2\partial_{(i}v_{j)}$ with three derivatives is left. This seemingly implies that the Petrov type I condition will be violated at the third order $\partial^3$ of the derivative expansion. As no explicit solution of vacuum Einstein equations is available up to $\partial^3$ in the literature, therefore we are here not able to show whether the Petrov type I condition holds at the third order and even arbitrary higher orders, although it is certainly of great interest to see this. In the following section, we will only consider the Petrov type I condition of the solution of vacuum Einstein equations up to the second order in the derivative expansion.

## III. PETROV TYPE I SPACETIME IN THE RELATIVISTIC HYDRODYNAMIC EXPANSION

Introduce the parameter $\mathbb{p} = (r_c - r_h)^{-1/2}$, which will turn out to be the pressure of the dual fluid, and $r_h$ is the location of the Rindler horizon of the equilibrium solution. Then keeping the induced metric flat and demanding the regularity on the horizon, regarding $u^a$ and $\mathbb{p}$ as two slowly varying functions of $x^a$, one can obtain the solution of vacuum Einstein equations to an arbitrary order by using the derivative expansion. Up to the second order, the solution can be written as [20]

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -2\mathbb{p}u_a\mathrm{d}x^a\mathrm{d}r + g_{ab}\mathrm{d}x^a\mathrm{d}x^b, \quad (6)$$

where $g_{ab} = g^{(0)}_{ab} + g^{(1)}_{ab} + g^{(2)}_{ab}$,

$$\begin{aligned}
g^{(0)}_{ab} &= -\mathbb{p}^2(r - r_c)u_a u_b + \gamma_{ab},\\
g^{(1)}_{ab} &= 2\mathbb{p}(r - r_c)(u^c\partial_c \ln\mathbb{p}u_a u_b + 2a_{(a}u_{b)}),\\
g^{(2)}_{ab} &= 2(r - r_c)[(\mathcal{K}_{cd}\mathcal{K}^{cd})u_a u_b - 2u_{(a}h^c_{b)}\partial_d\mathcal{K}^d{}_c\\
&\quad - \mathcal{K}_a{}^c\mathcal{K}_{cb} + 2\mathcal{K}_{c(a}\Omega^c{}_{b)} - 2h^c_a h^d_b u^e\partial_e\mathcal{K}_{cd}]\\
&\quad + \mathbb{p}^2(r - r_c)^2\bigg\{\left(\frac{1}{2}\mathcal{K}_{cd}\mathcal{K}^{cd} + a_c a^c\right)u_a u_b\\
&\quad + 2u_{(a}h^c_{b)}[\partial_d\mathcal{K}^d{}_c - (\mathcal{K}_{cd} + \Omega_{cd})a^d] - \Omega_{ac}\Omega^c{}_b\bigg\}\\
&\quad + \mathbb{p}^4(r - r_c)^3\left(\frac{1}{2}\Omega_{cd}\Omega^{cd}\right)u_a u_b.
\end{aligned} \quad (7)$$

Here the transverse projector $h^a_b = \gamma^a_b + u^a u_b$, tensors $\mathcal{K}_{ab} = h^c_a h^d_b \partial_{(c} u_{d)}$, $\Omega_{ab} = h^c_a h^d_b \partial_{[c} u_{d]}$, acceleration $a^a = u^b \partial_b u^a$. And the constraint equations are

$$\begin{aligned}\partial_a u^a &= 2\mathbb{p}^{-1}\mathcal{K}_{ab}\mathcal{K}^{ab} + O(\partial^3),\\ a_a + h^b_a \partial_b \ln \mathbb{p} &= 2\mathbb{p}^{-1} h^c_a \partial_b \mathcal{K}^b_c + O(\partial^3).\end{aligned} \tag{8}$$

Notice that $h^a_b$ can also be decomposed as $m_i{}^a m^i{}_b$, where

$$m_i{}^a = \delta_i{}^a + r_c^{-1/2} u_i \delta^a_\tau + (1 + r_c^{1/2}\gamma_v)^{-1} u_i u^j \delta^a_j, \tag{9}$$

$a, b, \ldots$ and $i, j, \ldots$ indices are raised (lowered) by $\gamma_{ab}$ and $\delta_{ij}$, respectively. Denote $\mathbf{n}$ being the spacelike unit normal of constant $r$ hypersurface, $\mathbf{u}$ being the normalized $p+2$ velocity, and $\mathbf{m}_i$ being the remaining orthonormal spatial vectors. One then has $g^{\mu\nu} = \mathbf{n}^\mu \mathbf{n}^\nu - \mathbf{u}^\mu \mathbf{u}^\nu + \delta^{ij}\mathbf{m}_i{}^\mu \mathbf{m}_j{}^\nu$, where $\mathbf{n} = \mathbf{n}^r \partial_r + \mathbf{n}^a \partial_a$, $\mathbf{u} = \mathbf{u}^a \partial_a$, $\mathbf{m}_i = \mathbf{m}_i{}^a \partial_a$, and

$$\begin{aligned}\mathbf{n}^r &= \mathbb{p}^{-1}[1 + \mathbb{p}(r - r_c)(\mathbb{p} - 2u^c \partial_c \ln \mathbb{p})\\ &\quad + (-g^{(2)}_{cd} + g^{(1)}_{ac} g^{(1)}_{bd} h^{ab}) u^c u^d]^{1/2},\\ \mathbf{n}^a &= (\mathbb{p}\mathbf{n}^r)^{-1}[u^a + 2\mathbb{p}(r - r_c)a^a + g^{(2)}_{bc} u^b h^{ca}],\\ \mathbf{u}^a &= \mathbf{n}^a, \qquad \mathbf{m}_i{}^a = m_i{}^a - \frac{1}{2} m_i{}^b g^{(2)}_{bc} h^{ca}.\end{aligned} \tag{10}$$

Further one can construct the two null vectors as

$$\sqrt{2}\boldsymbol{\ell}^\mu = -\mathbf{n}^\mu + \mathbf{u}^\mu, \qquad \sqrt{2}\mathbf{k}^\mu = -\mathbf{n}^\mu - \mathbf{u}^\mu, \tag{11}$$

which obey $\boldsymbol{\ell}_\mu \mathbf{k}^\mu = 1$ and all other products with $\mathbf{m}_i{}^\mu$ vanish. Along with the condition $g_{\mu\nu}\mathbf{m}^\mu_i \mathbf{m}^\nu_j = \delta_{ij}$ up to order $\partial^2$, one can obtain the $p+2$ Newman-Penrose-like vector fields $\boldsymbol{\ell}, \mathbf{k}, \mathbf{m}_i$ such that

$$g_{\mu\nu} = \boldsymbol{\ell}_\mu \mathbf{k}_\nu + \boldsymbol{\ell}_\nu \mathbf{k}_\mu + \delta_{ij}\mathbf{m}^i{}_\mu \mathbf{m}^j{}_\nu. \tag{12}$$

In this frame, $\sqrt{2}\boldsymbol{\ell} = \mathbf{n}^r \partial_r$ leads to the expression

$$\mathrm{P}^{(r)}_{ij} \equiv 2C_{(\boldsymbol{\ell})i(\boldsymbol{\ell})j} = \mathbf{m}_i{}^a \mathbf{m}_j{}^b \mathbb{P}^{(r)}_{ab}, \tag{13}$$

where $\mathbb{P}^{(r)}_{ab} \equiv \mathbf{n}^r h^c_a \mathbf{n}^r h^d_b C_{rcrd}$. With the metric (6), we find

$$\mathbb{P}^{(r)}_{ab} = -(\mathbf{n}^r)^2\left(\frac{1}{2} h^c_a h^d_b \partial^2_r g^{(2)}_{cd} + \mathbb{p}^2 \Omega_{ac}\Omega^c{}_b\right) + O(\partial^3), \tag{14}$$

and considering $g^{(2)}_{ab}$ in (7), we conclude $\mathbb{P}^{(r)}_{ab} = O(\partial^3)$, which also indicates $\mathrm{P}^{(r)}_{ij} = O(\partial^3)$. As a result, we have shown that the solution (6) of vacuum Einstein equations is Petrov type I at each point up to the second order $\partial^2$ in the derivative expansion.

## IV. PETROV TYPE I CONDITION ON THE CUTOFF SURFACE

We can project the Weyl tensor on the hypersurface $\Sigma_c$ and define $\mathrm{P}_{ij} \equiv 2C_{(\boldsymbol{\ell})i(\boldsymbol{\ell})j}|_{\Sigma_c}$. In [23], $\mathrm{P}_{ij} = 0$ is named as Petrov type I condition and $\mathrm{P}_{ij}$ can be rewritten in terms of the extrinsic curvature $K_{ab}$ of $\Sigma_c$ by employing the Gauss-Codazzi equations. Notice that $K_{ab}$ can be expressed in terms of the Brown-York stress tensor through $T_{ab} = 2(K\gamma_{ab} - K_{ab})$. We have $\mathrm{P}_{ij} = m_i{}^a m_j{}^b \mathbb{P}_{ab}$ where

$$\begin{aligned}4\mathbb{P}_{ab} &= h^m_a h^n_b[(T_{mc}T_{nd} - T_{mn}T_{cd})u^c u^d - T_{mc}T^c{}_n\\ &\quad - 4u^c \partial_c T_{mn} + 4u^c \partial_{(m} T_{n)c}]\\ &\quad + p^{-2}[T(T + pT_{cd}u^c u^d) + 4pu^c \partial_c T]h_{ab}.\end{aligned} \tag{15}$$

With the bulk metric in (6), the dual stress tensor can be expanded in the following form:

$$T_{ab} = T^{(0)}_{ab} + T^{(1)}_{ab} + T^{(2)}_{ab} + O(\partial^3), \tag{16}$$

and these terms are obtained in [20] as

$$\begin{aligned}T^{(0)}_{ab} &= \mathbb{p}h_{ab},\\ T^{(1)}_{ab} &= \zeta'(u^c \partial_c \ln \mathbb{p})u_a u_b - 2\eta\mathcal{K}_{ab},\\ T^{(2)}_{ab} &= \mathbb{p}^{-1}\{[d_1\mathcal{K}_{ab}\mathcal{K}^{ab} + d_2\Omega_{ab}\Omega^{ab} + d_3(u^c\partial_c \ln \mathbb{p})^2\\ &\quad + d_4 u^c \partial_c(u^d \partial_d \ln \mathbb{p}) + d_5 h^{cd}(\partial_c \ln \mathbb{p})(\partial_d \ln \mathbb{p})]u_a u_b\\ &\quad + [c_1\mathcal{K}_{ac}\mathcal{K}^c{}_b + c_2\mathcal{K}_{c(a}\Omega^c{}_{b)} + c_3\Omega_{ac}\Omega^c{}_b\\ &\quad + c_4 h^c_a h^d_b \partial_c \partial_d \ln \mathbb{p} + c_5\mathcal{K}_{ab}(u^c \partial_c \ln \mathbb{p})\\ &\quad + c_6(h^c_a \partial_c \ln \mathbb{p})(h^d_b \partial_d \ln \mathbb{p})]\}.\end{aligned} \tag{17}$$

Here the first and second order transport coefficients are

$$\begin{aligned}&\zeta' = 0, \qquad \eta = 1,\\ &d_1 = -2, \qquad d_2 = d_3 = d_4 = d_5 = 0,\\ &c_1 = -2, \qquad c_2 = c_3 = c_4 = c_5 = -c_6 = -4.\end{aligned} \tag{18}$$

The momentum constraint $2G_{\mu b}\mathbf{n}^\mu|_{\Sigma_c} = 0$, which leads to the conservation of the stress tensor $\partial^a T_{ab} = 0$, gives the constraint equations (8), while the Hamiltonian constraint $2G_{\mu\nu}\mathbf{n}^\mu \mathbf{n}^\nu|_{\Sigma_c} = 0$ leads to $4\mathbb{H} \equiv pT_{ab}T^{ab} - T^2 = 0$, which can be viewed as the equation of state for the dual fluid. In addition, one can show that the trace of the stress tensor satisfies $T = p\mathbb{p} + O(\partial^3)$. Putting the stress tensor (16) into the expression (15), we then obtain $\mathbb{P}_{ab} = O(\partial^3)$, which of course implies $\mathrm{P}_{ij} = O(\partial^3)$. Thus we have shown again that the Petrov type I condition $\mathrm{P}_{ij} = 0$ is satisfied up to $\partial^2$ by using the stress tensor of the dual relativistic fluid.

## V. FROM PETROV TYPE I CONDITION TO DUAL RELATIVISTIC FLUID

In this subsection we turn the logic around. Assuming the Hamiltonian constraint and Petrov type I condition on a finite cutoff surface, we will show that the stress tensor of the dual fluid can be fixed up to the second order of the derivative expansion, without using the details of the bulk metric. The resulting stress tensor exactly matches the one from the solution of vacuum Einstein equations.

Firstly, one can introduce an undetermined symmetric stress tensor $\hat{T}_{ab}$, and it satisfies $h_a^b \hat{T}_{bc} u^c = 0$, where $u^a$ is regarded as the relativistic fluid velocity. Then the stress tensor can be decomposed as $\hat{T}_{ab} = \mathbb{e} u_a u_b + \Pi_{ab}$, where

$$\mathbb{e} \equiv \hat{T}_{ab} u^a u^b, \qquad \Pi_{ab} \equiv h_a^c h_b^d \hat{T}_{cd}. \tag{19}$$

The Hamiltonian constraint becomes $\mathbb{H} = 0$, where

$$4\mathbb{H} \equiv p(\mathbb{e}^2 + \Pi_{ab}\Pi^{ab}) - \hat{T}^2, \tag{20}$$

and $\hat{T} = -\mathbb{e} + \Pi_{ab} h^{ab}$. The Petrov type I condition can be generalized as $\mathbb{P}_{ab} = 0$, where

$$\begin{aligned} 4\mathbb{P}_{ab} \equiv & -\mathbb{e}\Pi_{ab} - \Pi_{ac}\Pi^c{}_b - 4h_a^c h_b^d (u^e \partial_e \Pi_{cd}) - 4\Pi_{(a}{}^c h_{b)}^d \partial_d u_c \\ & - 4\mathbb{e}\mathcal{K}_{ab} + p^{-2}[\hat{T}(\hat{T} + p\mathbb{e}) + 4pu^c \partial_c \hat{T}] h_{ab}. \end{aligned} \tag{21}$$

Expanding the stress tensor in terms of the derivative expansion parameter $\partial$ as

$$\begin{aligned} \mathbb{e} &= \mathbb{e}^{(0)} + \mathbb{e}^{(1)} + \mathbb{e}^{(2)} + O(\partial^3), \\ \Pi_{ab} &= \Pi_{ab}^{(0)} + \Pi_{ab}^{(1)} + \Pi_{ab}^{(2)} + O(\partial^3), \end{aligned} \tag{22}$$

and we identify $\mathbb{e}^{(0)} = 0$, $\Pi_{ab}^{(0)} = \mathbb{p} h_{ab}$ from the zeroth order Brown-York stress tensor in (17). Then through

$$\mathbb{H}^{(1)} = 0 \Rightarrow \mathbb{e}^{(1)} = 0, \tag{23}$$

$$\mathbb{P}_{ab}^{(1)} = 0 \Rightarrow \Pi_{ab}^{(1)} = -2\mathcal{K}_{ab}, \tag{24}$$

we can fix the stress tensor at the first order. With these,

$$\mathbb{H}^{(2)} = 0 \Rightarrow \mathbb{e}^{(2)} = -2\mathbb{p}^{-1}\mathcal{K}_{ab}\mathcal{K}^{ab}, \tag{25}$$

$$\begin{aligned} \mathbb{P}_{ab}^{(2)} = 0 \Rightarrow \Pi_{ab}^{(2)} & \\ = \mathbb{p}^{-1}[ & -2\mathcal{K}_{ac}\mathcal{K}^c{}_b - 4\mathcal{K}_{c(a}\Omega^c{}_{b)} - 4\Omega_{ac}\Omega^c{}_b \\ & - 4h_a^c h_b^d \partial_c \partial_d \ln \mathbb{p} - 4\mathcal{K}_{ab}(u^c \partial_c \ln \mathbb{p}) \\ & + 4(h_a^c \partial_c \ln \mathbb{p})(h_b^d \partial_d \ln \mathbb{p})], \end{aligned} \tag{26}$$

we can then fix the second order terms in the stress tensor. In the above procedure we have chosen the isotropy gauge that there is no higher order corrections to the pressure $\mathbb{p}$.

Thus, up to the second order, we obtain the total stress tensor of the dual relativistic fluid as

$$\hat{T}_{ab} = \mathbb{e}^{(2)} u_a u_b + \mathbb{p} h_{ab} + \Pi_{ab}^{(1)} + \Pi_{ab}^{(2)}. \tag{27}$$

It is identical to the Brown-York stress tensor in (16) which is calculated from the whole metric (6).

## VI. NEAR-HORIZON EXPANSION

The relativistic hydrodynamic expansion can also be expressed in terms of the so-called alternative near-horizon expansion [20]. First take a Weyl rescaling $\mathrm{d}s^2 \to \lambda^2 \mathrm{d}s^2$, where the scaling parameter $\lambda$ is related to the cutoff $r_c$ as $\lambda = r_c^{1/2}$, then consider the relativistic hydrodynamic limit $\tilde{x}^a = \lambda x^a$ and the rescaled metric $\mathrm{d}\tilde{s}^2 = \lambda^2 \mathrm{d}s^2$, we can reach the metric in the near-horizon expansion with parameter $\lambda$ as

$$\begin{aligned} \mathrm{d}\tilde{s}^2 = \tilde{g}_{\mu\nu}\mathrm{d}\tilde{x}^\mu \mathrm{d}\tilde{x}^\nu = & -2\lambda^1 \tilde{\mathbb{p}} \tilde{u}_a \mathrm{d}\tilde{x}^a \mathrm{d}\tilde{r} \\ & + (\tilde{g}_{ab}^{(0)} + \lambda^1 \tilde{g}_{ab}^{(1)} + \lambda^2 \tilde{g}_{ab}^{(2)}) \mathrm{d}\tilde{x}^a \mathrm{d}\tilde{x}^b, \end{aligned} \tag{28}$$

where $\tilde{g}_{ab}^{(0)}, \tilde{g}_{ab}^{(1)}, \tilde{g}_{ab}^{(2)}$ are just obtained from (7) by mapping $(r_c, r, \mathbb{p}, u^a) \to (\tilde{r}_c, \tilde{r}, \tilde{\mathbb{p}}, \tilde{u}^a)$, and setting $\tilde{r}_c = 1$. With similar operation on the dual stress tensor in (17), the stress tensor $\tilde{T}_{ab}\mathrm{d}\tilde{x}^a \mathrm{d}\tilde{x}^b = \lambda^2 T_{ab}\mathrm{d}x^a \mathrm{d}x^b$ can be expressed as $\tilde{T}_{ab} = \tilde{T}_{ab}^{(0)} + \lambda^1 \tilde{T}_{ab}^{(1)} + \lambda^2 \tilde{T}_{ab}^{(2)}$. Then all the previous discussions can be redone in the near-horizon expansion formulism. In particular, the dynamic equations $\partial^{\tilde{a}} \tilde{T}_{ab}^{(0)} = 0$ for a perfect relativistic fluid appear as an attractor, when $\lambda \to 0$.

## VII. HIGHER CURVATURE GRAVITY

For asymptotically flat spacetime in higher curvature gravity, the effect of the Gauss-Bonnet term with coefficient $\alpha$ is studied in [35,36]. With the solutions found there, we find that $\mathrm{P}_{ij}^{(r)} = O(\partial^3)$, because the correction to the metric from the Gauss-Bonnet term appears only at order $\partial^2$, and the factor in front of the relevant terms $h_a^c h_b^d \delta g_{cd}^{(2)} \propto \alpha(r - r_c)$, the latter will not make any contribution to (14) up to order $\partial^2$. Furthermore the dual stress tensor whose second order transport coefficients with the Gauss-Bonnet term correction can also be recovered through the Petrov type I condition in the same way as in the present paper [37].

## VIII. WITH A NEGATIVE COSMOLOGICAL CONSTANT

In this case, the solution of Einstein equations will be asymptotically AdS [38–42], and we find that $\mathrm{P}_{ij}^{(r)} = g^{rr}\mathbf{m}_i^a \mathbf{m}_j^b C_{\mathrm{rarb}} \sim O(\partial)$ under a similar frame as that in this paper. However, notice that the near-horizon limit $g^{rr} \to 0$

leads to $P_{ij}^{(r)} \to 0$, which indicates a close relation between the Petrov type I condition and the membrane paradigm. In particular, the ratio of shear viscosity over entropy density, $\eta/s|_{r_h} = [1-2(p+1)(p-2)\alpha]/4\pi$ at the horizon, can also be extracted through imposing Petrov type I condition directly [37]. Here $\alpha$ is the Gauss-Bonnet coefficient. And higher order transport coefficients can also be obtained. In addition, a so-called AdS/Ricci flat correspondence has been proposed recently in [43,44], which can map asymptotically AdS black brane solutions [10] to asymptotically flat solutions [20], and the dual stress tenor of Rindler fluid (16) can be obtained exactly from the one of AdS fluid up to second order in derivative expansion. Thus, it would be interesting to see whether there exists a corresponding condition in the whole AdS or more general spacetime [45,46].

To summarize, we have shown that the whole spacetime is Petrov type I for the solution of vacuum Einstein equations in the nonrelativistic hydrodynamic expansion up to the third order $\epsilon^3$, but it is violated at $\epsilon^4$ unless some additional condition is imposed [32]. While in the relativistic hydrodynamic expansion, it holds at least up to the second order $\partial^2$. As no explicit solution of vacuum Einstein equations is available up to $\partial^3$ in the literature, we are here not able to show whether the whole spacetime is Petrov type I at the third order and even arbitrary higher orders in the derivative expansion. However, we can go a further step. The solution of vacuum Einstein equations up to $\epsilon^4$ in the nonrelativistic hydrodynamic expansion can be captured by that in the relativistic hydrodynamic expansion up to $\partial^3$ [19]. If the whole spacetime is Petrov type I at order $\partial^3$, it will also be Petrov type I at $\epsilon^4$ in the nonrelativistic hydrodynamic expansion. Our calculation in the nonrelativistic expansion indicates that in general, the Petrov type I condition will be violated at the third order $\partial^3$ of the relativistic hydrodynamical expansion parameter.

Turn the logic around, we have shown that imposing the Petrov type I condition and Hamiltonian constraint on a finite cutoff surface, the stress tensor of the dual relativistic fluid can be fixed up to the second order of the derivative expansion. The resulting stress tensor identically matches the one calculated from the solution of vacuum Einstein equations. As pointed out in [23], the Petrov type I condition is expected to be equivalent to the regularity condition on the future horizon of the spacetime, and it gives the constraint on the dual theory from gravity. We have indeed shown that imposing the Petrov type I condition is mathematically much simpler than imposing the regularity requirement, because one no longer needs to solve the perturbation equations of bulk gravity. Notice that the boundary condition on the horizon has to be imposed for the perturbations in the gravity/fluid duality, we therefore conclude that the Petrov type I condition would indeed play an important role in this aspect.

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[1] T. Damour, Phys. Rev. D **18**, 3598 (1978); R. H. Price and K. S. Thorne, Phys. Rev. D **33**, 915 (1986).
[2] P. Kovtun, D. T. Son, and A. O. Starinets, J. High Energy Phys. 10 (2003) 064.
[3] E. Gourgoulhon and J. L. Jaramillo, Phys. Rep. **423**, 159 (2006).
[4] C. Eling, I. Fouxon, and Y. Oz, Phys. Lett. B **680**, 496 (2009).
[5] C. Eling and Y. Oz, J. High Energy Phys. 02 (2010) 069.
[6] G. Policastro, D. T. Son, and A. O. Starinets, J. High Energy Phys. 09 (2002) 043.
[7] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, J. High Energy Phys. 02 (2008) 045.
[8] M. Van Raamsdonk, J. High Energy Phys. 05 (2008) 106.
[9] M. Haack and A. Yarom, J. High Energy Phys. 10 (2008) 063.
[10] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, and A. Sharma, J. High Energy Phys. 12 (2008) 116.
[11] S. Bhattacharyya, S. Minwalla, and S. R. Wadia, J. High Energy Phys. 08 (2009) 059.
[12] N. Iqbal and H. Liu, Phys. Rev. D **79**, 025023 (2009).
[13] D. Nickel and D. T. Son, New J. Phys. **13**, 075010 (2011).
[14] C. Eling and Y. Oz, Phys. Rev. Lett. **107**, 201602 (2011).
[15] I. Bredberg, C. Keeler, V. Lysov, and A. Strominger, J. High Energy Phys. 03 (2011) 141.
[16] I. Bredberg, C. Keeler, V. Lysov, and A. Strominger, J. High Energy Phys. 07 (2012) 146.
[17] I. Bredberg and A. Strominger, J. High Energy Phys. 05 (2012) 043.

[18] D. Anninos, T. Anous, I. Bredberg, and G. S. Ng, J. High Energy Phys. 05 (2012) 107.
[19] G. Compere, P. McFadden, K. Skenderis, and M. Taylor, J. High Energy Phys. 07 (2011) 050.
[20] G. Compere, P. McFadden, K. Skenderis, and M. Taylor, J. High Energy Phys. 03 (2012) 076.
[21] C. Eling, A. Meyer, and Y. Oz, J. High Energy Phys. 05 (2012) 116.
[22] A. Meyer and Y. Oz, J. High Energy Phys. 07 (2013) 090.
[23] V. Lysov and A. Strominger, arXiv:1104.5502.
[24] T.-Z. Huang, Y. Ling, W.-J. Pan, Y. Tian, and X.-N. Wu, J. High Energy Phys. 10 (2011) 079.
[25] T.-Z. Huang, Y. Ling, W.-J. Pan, Y. Tian, and X.-N. Wu, Phys. Rev. D **85**, 123531 (2012).
[26] C.-Y. Zhang, Y. Ling, C. Niu, Y. Tian, and X.-N. Wu, Phys. Rev. D **86**, 084043 (2012).
[27] X. Wu, Y. Ling, Y. Tian, and C. Zhang, Classical Quantum Gravity **30**, 145012 (2013).
[28] B. Wu and L. Zhao, Nucl. Phys. **B874**, 177 (2013).
[29] Y. Ling, C. Niu, Y. Tian, X.-N. Wu, and W. Zhang, arXiv:1306.5633.
[30] V. Lysov, arXiv:1310.4181.
[31] B. Wu and L. Zhao, Classical Quantum Gravity **31**, 105018 (2014).
[32] R.-G. Cai, L. Li, Q. Yang, and Y.-L. Zhang, J. High Energy Phys. 04 (2013) 118.
[33] A. Coley, R. Milson, V. Pravda, and A. Pravdova, Classical Quantum Gravity **21**, L35 (2004).
[34] A. Coley, Classical Quantum Gravity **25**, 033001 (2008).
[35] G. Chirco, C. Eling, and S. Liberati, J. High Energy Phys. 08 (2011) 009.
[36] C. Eling, A. Meyer, and Y. Oz, J. High Energy Phys. 08 (2012) 088.
[37] R.-G. Cai, Q. Yang, and Y.-L. Zhang (to be published).
[38] R.-G. Cai, L. Li, and Y.-L. Zhang, J. High Energy Phys. 07 (2011) 027.
[39] S. Kuperstein and A. Mukhopadhyay, J. High Energy Phys. 11 (2011) 130.
[40] D. Brattan, J. Camps, R. Loganayagam, and M. Rangamani, J. High Energy Phys. 12 (2011) 090.
[41] Y. Matsuo, M. Natsuume, M. Ohta, and T. Okamura, Prog. Theor. Exp. Phys. **2013**, 23B01 (2013).
[42] S. Kuperstein and A. Mukhopadhyay, J. High Energy Phys. 11 (2013) 086.
[43] M. M. Caldarelli, J. Camps, B. Goutéraux, and K. Skenderis, Phys. Rev. D **87**, 061502 (2013).
[44] M. M. Caldarelli, J. Camps, B. Goutéraux, and K. Skenderis, J. High Energy Phys. 04 (2014) 071.
[45] R.-G. Cai, L. Li, Z.-Y. Nie, and Y.-L. Zhang, Nucl. Phys. **B864**, 260 (2012).
[46] N. Pinzani-Fokeeva and M. Taylor, arXiv:1401.5975.