The structure of completely positive matrices according to their CP-rank and CP-plus-rank

Immanuel Bomze* Peter J.C. Dickinson† Georg Still‡

November 4, 2014

Abstract

We study the topological properties of the cp-rank operator $\mathrm{cp}(A)$ and the related cp-plus-rank operator $\mathrm{cp}^+(A)$ (which is introduced in this paper) in the set $\mathcal{S}^n$ of symmetric $n\times n$-matrices. For the set of completely positive matrices, $\mathcal{CP}^n$, we show that for any fixed p the set of matrices A satisfying $\mathrm{cp}(A) = \mathrm{cp}^+(A) = p$ is open in $\mathcal{S}^n \setminus \mathrm{bd}\,(\mathcal{CP}^n)$. By making use of the Perron-Frobenius vector we also prove that the set $\mathcal{A}^n$ of matrices with $\mathrm{cp}(A) = \mathrm{cp}^+(A)$ is dense in $\mathcal{S}^n$. By applying the theory of semi-algebraic sets we are able to show that membership in $\mathcal{A}^n$ is even a generic property. We furthermore answer several questions on the existence of matrices satisfying $\mathrm{cp}(A) = \mathrm{cp}^+(A)$ or $\mathrm{cp}(A) \neq \mathrm{cp}^+(A)$, and comment on genericity of having infinitely many minimal cp-decompositions.

1 Introduction

We define a symmetric matrix A to be completely positive if there exists nonnegative vectors $\mathbf{b}_1, \ldots, \mathbf{b}_p$ such that $A = \sum_{i=1}^p \mathbf{b}_i \mathbf{b}_i^T$. The set of completely positive matrices forms a proper cone, i.e. a cone which is closed, convex, pointed and full-dimensional. This cone plays an important role in the field of copositive optimisation (see, e.g., [Dür10, Bom12, Bur12]).

In this paper we investigate the cp- and cp-plus-ranks of matrices, which are closely related to complete positivity. These are defined below, where we let $\mathcal{S}^n$ be the set of symmetric $n \times n$ matrices, $\mathbb{N}$ be the set of nonnegative integers, $\mathbb{R}^n_+$ be the set of nonnegative real n-vectors and $\mathbb{R}^n_{++}$ be the set of strictly positive real n-vectors:

Definition 1.1. For $A \in \mathcal{S}^n$ we define its cp-rank and its cp-plus-rank respectively as:

$$\mathrm{cp}(A) := \min\left\{p \in \mathbb{N} \;\middle|\; \exists \mathbf{b}_1, \ldots \mathbf{b}_p \in \mathbb{R}^n_+ \text{ s.t. } A = \textstyle\sum_{i=1}^p \mathbf{b}_i \mathbf{b}_i^\mathsf{T}\right\},$$

$$\mathrm{cp}^+(A) := \min\left\{p \in \mathbb{N} \;\middle|\; \exists \mathbf{b}_1, \ldots \mathbf{b}_p \in \mathbb{R}^n_{++} \text{ s.t. } A = \textstyle\sum_{i=1}^p \mathbf{b}_i \mathbf{b}_i^\mathsf{T}\right\}.$$

*University of Vienna, Department of Statistics and Operations Research, 1090 Vienna, Austria. Email: immanuel.bomze@univie.ac.at

†University of Groningen, University of Vienna, and University of Twente, Dep. Appl. Mathematics, P.O. Box 217, 7500 AE Enschede, The Netherlands. Email: p.j.c.dickinson@utwente.nl

‡University of Twente, Dep. Appl. Mathematics, P.O. Box 217, 7500 AE Enschede, The Netherlands. Email: g.still@math.utwente.nl

Note that a matrix $A \in \mathcal{S}^n$ is completely positive if and only if $\mathrm{cp}(A) < \infty$.

One motivation for the study of the cp-plus-rank is given by the following theorem, where $\mathcal{CP}^n$ denotes the cone of completely positive matrices of order n, $\mathrm{int}\,(\mathcal{CP}^n)$ denotes its interior, and $\mathrm{rank}(A)$ denotes the standard linear rank of the matrix A:

Theorem 1.2. *For $A \in \mathcal{S}^n$ we have*

$$A \in \mathrm{int}\,(\mathcal{CP}^n) \quad \Longleftrightarrow \quad \mathrm{cp}^+(A) < \infty \;\; \text{and} \;\; \mathrm{rank}(A) = n.$$

Proof. This comes from [Dic10, Theorem 3.8], after noting that for any matrix $B \in \mathbb{R}^{m\times n}$ we have $\mathrm{rank}(B^\mathsf{T} B) = \mathrm{rank}(B)$. □

Another point of interest is when $\mathrm{cp}(M) = \mathrm{cp}^+(M)$. We then have the following two properties, where for $\varepsilon > 0$ and $M \in \mathcal{S}^n$ we define $N_\varepsilon(M) = \{X \in \mathcal{S}^n \mid \|M - X\| \le \varepsilon\}$, and for a matrix $A = (a_{ij}) \in \mathcal{S}^n$, by $\|A\|$ we mean the Frobenius norm, *i.e.*, $\|A\| = \sqrt{\sum_{i,j} a_{ij}^2}$:

Theorem 1.3. *Consider $M \in \mathcal{S}^n$ such that $2 \le \mathrm{cp}(M) = \mathrm{cp}^+(M) < \infty$. Then M has infinitely many minimal cp-decompositions, where a minimal cp-decomposition is a set $\{\mathbf{b}_1, \dots, \mathbf{b}_p\} \subseteq \mathbb{R}^n_+$ such that $p = \mathrm{cp}(M)$ and $M = \sum_{i=1}^p \mathbf{b}_i \mathbf{b}_i^\mathsf{T}$.*

Proof. This will follow directly from Lemma 2.10. □

Theorem 1.4. *Consider $M \in \mathrm{int}\,(\mathcal{CP}^n)$ such that $\mathrm{cp}(M) = \mathrm{cp}^+(M)$. Then there exists $\varepsilon > 0$ such that for all $X \in N_\varepsilon(M)$ we have $\mathrm{cp}(X) = \mathrm{cp}^+(X) = \mathrm{cp}(M)$.*

Proof. This will be shown in Theorem 2.7. □

The aim of the present paper is to study the topological properties of the functions $\mathrm{cp}(M)$ and $\mathrm{cp}^+(M)$.

In Section 2 we will look at some basic preliminary results on these ranks. In Section 3 we show how orthogonal matrices can be used in considering them. In Section 4 properties of the rank functions are analysed by using Perron-Frobenius vectors. In Section 5 we are interested in properties of the maximum cp- and cp-plus-ranks. Finally, in Section 6 of this paper we shall show that membership in the set $\{M \in \mathcal{S}^n \mid \mathrm{cp}(M) = \mathrm{cp}^+(M)\}$ is a generic property. This yields

Theorem 1.5. *The following properties are generic within the completely positive cone:*

1. *Having infinitely many minimal cp-decompositions,*
2. *The cp- and cp-plus- ranks being equal and locally constant.*

Proof. This will be shown in Corollary 6.8. □

Notation

In this paper we shall always consider n to be an integer which is strictly greater than one. In addition to the notation introduced earlier in this section, we shall let $\mathbb{R}^n$ denote the set of real n-vectors; $\mathcal{S}^n_+$ the set of positive semidefinite matrices of order n; $\mathcal{N}^n$ the set of nonnegative symmetric matrics of order n; and $\mathrm{bd}\,(\mathcal{CP}^n)$ the boundary of the set of completely positive matrices. For a vector $\mathbf{a} \in \mathbb{R}^n$, whenever we mention a norm we mean the Euclidean norm, *i.e.*, $\|\mathbf{a}\| = \sqrt{\sum_i a_i^2}$.

2 Preliminary results

In this section we shall consider some basic results connected to the cp- and cp-plus-ranks. We start with the following three trivial results.

Lemma 2.1. *For all $M \in \mathcal{S}^n$, we have* $\mathrm{cp}^+(M) \geq \mathrm{cp}(M) \geq \mathit{rank}(M)$.

Lemma 2.2. *If $M \in \mathcal{S}^n \setminus \{0\}$ such that $\mathrm{cp}^+(M)$ is finite, then $M \in \mathrm{int}\,(\mathcal{N}^n) \cap \mathcal{CP}^n$.*

Lemma 2.3. *For all $A, B \in \mathcal{S}^n$ and $\alpha, \beta > 0$ we have*

$$\mathrm{cp}(\alpha A + \beta B) \leq \mathrm{cp}(A) + \mathrm{cp}(B).$$

We shall now consider how the cp- and cp-plus-ranks vary in a neighbourhood of a matrix $M \in \mathcal{S}^n$.

Theorem 2.4. *Let $M \in \mathcal{S}^n$. Then there exists $\varepsilon > 0$ such that $\mathrm{cp}(P) \geq \mathrm{cp}(M)$ for all $P \in N_\varepsilon(M)$.*

Proof. This was shown in [SMBJS13, Proposition 2.4]. □

A similar result also holds for the cp-plus-rank, although with the inequality reversed.

Theorem 2.5. *Let $M \in \mathcal{S}^n \setminus \mathrm{bd}\,(\mathcal{CP}^n)$. Then there exists $\varepsilon > 0$ such that $\mathrm{cp}^+(P) \leq \mathrm{cp}^+(M)$ for all $P \in N_\varepsilon(M)$.*

Proof. If $M \notin \mathcal{CP}^n$ then there exists $\varepsilon > 0$ such that for all $P \in N_\varepsilon(M)$ we have $P \notin \mathcal{CP}^n$, and thus $\mathrm{cp}^+(P) = \infty = \mathrm{cp}^+(M)$.

If $M \in \mathrm{int}\,(\mathcal{CP}^n)$ then the result comes directly from considering the proof of [DS08, Theorem 2.3]. □

Remark 2.6. The result of the previous theorem does not in general hold when $M \in \mathrm{bd}\,(\mathcal{CP}^n)$. For example, if $M \in \mathrm{bd}\,(\mathcal{CP}^n)$ such that $\mathrm{cp}^+(M) \neq \infty$, then for all $\varepsilon > 0$ there exists $P \in N_\varepsilon(M) \setminus \mathcal{CP}^n$ and thus $\mathrm{cp}^+(P) = \infty > \mathrm{cp}^+(M)$.

Combining Lemma 2.1 and Theorems 2.4 and 2.5, we get the following result.

Theorem 2.7. *Let $M \in \mathcal{S}^n \setminus \mathrm{bd}\,(\mathcal{CP}^n)$ such that $\mathrm{cp}(M) = \mathrm{cp}^+(M) = p$. Then there exists $\varepsilon > 0$ such that $\mathrm{cp}^+(P) = \mathrm{cp}(P) = p$ for all $P \in N_\varepsilon(M)$.*

Corollary 2.8. *The following sets are open for all $p \in \mathbb{N}$:*

$$\left\{M \in \mathcal{S}^n \setminus \mathrm{bd}\,(\mathcal{CP}^n) \;\middle|\; \mathrm{cp}(M) = \mathrm{cp}^+(M) = p\right\},$$
$$\left\{M \in \mathcal{S}^n \setminus \mathrm{bd}\,(\mathcal{CP}^n) \;\middle|\; \mathrm{cp}(M) = \mathrm{cp}^+(M)\right\}.$$

We finish this section by considering some equivalent definitions of the cp- and cp-plus-ranks, which will be used regularly throughout the paper.

We begin with the following trivial result:

Lemma 2.9. *For all $A \in \mathcal{S}^n \setminus \{0\}$ we have*

$$\mathrm{cp}(A) = \min\left\{p \in \mathbb{N} \;\middle|\; \exists B \in \mathbb{R}^{p\times n}_+ \textit{ s.t. } A = B^\mathsf{T} B\right\},$$

$$\mathrm{cp}^+(A) = \min\left\{p \in \mathbb{N} \;\middle|\; \exists B \in \mathbb{R}^{p\times n}_{++} \textit{ s.t. } A = B^\mathsf{T} B\right\}.$$

Proof. This comes from noting that if we have a matrix $B \in \mathbb{R}^{p\times n}$ whose rows are given by $\mathbf{b}_1^\mathsf{T}, \dots, \mathbf{b}_p^\mathsf{T}$ then $B^\mathsf{T} B = \sum_{i=1}^p \mathbf{b}_i \mathbf{b}_i^\mathsf{T}$. □

We now consider another equivalent definition of the cp-plus-rank which is less trivial. These results come from [Dic13, Lemma 7.13] and its proof.

Lemma 2.10. *Consider $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, and for all $\theta \in \mathbb{R}$ let $\mathbf{c}_\theta = \mathbf{a}\sin\theta + \mathbf{b}\cos\theta$ and $\mathbf{d}_\theta = \mathbf{a}\cos\theta - \mathbf{b}\sin\theta$. Then we have:*

1. $\mathbf{a}\mathbf{a}^\mathsf{T} + \mathbf{b}\mathbf{b}^\mathsf{T} = \mathbf{c}_\theta \mathbf{c}_\theta^\mathsf{T} + \mathbf{d}_\theta \mathbf{d}_\theta^\mathsf{T}$ *for all* $\theta \in \mathbb{R}$, *and*
2. *if* $\mathbf{a} \in \mathbb{R}^n_{++}$ *and* $\mathbf{b} \in \mathbb{R}^n_+$ *then there exists* $\Theta > 0$ *such that* $\mathbf{c}_\theta, \mathbf{d}_\theta \in \mathbb{R}^n_{++}$ *for all* $\theta \in (0, \Theta]$.

Corollary 2.11. *For $A \in \mathcal{S}^n \setminus \{0\}$, we have*

$$\operatorname{cp}^+(A) = \min\left\{p \in \mathbb{N} \;\middle|\; \exists \mathbf{b}_1, \dots \mathbf{b}_p \in \mathbb{R}^n_+ \text{ s.t. } \mathbf{b}_1 \in \mathbb{R}^n_{++} \text{ and } A = \textstyle\sum_{i=1}^p \mathbf{b}_i \mathbf{b}_i^\mathsf{T}\right\}.$$

This result leads to an inequality for the function cp-plus-rank similar to Lemma 2.3, but note that here we have a mixture of cp- and cp-plus-ranks.

Corollary 2.12. *For all $A, B \in \mathcal{S}^n \setminus \{0\}$ and $\alpha, \beta > 0$ we have*

$$\operatorname{cp}^+(\alpha A + \beta B) \le \operatorname{cp}^+(A) + \operatorname{cp}(B).$$

3 Orthogonal matrices

The concept of cp-plus-rank connects to orthogonal matrices through the following lemma.

Lemma 3.1. *Let $A, B \in \mathbb{R}^{p\times n}$. Then $A^\mathsf{T} A = B^\mathsf{T} B$ if and only if there exists an orthogonal matrix $Q \in \mathbb{R}^{p\times p}$ such that $A = QB$.*

Proof. The reverse implication (which we will need below) is trivial. The forward implication is a well known result in linear algebra, and a sketch of the proof is presented in [Xu04, Lemma 1]. □

In the paper [SSMS13] the authors considered matrices $B \in \mathbb{R}^{p\times n}_+$ and defined such matrices to be *nearly positive* if there exist orthogonal matrices $\{Q_i \mid i \in \mathbb{N}\}$ such that $Q_i B > 0$ for all i and $\lim_{i\to\infty} Q_i = I$ (where I is the identity matrix). Using the lemma above we then get the following sufficient condition for when the cp-rank of a matrix is equal to its cp-plus rank.

Corollary 3.2. *Let $X \in \mathcal{CP}^n$ with $\operatorname{cp}(X) = p$, and let $B \in \mathbb{R}^{p\times n}_+$ such that $X = B^\mathsf{T} B$. If B is a nearly positive matrix, then $\operatorname{cp}^+(X) = p$.*

In [SSMS13, Example 7.4] it was shown that the reverse implication to this does not hold. In that example, for $n \ge 4$, they considered a family of $M = A^\mathsf{T} A \in \operatorname{int}(\mathcal{CP}^n)$, with $A \in \mathbb{R}^{n\times n}_+$ not being a nearly positive matrix, but $\operatorname{cp}^+(M) = \operatorname{cp}(M) = \operatorname{rank}(M) = n$.

In [SSMS13] the authors looked at many interesting results on nearly positive matrices, including the following:

Theorem 3.3. *Let $X \in \mathcal{CP}^n \cap \operatorname{int}(\mathcal{N}^n)$ and let $B \in \mathbb{R}^{p\times n}_+$ such that $X = B^\mathsf{T} B$. If either $n \le 3$ or $p \le 2$ (or both) then B is nearly positive.*

Translating this result for the cp-plus-rank we get the following corollary.

Corollary 3.4. *Let $X \in \mathcal{CP}^n \cap \operatorname{int}(\mathcal{N}^n)$. If $n \le 3$ or $\operatorname{cp}(X) \le 2$ (or both) then $\operatorname{cp}(X) = \operatorname{cp}^+(X)$.*

4 Perron-Frobenius Vectors

In this section we shall analyse the cp- and cp-plus-ranks using the theory of Perron-Frobenius vectors. We begin by recalling some basic definitions and results on Perron-Frobenius vectors, applied to matrices in $\mathcal{N}^n \setminus \{0\}$.

Theorem 4.1. *Let $M \in \mathcal{N}^n \setminus \{0\}$. Then there exists $\lambda \in \mathbb{R}_{++}$ such that:*

1. *λ is an eigenvalue of M,*

2. *the absolute values of all eigenvalues of M are less than or equal to λ,*

3. *there is an eigenvector $\mathbf{x} \in \mathbb{R}^n_+$, with $\|\mathbf{x}\| = 1$, corresponding to the eigenvalue λ. We refer to this as a* Perron-Frobenius *(P-F) vector of M.*

Furthermore, if $M \in \operatorname{int}(\mathcal{N}^n)$, then for λ and $\mathbf{x}$ given above we have:

4. *the absolute values of all eigenvalues of M, excluding λ, are strictly less than λ,*

5. *$\mathbf{x} \in \mathbb{R}^n_{++}$, and $\mathbf{x}$ is the unique eigenvector of M corresponding to λ, up to multiplication by a scalar (i.e. the eigenvalue λ has multiplicity one). We shall denote this eigenvector by $\mathbf{x}_M$.*

Remark 4.2. Note that any matrix $M \in \operatorname{int}(\mathcal{CP}^n)$ satisfies $M \in \operatorname{int}(\mathcal{N}^n)$. Also note that if in the theorem above we have $M \notin \operatorname{int}(\mathcal{N}^n)$, then we do not necessarily have a unique P-F vector. For example, consider M being equal to the identity matrix.

We now recall the following well known lemma on eigenvectors and eigenvalues.

Lemma 4.3. *Consider a matrix $A \in \mathcal{S}^n$ with eigenvectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$, whose corresponding eigenvalues are $\lambda_1, \lambda_2 \in \mathbb{R}$. If $\lambda_1 \neq \lambda_2$ then $\mathbf{x}_1^\mathsf{T}\mathbf{x}_2 = 0$.*

Proof. This comes from noting that $\lambda_1\mathbf{x}_1^\mathsf{T}\mathbf{x}_2 = \mathbf{x}_1^\mathsf{T}A\mathbf{x}_2 = \lambda_2\mathbf{x}_1^\mathsf{T}\mathbf{x}_2$. □

From this we then get the following result on P-F vectors.

Lemma 4.4. *Consider $M \in \mathcal{N}^n \setminus \{0\}$ and let $\mathbf{x} \in \mathbb{R}^n_{++}$ be an eigenvector of M such that $\|\mathbf{x}\| = 1$. Then $\mathbf{x}$ is a P-F vector of M.*

Proof. Assume for the sake of contradiction that the eigenvector $\mathbf{x}$ with corresponding eigenvalue μ is not a P-F vector. Then there exists a P-F vector $\mathbf{y} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ with eigenvalue $\lambda > \mu$. By Lemma 4.3 it would follow $\mathbf{y}^\mathsf{T}\mathbf{x} = 0$, a contradiction to $\mathbf{0} \neq \mathbf{y} \in \mathbb{R}^n_+$, $\mathbf{x} \in \mathbb{R}^n_{++}$. □

Another well known lemma on eigenvectors is the following.

Lemma 4.5. *Consider $P, Q \in \mathcal{S}^n$ and $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $P = Q + \mu\mathbf{x}\mathbf{x}^\mathsf{T}$ for some $\mu \in \mathbb{R}$. Then $\mathbf{x}$ is an eigenvector of P if and only if it is an eigenvector of Q.*

Proof. Without loss of generality let $\|\mathbf{x}\| = 1$. Then we have

$$Q\mathbf{x} = \lambda\mathbf{x} \quad \Leftrightarrow \quad P\mathbf{x} = (\lambda + \mu)\mathbf{x}.$$ □

Now combining Lemmas 4.4 and 4.5 we get the following result:

Lemma 4.6. *Consider $P, Q \in \mathcal{N}^n \setminus \{0\}$ and $\mathbf{x} \in \mathbb{R}^n_{++}$ such that $P = Q + \mu \mathbf{x}\mathbf{x}^\mathsf{T}$ for some $\mu \in \mathbb{R}$. Then $\mathbf{x}$ is a P-F vector of P if and only if it is a P-F vector of Q.*

We will now look at what P-F vectors can tell us about the cp- and cp-plus-ranks. In order to do this for $M \in \operatorname{int}(\mathcal{N}^n)$ and $\mu \in \mathbb{R}$ we let

$$M(\mu) := M + \mu \mathbf{x}_M \mathbf{x}_M^\mathsf{T}.$$

Note from Theorem 4.1 that this is well defined. Furthermore, from the definition, we have that $\|M - M(\mu)\| = |\mu|$ and thus $M(\mu) \in N_{|\mu|}(M)$. We also note the following basic result.

Lemma 4.7. *Let $M, P \in \operatorname{int}(\mathcal{N}^n)$ and $\mu \in \mathbb{R}$ such that $P = M(\mu)$. Then we have $M = P(-\mu)$.*

Proof. This comes directly from Theorem 4.1 and Lemma 4.6. □

We are now ready to present the main results of this section.

Theorem 4.8. *For all $M \in \operatorname{int}(\mathcal{N}^n)$ and all $\mu > 0$ we have $\operatorname{cp}(M) \geq \operatorname{cp}^+(M(\mu))$.*

Proof. This proof is an adaptation of one from [SMBJS13].

In this proof we will in fact prove the more general result that considers $M \in \mathcal{N}^n \setminus \{0\}$ with a P-F vector $\mathbf{x} \in \mathbb{R}^n_{++}$. Under these circumstances, we have

$$\operatorname{cp}(M) \geq \operatorname{cp}^+(M + \mu \mathbf{x}\mathbf{x}^\mathsf{T}) \qquad \text{for all } \mu > 0.$$

Indeed, if $M \notin \mathcal{CP}^n$ then we have $\operatorname{cp}(M) = \infty$ and the result is trivial. From now on we assume $M \in \mathcal{CP}^n \setminus \{0\}$ and consider an arbitrary $\mu > 0$.

Letting $p = \operatorname{cp}(M) \in (0, \infty)$, there exists $V \in \mathbb{R}^{p \times n}_+$ such that $M = V^\mathsf{T} V$. All rows of V are nonzero, otherwise $\operatorname{cp}(M) < p$. Therefore, letting $\mathbf{y} = V\mathbf{x}$, we have $\mathbf{y} \in \mathbb{R}^p_{++}$.

As $\mathbf{x}$ is a P-F vector of M, there exists $\lambda > 0$ such that $\lambda \mathbf{x} = M\mathbf{x} = V^\mathsf{T} V \mathbf{x} = V^\mathsf{T} \mathbf{y}$. We thus have that $\mathbf{y}^\mathsf{T}\mathbf{y} = \mathbf{x}^\mathsf{T} V^\mathsf{T} \mathbf{y} = \lambda \mathbf{x}^\mathsf{T}\mathbf{x} = \lambda$.

The proof is now completed by letting $\nu = \sqrt{1 + (\mu/\lambda)} - 1 > 0$, noting that we have $(V + \nu \mathbf{y}\mathbf{x}^\mathsf{T}) \in \mathbb{R}^{p \times n}_{++}$ and considering the following:

$$\begin{aligned}
(V + \nu \mathbf{y}\mathbf{x}^\mathsf{T})^\mathsf{T}(V + \nu \mathbf{y}\mathbf{x}^\mathsf{T}) &= V^\mathsf{T} V + \nu(V^\mathsf{T}\mathbf{y}\mathbf{x}^\mathsf{T} + \mathbf{x}\mathbf{y}^\mathsf{T} V) + \nu^2 \mathbf{x}\mathbf{y}^\mathsf{T}\mathbf{y}\mathbf{x}^\mathsf{T} \\
&= M + \nu\lambda(2 + \nu)\mathbf{x}\mathbf{x}^\mathsf{T} \\
&= M + \mu \mathbf{x}\mathbf{x}^\mathsf{T}.
\end{aligned}$$

□

Theorem 4.9. *Consider $M \in \operatorname{int}(\mathcal{CP}^n)$ with $p := \operatorname{cp}(M) \leq \operatorname{cp}^+(M) =: q$. Then there exists $\widehat{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \widehat{\varepsilon}]$ we have*

$$\operatorname{cp}(M(\varepsilon)) = \operatorname{cp}^+(M(\varepsilon)) = p \qquad \textit{and} \qquad \operatorname{cp}(M(-\varepsilon)) = \operatorname{cp}^+(M(-\varepsilon)) = q.$$

Proof. From Theorem 2.4, there exists $\varepsilon_+ > 0$ such that for all $\varepsilon \in (0, \varepsilon_+]$ we have $p \leq \operatorname{cp}(M(\varepsilon))$, and from Theorem 4.8 we have $\operatorname{cp}^+(M(\varepsilon)) \leq p$. We now note from Lemma 2.1 that $\operatorname{cp}(M(\varepsilon)) \leq \operatorname{cp}^+(M(\varepsilon))$, and combining these three inequalities together we get $\operatorname{cp}(M(\varepsilon)) = \operatorname{cp}^+(M(\varepsilon)) = p$ for all $\varepsilon \in (0, \varepsilon_+]$.

Similarly, from Theorem 2.5, there exists $\varepsilon_- > 0$ such that for all $\varepsilon \in (0, \varepsilon_-]$ we have $q \geq \operatorname{cp}^+(M(-\varepsilon))$. For such ε, as the cp-plus-rank is finite, we have $M(-\varepsilon) \in \operatorname{int}(\mathcal{N}^n)$, and thus from Lemma 4.7 and Theorem 4.8 we have $\operatorname{cp}(M(-\varepsilon)) \geq q$. We now note from Lemma 2.1 that $\operatorname{cp}^+(M(-\varepsilon)) \geq \operatorname{cp}(M(-\varepsilon))$, and combining these three inequalities together we get $\operatorname{cp}(M(-\varepsilon)) = \operatorname{cp}^+(M(-\varepsilon)) = q$ for all $\varepsilon \in (0, \varepsilon_-]$.

Now letting $\widehat{\varepsilon} = \min\{\varepsilon_-, \varepsilon_+\}$, this completes the proof. □

5 Maximum cp- and cp-plus-ranks

Let us define the numbers

$$p_n := \max\{\mathrm{cp}(M) \mid M \in \mathcal{CP}^n\},$$
$$p_n^+ := \max\{\mathrm{cp}^+(M) \mid \mathrm{cp}^+(M) < \infty\}.$$

In the following theorem we collect some known results on these numbers, along with a couple of new ones.

Theorem 5.1. *We have that*

$$\begin{aligned} p_n &= \max\{\mathrm{cp}(M) \mid M \in \mathrm{bd}\,(\mathcal{CP}^n)\}, &(1)\\ &= \max\{\mathrm{cp}(M) \mid M \in \mathrm{int}\,(\mathcal{CP}^n)\} &(2)\\ &= \max\{\mathrm{cp}^+(M) \mid M \in \mathrm{int}\,(\mathcal{CP}^n)\} &(3)\end{aligned}$$

$$p_n \le p_n^+ \le p_n + 1 \tag{4}$$

$$p_n = n \qquad \textit{for all } n = 2, 3, 4, \tag{5}$$

$$p_n \ge \lfloor n^2/4 \rfloor > n \qquad \textit{for all } n \ge 5, \tag{6}$$

$$p_n \ge \tfrac{1}{2}n(n+1) - 4 - \sqrt{2}n^{\frac{3}{2}} + \tfrac{3}{2}n > \lfloor n^2/4 \rfloor \qquad \textit{for all } n \ge 15, \tag{7}$$

$$p_n \le \tfrac{1}{2}n(n+1) - 4 \qquad \textit{for all } n \ge 5, \tag{8}$$

$$\begin{aligned}&\textit{for all } k \in \{1, \dots, n-1\}\ \exists M \in \mathcal{CP}^n \textit{ s.t. } \mathrm{cp}(M) = \mathrm{cp}^+(M) = k\\ &\qquad \textit{and we have } M \in \mathrm{bd}\,(\mathcal{CP}^n)\,, &(9)\end{aligned}$$

$$\textit{for all } k \in \{n, \dots, p_n\}\ \exists M \in \mathrm{int}\,(\mathcal{CP}^n)\ \textit{ s.t. } \mathrm{cp}(M) = \mathrm{cp}^+(M) = k, \tag{10}$$

$$\textit{for all } k \in \{n+1, \dots, p_n\}\ \exists M \in \mathrm{int}\,(\mathcal{CP}^n)\ \textit{ s.t. } k-1 = \mathrm{cp}(M) \ne \mathrm{cp}^+(M) = k. \tag{11}$$

Proof. (1) and (2) were proven in [SMBJS13], and (3) follows directly from (2) and Theorem 4.9. The leftmost inequality in (4) follows from (3). To prove the rightmost inequality in (4), we consider an arbitrary $M \in \mathcal{S}^n \setminus \{0\}$ such that $\mathrm{cp}^+(M) < \infty$. From the definitions, there exists $\mathbf{v} \in \mathbb{R}^n_{++}$ such that $M - \mathbf{v}\mathbf{v}^\mathsf{T} \in \mathcal{CP}^n \setminus \{0\}$ and from Corollary 2.12 we have $\mathrm{cp}^+(M) \le \mathrm{cp}^+(\mathbf{v}\mathbf{v}^\mathsf{T}) + \mathrm{cp}(M - \mathbf{v}\mathbf{v}^\mathsf{T}) \le 1 + p_n$.

While (5) and (6) are well known since long, see for example [BSM03], the bounds in (7) and (8) were established quite recently, namely in [SMBB⁺13] and in [BSU14b]. For $n = 5$ we have $p_n = \lfloor n^2/4 \rfloor$ [SMBJS13]. It was conjectured in [DJL94] that this equality holds for all $n \ge 5$, however counter examples to this conjecture for $n = 7, \dots, 11$ were recently presented in [BSU14a]. For $n \ge 15$ this conjecture is refuted by (7), and for $n = 12, 13, 14$ tighter lower bounds also refute it [BSU14b].

We shall now prove (9), (10) and (11). From Theorem 1.2 and Lemma 2.1, if $\mathrm{cp}(M) < n$ then $M \in \mathrm{bd}\,(\mathcal{CP}^n)$. From (2), (5), (6) and Theorem 4.9, there exists $M \in \mathrm{int}\,(\mathcal{CP}^n)$ such that $\mathrm{cp}(M) = \mathrm{cp}^+(M) = p_n \ge n$, and thus statement (10) holds for $k = p_n$. From Theorem 1.2 and using that $\mathrm{rank}(M) = \mathrm{rank}(B)$ holds for $M = B^\mathsf{T}B$, there exists $\mathbf{b}_1, \dots, \mathbf{b}_{p_n} \in \mathbb{R}^n_{++}$ such that $\mathrm{span}\{\mathbf{b}_1, \dots, \mathbf{b}_n\} = \mathbb{R}^n$ and $M = \sum_{i=1}^{p_n} \mathbf{b}_i\mathbf{b}_i^\mathsf{T}$. For all $k \in \{1, \dots, p_n\}$, $\theta \in [0, 1]$ we let $M_k(\theta) := \sum_{i=1}^{k-1} \mathbf{b}_i\mathbf{b}_i^\mathsf{T} + \theta\mathbf{b}_k\mathbf{b}_k^\mathsf{T}$. From Theorem 1.2 we have

$$\begin{aligned} M_k(\theta) &\in \mathrm{int}\,(\mathcal{CP}^n) \qquad \text{for all } k \in \{n, \dots, p_n\},\ \theta \in (0, 1],\\ M_k(\theta) &\in \mathrm{bd}\,(\mathcal{CP}^n) \qquad \text{for all } k \in \{1, \dots, n-1\},\ \theta \in [0, 1].\end{aligned}$$

Furthermore, for all $k \in \{1, \dots, p_n\}$, $\theta \in [0,1]$ we have $M = M_k(\theta) + (1-\theta)\mathbf{b}_k\mathbf{b}_k^\mathsf{T} + \sum_{i=k+1}^{p_n} \mathbf{b}_i\mathbf{b}_i^\mathsf{T}$, and thus by Lemma 2.3 we have

$$\begin{aligned} p_n = \mathrm{cp}(M) &\le \mathrm{cp}(M_k(\theta)) + \mathrm{cp}\left((1-\theta)\mathbf{b}_k\mathbf{b}_k^\mathsf{T} + \textstyle\sum_{i=k+1}^{p_n}\mathbf{b}_i\mathbf{b}_i^\mathsf{T}\right) \\ &\le \mathrm{cp}(M_k(\theta)) + 1 + p_n - k. \end{aligned}$$

It is also trivial to see from the definitions that $\mathrm{cp}(M_k(\theta)) \le \mathrm{cp}^+(M_k(\theta)) \le k$. Combining these inequalities together, we get

$$k - 1 \le \mathrm{cp}(M_k(\theta)) \le \mathrm{cp}^+(M_k(\theta)) \le k \qquad \text{for all } k \in \{1, \dots, p_n\},\ \theta \in [0,1].$$

For all $k \in \{1, \dots, p_n - 1\}$ we have $M_{k+1}(0) = M_k(1)$ and thus using the above we get $\mathrm{cp}(M_k(1)) = \mathrm{cp}^+(M_k(1)) = k$, which completes the proof for statements (9) and (10). Similar arguments can also be found in [SMBB+13, Prop.4.1, Thm.4.1].

For an arbitrary $k \in \{n+1, \dots, p_n\}$, we now let $\vartheta_k = \sup_{\theta\in[0,1]}\{\theta \mid \mathrm{cp}(M_k(\theta)) = k-1\}$ and note by Corollary 2.8 that $0 < \vartheta_k < 1$. For all $\theta \in (\vartheta_k, 1]$ we have $k = \mathrm{cp}(M_k(\theta)) = \mathrm{cp}^+(M_k(\theta))$. Therefore, by Theorem 2.5, we have $k \le \mathrm{cp}^+(M_k(\vartheta_k))$, and thus $\mathrm{cp}^+(M_k(\vartheta_k)) = k$. Additionally, for all $\varepsilon > 0$ there exists $\theta \in [\vartheta_k - \varepsilon, \vartheta_k]$ such that $k - 1 = \mathrm{cp}(M_k(\theta))$. Therefore, by Theorem 2.4, we have $k - 1 \ge \mathrm{cp}(M_k(\vartheta_k))$, and thus $\mathrm{cp}(M_k(\vartheta_k)) = k - 1$, which completes the proof. □

From the following lemma we get $p_n = p_n^+$ for $n = 2, 3, 4$. It is an open question whether this equality continues to hold for $n \ge 5$.

Lemma 5.2. *For $n = 2, 3, 4$ let $M \in \mathcal{CP}^n \cap \mathrm{int}\,(\mathcal{N}^n)$. Then $\mathrm{cp}^+(M) \le p_n = n$.*

Proof. From Corollary 3.4, for $n = 2, 3$ we already have $\mathrm{cp}(M) = \mathrm{cp}^+(M)$. However, the following proof will be for a general $n = 2, 3, 4$, as nothing is lost in doing this.

We begin by recalling that for $n = 2, 3, 4$ we have $\mathcal{CP}^n = \mathcal{S}_+^n \cap \mathcal{N}^n$ and thus $\mathcal{CP}^n \cap \mathrm{int}\,(\mathcal{N}^n) = \mathcal{S}_+^n \cap \mathrm{int}\,(\mathcal{N}^n)$, see [MM62].

Let $M \in \mathcal{CP}^n \cap \mathrm{int}\,(\mathcal{N}^n)$, with P-F vector $\mathbf{x}$. For $\varepsilon > 0$ small enough we have $P = (M - \varepsilon\mathbf{x}\mathbf{x}^\mathsf{T}) \in \mathrm{int}\,(\mathcal{N}^n)$ and thus from Lemma 4.6, $\mathbf{x}$ is also the P-F vector of P. When going from M to P, the only eigenvalue that we are affecting is the eigenvalue corresponding to $\mathbf{x}$, which remains strictly positive. Therefore we have $P \in \mathcal{S}_+^n$. This implies that $P \in \mathcal{CP}^n$, and thus $\mathrm{cp}(P) \le p_n$. Finally, from Theorem 4.8, we have $\mathrm{cp}^+(M) \le \mathrm{cp}(P) \le p_n$, completing the proof. □

For $n \ge 5$ this lemma no longer holds, consider for example the following:

Example. In [DS08, Example 2.2], the authors showed that the following matrix is on the boundary of the completely positive cone:

$$B = \begin{pmatrix} 8 & 5 & 1 & 1 & 5 \\ 5 & 8 & 5 & 1 & 1 \\ 1 & 5 & 8 & 5 & 1 \\ 1 & 1 & 5 & 8 & 5 \\ 5 & 1 & 1 & 5 & 8 \end{pmatrix}.$$

We have $\mathrm{rank}(B) = 5$, but $B \notin \mathrm{int}\left(\mathcal{CP}^5\right)$, and thus from Theorem 1.2 we have $\mathrm{cp}^+(B) = \infty$.

6 Genericity of the property $\mathrm{cp}^+(M) = \mathrm{cp}(M)$

6.1 Genericity vs. open and dense

In this section we consider the topological properties of the following set:

$$\mathcal{A}^n := \{M \in \mathcal{S}^n \mid \mathrm{cp}(M) = \mathrm{cp}^+(M)\} . \tag{12}$$

As usual, we say that a set $\mathcal{A} \subseteq \mathcal{S}^n$ is dense if for all $X \in \mathcal{S}^n$ and $\varepsilon > 0$ we have $N_\varepsilon(X) \cap \mathcal{A} \neq \emptyset$. From the results so far it follows that the set $\mathcal{A}^n$ contains an open and dense subset of $\mathcal{S}^n$:

Theorem 6.1. *The set $\mathscr{A}^n := \mathcal{A}^n \setminus \mathrm{bd}(\mathcal{CP}^n)$ is open and dense in $\mathcal{S}^n$.*

Proof. For an arbitrary $M \in \mathcal{S}^n$, we consider the following cases, which will complete the proof:

1. $M \notin \mathcal{CP}^n$: We have $M \in \mathscr{A}^n$, and as the set of completely positive matrices is closed, there exists $\varepsilon > 0$ such that $N_\varepsilon(M) \subseteq \mathcal{S}^n \setminus \mathcal{CP}^n \subseteq \mathscr{A}^n$.

2. $M \in \mathrm{bd}(\mathcal{CP}^n)$: We have $M \notin \mathscr{A}^n$, and for all $\varepsilon > 0$ there exists $M_\varepsilon \in N_\varepsilon(M)$ such that $M_\varepsilon \in \mathcal{S}^n \setminus \mathcal{CP}^n \subseteq \mathscr{A}^n$.

3. $M \in \mathrm{int}(\mathcal{CP}^n) \cap \mathscr{A}^n$: From Theorem 2.7, there exists $\varepsilon > 0$ such that $N_\varepsilon(M) \subseteq \mathscr{A}^n$.

4. $M \in \mathrm{int}(\mathcal{CP}^n) \setminus \mathscr{A}^n$: From Theorem 4.9, for all $\varepsilon > 0$ we have $N_\varepsilon(M) \cap \mathscr{A}^n \neq \emptyset$.

□

By this theorem we know that the set $\mathcal{A}^n$ contains an open and dense set. But it is well known that for a set $\mathcal{A} \subset \mathcal{S}^n$, being dense and open does not necessarily imply that the Lebesgue measure of $\mathcal{S}^n \setminus \mathcal{A}$, denoted $\mu_{\mathrm{L}}(\mathcal{S}^n \setminus \mathcal{A})$, is equal to zero. Indeed, the set of rational numbers, $\mathbb{Q} \subseteq \mathbb{R}$, is a well-known dense set with $\mu_{\mathrm{L}}(\mathbb{Q}) = 0$ [Bea04, p.133]. Considering approximations of measurable sets [Bea04, p.139], for all $\varepsilon > 0$ there exists an open set $\mathcal{A}_\varepsilon$ such that $\mathbb{Q} \subseteq \mathcal{A}_\varepsilon \subseteq \mathbb{R}$ and $\mu_{\mathrm{L}}(\mathcal{A}_\varepsilon) \leq \mu_{\mathrm{L}}(\mathbb{Q}) + \varepsilon = \varepsilon$. We then have that $\mathcal{A}_\varepsilon$ is an open and dense set in $\mathbb{R}$ with $\mu_{\mathrm{L}}(\mathbb{R} \setminus \mathcal{A}_\varepsilon) = \infty \neq 0$.

In what follows we wish to strengthen the statement of Theorem 6.1, and we will show that the membership in $\mathcal{A}^n$ is a generic property.

Recall that in topology for a subset $\mathcal{A} \subseteq \mathbb{R}^N$, we say that membership in $\mathcal{A}$ is generic in $\mathbb{R}^N$ if $\mathcal{A}$ contains a set $\mathcal{A}_0$ such that the following two statements hold:

1. the set $\mathcal{A}_0$ is open, and

2. the Lebesgue measure of $\mathbb{R}^N \setminus \mathcal{A}_0$ is equal to zero.

Statement (1) means that membership in $\mathcal{A}_0$ is stable for small variations. Statement (2) means that 'almost all' elements of $\mathbb{R}^N$ are in $\mathcal{A}_0$ (and thus also in $\mathcal{A}$).

In the next subsection we prove that indeed membership in $\mathcal{A}^n$ is a generic property.

6.2 Semi-algebraic sets

In order to show that membership in the set $\mathcal{A}^n$ is generic we make use of the theory of semi-algebraic sets and only need the density part of Theorem 6.1.

We prove that $\mathcal{A}^n$ is a semi-algebraic set and apply the fact that for a semi-algebraic set, being dense is a sufficient condition for membership in the set being generic. We note that similar arguments have been used recently to obtain genericity results in cone programming [BDL11].

The results on semi-algebraic sets will be stated for the space $\mathbb{R}^N$. The results can then be trivially applied to the space $\mathcal{S}^n \equiv \mathbb{R}^{(n+1)n/2}$.

We begin by recalling some preliminary definitions and results on semi-algebraic sets (see [BR90, GWdL76]).

Definition 6.2. A set $\mathcal{A} \subset \mathbb{R}^N$ is called semi-algebraic if it is given by a finite union of sets of the form

$$\{\mathbf{x} \in \mathbb{R}^N \mid p_i(\mathbf{x}) = 0 \text{ for all } i = 1, \dots, k, \quad q_j(\mathbf{x}) > 0 \text{ for all } j = 1, \dots, s\}$$

with $k, s \in \mathbb{N}$ and polynomial functions $p_i, q_j \in \mathbb{R}[\mathbf{x}]$.

Remark 6.3. Since $\{\mathbf{x} \mid p(\mathbf{x}) \geq 0\} = \{\mathbf{x} \mid p(\mathbf{x}) = 0\} \cup \{\mathbf{x} \mid p(\mathbf{x}) > 0\}$, also sets defined by polynomial inequalities $p(\mathbf{x}) \geq 0$ are semi-algebraic.

The following theorem states some well-known facts on semi-algebraic sets.

Theorem 6.4. *For $N, M \in \mathbb{N}$, consider semi-algebraic sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^N$ and a polynomial function $h : \mathbb{R}^N \to \mathbb{R}^M$ (e.g. a projection operator). Then the following sets are also semi-algebraic:*

$$\mathcal{A} \cup \mathcal{B}, \qquad \mathcal{A} \cap \mathcal{B}, \qquad \mathcal{A} \setminus \mathcal{B}, \qquad h(\mathcal{A}).$$

Proof. For a proof we refer to [BR90, Section 2.1–2.3]. □

We shall now show that the set $\mathcal{A}^n$ from (12) is a semi-algebraic set.

Lemma 6.5. *The set $\mathcal{A}^n$ from* (12) *is semi-algebraic.*

Proof. Recalling from the definition that the union of finitely many semi-algebraic sets is also semi-algebraic and recalling from Theorem 5.1 that we have $p_n < \frac{1}{2}n(n+1)$, it is sufficient to show that the following set is semi-algebraic for all $p \in \mathbb{N} \cup \{\infty\}$:

$$\mathcal{A}_p^n := \left\{ A \in \mathcal{S}^n \;\middle|\; \operatorname{cp}(A) = \operatorname{cp}^+(A) = p \right\}.$$

For all $p \in \mathbb{N}$ we have that the following sets are trivially semi-algebraic:

$$\mathcal{E} := \left\{ (X, V) \in \mathcal{S}^n \times \mathbb{R}^{p \times n} \;\middle|\; v_{ij} \geq 0 \text{ for all } i, j, \quad X = V^\mathsf{T} V \right\},$$
$$\mathcal{F} := \left\{ (X, V) \in \mathcal{S}^n \times \mathbb{R}^{p \times n} \;\middle|\; v_{ij} > 0 \text{ for all } i, j, \quad X = V^\mathsf{T} V \right\}.$$

From Theorem 6.4, the projections $\operatorname{proj}_X(\mathcal{E})$ and $\operatorname{proj}_X(\mathcal{F})$ are also semi-algebraic for all $p \in \mathbb{N}$; but obviously

$$\operatorname{proj}_X(\mathcal{E}) = \{X \in \mathcal{S}^n \mid \operatorname{cp}(X) \leq p\} \quad \text{and} \quad \operatorname{proj}_X(\mathcal{F}) = \{X \in \mathcal{S}^n \mid \operatorname{cp}^+(X) \leq p\}.$$

Therefore again considering Theorem 5.1 and Theorem 6.4, the following sets are semi-algebraic for all $p \in \mathbb{N}$:

$$\begin{aligned}
\{X \in \mathcal{S}^n \mid \mathrm{cp}(X) = \infty\} &= \mathcal{S}^n \setminus \{X \in \mathcal{S}^n \mid \mathrm{cp}(A) \le p_n\}\,, \\
\{X \in \mathcal{S}^n \mid \mathrm{cp}^+(X) = \infty\} &= \mathcal{S}^n \setminus \{X \in \mathcal{S}^n \mid \mathrm{cp}^+(A) \le p_n^+\}\,, \quad \text{as well as} \\
\{X \in \mathcal{S}^n \mid \mathrm{cp}(X) = p\} &= \{X \in \mathcal{S}^n \mid \mathrm{cp}(X) \le p\} \setminus \{X \in \mathcal{S}^n \mid \mathrm{cp}(X) \le p-1\}, \\
\{X \in \mathcal{S}^n \mid \mathrm{cp}^+(X) = p\} &= \{X \in \mathcal{S}^n \mid \mathrm{cp}^+(X) \le p\} \setminus \{X \in \mathcal{S}^n \mid \mathrm{cp}^+(X) \le p-1\}.
\end{aligned}$$

Since

$$\mathcal{A}_p^n = \{X \in \mathcal{S}^n \mid \mathrm{cp}(X) = p\} \cap \{X \in \mathcal{S}^n \mid \mathrm{cp}^+(X) = p\}\,,$$

this finally implies that also $\mathcal{A}_p^n$ is semi-algebraic for all $p \in \mathbb{N} \cup \{\infty\}$. □

We can also combine Theorem 6.4 with other well-known results to obtain the following which may be of general interest in algebraic geometry:

Theorem 6.6. *Let $\mathcal{A} \subseteq \mathbb{R}^N$ be a semi-algebraic set. Then the membership in $\mathcal{A}$ is generic if and only if $\mathcal{A}$ is dense in $\mathbb{R}^N$.*

Proof. The forward implication is trivial. To prove the reverse implication we make use of the following facts on semi-algebraic sets.

From [GWdL76, 2.7], we have that any semi-algebraic set $\mathcal{A} \subset \mathbb{R}^N$ admits a (stratification) partition $\mathcal{A} = \bigcup_{i=0}^{d} \mathcal{S}_i$ with some $d \in \mathbb{N}$ such that

1. $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ for $i \neq j$ and
2. the sets $\mathcal{S}_i$ are smooth manifolds of $\mathbb{R}^N$ of dimension i (or are empty).

It is a well-known result, see for example [GP74], that the manifolds of dimension N in $\mathbb{R}^N$ are precisely the open sets in $\mathbb{R}^N$. Furthermore, manifolds of dimension $k < N$ in $\mathbb{R}^N$ have Lebesgue measure zero (*cf., e.g.,* [GP74, p.45]).

We first consider the set $\mathcal{M} = \mathbb{R}^N \setminus \mathcal{A}$, and note from Theorem 6.4 that this set is semi-algebraic. So $\mathcal{M}$ allows a stratification $\mathcal{M} = \bigcup_{i=0}^{d} \mathcal{S}_i$ with some $d \in \mathbb{N}$. As $\mathcal{A}$ is dense, $\mathcal{M}$ cannot contain any open sets. This implies that for all $0 \le i \le d$ we have $\dim \mathcal{S}_i < N$ and thus $\mu_{\mathrm{L}}(\mathcal{S}_i) = 0$, implying:

$$0 \le \mu_{\mathrm{L}}(\mathcal{M}) = \mu_{\mathrm{L}}\left(\bigcup_{i=0}^{d} \mathcal{S}_i\right) \le \sum_{i=0}^{d} \mu_{\mathrm{L}}(\mathcal{S}_i) = 0.$$

Now we take the semi-algebraic set $\mathcal{A}$ and a stratification $\mathcal{A} = \bigcup_{i=0}^{q} \tilde{\mathcal{S}}_i$ with some $q \in \mathbb{N}$. We claim that the manifold $\tilde{\mathcal{S}}_q$ with (highest) dimension q must be of dimension $q = N$. So by the remark above, $\tilde{\mathcal{S}}_q$ must be an open set. Indeed, the condition $\dim \tilde{\mathcal{S}}_q < N$ would also imply $\mu_{\mathrm{L}}(\mathcal{A}) = 0$ and then

$$\mu_{\mathrm{L}}(\mathbb{R}^N) = \mu_{\mathrm{L}}\big((\mathbb{R}^N \setminus \mathcal{A}) \cup \mathcal{A}\big) \le \mu_{\mathrm{L}}\big((\mathbb{R}^N \setminus \mathcal{A})\big) + \mu_{\mathrm{L}}(\mathcal{A}) = 0,$$

a contradiction. Altogether we have shown that the set $\mathcal{A}$ contains the open set $\mathcal{A}_0 := \tilde{\mathcal{S}}_N$ with complement

$$\mathbb{R}^N \setminus \mathcal{A}_0 = \big(\mathcal{A} \setminus \tilde{\mathcal{S}}_N\big) \cup \big(\mathbb{R}^N \setminus \mathcal{A}\big) = \big(\bigcup_{i=0}^{N-1} \tilde{\mathcal{S}}_i\big) \cup \big(\mathbb{R}^N \setminus \mathcal{A}\big)$$

of Lebesgue measure

$$\mu_{\rm L}\big(\mathbb{R}^N \setminus \mathcal{A}_0\big) \leq \sum_{i=0}^{N-1} \mu_{\rm L}(\tilde{\mathcal{S}}_i) + \mu_{\rm L}\big(\mathbb{R}^N \setminus \mathcal{A}\big) = 0\ .$$

So membership in the set $\mathcal{A}$ is a generic property. □

We are now ready to present the main result of this section.

Theorem 6.7. *Membership in the set $\mathcal{A}^n$ from* (12) *is generic in $\mathcal{S}^n$.*

Proof. By Theorem 6.1 the set $\mathcal{A}^n$ is dense in $\mathcal{S}^n$. The result then follows by Lemma 6.5 and Theorem 6.6. □

Corollary 6.8. *The following properties are generic within the completely positive cone:*

1. *Having infinitely many minimal cp-decompositions,*
2. *The cp- and cp-plus-ranks being equal and locally constant.*

Proof. From Theorems 1.2 to 1.4 and Lemma 2.1 it is sufficient to show that membership of the open set $\mathcal{A}^n \cap \operatorname{int}(\mathcal{CP}^n)$ is generic in $\mathcal{CP}^n$. Since $\mathcal{CP}^n$ is convex we have $\mu_{\rm L}(\operatorname{bd}(\mathcal{CP}^n)) = 0$ (see e.g. [Lan86]), and from Theorem 6.7 we have $\mu_{\rm L}(\mathcal{S}^n \setminus \mathcal{A}^n) = 0$. The proof is then completed by noting the following:

$$\begin{aligned} \mu_{\rm L}(\mathcal{CP}^n \setminus (\mathcal{A}^n \cap \operatorname{int}(\mathcal{CP}^n))) &= \mu_{\rm L}((\mathcal{CP}^n \setminus \mathcal{A}^n) \cup \operatorname{bd}(\mathcal{CP}^n)) \\ &\leq \mu_{\rm L}(\mathcal{S}^n \setminus \mathcal{A}^n) + \mu_{\rm L}(\operatorname{bd}(\mathcal{CP}^n)) = 0. \end{aligned}$$ □

7 Concluding Remarks

In this paper we studied the distribution of completely positive matrices according to their cp- and cp-plus-ranks. One interesting result found was that whereas it was previously known that in a sufficiently small neighbourhood of a matrix $M \in \mathcal{S}^n$ the cp-rank cannot go down, we have shown that in a sufficiently small neighbourhood of a matrix $M \in \mathcal{S}^n \setminus \operatorname{bd}(\mathcal{CP}^n)$ the cp-plus-rank can not go up. As the cp-plus-rank of a matrix is an upper bound on its cp-rank, this means that for a matrix $M \in \mathcal{S}^n \setminus \operatorname{bd}(\mathcal{CP}^n)$ with its cp-rank equal to its cp-plus rank, in a sufficiently small neighbourhood of the matrix, neither the cp-rank nor the cp-plus-rank will change.

Motivated by this result we considered the open sets

$$\{M \in \mathcal{S}^n \setminus \operatorname{bd}(\mathcal{CP}^n) \mid \operatorname{cp}(M) = \operatorname{cp}^+(M) = p\},$$

which were shown to be nonempty for all $p \in \{n, \ldots, p_n, \infty\}$. An interesting open question is whether these are also connected sets.

We have also established that the sets

$$\{M \in \operatorname{int}(\mathcal{CP}^n) \mid k-1 = \operatorname{cp}(M) < \operatorname{cp}^+(M) = k\}$$

are nonempty for all $k \in \{n+1, \ldots, p_n\}$.

Considering the set $\mathcal{A}^n = \{M \in \mathcal{S}^n \mid \text{cp}(M) = \text{cp}^+(M)\}$, we have shown that this is dense in $\mathcal{S}^n$ and open in $\mathcal{S}^n \setminus \text{bd}(\mathcal{CP}^n)$. By applying the theory of semi-algebraic sets we in addition established that membership in $\mathcal{A}^n$ is a generic property in $\mathcal{S}^n$.

Some interesting questions are still open: For example, is the set $\mathcal{A}^n \setminus \{0\}$ open in $\mathcal{S}^n$? Note that around the zero matrix the set of matrices M satisfying $\text{cp}(M) = \text{cp}^+(M)$ is not open. Indeed, take any matrix $B \in \mathcal{CP}^n$ with $\text{cp}(B) \neq \text{cp}^+(B)$. For all $\lambda > 0$ we have $\text{cp}(\lambda B) = \text{cp}(B) \neq \text{cp}^+(B) = \text{cp}^+(\lambda B)$, but for $A = 0$ we have $A = \lim_{\lambda \searrow 0} \lambda B$ and $\text{cp}(A) = \text{cp}^+(A) = 0$.

On the other hand in contrast to the behaviour on $\mathcal{S}^n$ the set of matrices $\mathcal{A}^n$ is not dense on $\text{bd}(\mathcal{CP}^n)$. Indeed there exist matrices $A \in \text{bd}(\mathcal{CP}^n)$ and $\varepsilon > 0$, such that for all $M \in \text{bd}(\mathcal{CP}^n) \cap N_\varepsilon(A)$ we have $\text{cp}(M) \neq \text{cp}^+(M) = \infty$. Take for example the identity matrix $I \in \text{bd}(\mathcal{CP}^n)$. Since I has full rank n, by Theorem 1.2 we must have $\text{cp}^+(M) = \infty$ (otherwise I would be in the interior of $\mathcal{CP}^n$) and this argument holds for all $M \in \text{bd}(\mathcal{CP}^n) \cap N_\varepsilon(I)$ for some $\varepsilon > 0$.

Acknowledgements. I.M. Bomze and P.J.C. Dickinson want to thank the Isaac Newton Institute at Cambridge University where this paper was discussed during their participation as visiting fellows in the Polynomial Optimization Programme 2013, organized by Joerg Fliege, Jean-Bernard Lasserre, Adam Letchford and Markus Schweighofer. The authors are grateful for stimulating atmosphere and interesting events that were provided.

This research was started whilst P.J.C. Dickinson was at the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, The Netherlands. It was then continued whilst the author was at the Department of Statistics and Operations Research, University of Vienna, Austria, and then again whilst the author was at the Department of Applied Mathematics, University of Twente, The Netherlands. The author would like to thank all of these institutes for the support that they provided.

References

[BDL11] Jérôme Bolte, Aris Daniilidis, and Adrian S. Lewis. Generic optimality conditions for semialgebraic convex programs. *Mathematics of Operations Research*, 36(1):55–70, 2011.

[Bea04] Richard Beals. *Analysis.* Cambridge University Press, 2004.

[Bom12] Immanuel M. Bomze. Copositive optimization – recent developments and applications. *European J. Oper. Res.*, 216:509–520, 2012.

[BR90] Riccardo Benedetti and Jean-Jacques Risler. *Real algebraic and semi-algebraic sets.* Hermann, Éditeurs des Sciences et des Arts, Paris, 1990.

[BSM03] Abraham Berman and Naomi Shaked-Monderer. *Completely Positive Matrices.* World Scientific, 2003.

[BSU14a] Immanuel M. Bomze, Werner Schachinger, and Reinhard Ullrich. From seven to eleven: Completely positive matrices with high cp-rank. *Linear Algebra Appl.*, 459:208–221, 2014.

[BSU14b] Immanuel M. Bomze, Werner Schachinger, and Reinhard Ullrich. New lower bounds and asymptotics for the cp-rank. *SIAM Journal on Matrix Analysis and Applications*, to appear. Also available as: INI14048–POP, Isaac Newton Institute, Cambridge UK, 2014.

[Bur12] Samuel Burer. Copositive programming. In Miguel F. Anjos and Jean Bernard Lasserre, editors, *Handbook of Semidefinite, Cone and Polynomial Optimization: Theory, Algorithms, Software and Applications*, International Series in Operations Research and Management Science, pages 201–218. Springer, New York, 2012.

[Dic10] Peter J.C. Dickinson. An improved characterisation of the interior of the completely positive cone. *Electronic Journal of Linear Algebra*, 20:723–729, 2010.

[Dic13] Peter J.C. Dickinson. *The Copositive Cone, the Completely Positive Cone and their Generalisations*. PhD thesis, University of Groningen, 2013.

[DJL94] John H. Drew, Charles R. Johnson, and Raphael Loewy. Completely positive matrices associated with M-matrices. *Linear and Multilinear Algebra*, 37:303–310, 1994.

[DS08] Mirjam Dür and Georg Still. Interior points of the completely positive cone. *Electronic Journal of Linear Algebra*, 17:48–53, 2008.

[Dür10] Mirjam Dür. Copositive programming — a survey. In Moritz Diehl, Francois Glineur, Elias Jarlebring, and Wim Michiels, editors, *Recent Advances in Optimization and its Applications in Engineering*, pages 3–20. Springer, Berlin Heidelberg New York, 2010.

[GP74] Victor Guillemin and Alan Pollack. *Differential Topology*. Prentice-Hall, Englewood Cliffs, N.J., 1974.

[GWdL76] Christopher G. Gibson, Klaus Wirthmüller, Andrew A. du Plessis, and Eduard J. N. Looijenga. *Topological Stability of Smooth Mappings*. Springer, 1976.

[Lan86] Robert Lang. A note on the measurability of convex sets. *Archiv der Mathematik*, 47(1):90–92, 1986.

[MM62] John E. Maxfield and Henryk Minc. On the matrix equation $X'X = A$. *Proceedings of the Edinburgh Mathematical Society (Series 2)*, 13(02):125–129, 1962.

[SMBB+13] Naomi Shaked-Monderer, Abraham Berman, Immanuel M. Bomze, Florian Jarre, and Werner Schachinger. New results on the cp rank and related properties of co(mpletely)positive matrices. *Linear Multilinear Algebra*, to appear. Also available at: arxiv.org/abs/1305.0737, 2013.

[SMBJS13] Naomi Shaked-Monderer, Immanuel M. Bomze, Florian Jarre, and Werner Schachinger. On the cp-rank and minimal cp factorizations of a completely positive matrix. *SIAM Journal on Matrix Analysis and Applications*, 34(2):355–368, 2013.

[SSMS13] Bryan Shader, Naomi Shaked-Monderer, and Daniel B. Szyld. Nearly positive matrices. *Preprint*, 2013.

[Xu04] Changqing Xu. Completely positive matrices. *Linear Algebra Appl.*, 379:319–327, 2004.