The net worth trap: investment and output dynamics in the presence of financing constraints*

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Abstract

We study the impact of financing constraints on investment and output dynamics, in a continuous time setting with output a linear function of capital. Decline of net worth reduces investment and, if firms can rent capital to unconstrained outside investors, can create a 'net worth trap' with both investment and output falling below normal levels for long time periods. We provide a detailed account of our model solution and discuss both the economic intuition underpinning our results and the implications for macroeconomic modeling.[83 words].

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1 Introduction

The global financial crisis of 2008-2009 has prompted fundamental reassessment of our understanding of both macroeconomic dynamics and of the impact of capital market frictions on corporate behaviour. Two gaps in our understanding can be highlighted: (a) Capital market frictions have conventionally been thought of as impacting primarily on smaller and less credit worth firms, not on large corporations, but as it turned even some of the largest companies and financial institutions in the world were affected by the major contraction in intermediated credit during the crisis; (b) The workhorse linearised general equilibrium (DSGE) models routinely employed for macromonetary forecasting and simulation proved inadequate for capturing the resulting sharp contraction in economic activity.

These shortcomings have motivated a renewed interest in the effect of financing constraints on both corporate decisions and macroeconomic dynamics. The impact of financing constraints on corporate behaviour has of course been studied in prior literature, but the dominant approach before the crisis was a relatively narrow one, recognising their impact only on smaller firms that cannot access public debt markets and with a focus one period operational and financial decision making. Our paper, adopting an approach that has recently become popular, studies the impact of financing constraints on firm operations and finances over multiple periods. This explicitly dynamic approach yields a range of novel insights into corporate behaviour.

Our work is closely related to the analyses of (Bolton, Chen, and Wang 2011) and (Brunnermeier and Sannikov 2014). We assume convex costs of adjusting the stock of physical capital and a constraint in the supply of external finance determined by the valuation placed on productive capital by uninformed outside investors. Solution can be expressed in terms of a single state variable the ratio of internal cash to fixed capital with an exogenous lower boundary (the financing constraint) and an endogenous upper boundary (where dividends are paid). The response of output and investment to exogenous shocks is non-linear and, following a sufficiently large negative shock, it is possible for producer net worth, output and investment to remain trapped below normal levels for an extended period of time (a 'net worth trap') as reported in (Brunnermeier and Sannikov 2014) (henceforth BS). We show that this net-worth trap is though model dependent, arising for some parameter values and specifications but not for others.

A further contribution of our paper is clarifying the mechanisms through which financial constraints impact on corporate decisions. One focus of research, following the financial crisis, focuses on credit supply and in particular on the output impact of bank balance sheets and bank capital regulation.

We pursue a complementary approach in which the supply of credit is fully elastic, up to some maximum level (the financing constraint), but in which corporate net worth affects the demand for credit (through output and investment decisions).

An analogy for this mechanism is provided by (Whittle 1982) pages 287-288:

"This might be termed the 'fly-paper' effect....A deterministic fly, whose path is fully under its own control, can approach arbitrarily closely to the fly-paper with impunity, knowing he can avoid entrapment....A stochastic fly cannot guarantee his escape; the nearer he is to the paper, the more certain it is that he will be carried onto it. This also explains why the fly tries so much harder in the stochastic case than in the deterministic case to escape the neighbourhood of the fly-paper....One may say that the penalty of ending on the fly-paper 'propagates' into the free-flight region in the stochastic case, causing the fly to take avoiding action while still at a distance from the paper."

Whittle is pointing out that in dynamic settings with: (i) uncertainty in the equations of motion for state variables (the position of they fly); and (ii) constraints on state (the fly paper); then (in the language of economic theory) *non-zero shadow prices appear even for values of the state where the constraints are not currently binding.*

In our setting we find two such shadow prices affecting firm decisions:

1. A shadow price of internal funds creating a wedge between the internal and external cost of capital. This reduces the marginal valuation of investment in terms of internal funds (Tobin's marginal-q). As a result firms invest less and less as their net worth declines. The consequence is corporate prudential saving, analogous to the household prudential saving extensively discussed in the literature on the consumption function.

2. A shadow price of risk creating an 'induced risk aversion', with firms reducing their risk exposure by renting out more and more of their capital as net worth declines below a threshold level. It is this mechanism which, if sufficiently powerful, creates the 'net worth trap'.

The resulting dynamics of corporate output and investment and the demand for credit are non-linear, with an accumulation of negative shocks to net worth, having a larger proportionate impact on these shadow prices than an isolated negative shock and shocks (positive or negative) having a relatively larger impact when net worth is already low and shadow prices are elevated.

Can these predictions of corporate behaviour incorporating financing constraints into a model of multiperiod corporate decision making provide an explanation of extended macroeconomic downturns? Our simulations suggest yes, sometimes. But this conclusion must be qualified. What stops households, firms and financial intermediaries insulating themselves against the risk of macroeconomic propagation by writing conditional contracts? These could depend on macroeconomic observables, such as the state of the economy, which are contractible and cannot be manipulated by any party to the contract. If macroeconomic propagation via financing constraints is so important, why do freely contracting parties not take steps to protect themselves and eliminate this risk? This criticism is especially obvious in the context of our own model because the characterisation of financial constraints is rather stark, but it applies to much of the literature on the financial propagation of macroeconomic shocks.

Macroeconomic shocks are rare and their impacts, when they do materialise, are neither clearly anticipated nor well understood by economic agents. As we argue in our conclusion macroeconomic consequences from financial contracts arise when the underlying uncertainty is Knightian, i.e. unquantifiable and hence unhedgeable. Hence the important implications of our model are not its quantitative predictions but rather the elucidation, in stylised fashion, of the dynamics of output and investment in periods of extreme uncertainty.

Our modeling also makes some technical contributions. We introduce one further parameter representing a fixed cost of recapitalisation (as in (Milne and Whalley 2002), itself an extension of MR to analyse bank capital regulation). We obtain a sufficient condition for the existence of a unique equilibrium with positive expected dividend payments that appears also to apply to BS. We improve on the solution method of BS by obtaining asymptotic expansions of both the value function and the ergodic density function describing the unconditional distribution of net worth, on those occasions where singularities emerge at the financing constraint. Our dynamic programming solution is a rapidly solved single first order ordinary differential equation (often without iteration) supporting full exploration of the parameter space.{ We have created a standalone Mathematica solution module which can be used by any interested reader to explore the impact of parameter choice on model outcomes. This solution module and the underlying Mathematica notebooks can be found at www.leveragecycles.lboro.ac.uk

The remainder of our paper is set out as follows. Section 2 locates our work in the economics, finance and mathematical insurance literatures. Section 3 presents a simplified version of our model in which capital cannot be rented out. For high values of the fixed cost of recapitalisation firms do not

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recapitalise and the solution is similar to that of MR with liquidation on the lower net worth boundary; but for lower values firms choose to exercise their option to recapitalise on the lower boundary and so avoid liquidation. In either case investment is reduced below unconstrained levels by the state dependent shadow price of internal funds. Section 4 then introduces the possibility that firms, by renting capital to outsiders, are able to reduce their risk exposure, but at the expense of a decline in their expected output. The extent to which this is done depends on the extent to which a shadow price of risk induces a higher effective level of risk aversion to cash flow risks. Section 5 provides a concluding discussion. Four appendices contain supporting technical detail.[1]

2 Related literature

The present paper is one of a number of recent studies of the dynamics of corporate behaviour subject to financing constraints. These analyses are rooted in earlier literature examining firm operations, financing and risk management over multiple periods. Central to all this work is the inventory theoretic modelling of both financial (cash, liquidity and capitalisation) and operational (inventory, employment, fixed capital investment) decisions subject to fixed (and sometimes also proportional or convex) costs of replenishment or investment.

Most dynamic models of corporate behaviour focus either on financial or on operational decisions without considering their interaction. Well known contributions include the work of (Jorgenson 1963, Lucas Jr and Prescott 1971) and others on the dynamics of fixed capital investment in the presence of adjustment costs; and that of ((Miller and Orr 1966, Constantinides 1976, Frenkel and Jovanovic 1980)) applying standard tools of inventory modelling (drawn from (Arrow, Harris, and Marschak 1951, Scarf 1960)) to study corporate cash holdings and money demand. Dynamic modelling methods are also employed in the contingent claims literature, to examine both the pricing of corporate liabilities ((Merton 1974)) and the possibility

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[1]

Appendix A solves the situation when there is no non-negativity constraint on dividend payments; or equivalently when uncertainty vanishes. Appendix B provides proofs of the propositions in the main text. Appendix D derives the asymptotic approximations used to incorporate the singularities that arise in the model with rental. Appendix C details our numerical solution, noting how this must be handled differently in the two possible cases, where a 'no Ponzi' condition applies to the unconstrained model of Appendix A; and when this condition does not.

of strategic debt repudiation ((Anderson and Sundaresan 1996, MellaBarral and Perraudin 1997)). This work has been extended to examine the interaction of the choice of asset risk and capital structure, taking account of the implications for the cost of debt (Leland 1998). But this line of research says nothing about the dynamic interaction of financing and investment.

The interaction of financial and operational decisions is often considered in a static framework. This allows an explicit statement of the informational asymmetries and strategic interactions that lead to departure from the (Modigliani and Miller 1958) irrelevance proposition (for a unified presentation of much of this literature see (Tirole 2006)). This focus yields valuable insight into issues of corporate governance, managerial incentives and contractual design, and can also be used to support the standard tradeoff theory of optimal capital structure theory, which in turn justifies the employment of the weighted average cost of capital as a hurdle rate for investment decisions. A static analysis also supports the pecking order theory of capital structure in which costs of equity issuance, resulting in discrepancies between the costs of inside funds (retained earnings), debt and outside equity ((Myers 1984, Myers and Majluf 1984)). (Froot, Scharfstein, and Stein 1993) apply this framework to develop a joint framework for the determination of investment and risk management decisions.

Progress has been made more recently on analysing optimal financial contracts in a dynamic principal agent context (see(Sannikov 2008, De Marzo and Sannikov 2006) and references therein), yielding similar divergencies between the cost of funds as in static pecking order theory. (De Marzo and Sannikov 2006) show how it can be optimal for a firm to use simultaneously both long term debt and short term lines of credit, in order to create incentives for managerial effort, but this work has not been extended to model the interaction of financial and operational decisions. One paper that does provide insight into the interaction with real investment decisions is (Gertler 1992), who extends the costly state verification problem of (Townsend 1979) into a recursive model of dynamic stochastic control where one period debt contract can be refinanced through a new debt contract. His analysis does not establish an optimal contract (as discussed by (Sannikov 2008) in a discrete time setting the optimal contract is a complicated function of current and past observable states), but it does show how if debt contracts are used to dynamically finance a productive investment opportunity then the value function has a ′characteristic′ convex shape, with a negative second derivative with respect to net worth, reflecting departure from (Modigliani and Miller 1958) capital structure irrelevance and a resulting shadow price of internal funds. In consequence as net worth declines so does investment and output.

Because of the technical difficulty of obtaining a fully microfounded solution, work on the dynamic interaction of financial and operational decisions has instead typically proceeded by imposing (rather than modelling) costly financial frictions, i.e. from an inventory theoretic perspective similar to that which we adopt. Most of these papers, like our own, employ continuous time modelling techniques. An early example is (Mauer and Triantis 1994) who explores the bond financing of a project subject to fixed costs both of opening and shutting the project (hence creating real option values) and of altering capital structure through bond issue. Four papers written independently (Radner and Shepp 1996, Milne and Robertson 1996, Jeanblanc-Picqué and Shiryaev 1995, Asmussen and Taksar 1997) explore cash flow management and dividend policy in a context where cash holdings evolve stochastically (as a continuous time diffusion) resulting in a need for liquidity management. This leads to the simple boundary control for dividends that is inherited by the model of the present paper: paying no dividends when net cash holdings are below a target level and making unlimited dividend payments on this boundary.

This set up has been employed to examine the risk exposure decisions of both insurance companies and non-financial corporates (see (Taksar and Zhou 1998, Jgaard and Taksar 1999, Asmussen, Højgaard, and Taksar 2000)). Recent and closely related work exploring the interaction of financing, risk management and operational decisions includes (Rochet and Villeneuve 2011, Bolton, Chen, and Wang 2011, Bolton, Chen, and Wang 2013, Rampini and Viswanathan 2013, Palazzo 2012, Anderson and Carverhill 2011) While their specific assumptions and focus of analysis differ, these papers have a great deal in common. The resulting dynamic optimisation yields a value function with the 'characteristic' convex shape reported by (Gertler 1992) and appearing also in our own work (see the main upper panel of Figures 1 and 2 below) and hence a motive to reduce risk, output and investment as net worth or cash holdings decline. Related work employing discrete time techniques is that of (Gamba and Triantis 2008, Gamba and Triantis 2014) who consider risk management and firm decision making in the presence of taxation and imposed costs of financial transactions. They incorporate a wide range of determining factors and firm decision variables, again finding that a reduction in net worth leads to reduced of risk exposure and increased incentives to hedge risks.

While the literature offers a consistent account of the dynamic interactions of corporate financing and operational decisions, the implications for macroeconomic dynamics are less clearly established. Capital market frictions, in particular the high costs of external equity finance and the role of collateral values, have been proposed as an explanation of macroeco-

nomic dynamics (see e.g. (Greenwald, Stiglitz, and Weiss 1984, Kiyotaki and Moore 1997)). The most widely used implementation of these ideas is the 'financial accelerator' introduced into macroeconomics by (Bernanke and Gertler 1989, Bernanke, Gertler, and Gilchrist 1999). This is based on a static model of underlying capital market frictions, in which the macroeconomic impact of financing constraints comes through costly state verification and the resulting the difficulties entrepreneurs face in obtaining external finance for the creation of new investment projects. The resulting propagation mechanism operates through an 'external financing premium', i.e. a additional cost that must be paid by investors in fixed capital projects in order to overcome the frictional costs of external monitoring whenever they raise external funds, not an internal shadow price. An alternative perspective on the propagation of macroeconomic shocks is found in the literature on endogenous risk in traded asset markets (see (Danielsson, Shin, and Zigrand 2004, Brunnermeier and Pedersen 2008) in which asset price volatility rather than unobserved shadow prices limits access to external finance (a recent model of this kind is (Adrian and Boyarchenko 2013)).

Finally some other related literature on bank balance sheets and financial market pricing deserves brief mention. (Milne and Whalley 1999), (Milne and Whalley 2002), (Milne 2004) and (Peura and Keppo 2006) use the continuous time framework of (Milne and Robertson 1996) to analyse bank capital regulation and bank behaviour. Other papers have used continuous time methods to model intervention in exchange rates and in money markets, and how intervention rules affect market pricing in these markets, for example (Krugman 1991) and (Mundaca and Ø ksendal 1998). (Korn 1999) provides a useful survey article linking this work to that on both optimal portfolio allocation subject to transaction costs and the modeling of cash management problems faced by companies and insurance firms.

3   A basic model

This section presents a first version of our model in which firms decide only on investment and dividend payments, postponing until Section 4 the possibility that firms reduce their risk exposure by selling or renting capital to outsiders. Section 3.1 sets out the model assumptions. Section 3.2 discusses the solution method. Section 3.3 presents some simulation results.

3.1 Model assumptions

Firms manage two 'state' variables, net cash c and capital k (we later show that the model collapses to a single state variable $\eta = c/k$). These evolve according to:

$$\mathrm{d}c = \left[-\lambda + ak + rc - ik - \frac{1}{2}\theta \left(i - \delta \right)^2 k \right] \mathrm{d}t + \epsilon\left(\tau \right) + \sigma k \, \mathrm{d}z \qquad \text{(1a)}$$

$$\mathrm{d}k = (i - \delta)k \, \mathrm{d}t \qquad \text{(1b)}$$

- r is both the rate of interest paid on borrowing $(c < 0)$ and the rate of interest received on cash deposits $(c > 0)$.

- Output ak is a linear function of the capital stock.

- The coefficient θ captures costs of adjustment of the capital stock increasing with the net rate of investment i-δ.

- λ is the rate of dividend payments, subject to a non-negativity constraint $\lambda \geq 0$;

- $\epsilon\left(\tau \right)$ are non-infinitesimal re-capitalisations at times τ chosen by the firm.

Cash holding and borrowing by the firm are represented by the same variable c. When $c > 0$ the firm is holding cash. When $c < 0$ the firm is borrowing. Firms choose rules for two control variables, the investment rate i and dividend payout rate λ, together with times τ_t and amounts ϵ_τ of recapitalisations, in order to maximise the objective:

$$\Omega = \max_{\{i_t\}, \{\lambda_t\}, \{\epsilon_\tau\}} \mathbb{E} \int_{t=0}^{\infty} \mathrm{e}^{-\rho t} \lambda \, \mathrm{d}t - \sum_{\tau = \tau_1}^{\tau_\infty} \mathrm{e}^{-\rho\tau} \left(\epsilon_\tau + \chi k \right) \qquad \text{(2)}$$

χ represents the cost to shareholders of recapitalisation, arising from any associated due-diligence or dilution of interests, assumed proportional to k.

The only other agents are outside investors ('households' in the terminology of (Brunnermeier and Sannikov 2014)) who lend to firms, but they do not take credit risk, instead they require that lending is secured against the firm's assets, limiting the amount of credit available to the firm, and they become the residual acquirer of the firm's assets if and when the debt is not serviced. Like firms these investors are risk-neutral and seek to maximise the present discounted value of current and future consumption. Unlike firms there is no non-negativity constraint on their consumption. Since they are

the marginal suppliers of finance, and there is no risk of credit losses, they lend to or borrow from firms at a rate of interest r reflecting their rate of time discount. We further assume that investors are more patient than firms i.e. $r < \rho$ (without this assumption firms will build up unlimited cash holdings instead of paying dividends). Fixed capital held directly by outside investors generates an output of $\bar{a}k$.

In order to obtain a meaningful solution we require: (i) that capital is less productive in the hands of outside investors than when held by firms (otherwise firms will avoid using capital for production), $\bar{a} < a$; (ii) upper bounds on both a and $\bar{a}$ to ensure that the technology does not generate sufficient output to allow self-sustaining growth faster than the rates of shareholder or household discount; (iii) a further technical condition (a tighter upper bound on a) ensuring that there is a solution in which dividends are paid to firm shareholders.

This model, and its subsequent generalisation in Section 4, are representative agent models. They can be interpreted as a large number of firms, each with the same preferences and production technology and hit by the same distribution of shocks, who can trade fixed capital and cash amongst themselves and as a result each choose the same ratio of cash to capital ck^{-1} and so are unaffected by idiosyncratic shocks. In the rest of the paper we neglect discussion of idiosyncratic shocks and refer to 'the firm'. σ is thus the instantaneous standard error of remaining aggregate economic shocks whose impact cannot be diversified away.

3.2 Solution

3.2.1 Characterisation of solution

Solution is summarised by the following propositions:

Proposition 1 *The maximum amount of borrowing available to the firm from outside investors is:*

$$c > \bar{\eta}k = -\left[1 + \theta\left(r - \sqrt{r^2 - 2\theta^{-1}[\bar{a} - \delta - r]}\right)\right]k \qquad (3)$$

Proof: Appendix A.

If c falls to this bound then the firm has a choice: either to liquidate (in which case its assets are acquired by the lenders and there is no further payment to shareholders); or to recapitalise (at a cost to shareholders of χk).

Proposition 2 *Sufficient conditions for an optimal policy for choice of $\{i_t\}$, $\{\lambda_t\}$, $\{\epsilon_\tau\}$ as functions of the single state variable $\eta = ck^{-1}$ to exist and satisfy $i_t - \delta < \rho$, $\forall t$ are*

$$a - \delta < \rho + \frac{1}{2}\theta\rho^2 - (\rho - r)\left[1 + \theta\left(r - \sqrt{r^2 - 2\theta^{-1}[\bar{a} - \delta - r]}\right)\right], \quad (4a)$$

$$\bar{a} - \delta < r + \frac{1}{2}\theta r^2. \quad (4b)$$

Further, if Eq. (4a) is satisfied, the growth rate of the capital stock $g(\eta)$ and the optimal investment rate $i(\eta)$ always satisfies the constraints

$$\bar{g} = \left(r - \sqrt{r^2 - 2\theta^{-1}[\bar{a} - \delta - r]}\right) \le g(\eta) = i(\eta) - \delta < \rho.$$

Proof: Appendix B.

If a solution exists then it is characterised by the following further proposition:

Proposition 3 *An optimal policy choice for $\{i_t\}, \{\lambda_t\}, \{\epsilon_\tau\}$ as functions of the single state variable $\eta = ck^{-1}$, if it exists, takes the following form: (i) making no dividend payments a long as $\bar{\eta} < \eta < \eta^*$ for some value η^* of η, while making dividend payments at an unlimited rate if $\eta > \eta^*$; (ii) investing at a rate*

$$i = \delta + \theta^{-1}(q - 1)$$

where $W(\eta)k$ is the value of Ω under optimal policy; and $q(\eta)$ representing the valuation of fixed assets by the firm (the cash price it would be willing to pay for a small increase in k) is given by:

$$q = \frac{W}{W'} - \eta, \qquad q' = -\frac{WW''}{W'W'}, \quad (5)$$

with $q' > 0$ whenever $\eta < \eta^*$; and $W(\eta)$ is the unique solution to the second order differential equation over $\eta \in [\bar{\eta}, \eta^*]$:

$$\rho\frac{W}{W'} = a - \delta + r\eta - \frac{1}{2}\sigma^2\left(-\frac{W''}{W'}\right) + \frac{1}{2}\theta^{-1}\left(\frac{W}{W'} - \eta - 1\right)^2 \quad (6)$$

obtained subject to three boundary conditions: (i) an optimality condition for payment of dividends at η^* $W''(\eta^*) = 0$ (ii) a scaling condition $W'(\eta^*) = 1$; and (iii) the matching condition:

$$W(\bar{\eta}) = \max\left[W(\eta^*) - (\eta^* - \bar{\eta} + \chi), 0\right]. \quad (7)$$

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Finally the firm recapitalises only on the lower boundary and only if $W(\bar{\eta}) > 0$
in which case it recapitalises by increasing η *immediately to* η^*.

Proof: Appendix B.

A key feature of this solution is that the firm pursues a policy of targeting a level of cash holding/borrowing $\eta^* k$, retaining all earnings when below this level and paying out all earnings that would take it beyond this level (a form of barrier control). It never holds more cash (or conducts less borrowing) than this targeted amount and below this level no dividends are paid (as discussed in Section 2 this feature is shared by many related papers).

The rate of investment declines the further η falls below the target, the firm reducing investment in order to realise cash and stave off costly liquidation or recapitalisation.[2] It only recapitalises if shocks drive it to the lower boundary of maximum borrowing and the cost to shareholders of recapitalisation is less than their valuation of the recapitalised firm. If this is not possible then it liquidates and the value obtained by shareholders is zero.

The mechanism driving this decline of investment is a rise in the marginal or internal cost of cash (V_c) relative to the marginal benefits of capital (V_k) i.e. the familiar Tobin's-q mechanism where investment depends on $q = V_k/V_c$. q falls the closer the firm is to its maximum borrowing limit because cash becomes increasingly desirable as a means of avoiding liquidation or costly recapitalisation).

This increasing marginal valuation of cash as the firm comes closer to liquidation is reflected in a curvature of the value function $V = kW(\eta)$ characteristic of dynamic models of financing constraints (see the upper panel of Figure 1 and discussion in Section 2). In the absence of financing constraints (as discussed in Appendix A) the value function is linear in η and given by $V = k(W_0 + \eta)$, $W' = 1$ and $q = W/W' - \eta = W_0$ is inaffected by leverage. In the presence of financing constraints the value function is distorted downward, the closer η is to the maximum level of borrowing, and the increasing marginal valuation of cash (the slope of W) results in the corresponding fall of q. Note that neither an external financing premium or endogenous risk plays any role in the operation of the model. External finance is always provided upto the borrowing limit at the rate of interest r.

In order to explore the implications of the model for dynamic behaviour, we also solve for the steady state 'ergodic distribution'.[3] This is a probability

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[2]Because $W'' < 0$; see MR for further discussion.

[3] The ergodic distribution, if it exists, represents both the cross-sectional distribution of many firms subject to independent shocks to cash flow and the unconditional time distribution of a single firm across states. Since we are investigating a model of a representative firm (or many small firms each the same) it is the interpretation as an unconditional time

density over η indicating the relative amount of time in which the economy stays in any particular state: when this is high then it visits this state often, when it is low then it visits this state rarely. We wish to investigate if the model produces a 'net worth trap' appearing as an ergodic distribution with two peaks, one at the lower boundary $\bar{\eta}$ and the other at the upper boundary η^*, indicating that if the economy is affected by a large shock moving it close to the lower boundary then it will remain there for an extended period of time.

If recapitalisation takes place then firms absorbed at the lower boundary move immediately to the upper boundary. If liquidated firms are never replaced then no ergodic density exists, so for comparability with the case of recapitalisation we assume that liquidated firms are replaced at the upper dividend paying boundary. We then have the following:

Proposition 4 *The pdf of the ergodic distribution is described the following first-order ode:*

$$\frac{1}{2}\sigma^2 f' - \left[a + r\eta - \delta - \theta^{-1}(1+\eta)(q-1) - \frac{1}{2}\theta^{-1}(q-1)^2\right]f = -d. \quad (8)$$

and can be computed subject to the boundary conditions

$$f(\bar{\eta}) = 0 \quad (9)$$

and $F(\eta^*) = 1$ where $F(\eta) = \int_{u=\bar{\eta}}^{\eta} f(u)\, du$.

Proof: Appendix B.

Here d is a constant representing the net flow of companies through the non-dividend paying region, until they exit at the lower boundary $\bar{\eta}$ through liquidation or recapitalisation and are replaced at the upper boundary η^*.

3.2.2 Numerical solution

Our numerical solution methods are presented in Appendix C. In outline these are as follows. We choose to work with the function $q(\eta)$. Eq (6) can be written as:

$$q' = \frac{2}{\sigma^2}\left[a - \delta - (\rho - r)\eta - \rho q + \frac{1}{2}\theta^{-1}(q-1)^2\right](q + \eta). \quad (10)$$

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distribution that is of interest.

requiring only two boundary conditions for solution: the optimality condition locating the upper boundary $q'(\eta^*) = 0$ together with the condition on the lower boundary Eq. (7).

In the case of liquidation no iteration is necessary. This is because $W(\bar\eta) = 0$ implying from Eq. (5), that $q(\bar\eta) = -\bar\eta$, i.e. the maximum amount of lending is the valuation of capital by outsiders and this determines the value of q on the lower boundary. Eq. (10) is simply computed directly beginning from the lower boundary with $q = -\eta$ and continuing for higher values of η until $q' = 0$ and the upper boundary, it is exists, is located.

Iteration is required when there is recapitalisation rather than liquidation. This is because in this case $q(\bar\eta)$ is not known, but must be determined from the matching condition $W(\bar\eta) = W(\eta^*) - (\eta^* - \bar\eta + \chi)$. Given any initial starting value for $q(\bar\eta)$ it is possible to jointly compute both $q(\eta)$ and the accompanying value function $W(\eta)$. Iteration on the starting value $q(\bar\eta)$ then yields the solution with recapitalisation (if one exists) with $W(\bar\eta) > 0$.[4]

3.3   Simulation results

We have performed extensive simulations of the model equations, focussing on the shape of the ergodic distribution $f(\eta)$ and whether it has two peaks and can therefore help explain a transition from a high output boom to a low out slump, or instead has a single peak. In this first version of our model there is always a single peak located at the maximum value η^* i.e. our model without rental or sale of capital does not create long lasting periods with output and investment below normal levels.

Typical value functions W together with the corresponding ergodic densities f are presented in Fig. 1. Here, the chosen parameters are:

$$\rho = 0.06, \quad r = 0.05, \quad \sigma = 0.2,$$
$$\theta = 15.0, \quad \chi = 0.75, \tag{11}$$
$$a = 0.1, \quad \bar a = 0.04, \quad \delta = 0.02.$$

The shape of these plots are typical of what we find, with a monotonically

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[4] While numerical solution is straightforward, it may fail to locate an upper boundary η^* for some combinations of parameters. This happens for example when the productivity of capital a is so high, and the adjustment costs of capital increase θ so low, that output can be reinvested to increase the stock of capital faster than the discount rate of firms (See Appendix (A) for a discussion of the parameter restrictions required to prevent this in the deterministic case $\sigma = 0$). In this case the value function is unbounded and there is no meaningful solution. Extreme parameter values, for example very low values of σ, can also result in numerical instability and failure to find a solution.

rising value function W, with single η^* –peaked ergodic densities f. This feature appears to persistent. In a wide parameter space search, we have found only single peaked distributions of this kind.[5]

4   An extended model

This section extends the model of Section 4 by assuming that capital can be rented by firms to outside investors. We also introduce an additional diffusion term affecting the productivity of capital. The structure of this section parallels that of Section 3, with subsections on assumptions (4.1), solution (4.2) and simulation results (4.3).

4.1   Additional assumptions

In this extended setting, firms continue to manage the same two 'state' variables, net cash c and capital k, but these now evolve according to:

$$\mathrm{d}c = \left\{ -\lambda + [\psi a + (1 - \psi)\bar{a}]k + rc - ik - \frac{1}{2}\theta(i - \delta)^2 k \right\} \mathrm{d}t \qquad (12\text{a})$$
$$+ \epsilon\,(\tau) + \psi\sigma_1 k\,\mathrm{d}z_1,$$
$$\mathrm{d}k = (i - \delta)k\,\mathrm{d}t + \sigma_2\psi k\,\mathrm{d}z_2. \qquad (12\text{b})$$

There are now two independent diffusion terms ($\psi\sigma_1 k\,\mathrm{d}z_1$ and $\sigma_2\psi k\,\mathrm{d}z_2$) and an additional third control variable, the proportion of capital ψ firms themselves manage (with remaining capital $1 - \psi$ rented to households). Because of competition amongst households for this capital – this provides a rental income of $\bar{a}$, the productivity of capital when managed by households. The introduction of the additional diffusion term is a relatively small change to the model; but the introduction of rental is a fundamental change, leading to the possibility of a double-peaked ergodic density and thus a prediction of possible persistence of large shocks (the 'net worth trap'). All the other assumptions of Section 3 continue to apply.

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[5] Although double peaks were not found, in some simulations the main peak normally at η^* can migrate into the central part of the η range. This occurs when choosing parameters for which cash-flows are non-positive ($\mathrm{d}\eta \le 0$). We do not report these simulations.
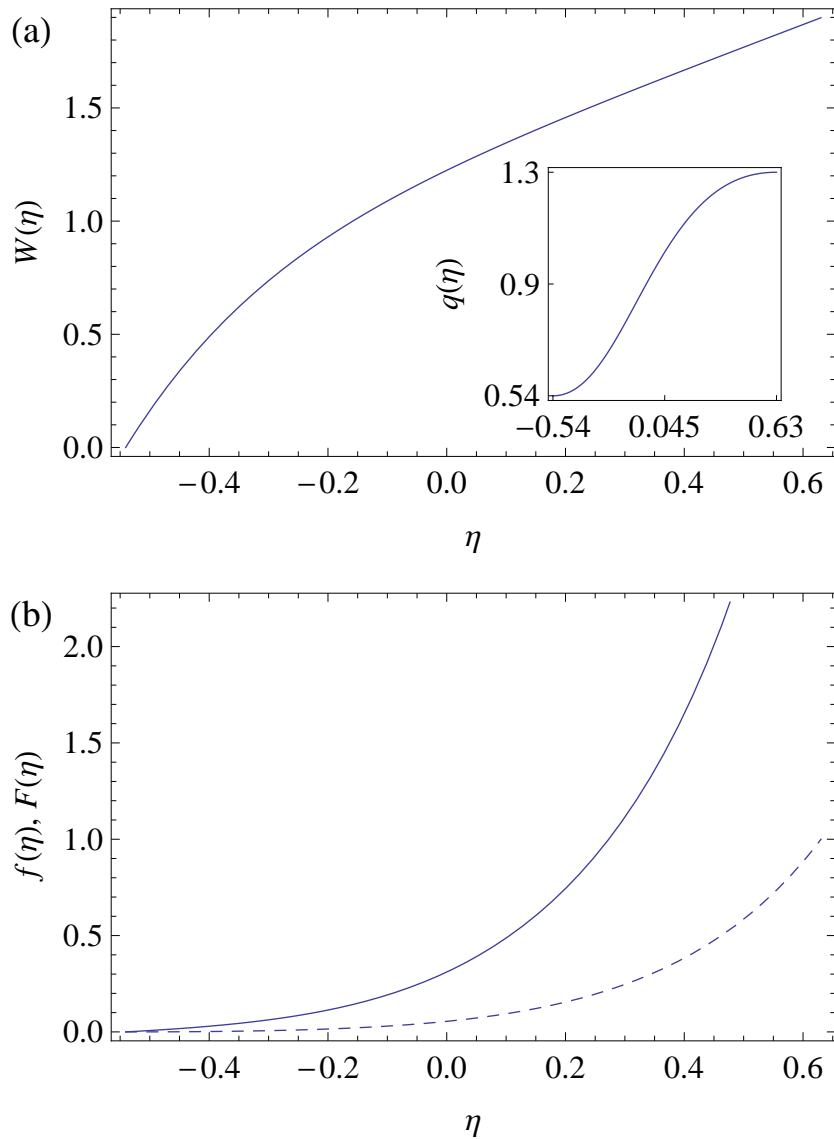
Figure 1: Solutions of the model equations of Section 3 for baseline parameters $\rho = 0.06$, $r = 0.05$, $\sigma = 0.2$, $a = 0.1$, $\bar{a} = 0.04$, $\delta = 0.02$, $\theta = 15$, and $\chi = 0.75$. Subfigure (a): value function W, inset shows the function q over the same η range; (b) the ergodic density f (solid curve) and the cumulative density function F (dashed).

4.2 Solution

4.2.1 Characterisation of solution

Propositions 1 and 2 apply to the generalised model with rental. Proposition 3 applies in the following amended form:

Proposition 5 *An optimal policy choice for $\{i_t\}, \{\psi_t\}, \{\lambda_t\}, \{\epsilon_\tau\}$ as functions of the single state variable $\eta = ck^{-1}$, if it exists, takes the following form. The rules for $i(\eta), \lambda(\eta)$ are exactly as stated in Proposition 3; optimal policy for $\psi(\eta)$ renting of fixed capital is that for some intermediate range of η $\bar{\eta} \leq \eta \leq \tilde{\eta}$ where $\bar{\eta} \leq \tilde{\eta} < \eta^*$, firms retain a varying proportion ψ of fixed capital given by:*

$$\psi = \frac{a - \bar{a}}{\sigma_1^2 + \sigma_2^2 \eta^2} \left[-\frac{W''}{W'} \right]^{-1} \tag{13}$$

and rent the remainder to firms; $\psi(\tilde{\eta}) = 1$; and for $\eta \geq \tilde{\eta}$, $\psi = 1$ and no fixed capital is rented out.

  $W(\eta)$, the unique solution to the second order differential equation over $\eta \in [\bar{\eta}, \eta^*]$, now obeys:

$$\rho \frac{W}{W'} = \bar{a} + (a - \bar{a})\psi - \delta + r\eta - \frac{\sigma^2(\eta)}{2}\psi^2 \left[-\frac{W''}{W'} \right]$$

$$+ \frac{1}{2\theta} \left[\frac{W}{W'} - 1 - \eta \right]^2 \tag{14}$$

where $\sigma_1^2 + \sigma_2^2 \eta^2 = \sigma^2(\eta)$ and solution is found subject to same boundary conditions as in Proposition 3

Proof Appendix B

  Here $-W''/W'$ expresses the induced risk aversion created by the presence of financing constraints in terms of the single state value function W. [6] The greater this induced risk-aversion the lower the proportion of capital that is managed by firms instead of being rented out to households.

  Comparison with corresponding equation Eq (6) of Proposition 3 is informative. There are two differences: volatility σ^2 is no longer a constant but because of the shocks to the productivity of capital increases in the absolute

---

[6] see (Milne and Robertson 1996) section 4 for further discussion of this induced risk aversion and comparison with the risk loving behaviour that emerges in many standard discrete time models as a result of moral hazard.

magnitude of η; also reducing ψ to less than 1 (i.e. renting out productive capital to households) reduces both the drift and the diffusion of η.

The ergodic density can now be computed using:[7]

Proposition 6 *The pdf of the ergodic distribution is described by the following first-order ode:*

$$
\phi' - \left[\frac{1}{2} \psi^2 \sigma^2(\eta) \right]^{-1} \left[a + (a - \bar{a})\psi + r\eta - \delta \right.
$$
$$
\left. - \theta^{-1}(1 + \eta)(q - 1) - \frac{1}{2}\theta^{-1}(q - 1)^2 \right] \phi = -d \quad (15)
$$

where $\phi = \psi^2 \sigma^2 f / 2$ and satisfies the boundary conditions

$$
\begin{cases}
f(\bar{\eta}) = 0, & \text{if } W(\bar{\eta}) > 0 \\
d = 0 & \text{if } W(\bar{\eta}) = 0
\end{cases}
\quad (16)
$$

and $F(\eta^*) = 1$ where $F(\eta) = \int_{u=\bar{\eta}}^{\eta} f(u)\, du$.

Proof Appendix B.

4.2.2 Numerical calculation

Our numerical solution methods are again detailed in Appendix C. This proceeds in the same way as for the first model without rental of Section (3), by re-expressing Eq (14) as a differential equation in q. Over the lower region $\eta < \tilde{\eta}$ Eq. (14) becomes:

$$
q' = -\frac{1}{2} \frac{(a - \bar{a})^2}{\sigma_1^2 + \eta^2 \sigma_2^2} \frac{q + \eta}{\bar{a} - \delta + r\eta - \rho(q + \eta) + \frac{1}{2}\theta^{-1}(q - 1)^2} \quad (17)
$$

while in the upper region Eq (10) continues to apply (except that now $\sigma^2 = \sigma_1^2 + \sigma_2^2 \eta^2$ is a function of η).

If there is no recapitalisation then the model can again be solved without iteration, commencing the calculation at $\eta = \bar{\eta}$ and continuing until the intermediate values $\eta = \tilde{\eta}$ and $\eta = \eta^*$ are located. However in this case $q(\bar{\eta}) = -\bar{\eta}$ and hence $\psi(\bar{\eta}) = 0$, with the consequence that there are singularities in f, q, and W at $\bar{\eta}$. We incorporate these singularities using asymptotic approximations summarised in the following further proposition.

---

[7] We use this indirect statement because of the dependency of ψ and σ on η. While ϕ can be substituted out from Eq. (15) the resulting ODE for f is rather cumbersome.

Proposition 7 *W, q and ϕ close to $\bar{\eta}$ are described by:*

$$W = C_W(\eta - \bar{\eta})^\beta(1 + \mathcal{O}(\eta - \bar{\eta})), \tag{18}$$

where C_W is a constant and $\beta = 1/(1 + q'(\bar{\eta})) \in (0, 1]$;

$$q = \bar{q} + q'(\bar{\eta})(\eta - \bar{\eta}). \tag{19}$$

and;

$$\phi = C_\phi(\eta - \bar{\eta})^\alpha, \tag{20}$$

where α given by Eq. (56) of Appendix (D) and C_ϕ is another constant.

This further implies that $-W''/W'$ (our measure of induced risk aversion) is divergent at $\bar{\eta} = -\bar{q}$,

$$-\frac{W''}{W'} \simeq \frac{1 - \beta}{\eta - \bar{\eta}}.$$

(consistent with $\psi(\bar{\eta}) = 0$), and the ergodic density is approximated by

$$f \propto (\eta - \bar{\eta})^{\alpha - 2} \tag{21}$$

and thus diverges if $\alpha < 2$ and becomes degenerate, with the entire probability mass at $\bar{\eta}$ if $\alpha \leq 1$.

Proof: Appendix D.

In the case of recapitalisation $q(\bar{\eta}) > -\bar{\eta}$ and there is are no singularities in the solution; so, while iteration again required to determine $q(\bar{\eta})$, this can be conducted in exactly the same way as described in Sub-section 3.2.2 for the model without rental.

4.3   Simulation results

As expected from the power-law shape of f, Eq. (21), the option to rent can have a strong impact on the shape of the ergodic density. As an example of this, in Fig. 2 we have plotted the value function W together with q, and the probability and cumulative densities using again our baseline parameters $\rho = 0.06$, $r = 0.05$, $\sigma_1 = 0.2$, $\sigma_2 = 0.0$, $a = 0.1$, $\bar{a} = 0.04$, $\delta = 0.02$, $\theta = 15$, and $\chi = 0.75$ (identical parameters to those used in Fig. 1). Whereas the value function W and q show little change when rental is introduced, the density function f changes dramatically. This time a second peak is clearly present near the left-hand side range of η values.[8]

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[8]Note that with these particular parameter values the firm chooses to recapitalise, with χ slightly less than a critical value of around 0.55 at which recapitalisation is not worthwhile,
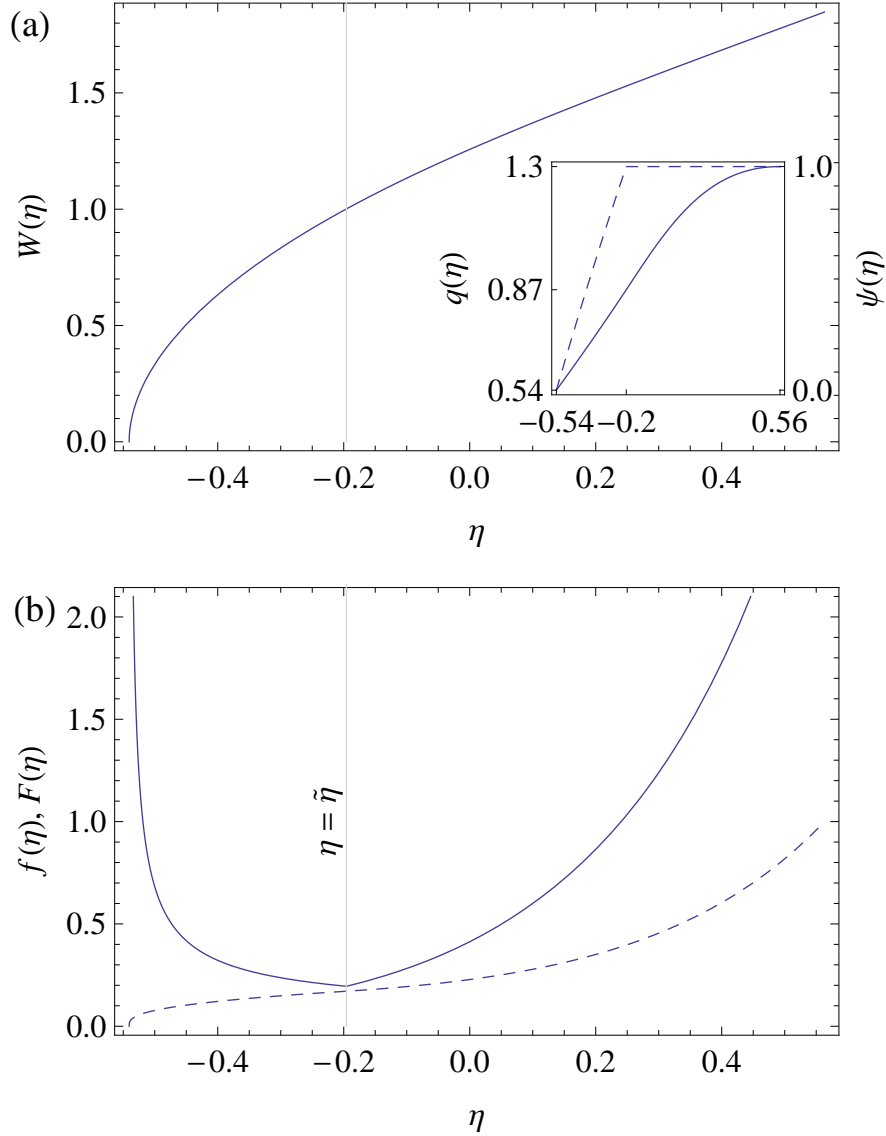
Figure 2: Solutions of the model of 4 with option to rent, using baseline parameters $\rho = 0.06$, $r = 0.05$, $\sigma_1 = \sigma = 0.2$, $\sigma_2 = 0.0$, $a = 0.1$, $\bar{a} = 0.04$, $\delta = 0.02$, $\theta = 15$, and $\chi = 0.75$. Contrast this to Fig. 1 where identical parameters were used, but without rental. Subfigure (a): value function W, inset shows the functions q and ψ over the same η range; (b) the ergodic density f (solid curve) and the cumulative density function F (dashed). Notice the prominent peak in f towards the left-hand side boundary.

Our results reveal the parameter dependence of this ergodic instability (a second peak towards in the ergodic density associated with low values of the state variable η representing the ratio of cash-to-capital). This parameter dependence emerges in two different ways: (i) through the power-law exponent α, and (ii) dependence on the cost of recapitalisation χ. The ability to recapitalise or not has a major impact on the ergodic distribution. For any given parameters, there is a threshold χ, $\bar{\chi}$, above which recapitalisation is no longer worthwhile. If χ is equal to or greater than this value, then$\psi(\bar{\eta}) = 0$, and the density diverges and the ergodic density follows the power-law $f \propto (\eta - \bar{\eta})^{\alpha-2}$ near $\bar{\eta}$, which in turn can lead to infinite densities. Hence, the strength of the instability (*i.e.* the amount of probability mass near $\bar{\eta}$) is strongly controlled by the parameter χ.

This is illustrated in Fig. 3 where we show how the ergodic density changes as χ is varied. For low values of χ, there is no left-hand side peak in the model with rental (Fig. 3a) and f largely resembles that of the model without the option to rent (Fig. 3b). As χ approaches $\bar{\chi}$ (indicated by the dotted lines on the floor of the two panels of this figure, where$\bar{\chi} \simeq 0.55$ with rental, $\bar{\chi} \simeq 0.54$ without), then in the model with rental a probability mass starts to appear near $\bar{\eta}$. Crossing $\bar{\chi}$, recapitalisation becomes no longer an option, and the density at $\bar{\eta}$ diverges. Above $\bar{\chi}$ there is no longer χ dependence. Note that the distribution f changes quite sharply approaching $\bar{\chi}$ is crossed, with a second peak of the distribution emerging close to $\eta = \bar{\eta}$, a robust result across a variety of simulations.

To further explore this parameter dependence we have investigated how the median of f depends on various parameters. Since the value of $\bar{\eta}$ and η^*, the range on which the distribution is defined, also varies with the parameters, it is convenient to scale the median on to the interval $[0, 1]$: Let m be the median, then the scaled median is defined as

$$\tilde{m} = \frac{m - \bar{\eta}}{\eta^* - \bar{\eta}}, \qquad F(m) = \frac{1}{2}. \tag{22}$$

A value of $\tilde{m} \sim 0$ implies that most of the probability mass is concentrated near $\bar{\eta}$, while $\tilde{m} \sim 1$ suggests that firms are more probably found near η^*. While this is a somewhat crude measure (e.g. the median cannot distinguish between distributions that are $\cup-$ or $\cap$-shaped), nonetheless, $\tilde{m} \lesssim 1/2$ is a strong indicator of large mass of probability near the lower boundary and hence of the long lasting response to a large initial shock found by BS.

---

and $W(\bar{\eta}) \approx 0.05$. The interested reader can observe, using our standalone application, how increasing χ to above this critical level results in the emergence of singularities and the divergence of $f(\eta)$ to $+\infty$ at $\eta = \bar{\eta}$.
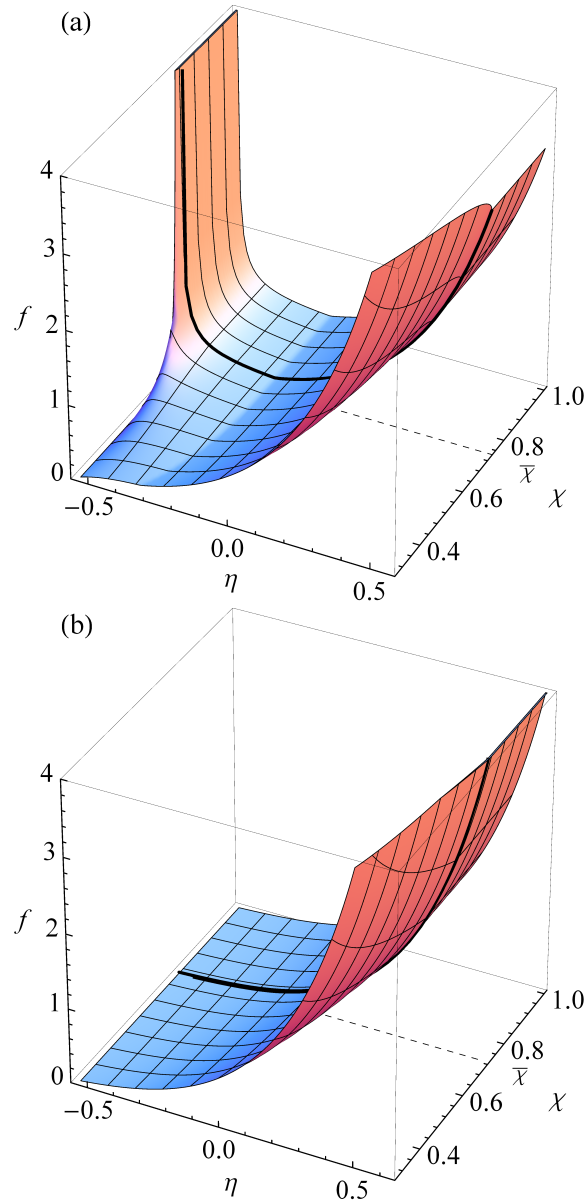
Figure 3: Comparison of ergodic densities f between the option to rent (a) and no option to rent (b) as the financing constraint χ is varied. Other parameters are set to baseline. The lower boundary is recapitalising upto $\chi = \bar{\chi}$ ($\bar{\chi} \simeq 0.64$ in (a), 0.62 in (b)), indicated by the thick solid line on the graph and dashed line on the axis. In (a) a left-boundary peak emerges for χ just less than $\bar{\chi}$. Density is infinite at $\bar{\eta}$ for $\chi > \bar{\chi}$. Note the complete absence of the left-hand side peak in (b).

In Fig. 4 we present a contour plot $\tilde{m}$ as a function of the financing constraint χ and the volatility σ (note that Fig. 3 represents a small slice of data presented in this figure). The solid heavy line represents the critical value $\bar{\chi}(\sigma)$, the firm choosing to recapitalise only when $\chi < \bar{\chi}(\sigma)$. Three roughly distinct regimes can be seen:

(i) the low volatility range $\sigma \lesssim 0.2$, in which the firm always prefers to recapitalise and where $\tilde{m} \gtrsim 0.8$ and so most of the probability is found near the dividend paying boundary.

(ii) a region where $\sigma \gtrsim 0.3$ and at the same time $\chi \gtrsim 0.5$, i.e. red region to the top right, where $\tilde{m} \sim 0$, and much of the probability mass is located near the left hand boundary

(iii) an intermediate transition range where small changes in either σ or in χ, result in a very substantial change in $\tilde{m}$. This transition is especially abrupt for high values of σ.

We have examined the behaviour of $\tilde{m}$ as a function of other model parameters, obtaining remarkably similar contour plots. For example as the relative impatience of shareholders $\rho - r$ is increased from relatively low to high values, there are also two distinct regions similar to those of Figure 4, with a relatively sharp transition in the balance of the probability distribution from near the upper boundary η^* to the lower boundary $\bar{\eta}$.

We report one further finding on our measure of induced aversion to cash flow risk $-\frac{W''}{W'}$. Induced risk aversion is, like ergodic instability, strongly parameter and model structure dependent. In the model with rental when firms do not recapitalise they become extremely risk-averse close to the lower boundary $\bar{\eta}$. This is revealed by an analysis of power-law behaviour of W at the lower boundary $\bar{\eta}$ (see Proposition 7). This extreme risk aversion does not arise in the model with rental or if recapitalisation is not costly.

This finding is illustrated in Figure (5) which compares induced risk-aversion for the two version of the model, with and without the option to rent. The parameters here are the same as in Figures (1) and (2). For relatively large values of η close to η^* the option to rent provides protection against cash flow risk and induced risk aversion $-\frac{W''}{W'}$ is lower for the model with rental; but as η falls down towards $\bar{\eta}$ then in the model with rental induced risk aversion $-\frac{W''}{W'}$ diverges upwards, rising increasingly rapidly as η approaches $\bar{\eta}$, whereas it rises only slightly in the model without rental.

We offer the following intuition for this result. It seems that, without an option to rent, the firm is rather like a boat in a stormy sea near a rocky shore, the probability of disaster is already very high, so it is worth taking some additional risk of shipwreck in order to escape the danger. But with the option to rent the situation is more like walking on a slippery slope near a cliff edge, by being very cautious and taking little risk eventually it is possible
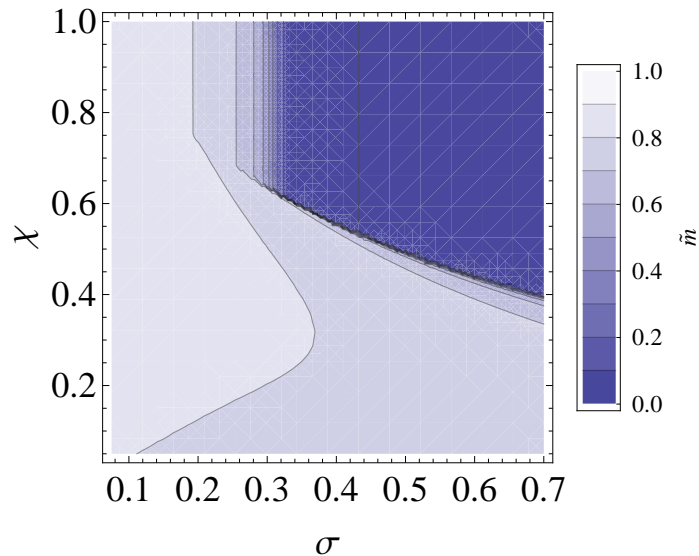
Figure 4: The scaled median $\tilde{m}$ as a function of χ and σ. Other parameters are set to baseline values, $\rho = 0.06$, $r = 0.05$, $a = 0.1$, $\bar{a} = 0.04$, $\delta = 0.02$, and $\theta = 15$. Contours are plotted at level values of $\tilde{m}$ and are spaced at intervals of 0.1.

to get away from the danger. Rental provides a slow and steady route away from the danger and risk must be radically reduced so as not to hamper this escape.

5   Conclusions

This paper investigates the possibility that a fall in corporate net worth can lead to a decline of output and investment that then remain below normal levels for extended periods of time (a 'net worth trap'). This possibility has been proposed by (Brunnermeier and Sannikov 2014) in a model closely related to our own, as an explanation of extended macroeconomic downturns. We reproduce this finding in a simple and tractable framework incorporating in one set up many of the insights of recent literature on corporate financing constraints (including for example (Bolton, Chen, and Wang 2011), Sec. 2 provides fuller references). One insight is the importance of 'corporate prudential saving', analogous to the household prudential saving extensively discussed in the literature on the consumption function (see (Carroll 2001)) as their net worth declines firms invest less and less (marginal q). The second is what we call 'induced risk aversion': firms with sufficiently high net worth
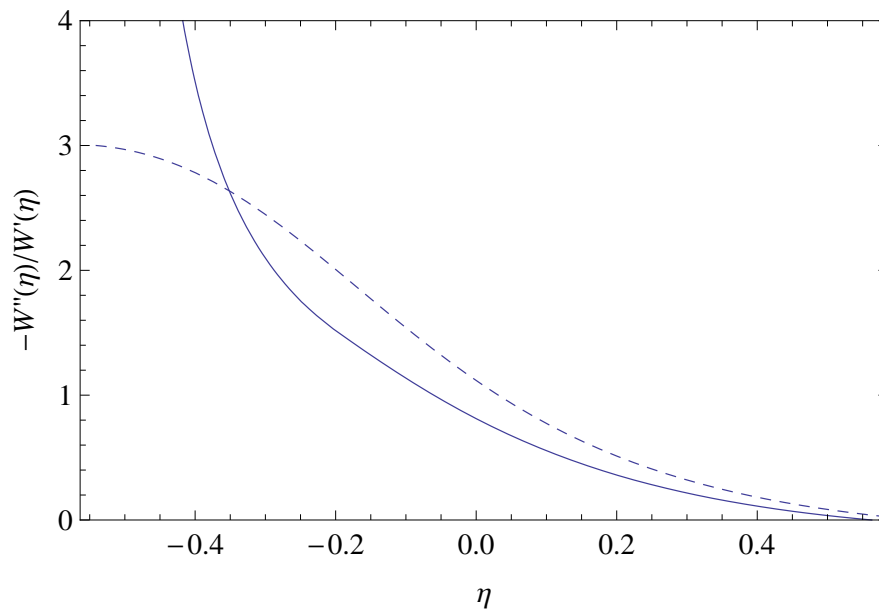
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Figure 5: Induced risk aversion $-W''/W'$ as a function of η. Parameters are set to baseline. Solid curve: model with option to rent; dashed curve: model without option to rent. Significantly, the risk aversion diverges strongly near $\bar{\eta}$ in the model without rent, in contrast to the model without rental.

have the same attitude to risk as their share holders (assumed for simplicity to be risk-neutral); but as net worth declines then firms behave increasingly as if they were averse to risk in order to reduce the probability of future liquidation or costly recapitalisation. As in the literature on corporate financial distress this in turn leads them to take actions that reduce their risk exposure, here by renting out more and more of their capital.

In our model it is this 'induced risk aversion', and the resulting incentive to rent out capital, that creates the 'net worth trap' found in (Brunnermeier and Sannikov 2014). Firms, following a large shock, retain only a small proportion of their productive capacity and can thus only very slowly build up their net worth and escape from the impact of financing constraints. We find though that the existence of this net-worth trap is both parameter and model structure dependent. It plays a role only when firms are able to rent capacity (compare our Figures 1 and 2); and it does so then only when the volatility of earnings or capital productivity are comparatively high, or the alternative policy of raising new external equity to reduce debt is relatively costly (as our Figures 3 and 4 illustrate).

We can complete our paper with a short discussion of the implications of these new perspectives on the dynamic interaction of corporate financing and operating decisions for macroeconomic modelling and policy. The mechanisms we capture have a similar impact on firm decisions as the standard model of the financial accelerator routinely employed in many macroeconomic models. The mechanism though is subtly different, operating not through an external financing premium placed on risky investment projects but instead because financing constraints result in shadow prices placed both on risk exposure and on use of internal finds. These shadow prices can impact on behaviour throughout the state space, not just on the constrained boundary (Whittle's 'fly paper effect' discussed in our introduction). The resulting behaviour varies qualitative and quantitatively both with parameterisation (Figure 4 illustrates the sensitivity to parameterisation) and model specification (Figure 5 reports our measure of induced risk aversion and how this can alter substantially depending upon whether capital can be rented out or not).

This possibility of prudential corporate saving and of induced risk aversion in a dynamic setting suggests an impact of indebtedness on a wider range of firm decision making (for example capacity utilisation, employment, pricing, inventory holding) than is usually considered in the standard theory of the financial accelerator; and also that such financial constraints may affect a relatively wide range of firms, not just start-up companies engaged in technological innovation.

Our modelling also highlights how constraints in the access to finance

mean that firm decisions can be highly *non-linear* functions of their indebtedness. This in turn helps to clarify when a linearisation – of the kind routinely employed in new Keynsian macroeconomic models – provides a reasonable approximation to the fully dynamic optimal behavior. Allowing for non-linear behavior can be necessary, either when firms are financially weak (generating comparatively low expected earnings or compared to cash flow uncertainty) or when there is an unexpectedly large negative shock driving many firms well below their desired ratio of debt to fixed capital.

In normal times the impact is likely to be relatively small and sufficiently well captured in standard linearised specifications. This is because volatility of earnings or output are within anticipated ranges, so firms are adequately hedged against both aggregate and idiosyncratic risk; and external equity capital can, if necessary, be raised. While firms may well increase their borrowing following a shock, it is not too difficult for them to pay down this additional debt relatively quickly.

In times of financial and economic stress the situation is quite different. In such periods the volatility of earnings and output can rise substantially, to the point where firms are inadequately hedged against risks, and recapitalisation may also become difficult or impossible. As our Figure 4 illustrates such an adverse change in the economic environment can lead to a 'phase change': a shift to a regime where the net worth trap emerges, with firms struggle to rebuild net worth; and the response to large aggregate shocks is then as reported by BS a deep and long lasting reduction in output and investment.

A further implication is that we should not expect to be able to build a single macroeconomic model that precisely captures the financial accelerator both in normal times and its magnification in periods of stress. The essence of the 'net worth trap' is its unpredictability, it is a trap precisely because firms and households do not properly anticipate the danger of falling into it or the extent to which it can emerge as a consequence of economy wide problems.

This also suggests caution about claims for the effectiveness of policy at averting the systemic risk associated with the net-worth trap and responding if and when it materialises. It is difficult to effectively employ an activist macroprudential policy to avert a 'net worth' trap. There is no easy way of quantifying the risk of it arising; and should a net worth trap emerge there are then no easy policy options available in order to escape it. For example, our modelling suggests that compulsory aggregate recapitalisation (swapping debt for equity) might be an effective response in a crisis situation; but this will not be so easy to implement in practice since, like any action to reduce debt, it will have substantial distributional impacts not incorporated at all into our representative agent modelling.

Perhaps the most important lesson is the need for macroeconomic policy makers remain vigilant and open minded; ensuring that they employ a variety of tools (ranging from formal quantitative models of many different kinds to historically informed qualitative data analysis) to alert them to danger of a 'net worth trap'; and that they use these same tools to ensure firms and households are fully aware of the potential impact of high levels of debt in economic downturns and hence encourage them to take steps, when the economic environment is comparatively benign, to reduce their own indebtedness to prudent levels. Formal modelling of the kind we explore here is an important input to this process, but is only one of many relevant sources of insight and information.

A   Solution in the absence of the non-negativity constraint on dividends

This appendix considers the solution to the model of this paper in the baseline case where dividend payments can be negative or, equivalently, there is no uncertainty. This provides a benchmark for studying and solving the case of the constrained firm for which there is uncertainty and dividends are required to be non-negative. It also yields a convenient formula for the maximum amount of borrowing provided by households to firms.

A crucial intuition emerges from this benchmark model, one that we need to keep in mind when we solve the model with a non-negativity constraint on dividend payments. The rate of growth preferred by firms is an increasing function of the ratio of debt to capital (this is because debt increases at the same rate of growth of capital, creating an additional cash flow that can be used for investment, and the higher the ratio of debt to capital the greater this cash flow). If the financing constraint is sufficiently lax then it is possible for firms can achieve a growth rate equal to their own rate of discount while still being able to pay dividends. In this case the objective of this firm (expected discounted dividend payments) is unbounded and the solution is no longer meaningful. Therefore some financing constraint is required in order for the model to have a meaningful solution.

To solve this benchmark note that, since firm owners can freely transfer funds into or out of the firm, optimal policy is to maintain the ratio of cash balances $\eta = c/k$ at whatever rate is preferred by borrowers, subject to the highest level of indebtedness allowed by lenders $\eta \geq \bar{\eta}$. If the initial time $t = 0$ ratio η_0 differs from the desired ratio η then an instantaneous dividend payment of $(\eta_0 - \eta) k$ is immediately made to bring the cash to capital ratio to the desired value of η.

There is therefore now only a single state variable k. The value function (the value of the objective function under optimal policy) is linearly homogeneous in k and so can be written $V = kW$ where W is a constant that depends on the parameters representing preferences and the evolution of the state variable k. This in turn implies that $V_k = W$ and $V_{kk} = 0$. Expected dividend payments will be determined by the expected net cash flow of the firm plus any additional borrowing possible because k and hence c are growing. The remaining policy decision is to choose a rate of investment i and hence expected growth of the capital stock $g = i - \delta$ to maximise Ω, Eq. (2)

The solution can be summarised in the following proposition.

Proposition 8 *Assuming $\rho > r$ then an optimal policy yielding positive payoffs for the owners of the firm can be found provided that:*

$$-2\frac{\rho - (a - \delta) + (\rho - r)\bar{\eta}}{\rho^2} < \theta < \begin{cases} \infty & \text{if } a + r\bar{\eta} \geq \delta \\ \dfrac{1}{2}\dfrac{(1 + \bar{\eta})^2}{\delta - r\bar{\eta} - a} & \text{if } a + r\bar{\eta} < \delta \end{cases} \quad (23)$$

in which case an instantaneous dividend payment of $\eta_0 - \bar{\eta}$ is made so that $\eta = \bar{\eta}$, the growth rate of the capital stock is constant (state independent) and is given by:

$$g = \rho - \sqrt{\rho^2 - 2\theta^{-1}[a - \delta - \rho + (r - \rho)\bar{\eta}]} < \rho, \qquad (24)$$

while the value of the maximised objective is given by:

$$V(\eta_0, k) = (\eta_0 - \bar{\eta})\, k + \frac{(a - \delta) + (r - g)\bar{\eta} - g - \frac{1}{2}\theta g^2}{\rho - g} k = (1 + \eta_0 + g\theta)\, k \quad (25)$$

where $\left[(a - \delta) + (r - g)\bar{\eta} - g - \frac{1}{2}\theta g^2\right] k$ is the expected flow of dividends per period of time paid to shareholders.

Proof. The firm has two choice variables η and g (with investment expenditure given by $ik = (g + \delta)\, k$ and associated quadratic adjustment costs of $\frac{1}{2}\theta\,(i - \delta)^2 = \frac{1}{2}\theta g^2$). The equations of motion (1) still apply and dividends are paid according to:

$$\lambda\, \mathrm{d}t = \left[(a - \delta) + (r - g)\,\eta - g - \frac{1}{2}\theta g^2\right] k\, \mathrm{d}t + \sigma k\, \mathrm{d}z$$

Substituting for λ the discounted objective can be written as:

$$\Omega = \max_{\eta, g} \left\{ \mathbb{E} \int_0^\infty \mathrm{e}^{-\rho t} \left[(a - \delta) + (r - g)\,\eta - g - \frac{1}{2}\theta g^2\right] k\, \mathrm{d}t \right.$$
$$\left. + (\eta_0 - \eta)\, k_0 + \int \mathrm{e}^{-\rho t}\sigma k\, \mathrm{d}z \right\} \quad (26)$$

yielding, since $\mathbb{E}[k] = k(0)\exp(gt)$ and $\mathrm{e}^{-\rho t}\sigma k = 0$:

$$\Omega = k(0) \max_{\eta, g} \left[\eta_0 - \eta + \frac{(a - \delta) + (r - g)\,\eta - g - \frac{1}{2}\theta g^2}{\rho - g} \right]. \qquad (27)$$

The growth rate g that maximises the right hand side of this expression is determined by the first order condition w.r.t. g

$$\frac{1}{2}g^2 - \rho g - \theta^{-1} \left[\rho - (a - \delta) + (\rho - r)\eta\right] = 0 \qquad (28)$$

yielding the solution (the positive root of the quadratic can be ruled out because we require that $g < \rho$; this ensures that the value function is finite and that the second order condition for maximisation is satisfied):

$$g = \rho - \sqrt{\rho^2 + 2\theta^{-1}[\rho - (a - \delta) + (\rho - r)\eta]}. \qquad (29)$$

Writing $\rho - g = \sqrt{\rho^2 + 2\theta^{-1}[\rho - (a - \delta) + (\rho - r)\eta]} = R$, implying $g^2 = \rho^2 - 2\rho R +$

R^2, and substituting into Eq. (27) then yields: (25).

The indebtedness is determined by the first order condition in (27) w.r.t η:

$$\frac{r-g}{\rho-g} - 1 = \frac{r-\rho}{\rho-g} < 0$$

establishing that the firm will seek to borrow as much as it possibly can. Hence the firm will make an instantaneous dividend at time $t = 0$ to reduce η as far as possible, until the borrowing constraint binds so $\eta = \bar{\eta}$. The first inequality on θ in the proposition ensures that the borrowing constraint does indeed bind at a level of borrowing at which Eq. (28) has real roots.

The remaining inequality conditions on θ ensure that it is possible to achieve positive dividends per unit of capital (these are relatively weak conditions since normally we would expect $a > \delta$ in which case a policy of zero growth $g = 0$ and no indebtedness will always yield positive dividends; but if depreciation is larger than the productivity of capital then a further restriction on θ is required). To establish these further conditions note that expected dividends per unit of capital $a - \delta + r\bar{\eta} - (1 + \bar{\eta})g - \frac{1}{2}\theta g^2$ are maximised by choosing $g = -(1 + \bar{\eta})\theta^{-1}$ resulting in expected dividend payments of $\lambda = a - \delta + r\bar{\eta} + \frac{1}{2}\theta^{-1}(1 + \bar{\eta})^2$. This is always greater than zero if $a > \delta$, otherwise this requires that $\theta < (1 + \bar{\eta})^2/2(\delta - a - r\bar{\eta})$. ∎

Finally note that the fundamental valuation of a firm's capital by outside investors can be obtained by substituting $r = \rho$, $a = \bar{a}$ and $\bar{\eta} = 0$ into this solution. A finite positive valuation is obtained provided the parameters satisfy:

$$2\frac{\bar{a} - \delta - r}{r^2} < \theta < \begin{cases} \infty & \text{if } \bar{a} \geq \delta \\ \dfrac{1}{2}\dfrac{1}{\delta - \bar{a}} & \text{if } \bar{a} < \delta \end{cases}$$

in which case the growth rate (when held by outside investors) is given by

$$\bar{g} = r - \sqrt{r^2 - 2\theta^{-1}[\bar{a} - \delta - r]},$$

and the value of the maximised objective by

$$V = \frac{\bar{a} - \bar{g} - \delta - \frac{1}{2}\theta\bar{g}^2}{r - \bar{g}}k = (1 + \theta\bar{g})\,k.$$

With this background we have an immediate proof of Proposition 1 in Section 3

Proof of Proposition 1. This valuation of the firm's assets by outside investors is also the maximum amount of debt that it can borrow from these investors, implying that the lower boundary for η is given by Eq (3) ∎

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B Proofs of propositions in Sections 3 and 4

Proof of Proposition 5. (Proposition 3 require no separate proof, since it is the special case when $\sigma_2 = 0$ and $\psi = 1$). While uniqueness of solution can be established using standard arguments based on the non-convexity of the optimisation program, we instead prefer a geometric proof which offers some additional insight into both the existence of solution and its numerical calculation.

Applying standard methods of stochastic dynamic programming, with two state variables k and c, the optimal policy by firms, at times when there is no recapitalisation ($\epsilon_t = 0$) satisfies the Hamilton-Jacobi-Bellman equation:

$$\rho V = \max_{i,\lambda,\psi} \left\{ \lambda + \left[-\lambda + (\bar{a} + (a - \bar{a})\psi)\,k + rc - ik - \frac{1}{2}\theta(i - \delta)^2 k \right] V_c \right. \tag{30}$$

$$\left. + (i - \delta)kV_k + \frac{1}{2}\sigma_1^2\psi^2 k^2 V_{cc} + \frac{1}{2}\sigma_2^2\psi^2 k^2 V_{kk} \right\}, \tag{31}$$

with three first order conditions for maximisation. The first is:

$$\begin{cases} \lambda \geq 0 \text{ of unbounded magnitude,} & V_c = 1 \\ \qquad\qquad \lambda = 0, & V_c > 1 \end{cases},$$

there is 'bang-bang' control with two distinct regions of dividend behaviour: one when $c \geq c^*(k)$ with $V_c = 1$ in which policy is to payout a discrete dividend to reduce cash holdings immediately to the dividend paying boundary c^*; the other when $c < c^*(k)$ where there is no payment of dividends and $V_c > 1$. The second first-order condition is:

$$(1 + \theta(i - \delta))\,V_c = V_k$$

yielding the investment rule:

$$i = \delta + \theta^{-1}\left(\frac{V_k}{V_c} - 1\right). \tag{32}$$

The third first order condition for maximisation (subject to the constraint $0 \leq \psi \leq 1$) is:

$$(a - \bar{a})kV_c + \psi k^2\left(\sigma_1^2 V_{cc} + \sigma_2^2 V_{kk}\right) = 0$$

yielding the final control rule:

$$\psi = \max\left\{\min\left\{(a - \bar{a})\left[-k\frac{\sigma_1^2 V_{cc} + \sigma_2^2 V_{kk}}{V_c}\right]^{-1}, 1\right\}, 0\right\}. \tag{33}$$

Because of the linearity of production the value function is linearly homogeneous in

k and so we can work with the value function W of a single state variable $\eta = c/k$:

$$W(\eta) = k^{-1}V(c,k) = V(\eta, 1) \tag{34}$$

implying the substitutions $V = kW$, $V_c = W'$, $V_k = W - \eta W'$, $q = V_k/V_c = W/W' - \eta$, $V_{cc} = k^{-1}W''$, $V_{ck} = -k^{-1}\eta W''$ and $V_{kk} = k^{-1}\eta^2 W''$

Substituting for both optimal policy and for V and its derivatives yields Eq. 14. The maximisation in this second boundary condition reflects the choice available to the firm when η falls to $\bar{\eta}$, it may choose either to liquidate in which case $W(\bar{\eta}) = 0$, or to recapitalise which is worth doing if it can achieve a higher valuation after paying the fixed cost of recapitalisation χk. Turning to the uniqueness of this solution, note that as discussed in Appendix C solution of the upper boundary η^* is characterised by Eq. (10) (itself obtained from 6 using $q = W/W' - \eta$ which yields $q' = -WW''/(W')^2$). Eq (10) can be written (allowing for dependencies of σ on η) as:

$$q' = \frac{2}{\sigma_1^2 + \eta^2 \sigma_2^2} Q(q, \eta)(q + \eta)$$

from which, since $\sigma_1^2 + \eta^2 \sigma_2^2 > 0$ and $q^* + \eta^* > \bar{q} + \bar{\eta} \geq 0$ and $Q(q, \eta) = a - \delta - (\rho - r)\eta - \rho q + \frac{1}{2}\theta^{-1}(q-1)^2$ is a quadratic function of q and linear function of η. This in turn implies that the possible locations of q^* is given by $Q(q, \eta) = 0$ i.e. a parabola in (q, η) space, which solves to yield the location of η on the dividend paying boundaries as a function of q^*:

$$\eta^* = \frac{a - \delta - \rho q^* + \frac{1}{2}\theta^{-1}(q^* - 1)^2}{\rho - r} \tag{35}$$

and we can invert this equation to solve for $q = q^*$ on the dividend paying boundary, yielding:

$$q^* = 1 + \theta\left(\rho \pm \sqrt{\rho^2 - 2\theta^{-1}\{a - \delta - \rho - \eta^*(\rho - r)\}}\right) \tag{36}$$

Uniqueness of solution then follows (assuming continuity of $q(\eta)$) from noting that the value of $q = q^*$ is a function of the value of $q(\bar{\eta}) = \bar{q}$ on the lower boundary. Given any starting value $\bar{q}$ the ODE characterising the solution can be computed (with $q' > 0$) until it meets $Q(q, \eta) = 0$. There can only be one such intersection. Having crossed $Q(q, \eta) = 0$, $q' < 0$ until there is another intersection, and this means any potential second intersection can only take place on the lower branch of $Q(q, \eta) = 0$. But in order for there to be an intersection on this lower branch it is necessary that the $q-$curve falls faster than the lower branch i.e. that on the point of intersection:

$$q' < \left.\frac{\partial q}{\partial \eta}\right|_{Q(q,\eta)=0} < 0$$

which contradicts the requirement that $q' = 0$ on $Q(q, \eta) = 0$. This contradiction shows that any solution of the ODE has at most one, unique, intersection with
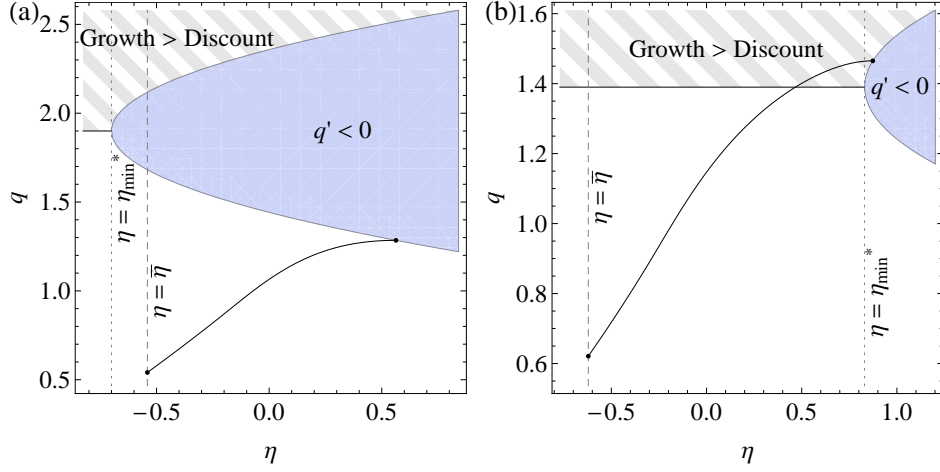
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Figure 6: Illustration of Proof of Prop. 2. In subfigure (a), the 'no-Ponzi' condition, Eq. (37) holds: the initial η is below the dividend paying boundary, and so the solution is guaranteed to hit it. Subfigure (b) shows a scenario where the condition does not hold: The solution starts from a point to the left of $\eta^*_{\min}$, and grows fast enough to miss the lower branch of $q' = 0$ curve, entering a region where growth exceeds the discount rate. In (a), parameters are set to baseline, $\rho = 0.06$, $r = 0.05$, $\sigma_1 = 0.2$, $\sigma_2 = 0.0$, $a = 0.1$, $\bar{a} = 0.04$, $\delta = 0.02$, $\theta = 15$, and $\chi = 0.75$; in (b), parameters are the same, except that $\theta = 6.5$.

$Q(q, \eta) = 0$. ∎[9]

This proof does not establish existence. While there can only be one solution to the ODE for W satisfying the boundary conditions of Proposition 5, the existence of this solution is dependent on parameter values. Prop. 2 gave sufficient conditions for a solution to exist.

Proof of Proposition 2. First note that Eq. (4a) is equivalent to

$$\bar{\eta} > \eta^*_{\min} = -\frac{\rho - (a - \delta) + \frac{1}{2}\theta\rho^2}{\rho - r}, \tag{37}$$

where $\eta^*_{\min}$ is the minimum value of η on the dividend paying boundary. We refer to this condition as the 'no-Ponzi' condition because as stated in Proposition 8 in Appendix A when $\bar{\eta} > \eta^*_{\min}$ is satisfied, then the solution to the problem in the deterministic limit $\lim_{\sigma \downarrow 0}$ exists in which the growth of the fixed capital stock is less than the discount rate of firm shareholders $g < \rho$ and the value to shareholders comes from both growth of the capital stock and dividend payments.

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[9] The firm will never choose to recapitalise to a value of $\eta < \eta^*$. This is because when $\eta < \eta^*$ $W'(\eta) > 1$, so the maximum possible value of $W(\bar{\eta})$ in 7 is achieved by a full recapitalisation up to η^*.

The idea of proof is illustrated in Fig. 6. Consider possible solutions of the ODE for $q(\eta)$. In the case of no recapitalisation $\bar{q} = -\bar{\eta}$ and the lower intersection of $Q(q, \eta) = 0$ with $\eta = \bar{\eta}$ is at $q = q_-^* = 1 + \theta(\rho - \sqrt{\rho^2 - 2\theta^{-1}\{a - \delta - \rho - \bar{\eta}(\rho - r)\}})$. This implies (using Eq. (3)) that:

$$q_-^* - \bar{q} = 1 + \theta\left(\rho - \sqrt{\rho^2 - 2\theta^{-1}\{a - \delta - \rho - \bar{\eta}(\rho - r)\}}\right) + \bar{\eta} > 0$$

This shows that a solution with no recapitalisation exists, because the ODE begins at a point strictly below q_-^* and since $q' > 0$ must eventually intersect with $Q(q, \eta) = 0$. This in turn implies the existence of solutions with recapitalisation, since these are associated with higher values of $\bar{q}$ satisfying $-\bar{\eta} < \bar{q} < q_-^*$, in all cases with the ODE eventually intersecting with the lower branch of $Q(q, \eta) = 0$; and with values of $\chi > 0$. Eventually in the limit $\lim_{\chi \downarrow 0} \bar{q} = q_-^*$ ∎

Some additional intuition into the factors that determine if the 'no-Ponzi' condition is satisfied or not can be obtained by re-expressing Eq. (37) as

$$\bar{g}^* < \left[\frac{1}{2}(\rho - g^*)^2 / (\rho - r) - \theta^{-1}\right]$$

where

$$\bar{g}^* = \left(r - \sqrt{r^2 - 2\theta^{-1}[\bar{a} - \delta - r]}\right) < r$$

is the rate of growth when capital stock is owned by external investors and

$$g^* = \left(\rho - \sqrt{\rho^2 - 2\theta^{-1}[\bar{a} - \delta - \rho]}\right) < \rho$$

the rate of growth of the capital stock in the situation where firms can costlessly issue equity ($\chi = 0$) but are unable to borrow (see Appendix A). This expression indicates that in order for the 'no-Ponzi' to be satisfied requires either that the difference between the discount rates of firms and outside investors $\rho - r$ is comparatively small or the net productivity of capital either in the hands of firms or investors ($a - \delta$, $\bar{a} - \delta$) relative to the maximum values given by the constraints of Eqs. (4a,4b) are comparatively small or the costs of adjustment of capital θ are comparatively high.

What about solution in the stochastic case if the 'no-Ponzi' condition is not satisfied? Our numerical computations can still be applied and indicate that an optimal policy for choice of $\{i_t\}, \{\lambda_t\}, \{\epsilon_\tau\}$, satisfying the conditions of Proposition 3, i.e. with future dividend payments after any initial dividend payment to reduce η to the desired target level η^*, may still exist, provided that g^* is not too close to ρ.

Finally we prove the propositions about the ergodic density.

Proof of Proposition 4. (Proposition of Section 3 can again be obtained by imposing appropriate parameter restrictions.) We denote the density function for the location of firms across the possible values of η at the moment t by $f(t, \eta)$,

with the corresponding cumulative density:

$$F(t, \eta) = \int_{\bar{\eta}}^{\eta} f(t, \eta') \, \mathrm{d}\eta', \quad F(t, \eta^*) = 1$$

The evolution of $f(t, \eta)$ is then determined by the Kolmogorov forward, or Fokker-Planck, equation:

$$\frac{\partial f}{\partial t}(t, \eta) = -\frac{\partial}{\partial \eta} \left[\mu^\eta(\eta) f(t, \eta) \right] + \frac{1}{2} \frac{\partial^2}{\partial \eta^2} \left[\sigma^\eta(\eta)^2 f(t, \eta) \right], \tag{38}$$

where η follows the equation of motion

$$\mathrm{d}\eta = \mu^\eta \, \mathrm{d}t + \sigma^\eta \, \mathrm{d}z.$$

with coefficients obtained simply by Itô differentiating $\eta = c/k$:

$$\mathrm{d}\eta = \frac{1}{k} \left[\mu^c - \eta\mu^k + \frac{\eta}{k}(\sigma^k)^2 \right] \mathrm{d}t + \frac{\sigma^c}{k} \, \mathrm{d}z_1 - \eta\frac{\sigma^k}{k} \, \mathrm{d}z_2, \tag{39}$$

where $\mu^{c,k}$ and $\sigma^{c,k}$ are respectively the drift and diffusion terms for c and k: $\mathrm{d}c = \mu^c \mathrm{d}t + \sigma^c \mathrm{d}z_1$, $\mathrm{d}k = \mu^k \mathrm{d}t + \sigma^k \mathrm{d}z_2$ i.e.:

$$\mu^c = [\psi a + (1 - \psi)\bar{a} - \delta + r\eta - \theta^{-1}(q - 1) - \frac{1}{2}\theta^{-1}(q - 1)^2]k,$$
$$\mu^k = \theta^{-1}(q - 1)k,$$
$$\sigma^c = \sigma_1\psi k, \quad \sigma^k = \sigma_2\psi k.$$

The increments $\mathrm{d}z_1$, $\mathrm{d}z_2$ are independent and normal distributed, and so the noise sources in $\mathrm{d}\eta$ can be combined into a single term,

$$k^{-1}\sigma^c \mathrm{d}z_1 - k^{-1}\eta\sigma^k \mathrm{d}z_2 = k^{-1}\sqrt{(\sigma^c)^2 + \eta^2(\sigma^k)^2} \, \mathrm{d}z.$$

Substituting in the expressions for $\mu^{c,k}$ and $\sigma^{c,k}$ we have

$$\mathrm{d}\eta = \left[\bar{a} + (a - \bar{a})\psi - \delta + r\eta - (1 + \eta)\theta^{-1}(q - 1) \right.$$
$$\left. - \frac{1}{2}\theta^{-1}(q - 1)^2 + \eta\sigma_2^2\psi^2 \right] \mathrm{d}t + \sqrt{\sigma_1^2 + \eta^2\sigma_2^2}\psi \, \mathrm{d}z. \tag{40}$$

The ergodic probability density is then the stationary, $\partial f/\partial t = 0$, solution of Eq. (38), which we also denote by $f(\eta)$. Integration of the Kolmogorov forward equation in η yields

$$d = \mu^\eta(\eta)f(\eta) - \frac{1}{2}\frac{\partial}{\partial \eta} \left[\sigma^\eta(\eta)^2 f(\eta) \right]. \tag{41}$$

We find it convenient to write in terms of ϕ,

$$\phi = \frac{(\sigma^\eta)^2}{2} f,$$

so this becomes:

$$d = \left[\frac{(\sigma^\eta)^2}{2} \right]^{-1} \mu^\eta(\eta)\phi(\eta) - \frac{1}{2}\frac{\partial}{\partial \eta}\phi(\eta). \tag{42}$$

and this yields Eq. (15). ∎

C   Numerical solution

C.1   Preliminary considerations and some economic intuition

The ordinary differential equation governing q (Eq. (17) for $\eta \leq \tilde{\eta}$ and Eq. (10) for $\eta \geq \tilde{\eta}$) can be solved by forward integration using standard methods starting from any given initial condition $q(\bar{\eta}) = \bar{q}$. Solution is completed by finding an intersection with $Q(q, \eta) = 0$ on which $q' = 0$, if one exists, or establishing that there is no such intersection (moreover any solution with an intersection with $Q(q, \eta) = 0$. Any initial value $\bar{q} \geq 1 + \theta\rho$ can be ruled out, since it implies that $g > \rho$ for all η.

The inequality in Eq. (37), which is required if the special case of the model with no uncertainty ($\sigma = 0$) is to a be one with no 'Ponzi-borrowing' and also ensures the existence of solution, leads to extremely straightforward numerical solution, since intersection with the lower branch of $Q(q, \eta) = 0$ is guaranteed.

If however this inequality is not satisfied then for some values of $\bar{q}$, a value of q^* where the ODE interacts with $Q(q, \eta) = 0$ may be located on the upper boundary (if the ODE 'misses' the lower branch in which case it may or may not hit the upper branch). This considerably complicates the search for numerical solution because it is no longer possible to restrict the initial values $\bar{q}$ to a range of values for which intersection with $Q(q, \eta) = 0$ is guaranteed.

Such solutions with upper branch intersections are of less economic interest than those where intersection is on the lower branch. It is possible that investment close to η^* is so high that the firm has negative cash flow. This can be seen by substituting Eq. (35) ,$\psi = 1$, $q = q^*$ and $\eta = \eta^*$ into Eq. (40), yielding the following expression for cash flow on the dividend paying boundary:

$$\mu^\eta = \frac{(a - \delta)\rho - \rho r + \left[(r - (a - \delta))\theta^{-1} - \rho r \right](q^* - 1)}{\rho - r}$$
$$+ \frac{\frac{1}{2}(2r + \rho)\theta^{-1} - \frac{1}{2}\theta^{-2}(q^* - 1)}{\rho - r}(q^* - 1)^2 + \eta^*\sigma_2^2 \tag{43}$$

which, for sufficiently high q^*, is negative. The economic intuition in this case is similar to that applicable to the 'Ponzi' solution of the model with no non-

negativity constraint on dividends in Appendix A. The firm creates most value not by dividend payments but from growing the capital stock at a rate close to and often above the shareholder rate of discount and this very high rate of investment can the generates negative expected cash flows.

We can find the function $W(\eta^*)$ by substituting into Eq. (36) on the boundary η^*, using the boundary conditions $W' = 1$, to yield:

$$W^* = (q^* + \eta^*) W' = 1 + \eta^* + \rho\theta \pm \sqrt{2\theta \left\{ (\eta^* - \eta^*_{\min}) (\rho - r) \right\}} \qquad (44)$$

with the positive root applying on the upper branch of q^* and the negative root on the lower branch.

This in turn results in some useful insights into the solution. In the case of recapitalisation Eq. (7) can be written:

$$0 < W^* - (\eta^* - \bar{\eta} + \chi) = 1 + \rho\theta + \bar{\eta} - \chi \pm \sqrt{2\theta \left\{ (\eta^* - \eta^*_{\min}) (\rho - r) \right\}} < 1 + \rho\theta + \bar{\eta} \quad (45)$$

where the first inequality is required by the maximisation in Eq. (7) and the second because the presence of financing constraints must lower value $W(\bar{\eta})$ relative to the valuation for the case of no non-negativity constraint on dividend payments given by Eq. (25)). This establishes the following further proposition:

Proposition 9 *A solution with recapitalisation (for some sufficiently low value of χ) exists. Let $\chi = \chi_0$ be the critical value of χ at which the firm is indifferent between recapitalisation or liquidation. Then: (i) if $\eta^*_{min} \leq \bar{\eta}$ a solution exists with η^* on the lower branch of $Q(q, \eta) = 0$; $\bar{q}$ satisfies:*

$$-\bar{\eta} \leq \bar{q} < \bar{q}max = 1 + \theta \left(\rho - \sqrt{\rho^2 - 2\theta^{-1} \left\{ a - \delta - \rho - \bar{\eta}(\rho - r) \right\}} \right) > \bar{q} \quad (46)$$

and the maximum possible value of η^* satisfies:

$$\eta^*_{min} \leq \eta^* < \eta^*_{min} + \frac{\left((\rho - r) + \sqrt{r^2 - 2\theta^{-1}[\bar{a} - \delta - r]} - \theta^{-1}\chi_0 \right)^2}{2(\rho - r)} \quad (47)$$

(ii) If instead $\eta^*_{min} > \bar{\eta}$ then $-\bar{\eta} \leq \bar{q} < 1 + \rho\theta$; a solution may or may not exist solution may be on the upper branch of $Q(q, \eta) = 0$ in which case η^* satisfies :

$$\eta^*_{min} < \eta^* \leq \eta^*_{min} + \frac{\chi^2}{2\theta(\rho - r)} \quad (48)$$

Proof. The existence of a solution with recapitalisation is guaranteed because $1 + \rho\theta + \bar{\eta} = (\rho - r)\theta + \theta\sqrt{r^2 - 2\theta^{-1}[\bar{a} - \delta - r]} > 0$. As noted above we can rule out solutions for which $\bar{q} > 1 + \theta\rho$ and hence all possible solutions, with recapitalisation or without, are with an intersection of the ODE for q on the lower branch of $Q(q, \eta) = 0$; and (the value that applies when cost of recapitalisation

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$\chi = 0$ and hence the maximum possible value of $\bar{q}$) is given by the intersection of this lower branch of Eq. (36) with $\eta = \bar{\eta}$. If solution is on the lower branch then the largest possible value of η^* and the smallest value of $\bar{q}$ ($\bar{q} = -\bar{\eta}$) arises when $\chi = \chi_0$ (the same solution also applies if χ is higher than this critical value and no-recapitalisation takes place). If $\chi = \chi_0$ then the first inequality in Eq. (45) binds and this implies the second inequality in Eq. (47). If instead solution is on the upper branch then the largest possible value is when $\chi < \chi_0$, so that Eq. (45) binds and this implies the second inequality in Eq. (48). ∎

Proposition 9 helps guide the numerical solution. If $\eta^*_{\min} \leq \bar{\eta}$ then a solution with a bounded value function exists and an intersection is guaranteed on the lower branch of $Q(q, \eta) = 0$. A first calculation of the case with no recapitalisation determines χ_0 and this can then be used to limit the scope of iteration on $\bar{q}$ (using Eq. (46)) in the search for solution in the case of recapitalisation. If instead $\eta^*_{\min} > \bar{\eta}$ then a solution with a bounded value function and finite η^* may not exist. The existence of a solution for the case of no-recapitalisation can be established by computing the ODE Eq. (10) upwards. If η exceeds the upper bound given by Eq. (48) then there is no intersection and no solution) is not satisfied, then while there is an intersection there is no finite solution to the value function. A solution with recapitalisation will exist for at least some values of χ if there is a solution for no-recapitalisation. The proposition then provides a slightly different limits on the scope of iteration on $\bar{q}$ and the same criteria can be applied to establish if there is an intersection with $Q(q, \eta) = 0$, and if so whether this represents a finite value for the value function.

C.2   Model without option to rent

For any given $\bar{q}$ the right-hand side boundary at $\eta = \eta^*$ where $q'(\eta^*) = 0$ is found by evaluating the function $q'(\eta)$ during the integration. After a single integration step is found to bracket a root of $q'(\eta)$, the critical value of η is pin-pointed using standard root finding methods, here the Brent's method.

The value function W can be solved from $W' = W/(\eta + q)$ parallel to integrating the equation for q. The boundary condition $W''(\eta^*) = 0$ will be satisfied since the q variable integration is stopped at $q' = 0$. In order to also satisfy the boundary condition $W'(\eta^*) = 1$, we solve W for an arbitrary initial value at $\bar{\eta}$. Let the resulting solution be $\tilde{W}$. Since the ODE for W is linear and homogeneous, we can simply multiply $\tilde{W}$ ex post by $[\tilde{W}'(\eta^*)]^{-1}$ to get a solution for which $W'(\eta^*) = 1$.

In the case of liquidation, the lower boundary is $\bar{\eta} = -\bar{q}$, and consequently, the derivative of W, $W' = W/(\eta + q)$ cannot be evaluated. In Appendix D we have shown that $W \propto \eta - \bar{\eta}$, and so $W'(\bar{\eta})$ is finite. If $\bar{\eta}$ is indeed liquidating, we simply set $\tilde{W}'(\bar{\eta}) = 1$ and $W(\bar{\eta}) = 0$.

We are solving for the ergodic density with an absorbing boundary which means we need to determine the constant of integration (the rate of flow across the boundary) d and this requires two boundary conditions. These conditions are that the

absorbing boundary must have a zero density i.e. $f(\bar{\eta}) = 0$ and the cumulative density must satisfy $F(\eta^*) = 1$.

To enforce these conditions we use the following method. We solve two independent differential equations for two densities f_0 and f_1 satisfying:

$$f_0'(\eta) = \frac{2\mu(\eta)}{\sigma^2} f_0(\eta), \quad f_1'(\eta) = \frac{2\mu(\eta)}{\sigma^2} f_1(\eta) + 1. \tag{49}$$

These are obtained by integration starting from arbitrary non-zero initial conditions. Let F_0 and F_1 be the resulting corresponding cumulative functions, with $F_0' = f_0$, $F_1' = f_1$ and $F_0(\bar{\eta}) = F_1(\bar{\eta}) = 0$. This determines values for $F(\eta^*)$ and $F_1(\eta^*)$.

We then find the ergodic density by choosing appropriate constants a_0 and a_1 in the following function f:

$$f(\eta) = a_0 f_0(\eta) + a_1 f_1(\eta), \tag{50}$$

These coefficients a_0, a_1 are determined by the conditions $f(\bar{\eta}) = 0$ and $F(\eta^*) = 1$ as follows. Upon substituting the trial solution (50), one obtains

$$a_0 f_0(\bar{\eta}) + a_1 f_1(\bar{\eta}) = 0, \quad a_0 F_0(\eta^*) + a_1 F_1(\eta^*) = 1.$$

yielding a pair of linear equations that can be solved for a_0 and a_1. To obtain d differentiate (50) and use Eq. (49), to get:

$$f'(\eta) = \frac{2\mu}{\sigma^2} f(\eta) + a_1,$$

so $a_1 = -2d/\sigma^2$ (cf. Eq. (41)).

The possibility for recapitalisation is tested by finding roots of

$$G(\bar{q}) = W[\bar{q}, \eta^*(\bar{q})] - W(\bar{q}, \bar{\eta}) - [\eta^*(\bar{q}) - \bar{\eta}] - \chi.$$

where we have made explicit the dependence of the location of the upper dividend paying boundary $\eta = \eta^*$ and the function $W(\eta)$ on the value of q on the lower boundary $q(\bar{\eta}) = \bar{q}$ explicit. Clearly $G = 0$ is equivalent to achieving Eq. (7). Functions η^*, q and W are all obtained using the same method outlined above (i.e. jointly computing the two odes for q and W using $\bar{q}$ and arbitrary value of W on $\bar{\eta}$, locating η^* from $q' = 0$, and rescaling W to enforce $W' = 1$.)

The task then is to iterate on the starting value $\bar{q}$ to find the root of $G(\bar{q})$. First a coarse root bracketing is attempted by evaluating G at $\bar{q}_i = -\bar{\eta} + (q_1 + \bar{\eta})i/n_q$, where $i = 0 \ldots n_q$ and n_q an integer (we use $n_q = 10$), and q_1 is q as given by Eq. (36) if that value is real, or $1 + \theta\rho$ if it is not. If sign of G changes across a bracketing interval $(\bar{q}_i, \bar{q}_{i+1})$, the root is pin-pointed using standard root finding algorithms. This locates a recapitalisation solution. If no roots are found, or a root is found with $\bar{q} < -\bar{\eta}$ or $q^* > 1 + \theta\rho$ then the solution is identified as liquidation

with $\bar{q} = -\bar{\eta}$.

C.3 Model with option to rent

The algorithm outline is same as in the model without the rental option. However, the solution near the lower boundary is more involved when recapitalisation is not undertaken and so $\psi(\bar{\eta}) = 0$.

The differential equations for q can again be solved by simple forward integration starting from $q(\bar{\eta}) = \bar{q}$. If recapitalisation is available ($\bar{q} > -\bar{\eta}$), no singularities are present, and the equation for q, Eq. (17), can be integrated directly to obtain $q(\eta)$, η^*, and now also $\widetilde{\eta}$. The point $\tilde{\eta}$ is found in the same way as η^*, *i.e.* by monitoring the function $\psi - 1$ as integration advances and polishing the root after a coarse approximation is found. Initial $\bar{q}$ is found the same way as for the model without rental (but with q, W computed slightly differently as described below).

If $\bar{q} = -\bar{\eta}$, then $\psi = 0$ and singularities appear. As is shown in Appendix D, the derivative $q'(\bar{\eta})$ is finite. In order to evaluate it numerically, we use Eq. (54) since Eq. (17) is indeterminate at $\bar{\eta}$ (in practice, numerical round-off would cause significant error in $\bar{q}$). Otherwise the solution of q proceeds the same way as with a recapitalising lower boundary.

Using ϕ, and expanding the resulting equation in the renting ($0 < \psi < 1$) and not renting regimes ($\psi = 1$), we have

$$\phi' = \begin{cases} \dfrac{\bar{a} + (a - \bar{a})\psi - \delta + r\eta + \sigma_2^2\psi^2\eta - \theta^{-1}(q-1)[\eta + \frac{1}{2}(q+1)]}{\frac{1}{2}(\sigma_1^2 + \eta^2\sigma_2^2)\psi^2}\phi - d, \\ \qquad \text{when } \psi \in (0,1), \\ \dfrac{a - \delta + r\eta + \sigma_2^2\eta - \theta^{-1}(q-1)[\eta + \frac{1}{2}(q+1)]}{\frac{1}{2}(\sigma_1^2 + \eta^2\sigma_2^2)}\phi - d, \\ \qquad \text{when } \psi = 1. \end{cases} \tag{51}$$

When there is no recapitalisation equations for f' and W' , unlike that for q', do not tend to finite values at $\bar{\eta}$, since $\psi(\bar{\eta}) = 0$ if $q(\bar{\eta}) = -\bar{\eta}$, $q'(\bar{\eta}) > 0$. Due to this divergence, the point $\bar{\eta}$ cannot be reached by directly integrating the model equations, which in principle could be done backwards from, say, $\tilde{\eta}$ down to $\bar{\eta} + \epsilon$, $0 < \epsilon \ll 1$. Cutting the integration short in this way would lead to severe underestimation of the probability mass near $\bar{\eta}$ if f diverges fast enough at this edge.

To resolve this issue, we use the analytically obtained power-law solutions, $f_{\rm a} \propto (\eta - \bar{\eta})^{\alpha-2}$ (Eq. (21)), and $W_{\rm a} \propto (\eta - \bar{\eta})^{\beta}$ (Eq. (18)), from $\bar{\eta}$ up to a cross-over value $\eta_{\times}$. Numerical solutions are matched to the analytic ones so that the resulting functions are continuous. The cross-over point can determined by requiring that

$$\left| \frac{f_{\rm a}'(\eta_{\times})}{f_{\rm a}(\eta_{\times})} \right| = \varepsilon^{-1}, \tag{52}$$

where $0 < \epsilon \ll 1$, implying that the divergent terms dominate the expression for the derivative of f. However, since W' also tends to infinity, we write the same condition for W_a as well. This gives two different cross-over values, of which we will choose the smallest:

$$\eta_\times = \varepsilon \min(|\alpha|, \beta) + \bar{\eta}, \tag{53}$$

where α is given by Eq. (56) and $\beta = 1/(1 + q'(\bar{\eta}))$, with $q'(\bar{\eta})$ from Eq. (54). We typically use the value $\varepsilon = 1.0 \times 10^{-3}$. Naturally, we use the analytic solution for f to obtain the cumulative density F below $\eta_\times$.

If the lower boundary is at $\bar{q} = -\bar{\eta}$, we can then directly integrate Eq. (51) with $d = 0$ from $\eta_\times$ to η^*. The obtained solution can then be multiplied by a constant to make the cumulative distribution satisfy $F(\eta^*) = 1$. If $\bar{\eta}$ is absorbing (recapitalisation), we use the same trick as in the model without rent: we solve for ϕ_0 and ϕ_1 satisfying Eq. (51) with $d = 0$ and $d = 1$, respectively. The final ϕ is then constructed as a superposition of these two, $\phi = a_0\phi_0 + a_1\phi_1$. Coefficients a_0 and a_1 are determined from

$$\phi(\bar{\eta}) = 0, \quad \int_{\bar{\eta}}^{\eta^*} \frac{2}{(\sigma^\eta(\eta))^2} \phi(\eta) \, d\eta = 1.$$

When needed, the same analytic solution, Eq. (21), can be used for both ϕ_0 and ϕ_1 ($\phi_{0,1} \propto (\eta - \bar{\eta})^\alpha/(\sigma^\eta)^2$), since d term is negligible near $\bar{\eta}$.

Note that reverting to the analytic solution for f is equivalent to using a truncated integration range with an additional correction term coming from the analytical solution near $\bar{\eta}$. Numerical simulations confirm that this approach is sound: (i) the analytical and numerical solutions are in very good agreement across a wide range of η, (ii) the obtained solutions are independent of ϵ provided it is small enough while keeping the numerical solution from reaching the singularity, and (iii) qualitative features of the solution do not change if the analytical correction is omitted.

D   Behaviour of solutions near boundaries

This Appendix provides the derivation of the asymptotic approximations summarised in Proposition 7.

D.1   Model without option to rent

While no singularities emerge in the model with no option to rent, it is still useful to begin with this simple case. The evolution of the value function W is given by $W'/W = 1/(q + \eta)$, which in the case of liquidation tends to infinity as the point of maximum borrowing where $q(\bar{\eta}) = -\bar{\eta}$ is approached. This means there is a potential singularity in W at $\bar{\eta}$. We can however show that in the model without

the option to rent this does not occur and W is linear close to $\eta = \bar{\eta}$.

Suppose now that q is of the form $q(\eta) = -\bar{\eta} + q'(\bar{\eta})(\eta - \bar{\eta}) + \mathcal{O}((\eta - \bar{\eta})^2)$. Near the boundary, W follows

$$W' = \frac{1}{1 + q'(\bar{\eta})} \frac{W}{\eta - \bar{\eta}} + \mathcal{O}(\eta - \bar{\eta}).$$

The solution is then given by Eq. (18) in the main text. In the case of the model without rental, it is clear from Eq. (10) that $q'(\bar{\eta}) = 0$ and so W is linear near $\bar{\eta}$.

D.2 Model with option to rent

Turning to the model with option to rent, again a singularity can occur only on the lower boundary and only when there is no recapitalisation i.e. when $q(\bar{\eta}) = -\bar{\eta}$ and $\psi(\bar{\eta}) = 0$.

Note now that in the equation for q', Eq. (17), both the numerator and the denominator vanish. Applying the l'Hopital's rule, the derivative can be solved as:

$$q'(\bar{\eta}) = \frac{-(\rho - r - \gamma) \pm \sqrt{(\rho - r - \gamma)^2 + 4\gamma\theta^{-1}[1 + \theta\rho - \bar{q}]}}{2\theta^{-1}[1 + \theta\rho - \bar{q}]}, \tag{54}$$

where $\gamma = (a - \bar{a})^2/2(\sigma_1^2 + \bar{\eta}^2\sigma_2^2)$. Above, only the plus sign applies. This can be seen by recalling that $\bar{q} < q_{\max} = 1 + \rho\theta$ must apply (see above for the reasoning), in which case only the plus sign gives a positive q'. Thus, the solution near $\bar{\eta}$ is given by Eq. (19) in the main text.

The power-law form of W given in Eq. (18) holds here as well. Since now $q'(\bar{\eta}) > 0$, the exponent $\beta = 1/(1 + q'(\bar{\eta}))$ is always less than one, in contrast to the model without option to rent, implying that $\lim_{\eta \downarrow \bar{\eta}} W' = \lim_{\eta \downarrow \bar{\eta}} (-W''/W') = +\infty$.

To find the behaviour of the ergodic density near $\bar{\eta}, \bar{q}$, we first need ψ. This time $\eta - \bar{\eta}$ is not negligible compared to $q - \bar{q}$. A straight-forward calculation gives:

$$\psi = \psi'(\bar{\eta})(\eta - \bar{\eta}) \tag{55}$$

where

$$\psi'(\bar{\eta}) = \frac{2}{a - \bar{a}} \left\{ \rho - r + \theta^{-1} [1 + \theta\rho - \bar{q}] q'(\bar{\eta}) \right\}.$$

Next, the $\eta \to \bar{\eta}$ limiting forms of q and ψ are substituted into Eq (42), and only terms upto $\mathcal{O}(\eta - \bar{\eta})$ are kept. Notice that the numerator vanishes in the leading order, and hence $\phi' \propto (\eta - \bar{\eta})^{-1}$ and not $\propto (\eta - \bar{\eta})^{-2}$:

$$\phi' = \alpha \frac{\phi}{\eta - \bar{\eta}},$$

where

$$\alpha = \frac{(a - \bar{a})\psi'(\bar{\eta}) + r - \frac{1}{2}\theta^{-1}q'(\bar{\eta})(\bar{\eta} + 1) + \theta^{-1}(\bar{\eta} + 1)(1 + q'(\bar{\eta})/2)}{\frac{1}{2}(\sigma_1^2 + \bar{\eta}^2\sigma_2^2)\psi'(\bar{\eta})^2}. \qquad (56)$$

This gives the power-law solution Eq. (20) in the main text. Finally using Eq. (55) yields Eq. (21) of the main text.

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