

Leverage constraints and real interest rates*

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Abstract

This paper investigates the macroeconomics of real interest rates when there are constraints on debt finance, used both for insurance against income shocks and transferability of resources over time. We amend a standard continuous-time deterministic model of international exchange, with patient and impatient countries, introducing country level shocks fully diversifiable at the global level. A series of shocks that push one country towards its leverage limit induces substantial precautionary saving and a collapse of real interest rate relative to the deterministic benchmark. We discuss the resulting dynamics of interest rates and the broader implications for macroeconomic modelling. [100 words]

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1 Introduction

A central challenge in contemporary monetary and macroeconomics is allowing for the impact of leverage on household consumption, corporate investment and other spending decisions. The unsustainable build up of household debt in the UK and other countries, and subsequent deleveraging, has been a prominent feature of the macroeconomic environment since the global financial crisis of 2007-2009. High levels of household debt are one reason why consumer spending has shown relatively modest response to the substantial monetary and fiscal stimulus since the crisis.

The objective of this paper is to explore one aspect of this broader modelling challenge, the relationship between household leverage and real interest rates, in a fully general equilibrium but non-monetary setting (so nominal prices are completely flexible and goods markets always clear). We present a simple model of patient and impatient households, in which shocks can potentially increase household indebtedness towards maximum levels of possible borrowing. A temporary negative shock to the income of indebted households can result in a large but temporary declines in real interest rates. Households seek to avoid reaching their leverage constraints because they are then unable to smooth consumption over time. In consequence the precautionary motive saving grows increasingly stronger the closer households are to their leverage constraints. In equilibrium this increased desire to save results in shortlived but substantial reductions of the real interest rate. Thus in our setting real interest rates are determined not only by tastes and productive technology, but also by household balance sheets, and can vary substantially over time as household net worth varies.

Our quantification of this phenomena is based on a model with some strong assumptions. There are two ‘countries’ (or alternatively groups of

households). Within each of these countries all households have the same preferences and experience the same shocks to their income. For simplicity we assume that income shocks are diversifiable at the global level, with a loss in one country fully offset by a gain in the other.

The key difference between the two countries is that households in one are relatively impatient while households in the other are relatively patient. In the absence of uncertainty this leads to a very standard outcome, described in several macroeconomic textbooks (usually presented in the context of an open economy extension of the Ramsey-Cass-Koopmans model of capital accumulation). In this deterministic baseline there is a continual increase in indebtedness of impatient households. At the same time patient households save and acquire claims on impatient households. The equilibrium real interest rate is then a weighted average of the rate of time preference of different households. Overtime the share of consumption of the patient country increases and the wealth of the impatient country declines. As a result global real interest rates fall gradually towards the rate of time preference of the most patient ‘country’.

We introduce income uncertainty, with the crucial additional assumption that financial markets are incomplete: there is no mechanism for direct insurance of the diversifiable income shocks.¹ As a consequence debt here must do double duty: first for transfer of resources over time; and second for management of temporary income shocks. The outcome is then a capital structure tradeoff. Rather than allowing debt to accumulate indefinitely over time, impatient households seek to reach a target level of debt which balances the benefits of greater flexibility in managing future income shocks against the desire to consume today rather than tomorrow. We do not seek to explain

¹In our setting, in which income uncertainty is fully diversifiable, the combination of income uncertainty and complete financial markets is equivalent to a deterministic model.

this market incompleteness, but it is consistent with both casual observation and standard microeconomic theory suggesting that informational, contractual and other frictions make it difficult to insure against temporary income loss.

A final element in our model is the presence of a constraints on leverage i.e. some maximum level of household borrowing. Some such constraint is required (even in the deterministic baseline) because otherwise impatient households would borrow unlimited amounts and service this debt with further borrowing. Rather than seek to determine this maximum level of leverage endogenously, we simply impose it as an extra parameter. As it turns out the exact level of maximum borrowing maks little difference to the predictions of the model about interest rate dynamics. Instead the gap between the maximum possible debt and the desired debt levels that arises from the tradeoff between time-preference and the management of income shocks is relatively constant (when measured in standard deviations of the time distribution of household net worth). Tightening or relaxing the debt constraint does not alter predictions about the impact of income shocks on real interest rates.

With incomplete insurance of income shocks leverage affects real interest rates through a potentially large precautionary motive for saving. If households reach their maximum leverage then they are no longer protected against loss of income. They therefore seek to avoid this outcome, saving more and consuming less the closer they are to their maximum possible leverage. Capital market equilibrium requires a lower real rate of interest, compared to the baseline without leverage constraints, in order to persuade both patient and impatient households to consume enough to offset this increased saving. In our setting, close to the leverage constraint, this precautionary motive

for saving can become very strong indeed, driving down global real interest rates to well below the rate of time preference of even the most patient households. Real interest rates can easily become negative, and in the asymptotic but unattained limit of unlimited negative shocks to income over a small time period they fall to $-\infty$.

Our results are different – both quantitatively and qualitatively – from what would emerge from the routine practice of exploring a linearised solution around a steady state (linearisation around a deterministic steady state is not in fact possible for our model, because no deterministic steady state exists; and as we show linearisation around some measure of average steady state debt, e.g. the median of the time-distribution of debt, substantially understates the impact of income shocks on real interest rates). We obtain fully non-linear solution using the relatively sophisticated mathematics of continuous time stochastic dynamic programming. These methods, widely known and applied in the physical sciences, have only recently found favour in the economics literature.

While we can relegate all the technicalities of solution to an Appendix, the modelling assumptions that we use in applying these methods are not entirely innocuous. In particular, we assume that household income is memoryless, with expected income fixed and accumulated income following a brownian motion with drift. The reasons for this choice are technical. It allows us to obtain numerical solution from two second order ordinary differential equations in a single state variable, the financial claims of one country on another; but this simplicity comes with some cost. Our model has no 'autarkic' baseline solution (if no borrowing is allowed at all then there is no meaningful solution). Also we are subject to the (technical) criticism that such an income process should result in household wealth breaching the borrowing

constraint almost certainly in finite time. In fact we can show that the condition that the borrowing constraint is never attained in finite time serves to identify a unique solution to our model. Our model is unusual because while no partial equilibrium solution exists there is a general equilibrium solution.

Generalisations the present model could include some or all features of a complete (two country) Ramsey model: per country output that depends on both capital and labour input, together with investment and wage rates. In this paper, however, we ignore all these extensions; our goal is to study a minimal model where the combined effects of income uncertainty and leverage constraints on global real interest rates can be explored. In our conclusions we discuss the relevance of the strong prudential savings impact on real interest rates that we uncover in richer and more realistic settings.

Further technicalities of our solution are described in the Appendix. As is typical of many intertemporal optimisation problems with uncertainty, there are no closed form expressions of model solution, in our case for interest rates and consumption. We also capture the singularities that emerge on the boundaries of maximum borrowing using asymptotic approximations. In order to simultaneously enforce these boundary conditions at both the lower and upper boundaries, the solution is then calculated using a standard partial differential solution technique (the pseudo-spectral method).

The paper is organised as follows. Section 2 locates our work in relation to previous literature. Section 3 sets out our model, distinguishing the baseline deterministic special case and outlining our solution method. Section 4 presents numerical calculations of our solution and investigates the resulting dynamics of interest rates and consumption. Section 5 concludes. The technical Appendix appears at the end of paper.

2 Related literature

= Our analysis is related to previous work on both the microeconomics of prudential savings and the macroeconomics of real interest rates. The prudential savings motive has long been recognised to result in sometimes substantial departures from complete market or certainty equivalent formulations of the consumption/ saving decision (for discussion see Browning and Lusardi [1996]). There is an extensive theoretical and empirical analysis, at the level of the individual household, on the particular mechanism that supports our paper, a combination of incomplete markets and borrowing constraints that increases incentives for prudential saving. This is especially useful for understanding the observed empirical regularities of consumption function (an accessible discussion, with extensive review of the literature, is provided by Carroll [2001]).

Such constraints have also been incorporated in models of general equilibrium, most notably by Aiyagari [1994] who in a set up closely related to our own shows how prudential saving in order to cope with uninsured income risk can lower the real interest rate relative to a complete markets benchmark. His analysis focuses on purely idiosyncratic risk. We instead focus on aggregate risk that is diversifiable at the global level, and investigate the consequences for the cross-sectional distribution of household net worth. In his setting, since there are no aggregate disturbances, the real interest rate is fixed; in ours the real interest rate, while also always lying below the level predicted by the complete markets benchmark, responds dynamically to aggregate income shocks.

These models of prudential saving are one part of a much broader literature exploring the implications of market incompleteness (see amongst others Magill and Shafer [1991], Magill and Quinzii [1994, 1996]). Much

of this literature explores relatively general abstract specifications and investigates variety of other issues, including the existence and uniqueness of equilibrium, Pareto optimality and efficiency and potential departures from fundamentals ('bubbles') in the prices of financial assets. Our model is also one of incomplete markets, but with a simple and defined structure developed in order to focus on the specific issue of the relationship between household leverage and real interest rates.

There are other contributions to the literature examining the choice between consumption and savings and the level of real interest rates at the global level. Caballero et al. [2008] develop a model to explain the combination of large global savings imbalances and declining real interest rates that emerged in the early 2000s following the Asian crisis. Their basic model is deterministic and hence excludes the precautionary savings motive central to our own analysis. They examine the impact of a fall in the level of 'pledgability' of productive assets in one country (R interpreted as emerging market countries) relative to that of productive assets in the other country (U interpreted as advanced countries such as the US). This pledgability might be determined for example by the perceived effectiveness of institutions of macroeconomic management, law, investor protection and corporate governance. An reduction in pledgability in (R) (which they suggest was triggered by the Asian crisis) leads to a fall in global real interest rates (the demand for financial assets is unchanged, the supply has fallen, so for asset market equilibrium to be maintained global real interest rates must fall); and also the emergence of a permanent current account surplus in R and permanent current account deficit in U (because savers in R now acquire assets from U).

Related work by Caballero and Krishnamurthy [2009] is closer to our own analysis. They also investigate how a change in demand for assets impacts on

real interest rates, but in their case the focus is on the emergence of a demand for safe assets issued in the US and held by international investors. Unlike our own model both domestic and overseas investors have the same rate of time preference. They further assume logarithmic preferences so consumption is always the same fixed proportion of domestic wealth. Two substantive differences from our own analysis can be highlighted: first that they assume a diffusion process (a random walk with drift) for the level of cash flows produced by productive assets, whereas we assume a similar diffusion process for *accumulated* cash flows; and second that their stochastic shocks cannot be diversified away by lending and borrowing between countries, instead shocks to the level of cash flows are reflected in changes in asset prices and wealth. They also avoid a full solution of the optimal decisions of overseas investors, instead assuming an exogenous stochastic inflow of investment funds and exogenous deterministic withdrawal of external investment. They then show how an increase in demand for safe assets from overseas investors pushes down domestic interest rates, reduces the premia on risky assets held by US investors, and leads to greater leverage by the intermediaries that issue US financial securities. Leverage is thus related to real interest rates but there is no leverage constraint the approach to which triggers a fall of real interest rates.

Another recent body of work examines the implications of investor leverage for asset pricing, investment and how changes in the monetary conditions and in collateral requirements can create a ‘leverage cycle’, mechanisms which seem to have played a central role in the global financial crisis (Geanakoplos [2009, 2010] reviews much of this work). Our own work has a complementary focus on the role of leverage in income and consumption, while ignoring the role of traded financial assets.

From the technical perspective, we build on a number of other models using continuous time dynamic stochastic optimisation to analyse corporate operational and financing decisions in the presence of financing constraints (see for example Brunnermeier and Sannikov [2014], Bolton et al. [2011]; a further contribution to and fuller review of this line of research is provided by Isohätälä et al. [2014]). These avenues of research have some parallel with the literature on the financial accelerator, established as a standard approach for macroeconomic modelling by Bernanke et al. [1999]. These newer contributions though suggest that similar balance sheet mechanisms can affect a wide range of household and corporate decisions, not just fixed capital investment as in the standard financial accelerator, and that these balance sheet impacts can be highly non-linear and therefore not well captured by the linearisation methods routinely used in macroeconomic modelling.

3 The model

3.1 Assumptions

There are two countries each consisting of a large number of identical households. Within each country every households receives the same income, has the same preferences and so makes the same consumption decisions. This allows us to work with a representative household for each country. The households maximise the expected discounted utility of future consumption. Normalising the ‘mass’ of households in each country to unity, and using a bar over variables and parameters to distinguish the more patient country from the less patient, the objective for a representative household in each

country can be written as:

$$\int_{\tau=t}^{\infty} e^{-\rho(\tau-t)} u(c) d\tau \quad \text{and} \quad (1)$$

$$\int_{\tau=t}^{\infty} e^{-\bar{\rho}(\tau-t)} \bar{u}(\bar{c}) d\tau \quad (2)$$

with $\bar{\rho} < \rho$.

Output of the two countries (and of each household in the two countries) is given by:

$$a dt + s dz \quad \text{and} \quad (3)$$

$$\bar{a} dt - s dz \quad (4)$$

i.e. expected output is fixed and the only uncertainty is a idiosyncratic Wiener process fully diversified at the global level. The goods market always clears, so:

$$c + \bar{c} = a + \bar{a}. \quad (5)$$

The sole asset are debt claims w denominated in units of the single good, which we assume are claims of the impatient country on the patient country (the claims of the patient country on the impatient country are given by $\bar{w} = -w$). The interest rate on these claims is given by r . The level of consumption is determined given the current wealth level w . The change in claims therefore satisfies the stochastic differential equation:

$$dw = (a + rw - c) dt + s dz = (-\bar{a} + rw + \bar{c}) dt - s dz \quad (6)$$

Completing the model specification, there are limits on borrowing

$$w^* < w < \bar{w}^* \tag{7}$$

with $-w^* > 0$ representing the maximum borrowing of the impatient country and $\bar{w}^* > 0$ the maximum borrowing of the patient country.

The model is parsimonious with only two standard instantaneous utility functions $(u(c), \bar{u}(\bar{c}))$ and seven other parameters $(a, \bar{a}, \rho, \bar{\rho}, w^*, \bar{w}^*, \text{ and } s)$.

3.2 Solving the model

Here we provide an overview of the solution of the model, emphasising the supporting economic intuition and comparing the stochastic specification $s > 0$ with the deterministic baseline $s = 0$. Technical details regarding the solution are described in the Appendix. With $s = 0$ we have a standard deterministic model to which the maximum principle would normally be applied yielding a semi-closed form solution. With $s > 0$ the model can be solved using dynamic programming; the solution however is no longer closed form, rather it is characterised by a second order ordinary differential equation for $c(w)$ which must be numerically solved.

In the $s \rightarrow 0$ limit of our stochastic model, and in a textbook deterministic model, the consumption in the impatient country is described by the standard Euler equation:

$$\dot{c} = (r - \rho) \left[-\frac{u''(c)}{u'(c)} \right]^{-1}. \tag{8}$$

We have assumed $\rho > r > \bar{\rho}$, so that in the deterministic case consumption in the impatient country declines continuously over time at a rate that de-

depends on the intertemporal elasticity of consumption $(-u''(c)/u'(c))^{-1}$ (and consumption in the patient country correspondingly increases).

Accounting for a non-zero noise, the time-evolution of the consumption follows the stochastic differential equation or SDE (with a corresponding SDE for consumption in the patient country):

$$dc = \left\{ (r - \rho) \left[-\frac{u''(c)}{u'(c)} \right]^{-1} + \underbrace{\frac{1}{2} s^2 \left[-\frac{u'''(c)}{u''(c)} \right] c'(w)^2}_{\text{precautionary saving term}} \right\} dt + \underbrace{sc'(w) dz}_{\text{from income uncertainty}} \quad (9)$$

In the stochastic case $s > 0$ there are two additional terms which do not appear in the deterministic Euler equation. The additional term in dt , for any given level of interest rates r and consumption $c(w)$, reduces the rate of decline in consumption of the impatient household compared to the deterministic case. The second additional terms is the diffusion in dz representing the impact of income uncertainty on the level of consumption.

The dependency of consumption $c(w)$ on the state variable (the level of wealth w) is then described by the following second-order ordinary differential equation:²

$$(\rho - r) u'(c) = (a + rw - c) u''(c) c' + \frac{1}{2} s^2 \left[u'''(c) (c')^2 + u''(c) c'' \right] \quad (10)$$

In the deterministic case this becomes

$$c'(w) = \frac{r - \rho}{a + rw - c} \left[-\frac{u''(c)}{u'(c)} \right]^{-1}$$

an equation which can also be obtained directly from (9) and (6).

²This equation is a transformation of the equation of optimality, or Hamilton-Jacobi-Bellman equation, which characterises the consumption policy that maximises the expected discounted utility objective equation (1). See Appendix for details.

The global interest rate r is the only market price and this adjusts to ensure aggregate demand equals aggregate supply (i.e. goods market clearing given by (5)). This requires (as shown in the Appendix) that:

$$r(w) = \frac{\left[-\frac{u''(c)}{u'(c)}\right]^{-1} \rho + \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})}\right]^{-1} \bar{\rho}}{\left[-\frac{u''(c)}{u'(c)}\right]^{-1} + \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})}\right]^{-1}} - \frac{1}{2} s^2 c'^2 \frac{\left[-\frac{u'''(c)}{u''(c)}\right] + \left[-\frac{\bar{u}'''(\bar{c})}{\bar{u}''(\bar{c})}\right]}{\left[-\frac{u''(c)}{u'(c)}\right]^{-1} + \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})}\right]^{-1}}. \quad (11)$$

In the deterministic case ($s = 0$) the global interest rate is a weighted average of the time preference of the two households, with the weight on the impatient country's time preference,

$$\left[-\frac{u''(c)}{u'(c)}\right]^{-1} / \left\{ \left[-\frac{u''(c)}{u'(c)}\right]^{-1} + \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})}\right]^{-1} \right\}$$

increasing with its consumption share. In the stochastic case ($s > 0$) there is an additional term, generated by the prudential saving of the two countries, correcting the rate of interest downwards to restore goods market equilibrium.

Equations (10), (11) and (5) together yield a somewhat complicated looking but fairly easily numerically solved ordinary differential equation for $c(w)$. In the deterministic case this ODE is first order and requires one boundary condition for solution, which in the presence of the leverage constraint is simply $c(w^*) = a + r(w^*) w^*$ (the deterministic model can then be solved as a function of state using Eq. (10) or as a function of time using the maximum principle). The wealth and the consumption of impatient country decline continuously over time until eventually the lower boundary w^* is reached at which point c declines to $a + r w^*$.

In the stochastic setting with uncertainty of output the ODE is second order and two boundary conditions are required for solution. We establish the existence of unique solution by considering the behaviour of the steady state

or ‘ergodic’ probability density function $f(w)$ of w . This ergodic distribution can be thought of as the steady state cross-sectional probability density of state variables for a large population of all of which are governed by the same dynamic stochastic equations of motion. However this is not an appropriate interpretation for our model because we assume all households are hit by the same shocks, so at any point in time all households in a given country have the same wealth. Instead this ergodic distribution should be thought of as the time-independent or unconditional distribution of w obtained by sampling the economy at a random points of time.

The ergodic distribution can be obtained from the solution of a further first-order ordinary differential equation (the stationary limit of the Fokker-Planck equation describing the evolution of the density function $f(w, t)$ over time):

$$\frac{1}{2}s^2 f''(w) = [r'(w)w + r(w) - c'(w)] f(w) + [a + r(w)w - c] f'(w). \quad (12)$$

This is solved with two boundary conditions. The first is the normalisation condition appropriate for a probability density function that $F(\bar{w}^*) = 1$. The second, reflecting our assumption that households never attain the boundary w^* , is that $f(w^*) = 0$ (our asymptotic approximation for $c(w)$ is the only solution for $c(w)$ consistent with $f(w^*) = 0$). As shown in the Appendix, this implies the further asymptotic expansion:

$$f(w) = (w - w^*)^{\frac{1}{2}} + o((w - w^*)^{\frac{1}{2}}).$$

An acceptable solution for our model, if it exists, requires two conditions on the ergodic density function, that $f(w^*) = 0$ and $f(\bar{w}^*) = 0$. These conditions ensure that while households may approach the boundaries for

maximum borrowing, they never actually reach these boundaries (if instead the boundaries were attainable then once on the boundary the household could no longer smooth consumption and the infinite local variance in income creates an unlimited penalty in terms of the objective Eq. (1) i.e. the optimisation is no longer well defined.). The Appendix shows that such a solution with $f(w^*)$ requires a singularity in consumption/ utility at this boundary given by:

$$\lim_{w \rightarrow w^*} \frac{u'''[c(w)]}{u'[c(w)]} (c'(w))^2 = \infty,$$

with a corresponding singularity at the upper boundary. The Appendix further establishes the existence of unique asymptotic approximations to $c(w)$ satisfying these singularities. Thus the approximation at w^* derives from the boundary behaviour

$$c(w) = c_0 + c_1(w - w^*)^{\frac{1}{2}} + o((w - w^*)^{\frac{1}{2}}),$$

where c_0 and c_1 are constants and an equivalent formulation at \bar{w}^{ast} . This is the only solution of our model with an accompanying ergodic density satisfying $f(w^*) = 0$ and $f(\bar{w}^*) = 0$.

4 Numerical solution

We have calculated numerical solutions of the stochastic version of the model in the case of iso-elastic instantaneous utility.³ The parameter assumptions are as follows (using a bar to indicate parameters of the households in the impatient country):

³Mathematica notebooks for and standalone demonstrations of this numerical solution can be found at www.leveragecycles.lboro.ac.uk

Parameter	Impatient country	Patient country
Rate of time preference	$\rho = 0.05$	$\bar{\rho} = 0.03$
Inverse intertemporal elasticity of substitution	$\varepsilon = 1.5$	$\bar{\varepsilon} = 1.5$
Expected output per annum	$a = 1$	$\bar{a} = 1$
Standard deviation of output per annum		$0 \leq s \leq 0.5$
Leverage constraint	$w \geq w^* = -5$	$w \leq \bar{w}^* = +5$

Figures 1-10 report our numerical solutions with these assumed parameters (for some figures we use only the single value $s = 0.2$). Figure 1 shows the rate of interest $r(w)$ as a function of w for different levels of income uncertainty s . The inset to the figure shows a magnified section $r(w)$ near w^* . Close to the leverage constraint the real interest rate falls well below the rate of time preference of the patient country ($\bar{\rho} = 0.03$) and when the leverage constraint is very close becomes negative (thus creating a positive income for the borrowing impatient country).

Figure 2 shows the level of saving ($a+rw-c(w)$) as a function of w . It can be seen that for very low levels of wealth the precautionary motive leads to positive saving as the indebted households seek to reduce their leverage and increase w . For higher levels of wealth then the time preference dominates and there is dissaving.

Figure 3 shows the level of consumption $c(w)$ as a function of w . Consumption, as expected, increases with wealth. Close to the leverage constraint for the impatient country, consumption of the patient country rises

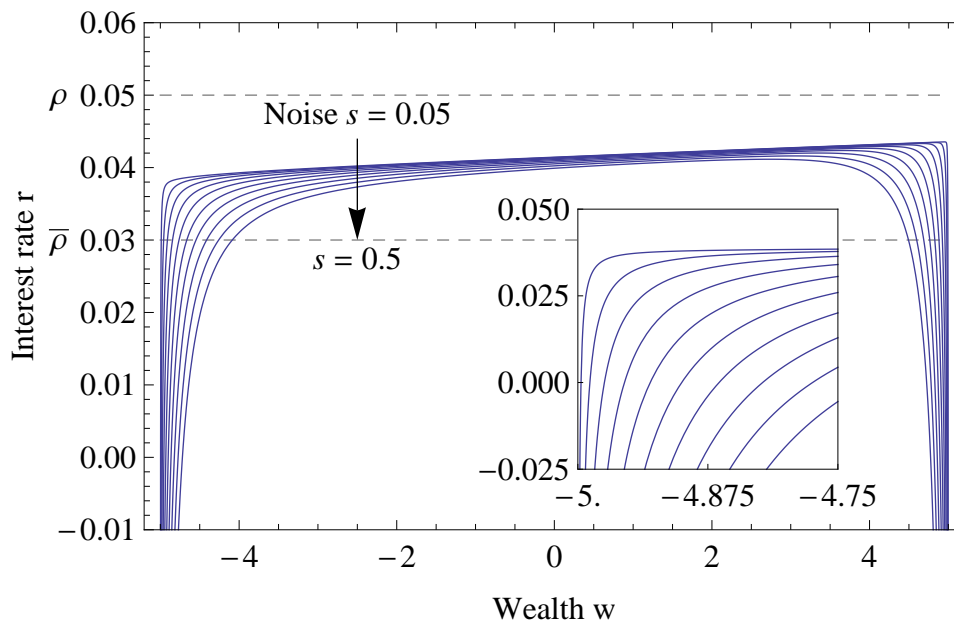


Figure 1: Interest rate r as function of wealth w .

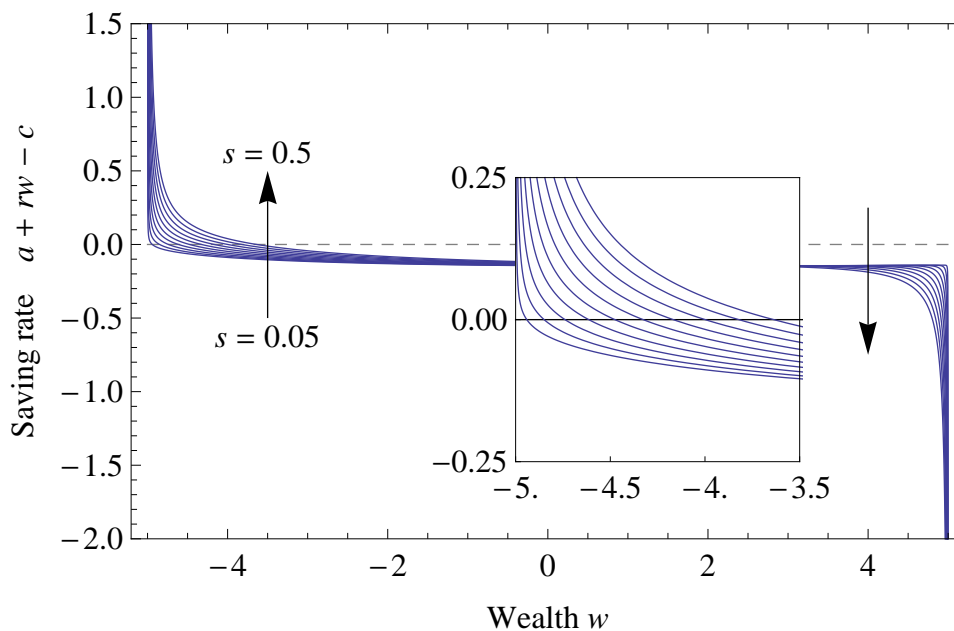


Figure 2: Saving rate $a + rw - c$ as a function of wealth w .

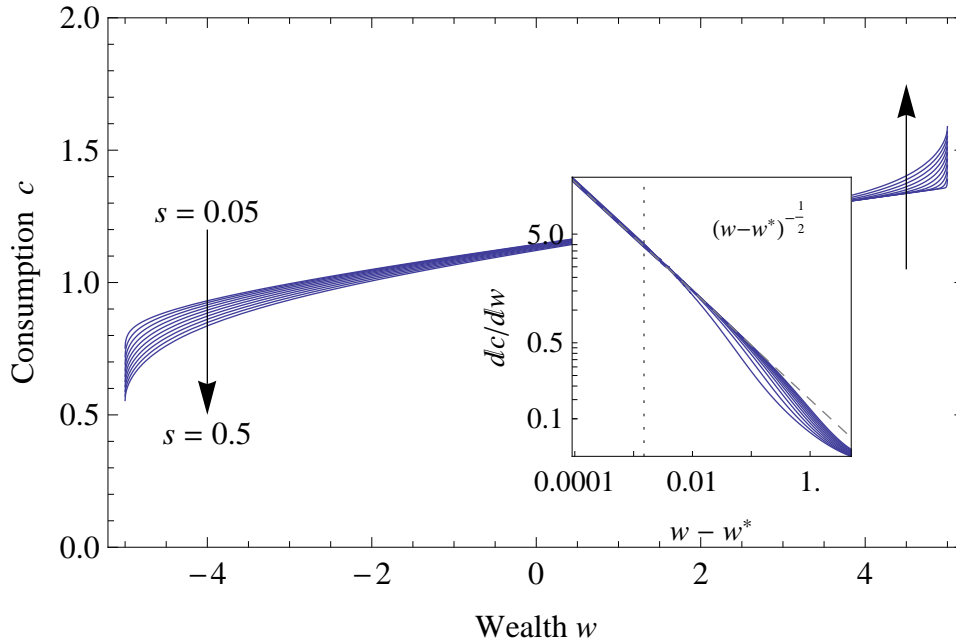


Figure 3: Consumption c as a function of wealth w . Inset: c' on logarithmic scale comparing to power law $(w - w^*)^{-1/2}$.

sharply and consumption of the impatient country fall sharply, due to the joint impact of lower interest rate and the prudential saving of the impatient country seeking to keep away from the leverage constraint. The decline of consumption is however relatively modest, compared to the fall of the interest rate shown in Figure 1. A technical issue is explored in the inset to the figure, which examines the accuracy of the asymptotic expansion for $c(w)$ applied when w is close to w^* : this can be seen to be very close to linear as required.

Figure 4 shows the ergodic density for different values of the standard deviation of income s . For low levels of s , 5% of annual income a , the ergodic density is concentrated near the maximum level of borrowing, the figure suggests that the density lies almost entirely between $w = -4$ and $w = w^* = -5$. As s increases the density both shifts to the right and widens.

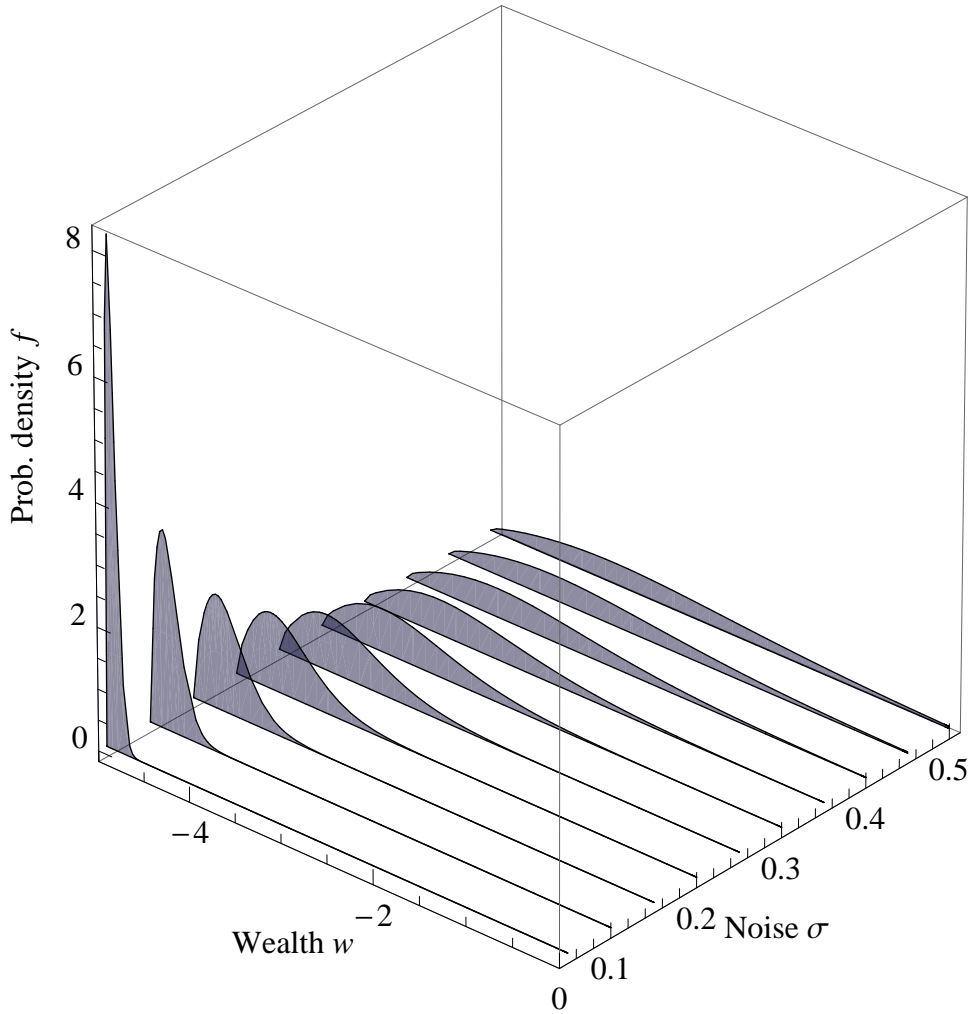


Figure 4: Probability density f as a function of wealth w and noise s .

For higher values of s , the figure runs upto 50% of annual income a , the density is visible over the range $w = 0$ to $w = w^* = -5$.

Figure 5 and Figure 6 show versions of Figures 4 and then Figure 1, using a rescaled measure of wealth with zero mean and a standard deviation of unity. Figure 5 shows the rescaled ergodic density function $f(w)$ (the pdf relative to the mean) is relatively little affected by the standard deviation of income s . Using this same rescaling of wealth, the interest rate function $r(w)$ (and consumption $c(w)$, though this is not shown here) vary only slightly with

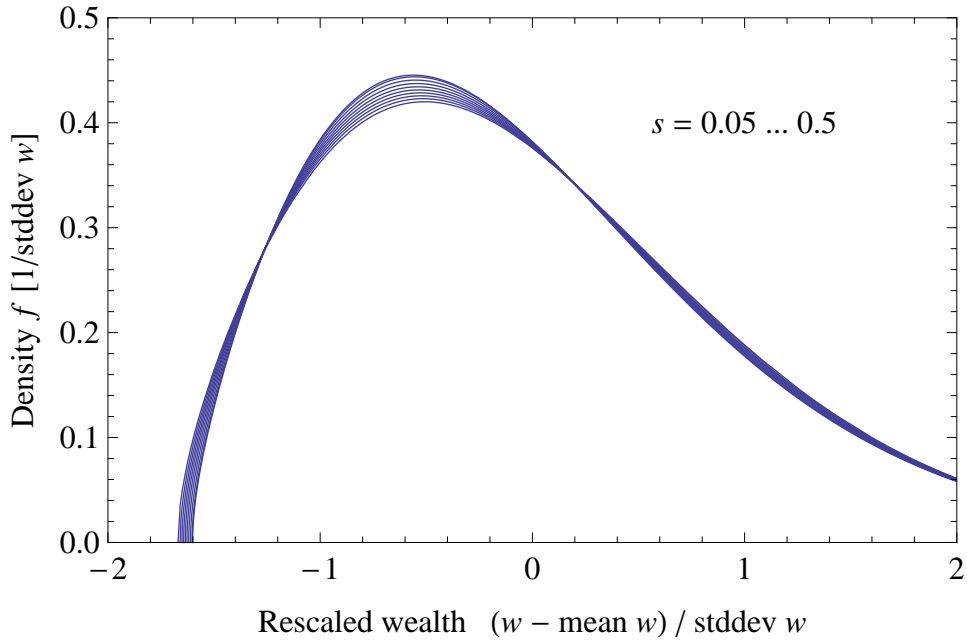


Figure 5: Wealth probability density as a function of normalized wealth.

changes in the standard deviation of income s . Note also that for all values of s shown here, from 5% to 50% of annual income, a sharp fall in r occurs once wealth w declines to about 1.2 to 1.6 standard deviations below its mean.

Figures 7 to 9 illustrate the impulse-response dynamics of wealth and interest rates following a fall in wealth to 1.5 standard deviations below its mean (these figures assume that the standard deviation of income is 20% of annual income i.e. $s = 0.2$). Because this is a stochastic model the dynamics are not a single line, rather starting from this initial point, the future evolution of wealth is a density function that gradually spreads out from the initial starting point. The heavy line in Fig. 7 and 8 show the median (50th percentile) of the distribution. Below this the figure shows the 5th and 25th percentiles, above this the 75th and 95th percentiles.

Figure 7 shows that to begin with, following a fall in wealth, with the density of w ($f(w, t)$) adjusts fairly rapidly towards the long run ergodic

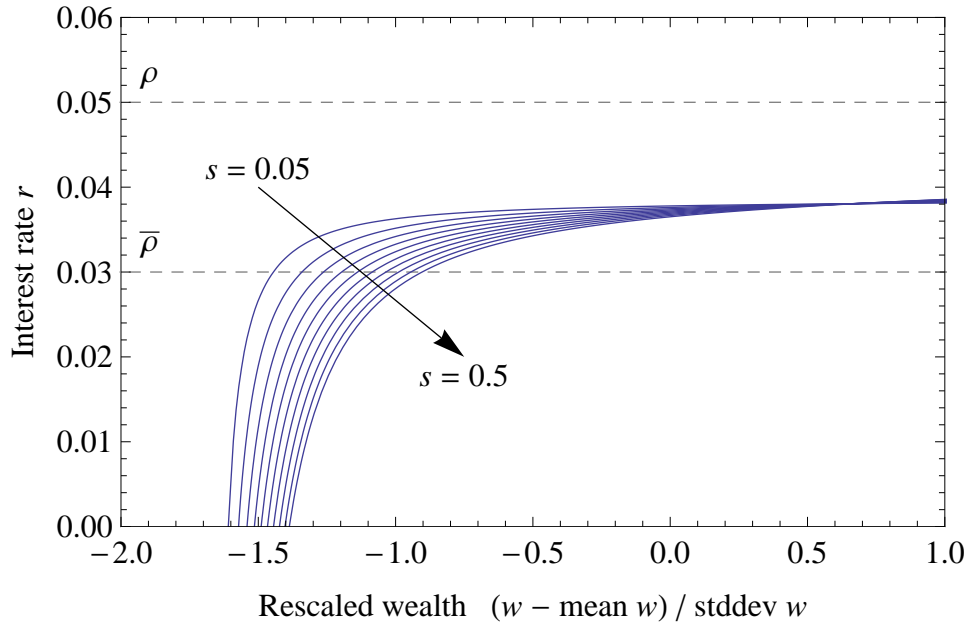


Figure 6: Interest rate as a function of normalized wealth.

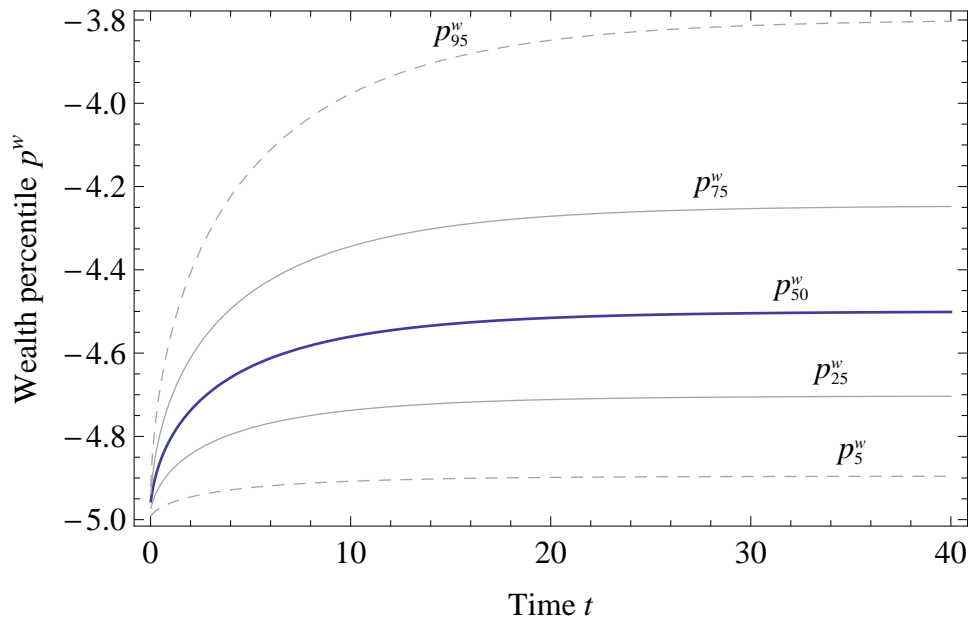


Figure 7: Select wealth percentiles as functions of time.

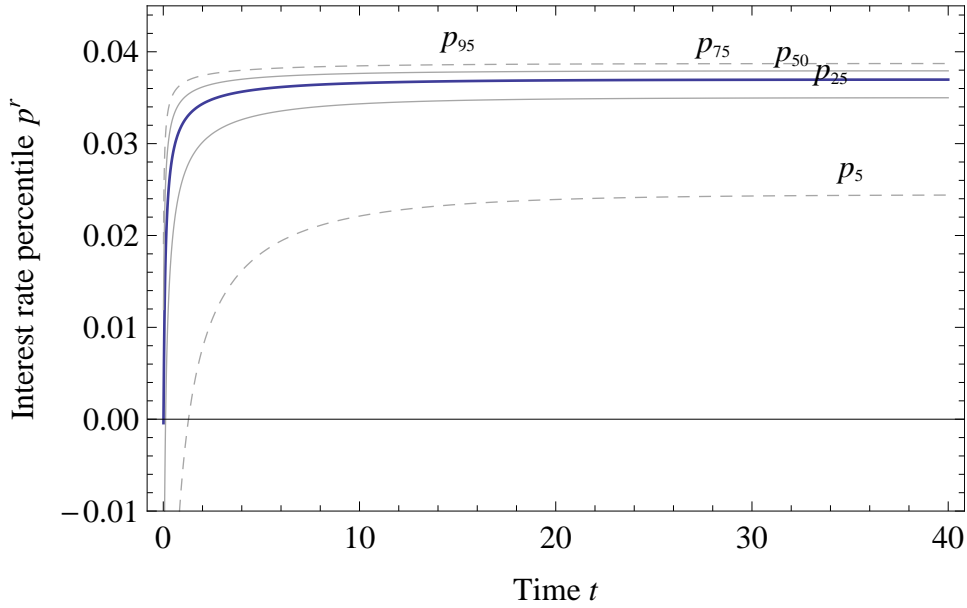


Figure 8: Select interest rate percentiles as functions of time.

density $f(w)$, so by the end of the second year ($t = 2$) the median has closed to within one standard deviation of its steady state level. Overtime the rate of adjustment slows down, it takes until about the end of the fourth year ($t = 4$) before the median is within half a standard deviation of its steady state level, and by about year twenty ($t = 20$) no difference can be seen in this chart between the median and its steady state level. There is a similar patten of adjustment for the other percentiles, fast to begin with and gradually slowing down. The adjustment of the higher percentiles is rather slower than the lower percentiles, taking until around year 30 before there is no further change in the 95th percentile of the distribution.

Figure 8 shows the corresponding impulse response for real interest rates, if wealth falls to 1.5 standard deviations below its mean. Because of the highly non-linear dependency of interest rates $r(w)$ on wealth w (Figure 1), there is an even greater change in the rate of adjustment of interest rates over time, extremely rapid during the first year (upto $t = 1$), slowing substantially

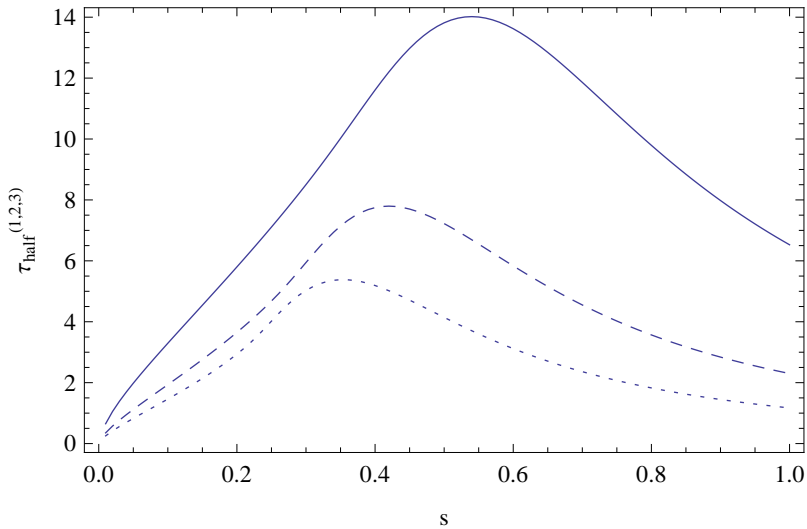


Figure 9: Half-life times $\tau_{\text{half}}^{(i)}$ as functions of noise.

over the next three years (until $t = 4$) and gradually correcting back to the long run ergodic density, to the extent discernible in the figure has been fully restored after twenty years ($t = 20$).

Figure 9 reports measures of three different ‘half-lives’ of the dynamics of $f(w, t)$, providing some further insight into these non-linear dynamics, and how these vary with the magnitude of the standard deviation s . Because the system is non-linear there is no single half life summarising the dynamics of the system so instead the figure reports three measures. These are the three different adjustment speeds (half lives) for the three most slow moving components of the series of exponential functions in the frequency domain used to approximate the dynamics of $f(w, t)$ by our pseudo spectral partial differential equation solver employed for numerical solution. As the density approaches the steady state $f(w)$ the slowest moving dynamics (the heavy top line in this figure) dominate; but further from steady state more rapid dynamics, represented by the lower lines, become more important. The figure indicates that all elements of the dynamics slow down approximately linearly

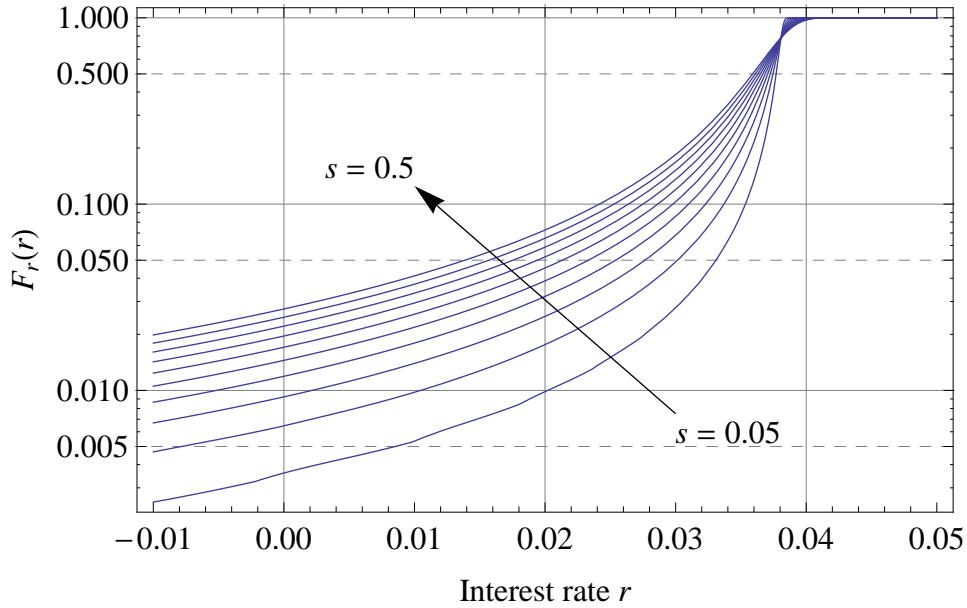


Figure 10: Interest rate cumulative density F_r .

as s increases, at least upto when is around s forty percent of annual income.

Finally Figure 10 reports the cumulative density of r , presented with a log scale on the vertical axis to make it easy to read off the different percentiles of the steady state density. As can be seen by reading across from the left-hand axis, for the heavy line in the figure, the 5th percentile of the distribution for our baseline choice of $s = 0.2$ is at about 2.5% – i.e. about one twentieth of the time real interest rates will be below 2.5% – compared to a median value of around 3.7%. Increasing the value of s has little impact on the median but has a quite substantial impact on the lower percentiles of the distribution, with 5th percentile falling to around 1.5% as s rises to 50%.

5 Conclusions

This paper has investigated the impact of aggregate income shocks on real interest rates in a simple setting with two groups of households (or countries).

One are impatient households who prefer to operate near to their maximum levels of borrowing, the other patient households who provide the loans to impatient households.

The previous research most closely related to ours, that of Aiyagari [1994], focusses on the microeconomic implications of leverage constraints and uninsured income shocks, investigating the resulting cross-sectional distribution of household wealth and the effect on the equilibrium interest rate. The competing forces of time preference, which encourages impatient households to gradually run down their wealth in order to consume now instead of in the future, and prudential saving, to avoid the constraint of the maximum borrowing limit, result in a distribution of household net worth over the range of net worth. Our modelling framework is similar but we focus instead on the macroeconomic implications and the resulting time distribution of household wealth and aggregate interest rates. With this focus in mind we assume away all idiosyncratic risk and require instead that all risk is aggregate and affects our two countries (or groups of households) to an equal but opposite extent.

The main prediction that emerges from this set up is that changes in household wealth (leverage) have a non-linear and sometimes very substantial impact on real interest rates. Increased prudential saving when impatient households are pushed close to the borrowing limit requires a large drop in real interest rates to maintain goods market equilibrium, with aggregate consumption equal to aggregate income. Even though the set up is simple the non-linearities in the relationship between wealth, interest rates and consumption mean that the responses to such shocks are quite varied. Small or positive shocks have relatively little affect. In contrast a large decline in income and wealth (big enough to reduce wealth to one and half standard deviations below the mean or more) has a substantial impact on real interest

rates. Larger shocks than this can result in negative real interest rates, four per cent or more below median levels. These interest rate responses to income shocks are remarkably robust, arising even with quite substantial variations in parameter values.

We also find that interest rates climb back fairly rapidly following a downward disturbance. This return of interest rates to normal levels is especially fast for large initial shocks. The fall of interest rates reduces the burden of debt repayment (and even becomes a source of income if real interest rates turn negative) in turn allowing indebted households both to rebuild their net worth relatively rapidly and to maintain levels of consumption. These transitory income gains for highly indebted households also means that the impact on consumption of these income shocks is relatively modest compared to the impact on real interest rates; even when pushed very close to their borrowing limits households can maintain relatively high consumption levels. Dynamics are comparatively slow closer to normal levels of leverage.

Our analysis complements the analysis of low global real interest rates by Caballero and Krishnamurthy [2009]. As in their setting a savings imbalance can push down interest rates real interest rates. Their analysis, based on differences in preferences amongst assets, speaks more to long term equilibrium level of interest rates and the spread between low risk and high risk assets. We instead investigate the impact of leverage on the dynamics of real interest rates.

Our characterisation of the financing constraints that underpin these results are, admittedly, very sharp. There are no mechanisms (such as social security, concessional international lending or debt forgiveness) to allow our households greater flexibility in coping with income uncertainty. Higher prudential saving is the only available response. But we do demonstrate how

allowing for leverage constraints and incomplete financial markets can result in very different predictions than of a standard deterministic macroeconomic model. The open question for future research is therefore exploring how leverage can impact on other aspects of macroeconomic behaviour. Brunnermeier and Sannikov [2014] and parallel work by three of the present authors Isohätälä et al. [2014] show how similar dramatic non-linearities in behaviour can arise when there are constraints on firm financing, leading potentially to a 'net worth trap' with leverage stuck at high levels for an extended period. There is thus a substantial agenda for future work, extending the analysis of the macroeconomics of leverage to include a range of other macroeconomic mechanisms, including price dynamics and hence a role for monetary policy, allowing for asset price dynamics, generalising from our pure exchange based economy to one with a production and investment dynamics, and allowing for potentially costly transfer of productive resources from one sector to another. Our paper is one relatively small step towards a full understanding of the macroeconomics of leverage.

A Appendix: Model Solution

This appendix contains technical details regarding the model solution that were omitted in the main text. It first states the stochastic model (the deterministic model is the special case $s = 0$). It then examines the solution to both versions of the model, beginning first with the deterministic case $s = 0$ for which solution is well known and then turning second to the stochastic case $s > 0$.

The deterministic case is solved using the Maximum Principle, the standard method for solving such problems in the macroeconomic textbooks.

This yields (i) well known ‘Euler’ equations in which the rate of change of consumption in each country depends on the difference between world interest rates and their rate of time preference, and also on their intertemporal elasticity of substitution over time (ii) the Euler equations yield a simple expression for the world interest rate as a weighted average of the rates of time preference of the two countries; and using these results (iii) closed form or semi-closed form solution for consumption as a function of time.

The stochastic case is analysed using dynamic programming. This yields (i) stochastic differential equations for consumption (these are the counterparts to the deterministic Euler ordinary differential equations) with an additional term reflecting that country’s absolute prudence; (ii) a generalisation of the simple expression for world interest rates; (iii) a simple second order ordinary differential equation for $c(w)$ with the possibility of numerical solution determining the consumption as a function of borrowing w (in the deterministic case this is a first order ODE).

The full solution requires imposing boundary conditions, either a transversality condition (in the case of the deterministic model without leverage constraints), or constraints imposed at the values of w where the leverage constraints bind (in the stochastic case these can be obtained by a simple application of the Ito calculus; in the deterministic case from the transversality condition).

A.1 Solution to the deterministic version of the model

In this case the equation of motion for w becomes:

$$\dot{w} = (a + rw - c).$$

To apply the Maximum Principle we can write the Hamiltonian (where $v(t)$ are the co-state variable for the evolution of wealth w ; this can be interpreted either as a Lagrangian multiplier for the constraint implied by the equation of motion for w at each point of time t or as the derivative of the dynamic programming value function with respect to w).

$$J = u(c) + v(t)(a + rw - c)$$

Optimal control is the choice of time path for c that maximises the Hamiltonian at each point in time, implying the first order condition:

$$\frac{\partial J}{\partial c} = u'(c) - v(t) = 0 \quad (13)$$

and is subject to the dynamic constraint:

$$\dot{v}(t) = \rho v(t) - \frac{\partial J}{\partial w} = (\rho - r)v(t) \quad (14)$$

together with the additional *transversality condition*:

$$\lim_{t \rightarrow \infty} e^{-\rho t} v(t) w(t) = \lim_{t \rightarrow \infty} e^{-\rho t} u'(c) w(t) = 0 \quad (15)$$

Note that we treat $r(t)$ as a function of time t but *not* as a function of household debt w . The reason for this is because we are assuming that the population of each country consists of a large number of identical households. The interest rate depends upon aggregate debt of all households, not the debt of any individual household. Even though all households are the same and make the same decisions – they do not take account of the fact that their (collective) consumption decisions will have an affect on the interest rate r . If instead households were to co-ordinate their decisions then they could take

the dependency of r on w into account and the constraint on their decision making becomes:

$$\dot{v}(t) = \rho v(t) - \frac{\partial J}{\partial w} = [\rho - r(w) - r'(w)w] v(t)$$

Assuming that such co-ordination of decision making is not possible, differentiating the first order condition (13) w.r.t. t and substituting into (14) we obtain:

$$\dot{v}(t) = \dot{c}u''(c) = (\rho - r)v(t) = (\rho - r)u(c)$$

so:

$$\dot{c} = \left[-\frac{u''(c)}{u'(c)} \right]^{-1} (r - \rho) \quad (16)$$

This is the familiar ‘Euler equation’ determining the dynamics of consumption with a corresponding ‘Euler equation’ for the other country:

$$\dot{\bar{c}} = \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right]^{-1} (r - \bar{\rho}) \quad (17)$$

A challenge to solving for c is that $r(t)$ is unknown. If we combine Eqs. (16) and (17) and with the first derivative of the resource constraint $c + \bar{c} = a + \bar{a}$, we obtain:

$$\dot{c} + \dot{\bar{c}} = \left[-\frac{u''(c)}{u'(c)} \right]^{-1} (r - \rho) + \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right]^{-1} (r - \bar{\rho}) = 0$$

revealing that the interest rate is a weighted average of the time preference of the two households in the two countries with the weights depending on

their intertemporal elasticity of consumption:

$$r = \frac{\left[-\frac{u''(c)}{u'(c)} \right] \rho + \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right] \bar{\rho}}{\left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right] + \left[-\frac{u''(c)}{u'(c)} \right]},$$

and so:

$$\dot{c} = \frac{\left[-\frac{u''(c)}{u'(c)} \right] \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right]}{\left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right] + \left[-\frac{u''(c)}{u'(c)} \right]} (\bar{\rho} - \rho) = -\dot{\bar{c}}. \quad (18)$$

A.2 Solution of stochastic problem using dynamic programming

Using a dynamic programming approach, the goal is to solve for consumption $c(w)$ as function of net lending/borrowing w (rather than as in the deterministic case for $c(t)$ as a function of time t). We begin by finding the consumption of an individual household, and then proceed to make the representative agent assumption, equating the consumption of the individual with that of the aggregate. In this subsection we use the Hamilton-Jacobi-Bellman equation and Itô calculus to derive the equations for optimality for household decision making.

The goal of the impatient household is to maximise the objective function, Eq. (1) (the derivation for the patient household entirely parallels that for the impatient household, so is suppressed here). Let V be the function for which the objective is maximal:

$$V(w_0) = \max_c \mathbb{E} \int_0^\infty e^{-\rho t} u[c(t)] dt, \quad (19a)$$

In order to solve for the optimal consumption rule we must be careful to distinguish between the wealth and consumption of an individual household

indexed by j (w_j, c_j) from the wealth and consumption of the representative agent (w, c). The value function $V(w_j, w)$ for the individual agent satisfies the Hamilton-Jacobi-Bellman equation (note that the interest rate r is unaffected by individual household wealth i.e. each household is a price taker in the capital market):

$$\rho V(w_j, w) = \max_c \left[u(c) + \begin{pmatrix} a + r(w)w_j - c_j & a + r(w)w - c \end{pmatrix} \begin{pmatrix} V_{w_j} \\ V_w \end{pmatrix} + \frac{1}{2} s^2 \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} V_{w_j w_j} & V_{w_j w} \\ V_{w_j w} & V_{w w} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right], \quad (20)$$

Maximising the right-hand sides of Eq. (20), the first order conditions for the choice of c_j is

$$V_{w_j} = u'(c_j) = u'_j. \quad (21)$$

To obtain an ODE for the optimal consumption c we differentiate the HJB, Eq. (20), with respect to w and substitute for optimal consumption obtaining:

$$(\rho - r) V_{w_j} = \begin{pmatrix} a + r(w)w_j - c_j & a + r(w)w - c \end{pmatrix} \begin{pmatrix} V_{w_j w_j} \\ V_{w_j w} \end{pmatrix} + \frac{1}{2} s^2 \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} V_{w_j w_j w_j} & V_{w_j w_j w} \\ V_{w_j w_j w} & V_{w_j w w} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (22)$$

We next proceed to aggregation. In equilibrium, since all households are alike, $w_j = w$ and $c_j = c$ i.e. the j subscripts can be dropped. We need though to consider small departures of consumption out of equilibrium in order to evaluate the derivatives in Eq. (22). For this purpose, since only small deviations are of interest, we can assume that the consumption decision rule, when individual household wealth w_j departs slightly from the wealth of the representative household w , takes the linearised form:

$$c(w, w_j) = c(w) + r(w_j - w)$$

i.e. the optimal consumption rule is to consume just enough so as to maintain a constant difference between w_j and w now and in the future.

With this assumption we then find (from differentiation of Eq. (21) and taking the limit $w_j \rightarrow w$) that:

$$\begin{aligned} V_{w_j w_j} &= r u''(c) \\ V_{w_j w} &= (c'(w) - r) u''(c) \\ V_{w_j w_j w_j} &= r^2 u'''(c) \\ V_{w_j w_j w} &= r(c'(w) - r) u'''(c) + r'(w) u''(c) \\ V_{w_j w w} &= (c'(w) - r)^2 u'''(c) + (c''(w) - 2r'(w)) u''(c) \end{aligned}$$

and substitution back into Eq. (22) then yields the following ODE for $c(w)$:

$$[\rho - r(w)] \frac{u'(c)}{u''(c)} = [a + r(w)w - c]c' + \frac{1}{2}s^2 \left[\frac{u'''(c)}{u''(c)} c'^2 + c'' \right] \quad (23)$$

Just as in the deterministic case we can obtain an equation for r in terms of consumption and the utility function. Differentiating $c(w)$ using Itô's lemma,

we get

$$\begin{aligned} dc &= \left\{ [a + r(w)w - c]c'(w) + \frac{1}{2}s^2c''(w) \right\} dt + sc'(w) dz \\ &= \left\{ [r(w) - \rho] \left[-\frac{u''(c)}{u'(c)} \right]^{-1} + \frac{1}{2}s^2 \left[-\frac{u'''(c)}{u''(c)} \right] c'(w)^2 \right\} dt + sc'(w) dz, \end{aligned}$$

where on the second line we have used Eq. (23).

The equivalent equation for the patient country reads (prime on \bar{c} denoting w derivatives so that $\bar{c}_w = -\bar{c}'$):

$$d\bar{c} = \left\{ [r(w) - \bar{\rho}] \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right]^{-1} + \frac{1}{2}s^2 \left[-\frac{\bar{u}'''(\bar{c})}{\bar{u}''(\bar{c})} \right] c'(w)^2 \right\} dt - s\bar{c}'(\bar{w}) dz.$$

These two equations are the stochastic counterparts to the Euler equations describing the evolution of consumption in the deterministic case. Adding them and using $c'(w) = -\bar{c}'(w)$ yields:

$$\begin{aligned} \frac{d(c + \bar{c})}{dt} &= [r(w) - \rho] \left[-\frac{u''(c)}{u'(c)} \right]^{-1} + [r(w) - \bar{\rho}] \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right]^{-1} \\ &\quad + \frac{1}{2}s^2 \left\{ \left[-\frac{u'''(c)}{u''(c)} \right] + \left[-\frac{\bar{u}'''(\bar{c})}{\bar{u}''(\bar{c})} \right] \right\} c'(w)^2. \quad (24) \end{aligned}$$

and this results in the following expression for the world real interest

$$r(w) = \frac{\left[-\frac{u''(c)}{u'(c)} \right]^{-1} \rho + \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right]^{-1} \bar{\rho}}{\left[-\frac{u''(c)}{u'(c)} \right]^{-1} + \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right]^{-1}} - \frac{1}{2}s^2 \frac{\left[-\frac{u'''(c)}{u''(c)} \right] + \left[-\frac{\bar{u}'''(\bar{c})}{\bar{u}''(\bar{c})} \right]}{\left[-\frac{u''(c)}{u'(c)} \right]^{-1} + \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right]^{-1}} c'^2. \quad (25)$$

The first term here is the expression for r that applies in the deterministic case. The second additional term in $\frac{1}{2}s^2c'(w)^2$ is a correction that depends

on the absolute prudence,

$$\left[-\frac{u'''(c)}{u''(c)} \right], \quad \left[-\frac{\bar{u}'''(\bar{c})}{\bar{u}''(\bar{c})} \right],$$

and absolute risk aversion

$$\left[-\frac{u''(c)}{u'(c)} \right], \quad \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right]$$

of the two countries. Eq. (23) together with $r(w)$ as given by Eq. (25) form an ordinary differential equation from where the aggregate consumption c can be solved.

As a final point, the equations for the consumption can be written in a compact way that also gives insight as to what effects are controlling the interest rate. We introduce effective time-constants ρ_{eff} and $\bar{\rho}_{\text{eff}}$, defined as

$$\rho_{\text{eff}}(w) = \rho - \frac{1}{2}s^2 \frac{u'''}{u''}(c)^2, \quad (26a)$$

$$\bar{\rho}_{\text{eff}}(w) = \bar{\rho} - \frac{1}{2}s^2 \frac{\bar{u}'''}{\bar{u}''}(\bar{c})^2. \quad (26b)$$

A simple rearrangement of terms and factors gives then

$$(a + rw - c)c' = \frac{\bar{\rho}_{\text{eff}} - \rho_{\text{eff}}}{-u''/u' - \bar{u}''/\bar{u}'} - \frac{1}{2}s^2 c'', \quad (27)$$

where the interest rate is now the weighted average of the effective time constants:

$$r(w) = \frac{\left[-\frac{u''(c)}{u'(c)} \right]^{-1} \rho_{\text{eff}}(w) + \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right]^{-1} \bar{\rho}_{\text{eff}}(w)}{\left[-\frac{u''(c)}{u'(c)} \right]^{-1} + \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right]^{-1}}. \quad (28)$$

Comparing to the deterministic limit,

$$(a + rw - c)c' = \frac{\bar{\rho} - \rho}{-u''/u' - \bar{u}''/\bar{u}'},$$

$$r(w) = \frac{\left[-\frac{u''(c)}{u'(c)}\right]^{-1} \rho + \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})}\right]^{-1} \bar{\rho}}{\left[-\frac{u''(c)}{u'(c)}\right]^{-1} + \left[-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})}\right]^{-1}}.$$

we see that in the stochastic case, the effective time-constants simply take the place of ρ and $\bar{\rho}$. In addition, the diffusion term $(1/2)s^2c''$ appears in the equation, effectively adding consumption inertia. The new time-constants can then be viewed as household prudence adjusted $\rho, \bar{\rho}$, where the corrections $(1/2)s^2u'''/u'(c)^2$, $(1/2)s^2\bar{u}'''/\bar{u}'(\bar{c})^2$ take account the noise and the rate of change of consumption, effectively increasing household patience (of course, there is no real change in $\rho, \bar{\rho}$, rather, the behaviour of the households roughly matches the deterministic case with lower time-constants).

A.3 Boundary conditions and asymptotic expansions

The problem specification is not yet complete, as we have not yet fixed the two boundary conditions needed to solve Eq. (23). We look for a joint solution of the ODE for $c(w)$ Eq. (23) and the steady state ergodic density Eq. (12) satisfying the zero density boundary condition

$$\lim_{w \rightarrow w^*} f(w) = 0,$$

which we interpret as requiring that household behaviour is such that it completely avoids the borrowing limit. We only consider the behaviour at the lower boundary w^* , and attempt to find the shape of c and f that are consistent with the above. The calculation for the upper boundary is essentially

identical.

A formal solution to the Fokker-Planck equation, Eq. (12) gives the ergodic density

$$f(w) = C \exp \left\{ \frac{2}{s^2} \int_0^w [a + r(w')w' - c(w')] dw' \right\},$$

where C is a constant of integration. In order for the zero density boundary condition to be satisfied, the integral in the exponent cannot be finite, but rather must tend to infinity. Within the integral, only $r(w)$ does not have strict bounds, and inspection of Eq. (28) shows that infinite interest rate is only possible if $\rho_{\text{eff}}(w) \rightarrow \pm\infty$ or $\bar{\rho}_{\text{eff}}(w) \rightarrow \pm\infty$ as $w \rightarrow w^*$. This is the case regardless of the choice of u, \bar{u} , as the weighted average is always in the closed interval $[\min(\rho_{\text{eff}}, \bar{\rho}_{\text{eff}}), \max(\rho_{\text{eff}}, \bar{\rho}_{\text{eff}})]$ even when $-u''/u'$ or $-\bar{u}''/\bar{u}' \rightarrow \infty$. Therefore, either ρ_{eff} or $\bar{\rho}_{\text{eff}}$ must be infinite, or by Eq. (26),

$$\lim_{w \rightarrow w^*} \frac{u'''[c(w)]}{u'[c(w)]} (c'(w))^2 = \infty.$$

Clearly this cannot be satisfied for quadratic utility, and so we assume $u''' \neq 0$. Next task is to find solutions (if any) to Eq. (23) such that the above boundary scaling is realised.

Let us suppose that $0 < c(w^*) < \infty$. This avoids the possibility of the utility functions becoming also infinite, complicating the analysis. We look for a solution that has the form

$$c(w) = c_0 + c_1(w - w^*)^\alpha + o((w - w^*)^\alpha), \quad (29)$$

$$0 < \alpha < 1.$$

Near the boundary, c derivatives read

$$\begin{aligned} c'(w) &= \alpha c_1 (w - w^*)^{\alpha-1} + o((w - w^*)^{\alpha-1}), \\ c''(w) &= \alpha(\alpha - 1)c_1 (w - w^*)^{\alpha-2} + o((w - w^*)^{\alpha-2}), \end{aligned}$$

and the effective time-constants become, denoting $u_* = u(c_0)$, $\bar{u}_* = \bar{u}(a + \bar{a} - c_0)$ and analogously for the derivatives,

$$\begin{aligned} \rho_{\text{eff}} &= -\frac{1}{2}s^2 \frac{u_*'''}{u_*''} \alpha^2 c_1^2 (w - w^*)^{2\alpha-2} + o((w - w^*)^{2\alpha-2}), \\ \bar{\rho}_{\text{eff}} &= -\frac{1}{2}s^2 \frac{\bar{u}_*'''}{\bar{u}_*''} \alpha^2 c_1^2 (w - w^*)^{2\alpha-2} + o((w - w^*)^{2\alpha-2}), \end{aligned}$$

and the interest rate

$$\begin{aligned} r &= -\frac{1}{2}s^2 \frac{(u_*'''/u_*'')(-\bar{u}_*''/\bar{u}_*') + (\bar{u}_*'''/\bar{u}_*'')(-u_*''/u_*')}{-\bar{u}_*''/\bar{u}_*' - u_*''/u_*'} \alpha^2 c_1^2 (w - w^*)^{2\alpha-2} \\ &\quad + o((w - w^*)^{2\alpha-2}) \\ &= -\frac{1}{2}s^2 \gamma \alpha^2 c_1^2 (w - w^*)^{2\alpha-2} + o((w - w^*)^{2\alpha-2}), \end{aligned}$$

where

$$\gamma = \frac{-u_*'''/u_*'' - \bar{u}_*'''/\bar{u}_*''}{-u_*''/u_*' - \bar{u}_*''/\bar{u}_*'}$$

We can now write Eq. (27), picking for each term just the terms and factors of lowest power in $(w - w^*)$, up to leading order as

$$\begin{aligned} &\frac{1}{2}s^2 \alpha(\alpha - 1)c_1 (w - w^*)^{\alpha-2} - \frac{1}{2}s^2 \gamma \alpha^3 c_1^3 (w - w^*)^{3\alpha-3} w^* \\ &= -\frac{1}{2}s^2 \frac{\bar{u}_*'''/\bar{u}_*' - u_*'''/u_*'}{-u_*''/u_*' - \bar{u}_*''/\bar{u}_*' } \alpha^2 c_1^2 (w - w^*)^{2\alpha-2} + o((w - w^*)^{\min\{\alpha-2, 3\alpha-3\}}) \quad (30) \end{aligned}$$

In order for the asymptotic trial function to be correct, two of the terms

here must be of equal order, that is

$$\alpha - 2 = 3\alpha - 3 \quad \vee \quad \alpha - 2 = 2\alpha - 2 \quad \vee \quad 3\alpha - 3 = 2\alpha - 2.$$

Only the first possibility gives $0 < \alpha < 1$, specifically

$$\alpha = \frac{1}{2}.$$

Substituting this into Eq. (30) and neglecting the higher order terms, we are left with

$$c_1 + \frac{1}{2}\gamma c_1^3 w^* = 0,$$

which gives as the only valid solution (assuming $c'(w) > 0$ near boundary; note that $\gamma > 0$ and $w^* < 0$)

$$c_1 = \sqrt{\frac{2}{|w^*|\gamma}} = \sqrt{\frac{2}{|w^*|} \frac{-u'_*/u''_* - \bar{u}'_*/\bar{u}''_*}{-u'''_*/u''_* - \bar{u}'''_*/\bar{u}''_*}}. \quad (31)$$

This relates c_0 to c_1 , and therefore gives us the boundary condition at the lower edge.

Finally, we can find the interest rate and the ergodic density. Substituting α and c_1 into the expansion for r we get

$$\begin{aligned} r &= -\frac{1}{8}s^2\gamma c_1^2(w - w^*)^{-1} + o((w - w^*)^{-1}) \\ &= -\frac{1}{4}s^2\frac{1}{|w^*|}(w - w^*)^{-1} + o((w - w^*)^{-1}). \end{aligned}$$

The probability density function then follows

$$\begin{aligned} f(w) &\propto \exp \left\{ \frac{2}{s^2} \int_{w^*}^w w^* \left[-\frac{1}{4} s^2 \frac{1}{|w^*|} (w - w^*)^{-1} + o((w - w^*)^{-1}) \right] dw' \right\} \\ &= (w - w^*)^\beta + o((w - w^*)^\beta), \end{aligned}$$

where the exponent

$$\beta = \frac{1}{2}. \quad (32)$$

As f follows square-root law near the boundary, the zero density boundary condition is satisfied.

Note that the calculation has not assumed anything about the utility function, other than that it is not quadratic (we do have to suppose that $\gamma > 0$, but this is satisfied by common utility functions). Furthermore, if $u[c(w^*)]$ and derivatives are not infinite, then the calculation goes through with $c_0 = 0$ just as well.

Finally, let us consider the case that $c(w^*) = 0$, and let us take the iso-elastic utility function $u = (1 - \varepsilon)^{-1} c^{1-\varepsilon}$. Again assume that c has the form (29), but with $c_0 = 0$ and $\alpha > 0$. Repeating the above calculation gives in the lowest order of $(w - w^*)$:

$$\begin{aligned} c_1 \alpha (\alpha - 1) (w - w^*)^{\alpha-2} - (1 + \varepsilon) c_1 \alpha^2 (w - w^*)^{\alpha-2} \\ - 2c_1^2 \alpha^3 (w - w^*)^{2\alpha-3} w^* \frac{\bar{\varepsilon} \varepsilon (1 + \varepsilon)}{a + \bar{a}} + o((w - w^*)^{\alpha-2}) = 0. \end{aligned} \quad (33)$$

If $2\alpha - 3 \neq \alpha - 2$, or $\alpha \neq 1$, we get

$$\alpha(\alpha - 1) = (1 + \varepsilon)\alpha^2,$$

yielding $\alpha = 0 \vee \alpha = -1/\varepsilon$, neither of which is valid. The other possibility is that $\alpha = 1$, which in turn gives

$$c_1 = -\frac{1}{2} \frac{a + \bar{a}}{w^* \bar{\varepsilon} \varepsilon}.$$

Thus, there exists a linear solution. Note, however, that this binds both the value and derivative of c at $w = w^*$. The interest rate is still negative infinity at the boundary:

$$\begin{aligned} r &= -\frac{1}{2} s^2 c_1 \frac{\bar{\varepsilon}(1 + \varepsilon)}{a + \bar{a}} (w - w^*)^{-1} + o((w - w^*)^{-1}) \\ &= \frac{1}{4} s^2 \frac{1 + \varepsilon}{w^* \varepsilon} (w - w^*)^{-1} + o((w - w^*)^{-1}). \end{aligned}$$

So being, the density is as wanted a power law tending to zero at w^* :

$$\begin{aligned} f(w) &\propto \exp \left\{ \frac{2}{s^2} \int_{w^*}^w w^* \left[-\frac{1}{4} s^2 \frac{1 + \varepsilon}{|w^*| \varepsilon} (w - w^*)^{-1} + o((w - w^*)^{-1}) \right] dw' \right\} \\ &= (w - w^*)^\beta + o((w - w^*)^\beta), \end{aligned}$$

where the exponent β is now

$$\beta = \frac{1}{2} \frac{1 + \varepsilon}{\varepsilon}. \quad (34)$$

Again, this is positive, and zero density boundary condition is satisfied.

A.4 Numerical solution of the model equation

For the numerical solution of the model equations, we used two different, but complementary methods: standard ODE initial value integrator to iteratively find the solution satisfying the boundary conditions, and a pseudospectral

method.

A.4.1 Initial value iterations

Let us write the model equations in the form

$$F(w, c(w), c'(w), c''(w)) = 0, \quad (35)$$

and suppose boundary conditions of the form

$$g^*(c(w^*), c'(w^*)) = 0, \quad (36a)$$

$$\bar{g}^*(c(\bar{w}^*), c'(\bar{w}^*)) = 0. \quad (36b)$$

Let $c(w; c_0, c'_0)$ stand for the solution of the Cauchy problem consisting of Eq. (35) with initial values

$$c(w^*; c_0, c'_0) = c_0, \quad (37a)$$

$$c'(w^*; c_0, c'_0) = c'_0. \quad (37b)$$

Let now x_0 stand for either the values of c_0 or c'_0 , and let the other be chosen as a solution of Eq. (36a). The solution to the initial value problem can then be parametrized in terms of x_0 . Let that solution be $c(w; x_0)$. The upper boundary condition now becomes

$$g^*(x_0) = 0,$$

$$\bar{g}^*(x_0) = \bar{g}^*[c(\bar{w}^*; x_0), c'(\bar{w}^*; x_0)]$$

Standard root-finding methods can be used to solve this equation.

For the Neumann boundary conditions $g^*(c_0, c'_0) = \bar{g}^*(c_0, c'_0) = c'_0 - 1$, we

have numerically solved the Cauchy problem using standard Runge-Kutta adaptive step-size integrators, here usually the Prince-Dormand embedded 8(7) pair. We found that the upper boundary values were rather sensitive to the initial data, which meant that initial bracketing of a root x_0 required a rather dense search grid, upto 10^4 points over the interval $[0, a + \bar{a}]$, if a uniform mesh was used and no initial guess was available. For the same reason, relatively stringent bounds for local error estimates were needed for the adaptive step-size control: Letting y_{err} be the stepper method supplied error estimate and $y = (c(w), c'(w))$ the corresponding numerically obtained state, the bound

$$\|y_{\text{err}}\| < \varepsilon_{\text{rel}}\|y\| + \varepsilon_{\text{abs}},$$

with $\varepsilon_{\text{rel}}, \varepsilon_{\text{abs}} \sim 10^{-11}$ was enforced to keep accumulated numerical error in check and to reliably resolve the root x_0 .

A.4.2 Pseudospectral method

The nonlinear problem, Eq. (35), is solved using the Newton's method: We attempt to find a sequence of approximations $(c^{(k)})_{k=0}^{\infty}$ converging to c , from which we accept the first $c^{(k)}$ satisfying our convergence criterion as the solution. The approximations are constructed as follows. Let us write the nonlinear problem in the form

$$\begin{aligned} F[\omega, c(\omega), c'(\omega), c''(\omega)] &= 0, \\ w &= \frac{-w^* + \bar{w}^*}{2}\omega + \frac{w^* + \bar{w}^*}{2}. \end{aligned} \tag{38}$$

For reasons that will become obvious, we have rescaled the independent variable $w \in [w^*, \bar{w}^*]$ to $\omega \in [-1, 1]$. Let us here suppose Robin boundary

conditions

$$\alpha c(-1) + \alpha' c'(-1) = \gamma, \quad (39a)$$

$$\beta c(+1) + \beta' c'(+1) = \bar{\gamma}. \quad (39b)$$

The algorithm can be adapted to nonlinear conditions as given in Eq. (36), however, for simplicity we only consider (39).

The $(k + 1)$ th approximation $c^{(k+1)}$ is given by

$$c^{(k+1)} = c^{(k)} + \zeta^{(k)},$$

where the function $\zeta^{(k)}$ solves the linear boundary value problem

$$\mathcal{L}^{(k)} \zeta^{(k)} = F^{(k)}, \quad (40a)$$

$$\alpha \zeta^{(k)}(-1) + \alpha' \zeta^{(k)' }(-1) = \gamma - \alpha c^{(k)}(-1) - \alpha' c^{(k)'}(-1), \quad (40b)$$

$$\beta \zeta^{(k)}(+1) + \beta' \zeta^{(k)' }(+1) = \bar{\gamma} - \beta c^{(k)}(+1) - \beta' c^{(k)'}(+1),$$

and where the operator $\mathcal{L}^{(k)}$ and vector $F^{(k)}$ are

$$\begin{aligned} \mathcal{L}^{(k)}(\omega) &= \frac{\partial F}{\partial c''}[\omega, c^{(k)}(\omega), c^{(k)'}(\omega), c^{(k)''}(\omega)] \frac{\partial^2}{\partial \omega^2} \\ &+ \frac{\partial F}{\partial c'}[\omega, c^{(k)}(\omega), c^{(k)'}(\omega), c^{(k)''}(\omega)] \frac{\partial}{\partial \omega} \\ &+ \frac{\partial F}{\partial c}[\omega, c^{(k)}(\omega), c^{(k)'}(\omega), c^{(k)''}(\omega)], \end{aligned} \quad (41a)$$

$$F^{(k)}(\omega) = F[\omega, c^{(k)}(\omega), c^{(k)'}(\omega), c^{(k)''}(\omega)]. \quad (41b)$$

This equation can be easily derived by supposing that $c = c^{(k)} + \lambda^{(k)} \zeta_*^{(k)}$, $\lambda^{(k)} = |c - c^{(k)}|$, and substituting that into Eq. (38). By neglecting $\mathcal{O}(\lambda^{(k)2})$ terms, the above is obtained. Informally, if the norm of $\zeta_*^{(k)}$ is sufficiently

small, then $\zeta^{(k)} \approx \lambda^{(k)} \zeta_*^{(k)}$.

To solve Eq. (40), we use a Chebyshev pseudospectral method. We seek a solution, again approximate, in the subspace spanned by the N first Chebyshev polynomials $T_i : [-1, 1] \mapsto [-1, 1]$, $T_i(\omega) = \cos(i \arccos \omega)$, $i = 0 \dots N$,

$$\zeta^{(k)} = \xi^{(k)} + E^{(k)}, \quad \xi^{(k)} = \sum_{i=0}^N \bar{\xi}_i^{(k)} T_i(\omega).$$

Here $E^{(k)}$ stands for the truncation error, and the integer N , the order of the approximation, is naturally chosen high enough for the error to become negligible.

To determine the superposition or *modal* coefficients $\bar{\xi}_i^{(k)}$, $i = 0, \dots, N$, we ask that the residual $R^{(k)}$,

$$R^{(k)} = \mathcal{L}^{(k)} \xi^{(k)} - F^{(k)}$$

vanishes at specially chosen collocation points ω_i . We chose the Chebyshev-Gauss-Lobatto (CGL) -nodes, the extremal points of T_N ,

$$\omega_i = \cos \frac{\pi(N-i)}{N}, \quad i = 0, \dots, N.$$

Let $\hat{\xi}_i^{(k)}$, $i = 0, \dots, N$, stand for the *nodal* values of $\xi^{(k)}$, that is, $\xi^{(k)}$ evaluated at the CGL-points,

$$\hat{\xi}_i^{(k)} = \xi^{(k)}(\omega_i).$$

In the following, we will denote the vectors $(\hat{\xi}_0^{(k)}, \dots, \hat{\xi}_N^{(k)})^T$ and $(\bar{\xi}_1^{(k)}, \dots, \bar{\xi}_N^{(k)})^T$ simply by $\hat{\xi}^{(k)}$ and $\bar{\xi}^{(k)}$.

Rather than actually solving for the modal coefficients $\bar{\xi}^{(k)}$, the problem

is turned into an equation for the nodal vector $\hat{\xi}^{(k)}$. The main ingredient needed to achieve this is the differentiation operator $\partial/\partial\omega$ for the \mathbb{R}^{N+1} space of nodal values, *i.e.* the $(N+1) \times (N+1)$ matrix \hat{D} that (letting f stand for any polynomial of order N or less) maps a vector $\hat{f} = (f(\omega_0), \dots, f(\omega_N))^T$ to $\hat{f}' = (f'(\omega_0), \dots, f'(\omega_N))^T$,

$$\hat{f}' = \hat{D}\hat{f}.$$

An explicit, closed form expression for \hat{D} can be readily found, e.g. by forming the Lagrange interpolating polynomial for data points \hat{f} and differentiating that; we will omit writing \hat{D} down as it is not relevant for the present discussion.

As the solution for the linear problem is now sought at the collocation points, we represent the approximate solution $c^{(k)}$ using its nodal values $\hat{c}_i^{(k)}$,

$$\hat{c}_i^{(k)} = c^{(k)}(\omega_i) - E_c^{(k)}.$$

Here $E_c^{(k)}$ is the truncation error carried from solving the linear problem. We can forget the functions $c^{(k)}$ and simply seek approximations to c in the form of the nodal vectors $\hat{c}^{(k)} = (\hat{c}_0^{(k)}, \dots, \hat{c}_N^{(k)})$, and define

$$\hat{c}^{(k)'} = \hat{D}\hat{c}^{(k)}, \quad \hat{c}^{(k)''} = \hat{D}^2\hat{c}^{(k)}.$$

The sequence of approximations we wish to find is now generated by

$$\hat{c}_i^{(k+1)} = \hat{c}_i^{(k)} + \hat{\xi}_i^{(k)}, \quad i = 0, \dots, N. \quad (42)$$

Substituting $\xi^{(k)}$ into Eq. (40a) and evaluating the result at the collocation

points, we recover the matrix equation

$$\hat{L}^{(k)}\hat{\xi}^{(k)} = \hat{F}^{(k)}, \quad (43)$$

$$(44)$$

where the matrix $\hat{L}^{(k)}$ and vector $\hat{F}^{(k)}$ are defined as

$$\hat{L}^{(k)} = \hat{G}^{(k)''} \hat{D}^2 + \hat{G}^{(k)'} \hat{D} + \hat{G}^{(k)}, \quad (45)$$

$$\hat{G}^{(k)'\dots'} = \text{diag} \left\{ \frac{\partial F}{\partial c^{(k)'}}[\omega_0, \hat{c}_0^{(k)}, \hat{c}_0^{(k)'}, \hat{c}_0^{(k)''}], \dots, \frac{\partial F}{\partial c^{(k)'}}[\omega_N, \hat{c}_N^{(k)}, \hat{c}_N^{(k)'}, \hat{c}_N^{(k)''}] \right\}, \quad (46)$$

$$\hat{F}^{(k)} = (F^{(k)}(\omega_0), \dots, F^{(k)}(\omega_N))^T. \quad (47)$$

The above does not account for the boundary values. These are incorporated into the problem by replacing the matrix $\hat{L}^{(k)}$ and vector $\hat{F}^{(k)}$ by $\hat{L}_{\text{BV}}^{(k)}$ and $\hat{F}_{\text{BV}}^{(k)}$, respectively. The matrix $\hat{L}_{\text{BV}}^{(k)}$ is equal to $\hat{L}^{(k)}$ except for:

$$\text{First row of } \hat{L}_{\text{BV}}^{(k)} = \text{First row of } \alpha I + \alpha' \hat{D},$$

$$\text{Last row of } \hat{L}_{\text{BV}}^{(k)} = \text{Last row of } \beta I + \beta' \hat{D},$$

where I is the $(N+1) \times (N+1)$ identity matrix. Likewise, the vector $\hat{F}_{\text{BV}}^{(k)}$ is equal to $\hat{F}^{(k)}$ except for:

$$\text{First element of } \hat{F}_{\text{BV}}^{(k)} = \gamma - \alpha \hat{c}_0^{(k)} - \alpha' \hat{c}_0^{(k)'},$$

$$\text{Last element of } \hat{F}_{\text{BV}}^{(k)} = \bar{\gamma} - \beta \hat{c}_N^{(k)} - \beta' \hat{c}_N^{(k)'}$$

The equation we need to solve is then

$$\hat{L}_{\text{BV}}^{(k)} \hat{\xi}^{(k)} = \hat{F}_{\text{BV}}^{(k)}. \quad (48)$$

It is easy to see that the first and last equations in this set are just the Robin boundary conditions (41a). The (approximate) solution of the differential equation Eq. (40) has now been reduced to solving an ordinary set of linear equations, which can be done using any standard methods.

Eq. (42) together with Eq. (48) now give the sequence of approximations to c (at the CGL-nodes). Convergence is checked by monitoring the Euclidean norm of the (nodal) residual of the nonlinear equation, $\hat{F}^{(k)}$, and stopping the iterations at first k^* such that

$$\|\hat{F}^{(k^*)}\| < \delta,$$

where $\delta > 0$ is some number deemed sufficiently small, typically $\delta \sim 10^{-6}$. We seed the algorithm with some initial guess $\hat{c}^{(0)}$ that is an increasing sequence in the range $[0, a + \bar{a}]$. For the baseline parameters and boundary conditions $c'(w^*) = 1$, $c'(\bar{w}^*) = 1$, we were able to get convergence using a piecewise cubic, twice continuously differentiable initial guess such that $\hat{c}_0^{(0)} = 0.05 \times (a + \bar{a})$, $\hat{c}_N^{(0)} = 0.95 \times (a + \bar{a})$ and $\hat{c}_0^{(0)'} = \hat{c}_N^{(0)'} = 2/(-w^* + \bar{w}^*)$. The initial guess was taken to be linear for central part of the range. It should be emphasized that the Newton's method iterations are very sensitive to the initial guess, and convergence may not be achieved if initial guess is not good enough.

With regard to the order of the approximation, we found that $N \sim 40$ is typically sufficient, although values ~ 200 may be needed if the solution is changing rapidly in some parts of the interval $[w^*, \bar{w}^*]$.

A.4.3 Performance considerations

The pseudospectral solver has proved to be significantly faster for this particular problem, mainly owing to the fact that the Cauchy solution $c(w; c_0, c'_0)$ is sensitive to the initial data, and thus requiring dense search grids for the free lower boundary variable. Furthermore, the pseudospectral solver can be very easily adapted to ODE eigenvalue problems such as the time-dependent Fokker-Planck equation, and extensions of the model that turn the HJB into a partial differential equation.

The drawback, on the other hand, of the pseudospectral method is first the effort of coding the algorithm (as few numerical libraries offer pseudospectral ODE solvers, further these tend to be for specialized applications); second, more importantly, the necessity of being able to supply the algorithm with a good enough initial guess $c^{(0)}$, and third, the inability to easily check for presence of multiple solutions (as this would require searching the space of initial guess *functions*, rather than just initial *values*).

A standard iterative solver, although slower, is therefore useful in searching for plausible initial guesses to initialize the pseudospectral algorithm, and helps also to detect presence of multiple solutions.

A.5 Time-dependent Fokker-Planck equation

In order to compute the time-dependent distribution function $f(w, t)$, we need to solve the Fokker-Planck equation with time derivative term retained. Writing in terms of the probability current j , this reads

$$\partial_t f(w, t) = -\partial_w j(w, t), \quad (49a)$$

$$j(w, t) = [a + r(w)w - c(w)] f(w, t) - \frac{1}{2} s^2 \partial_w^2 f(w, t). \quad (49b)$$

This is to be solved with the boundary and initial conditions

$$j(w, t)|_{\text{boundaries}} = 0 \quad \forall t \quad (50a)$$

$$f(w, 0) = f_{\text{init}}(w), \quad (50b)$$

where f_{init} is some probability distribution function.

Theorem 1 *Solutions of Eq. (49) with boundary conditions and initial data of Eq. (50) can be written in the form*

$$f(w, t) = \sum_{i=0}^{\infty} c_i e^{-\lambda_i t} f_i(w), \quad (51)$$

where $0 = \lambda_0 < \lambda_1 < \dots$, $c_i \in \mathbb{R}$,

$$c_i = \int_{w^*}^{\bar{w}^*} \frac{f_i(w) f_{\text{init}}(w)}{f_0(w)} dw, \quad (52)$$

where f_0 again is the solution to the time-independent Fokker-Planck equation. The functions f_i satisfy the orthogonality

$$\int_{w^*}^{\bar{w}^*} \frac{f_i(w) f_j(w)}{f_0(w)} dw = \delta_{ij}. \quad (53)$$

Function f_i has i roots on the open interval (w^*, \bar{w}^*) .

Proof. Let us try to find a solution of Eq. (49) of the form

$$f(w, t) = X(w)T(t). \quad (54)$$

Substituting this into Eq. (49) and rearranging, one obtains

$$\frac{T'(t)}{T(t)} = -\frac{1}{X(w)} \frac{d}{dw} \left\{ [a + r(w)w - c(w)] X(w) - \frac{1}{2} s^2 X'(w) \right\}. \quad (55)$$

This reduces to two ODEs using the standard argument: Left-hand side depends only on t while on the right only w appears. Therefore both sides of the equation are constants. Let us denote that constant by $-\lambda$. We get:

$$T'(t) = -\lambda T(t), \quad (56)$$

$$\lambda X(w) = \left\{ [a + r(w)w - c(w)] X(w) - \frac{1}{2} s^2 X'(w) \right\}'. \quad (57)$$

The boundary condition Eq. (50a) now reads

$$[a + r(w)w - c(w)] X(w) - \frac{1}{2} s^2 X'(w) \Big|_{w=\bar{w}^*, w^*} = 0. \quad (58)$$

Together with these boundary conditions, the second ODE forms an eigenvalue problem for the eigenvalue λ and eigenfunctions X . We will use the label i to identify different λ, X, T tuples. Values of i come from an index set that will turn out to be the natural numbers.

The first ODE, Eq. (55), is readily solved to give the time-dependence

$$T_i(t) = T_i(0) e^{-\lambda_i t}. \quad (59)$$

For the second, boundary value problem Eq. (57, 58), we substitute

$$X_i(w) = \Xi_i(w) f_0(w), \quad (60)$$

which yields after easy manipulations

$$-\frac{1}{2} s^2 [f_0(w) \Xi_i'(w)]' = \lambda_i f_0(w) \Xi_i(w), \quad (61a)$$

$$\Xi_i(w) \Big|_{w=w^*, \bar{w}^*} = 0. \quad (61b)$$

These amount to a regular Sturm-Liouville problem, and from there, the claims of the theorem immediately follow. ■

Intuition regarding functions f_i is that, for increasing i , they describe finer and finer detail in the distribution f . Since $(\lambda_i)_{i=0}^\infty$ is increasing, the finer details decay faster. Ultimately, the time-dependent distribution relaxes to the steady state when the slowest of f_i 's, that is $f_1 \propto \exp(-\lambda_1 t)$, effectively vanishes. Thus, we can estimate the relaxation time by the inverse of λ_1 ,

$$\tau_{\text{relax}} = \frac{1}{\lambda_1}. \quad (62)$$

Generalizing, we can define half-lives $\tau_{\text{half}}^{(k)}$ as the times it takes for the k th term in the series of Eq. (51) to reduce by one half:

$$\tau_{\text{half}}^{(k)} = \frac{\ln 2}{\lambda_k}. \quad (63)$$

A.5.1 Numerical solution

To obtain a numerical solution to Eq. (49), we solve the eigenvalue problem Eq. (57, 58) and use the series expansion of Eq. (51), together with coefficients from Eq. (52), to evaluate $f(w, t)$. Only N_{modes} first eigenvalue/function pairs λ_i, X_i are computed, where the number of modes N_{modes} is chosen depending on the initial data f_{init} .

In practice it has proved useful to solve for functions Φ_i , defined via,

$$f_i(w) = \sqrt{f_0(w)}\Phi_i(w). \quad (64)$$

From Eq. (53) one sees that the Φ_i are orthogonal with respect to unity

weight:

$$\int_{w^*}^{\bar{w}^*} \Phi_i(w) \Phi_j(w) dw = \delta_{ij}. \quad (65)$$

Equations (57, 58) now become

$$\mathcal{L}^{\text{FP}} \Phi_i(w) = \lambda_i \Phi_i(w), \quad (66a)$$

$$-\mu(w)\Phi(w) + s^2\Phi'(w) \Big|_{w=w^*, \bar{w}^*} = 0, \quad (66b)$$

where we have defined the linear operator \mathcal{L}^{FP} as

$$\mathcal{L}^{\text{FP}} = -\frac{1}{2}s^2 \frac{\partial^2}{\partial w^2} + \left[\frac{\mu(w)^2}{2s^2} + \frac{1}{2}\mu'(w) \right]. \quad (67)$$

The pseudospectral solver used to solve the HJB is repurposed to solve the above boundary value problem. The differential operators in \mathcal{L}^{FP} are replaced by their pseudospectral equivalents using the same approximation order N as was used in the solution of the HJB. This yields the matrix $\hat{\mathcal{L}}^{\text{FP}}$. Replacing \mathcal{L}^{FP} by $\hat{\mathcal{L}}^{\text{FP}}$, Eq. (66) becomes a usual matrix eigenvalue problem, which can be solved using standard methods in numerical linear algebra.

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