A new isoperimetric inequality for the elasticae

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Abstract

For a smooth curve γ, we define its elastic energy as $E(\gamma) = \frac{1}{2}\int_\gamma k^2(s)ds$ where $k(s)$ is the curvature. The main purpose of the paper is to prove that among all smooth, simply connected, bounded open sets of prescribed area in $\mathbb{R}^2$, the disc has the boundary with the least elastic energy. In other words, for any bounded simply connected domain Ω, the following isoperimetric inequality holds: $E^2(\partial\Omega)A(\Omega) \geq \pi^3$. The analysis relies on the minimization of the elastic energy of drops enclosing a prescribed area, for which we give as well an analytic answer.

1 Introduction

Let Ω be a smooth, bounded simply connected open set in the plane (the exact smoothness which is required will be made precise in Section 2) and let us denote by $\partial\Omega$ its boundary. Following L. Euler, we define its elastic energy as

$$E(\partial\Omega) = \frac{1}{2}\int_{\partial\Omega} k^2(s)ds \tag{1}$$

where s is the curvature abscissa and k is the curvature. We will denote by $A(\Omega)$ the area of Ω and $L(\Omega)$ its perimeter. The aim of this paper is to prove the following isoperimetric inequality.

Theorem 1.1 *For any bounded, smooth, simply connected open set* $\Omega \subseteq \mathbb{R}^2$

$$E^2(\partial\Omega)A(\Omega) \geq \pi^3 \tag{2}$$

where equality holds only for the disc.

In other words, using the behavior of the elastic energy on rescaling, we get that for every $A_0 > 0$, the disc is the unique solution for the minimization problem

$$\min\{E(\partial\Omega) : A(\Omega) \leq A_0, \Omega \text{ bounded, smooth, simply connected open set of } \mathbb{R}^2\}.$$

More precisely, if we perform any scaling of ratio t, we have $E(t\partial\Omega) = t^{-1}E(\partial\Omega)$ and $A(t\partial\Omega) = t^2A(\partial\Omega)$. Therefore, it is classical to prove that the following three minimization problems are equivalent (in the sense that any solution of one gives a solution of the others after a suitable scaling):

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(i) $\min E^2(\partial\Omega)A(\Omega)$

(ii) $\min\{E(\partial\Omega) : A(\Omega) \leq A_0\}$

(iii) $\min E(\partial\Omega) + A(\Omega)$

Let us make some comments. For a detailed bibliography on closed elasticae, we refer to the classical [7] or the more recent [8]. Inequality (2) was already known for convex domains. Indeed, by a famous inequality due to M. Gage [5], for any bounded convex domain

$$\frac{E(\partial\Omega)A(\Omega)}{L(\Omega)} \geq \frac{\pi}{2}$$

with equality for the disc. Therefore,

$$E^2(\partial\Omega)A(\Omega) \geq E^2(\partial\Omega)A(\Omega)\frac{4\pi A(\Omega)}{L^2(\Omega)} \geq \frac{\pi^2}{4} \times 4\pi = \pi^3,$$

the first inequality being the classical isoperimetric inequality, and the second the Gage inequality. If the convexity hypothesis is dropped, then the Gage inequality is false (as shown by the counter-example of Figure 1).

The simply connectedness assumption is necessary. Indeed, if we take as a domain Ω the ring

$$\Omega_R = \{(x,y) : R < \sqrt{x^2+y^2} < R + \frac{1}{R}\},$$

we get $E(\partial\Omega_R) = \frac{\pi}{R} + \frac{\pi R}{R^2+1}$, while

$$A(\Omega_R) = \pi(R+\frac{1}{R})^2 - \pi R^2 = 2\pi + \frac{\pi}{R^2}$$

showing that $E^2(\partial\Omega_R)A(\Omega_R) \to 0$ when $R \to +\infty$.

In the same way, the boundedness assumption is also necessary. Let us consider the following unbounded domain, subgraph of a Gaussian function, but with finite area and elastic energy:

$$\Omega_\alpha = \{(x,y) \in \mathbb{R}^2 : -\infty < x < +\infty, 0 < y < e^{-\alpha x^2/2}\}.$$

We have

$$A(\Omega_\alpha) = \int_{-\infty}^{+\infty} e^{-\alpha x^2/2}dx = \sqrt{\frac{2\pi}{\alpha}},$$

while

$$E(\partial\Omega_\alpha) = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{(\alpha^2x^2-\alpha)^2e^{-\alpha x^2}}{(1+\alpha^2x^2e^{-\alpha x^2})^{\frac{5}{2}}}dx = \frac{\alpha^{\frac{3}{2}}}{2}\int_{-\infty}^{+\infty} \frac{(u^2-1)^2e^{-u^2}}{(1+\alpha u^2e^{-u^2})^{\frac{5}{2}}}du,$$

and we see that $E^2(\partial\Omega_\alpha)A(\Omega_\alpha) \to 0$ as $\alpha \to 0$.

This shows that the assumptions in Theorem 1.1 can not be weakened. The proof of Theorem 1.1 is a classical variational proof (existence, regularity, analysis of the optimality conditions), but the existence part is by no means easy since we need a control on the perimeter of a minimizing sequence. The boundedness constraints on $E(\Omega)$ and $A(\Omega)$ do not ensure that the perimeter is uniformly bounded, as shown by a counter-example like a dumbell, see Figure 1.

One has to be particularly careful that a minimizing sequence may a priori have a diameter going to infinity. The key point of our strategy is to analyze first the minimization of the elastic energy

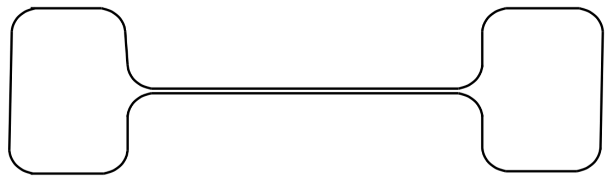

Figure 1: A dumbbell with bounded area and elastic energy with a large perimeter

of drops enclosing a fixed area, i.e. closed loops without self-intersection points, which are smooth except one point, where the tangents are opposite. The result for drops, will straight forward imply the conclusion of Theorem 1.1, relying on a result of B. Andrews (see [1] and Theorem 4.1 in Section 4).

Here is our plan.

- We solve the minimization problem
$$\min\{E(\partial\Omega) + A(\Omega) : \Omega \text{ open, smooth, bounded, simply connected}\}, \tag{3}$$
which is equivalent to (2).

- We fix some radius $R > 0$ and replace problem (3) by
$$\min\{E(\partial\Omega) + A(\Omega) : \Omega \subseteq B_R \text{ open, smooth, simply connected}\}, \tag{4}$$
where B_R is the ball centered at 0 of radius R. We prove that every simply connected domain in B_R satisfies $L(\Omega) \leq R^2 E(\partial\Omega)$. This is a key point for proving existence.

- In order to be able to exploit the optimality conditions we have to deal with self-intersection points and with the points where the optimal set is touching the boundary of the ball B_R. For this reason, we analyze the problem
$$\min\{E(\partial\Omega) + A(\Omega) : \Omega \subseteq B_R \text{ open, smooth, simply connected } drop\}. \tag{5}$$
We refer to Section 3 for a precise definition of drops. We prove that an optimal drop does not have self-intersection points and, if R is large enough, it does not touch the boundary of the ball B_R (up to a translation, inside the ball). Henceforth, optimality conditions allow us to exhibit precisely the optimal drop and to evaluate its energy. This drop turns out to be *unique*.

- We come back to problem (4) and prove that a limit of minimizing sequence can not have self-intersection points and can not touch the boundary of B_R, provided that R is large enough. Consequently, optimality conditions can be written on all its boundary. The elimination of self-intersection points relies on the previous result on drops, since the presence of at least one such point would make the energy not smaller than the double of the energy of the optimal drop, which turns out to be larger than the energy of a disc of radius $2^{-1/3}$. As optimality conditions can be written on the full loop, the shape is a circle (of radius $2^{-1/3}$) as a consequence of the result of Andrews [1]), solving in this way (3).

2 Preliminaries

All curves $\gamma : [0, L] \to \mathbb{R}^2$ are parametrized by the arc-length. We denote θ the angle of the tangent to γ with respect to the axis Ox. The curvature of γ at the point $\gamma(s)$ will be denoted $k(s)$ and it is equal to $\theta'(s)$. Since we shall work with curves with finite elastic energy, the function θ belongs to the Sobolev space $H^1(0, L)$. Using the embedding $H^1(0, L) \subseteq C^{0,\alpha}[0, L]$, for any $\alpha < 1/2$, the function θ is, in particular, continuous.

All curves we work in this paper have finite elastic energy

$$E(\gamma) = \frac{1}{2} \int_{[0,L]} |\theta'(s)|^2 ds < +\infty.$$

Lemma 2.1 *Let $M > 0$ and $\gamma : [0, L] \to \mathbb{R}^2$ be a curve parametrized by the arc length such that $E(\gamma) \leq M$. There exists $l = l(M) > 0$ such that for every $s_0 \in [0, L - l]$ the curve is a graph in a local system of coordinates with a first axis aligned on $\theta(s_0)$, on a segment $[0, \frac{l}{\sqrt{2}}]$, of a function $g : [0, \frac{l}{\sqrt{2}}] \to \mathbb{R}$ with $g(0) = \gamma(s_0)$, $g'(0) = 0$ and such that $\forall t \in (0, L)$ $|g'(t)| \leq 1$.*

Proof Let us fix s_0 and consider the smallest $s_1 > s_0$ such that $|\theta(s_1) - \theta(s_2)| = \frac{\pi}{4}$. If s_1 does not exist, then the conclusion follows directly.

We reproduce the curve $\gamma_{[s_0,s_1]}$ eight times, taking successively a reflection with respect to the line passing trough the point $\gamma(s_1)$ and orthogonal to the tangent at $\gamma(s_1)$, then the same procedure for the image of $\gamma(s_0)$ and a last reflection in order to close the loop.

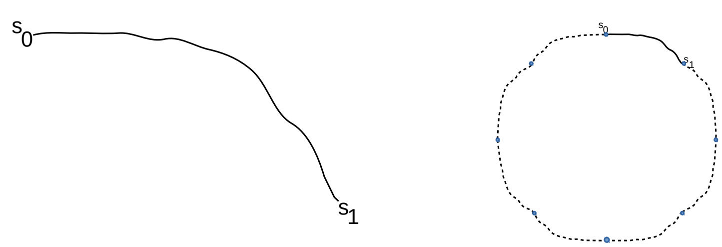

Figure 2: Initial curve $\gamma|_{[s_0,s_1]}$ and the (rescaled) loop built from the curve

Let us denote by C the curve which is the boundary of the convex envelope of the loop. From [5], we have

$$\int_C k^2 ds \geq \frac{\pi |\mathcal{H}^1(C)|}{Area(C)} \geq \frac{\pi |\mathcal{H}^1(C)|}{\frac{|\mathcal{H}^1(C)|^2}{4\pi}} \geq \frac{4\pi^2}{8(s_1 - s_0)}.$$

But

$$8\int_{\gamma_{[s_0,s_1]}}|\theta'|^2ds \geq \int_C k^2ds$$

thus

$$128M \geq \frac{4\pi^2}{s_1 - s_0},$$

hence

$$s_1 - s_0 \geq \frac{\pi^2}{32M}.$$

Denoting $l = \frac{\pi^2}{32M}$, we conclude the proof.

□

Remark 2.2 The assertion of this lemma is of course available on backwards, so that the curve is locally a graph in a neighborhood of each point, over an interval of controlled length.

Lemma 2.3 *Let $\gamma : [0, L] \to \mathbb{R}^2$ be a curve parameterized by the arc length such that $E(\gamma) < +\infty$. If $\varepsilon > 0$ is given and $0 \leq s < t \leq L$ are such that*

$$|\theta(s) - \theta(t)| = \varepsilon,$$

then

$$\int_{[s,t]}|\theta'|^2ds \geq \frac{\varepsilon^2}{L}.$$

Proof As $\int_{[0,L]}|\theta'|^2ds < +\infty$, we write

$$|\theta(s) - \theta(t)| = \left|\int_s^t \theta'(u)du\right| \leq |t-s|^{\frac{1}{2}}(\int_{[s,t]}|\theta'|^2)^{\frac{1}{2}}$$

which gives the result. □

Remark 2.4 The idea coming out of the lemma is that if there is an ε-variation of the angle, the elastic energy on that section of the curve is at least a constant times ε^2, the constant depending on the global length of the curve.

Let B_R be a ball of radius R.

Lemma 2.5 *Let $\gamma : [0, L] \to \mathbb{R}^2$ be a smooth loop parameterized by the arc length such that $E(\gamma) < +\infty$ and $\gamma([0, L]) \subseteq B_R$. Then*

$$L \leq 2R^2E(\gamma).$$

Proof Denoting $\gamma(s) = (x(s), y(s))$, we have

$$L = \int_0^L x'^2(s) + y'^2(s) = -\int_0^L x(s)x''(s) + y(s)y''(s)ds.$$

But $|x(s)x''(s)+y(s)y''(s)| \leq (x^2(s)+y^2(s))^{\frac{1}{2}}(x''^2(s)+y''^2(s))^{\frac{1}{2}} \leq R|k(s)|$. Therefore, the conclusion of the lemma follows from the Cauchy-Schwarz inequality

$$L^2 \leq R^2L\int_0^L k^2(s)ds.$$

□

Assume that a simply connected open set Ω is bounded by a loop γ which does not have self intersections on $(s_0, s_0 + L)$. We shall call this piece of curve, a free branch. Let us first give the optimality conditions satisfied by any free branch of an optimal domain. On such a free branch, we can perform any (small and compactly supported, smooth) perturbation.

Theorem 2.6 (Optimality conditions) *Let γ be any free branch of a minimizer Ω of the energy $E(\partial\Omega) + A(\Omega)$. Then $s \mapsto k(s)$ is C^∞ on γ and satisfies:*

(B1) $k'' = -\frac{1}{2}k^3 + 1$

(B2) $k'^2 = -\frac{1}{4}k^4 + 2k + 2C$*, for some constant* C

(B3) $\exists Q \in \mathbb{R}^2$*, such that* $\forall M \in \gamma$*:* $QM^2 = 2k + 2C$*, for some constant* C

(B4) $\exists Q \in \mathbb{R}^2$*, such that* $\forall M \in \gamma$*:* $QM.\nu = \frac{1}{2}k^2$ *where* ν *is a normal vector to* γ*.*

Remark 2.7 *The point Q in (B3), (B4) is the same (see the proof below). The constant C in (B2), (B3) is also the same. To see that, take a point M_M on γ where the curvature k is maximum. If this point does not exist, just extend the curve with the same ODE. Then, according to (B3), QM_M is also maximum and the normal derivative of the boundary at this point is $QM_M/|QM_M|$. Therefore (B4) yields $QM_M = \frac{1}{2}k^2$ and plugging into (B3) gives (B2), because $k' = 0$ at this point, with the same constant.*

Proof The C^∞ regularity of $k(s)$ (and $\theta(s)$) comes from a bootstrap argument and equation (8) below. The first condition (B1) comes from the classical *shape derivative* of the elastic energy (under small perturbation of the boundary driven by some smooth vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$), see [6, chapter 5] for more details on the shape derivative. Following e.g. the Appendix in [2], we see that it is given by

$$dE(\partial\Omega, V) = -\int_\gamma \left(\frac{1}{2}k(s)^3 + k''(s)\right) \langle V, \nu\rangle \, ds$$

while the derivative of the area is classically

$$dA(\Omega, V) = \int_\gamma \langle V, \nu\rangle \, ds$$

Condition (B1) follows since the derivative of $E + A$ must vanish for any V. We obtain condition (B2) multiplying (B1) by k' and integrating.

To get condition (B3), we use another expression of the elastic energy and the area. Namely, with the parametrization with the angle θ we have (see [2] for more details):

$$E(\gamma) = \frac{1}{2}\int_\gamma \theta'^2 ds := e(\theta), \quad A(\Omega) = \int\int_T \cos\theta(u)\sin\theta(s)\, du\, ds := a(\theta)$$

where T is the triangle $T = \{(u, s) \in \mathbb{R}^2 \ ; \ 0 \le u \le s \le L(\Omega)\}$. We note L for $L(\Omega)$. Thus we are led to minimize the sum $e(\theta) + a(\theta)$ with the following constraints (the starting and the ending point of the branch γ are fixed).

$$\int_0^L \cos(\theta(s))\, ds = x(L) - x(0), \qquad \int_0^L \sin(\theta(s))\, ds = y(L) - y(0). \tag{6}$$

The derivative of $e(\theta)$ is (for a perturbation v compactly supported)

$$\langle de(\theta), v\rangle = \int_0^L \theta' v' ds = -\int_0^L \theta'' v ds$$

while the derivative of $a(\theta)$ is given by

$$\langle da(\theta), v\rangle = \int\int_T \cos\theta(u)\cos\theta(s)v(s) - \sin\theta(s)\sin\theta(u)v(u)duds.$$

Using (6) and Fubini, we can write

$$\int\int_T \sin\theta(s)\sin\theta(u)v(u)duds = (y(L)-y(0))\int_0^L \sin\theta(s)v(s)ds - \int\int_T \sin\theta(u)\sin\theta(s)v(s)duds.$$

Therefore, the optimality condition for the constrained problem reads: there exists Lagrange multipliers λ_1, λ_2 such that, for any v:

$$-\int_0^L \theta'' v ds + \int_0^L \left(\cos\theta(s)\int_0^s \cos(\theta(u)du + \sin\theta(s)\int_0^s \sin(\theta(u)du\right)v(s)ds = \tag{7}$$
$$= (y(L)-y(0))\int_0^L \sin\theta(s)v(s)ds - \lambda_1\int_0^L \sin\theta(s)v(s)ds + \lambda_2\int_0^L \cos\theta(s)v(s)ds.$$

which implies (thanks to $x'(s) = \cos\theta(s), y'(s) = \sin\theta(s)$)

$$-\theta'' + x'(x-x(0)) + y'(y-y(0)) = (y(L)-y(0)-\lambda_1)y' + \lambda_2 x' \tag{8}$$

By integration, we get (B3) setting $Q = (x(0)+\lambda_2, y(L)-\lambda_1)$.

At last, differentiating twice (B3) we get $k' = QM.\tau$ (where τ is the tangent vector) and $k'' = 1 - kQM.\nu$. Using (B1) we see that $\frac{1}{2}k^3 = kQM.\nu$, so (B4) holds where $k \neq 0$. Since k is a solution of the ODE (B1), and therefore can be written with elliptic functions, it can only vanish on isolated points and thus (B4) holds everywhere by continuity of both members. □

In the following lemma, we assume that the simply connected open set Ω is a minimizer of the energy $E(\partial\Omega) + A(\Omega)$.

Lemma 2.8 *Any free branch of a minimizer Ω has a length L uniformly bounded by*

$$L \leq 146.$$

Proof We work with a free branch of γ on $s \in (s_0, s_0 + L)$ and use the optimality conditions above. We also know that the elastic energy of this branch is less than the total energy of the best disk B, so that

$$E(\gamma) \leq E(\partial B) + A(B) = 3\pi 2^{-\frac{2}{3}}. \tag{9}$$

We consider two cases. Assume first that $C \leq 1$ on this branch (C is defined above in (B2), (B3)). Then we know from (ODE3) in the Appendix that

$$k(s) \leq k_M(C) \leq k_M(1) \leq \frac{7}{3} \tag{10}$$

Then, from (B3)

$$QM^2 \leq \frac{14}{3} + 2 = \frac{20}{3},$$

hence the arc is contained in the disc centered at Q with radius $R_0 = \sqrt{\frac{20}{3}}$.

On the other hand, if we put the origin at Q

$$L(\gamma) = L = \int_0^L x'^2 + y'^2 dx = (xx' + yy')|_0^L - \int_0^L xx'' + yy'' ds.$$

But $|x(L)x'(L) + y(L)y'(L)| \leq R_0$ and $|x(0)x'(0) + y(0)y'(0)| \leq R_0$ while by Cauchy-Schwarz and (9)

$$|\int_0^L xx'' + yy'' ds| \leq R_0 \int_0^L |k| ds \leq R_0 \sqrt{L 2E(\gamma)} \leq R_0 \sqrt{L 3\pi 2^{\frac{1}{3}}}.$$

Therefore, L satisfies

$$L \leq 2\sqrt{\frac{20}{3}} + \sqrt{\frac{20}{3} \times 3\pi \times 2^{\frac{1}{3}}} \sqrt{L}, \tag{11}$$

which implies (as soon as $C \leq 1$)

$$L \leq 90. \tag{12}$$

Second case: $C \geq 1$ for this branch. In this case we have from (ODE3) in the Appendix

$$k_M(C) \geq k_M(1) \geq \frac{9}{4},$$

$$k_m(C) \leq k_m(1) \leq -\frac{9}{10}.$$

We decompose the interval $I = (s_0, s_0 + L)$ in three parts (some could be empty), $I = I_- \cup I_0 \cup I_+$ where

$$I_- = \{s \in I : k(s) \leq 0\}$$

$$I_0 = \{s \in I : 0 < k(s) < 2^{\frac{1}{3}}\}$$

$$I_+ = \{s \in I : 2^{\frac{1}{3}} \leq k(s)\}$$

and we are going to prove that the length of each part is uniformly bounded, by a controlled constant. First of all, we have seen that the integral of k^2 on a period satisfies (see (ODE4) in the Appendix)

$$\frac{1}{2} \int_0^T k^2 ds \geq \frac{\pi}{4} \sqrt{\frac{22}{3}}.$$

Following (9), this implies that we can not have more than 3 periods on each free branch. We begin with I_+. Obviously

$$E(\gamma) \geq \frac{1}{2} \int_{I_+} k^2 ds \geq \frac{1}{2} 2^{\frac{2}{3}} |I_+|,$$

therefore

$$|I_+| \leq 3\pi \times 2^{-\frac{2}{3}} \times 2^{\frac{1}{3}} \leq 8. \tag{13}$$

For I_0, we consider one of its connected components, say (α, β). Since $k_M(C) \geq \frac{9}{4} > 2^{\frac{1}{3}}$ and $k_m(C) \leq -\frac{9}{10} < 0$, we cannot have any local minimum or local maximum of k in I_0 according to (ODE2) form the Appendix. Therefore, k is either increasing from $k(\alpha)$ to $k(\beta)$, or decreasing from $k(\alpha)$ to $k(\beta)$. Moreover, there are at most 6 such connected components because there are at most 3 periods of k. Let us consider the case of k increasing from $k(\alpha)$ to $k(\beta)$, the other one being similar. We have $0 \leq k(\alpha) \leq k(\beta) \leq 2^{\frac{1}{3}}$. By (B1), $k'' \geq 0$ on (α, β), so that k is convex. Therefore

$$k(\alpha) + k'(\alpha)(s - \alpha) \leq k(s) \tag{14}$$

which implies

$$k(\alpha) + k'(\alpha)(\beta - \alpha) \leq k(\beta) \leq 2^{\frac{1}{3}}.$$

Now $k(\alpha) \geq 0$ and $k'(\alpha) = \sqrt{2C + 2k(\alpha) - \frac{1}{4}k^4(\alpha)} \geq \sqrt{2C}$ thus $\sqrt{2}(\beta - \alpha) \leq \sqrt{2C}(\beta - \alpha) \leq 2^{\frac{1}{3}}$ or $\beta - \alpha \leq 2^{-\frac{1}{6}}$.

Since, there are at most 6 such intervals, we have

$$|I_0| \leq 6 \times 2^{-\frac{1}{6}} \leq 6. \tag{15}$$

At last we consider the case of I_-. The set I_- is not empty only when $C > 0$ and $k_m < 0$. The set I_- is composed of connected components $[\alpha, \beta]$ such that $k(\alpha) = k(\beta) = 0$ or is included in such connected components. Since we want to estimate from above the length of I_-, it suffices to look for the length of these connected components. There are at most 3 of these (identical) components and $k(\frac{\alpha+\beta}{2}) = k_m$ by symmetry.

Now, the elastic energy of such a component satisfies

$$E(\gamma_{\alpha_1,\beta_1}) = \frac{1}{2}\int_{\alpha_1}^{\beta_1} k^2 ds = \int_{\frac{\alpha_1+\beta_1}{2}}^{\beta_1} k^2 ds = \int_{\alpha_1}^{\frac{\alpha_1+\beta_1}{2}} k^2 ds. \tag{16}$$

We denote $L_- = \beta_1 - \alpha_1$ the length of this component. By convexity, on $(\alpha_1, \frac{\alpha_1+\beta_1}{2})$ we have

$$k(s) \leq \frac{2k_m}{L_-}(s - \alpha_1) \leq 0,$$

thus

$$E(\gamma_{\alpha_1,\beta_1}) \geq \int_{\alpha_1}^{\alpha_1+\frac{L_-}{2}} \frac{4k_m^2}{L_-^2}(s - \alpha_1)^2 ds = \frac{k_m^2}{6} L_-.$$

Now, for $C \geq 1$ we have (see (ODE3) in the Appendix) $k_m^2 \geq k_m^2(1) \geq \frac{81}{100}$ and $E(\gamma_{\alpha_1,\beta_1}) \leq 3\pi 2^{-\frac{2}{3}}$. Therefore

$$L_- \leq \frac{600}{81} \times 3\pi 2^{-\frac{2}{3}} \leq 44,$$

and the total length of

$$|I_-| \leq 3L_- \leq 132. \tag{17}$$

In conclusion, for $C \geq 1$ the total length is less than (by gathering (13), (15), (17))

$$L \leq 132 + 8 + 6 = 146.$$

□

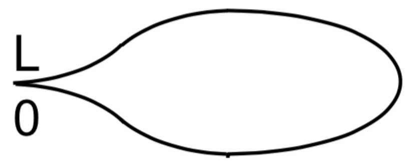

Figure 3: A drop

3 The optimal drop

In this section we prove the existence of a best *drop* minimizing the sum of the elastic energy and the area enclosed. We introduce the class of admissible *Jordan drops* consisting of simply connected open sets Ω bounded by a Jordan curve γ of finite length, which satisfies

$$\theta(0) = \theta(L_\gamma) - \pi, \quad E(\gamma) < +\infty,$$

where L_γ is the length of γ. A drop will be denoted (Ω, γ), Ω being the open set enclosed by the Jordan curve γ (all Jordan curves are oriented in the positive sense).

For some $R > 0$, we consider the problem

$$\inf\{E(\gamma) + A(\Omega) : (\Omega, \gamma) \text{ is a drop}, \Omega \subseteq B_R\}. \tag{18}$$

Note that by a similar argument as in Lemma 2.5, the length of Jordan drop γ can not exceed $8R^2 E(\gamma)$. Indeed, the same argument works for the drop, if the singularity lies at the origin, we have $x^2 + y^2 \leq 4R^2$ since the diameter of the drop is less than $2R$.

Here is the main result.

Theorem 3.1 *Problem* (18) *has at least one solution.*

Remark 3.2 With no assumptions on the radius R, it could be possible that the optimal drop (Ω, γ) touches the boundary of the ball but it may not have self intersections.

Proof For simplicity of the notation, the ball B_R will be denoted B and the area of Ω will be denoted by $|\Omega|$. We start with the following.

Lemma 3.3 *Let (Ω, γ) be a drop contained in B. If for some $\varepsilon > 0$ there exists $0 \leq s < t \leq L_\gamma$ with*

$$\theta(t) = \theta(s) - \pi - \varepsilon$$

then there exists a new drop $(\tilde\Omega, \tilde\gamma)$ in B such that

$$\int_{\tilde\gamma} |\tilde\theta'|^2 \leq \int_\gamma |\theta'|^2 - \frac{\varepsilon^2}{2L_\gamma} \ \text{ and } \ |\tilde\Omega| \leq |\Omega|.$$

[Proof of the Lemma] Assume s and t satisfy the hypotheses. Then, from continuity of θ, there exists $s < \overline{s} < \overline{t} < t$ such that

$$\theta(\overline{s}) = \theta(s) - \frac{\varepsilon}{2} \ \ and \ \ \theta(\overline{t}) = \theta(t) + \frac{\varepsilon}{2}.$$

Moreover, there exists $\overline{s} \leq s' < t' \leq \overline{t}$ such that

$$\theta(t') = \theta(\overline{t}), \theta(s') = \theta(\overline{s})$$

and for every $u \in (s', t')$

$$\theta(u) \in (\theta(t'), \theta(s')).$$

Indeed, we define

$$t' = \inf\{t : t > \overline{s}, \theta(t) = \theta(\overline{t})\},$$

and

$$s' = \sup\{s : s < t', \theta(s) = \theta(\overline{s})\}.$$

Then we notice that the curve $\gamma_{[s',t']}$ is a graph in the direction $\theta(s')$, otherwise it would contradict the choice of s' and t'. Setting the orientation of the curve in the trigonometric sense, we are in

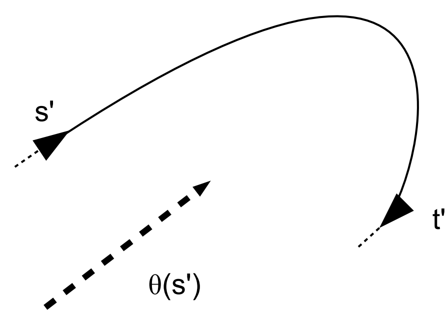

Figure 4: The curve is a graph in the direction $\theta(s')$

configuration similar to Figure 5. Using the graph property, we can translate continuously the piece of the curve $\gamma|_{[s',t']}$ in a parallel way in the direction $\theta(s')$ until this piece touches again γ.

We denote $s_\alpha \in [s', t']$ and $t_\alpha \in [0, L] \setminus [s', t']$ the couples of touching points. We denote s_1, respectively s_2, the minimal and maximal values of s_α. Then, one of the curves starting with s_2 and ending in t_2, or starting in t_1 and ending in s_1 is a drop. Precisely, it is the one which does not contain the point $\gamma(0)$. Without loosing generality we can assume it is the curve $s_2 \to t_2$ and rename the point $(s_2, t_2) = (s^*, t^*)$ and denote this curve $\tilde{\gamma}$. We notice that $\tilde{g}$ can not touch any the piece of curve $\gamma|_{[\overline{s},s']}$. If there would be a contact point, this contact is generated by the translation of $\gamma|_{[s',t']}$ and has to be precisely (s^*, t^*). But in this case, t^* lies in the interval $[\overline{t}, s']$, so the curve starting at t^* and ending at s^* is a drop, which does not touch the piece of curve $\gamma|_{[t',\overline{t}]}$.

In this way, we built a new drop $(\tilde{\Omega}, \tilde{\gamma})$, which encloses a domain contained in Ω and, in view of Lemma 2.3 has an elastic energy smaller by at least an increment of $\frac{\varepsilon^2}{4L_\gamma}$.

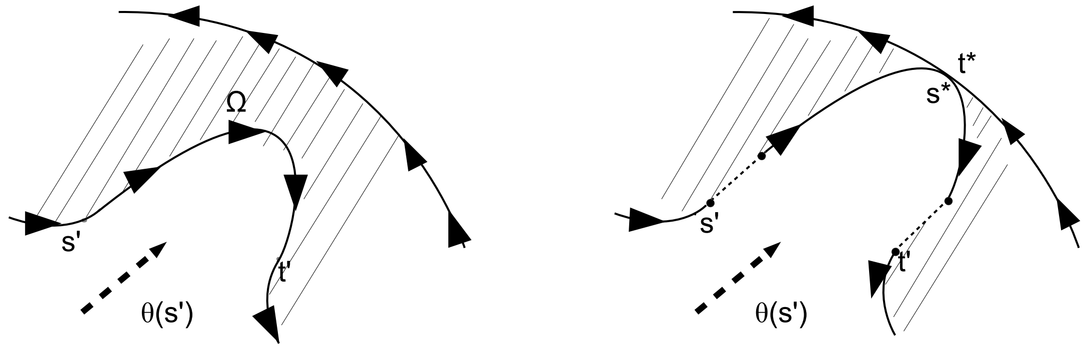

Figure 5: Translation of $\gamma|_{[s',t']}$ in the direction $\theta(s')$

[Proof of the Theorem 3.1 (continuation)] Coming back to the proof of Theorem 3.1, les us consider (Ω_n, γ_n) be a minimizing sequence of drops. We may assume that $E(\gamma_n)$, $|\Omega_n|$ and L_{γ_n} are convergent. Assume that for every n we have $L_{\gamma_n} \leq L^*$. In order to work on a fixed Sobolev space $H^1(0, L^*)$, we assume that θ_n is formally extended by the constant $\theta_n(L_{\gamma_n})$ on $(L_{\gamma_n}, L^*]$. Up to a subsequence, we can assume that θ_n converges uniformly on $[0, L^*]$ to some function θ. We define the limit curve γ in the following way: $L_\gamma = \lim_{n\to\infty} L_{\gamma_n}$ and $\gamma : [0, L_\gamma] \to \mathbb{R}^2$, $\gamma(s) = \int_0^s e^{i\theta(s)} ds + a$, where $a = \lim_{n\to\infty} \gamma_n(0)$.

Let us fix $\varepsilon > 0$. Then, from the previous lemma, for every $s < t$ and n large enough we have

$$\theta_n(t) \geq \theta_n(s) - \pi - \varepsilon.$$

Indeed, otherwise we would replace (Ω_n, γ_n) by $(\tilde{\Omega}_n, \tilde{g}_n)$ decreasing the energy by a fixed increment $\frac{\varepsilon^2}{4L^*}$, where L^* is a bound of the lengths. This is in contradiction with the minimality of the sequence.

In particular, passing to the limit we get that for every $\varepsilon > 0$ and for every $s < t$

$$\theta(t) \geq \theta(s) - \pi - \varepsilon.$$

Since ε is arbitrary, we get

$$\theta(t) \geq \theta(s) - \pi. \tag{19}$$

From the compactness of the class of closed subsets of $\overline{B}_R$ endowed with the Hausdorff metric, and the embedding of $H^1(0, L^*)$ into $C^{0,\alpha}[0, L^*]$, we may assume that for some open set $\Omega \subseteq B_R$

$$\Omega_n^c \xrightarrow{H} \Omega^c,$$

and the convergence of θ_n leads to

$$\gamma_n([0, L_{\gamma_n}]) \xrightarrow{H} \gamma([0, L_\gamma]).$$

We refer to [3] or [6] for precise properties of the Hausdorff convergence. We know that in general $1_\Omega \leq \liminf_{n\to\infty} 1_{\Omega_n}$, so that $|\Omega| \leq \lim_{n\to\infty} |\Omega_n|$. Nevertheless, in our situation the perimeters being uniformly bounded, we get $1_{\Omega_n} \to 1_\Omega$ in $L^1(B_R)$. Moreover, $\partial\Omega \subseteq \gamma([0, L_\gamma])$ and Ω is simply connected (i.e. any loop contained in Ω is homotopic to a point in Ω), but not necessarily connected. The curve γ is possibly self-intersecting, but not crossing, i.e. at every self-interesting point, the tangent line is the same, while looking locally around the point, the pieces of curve passing through it are (in view of Lemma 2.1) graphs of functions. From the simple connectedness hypothesis, these functions are necessarily ordered. From Lemma 2.3 and the fact that the elastic energy is finite, the number of pieces of curve passing through the touching point is uniformly finite. The situation displayed in Figure 6 may occur.

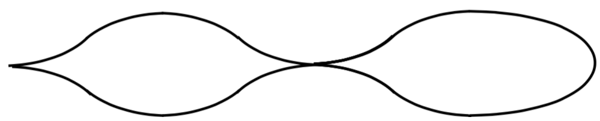

Figure 6: Self touching curve, disconnecting the limit

We shall prove that γ can not have self intersection points, other that the type above. The key ingredients are the local representation of the curve as a graph and inequality (19). We shall analyze the different contact types between two pieces of γ. Since the curves are graphs on an interval $[-\frac{l}{\sqrt{2}}, \frac{l}{\sqrt{2}}]$, and the representing functions are ordered, we shall look to the orientation of each piece of curve.

Case 1. Opposite orientation, not disconnecting. Two branches of γ touching at some

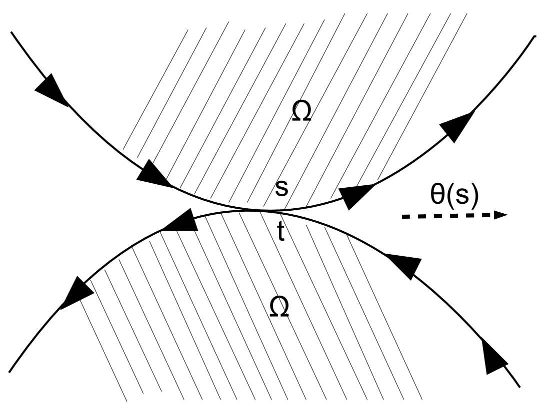

Figure 7: Case 1: opposite orientation, not disconnecting.

point $\gamma(s) = \gamma(t)$, are represented as graphs of the functions g_s, g_t, on $[-\frac{l}{\sqrt{2}}, \frac{l}{\sqrt{2}}]$. We assume that $g_s(0) = \gamma(s) = \gamma(t) = g_t(0)$ and choose the couple (s, t) such that for some $\varepsilon > 0$ we have

$$\forall u \in (0, \varepsilon) \;\; g_s(u) > g_t(u),$$

otherwise we change the contact point. This inequality would imply the existence of points $s' > s$ and $t' < t$ such that $\theta(t') < \theta(s') - \pi$, which is in contradiction with (19), so that this situation can not occur.

Case 2. Contact of two branches of the same orientation. From the simple connectedness,

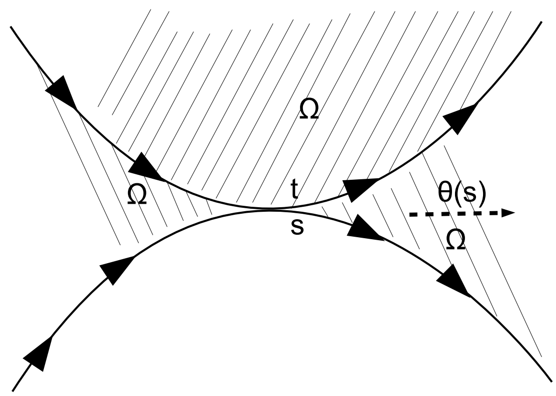

Figure 8: Case 2: same orientation

this situation implies that the touching point $\gamma(s)$ belongs to at least three branches, in particular between the graphs of g_s and g_t, there is a graph corresponding to piece of curve with opposite orientation. There are two possibilities: either this new contact corresponds to a point $t' \in (t, L)$ or to $s' \in (0, s)$. The first situation is in fact the case 1 between the contact points s and t'. The second situation leads also to Case 1, but for the contact points s' and t', so we conclude that the second case can not hold.

Case 3. Opposite orientation, disconnecting. This is the only remaining possibility for self-

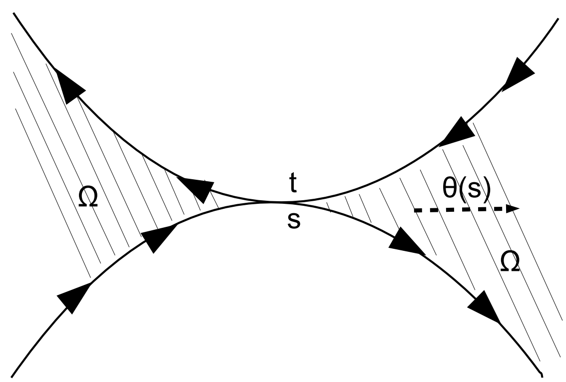

Figure 9: Case 3: simple touch, disconnecting

intersections. There may be several contact points, but every contact point is simple, otherwise we would fall in Case 2. So let us denote $\{(s_\alpha, t_\alpha)\}_\alpha$ the couple of parameters corresponding to the contact points. Because of the simple connectedness and of the absence of contact poins as in cases 1 and 2, we have that if $s_\alpha < s_\beta$ then $t_\beta < t_\alpha$. Consequently, we can identify the contact point (s^*, t^*) such that between s^* and t^* there is no other contact, by setting $s^* = \sup_\alpha s_\alpha$ and $t^* = \inf_\alpha t_\alpha$. Of course, s^* and t^* can not collapse. Indeed, in view of Lemma 2.3 applied to $\gamma|_{[s_\alpha, t_\alpha]}$ if collapse occurs then the elastic energy would blow up. So $\gamma|_{[s^*, t^*]}$ is a Jordan curve for which all the area enclosed is part of Ω, otherwise, because of the simple connectedness, a branch of the curve must pass through the contact point, bringing it to the case 2.

So $\gamma|_{[s^*,t^*]}$ is a drop, with lower elastic energy than γ and enclosing a surface less than or equal to $|\Omega|$. This means that $\gamma|_{[s^*,t^*]}$ is a solution for problem (18).

$\square$

Lemma 3.4 *There exists R_0, such that if the radius R of the ball B_R in Theorem 3.1 satisfies $R \geq R_0$, then there exists a translation of the optimal drop which does not touch the boundary of B_R.*

Proof The proof relies on Lemma 2.8. Let $R \geq R_0$ (the value of R_0 will be precised at the end of the proof). Assume that (Ω^*, γ^*) is an optimal drop for problem (18) which touches the boundary, such that there is no translation moving the drop at positive distance from the boundary. This means that the touching points between γ^* and B_R are distributed in such a way that they do not fit in an arc of length less than πR. Using Lemma 2.8, if R is large, e.g. $R \geq 300$, then the center of B_R has to be inside Ω^* together with a disc of radius 150. Indeed, the longest piece of curve between two contact points or between a contact point and the singularity has a length less than 146. Consequently, the energy of (Ω^*, γ^*) is larger than the area of the disc of radius 150: $\pi \cdot 150^2$, in contradiction with its optimality. Taking $R_0 = 300$, the lemma is proved. $\square$

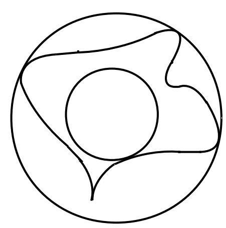

Figure 10: An optimal drop touching the boundary

Theorem 3.5 *There exists a unique optimal drop (Ω^*, γ^*) which minimizes the energy $E(\gamma)+A(\Omega)$ among all drops in $\mathbb{R}^2$. This one is fully characterized by the optimality conditions (B1)-(B2) with a unique constant C which can be determined.*
Moreover

$$E(\gamma^*) + A(\Omega^*) > \pi > 3\pi 2^{-\frac{5}{2}} = \frac{1}{2}[E(\partial B_{2^{-1/3}}) + A(B_{2^{-1/3}})].$$

Remark 3.6 *Figure 11 gives the representation of the optimal drop.*

Proof The proof of existence follows from Theorem 3.1 and Lemma 3.4. The optimality conditions (B1)-(B4) can be written on the whole γ (except at the singularity) according to Theorem 2.6. We start for $s = 0$ at the origin which is the singular point with an horizontal tangent ($\theta(0) = 0$).

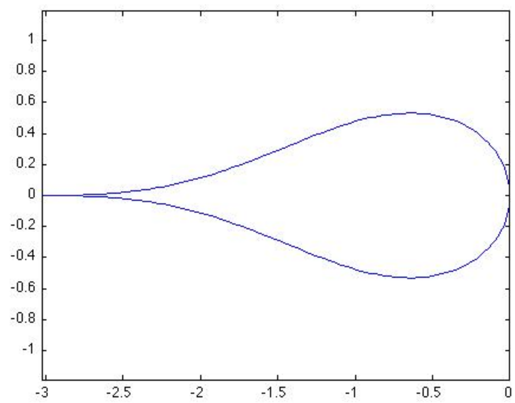

Figure 11: The optimal drop

By (B4) and starshaped property, the point Q is necessarily on the x-axis, the curvature $k(s)$ is negative for $s > 0$ small and $k(s) \to 0$ when $s \to 0$. The function $k(s)$ is periodic but we will prove below (see the end of the proof) that we have only one period for the optimal drop and the curve is symmetric around the x-axis. Therefore to characterize the optimal drop, we can proceed in the following way: for any constant $C > 0$, we solve the ODE

$$\begin{cases} k'' = -\frac{1}{2}k^3 + 1 \\ k(0) = 0 \\ k'(0) = -\sqrt{2C} \end{cases} \tag{20}$$

which has a unique solution. Let us denote by s_M the value where k is maximum with $k(s_M) = k_M$ (respectively s_m and $k_m = k(s_m)$ for the minimum). The point M_M of abscissa s_M is necessarily on the x-axis and its tangent is vertical. Thus, we look for the value of C for which $\theta(s_M) = \int_0^{s_M} k(s)ds = \pi/2$.

We claim that conversely, if we find a value of C for which $\int_0^{s_M} k(s)ds = \pi/2$, then we have found the optimal drop. Indeed, since it satisfies the optimality conditions, it suffices to check that the curve we obtain by $x(s) = \int_0^s \cos\theta(t)dt$ and $y(s) = \int_0^s \cos\theta(t)dt$ with $\theta(s) = \int_0^s k(t)dt$ is an admissible drop. Since M_M is the point where the curvature is maximum, according to (B3), it is the point on γ which is the farthest to Q. But since the tangent is vertical at this point it is necessarily on the x-axis: $y(s_M) = 0$ and the total length of the curve is $2s_M$. Now, since k is symmetric with respect to s_M (see (ODE1) in the Appendix), $k(s_M+t) = k(s_M-t)$ which provides after integration: $\theta(s_M + t) = \pi - \theta(s_M - t)$. This identity gives $\theta(2s_M) = \pi$ and

$$x(2s_M) = \int_0^{s_M} \cos\theta(t)dt + \int_{s_M}^{2s_M} \cos\theta(t)dt = \int_0^{s_M} \cos\theta(t) + \cos(\pi - \theta(t))dt = 0$$

$$y(2s_M) = \int_0^{s_M} \sin\theta(t)dt + \int_{s_M}^{2s_M} \sin\theta(t)dt = \int_0^{s_M} \sin\theta(t) + \sin(\pi - \theta(t))dt = 2y(s_M) = 0$$

which shows that the curve γ is a drop.

Thus to prove uniqueness of the optimal drop, we need to prove that we can find only one $C > 0$ for which $I(C) := \int_0^{s_M} k(s)ds = \pi/2$. Let us write

$$\int_0^{s_M} k(s)ds = \int_0^{2s_m} k(s)ds + \int_{2s_m}^{s_M} k(s)ds = 2\int_0^{s_m} k(s)ds + \int_{2s_m}^{s_M} k(s)ds$$

where we used the symmetry of k with respect to s_m, see (ODE1). This symmetry also shows that $k(2s_m) = 0$. We are going to prove uniqueness of C (and therefore of the optimal drop) by proving that the function $C \mapsto \int_0^{s_M} k(s)ds$ is strictly decreasing. Let us perform the change of variable $u = k(s)$ in each above integral. It comes, using (B2) to express k':

$$\begin{aligned}\int_{2s_m}^{s_M} k(s)ds &= \int_0^{k_M} \frac{u}{\sqrt{2C + 2u - u^4/4}}\,du \\ \int_0^{s_m} k(s)ds &= -\int_0^{k_m} \frac{u}{\sqrt{2C + 2u - u^4/4}}\,du\end{aligned} \tag{21}$$

Now to compute the derivative of the first integral $I_1(C)$ with respect to C, we make the change of variable $u = k_M x$, it comes

$$I_1(C) = \int_0^1 \frac{k_M^2 x}{\sqrt{2C + 2k_M x - k_M^4 x^4/4}}\,dx$$

We compute the derivative of I_1 using $\frac{dk_M}{dC} = 2/(k_M^3 - 2)$ (see (ODE3) in the appendix) and an easy computation gives

$$\frac{dI_1}{dC} = \int_0^1 \frac{6k_M^2 x(x-1)}{(k_M^3 - 2)\left(2C + 2k_M x - k_M^4 x^4/4\right)^{3/2}}\,dx$$

which is clearly negative. In the same way, we get for the second integral $I_2(C) = \int_0^{s_m} k(s)ds$:

$$\frac{dI_2}{dC} = -\int_0^1 \frac{6k_m^2 x(x-1)}{(k_m^3 - 2)\,(2C + 2k_m x - k_m^4 x^4/4)^{3/2}}\,dx$$

which is also negative, proving the uniqueness of a solution C for the equation $I_1(C) + 2I_2(C) = \pi/2$. Let us remark that a simple computation yields $I(0) = \frac{2\pi}{3}$ while the limit of $I(C)$ when C goes to $+\infty$ is $-\frac{\pi}{2}$ confirming that there exists a solution to our problem.

Let us estimate from below the energy of the optimal drop. Denote by $s_1 = 2s_m$ the first positive zero of k, we recall that s_m is the first minimum of k and $k_m = k(s_m)$, s_M the first maximum of k and $k_M = k(s_M)$. From (B2) k_m and k_M are the real roots of the polynomial (which is concave)

$$P_C(X) = -\frac{1}{4}X^4 + 2X + 2C. \tag{22}$$

The maximum of P_C is at $X = 2^{\frac{1}{3}}$ and $P_C(0) = 2C$. We have

$$k_m < 0 \leq 2^{\frac{1}{3}} \leq k_M \tag{23}$$

(k_m can not be nonnegative, otherwise the set Ω^* would be convex).

Moreover, when C increases, $k_M(C)$ is increasing while $k_m(C)$ is decreasing (with increasing absolute value $|k_m(C)|$), because we translate the curve $y = -\frac{1}{4}x^4 + 2x$ up).

If we denote $S = k_M + k_m$ and $P = k_m k_M$ the sum and the product of those two roots, classical elimination and relation between roots provide

$$S^2 = P - \frac{8C}{P} \qquad -\frac{8}{S} = P + \frac{8C}{P} \tag{24}$$

while the two complex roots $z_0, \overline{z}_0$ satisfy $z_0 + \overline{z}_0 = -S$, $z_0\overline{z}_0 = -\frac{8C}{P}$.

Since $P \leq 0$ and $C > 0$, the last equation gives $S > 0$. Let us come back to the computation of the elastic energy of the optimal drop (Ω^*, γ^*)

$$E(\gamma^*) = \int_0^{s_M} k^2 ds.$$

Now $\int_0^{s_M} k^2 ds \geq \int_{s_m}^{s_M} k^2 ds$ and k is increasing from s_m to s_M (since k' can only vanish at zeroes of $P_C(X)$, which only correspond to maxima k_M and minima k_m). We perform the change of variable $x = k(s)$ on this interval $dx = k'(s)ds = \sqrt{2C + 2k - \frac{1}{4}k^4} ds$. Therefore

$$E(\gamma^*) \geq \int_{s_m}^{s_M} k^2 ds = \int_{k_m}^{k_M} \frac{x^2}{\sqrt{2C + 2x - \frac{1}{4}x^4}} dx.$$

We want to find a lower bound of this integral. For this purpose, we write (following (22))

$$P_C(x) = \frac{1}{4}(k_M - x)(x - k_m)(x^2 + Sx - \frac{8C}{P}).$$

Now, the parabola $y = \frac{1}{4}(x^2 + Sx - \frac{8C}{P})$ is symmetric with respect to $-\frac{S}{2}$, an since $\frac{k_M + k_m}{2} = \frac{S}{2} \geq -\frac{S}{2}$, the maximum of y on the interval $[k_m, k_M]$ is equal to

$$F^2 = \frac{1}{4}(k_M^2 + Sk_M - \frac{8C}{P}) = \frac{1}{4}(2k_M^2 + k_m k_M - \frac{8C}{P}) = \frac{1}{4}(3k_M^2 + 2k_m k_M + k_m^2), \tag{25}$$

where we have used (24) for the last equality. Thus

$$\int_{k_m}^{k_M} \frac{x^2}{\sqrt{2C + 2x - \frac{1}{4}x^4}} dx \geq \frac{1}{F} \int_{k_m}^{k_M} \frac{x^2}{\sqrt{(k_M - x)(x - k_m)}} dx.$$

This last integral can be computed explicitly and gives

$$E(\gamma^*) \geq \frac{1}{F} \frac{3k_M^2 + 2k_m k_M + 3k_m^2}{4} \frac{\pi}{2}. \tag{26}$$

We have $F \leq \frac{1}{2}\sqrt{3k_M^2 + 2k_m k_M + 3k_m^2}$ and (26) gives

$$E(\gamma^*) \geq \frac{\pi}{4}\sqrt{3k_M^2 + 2k_m k_M + 3k_m^2}. \tag{27}$$

It remains to get a bound for the quantity $H = 3k_M^2 + 2k_m k_M + 3k_m^2$ which depends only on C. We discuss two cases.

Case A. If $C \geq 1$, $H = k_M^2 + 2k_M(k_M + k_m) + 3k_m^2 \geq k_M^2 + 3k_m^2$. Both mappings $C \mapsto k_m^2$, $C \mapsto k_M^2$ are increasing, thus $C \geq k_M^2(1) + 3k_m^2(1)$. We study $P_1(X) = -\frac{1}{4}X^4 + 2X + 2$

$$P_1(\frac{7}{3}) = -\frac{241}{324} \text{ and } P_1(\frac{9}{4}) = \frac{95}{1024},$$

we get

$$\frac{9}{4} \leq k_M(1) \leq \frac{7}{3}. \tag{28}$$

While from $P_1(-1) = -\frac{1}{4}$ and $P_1(-\frac{9}{10}) = \frac{1439}{40000}$, we get

$$-1 \leq k_m(1) \leq -\frac{9}{10}. \tag{29}$$

It follows that $H \geq \left(\frac{9}{4}\right)^2 + 3(\frac{9}{10}\Big)^2 = \frac{2997}{400} \approx 7.4925$.

Case B. In the case $0 \leq C \leq 1$, we use $k_M^2(C) \geq k_M^2(0) = 4$, $k_m^2(C) \geq 0$ and $|k_M(C)k_m(C)| \leq |k_M(1)k_m(1)| \leq \frac{7}{3}$ to get

$$H = 3k_M^2 + 2k_m k_M + 3k_m^2 \geq 12 - \frac{14}{3} = \frac{22}{3} = 7.333...$$

So in any case, $H \geq \frac{22}{3}$. It follows from (26) that

$$E(\gamma^*) \geq \frac{\pi}{4}\sqrt{\frac{22}{3}}. \tag{30}$$

Now, integrating (B4) on the curve, we get $2A(\Omega^*) = \int_{\gamma^*} \overrightarrow{QM} \cdot \vec{\nu} ds = \frac{1}{2}\int_{\gamma^*} k^2 ds = E(\gamma^*)$.

Therefore

$$E(\gamma^*) + A(\Omega^*) = \frac{3}{2}E(\gamma^*) \geq \frac{3\pi}{8}\sqrt{\frac{22}{3}} > \pi > 3\pi 2^{-\frac{5}{3}}. \tag{31}$$

Let us now conclude by proving that the optimal drop has only one period of the function $k(s)$. The estimate (30) we get is actually true on any possible period. Therefore, if we have a solution (γ_2^*, Ω_2^*) with at least two periods, we would have $E(\gamma_2^*) \geq \frac{\pi}{2}\sqrt{\frac{22}{3}}$, therefore like in (31) its total energy would satisfy $E(\gamma_2^*) + A(\Omega_2^*) > 2\pi$. Now, proceeding in a similar way as we did for the estimate from below, we can get (details omitted) an estimate from above for an optimal drop with only one period which is

$$E(\gamma^*) + A(\Omega^*) \leq 2\pi$$

(the exact value is $E(\gamma^*) + A(\Omega^*) \simeq 4.6823$) therefore, any critical point with more than one period cannot be optimal.

$\square$

4 Proof of Theorem 1.1

With the notation settled in Sections 2 and 3 we return to problem (3), and write

$$\inf\{E(\gamma) + |\Omega| : \Omega \text{ smooth, bounded, simply connected set }, \partial\Omega = \gamma\}. \tag{32}$$

First of all we recall that among all circles, the optimal one has the radius $r = 2^{-\frac{1}{3}}$. Let us consider $R \geq 300$ and solve the problem

$$\inf\{E(\gamma) + |\Omega| : \Omega \text{ smooth, bounded, simply connected set}, \Omega \subseteq B_R, \partial\Omega = \gamma\}. \tag{33}$$

Using the same arguments as in Section 3, a minimizing sequence will converge to a couple (Ω, γ). Two possibilities occur. Assume first that there are self intersections. In this case the limiting couple (Ω, γ) contains at least two drops, as in Case 3 of Theorem 3.1. Following Theorem 3.5, this configuration can not be optimal since the energy of Ω is larger than the double of the optimal energy of a drop, so it is excluded.

The second situation is that (Ω, γ) does not have self-intersections. Since the radius is large enough, for a suitable translation the loop does not touch the boundary of the ball, as in Lemma 3.4. Moreover, in this case the optimality conditions $OM.\nu = \frac{1}{2}k^2$ can be written on the full boundary. We use now the following result of Ben Andrews see Theorem 1.5 in [1]:

Theorem 4.1 *[Andrews] The only curves satisfying*

$$QM.\nu = \lambda k^\alpha \quad \textit{with } \lambda > 0 \textit{ and } \alpha > \tfrac{1}{3}$$

are circle with center at Q.

which allows us to conclude that the curve is a circle. Its radius is equal to $2^{-\frac{1}{3}}$ by direct computation. This proves Theorem 1.1.

5 Appendix: analysis of the ODE issued from optimality conditions

In this section, we give several properties of the following ODE in nonstandard form

$$k'^2 = -\frac{1}{4}k^4 + 2k + 2C,$$

where $C \in \mathbb{R}$ is a constant. This ODE is issued from the optimality conditions on a free branch of a minimizer for our problem, see Theorem 2.6. We also refer the reader to reference [2] for related analysis.

Clearly, $C \geq -\frac{3}{4}2^{\frac{1}{3}} \approx -0.944$, otherwise the right hand side is negative. We denote $k_m(C) \leq k_M(C)$ the two real roots of the polynomial $P_C(X) = -\frac{1}{4}X^4 + 2X + 2C$, or if there is no ambiguity simply k_m, k_M.

Here we gather some immediate facts concerning this ODE.

(ODE1) The solution of the ODE is periodic (the period is denoted by T), symmetric with respect to its minimum or maximum.

(ODE2) The only local minima (maxima) are actually global minima (maxima, respectively) and correspond to $k = k_m$ ($k = k_M$, respectively), and k is monotone between these two values.

(ODE3) The mapping $C \mapsto k_M(C)$ is increasing and its range is from $2^{\frac{1}{3}}$ to $+\infty$, while the mapping $C \mapsto k_m(C)$ is decreasing and its range is from $-\infty$ to $2^{\frac{1}{3}}$. Moreover, $k_m(C) < 0$ when $C > 0$, $\frac{9}{4} \leq k_M(1) \leq \frac{7}{3}$, $-1 \leq k_m(1) \leq -\frac{9}{10}$, $-C \leq k_m(C)$. As well, $k_M(C) \geq 2 + C$ for $-\frac{3}{2} \times 2^{\frac{1}{3}} \leq C \leq 0$.

(ODE4) The integral $\frac{1}{2}\int_0^T k^2 ds$ on one period is estimated from below

$$\frac{1}{2}\int_0^T k^2 ds \geq \frac{\pi}{4}\sqrt{\frac{22}{3}}.$$

The proof of (ODE1) is classical, either working with the closed orbit, or using an explicit form of the solution thanks to elliptic functions.

The proof of (ODE2) is easy since k' can vanish only at the zeroes of P_C.

For the proof of (ODE3) we notice that $\frac{dk_M}{dC} = \frac{2}{k_M^3-2} > 0$ and $\frac{dk_m}{dC} = \frac{2}{k_m^3-2} < 0$, $k_m(0) = 0, k_M(0) = 2$, $P_C(-C) < 0 \Longrightarrow k_m(C) \geq -C$, $P_C(2+C) = -C[\frac{1}{4}C^3 + 2C^2 + 6C + 4] \geq 0$ and the bounds for $k_m(1), k_M(1)$ have been obtained in (28), (29).

The proof of (ODE4): we have already proved this inequality in Section 3, when $C \geq 0$. It remains the case $-\frac{3}{4}2^{\frac{1}{3}} \leq C \leq 0$. In this case, we have $k_m \geq C$ and $k_M \geq 2 + C$, so $3k_M^2 + 2k_m k_M + 3k_m^2 \geq 4C^2 + 8C + 2 \geq 8 \geq \frac{22}{3}$, and the result follows in the same way.

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