

# SPECTRAL HOMOGENEITY OF LIMIT-PERIODIC SCHRÖDINGER OPERATORS

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ABSTRACT. We prove that the spectrum of a limit-periodic Schrödinger operator is homogeneous in the sense of Carleson whenever the potential obeys the Pastur–Tkachenko condition. This implies that a dense set of limit-periodic Schrödinger operators have purely absolutely continuous spectrum supported on a homogeneous Cantor set. When combined with work of Gesztesy–Yuditskii, this also implies that the spectrum of a Pastur–Tkachenko potential has infinite gap length whenever the potential fails to be uniformly almost periodic.

## 1. INTRODUCTION

We study continuum Schrödinger operators, that is, operators on  $L^2(\mathbb{R})$  of the form  $H = H_V$ , where

$$H_V \phi = -\phi'' + V\phi \tag{1}$$

and  $V$  is a real-valued function on  $\mathbb{R}$ . It is well known that (1) defines a self-adjoint operator on a dense linear subspace of  $L^2(\mathbb{R})$  under appropriate restrictions on  $V$ . For operators of the form (1), there are essentially two broad classes of spectral-theoretic results: *direct spectral theory*, which deduces characteristics of the spectrum and spectral measures of  $H_V$  for a given  $V$  or a given class of  $V$ 's, and *inverse spectral theory*, which fixes spectral information and attempts to describe the space of potentials which exhibit such spectral data. In rare instances, one is able to obtain a result which goes in both directions simultaneously – such results are called gems of spectral theory in [16]. In this paper, we will consider the class of Pastur–Tkachenko potentials, which have been heavily studied from the inverse point of view [12, 13]. To begin, let us recall the definition of the Pastur–Tkachenko class:

**Definition.** The *Besicovitch norm* of a measurable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$\|f\|_{\mathbb{B}} = \lim_{T \rightarrow \infty} \left( \frac{1}{2T} \int_{-T}^T |f(t)|^2 dt \right)^{1/2}.$$

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We say that a potential  $V : \mathbb{R} \rightarrow \mathbb{R}$  is (*Besicovitch*) *limit-periodic* if there exist periodic  $V_n \in L^2_{\text{loc}}(\mathbb{R})$  such that

$$\lim_{n \rightarrow \infty} \|V - V_n\|_{\text{B}} = 0.$$

We will denote the class of all Besicovitch limit-periodic functions on  $\mathbb{R}$  by  $\text{LP}(\mathbb{R})$ . We say that a (Besicovitch) limit-periodic potential  $V$  satisfies the *Pastur–Tkachenko condition* if its periodic approximants can be chosen such that

$$\lim_{n \rightarrow \infty} e^{bT_{n+1}} \|V - V_n\|_{\text{B}} = 0 \text{ for every } b > 0, \quad (2)$$

where  $T_n$  denotes the period of  $V_n$ , and  $T_n$  divides  $T_{n+1}$  for all  $n$ ; compare [12, 13]. We denote by  $\text{PT}(\mathbb{R})$  the set of all potentials  $V \in \text{LP}(\mathbb{R})$  which obey the Pastur–Tkachenko condition.

Our main goal in this paper is to prove that the spectrum of every potential in the class  $\text{PT}(\mathbb{R})$  is homogeneous in the sense of Carleson. Loosely speaking, this says that the spectrum has a uniform positive density with respect to Lebesgue measure around each of its points. The precise formulation follows.

**Definition.** We say that a set  $\Sigma \subseteq \mathbb{R}$  is  $\tau$ -*homogeneous* if there exists  $\delta_0 > 0$  such that

$$|B_\delta(x) \cap \Sigma| \geq \tau\delta \text{ for all } x \in \Sigma \text{ and } 0 < \delta \leq \delta_0,$$

where  $B_\delta(x) = (x - \delta, x + \delta)$  denotes the  $\delta$ -neighborhood of  $x$  and  $|\cdot|$  denotes Lebesgue measure (compare [3]).

**Theorem 1.1.** *If  $V \in \text{PT}(\mathbb{R})$ , then  $\sigma(H_V)$  is  $\tau$ -homogeneous for every  $\tau \in (0, 1)$ .*

As a consequence of Theorem 1.1, we easily recover a continuum Schrödinger analog of a strengthened version of the main result of [7].

**Corollary 1.2.** *For each  $0 < \tau < 1$ , denote by  $\mathcal{H}_\tau$  the set of  $V \in \text{LP}(\mathbb{R})$  such that  $H_V$  has purely absolutely continuous spectrum and  $\sigma(H_V)$  is a  $\tau$ -homogeneous Cantor set. Then  $\mathcal{H}_{1-} = \bigcap_{0 < \tau < 1} \mathcal{H}_\tau$  is dense in  $\text{LP}(\mathbb{R})$  (with respect to the topology induced by  $\|\cdot\|_{\text{B}}$ ).*

*Proof.* It is easy to see that  $\text{PT}(\mathbb{R})$  is dense in  $\text{LP}(\mathbb{R})$ . In [11, Theorem 17.2], the spectrum is encoded in terms of Marchenko–Ostrovski heights  $|\kappa_\tau|$  and it is easy to see that for a dense set of  $V \in \text{PT}(\mathbb{R})$ , all  $|\kappa_\tau|$  are strictly positive, and therefore all gaps are open and the spectrum is a Cantor set. By Theorem 1.1 and [12, 13],  $\text{PT}(\mathbb{R}) \subseteq \mathcal{H}_{1-}$ . Thus,  $\mathcal{H}_{1-}$  is dense in  $\text{LP}(\mathbb{R})$ , as desired.  $\square$

*Remark.* Theorem 1.1 and Corollary 1.2 improve the results of [7] in two senses. First, for each fixed  $\tau < 1$ , [7] constructs a dense set of examples with  $\tau$ -homogeneous spectrum, while the previous result constructs a dense set which works for all  $\tau < 1$  simultaneously. Secondly, [7] does not construct explicit examples of aperiodic limit-periodic potentials with homogeneous spectrum, while the Pastur–Tkachenko class is very explicit.

The motivation for studying spectral homogeneity arises from inverse spectral problems in the regime of absolutely continuous (a.c.) spectrum. Specifically, it is well-known that the presence of nontrivial a.c. spectrum in one dimension places rather strict restrictions on the potential; compare [10, 15]. This led to the popularity of the following conjecture:

**Conjecture** (Kotani-Last). *If an ergodic Schrödinger operator has nonempty absolutely continuous spectrum, then the potential is almost-periodic.*

The conjecture is false – Avila constructed counterexamples in the discrete setting [1]. Damanik, Volberg and Yuditskii constructed counterexamples using different techniques [5, 20].

On the other hand, if  $\Sigma$  is a homogeneous closed subset of  $\mathbb{R}$ , then the space of potentials which have spectrum  $\Sigma$  and are reflectionless thereupon is known to consist of almost-periodic operators by a theorem of Sodin and Yuditskii [17] (provided  $\Sigma$  has finite total gap length, where a gap of  $\Sigma$  is defined to be a bounded component of  $\mathbb{R} \setminus \Sigma$ ); moreover, Gesztesy and Yuditskii have proved that the spectral measures of reflectionless Schrödinger operators with homogeneous spectrum are purely absolutely continuous [8]. There are analogous results for the inverse spectral theory of Jacobi and CMV matrices in [14, 18] and [9], respectively. One should note that results of this form might or might not hold if the spectrum is not homogeneous; compare [5, 20, 21]. The inverse spectral results merely require that  $\mathbb{C} \setminus \Sigma$  is a Widom domain with a direct Cauchy theorem. Homogeneity of  $\Sigma$  is a pleasantly explicit sufficient condition which guarantees that this happens; see [21].

In light of this, one wonders what kinds of restrictions the presence of absolutely continuous spectrum places on the spectrum itself.

**Question.** *If  $V$  is almost-periodic, and  $H_V$  has nonempty absolutely continuous spectrum, must  $\sigma(H_V)$  be homogeneous?*

Although this is a question about inverse spectral theory, one can attempt to study it from a direct spectral perspective by seeing whether or not examples known to have a.c. spectrum also have homogeneous spectrum, which is what we do here in the limit-periodic class. This has also been pursued for quasi-periodic potentials in the small coupling regime; see [4].

While the work of Pastur–Tkachenko does not explicitly conclude that potentials in  $\text{PT}(\mathbb{R})$  are reflectionless, this follows readily from their work. For instance, denoting Weyl solutions of  $H_V$  at  $z \notin \sigma(H_V)$  by  $\psi_{\pm}(\cdot, z)$ , with the normalization  $\psi_{\pm}(0, z) = 1$ , it follows from the arguments in [11, Section 17] that for almost every  $E \in \sigma(H_V)$ , the boundary values  $\psi_{\pm}(x, E + i0)$  exist and obey

$$\psi_{-}(x, E + i0) = \overline{\psi_{+}(x, E + i0)} \quad (3)$$

(since these limits are also limits of Weyl solutions for the periodic approximants, and those obey the condition (3)). Property (3) implies that  $H_V$  is reflectionless, as noted by Breuer–Ryckman–Simon [2].

Potentials which satisfy a Pastur–Tkachenko condition are Besicovitch almost-periodic, but not necessarily uniformly almost periodic. By combining our theorem with work of Gesztesy–Yuditskii [8], we harvest the following interesting corollary:

**Corollary 1.3.** *Given  $V \in \text{PT}(\mathbb{R})$ , denote*

$$\Sigma = \sigma(H_V) = [E_0, \infty) \setminus \bigcup_j (a_j, b_j).$$

*If  $V$  is not uniformly almost periodic (and in particular, if  $V$  is not continuous), then the spectrum has infinite gap length, i.e.*

$$\sum_j (b_j - a_j) = \infty.$$

*Proof.* By way of contraposition, assume  $\Sigma$  has finite gap length. Since  $V$  is reflectionless by the foregoing remarks and  $\Sigma$  is homogeneous by Theorem 1.1, then [8, Theorem 2.5] implies that  $V$  is uniformly almost periodic.  $\square$

The organization of the paper is as follows. In Section 2, we recall some relevant aspects of Floquet Theory for Hill's equation. In particular, we prove an estimate on the discriminant which gives us a useful lower bound on the length of a band of the spectrum of a periodic potential. Next, in Section 3, we discuss the stability of periodic spectra under  $L^2$  perturbations. These stability estimates replace the  $L^\infty$  estimates that are exploited in [7], and they comprise the key ingredient in proving effective step-by-step estimates on the measure of the spectra of periodic approximants. In Section 4, we describe the sense in which the periodic spectra converge to the spectrum associated with  $V \in \text{PT}(\mathbb{R})$ , which enables us to push the step-by-step estimates through to the limit. Section 5 combines these ingredients and proves Theorem 1.1. Finally, we discuss the extension of Theorem 1.1 to Jacobi and CMV matrices in Section 6.

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#### 2. FLOQUET-BLOCH THEORY FOR PERIODIC POTENTIALS

Suppose  $V \in L^2_{\text{loc}}(\mathbb{R})$  is  $T$ -periodic. Given  $z \in \mathbb{C}$ , let  $y_D$  and  $y_N$  denote the Dirichlet and Neumann solutions of the Schrödinger equation

$$-y'' + Vy = zy. \quad (4)$$

Specifically,  $y_D$  and  $y_N$  solve (4) subject to the initial conditions

$$y_D(0) = y'_N(0) = 0, \quad y'_D(0) = y_N(0) = 1.$$

The *monodromy matrix* is defined by

$$M_z = \begin{pmatrix} y_N(T) & y_D(T) \\ y'_N(T) & y'_D(T) \end{pmatrix}$$

and the *discriminant* is given by  $\Delta(z) = \text{tr}(M_z)$ . One has

$$\sigma(H_V) = \{E \in \mathbb{R} : |\Delta(z)| \leq 2\} = \bigcup_{j=1}^{\infty} [\alpha_j, \beta_j],$$

where  $\alpha_1 < \beta_1 \leq \alpha_2 < \dots$  denote the solutions of  $\Delta = \pm 2$ . We call the intervals  $[\alpha_j, \beta_j]$  the *bands* of the spectrum, while the intervals  $(\beta_j, \alpha_{j+1})$  are called *gaps*. Whenever  $\beta_j = \alpha_{j+1}$ , we say that the  $j$ th gap is *closed*. Otherwise,  $\beta_j < \alpha_{j+1}$  and the  $j$ th gap is said to be *open*. The discriminant is strictly monotone on each band.

We prove an estimate on the discriminant, and the corresponding estimate on band lengths. These estimates are likely well-known, but we could not find a precise reference.

**Lemma 2.1.** *Let  $V \in L^2_{\text{loc}}(\mathbb{R})$  be a  $T$ -periodic potential with corresponding discriminant  $\Delta$ , and let  $E_0 = \min(0, \inf \sigma(H_V))$ . Then*

$$|\Delta'(E)| \leq \frac{CT^3}{T + |E|^{1/2}} \exp\left(CT \left(\|V\|_{\text{B}}^{1/2} + |E_0|^{1/2}\right)\right) \quad (5)$$

for all  $E \in \sigma(H_V)$ , for some universal constant  $C > 0$  which does not depend on  $E$ ,  $T$ , or  $V$ .

*Proof.* In this proof,  $C$  will stand for different universal constants. Its value will increase only finitely many times over the course of the argument.

We begin by proving this for  $T = \pi$ . Denote  $Q = \|V\|_{\mathbb{B}}$ . It follows from [11, Corollary 17.8] that

$$|\Delta(\mu^2)| \leq C \exp\left(CQ^{1/2} + \pi|\operatorname{Im} \mu|\right). \quad (6)$$

From the Cauchy differentiation formula on a circle of radius 1, for  $f(\mu) = \Delta(\mu^2)$ ,

$$|f'(\mu)| \leq \int_0^{2\pi} |f(\mu + e^{i\theta})| \frac{d\theta}{2\pi}$$

so we conclude that

$$|2\mu\Delta'(\mu^2)| \leq C \exp\left(CQ^{1/2} + \pi|\operatorname{Im} \mu| + \pi\right). \quad (7)$$

We can also conclude from (6) that

$$|\Delta(z)| \leq C \exp\left(CQ^{1/2} + 2\pi\right), \quad |z| \leq 2,$$

and then by the Cauchy differentiation formula for  $\Delta(z)$ , similarly as above, that

$$|\Delta'(z)| \leq C \exp\left(CQ^{1/2} + 2\pi\right), \quad |z| \leq 1. \quad (8)$$

Using (7) for  $|z| > 1$  and (8) for  $|z| \leq 1$ , we conclude that for all  $z \in \mathbb{C}$ ,

$$|\Delta'(z)| \leq \frac{C}{1 + |z|^{1/2}} \exp\left(CQ^{1/2} + \pi|\operatorname{Im} \sqrt{z}|\right).$$

If  $E \in \sigma(H_V)$ , then  $|\operatorname{Im} \sqrt{E}| \leq |E_0|^{1/2}$ , so we conclude that the statement of the lemma holds for  $T = \pi$ .

For a potential  $V$  of arbitrary period  $T$ , introduce the rescaled  $\pi$ -periodic potential  $V_\pi(x) = (T/\pi)^2 V(Tx/\pi)$ . Its discriminant  $\Delta_\pi(E)$  obeys  $\Delta_\pi((T/\pi)^2 E) = \Delta(E)$ , so the conclusion of the lemma for  $V$  follows by applying the previous argument to  $V_\pi$  and noting that  $\|V_\pi\|_{\mathbb{B}} = (T/\pi)^2 \|V\|_{\mathbb{B}}$  and that  $\nu_0$  scales in the same way.  $\square$

**Corollary 2.2.** *Let  $C$  denote the universal constant from Lemma 2.1, and suppose  $V \in L^2_{\text{loc}}(\mathbb{R})$  is  $T$ -periodic with  $\|V\|_{\mathbb{B}} \leq Q$ . Then the length of any band  $[\alpha_j, \beta_j]$  of  $\sigma(H_V)$  is bounded from below by*

$$|\beta_j - \alpha_j| \geq 4C^{-1} e^{-CT(Q^{1/2} + |E_0|^{1/2})} (T + \lambda_0^{1/2}) T^{-3},$$

with  $E_0$  defined as in Lemma 2.1 and

$$\lambda_0 = \min_{E \in [\alpha_j, \beta_j]} |E|.$$

*Proof.* This is a consequence of Lemma 2.1 and the Mean Value Theorem, since  $|\Delta(\beta_j) - \Delta(\alpha_j)| = 4$ .  $\square$

## 3. STABILITY OF BAND EDGES

An important component of our proof of Theorem 1.1 will be control of the stability of bands of periodic Schrödinger operators under an  $L^2$  perturbation. Band edges of a  $T$ -periodic Schrödinger operator are precisely eigenvalues of the Schrödinger operator with periodic boundary conditions on  $[0, 2T]$ . In this section, we prove a stability statement for those eigenvalues under an  $L^2$  perturbation (note that, for  $L^\infty$  perturbations, this is trivial). Pastur–Figotin [11] contains an estimate of this form for the bottom of the spectrum, but we require an estimate for all eigenvalues. While estimates of this form are classical, we are not aware of this estimate appearing in the literature, so we provide a proof. Note that, for our application, it will be crucial that the estimate below features only the square root, and not a higher power, of the energy.

For  $V \in L^2[0, T]$ , denote by  $H_V^{\text{per}}$  the Schrödinger operator  $-\Delta + V$  on  $L^2[0, T]$  with periodic boundary conditions. Denote by  $E_{n,V}$  the  $n$ -th eigenvalue (including multiplicity) of  $H_V^{\text{per}}$ ; in particular,  $E_{1,V}$  is the ground state energy of  $H_V^{\text{per}}$ . Finally, let  $R_{\lambda,V}$  denote the resolvent of  $H_V^{\text{per}}$  at  $\lambda \in \mathbb{C} \setminus \sigma(H_V^{\text{per}})$ .

**Theorem 3.1.** *Fix  $Q > 0$ . If  $V_1, V_2 \in L^2[0, T]$  and  $\|V_j\|_{\text{B}} \leq Q$  for  $j = 1, 2$ , then*

$$E_{1,V_j} \geq -C_1(Q + T^2Q^2)$$

and

$$|E_{n,V_1} - E_{n,V_2}| \leq C_1(1 + T^2Q)(1 + T|E_{n,V_2}|^{1/2})\|V_1 - V_2\|_{\text{B}} \quad (9)$$

for some constant  $C_1$  independent of  $T, Q, V_1, V_2$ .

The proof of the theorem will start with a comparison with the free operator, so our first lemma concerns the integral kernel of the resolvent of  $H_0^{\text{per}}$ .

**Lemma 3.2.** *For  $\lambda \notin \sigma(H_0^{\text{per}})$ ,*

$$(R_{\lambda,0}f)(x) = \int_0^T G(\lambda; x, t)f(t) dt, \quad (10)$$

where

$$G(\lambda; x, t) = -\frac{1}{2\sqrt{\lambda}} \frac{\cos\left(\sqrt{\lambda}\left(\frac{T}{2} - |x - t|\right)\right)}{\sin\left(\sqrt{\lambda}\frac{T}{2}\right)}. \quad (11)$$

In particular, there is a constant  $C \in (0, \infty)$  (depending only on  $T$ ) such that for all  $\lambda < 0$  and all  $x, t \in [0, T]$ ,

$$|G(\lambda; x, t)| \leq CT^{-1/2}(|\lambda|^{-1/2} + |\lambda|^{-1}),$$

so  $R_{\lambda,0}$  is a bounded operator from  $L^2[0, T]$  to  $L^\infty[0, T]$ ,

$$\|R_{\lambda,0}\|_{2,\infty} \leq C(|\lambda|^{-1/2} + |\lambda|^{-1}). \quad (12)$$

*Proof.* For  $\lambda \neq 0$ , let

$$s_\lambda(x) = \lambda^{-1/2} \sin(\lambda^{1/2}x), \quad c_\lambda(x) = \cos(\lambda^{1/2}x)$$

The general solution of  $-y'' - \lambda y = f$  is

$$y(x) = As_\lambda(x) + Bc_\lambda(x) - \int_0^x f(t)s_\lambda(x-t) dt. \quad (13)$$

Applying periodic boundary conditions, solving the resulting system for  $A, B$  and inserting it back into  $y(x)$  and simplifying, we obtain (10) and (11). The estimates are then elementary.  $\square$

*Proof of Theorem 3.1.* We will prove the estimates for  $T = 1$ . After that, it is an immediate calculation to rescale the estimates to an arbitrary period  $T$ , as in the proof of Lemma 2.1.

Let  $A = 16C$ , where  $C$  is the constant from (12), and suppose

$$\lambda \leq -(AQ + A^2Q^2).$$

From (12) we see that  $VR_{\lambda,0}$  is a bounded operator on  $L^2[0, T]$ ,

$$\|VR_{\lambda,0}\|_{2,2} \leq \|V\|_{\mathbb{B}} \|R_{\lambda,0}\|_{2,\infty} \leq QC(|\lambda|^{-1/2} + |\lambda|^{-1}) \leq \frac{2C}{A} = \frac{1}{8},$$

so the Neumann series

$$\sum_{n=0}^{\infty} R_{\lambda,0}(-VR_{\lambda,0})^n$$

is convergent and equal to the inverse of  $H_V^{\text{per}} - \lambda$ . This implies that

$$\sigma(H_V) \subset (-(AQ + A^2Q^2), \infty).$$

Moreover, since

$$\left\| \sum_{n=0}^{\infty} (-VR_{\lambda,0})^n \right\|_{2,2} \leq \frac{1}{1-1/8} = \frac{8}{7},$$

composing this with  $R_{\lambda,0}$  and using  $\|R_{\lambda,0}\|_{2,2} = |\lambda|^{-1}$  and (12) proves the estimates

$$\|R_{\lambda,V}\|_{2,2} \leq \frac{8}{7} |\lambda|^{-1} \tag{14}$$

$$\|R_{\lambda,V}\|_{2,\infty} \leq \frac{8}{7} C(|\lambda|^{-1/2} + |\lambda|^{-1}) \tag{15}$$

The resolvent identity, together with (14), (15), gives

$$\begin{aligned} \|R_{\lambda,V_1} - R_{\lambda,V_2}\|_{2,2} &\leq \|R_{\lambda,V_1}\|_{2,2} \|V_1 - V_2\|_{\mathbb{B}} \|R_{\lambda,V_2}\|_{2,\infty} \\ &\leq \frac{64}{49} C \|V_1 - V_2\|_{\mathbb{B}} (|\lambda|^{-3/2} + |\lambda|^{-2}). \end{aligned}$$

By the spectral mapping,  $\sigma(R_{\lambda,V_j}) = \{(E_{n,V_j} - \lambda)^{-1} \mid n \in \mathbb{N}\} \cup \{0\}$  so by the variational principle,

$$\left| \frac{1}{E_{n,V_1} - \lambda} - \frac{1}{E_{n,V_2} - \lambda} \right| \leq \|R_{\lambda,V_1} - R_{\lambda,V_2}\|_{2,2} \leq \frac{64}{49} C \|V_1 - V_2\|_{\mathbb{B}} (|\lambda|^{-3/2} + |\lambda|^{-2}). \tag{16}$$

To turn this into an estimate for  $|E_{n,V_1} - E_{n,V_2}|$ , we set

$$\lambda = -\max\{2|E_{n,V_2}|, AQ + A^2Q^2\}.$$

Then

$$|E_{n,V_2} - \lambda| \frac{64}{49} C \|V_1 - V_2\|_{\mathbb{B}} (|\lambda|^{-3/2} + |\lambda|^{-2}) \leq \frac{3}{2} |\lambda| \frac{64}{49} C (2Q) |\lambda|^{-1} \frac{2}{AQ} < \frac{1}{2},$$

so (16) is easily seen to imply

$$|E_{n,V_1} - E_{n,V_2}| \leq 2|E_{n,V_2} - \lambda| \frac{64}{49} C \|V_1 - V_2\|_{\mathbb{B}} (|\lambda|^{-3/2} + |\lambda|^{-2})$$

$$\begin{aligned} &\leq 2\frac{9}{4}|\lambda|^2\frac{64}{49}C(|\lambda|^{-3/2} + |\lambda|^{-2})\|V_1 - V_2\|_{\mathbb{B}} \\ &\leq 9C(|\lambda|^{1/2} + 1)\|V_1 - V_2\|_{\mathbb{B}} \end{aligned}$$

which completes the proof by our choice of  $\lambda$ .  $\square$

#### 4. CONVERGENCE OF PERIODIC SPECTRA

The results of Section 3 control the stability of bands of periodic spectra under  $L^2$  perturbations. This will enable us to prove effective estimates on the measure of the spectrum in passing from  $V_\ell$  to  $V_{\ell+1}$ . The next key ingredient one needs to control is the relationship between the spectra of the periodic approximants and the spectrum of the limiting potential. This will enable us to push our step-by-step estimates through to the limit. If  $V_n \rightarrow V$  uniformly, then this is easy, since an unbounded version of Lemma 4.1 implies that  $\sigma(H_{V_n}) \rightarrow \sigma(H_V)$  in the Hausdorff metric. However, in our setting, we only have  $V_n \rightarrow V$  in a uniform local  $L^2$  sense, so the convergence statement on the spectra is weaker. Nevertheless, we still have norm convergence of resolvents, which enables us to establish a ‘‘pseudo local Hausdorff’’ convergence result for the spectra which is strong enough to get lower bounds on the limiting spectrum.

First, let us recall the definition of the Hausdorff metric on compact subsets of  $\mathbb{R}$ .

**Definition.** Given two compact sets  $F, K \subseteq \mathbb{R}$ , the *Hausdorff distance* between them is defined by

$$d_{\mathbb{H}}(F, K) = \inf\{r > 0 : F \subseteq B_r(K) \text{ and } K \subseteq B_r(F)\}. \quad (17)$$

In (17),  $B_r(X)$  denotes the open  $r$ -neighborhood of the set  $X \subseteq \mathbb{R}$ .

The following relationship between spectra of bounded, self-adjoint operators and the Hausdorff metric is well-known.

**Lemma 4.1.** *If  $A$  and  $B$  are bounded self-adjoint operators, then*

$$d_{\mathbb{H}}(\sigma(A), \sigma(B)) \leq \|A - B\|.$$

**Lemma 4.2.** *Given  $V \in \text{PT}(\mathbb{R})$  with periodic approximants  $(V_j)_{j=1}^\infty$ , let  $H = -\Delta + V$  and  $H_j = -\Delta + V_j$ , and put  $\Sigma = \sigma(H)$  and  $\Sigma_j = \sigma(H_j)$ . If  $(I_j)_{j=1}^\infty$  is a family of compact subsets of  $\mathbb{R}$  which converges to the compact set  $I \subseteq \mathbb{R}$  in the Hausdorff metric, then, for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{Z}_+$  such that*

$$I_j \cap \Sigma_j \subseteq B_\varepsilon(I \cap \Sigma)$$

whenever  $j \geq N$ . In particular,

$$|I \cap \Sigma| \geq \limsup_{j \rightarrow \infty} |I_j \cap \Sigma_j|, \quad (18)$$

where  $|\cdot|$  denotes Lebesgue measure on  $\mathbb{R}$ .

*Proof.* This follows from the spectral mapping theorem and norm-resolvent convergence. To begin, notice that the bottoms of the spectra converge; that is,

$$E_1 := \lim_{j \rightarrow \infty} E_{1,j}$$



exists, where  $E_{1,j} = E_{1,V_j} = \inf \Sigma_j$  for each  $j \geq 1$ . This is a consequence of the Pastur–Tkachenko condition (2) and the stability condition (9) from Theorem 3.1 with  $n = 1$ . In particular,

$$\lambda_0 := \inf_{j \geq 1} E_{1,j} - 1$$

is finite. Since  $\text{dist}(\lambda_0, \sigma(H_j)) \geq 1$  for every  $j$ , we may apply [11, Lemma 17.17] to see that  $H_j$  converges to  $H$  in the norm-resolvent sense, i.e.,

$$R = \lim_{j \rightarrow \infty} R_j,$$

where  $R = (H - \lambda_0)^{-1}$  and  $R_j = (H_j - \lambda_0)^{-1}$ . Consequently, Lemma 4.1 implies that

$$\sigma(R) = \lim_{j \rightarrow \infty} \sigma(R_j),$$

where the limit is taken in with respect to the Hausdorff metric on compact subsets of  $\mathbb{R}$ . By the spectral mapping theorem for unbounded operators, we have

$$\Sigma'_j := \sigma(R_j) = \overline{g(\Sigma_j)} \text{ for every } j \geq 1, \text{ and } \Sigma' := \sigma(R) = \overline{g(\Sigma)},$$

with  $g(z) = (z - \lambda_0)^{-1}$ ; see [19, Theorem 3.18]. Notice that  $g$  is invertible with inverse  $h(z) = z^{-1} + \lambda_0$ . Without loss of generality, we may assume  $I_j \subseteq [\lambda_0 + 1, \infty)$  for each  $j$ . It is easy to see that  $I'_j := g(I_j)$  is a compact subset of  $(0, 1]$  for each  $j$ . Given  $\delta > 0$  Hausdorff convergence of  $\Sigma'_j$  implies

$$I'_j \cap \Sigma'_j \subseteq B_\delta(I' \cap \Sigma')$$

for all sufficiently large  $j$ . Of course,  $h$  is Lipschitz on  $I'$ , so this implies (for some  $I$ -dependent constant  $\kappa > 0$ ):

$$I_j \cap \Sigma_j \subseteq B_{\kappa\delta}(I \cap \Sigma)$$

for all sufficiently large  $j$ . The semicontinuity statement (18) follows by sending  $\delta \rightarrow 0$ . □

## 5. PROOF OF THEOREM 1.1

We can now combine the results of the previous three sections to prove the main theorem. The general strategy is as follows:

- Use Corollary 2.2 to obtain a lower bound on the lengths of bands of the spectra of the periodic approximants.
- Use Theorem 3.1 to obtain an upper bound on the lengths of gaps of the spectra of the periodic approximants.
- These two items give us effective upper bounds on quantities of the form  $|I \cap \Sigma_\ell \setminus \Sigma_{\ell+1}|$  with  $I$  a compact interval.
- Use Lemma 4.2 to push the estimates through to the limit.

*Proof of Theorem 1.1.* Suppose  $V \in \text{PT}(\mathbb{R})$ , and let  $0 < \tau < 1$  be given. If  $V$  is periodic, the conclusion of the theorem is trivial, so assume that  $V$  is aperiodic. Let  $(V_n)_{n=1}^\infty$  be a sequence of periodic potentials, such that  $V_n$  is  $T_n$ -periodic,  $T_n$  divides  $T_{n+1}$  for each  $n$ , and  $\|V - V_n\|_B$  is  $o(e^{-bT_{n+1}})$  for every  $b > 0$ . Notice that this property (i.e. equation (2)) is preserved if one removes finitely many terms of the sequence  $(V_n)_{n=1}^\infty$  and consecutively renumbers the resulting sequence. In particular, it is no loss of generality to assume  $T_n \geq 1$  for every  $n$ . For each  $n \in \mathbb{Z}_+$ , denote  $\Sigma_n = \sigma(H_{V_n})$ , and put  $\Sigma = \sigma(H_V)$ . Repeating the argument from

the beginning of the proof of Lemma 4.2, we see that  $\sup_n |\inf \Sigma_n|$  is finite. Since spectral homogeneity is preserved by adding a constant to the potential, this means that we may assume without loss of generality that  $\Sigma_n \subseteq [1, \infty)$  for every  $n$ . Now, define

$$Q = \sup_{n \geq 1} \|V_n\|_{\mathbb{B}}$$

$$K = \max\left(C, C_1, Q, CQ^{1/2}, 8\right),$$

where  $C$  is the universal constant from Lemma 2.1 and  $C_1$  is the universal constant from Theorem 3.1. Notice that  $K$  depends only on  $Q$ . Using the Pastur–Tkachenko condition (2), it is easy to see that

$$\sum_{n=1}^{\infty} T_{n+1}^6 e^{KT_{n+1}} \|V_n - V_{n+1}\|_{\mathbb{B}} < +\infty,$$

Thus, by removing finitely many terms of the sequence  $(V_n)_{n=1}^{\infty}$  and consecutively renumbering once more, we may assume that

$$\sum_{n=1}^{\infty} T_{n+1}^6 e^{KT_{n+1}} \|V_n - V_{n+1}\|_{\mathbb{B}} < \frac{1-\tau}{3K^4}. \quad (19)$$

Put

$$\delta_0 = \min\left(K^{-1}T_1^{-3}e^{-KT_1}, \frac{1-\tau}{3}\right).$$

Following the general strategy of [7], we will prove the following estimate:

$$|B_\delta(x) \cap \Sigma_N| \geq \delta\tau \text{ for all } x \in \Sigma_N \text{ and every } 0 < \delta \leq \delta_0 \quad (20)$$

for all  $N \in \mathbb{Z}_+$ . To that end, fix  $N \in \mathbb{Z}_+$ ,  $x \in \Sigma_N$ , and  $0 < \delta \leq \delta_0$ . For notational simplicity, put  $s = s(x) := 1 + \sqrt{x}$ . If  $\delta \leq sK^{-1}T_N^{-3}e^{-KT_N}$ , (20) is an obvious consequence of Corollary 2.2. Specifically,  $\delta$  is less than the length of the band of  $\Sigma_N$  which contains  $x$ , so

$$|B_\delta(x) \cap \Sigma_N| \geq \delta$$

in this case. Notice that we have used the assumption  $\inf \Sigma_N \geq 0$  to get rid of  $E_0$  from the statement of Corollary 2.2. Otherwise,  $\delta > sK^{-1}T_N^{-3}e^{-KT_N}$ , and there is a unique integer  $n$  with  $1 \leq n \leq N-1$  such that

$$sK^{-1}T_{n+1}^{-3}e^{-KT_{n+1}} < \delta \leq sK^{-1}T_n^{-3}e^{-KT_n}. \quad (21)$$

This integer  $n$  is relevant, as it determines the periodic approximant corresponding to the length scale  $\delta$  near and above  $x$ . More precisely, by Corollary 2.2, any band of  $\Sigma_n$  which is contained in  $[x-1, \infty)$  has length at least  $\delta$ . By Theorem 3.1, there exists  $x_0 \in \Sigma_n$  with

$$|x - x_0| \leq sK^3 \sum_{\ell=n}^{N-1} T_{\ell+1}^3 \|V_\ell - V_{\ell+1}\|_{\mathbb{B}} < 1. \quad (22)$$

This bound follows by a straightforward backward induction on  $n$ , starting from  $n = N$ . The inductive step uses Theorem 3.1 and the estimate  $1 + |x_0|^{1/2} \leq 2s$ , which follows from the inductive assumption. Using (21), we deduce

$$|x - x_0| \leq sK^3 \sum_{\ell=n}^{N-1} T_{\ell+1}^3 \|V_\ell - V_{\ell+1}\|_{\mathbb{B}}$$

$$\begin{aligned}
&< \delta e^{KT_{n+1}} T_{n+1}^3 K^4 \sum_{\ell=n}^{N-1} T_{\ell+1}^3 \|V_\ell - V_{\ell+1}\|_{\mathbb{B}} \\
&< \delta K^4 \sum_{\ell=n}^{N-1} e^{KT_{\ell+1}} T_{\ell+1}^6 \|V_\ell - V_{\ell+1}\|_{\mathbb{B}} \\
&< \delta \frac{1-\tau}{3}.
\end{aligned} \tag{23}$$

Thus, there exists an interval  $I_0$  with  $x_0 \in I_0 \subseteq B_\delta(x) \cap \Sigma_n$  such that

$$|I_0| = \delta - \frac{1-\tau}{3}\delta = \frac{2+\tau}{3}\delta.$$

By standard measure theory, we have

$$|B_\delta(x) \cap \Sigma_N| \geq |I_0 \cap \Sigma_n| - \sum_{\ell=n}^{N-1} |I_0 \cap (\Sigma_\ell \setminus \Sigma_{\ell+1})|$$

By Corollary 2.2, the interval  $I_0$  completely contains at most  $\delta K s^{-1} T_{\ell+1}^3 e^{KT_{\ell+1}}$  bands of  $\Sigma_{\ell+1}$  for each  $\ell \geq n$ . Consequently,

$$\begin{aligned}
|I_0 \cap (\Sigma_\ell \setminus \Sigma_{\ell+1})| &\leq K^3 T_{\ell+1}^3 s (\delta K s^{-1} T_{\ell+1}^3 e^{KT_{\ell+1}} + 1) \cdot \|V_\ell - V_{\ell+1}\|_{\mathbb{B}} \\
&\leq 2\delta K^4 T_{\ell+1}^6 e^{KT_{\ell+1}} \cdot \|V_\ell - V_{\ell+1}\|_{\mathbb{B}}
\end{aligned}$$

for each  $\ell \geq n$ , by Theorem 3.1 and (21). Summing this over  $\ell$  and estimating the result with (19), we obtain

$$\begin{aligned}
\sum_{\ell=n}^{N-1} |I_0 \cap (\Sigma_\ell \setminus \Sigma_{\ell+1})| &\leq \sum_{\ell=n}^{N-1} 2\delta K^4 T_{\ell+1}^6 e^{KT_{\ell+1}} \cdot \|V_\ell - V_{\ell+1}\|_{\mathbb{B}} \\
&< 2\delta \frac{1-\tau}{3}.
\end{aligned}$$

Putting all of this together, we have

$$\begin{aligned}
|B_\delta(x) \cap \Sigma_N| &\geq |I_0 \cap \Sigma_n| - \sum_{\ell=n}^{N-1} |I_0 \cap (\Sigma_\ell \setminus \Sigma_{\ell+1})| \\
&> \frac{2+\tau}{3}\delta - \frac{2-2\tau}{3}\delta \\
&= \tau\delta.
\end{aligned}$$

This proves (20) for arbitrary  $N \in \mathbb{Z}_+$ . Using Lemma 4.2, we obtain

$$|B_\delta(x) \cap \Sigma| \geq \tau\delta \text{ for all } x \in \Sigma, \text{ and } 0 < \delta \leq \delta_0,$$

to wit,  $\Sigma$  is homogeneous.  $\square$

## 6. THE DISCRETE SETTING

In this section, we will describe how to extend Theorem 1.1 to Jacobi and CMV matrices. In this case, the discrete setting is markedly easier than the continuum setting, so we will only sketch the proof. Recall that a Jacobi matrix is an operator of the form  $J = J_{a,b}$  on  $\ell^2(\mathbb{Z})$ , defined by

$$(J\varphi)_n = a_{n-1}\varphi_{n-1} + a_n\varphi_{n+1} + b_n\varphi_n, \quad \varphi \in \ell^2(\mathbb{Z}), \quad n \in \mathbb{Z},$$



## REFERENCES

- [1] A. Avila, On the Kotani-Last and Schrödinger conjectures, to appear in *J. Amer. Math. Soc.*
- [2] J. Breuer, E. Ryckman, B. Simon, Equality of the spectral and dynamical definitions of reflection, *Comm. Math. Phys.* **295** (2010), 531–550.
- [3] L. Carleson, On  $H^\infty$  in multiply connected domains. Harmonic Analysis. Conference in honor of Antony Zygmund. vol. II, (1983), 349–382.
- [4] D. Damanik, M. Goldstein, M. Lukic, The spectrum of a Schrödinger operator with small quasi-periodic potential is homogeneous. to appear in *J. Spectral Theory*. arXiv:1408.4335.
- [5] D. Damanik, P. Yuditskii, Counterexamples to the Kotani-Last conjecture for continuum Schrödinger operators via character-automorphic Hardy spaces. Preprint (arXiv:1405.6343).
- [6] I. E. Egorova, Spectral analysis of Jacobi limit-periodic matrices, *Dokl. Akad. Nauk Ukrain. SSR Ser. A* **3** (1987), 7–9. (in Russian)
- [7] J. Fillman, Spectral homogeneity of discrete one-dimensional limit-periodic operators, to appear in *J. Spectral Theory*. arXiv:1409.7734.
- [8] F. Gesztesy, P. Yuditskii, Spectral properties of a class of reflectionless Schrödinger operators, *J. Func. Anal.* **351** (2006), 619–646.
- [9] F. Gesztesy, M. Zinchenko, Local spectral properties of reflectionless Jacobi, CMV, and Schrödinger operators, *J. Diff. Eq.* **246** (2009), 78–107.
- [10] S. Kotani, Jacobi matrices with random potentials taking finitely many values, *Rev. Math. Phys.* **1** (1989), 129–133.
- [11] L. Pastur, A. Figotin, *Spectra of Random and Almost-Periodic Operators*, Springer-Verlag, Berlin, 1991.
- [12] L. Pastur, V. A. Tkachenko, On the spectral theory of the one-dimensional Schrödinger operator with limit-periodic potential (Russian), *Dokl. Akad. Nauk SSSR* **279** (1984) 1050–1053.
- [13] L. Pastur, V. A. Tkachenko, Spectral theory of a class of one-dimensional Schrödinger operators with limit-periodic potentials (Russian), *Trudy Moskov. Mat. Obshch.* **51** (1988) 114–168.
- [14] A. Poltoratski, C. Remling, Reflectionless Herglotz functions and Jacobi matrices, *Commun. Math. Phys.* **288** (2009), 1007–1021.
- [15] C. Remling, The absolutely continuous spectrum of Jacobi matrices, *Ann. of Math.* **174** (2011), 125–171.
- [16] B. Simon, *Szegő's Theorem and its Descendants: Spectral Theory for  $L^2$  Perturbations of Orthogonal Polynomials*, M.B. Porter Lectures, Princeton University Press, Princeton, NJ 2011.
- [17] M. Sodin, P. Yuditskii, Almost periodic Sturm-Liouville operators with Cantor homogeneous spectrum, *Comment. Math. Helv.* **70** (1995), 639–658.
- [18] M. Sodin, P. Yuditskii, Almost periodic Jacobi matrices with homogeneous spectrum, infinite-dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions, *J. Geom. Anal.* **7** (1997), 387–435.
- [19] G. Teschl, *Mathematical Methods in Quantum Mechanics*, Graduate Studies in Mathematics, **99**, American Mathematical Society, Providence, RI, 2000.
- [20] A. Volberg, P. Yuditskii, Kotani-Last problem and Hardy spaces on surfaces of Widom type. To appear in *Invent. Math.* (arXiv:1210.7069).
- [21] P. Yuditskii, On the direct Cauchy theorem in Widom domains: Positive and negative examples, *Computational Methods and Function Theory*, **11** (2012), 395–414.

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