THE FRONT LOCATION IN BBM WITH DECAY OF MASS

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Abstract. We augment standard branching Brownian motion by adding a competitive interaction between nearby particles. Informally, when particles are in competition, the local resources are insufficient to cover the energetic cost of motion, so the particles' masses decay. In standard BBM, we may define the *front displacement* at time t as the greatest distance of a particle from the origin. For the model with masses, it makes sense to instead define the front displacement as the distance at which the local mass density drops from $\Theta(1)$ to $o(1)$. We show that one can find arbitrarily large times t for which this occurs at a distance $\Theta(t^{1/3})$ behind the front displacement for standard BBM.

1. Introduction

In this work, we propose a mathematical model of competition for resources within a single species, in a growing, spatially structured population, and provide an initial study of the front location in this new setting. The model is essentially standard branching Brownian motion (BBM), augmented with a destructive, local interaction between particles. We first briefly recall BBM: start from a single point $\mathbb{R}^d$, endowed with an Exp(1) "branching clock". The particle moves according to Brownian motion; when its clock rings, it splits in two (branches). The new particles receive independent Exp(1), clocks, and move independently (according to Brownian motion) starting from where the first particle splits, until their own clocks ring and they in turn split, *et cetera*.

Write $N(t)$ for the total number of particles at time t, and $\mathbf{X}(t) = (X_i(t), 1 \leq i \leq N(t))$ for the locations of such particles. We assume the particles are listed in a way that makes the vector $\mathbf{X}(t)$ exchangeable; one possible formalism is via the Ulam-Harris tree, with particles listed lexicographically acording to their label in the tree. We refer the reader to [10] for more details on such matters; but many different references are possible. We also write $N(t, x) = \#\{i : X_i(t) \geq x\}$ for the number of particles with position greater than x at time t.

We sometimes write $(X_i(t), i \geq 1)$, ignoring the fact that $\mathbf{X}(t)$ has finite length, for convenience. We adopt the convention that $X(t, k) = \partial$ for $k > N(t)$ (so ∂ is where new babies come from). We refer to "the particle $X_i(t)$" as shorthand for "the particle with position $X_i(t)$ at time t"; this is unambiguous at Leb-a.e. time t. We write $\mathbf{P}$ for the probability measure under which $(\mathbf{X}(t), t \geq 0)$ has the law of one dimensional BBM with initial individual at 0, $\mathbf{E}$ for the corresponding expectation, and $(\mathcal{F}_t, t \geq 0)$ for the filtration generated by the process.

We now add destructive interaction as follows. Informally, imagine that the particles are, say, amoeba. Motion has an energetic cost, but for a single particle in isolation, this cost is exactly accounted for by the resources (food) available in the environment. When particles are nearby, however – at distance less than one, say – they must share resources; in this case individuals do not consume enough to meet their energy expenditure, and their mass decreases. Finally, larger (more massive) individuals consume resources at a greater rate.

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Formally, we define a vector $\mathbf{M}(t) = (M_i(t), i \geq 0)$, and call $M_i(t)$ the *mass* of particle $X_i(t)$. By convention, if $X_i(t) = \partial$ then $M_i(t) = 0$. Write

$$\zeta(t,x) = \sum_{\{i:|X_i(t)-x|\in(0,1)\}} M_i(s)$$

for the total mass of particles within distance one of x at time t, excluding any particles at position x. Then at time t, $M_i(t)$ decays at rate $\zeta(t, X_i)$. In other words, $\mathrm{d}M_i(t) = -M_i(t)\cdot\zeta(t, X_i(t))\mathrm{d}t$, so

$$M_i(t) = \exp(-\int_0^t \zeta(s, X_i(s))\mathrm{d}s)\,.$$

This should be viewed as defining $(\mathbf{M}(t), t \geq 0)$ to be the solution of a system of differential equations; the definition makes sense since the system has a unique solution $\mathbf{P}^x$-almost surely. Furthermore, the process $(\mathbf{M}(t), t \geq 0)$ is clearly $\mathcal{F}_t$-adapted.

For later use, write $(X_{i,t}(s), 0 \leq s \leq t)$ for the ancestral path leading to $X_i(t)$, and let $\sigma_i(t)$ be the final branching time along this path. Also, let $j_{i,t}(s)$ for the index of $X_{i,t}(s)$ among the time-s population, so that $X_{i,t}(s) = X_{j_{i,t}(s)}(s)$. We also write $M_{i,t}(s)$ for the mass of the ancestor of $X_i(t)$ at time s (so $M_{i,t}(s) = M_{j_{i,t}(s)}(s)$).

Write

$$d(t,m) = \min\{x > 0 : \zeta(t,x) < m\}, \quad D(t,m) = \max\{x : \zeta(t,x) > m\},$$

for the leftmost (positive) location at which such a mass falls below m, and the rightmost location at which the total mass of nearby particles exceeds m, respectively. We prove the following theorem.

Theorem 1.1. *There exist constants $c-, c^+ \in (0,\infty)$ such that almost surely, for all $m < 1$,*

$$\limsup_{t\to\infty} \frac{\sqrt{2}t - d(t,m)}{t^{1/3}} \geq c^- \quad \text{and} \quad \limsup_{t\to\infty} \frac{\sqrt{2}t - D(t,m)}{t^{1/3}} \leq c^+$$

A well-known result of Bramson [4] states that the rightmost particle location $\max_{i\geq 1} X_i(t)$ has median $\mathrm{med}(t)$ satisfying

$$\mathrm{med}(t) = \sqrt{2}t - \frac{3}{2^{3/2}}\log t + O(1).$$

Furthermore, it turns out [12] that $|\max_{i\geq 1} X_i(t) - \mathrm{med}(t)|$ is almost surely $O(\log t)$, in that $|\max_{i\geq 1} X_i(t) - \mathrm{med}(t)|/\log t$ is a.s. finite. In view of this, the theorem states that (1) there are (large) times t at which the first low-density region lags at least distance $t^{1/3}$ behind the rightmost particle, and (2) there are also (potentially different, large) times t at which there is some high-density region within distance $t^{1/3}$ of the rightmost particle.

In fact, our proofs yield explicit values for c^- and c^+: these are stated in Propositions 3.1 and 5.1, respectively. We expect that in fact there exists $c^* \in (0,\infty)$ such that almost surely, for all $m < 1$,

$$\lim_{t\to\infty} \frac{\sqrt{2}t - d(t,m)}{t^{1/3}} = c^* = \lim_{t\to\infty} \frac{\sqrt{2}t - D(t,m)}{t^{1/3}}\,,$$

and that Proposition 5.1 predicts the correct value of c^*. We provide some justification for this belief in Section 6. That section also contains: a few open questions about the model; a discussion of various generalizations (some straightforward, some conjectural); and a heuristic but, I believe, compelling link with a family of integro-differential reaction-diffusion equations that have received substantial study in the PDE and mathematical biology literature.

2. Proof sketch

Here comes an outline of the key tools in our argument. The first is technical but important and also, we believe, provides important intuition when making heuristic predictions about the behaviour of the process. The remainder give a fairly detailed overview of the proof.

Density self-correction. It is not hard to see that when $\zeta(t,x)$ is small (much less than one), and this also holds in a region around x, then $\zeta(t,\cdot)$ will exhibit exponential growth near x, at least for a short time. Indeed, we heuristically have

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta(t,x) \approx \zeta(t,x) - \sum_{\{i:|X_i(t)-x|\in(0,1)\}} M_i(t)\cdot\zeta(t,X_i(t))\,.$$

This is not exactly correct since it ignores the effect of motion (particles may enter or leave the region near x), but it is a useful first approximation. In particular, it suggests that if $\zeta(t,y)$ is small (much less than one) for all y with $|y-x|<1$, then $\zeta(t,\cdot)$ will exhibit exponential growth near x, at least for a short time. This is indeed true; one important consequence is that if $\zeta(t,x)=\epsilon$ and $\zeta(t,\cdot)$ is not too wild then it is very likely that $\zeta(t',x)=\Theta(1)$ for some $t'=t+\Theta(\log(1/\epsilon))$. Similarly, when $\zeta(t,y)$ is much larger than 1 for y near x then $\zeta(t,x)$ will decrease exponentially quickly.

As an aside, we remark that if $\zeta(t,y)\approx\zeta(t,x)$ for $|y-x|<1$ then the above heuristic gives $\frac{\mathrm{d}}{\mathrm{d}t}\zeta(t,x)\approx\zeta(t,x)(1-\zeta(t,x))$, which is suggestive of the logistic control; we briefly revisit this connection in the conclusion.

Population + no competitors=mass. Fix $\beta>0$ and suppose that for some function $f:[0,\infty)\to\mathbb{R}$, for all $s\in[0,t]$, $D(s,\beta)\le f(s)$, or in other words $\zeta(s,x)\le\beta$ for all $x\ge f(s)$. In this case, particles that stay ahead of the moving barrier f are in a relatively sparse environment, so do not lose mass too quickly. More precisely, if $X_i(t)$ satisfies $X_{i,t}(s)\ge f(s)$ for all $s\in[0,t]$ then $M_i(t)\ge e^{-\beta t}$. It follows that for any $x\ge f(t)+1$,

$$\zeta(t,x)\ge e^{-\beta t}\cdot\#\{i:|X_i(t)-x|<1,\forall s\in[0,t],X_{i,t}(s)>f(s)\}\,.$$

For such x, if $\#\{i:|X_i(t)-x|<1,\forall s\in[0,t],X_{i,t}(s)>f(s)\}>\beta e^{\beta t}$ then $\zeta(t,x)\ge\beta$, contradicting the assumption that $D(t,\beta)\le f(t)$.

Surfing the wave. To exploit the above contradiction, we require that with high probability there are many particles staying ahead of some barrier. Such results are available: it follows fairly straightforwardly from recent studies of *consistent maximal displacement* for BBM [19] that for sufficiently large $C>0$, for all large times t there are $e^{\Theta(t^{1/3})}$ particles at time t with which have stayed ahead of the curve $f(s)=\sqrt{2}s-Cs^{1/3}$. This allows us to take $\beta=t^{-1}$ above and obtain that there is $s\in[0,t]$ and $x\ge f(s)$ such that $\zeta(s,x)\ge t^{-1}$. Since the local density grows exponentially in regions with small density, we will with high probability find s' with $\zeta(s',x)>b>0$ and $s'-s=O(\log t)$. For such s' we have $f(s')=f(s)+O(\log s)$ so $x\ge\sqrt{2}s'-C(s')^{1/3}-O(\log s')$. With high probability we thus have $d(s',b)\ge\sqrt{2}s'-C(s')^{1/3}-O(\log s')$.

The lower bound is practically complete, but we must rule out the possibility that $s'=O(1)$ for all t. To do so, we first establish that

$$\sup_{t>0}\frac{\max\{\zeta(t,x),x\in\mathbb{R}\}}{\log(t+2)}=:Z<\infty\quad\text{almost surely}\,.$$

Proving this is harder than might be expected; its proof, given in Section 4, occupies 8 pages and is the most technically challenging part of the paper.

Once we prove that $Z < \infty$, we then reprise the above argument, but with a variable mass bound

$$\beta = \beta(s) = \begin{cases} Z\log(s+2) & \text{for } s \le t^{1/4} \\ t^{-1} & \text{for } s \in (t^{1/4}, t]\,. \end{cases}$$

The loss of mass before time $t^{1/4}$ is insignificant compared with that which follows, so essentially the same argument as above yields that there is $s \in [0,t]$ and $x \ge f(s)$ such that $\zeta(s,x) \ge \beta(s)$. On the other hand, this can not happen for $s < t^{1/4}$ by the definition of Z, so it must happen later. This is enough to conclude the lower bound. The details of this argument appear in Section 5.

Competition implies decay. For the upper bound, we invert the above argument by contradiction. In brief: if all particles to the right of a given curve have spent large amounts of time in high-mass environments, then all such individuals will have very low weight; if furthermore there are not many of them, then their total weight is also small.

More precisely, suppose that for some $\alpha > 0$ and some function $g : [0,\infty) \to \mathbb{R}$, for all $s \in [\alpha t, t]$, we have $d(s,\beta) \ge g(s)$, so $\zeta(s,x) \ge \beta$ for all x with $x \in [0, g(s)]$. Recalling the notation $N(t,x)$ from the introduction, the preceding sentence gives that for all i,

$$M_i(t) \le \exp(-\beta \cdot \mathrm{Leb}(s \in [\alpha t, t] : |X_{i,t}(s)| \in [0, g(s)]))\,.$$

It follows that if all particles with $X_i(t) \ge g(t)$ have $\mathrm{Leb}(s \in [\alpha t, t] : |X_{i,t}(s)| \in [0, g(s)]) \ge \ell$ then for all $x \ge g(t) + 1$,

$$\zeta(t,x) \le e^{-\beta\ell} \cdot N(t, g(t))\,.$$

If $N(t, g(t)) \le \beta e^{\beta\ell}$, this is in contradiction with the assumption that $d(s,\beta) \ge g(s)$.

Whitecaps are just foam. Once again using estimates related to consistent maximal displacement for BBM, we show that for $\delta > 0$ sufficiently small, with $g(s) = \sqrt{2}s - \delta s^{1/3}$, with high probability every particle with $X_i(t) > g(t)$ indeed spends at least a constant proportion α of its time behind the curve g. Under the above assumption, it follows that the particles counted by $N(t, g(t))$ are as insubstantial as sea spray; for all $x \ge g(s) + 1$,

$$\zeta(t,x) \le e^{-\beta(\alpha t - t^{1/2})} \cdot \#\{i : |X_i(t) - x| < 1\} \le e^{-\beta(\alpha t - t^{1/2})} \cdot N(t, g(t)).$$

Standard and simple arguments for BBM show that $N(t, g(t)) = e^{O(ct^{1/3})}$ with high probability, so we obtain a contradiction if $\beta(\alpha t - t^{1/2})/(ct^{1/3})$ is sufficiently large. It follows that with high probability there is $s \in [t^{1/2}, t]$ and $x \in [0, g(s)]$ such that $\zeta(s,x) = O(t^{-2/3})$; for such s we have $d(s,b) \le g(s)$ for any fixed $b > 0$. This is explained in detail in Section 3.

Definitions

We sometimes need to consider the evolution of a subset of the particles starting at a time greater than zero, so it is useful to allow initial conditions other than a single mass-one particle at the origin. Generally, for $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and $\mathbf{m} = (m_1, \ldots, m_k) \in (0,\infty)^k$, we write $\mathbf{P}_{\mathbf{x},\mathbf{m}}$ for the probability measure corresponding to an initial condition with a particle of mass m_i at location x_i for each $1 \le i \le k$. We write $\mathbf{P} = \mathbf{P}_{(0),(1)}$ for the default initial condition.

We say a random variable X is *geometric with parameter* p, or is $\mathrm{Geom}(p)$-distributed, if $\mathbf{P}\{X = k\} = (1-p)^{k-1}p$ for positive integer p.

3. Upper bound

A sense of momentum helps with motivation, so we begin by proving the easier inequality of the theorem.

Proposition 3.1. *For any $m > 0$, almost surely*
$$\limsup_{t\to\infty} \frac{\sqrt{2}t - d(t,m))}{t^{1/3}} \geq 2^{-5/6}.$$

In the proof we use a straightforward bound on the probability that Brownian motion spends a constant proportion of its time in a narrow tube; we state the bound here and provide its easy proof in an appendix.

Lemma 3.2. *Let $(B_s, s \geq 0)$ is standard one-dimensional Brownian motion, and for $x > 0$ and $t > 0$ write $L_t(x) = \mathrm{Leb}(\{s \in [0,t] : |B_s| \leq x\})$. Then for all $t > 0$, $y > 0$, and $\alpha > 0$,*
$$\mathbf{P}\left\{L_t(y) \geq \alpha t\right\} \leq 5\exp(-t\cdot\alpha^2/2y^2)\,.$$

The next lemma is the key step of the proof.

Lemma 3.3 ("No one can surf g")**.** *For all $\delta \in (0, 2^{-5/6})$, there exists $c = c(\delta) > 0$ such that for all t sufficiently large,*
$$\mathbf{P}\left\{\exists i : X_i(t) \geq g(t), \mathrm{Leb}(\{s \in [0,t] : X_{i,t}(s) < \sqrt{2}s - \delta s^{1/3}\}) \leq ct\right\} \leq \exp(-ct^{1/3})\,.$$

For the remainder of the section we fix δ as in the lemma, and let $g(s) = \sqrt{2}s - \delta s^{1/3}$ for $s \geq 0$.

Proof of Lemma 3.3. Fix $\alpha \in (2^{1/4}\delta^{3/2}, 1/2)$; this is possible by our choice of δ. Suppose that $i \in N(t, g(t))$, or in other words that $X_i(t) \geq g(t)$. If the path $(X_{i,t}(s), 0 \leq s \leq t)$ spends less than $(1-2\alpha)t$ time below the curve g then it either spends time at least αt at distance at most $\delta s^{1/3}$ from $\sqrt{2}s$, or else it spends time at least αt with position above $\sqrt{2}s + \delta s^{1/3}$. In the latter case, there must be a time $s > \alpha t$ such that
$$X_{i,t}(s) \geq \sqrt{2}s + s^{1/3} \geq \sqrt{2}s + \delta(\alpha t)^{1/3}\,.$$

It follows that
$$\begin{aligned}
&\mathbf{P}\left\{\exists i \in N(t,g(t)) : \mathrm{Leb}(\{s \in [0,t] : X_{i,t}(s) < \sqrt{2}s - \delta s^{1/3}\}) \leq (1-2\alpha)t\right\}\\
\leq&\mathbf{P}\left\{\exists i \in N(t,g(t)) : \mathrm{Leb}(\{s \in [0,t] : |X_{i,t}(s) - \sqrt{2}s| \leq \delta s^{1/3}\}) > \alpha t\right\}\\
&+\mathbf{P}\left\{\exists i \in N(t,g(t)), s\in[\alpha t, t] : X_{i,t}(s) \geq \sqrt{2}s + \delta(\alpha t)^{1/3}\right\}\,.
\end{aligned} \tag{1}$$

Let $(B_s, s \geq 0)$ be a standard linear Brownian motion. By Markov's inequality and the a spinal change of measure ([7], equation (2.3); see also [10, 20])) the first probability on the right is at most
$$\begin{aligned}
&\mathbf{E}\left[\#\{i \in N(t,g(t)) : \mathrm{Leb}(\{s \in [0,t] : |X_{i,t}(s) - \sqrt{2}s| < s^{1/3}\}) > \alpha t\}\right]\\
=&\mathbf{E}\left[e^{-\sqrt{2}B_t}\mathbf{1}_{[\mathrm{Leb}(\{s\in[0,t]:|B_s|<\delta s^{1/3}\})>\alpha t, B_t \geq -\delta t^{1/3}\}]}\right]\\
\leq&e^{\sqrt{2}\cdot\delta t^{1/3}}\mathbf{P}\left\{\mathrm{Leb}(\{s \in [0,t] : |B_s| < \delta s^{1/3}\}) > \alpha t\right\}
\end{aligned}$$

Since $s^{1/3} \leq t^{1/3}$ for $s \leq t$ it follows by Lemma 3.2 that
$$\begin{aligned}
&\mathbf{P}\left\{\exists i \in N(t,g(t)) : \mathrm{Leb}(\{s \in [0,t] : |X_{i,t}(s) - \sqrt{2}s| \leq \delta s^{1/3}\}) > \alpha t\right\}\\
\leq&e^{\sqrt{2}\cdot\delta t^{1/3}}\mathbf{P}\left\{L_t(\delta t^{1/3}) > \alpha t\right\}\\
\leq&5e^{\sqrt{2}\cdot\delta t^{1/3} - \alpha^2 t^{1/3}/\delta^2}\\
=&5e^{-ct^{1/3}}\,,
\end{aligned}$$

where in the last line we have set $c = \sqrt{2}\delta - \alpha^2/\delta^2$, which is positive since $\alpha > 2^{1/4}\delta^{3/2}$. Next, since $\sup_{j\geq 1} X_j(s) \geq X_{i,t}(s)$, the second probability on the right of (1) is at most

$$\mathbf{P}\left\{\exists s \in [0,t] : \sup_{j\geq 1} X_j(s) \geq \sqrt{2}s + \delta(\alpha t)^{1/3}\right\} \leq \exp(-\delta(\alpha t)^{1/3}/2)\,,$$

the last inequality holding for t sufficiently large by [19], Proposition 11.[1] After decreasing c if necessary, the proof is complete. □

Proof of Proposition 3.1. Recall that we have fixed $\delta \in (0,^{-5/6})$ and that $g(s) = \sqrt{2}s - \delta s^{1/3}$. Let $c = c(\delta)$ be the constant from Lemma 3.3. Fix $m > 0$. It suffices to show that, as $t \to \infty$,

$$\mathbf{P}\{\exists s \in [ct/2, t] : d(s,m) \leq g(s) + 1\} \to 1\,.$$

So fix t large and let E be the event that $d(s,m) > g(s) + 1$ for all $s \in [ct/2, t)$. Let

$$A = \{i \in N(t, g(t)) : \mathrm{Leb}(\{s \in [0,t] : X_{i,t}(s) < g(s)\}) \leq ct\} \text{ and}$$
$$B = \{i \in N(t, g(t)) : \exists s \in [ct/2, t] : X_{i,t}(s) < 0\}\,.$$

On the event E, if $i \notin A \cup B$ then

$$\begin{aligned} M_i(t) &= \exp(-\int_0^t \zeta(s, X_i(s))\mathrm{d}s) \\ &\leq \exp(-m\mathrm{Leb}(\{s \in [ct/2, t) : X_{i,t}(s) \in (0, g(s))\}) \\ &\leq \exp(-mct/2)\,. \end{aligned}$$

Since all masses are at most one, It follows that on E,

$$\sum_{i\in N(t,g(t))} M_i(t) \leq |A \cup B| + \exp(-mct/2) \cdot |N(t, g(t))|. \tag{2}$$

By Lemma 3.3, $\mathbf{P}\{A \neq \emptyset\} = \exp(-ct^{1/3})$. Another spinal change of measure gives

$$\begin{aligned} \mathbf{P}\{B \neq \emptyset\} &\leq \mathbf{E}[|B|] \\ &\leq e^{\sqrt{2}\cdot\delta t^{1/3}}\mathbf{P}\left\{B_t \geq -\delta t^{1/3}, \exists s \in [ct/2, t] : B_s \leq -\sqrt{2}s\right\} \\ &\leq e^{\sqrt{2}\cdot\delta t^{1/3}}\mathbf{P}\left\{\inf_{s\in[0,t]} B_s \leq -ct/\sqrt{2}\right\} \\ &= e^{\sqrt{2}\cdot\delta t^{1/3}}\mathbf{P}\left\{|B_t| < -ct/\sqrt{2}\right\} \\ &\leq e^{\sqrt{2}\cdot\delta t^{1/3}} \cdot (2e^{-c^2t/4}) \\ &< e^{-ct/8} \end{aligned}$$

for t large. A final spinal tilt gives

$$\mathbf{P}\{|N(t, g(t))| > x\} \leq x^{-1}\mathbf{E}|N(t, g(t))| = x^{-1}\mathbf{E}\left[e^{-\sqrt{2}B_t}\mathbf{1}_{[B_t\geq -\delta t^{1/3}]}\right] \leq x^{-1}e^{\sqrt{2}\cdot\delta t^{1/3}}.$$

[1] In fact, this bound appears in the course of the proof of [19], Proposition 11: it is the bound on $\mathbf{P}\{\tau < t\}$ that we are using here.

Now note that for all $y \geq g(t)+1$ necessarily $\zeta(t,y) \leq \sum_{i \in N(t,g(t))} M_i(t)$. Using this, then (2), and finally the preceding probability bounds, we obtain that

$$\begin{aligned}
\mathbf{P}\{d(t,m) \geq g(t)+1, E\} &\leq \mathbf{P}\left\{\sum_{i \in N(t,g(t))} M_i(t) > m, E\right\} \\
&\leq \mathbf{P}\{|A \cup B| \neq \emptyset\} + \mathbf{P}\{|N(t,g(t))| \geq m \cdot \exp(mct/2)\} \\
&\leq e^{-ct^{1/3}} + e^{-c^2t/8} + m^{-1}\exp(\sqrt{2}\cdot\delta t^{1/3} - m \cdot ct/2) \\
&\to 0\,.
\end{aligned}$$

On the other hand, if E does not occur then there is $s \in [ct,t)$ with $d(s,m) \leq g(s)+1$. This completes the proof. □

4. The greatest overall particle density

Before moving to the lower bound, we first prove logarithmic upper bounds on how the greatest particle density grows over time; these are needed to ensure that particle masses can not *decay* too quickly. This may seem contradictory, but the point is that a particle may *a priori* quickly lose a large amount of mass if it finds itself in an extremely dense environment. The next proposition rules this out.

Proposition 4.1. *For all s sufficiently large,*

$$\mathbf{P}\{\sup\{\zeta(t,x) : 0 \leq t \leq s, x \in \mathbb{R}\} > 20\log s\} \leq s^{-4}.$$

Proving Proposition 4.1 turns out to be a fair amount of work; the reader who is willing to believe the proposition without proof – or who is impatient to see how it is used to prove the lower bound – could skip directly to Section 5.

Still here? Let's go on. There are a handful of technical estimates that are required to transform our approach from idea to proof. In order that the idea is not obscured by detail, however, we set up the heart of the argument right away.

Let $z(t,x) = \sum_{\{i:|X_i(t)-x|<1/2\}} M_i(t)$. The differences between z and ζ are that z only counts mass within distance $1/2$ of x, and does not ignore the mass of particles at x (should there be any).

Let $z(t) = \sup_x z(t,x)$, and define a sequence $(\tau_i, i \geq 0)$ of stopping times as follows. Fix s large and for the remainder of the section write $N = N(s) = 3\log s$. Let $\tau_0 = \inf\{t : z(t) \geq N-1\}$, and for $k \geq 0$ let $\tau_{k+1} = \inf\{t > \tau_k + 20/N : z(t) \geq N-1\}$. Then $\tau_k \geq 20k/N$, so with $I = I(s) = \inf\{k : \tau_k > s - 20/N\}$, we have $I \leq Ns/20$ and

$$\sup\{z(t), t \leq s\} \leq \sup\{z(t), t < \tau_I\}.$$

Notice that the sequence of stopping times "ignores" small time intervals $[\tau_k, \tau_k + 20/N]$. However in any time interval $[\tau_k + 20/N, \tau_{k+1})$, the function z nowhere exceeds N by the definition of the stopping time τ_{k+1}. We thus have

$$\sup\{z(t), t \leq s\} \leq \sup\{z(t), t < \tau_I\} \leq \max\left(N, \sup_{k<I}\sup_{t \in [\tau_k, \tau_k+20/N]} z(t)\right) \tag{3}$$

We prove the proposition by establishing the following facts. The first fact says that for $k < Ns/20$, if $z(\tau_k)$ is not too large then with high probability $z(t)$ is not too large for any $t \in [\tau_k, \tau_k + 20/N]$. The second says that for such k, with high probability $z(\tau_k + 20/N)$ is small.

Fact 4.2. *For s sufficiently large, for all $0 \leq k < Ns/20$,*

$$\mathbf{P}\{\sup\{z(t), t \in [\tau_k, \tau_k + 20/N]\} > 10N, z(\tau_k) \leq N, k < I\} < s^{-6}.$$

Fact 4.3. *For s sufficiently large, for all $0 \le k < Ns/20$,*

$$\mathbf{P}\left\{z(\tau_k + 20/N) \ge N-1, k < I\right\} < s^{-6}.$$

Assuming these two facts, the proposition follows easily.

Proof of Proposition 4.1. Fix $k \le Ns/20$. Note that if $z(\tau_{k-1}+20/N) < N-1$ then $z(\tau_k^-) < N-1$. Since mass only increases by branching, it follows that almost surely a single branching event at time τ_k causes z to increase above $N-1$. As all masses are at most 1 and branching is binary, it follows that in this case almost surely $z(\tau_k) \le z(\tau_k^-)+1 < N$. With Fact 4.3, this implies that

$$\begin{aligned}\mathbf{P}\left\{z(\tau_k) > N, k < I\right\} &\le \mathbf{P}\left\{z(\tau_{k-1}+20/N) \ge N-1, k<I\right\}\\ &\le \mathbf{P}\left\{z(\tau_{k-1}+20/N) \ge N-1, k-1<I\right\}\\ &< s^{-6}.\end{aligned}$$

We now use that for any events A, B, C we have $\mathbf{P}\left\{A \cap C\right\} \le \mathbf{P}\left\{A \cap B \cap C\right\} + \mathbf{P}\left\{B^c \cap C\right\}$. By Fact 4.2 and the preceding bound, we obtain that for $0 \le k < Ns/20$,

$$\mathbf{P}\left\{\sup\{z(t), t \in [\tau_k, \tau_k + 20/N]\} > 10N, k < I\right\} \le 2s^{-6}$$

A union bound and (3) then yield

$$\begin{aligned}\mathbf{P}\left\{\sup_{t\le s} z(t) > 10N\right\} &\le \mathbf{P}\left\{\sup_{k<I}\sup_{t\in[\tau_k,\tau_k+20/N]} z(t) > 10N\right\}\\ &\le \sum_{k=0}^{\lfloor Ns/20\rfloor} \mathbf{P}\left\{\sup_{t\in[\tau_k,\tau_k+20/N]} z(t) > 10N, k<I\right\}\\ &\le \left(1+\frac{Ns}{20}\right)\cdot 2s^{-6}\\ &< s^{-4},\end{aligned}$$

the last inequality holding for s large. Finally, it is easy to see that $\sup_x \zeta(t,x) \le 2z(t)$, so we obtain the same bound for $\mathbf{P}\left\{\sup_{t\le s}\sup_x \zeta(t,x) > 20N\right\}$, which proves the proposition. □

We now turn to the technical lemmas required for the proofs of Facts 4.2 and 4.3. The first shows that a fixed mass of particles is extremely unlikely to quickly increase its total mass. Recall the definition of $\mathbf{P}_{\mathbf{x},\mathbf{m}}$ from just before the start of Section 3.

Lemma 4.4. *Fix $\mathbf{x} = (x_1,\ldots,x_k) \in \mathbb{R}^k$ and $\mathbf{m} = (m_1,\ldots,m_k) \in [0,\infty)^k$. Under $\mathbf{P}_{\mathbf{x},\mathbf{m}}$, for $1 \le j \le k$ let $G_j(s) = \#\{i : j_{i,s}(0) = j\}$ be the number of time-s descendants of x_j. Then for any $J \subset \{1,\ldots,k\}$, any $x \ge \sum_{j\in J} m_j$, for all $t \le \log 2$ and all $\delta > 0$,*

$$\mathbf{P}_{\mathbf{x},\mathbf{m}}\left\{\sum_{j\in J} m_j G_j(t) \ge (1+\delta)x\right\} \le \left(2^{1+\delta}(1-e^{-t})^\delta\right)^x$$

Proof. We may clearly assume $J = \{1,\ldots,k\}$. Also, adding particles to increase the mass of the starting configuration can only increase the probability we aim to bound, so we may assume that $x = \sum_{i=1}^k m_i$. The random variables $(G_j(s), 1 \le j \le k)$ are iid and are $\mathrm{Geom}(e^{-s})$-distributed (see, e.g., [17]). Lemma A.2 provides upper tail bounds for weighted sums of geometric random variables where the individual coefficients are small compared with their sum. Using that lemma (with $\epsilon = 1-e^{-t}$ – this is where we require that $t < \log 2$), the result follows. □

Note that since masses decrease with time, we have

$$\sum_{j\in J} M_j(s) \le \sum_{j\in J} m_j G_j(s)\,.$$

Since $G_j(s)$ is non-decreasing in s, it follows that

$$\sup_{s\in[0,t]} \sum_{j\in J} M_j(s) \le \sup_{s\in[0,t]} \sum_{j\in J} m_j G_j(s) = \sum_{j\in J} m_j G_j(t)\,.$$

Combining this inequality with the preceding lemma thus also yields the following bound.

Corollary 4.5. *With the hypotheses and notation of Lemma 4.4,*

$$\mathbf{P}_{\mathbf{x},\mathbf{m}}\left\{\sup_{s\in[0,t]}\sum_{j\in J} M_j(s) \ge (1+\delta)N\right\} \le \left(2^{1+\delta}(1-e^{-t})^{\delta}\right)^N$$

The next proposition says that mass does not travel far in a short time, even once branching is taken into account.

Proposition 4.6. *There is $t_0 > 0$ such that the following holds. Fix $\mathbf{x} = (x_1, \ldots, x_k) \in [0,1]^k$ and $\mathbf{m} = (m_1, \ldots, m_k) \in (0,1]^k$. Then for all $x \ge \sum_{1\le i\le k} m_i$, for all $0 < t \le t_0$ and all $L \ge 1/2$, we have*

$$\mathbf{P}_{\mathbf{x},\mathbf{m}}\left\{\sum_{\{i:X_i(t)-X_{i,t}(0)>L\}} M_{i,t}(0) > \frac{x}{20L^2}\right\} \le 2e^{-5x}\,.$$

We require one lemma for the proof, which itself exploits the following, straightforwardly checked distributional identity. Fix $p, q \in (0,1)$, let $G \overset{\mathrm{d}}{=} \mathrm{Geom}(q)$, and let $G' \overset{\mathrm{d}}{=} \mathrm{Geom}(1-p(1-q)/(1-(1-p)(1-q)))$ and $B \overset{\mathrm{d}}{=} \mathrm{Ber}(p)$ be independent. Then $\mathrm{Bin}(G,p) \overset{\mathrm{d}}{=} G'-1+B$.

Lemma 4.7. *Fix $t > 0$ and $L > 0$, and let $p = \mathbf{P}\{\mathrm{N}(0,t) > L\}$, $q = e^{-t}$. Then*

$$\#\{1 \le i \le G : X_i(t) \ge L\} \preceq_{\mathrm{st}} \mathrm{Bin}(G,p) \overset{\mathrm{d}}{=} G' - 1 + B\,.$$

Proof. Let $G = \#\{i : X_i(t) \ne -\infty\}$ be the number of particles at time t, so $G \overset{\mathrm{d}}{=} \mathrm{Geom}(q)$. For $1 \le i < i' \le G$ let $\sigma_{i,i'} = \sup\{s : j_{i,t}(s) = j_{i',t}(s)\}$ be the time at which the trajectories of $X_i(t)$ and $X_{i'}(t)$ branched. Conditional on $(\sigma_{i,i'}, 1 \le i < i' \le G)$, the vector

$$(X_i(t), 1 \le i \le G),$$

containing the positions of all particles at time t, is Gaussian, with $\mathbf{E}\left[X_i^{(1)}(t)^2\right] = t$ and $\mathbf{E}\left[X_i^{(1)}(t)X_{i'}^{(1)}(t)\right] = \sigma_{i,i'}$. By Slepian's lemma [3, Theorem 13.3], it follows that for any $(y_1, \ldots, y_k) \in \mathbb{R}^k$,

$$\mathbf{P}\left\{\forall 1 \le i \le k, X_i^{(1)}(t) \le y_i \mid G = k\right\} \ge \mathbf{P}\{\forall 1 \le i \le k, N_i \le y_i\}\,,$$

where $N_1, \ldots, N_k$ are iid Normal$(0,t)$. The preceding inequality implies that given $\{G = k\}$, we have $\#\{i \le i \le k : X_i^{(1)}(t) \ge L\} \preceq_{\mathrm{st}} \mathrm{Bin}(k,p)$. Using the distributional identity from just before the lemma, the result follows. □

Proof of Proposition 4.6. Adding particles to the system at time 0 can only increase the probability we aim to bound, so we may assume that $x = \sum_{i=1}^k m_i$. Let p and q be as in Lemma 4.7. By that lemma, under $\mathbf{P}_{\mathbf{x},\mathbf{m}}$ we have

$$\sum_{\{i:X_i(t)\ge L\}} M_{i,t}(0) \preceq_{\mathrm{st}} \sum_{i=1}^k m_i(G_i - 1) + \sum_{i=1}^k m_i B_i,$$

where the G_i are $\mathrm{Geom}(1-p(1-q)/(1-(1-p)(1-q)))$ the B_i are $\mathrm{Ber}(p)$, and all random variables are independent. It follows that for any $C > 0$,

$$\mathbf{P}_{\mathbf{x},\mathbf{m}}\left\{\sum_{\{i:X_i(t)\geq L\}} M_i(t) > C\right\} \leq \mathbf{P}\left\{\sum_{i=1}^{k} m_i G_i > x + C_1\right\} + \mathbf{P}\left\{\sum_{i=1}^{k} m_i B_i > C_2\right\}. \quad (4)$$

In bounding the right-hand probabilities, we use that $p = \mathbf{P}\{\mathrm{N}(0,t) > L\} \leq e^{-L^2/2t}$ and that $1-q = 1-e^{-t} \leq t$, so

$$\frac{p(1-q)}{1-(1-p)(1-q)} \leq \frac{p(1-q)}{q} \leq \frac{te^{-L^2/2t}}{1-t} < 2te^{-L^2/2t}.$$

For the first probability on the right of (4), we now use the second bound of Lemma A.2 (the conditions of the lemma are satisfied for t small enough that $2te^{-L^2/2t} < 1/(2e^2)$, to obtain

$$\mathbf{P}\left\{\sum_{i=1}^{k} m_i G_i > x + C_1\right\} \leq 2^x (2te^{-L^2/2t})^{C_1}.$$

For the second note that $S = \sum_{i=1}^k m_i B_i$ satisfies the conditions of Bernstein's inequality, Theorem A.1, with $\mathbf{E}S = x$ and $\mathbf{Var}\{S\} = \sum_{i=1}^k m_i^2 p(1-p) < xe^{-L^2/2t}$. We conclude that

$$\mathbf{P}\left\{\sum_{i=1}^{k} m_i B_i > C_2\right\} \leq \left(\frac{xe^{1-L^2/2t}}{C_2}\right)^{C_2}.$$

Take $C_1 = C_2 = x/40L^2$. Then for t small the first of the two preceding inequalities gives

$$\mathbf{P}\left\{\sum_{i=1}^{k} m_i G_i > x + C_1\right\} < 2^x e^{-x/80t} < e^{-5x}.$$

Next, if t is small enough then for all $L \geq 1/2$ we have

$$\frac{xe^{1-L^2/2t}}{C_2} = aL^2 e^{1-L^2/2t} < e^{-L^2/4t};$$

note that this inequality only involves L and t, and not x (or $\mathbf{m}$ or $\mathbf{x}$). For t sufficiently small the inequality obtained with the Bernstein bound thus gives

$$\mathbf{P}\left\{\sum_{i=1}^{k} m_i B_i > C_2\right\} \leq e^{-x/160t} < e^{-5x},$$

and the result follows. □

The next corollary extends Proposition 4.6, considering all $t \leq t_0$ rather than a fixed $t < t_0$, at the cost of a slightly weaker bound.

Corollary 4.8. *Under the conditions of Proposition 4.6,*

$$\mathbf{P}_{\mathbf{x},\mathbf{m}}\left\{\sup_{t\leq t_0} \sum_{\{i:X_i(t)-X_{i,t}(0)\geq L\}} M_{i,t}(0) > \frac{x}{20L^2}\right\} \leq 4e^{-5x}.$$

Proof. Consider the stopping time

$$\tau = \inf\left\{t : \sum_{\{i:X_i(t)-X_{i,t}(0)\geq L\}} M_{i,t}(0) > \frac{x}{20L^2}\right\}.$$

By symmetry,

$$\mathbf{P}\left\{ \sum_{\{i:X_i(t_0)-X_{i,t_0}(0)\geq L\}} M_{i,t_0}(0) > \frac{x}{20L^2} \;\middle|\; \tau \leq t_0 \right\} \geq \frac{1}{2}\,,$$

and the corollary follows. □

The next lemma says that a large, concentrated mass will quickly decay; once we prove this we will have all the tools we need to establish Facts 4.2 and 4.3.

Lemma 4.9. *There exist $t_0 > 0$ and $C > 0$ such that the following holds. Fix $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and $\mathbf{m} = (m_1, \ldots, m_k) \in [0,1]^k$. Let $J = \{j : |x_j| < 1/4\}$, and suppose $A = \sum_{j\in J} m_j > C$. Then for all $t \in [400/A, t_0]$, setting $I = \{i : j_{i,t}(0) \in J\}$ we have*

$$\mathbf{P}_{\mathbf{x},\mathbf{m}}\left\{\sum_{i\in I} M_i(t) > A/24\right\} \leq 2e^{-200A}\,.$$

Proof. The proof is divided as follows. First, the total mass at time t of particles whose trajectory branches at least once is small. Next, among non-branching trajectories, the total mass which moves far from the origin is small. Finally, particles whose trajectories do not branch and stay near the origin will lose a large amount of mass since they are a dense environment. We now formalize this.

Write $I_b = \{i \in I : \exists i' \neq i, j_{i,t}(0) = j_{i',t}(0)\}$ for the indices of particles starting near (distance $< 1/4$) to the origin whose trajectories branch before time t. Then let $I \setminus I_b = I_f \cup I_n$, where

$$I_f = \left\{ i \in I \setminus I_b : |X_{i,t}(0)| < 1/4\}, \sup_{s\in[0,t]} |X_{i,t}(s)| > 1/2 \right\}$$

indexes non-branching trajectories that start near the origin but moves far (distance $> 1/2$) from the origin before time t, and where $I_n = I\setminus(I_f \cup I_b)$ indexes non-branching trajectories that stay near the origin. Then with $M_b = \sum_{i\in I_b} M_t(i)$ and M_f, M_n defined accordingly, we have

$$\sum_{i\in I} M_i(t) = M_b + M_f + M_n.$$

We begin by considering branching trajectories. For each $1 \leq j \leq k$, let $G_j = \#\{i \in I : j_{i,t}(0) = j\}$. Then $i \in I_b$ precisely if $j_{i,t}(0) \in J$ and $G_{j_{i,t}(0)} > 1$. Since masses decrease with time,

$$\begin{aligned}\sum_{i\in I_b} M_i(t) &\leq \sum_{i\in I_b} m_{j_{i,t}}(0)\\ &= \sum_{j\in J} m_j G_j \mathbf{1}_{[G_j>1]}.\end{aligned}$$

Next, since the G_j are integer-valued,

$$\sum_{j\in J} m_j G_j \mathbf{1}_{[G_j>1]} = \sum_{j\in J} m_j(G_j-1) + \sum_{j\in J} m_j \mathbf{1}_{[G_j>1]} < 2\sum_{j\in J} m_j(G_j-1),$$

which with the preceding bound gives

$$\sum_{i\in I_b} M_i(t) \leq 2(\sum_{j\in J} m_j G_j - A).$$

By Lemma 4.4, it follows that for any fixed $\delta > 0$,

$$\begin{aligned}\mathbf{P}\left\{\sum_{i\in I_b} M_i(t) > 2\delta A\right\} &\le \mathbf{P}\left\{\sum_{j\in J} m_j G_j \ge (1+\delta)A\right\} \\ &\le (2^{1+\delta}(1-e^{-t})^{\delta})^A \\ &< (2^{1+\delta}t^{\delta})^A \\ &\le e^{-200A}, \end{aligned} \tag{5}$$

the last bound holding for t sufficiently small that $2^{1+\delta}t^{\delta} < e^{-200}$. We next bound $\sum_{i\in I_n} M_i(t)$, the total final mass from "typical" trajectories, which do not branch and do not move far from their starting position by time t. Fix $c \in (0,1)$ and let E be the event that for all $s \in [0,t]$, $\sum_{\{i:|X_i(s)|<1/2\}} M_i(s) > cA$. On E, if $i \in I_n$ has $j_{i,t}(0) = j$ then $M_i(t) \le m_j \cdot e^{-tcA}$. We thus have

$$\sum_{i\in I_n} M_i(t)\mathbf{1}_{[E]} \le \sum_{j\in J} m_j \cdot e^{-tcA} \cdot \mathbf{1}_{[E]} = Ae^{-tcA}\mathbf{1}_{[E]}\,.$$

Next, let $I_n(s) = \{j_{i,t}(s) : i \in I_n\}$ be the indices of time-s ancestors of individuals in I_n. Since trajectories indexed by I_n do not branch, $\sum_{i\in I_n(s)} M_i(s)$ is decreasing for $s \in [0,t]$. Necessarily $|X_i(s)| < 1/2$ for $i \in I_n(s)$, so if E^c occurs then there is $s \in [0,t]$ such that $\sum_{i\in I_n(s)} M_i(s) \le cA$. We thus have

$$\sum_{i\in I_n} M_i(t)\mathbf{1}_{[E^c]} \le cA\mathbf{1}_{[E^c]}\,.$$

and the two preceding bounds together give

$$\sum_{i\in I_n} M_i(t) \le \max\left(cA, Ae^{-tcA}\right)\,. \tag{6}$$

Finally, we turn to the final mass of non-branching trajectories that move far from the origin, counted by $\sum_{i\in I_f} M_i(t)$. For any $i \in I$, If $j_{i,t}(0) = j$ and $|x_j| < 1/4$ then in order to have $\sup_{s\in[0,t]} |X_{i,t}(s)| > 1/2$ the trajectory leading to $X_i(t)$ wanders a distance of at least $1/4$ from its starting position. Let W denote one-dimensional Brownian motion started from the origin. By the reflection principle and the fact that $\mathbf{P}\{N > x\} \le e^{-x^2/2}/2$ for N a standard normal and for all $x > 0$, we have

$$\mathbf{P}\left\{\sup_{s\le t} |W_s| > 1/4\right\} \le 4\mathbf{P}\left\{W_t > \frac{1}{4}\right\} \le 2\exp(-1/(32t))\,.$$

Since an individual trajectory of X has the law of Brownian motion, for a particle starting at distance less than $1/4$ from the origin whose trajectory never branched, the above is a bound on the probability the trajectory attained distance $1/2$ from the origin. It follows that

$$\sum_{i\in I_f} M_i(t) \preceq_{\mathrm{st}} \sum_{j\in J} m_j \cdot \xi_j\,,$$

where the terms ξ_j are iid $\mathrm{Ber}(2\exp(-1/(32t)))$. The variance of the latter sum is bounded by $A \cdot 2\exp(-1/(32t))$, so Theorem A.1 yields that for any fixed $b > 0$,

$$\mathbf{P}\left\{\sum_{i\in I_f} M_i(t) > (b + 2\exp(-1/(32t)))A\right\} \le \left(\frac{2e^{1-1/(32t)}}{b}\right)^{bA} < e^{-200A}\,, \tag{7}$$

the final inequality for t sufficiently small.

We now combine (5), (6) and (7). This yields that for t sufficiently small, and in particular provided that $2^{1+\delta}t^\delta < e^{-2}$ and that

$$2\delta + \max(c, e^{-tcA}) + b + 2\exp(-1/(32t)) < \frac{1}{24}$$

we have

$$\mathbf{P}\left\{\sum_{i\in I} M_i \geq A/24\right\} \leq 2e^{-200A},$$

It can be checked that taking $\delta = b = c = 1/100$ does the job when $t > 100\log 100/A$ (so that $\max(c, e^{-tcA}) = 1/100$) and t is sufficiently small (it is in order to satisfy these simultaneously that we require a lower bound on A). This completes the proof. ☐

Proof of Fact 4.2. Let $\mathbb{Z}/2 = \{y/2 : y \in \mathbb{Z}\}$. Define the event

$$E = \{\max\{|X_i(r)|, i \geq 1, 0 \leq r \leq s\} \leq 3s\}.$$

Any unit interval $[x-1/2, x+1/2]$ is covered by at most two intervals from $\{[y-1/2, y+1/2] : y \in \mathbb{Z}/2\}$. It follows that on E, if $\tau_k + 20/N < s$ but $\sup\{z(t), t \in [\tau_k, \tau_k + 20/N]\} > 10N$ then there is y with $y \in [-3s, 3s] \cap \mathbb{Z}/2$ such that

$$\sup_{t\in[\tau_k,\tau_k+20/N]} \sum_{\{i:|X_i(t)-y|<1/2\}} M_i(t) > 5N.$$

When $k < I$ we have $\tau_k + 20/N < s$, so

$$\begin{aligned}&\mathbf{P}\left\{\sup_{t\in[\tau_k,\tau_k+20/N]} z(t) > 10N, z(\tau_k) \leq N, k < I\right\}\\ \leq&\mathbf{P}\{E^c\} + \sum_{y\in[-3s,3s]\cap\mathbb{Z}/2} \mathbf{P}\left\{\sup_{t\in[\tau_k,\tau_k+20/N]} z(t,y) > 5N, z(\tau_k) \leq N, k < I, E\right\}\end{aligned} \tag{8}$$

Our bound on the above summands works identically for each $y \in [-3s, 3s] \cap \mathbb{Z}/2$; we explain it for $y = 0$ to avoid notational overload. So we wish to bound

$$\mathbf{P}\left\{\sup_{t\in[\tau_k,\tau_k+20/N]} z(t,0) > 5N, z(\tau_k) \leq N, k < I, E\right\}.$$

Our strategy is as follows: we use Corollary 4.5 to show that with high probability, for all $t \in [\tau_k, \tau_k + 20/N]$ the total contribution to $z(t,0)$ from descendants of particles with $|X_i(\tau_k)| \leq 3/2$ is at most $4N$. We then use Proposition 4.6 to show that with high probability the contribution to $z(t,0)$ from descendants of further-off particles decreases quadratically (as a function of $|X_i(\tau_k)|$); since the quadratic series converges, this implies a bound on the total contribution from far-off particles. We now proceed to details.

For $n \in \mathbb{Z}$ let

$$Y_n = \sup_{t\in[\tau_k,\tau_k+20/N]} \sum_{\{i:|X_i(t)|\leq 1/2, |X_{i,t}(\tau_k)-n|\leq 1/2\}} M_i(t);$$

Y_n counts the greatest contribution at any time $t \in [\tau_k, \tau_k + 20/N]$, to the mass near 0 from particles that at time τ_k are near n. We clearly have

$$\sup_{t\in[\tau_k,\tau_k+20/N]} z(t,0) \leq \sum_{n\in\mathbb{Z}} Y_n.$$

Furthermore, on $\{k < I\}$ we have $\tau_k + 20/N \leq s$, so on $E \cap \{k < I\}$ we have $Y_n = 0$ whenever $|n| > 3s$. It follows that, on $E \cap \{k < I\}$, we have the strengthened bound

$$\sup_{t\in[\tau_k,\tau_k+20/N]} z(t,0) \leq \sum_{n\in\mathbb{Z}\cap[-3s,3s]} Y_n. \tag{9}$$

As sketched above, we bound the sum in two parts: the contribution from Y_{-1}, Y_0 and Y_1 is handled separately from the rest, and we do this first. Note that

$$Y_{-1} + Y_0 + Y_1 \le \sup_{t \in [\tau_k, \tau_k + 20/N]} \sum_{\{i : |X_{i,t}(\tau_k)| \le 3/2\}} M_i(t).$$

If $z(\tau_k) \le N$ then $\sum_{\{i:|X_i(\tau_k)| \le 3/2\}} M_i(\tau_k) \le 3N$; so by Corollary 4.5 and the strong Markov property,

$$\begin{aligned}\mathbf{P}\left\{Y_{-1} + Y_0 + Y_1 > 4N, z(\tau_k) \le N\right\} &\le \left(2^{1+1/3}(1 - e^{-20/N})^{1/3}\right)^{3N} \\ &\le (10/N)^N .\end{aligned}$$

Now consider $n \in \mathbb{Z}$ with $|n| \ge 2$, and assume by symmetry that $n > 0$. If $|X_i(t)| \le 1/2$ but $|X_{i,t}(\tau_k) - n| \le 1/2$ then $X_i(t) - X_{i,t}(\tau_k) \ge n - 1$. Assuming $z(\tau_k) \le N$, in particular we have $z(\tau_k, -n) \le N$. Furthermore,

$$Y_{-n} \le \sup_{t \in [\tau_k, \tau_k + 20/N]} \sum_{\{i : X_i(t) - X_{i,t}(\tau_k) > n-1\}} M_i(t) ,$$

so by Corollary 4.8, for $n \ge 2$,

$$\mathbf{P}\left\{Y_{-n} > \frac{N}{20(n-1)^2}, z(\tau_k) \le N\right\} \le 2e^{-5N} .$$

The same bound holds for Y_n by symmetry.

Using (9) and the two preceding probability bounds (and the fact that $(1/20)\sum_{|n|\ge 2}(n-1)^{-2} = \pi^2/60 < 1$), we thus have

$$\begin{aligned}&\mathbf{P}\left\{\sup_{t \in [\tau_k, \tau_k + 20/N]} z(t, 0) > 5N, z(\tau_k) \le N, k < I, E\right\} \\ \le &\mathbf{P}\left\{Y_{-1} + Y_0 + Y_1 > 4N, z(\tau_k) \le N\right\} + \sum_{\{n \in \mathbb{Z} \cap [-3s, 3s] : |n| \ge 2\}} \mathbf{P}\left\{Y_n \ge \frac{N}{20(n-1)^2}, z(\tau_k) \le N\right\} \\ \le &\left(\frac{10}{N}\right)^N + 12se^{-5N} . \qquad (10)\end{aligned}$$

The same argument yields the same bound with $z(t, y)$ in place of $z(t, 0)$, and (8) then gives

$$\begin{aligned}&\mathbf{P}\left\{\sup_{t \in [\tau_k, \tau_k + 20/N]} z(t) > 10N, z(\tau_k) \le N, k < I\right\} \\ \le &\mathbf{P}\left\{E^c\right\} + 12s \cdot \left(\left(\frac{10}{N}\right)^N + 12se^{-5N}\right) \\ \le &\mathbf{P}\left\{E^c\right\} + s^{-7} ,\end{aligned}$$

the latter bound holding for s large since $N = N(s) = 3\log s$. To conclude, we use the fact that

$$\mathbf{P}\left\{\max\{|X_i(s)|, i \ge 1\} \ge 3s \mid E^c\right\} \ge \frac{1}{2},$$

which follows by considering the stopping time $\tau = \inf\{s : \max\{|X_i(s)|, i \geq 1\} \geq 3s\}$ and using symmetry. This yields

$$\begin{aligned} \mathbf{P}\{E^c\} &\leq 2\mathbf{P}\{\max\{|X_i(s)|, i \geq 1\} \geq 3s\} \\ &\leq 4\mathbf{E}\left[\#\{i : X_i(s) \geq 3s\}\right] \\ &= 4e^s \mathbf{P}\{N(0,s) \geq 3s\} \\ &\leq e^{-8s} \qquad (11) \\ &< s^{-7}. \qquad \square \end{aligned}$$

Proof of Fact 4.3. The proof has aspects which will be familiar from the previous proof; we describe these first. We recycle the event E from the preceding proof. Note that on $E \cap \{k < I\}$ we have

$$z(\tau_k + 20/N) \leq 2 \sup_{y \in [-3s,3s] \cap \mathbb{Z}/2} z(\tau_k + 20/N, y)\,,$$

so

$$\begin{aligned} &\mathbf{P}\{z(\tau_k + 20/N) \geq N-1, z(\tau_k) \leq N, k < I, E\} \\ \leq &\sum_{y \in [-3s,3s] \cap \mathbb{Z}/2} \mathbf{P}\left\{z(\tau_k + 20/N, y) > \frac{N-1}{2}, z(\tau_k) \leq N, k < I, E\right\}. \qquad (12) \end{aligned}$$

We once again focus on the case $y = 0$ for notational simplicity. We write

$$Z_n = \sum_{\{i : |X_i(\tau_k + 20/N)| < 1/2, |X_{i,\tau_k + 20/N}(\tau_k) - n| < 1/2\}} M_i(\tau_k + 20/N)\,.$$

The indices of summation correspond to particles with position near 0 at time $\tau_k + 20/N$, whose time τ_k ancestor had position near n. On $E \cap \{k < I\}$ we have

$$z(\tau_k + 20/N) \leq \sum_{n \in \mathbb{Z} \cap [-3s,3s]} Z_n.$$

Now reprise the argument leading to (10); the argument actually is slightly easier here since the variables Z_n consider only one time, $\tau_k + 20/N$, whereas the variables Y_n involved a time supremum. We obtain

$$\begin{aligned} &\mathbf{P}\left\{z(\tau_k + 20/N, 0) \geq \frac{N-1}{2}, z(\tau_k) \leq N, k < I, E\right\} \\ \leq &\mathbf{P}\left\{Z_{-1} + Z_0 + Z_1 \geq \frac{N}{4}, z(\tau_k) \leq N\right\} + \sum_{\{n \in \mathbb{Z} \cap [-3s,3s] : |n| \geq 2\}} \mathbf{P}\left\{Z_n \geq \frac{N}{20(n-1)^2}, z(\tau_k) \leq N\right\} \\ \leq &\mathbf{P}\left\{Z_{-1} + Z_0 + Z_1 \geq \frac{N}{4}, z(\tau_k) \leq N\right\} + 12se^{-5N}. \qquad (13) \end{aligned}$$

We now bound $Z_{-1} + Z_0 + Z_1$ from above by the *total mass at time* $\tau_k + 20/N$ *of individuals whose time-*τ_k *ancestor lies in* $[-3/2, 3/2]$. More precisely, recall that $X_{i,t}(s)$ is the (location of) the time-s ancestor of $X_i(t)$, and write

$$D_\ell = \sum_{\{i : X_{i,\tau_k + 20/N}(\tau_k) \in [\ell/2, (\ell+1)/2]\}} M_i(\tau_k + 20/N).$$

Then

$$Z_{-1} + Z_0 + Z_1 \leq \sum_{\ell \in [-3,2] \cap \mathbb{Z}} D_\ell\,.$$

This holds because the time-τ_k ancestors of particles counted by $Z_{-1} + Z_0 + Z_1$ all lie in $[-3/2, 3/2] = \bigcup_{\ell \in [-3,2] \cap \mathbb{Z}} [\ell/2, (\ell+1)/2]$. The bound may be strict because particles counted by $Z_{-1} + Z_0 + Z_1$ are additionally required to lie near 0 at time $\tau_k + 20/N$.

Bounding each of the summands D_ℓ by the largest summand, we then have

$$Z_{-1}+Z_0+Z_1 \le 6 \max_{\ell\in[-3,2]\cap\mathbb{Z}} D_\ell\,,$$

so

$$\begin{aligned}&\mathbf{P}\left\{Z_{-1}+Z_0+Z_1 \ge \frac{N}{4}, z(\tau_k)\le N\right\}\\ \le& 6\max_{\ell\in[-3,2]\cap\mathbb{Z}} \mathbf{P}\left\{D_\ell > \frac{N}{24}, z(\tau_k)\le N\right\}\end{aligned}$$

The final probabilities are not hard to bound: if D_ℓ hearkens from a total time-τ_k mass which is very small then at time τ_k+20/N it is still rather small by Corollary 4.5. On the other hand, if the aggregate mass of its time-τ_k ancestors was larger (but still at most N) then by Lemma 4.9, at time τ_k+20/N that ancestral population has lost most of its mass.

More precisely, by Corollary 4.5 and the strong Markov property,

$$\mathbf{P}\left\{D_\ell > \frac{N}{24}\;\middle|\; \sum_{\{j:X_j(\tau_k)\in[\ell/2,(\ell+1)/2]\}} M_j(\tau_k)\le N/48\right\} \le (4(1-e^{-20/N}))^{N/48} \le \left(\frac{80}{N}\right)^{N/48}.$$

Now assume $N=N(s)=3\log s > 48C$, where C is the constant from Lemma 4.9. By that lemma,

$$\mathbf{P}\left\{D_\ell > \frac{N}{24}\;\middle|\; \sum_{\{j:X_j(\tau_k)\in[\ell/2,(\ell+1)/2]\}} M_j(\tau_k)\in[N/48,N]\right\} \le 2e^{-200N/48} < e^{-4N},$$

the latter inequality for $N=N(s)$ sufficiently large. This bound holds for each $\ell\in[-3,2]\cap\mathbb{Z}$. Under the assumption that $z(\tau_k)\le N$, one of the conditions in the above conditional probabilities must occur. It follows that

$$6\max_{\ell\in[-3,2]\cap\mathbb{Z}} \mathbf{P}\left\{D_\ell > \frac{N}{24}, z(\tau_k)\le N\right\} \le 6\max\left(\left(\frac{80}{N}\right)^{N/48}, e^{-4N}\right),$$

so

$$\mathbf{P}\left\{Z_{-1}+Z_0+Z_1 \ge \frac{N}{4}, z(\tau_k)\le N\right\} \le e^{-4N}.$$

Combined with (13) this gives

$$\mathbf{P}\left\{z(\tau_k+20/N,0)\ge \frac{N-1}{2}, z(\tau_k)\le N, k<I\right\}\le 13se^{-4N}.$$

The same bound holds for each $z(\tau_k+20/N,y)$, so using (12) and the bound $\mathbf{P}\{E^c\}\le e^{-8s}$ from (11), we obtain

$$\mathbf{P}\{z(\tau_k+20/N)\ge N-1, z(\tau_k)\le N, k<I\} \le 156s^2e^{-4N}+e^{-8s}\le 160s^2e^{-4N}.$$

The proof is almost complete; to finish it off we need to deal with the event $\{z(\tau_k)\le N\}$ in the preceding probability. To do so we use induction. First, for s large, since $N=N(s)=3\log s$ and since $\tau_0=1$ we always have $z(\tau_0)\le N$, so when $k=0$ we have

$$\mathbf{P}\{z(\tau_k+20/N)\ge N-1, k<I\} = \mathbf{P}\{z(\tau_k+20/N)\ge N-1, z(\tau_k)\le N, k<I\}\le 160s^2e^{-4N},$$

For larger k, recall that if $z(\tau_{k-1}+20/N)\le N-1$ then $z(\tau_k)\le z(\tau_k^-)+1\le N$ (this was explained in the proof of Proposition 4.1). We thus have

$$\begin{aligned}\mathbf{P}\{z(\tau_k+20/N)\ge N-1, k<I\} &\le \mathbf{P}\{z(\tau_k+20/N)\ge N-1, z(\tau_k)\le N, k<I\}\\ &\quad+\mathbf{P}\{z(\tau_k)>N, k<I\}\\ &\le 160s^2e^{-4N}+\mathbf{P}\{z(\tau_{k-1})\ge N-1, k-1<I\},\end{aligned}$$

so by induction and the hypothesis that $k \le Ns/20$,

$$\mathbf{P}\left\{z(\tau_k + 20/N) \ge N-1, k < I\right\} \le k \cdot 160 s^2 e^{-4N} < 8s^3 N e^{-4N} < s^{-6}. \qquad \square$$

5. Lower bound

The next proposition strengthens the second inequality of Theorem 1.1. Write $c^* = 3^{4/3}\pi^{2/3}/2^{7/6}$.

Proposition 5.1. *For any $m \in (0,1)$, almost surely*

$$\limsup_{t\to\infty} \frac{\sqrt{2}t - D(t,m))}{t^{1/3}} \le c^*.$$

Given a function $f : [0,\infty) \to \mathbb{R}$, for $t \ge 0$ let $I(t,f) = \{i \ge 1 : \forall s \in [0,t], X_{i,t}(s) \ge f(s)\}$ be the indices of particles whose ancestral trajectory stays above f up to time t. Note that $|I(t,f)|$ is decreasing in t: if a trajectory stays above f to time t then it also stays above f to time $t' < t$. It follows that $\mathbf{P}\left\{\forall t, I(t,g) \ne \emptyset\right\} = \lim_{t\to\infty} \mathbf{P}\left\{I(t,g) \ne \emptyset\right\}$, and this is a decreasing limit. We will use the following result of Roberts [19].

Lemma 5.2 ([19], Theorem 1)**.** *Write , and let $g(t) = \sqrt{2}t - c^* t^{1/3} + c^* t^{1/3}/\log^2(t+e) - 1$. Then*

$$\lim_{t\to\infty} \mathbf{P}\left\{I(t,g) \ne \emptyset\right\} = p^* > 0$$

The idea of the proof of Proposition 5.1 is if the density is always low beyond g then a particle staying beyond g will have reasonably large mass at time t; the lemma guarantees that such a particle has a reasonable chance p^* of existing. The next corollary implies that at the cost of a constant shift of the function g, we may increase p^* as close to one as we like. For $c \in \mathbb{R}$ write $g - c$ for the function with $(g-c)(x) = g(x) - c$.

Corollary 5.3. *Let $C^* = \inf\{c : \forall t, I(t, g-c) \ne \emptyset\}$. Then almost surely $C^* < \infty$.*

Proof. The proof technique is sometimes called an amplification argument. Consider the $N(t) \approx e^t$ independent copies of the BBM rooted at time-t particles, the i'th copy having initial individual at position $X_i(t)$. Suppose the "translate by $X_i(t)$" of the event from Lemma 5.2 occurs in the k'th copy; more precisely, suppose that for all $t' \ge t$ there is a descendant $X_j(t')$ of $X_k(t)$ such that for all $s \in [t,t']$,

$$X_{j,t'}(s) - X_k(t) \ge g(t'-t).$$

For $s \le t$ we also have

$$X_{j,t'}(s) \ge \inf_{i\ge1} X_i(s) \ge \inf_{s\le t}\inf_{i\ge1} X_i(s) \ge g(s) + \inf_{s\le t}\inf_{i\ge1} X_i(s) - \sup_{s\le t} g(s)\,,$$

so in this case

$$C^* \le -\inf_{s\in[0,t]}\inf_{i\ge1} X_i(s) + \sup_{s\le t} g(s).$$

By the branching property (i.e. the independence of the trajectories emanating from each of the particles $(X_i(t), i \ge 1)$), it follows that

$$\mathbf{P}\left\{C^* \le 3t + \sup_{s\in[0,t]} g(s)\right\} \le \mathbf{P}\left\{N(t) \le 2^t\right\} + \mathbf{P}\left\{\inf_{s\in[0,t]}\inf_{i\ge1} X_i(s) \le -3t\right\} + (1-p^*)^{2^t}, \tag{14}$$

where p^* is the constant from Lemma 5.2. Since $N(t)$ is $\mathrm{Geom}(e^{-t})$ we have $\mathbf{P}\left\{N(t) \le 2^t\right\} \le (2/e)^t$. Finally, let $\sigma = \inf\{\tau : \inf_{i\ge1} X_i(s) \le -3t\}$, so $\inf_{s\in[0,t]}\inf_{i\ge1} X_i(s) \le -3t$ if and only if $\sigma < t$. Considering the descendants of the first individual to reach position $-3t$, by symmetry we have

$$\mathbf{P}\left\{\inf_{i\ge1} X_i(t) \le -3t \,\middle|\, \sigma < t\right\} \ge \frac{1}{2}\,,$$

so

$$\mathbf{P}\left\{\sigma < t\right\} \le 2\mathbf{P}\left\{\inf_{i\ge 1} X_i(t) \le -3t\right\} \le 2e^t\mathbf{P}\left\{N(0,t) \le -3t\right\} \le e^{-8t}.$$

These bounds and (14) then yield

$$\mathbf{P}\left\{C^* \le 3t + \sup_{s\in[0,t]} g(s)\right\} \le (2/e)^t + e^{-8t} + (1-p^*)^{2^t}.$$

This can be made arbitrarily small by taking t large. □

In order to prove Proposition 5.1, we require one final lemma which shows that a small mass will quickly increase to form some region of constant density within a constant distance.

Lemma 5.4. *For all $\epsilon > 0$ and $m \in (0,1)$ there is $C > 0$ such that for all $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{m} \in (0,1]^k$, if $z := \sum_{\{i:|x_i|<1\}} > 0$ then*

$$\mathbf{P}_{\mathbf{x},\mathbf{m}}\left\{\exists t \in [0, C(1+\log(1/z))], x \in [-C\log(1/z, C\log(1/z)] : \zeta(t,x) \ge m\right\} \ge 1-\epsilon. \quad (15)$$

To prove the lemma we use the following fact, whose proof is a pleasant exercise and is left to the reader.

Fact 5.5. *For all $\epsilon > 0$, there is there is $t_0 = t_0(\epsilon)$ such that if $c > 0$ is chosen large enough then*

$$\mathbf{P}\left\{\forall t \ge t_0, \#\{i : \forall s \in [0,t], |X_{i,t}(s)| < c\} \ge (e-\epsilon)^t\right\} > 1-\epsilon. \quad (16)$$

One way to prove the fact is to first show that $p(c)$, the survival probability of branching Brownian motion with absorbing boundaries at $-c$ and c, started from the origin, satisfies $p(c) \to 1$ as $c \to \infty$; then use a suitable branching approximation. As an aside, we note the very nice recent work [11] on the asymptotics of this survival probability for c near the critical width $\hat{c}$ below which $p(c) = 0$.

Proof of Lemma 5.4. The claim is clearly true if $z \ge m$, and we hereafter assume $z \in (0,m)$. We also assume ϵ is small enough that $(e-\epsilon)e^{-m}(1-\epsilon^{1/2}) > (1+\epsilon)$; this can only make our job harder.

By relabelling, we may assume that for some $1 \le k' \le k$ we have $|x_i| < 1$ for $1 \le i \le k'$ and $|x_i| > 1$ for $i > k'$. We also assume $x_1, \ldots, x_k$ are ordered so that $(m_i, 1 \le i \le k')$ is decreasing.

For $1 \le i \le k'$ let $J_i(t)$ index the time-t descendants of x_i whose trajectory stays fairly near the origin, i.e.,

$$J_i(t) = \{\ell \ge 1 : j_{\ell,t}(0) = i, |X_{\ell,t}(s)| < c \; \forall s \in [0,t]\},$$

where c is chosen as in Fact 5.5. By that fact, we then have

$$\mathbf{P}\left\{\forall t \ge t_0, |J_i(t)| \ge (e-\epsilon)^t\right\} > 1-\epsilon.$$

For $1 \le n \le k'$ write

$$S_n = \#\{1 \le i \le n : \forall t \ge t_0, |J_i(t)| \ge (e-\epsilon)^t\}.$$

Then $S = (S_n)_{1\le n\le k'}$ stochastically dominates a random walk with Bernoulli$(1-\epsilon)$ steps. It follows by a ballot-type theorem ([13, Corollary 11.17], for example, is sufficient for our needs) that for any $A > 1$,

$$\mathbf{P}\left\{\exists n \le k' : S_n < (1-A\epsilon)n\right\} < A^{-1}. \quad (17)$$

We hereafter assume $t \geq t_0(\epsilon)$. Now suppose that $\zeta(s,x) < m$ for all $s \leq t$ and $|x| \leq c+1$. Then for each $1 \leq i \leq k$, for all $j \in J_i(t)$, $M_j(t) \geq m_i \cdot e^{-mt}$. Since the masses m_i are decreasing in $i \in \{1,2,\ldots,k'\}$, if $S_n > (1-\epsilon^{1/2})n$ for all $n \leq k'$ then it follows that

$$\sum_{1\leq i\leq k'}\sum_{j\in J_i(t)} M_j(t) \geq e^{-mt}(1-\epsilon^{1/2})\sum_{i=1}^{k'} m_i(e-\epsilon)^t = e^{-mt}(1-\epsilon^{1/2})\cdot z(e-\epsilon)^t.$$

Using that $(e-\epsilon)e^{-m}(1-\epsilon^{1/2}) > (1+\epsilon)$ this gives

$$\sum_{j:|X_j(t)|<c} \geq (1+2\epsilon)^{t-1}z > c\,,$$

the last inequality provided that $t \geq 1+\log_{1+\epsilon}(c/z)$. Since $[-c,c]$ can be covered by c intervals of radius 1, we see that in this case there is x with $|x| \leq c$ such that $\zeta(t,x) > 1$.

To sum up: assuming the random walk S behaves, and that $t \geq t_0(\epsilon)$ and $t \geq 1 + \log_{1+\epsilon}(c/z)$, either $\zeta(s,x) \geq m$ for some $s \leq t$ and $|x| \leq c+1$, or else $\zeta(t,x) > 1$ for some x with $|x| \leq c$. By taking $A = \epsilon^{-1/2}$ in (17) and choosing $C = C(\epsilon)$ appropriately, we obtain

$$\mathbf{P}_{\mathbf{x},\mathbf{m}}\left\{\exists s \in [0, C(1+\log(1/z)], x \in [-C,C] : \zeta(t,x) \geq m\right\} \geq 1-\epsilon^{1/2}. \qquad \square$$

We are now ready for the final proof of the paper.

Proof of Proposition 5.1. Fix $m \in (0,1)$. Let

$$t^* = \inf\{r \geq 0 : \forall t \geq r, \sup_{s\in[0,t]}\sup_{x\in\mathbb{R}}\zeta(s,x) \leq 20\log t\},$$

and note that $t^* < \infty$ almost surely by Proposition 4.1 and the first Borel-Cantelli lemma.

Fix $\epsilon > 0$ and choose $L > 1$ large enough that $\mathbf{P}\left\{\max(C^*,t^*) \geq \ell\right\} < \epsilon$. Fix t much larger than L (so that $\log\log t > L$, say).

Let $\sigma = \inf\{s \geq t^{1/4} : D(s,1/t) \geq g(s) - C^* - 1\}$. We first suppose that $\sigma > t$, so that for all $s \in [t^{1/4}, t]$ we have $D(s,1/t) < g(s) - C^* - 1$. Let i^* be such that $X_{i^*,t}(s) \geq g(t) - C^*$ for all $s \in [0,t]$; such i^* exists by the definition of C^*. If $t \geq t^*$ then we have

$$\begin{aligned}
-\log M_{i^*}(t) &= \int_0^t \zeta(s, X_{i^*,t}(s))\mathrm{d}s \\
&= \int_0^{t^{1/4}} \zeta(s, X_{i^*,t}(s))\mathrm{d}s + \int_{t^{1/4}}^t \frac{1}{t}\mathrm{d}s \\
&\leq 20t^{1/4}\log t + 1\,,
\end{aligned}$$

the last bound because when $t \geq t^*$ the integrand is at most $20\log t$.

Let $C = C(\epsilon, m)$ be the constant from Lemma 5.4. Then by that lemma (applied with $z = M_{i^*}(t) \geq \exp(-1 - 20t^{1/4}\log t)$) and the Markov property, given that $\{t^* \leq t\}$, with probability at least $1-\epsilon$ there is $s \in (t, t + C(2 + 20t^{1/4}\log t))$ and x with $|x| \leq C$ such that $\zeta(s, X_{i^*}(t) + x) \geq m$. If this occurs, and additionally $C^* \leq L$ we have

$$D(s,m) \geq X_{i^*}(t) - C \geq g(t) - C^* - C \geq g(s) - s^{1/4}\log^2 s\,,$$

the last bound holding for all t sufficiently large since $s-t \le C(2+20t^{1/4}\log t)$, and for s and t large we have $g(s)-g(t)=O(s-t)$. We thus have

$$\begin{aligned}
&\mathbf{P}\left\{\exists s\ge t : D(s,m)\ge g(s)-s^{1/4}\log^2 s \mid \sigma>t\right\}\\
\ge&\mathbf{P}\left\{\max(C^*,t^*)<L,\ \exists s\ge t : D(s,m)\ge g(s)-s^{1/4}\log^2 s \mid \sigma>t\right\}\\
&-\mathbf{P}\left\{\max(C^*,t^*)\ge L \mid \sigma>t\right\}\\
\ge&1-\epsilon-\mathbf{P}\left\{\max(C^*,t^*)\ge L \mid \sigma>t\right\}. \qquad (18)
\end{aligned}$$

Next suppose that $\sigma\le t$. Apply the strong Markov property at time σ, and by applying Lemma 5.4 just as above (but with a starting mass of at least $1/t=e^{-\log t}$ rather than $e^{-1-20\log^2 t}$). We obtain that with probability at least $1-\epsilon$ there is $s\in(\sigma,\sigma+C(1+\log t))$ such that

$$D(s,m)\ge X_{i^*}(\sigma)-C\ge g(\sigma)-C-C^*\ge g(s)-\log^2 s\,,$$

the last bound holding $s-\sigma\le C(1+\log t)$ and $\log t\le 4\log\sigma\le 4\log s$, and under the assumption $C^*\le L$.

Since $\sigma\ge t^{1/4}$ and $\log^2 s<s^{1/4}\log^2 s$, It follows that

$$\begin{aligned}
&\mathbf{P}\left\{\exists s\ge t^{1/4} : D(s,m)\ge g(s)-s^{1/4}\log^2 s \mid \sigma\le t\right\}\\
\ge&1-\epsilon-\mathbf{P}\left\{\max(C^*,t^*)\ge L \mid \sigma\le t\right\}.
\end{aligned}$$

Now combine this with (18) using the law of total probability. We chose L large enough that $\mathbf{P}\left\{\max(C^*,t^*)\ge L \mid \le\right\}\epsilon$, so obtain

$$\mathbf{P}\left\{\exists s\ge t^{1/4} : D(s,m)\ge g(s)-s^{1/4}\log^2 s\right\}>1-2\epsilon\,.$$

Finally, if $D(s,m)\ge g(s)-s^{1/4}\log^2 s$ then

$$\frac{\sqrt{2}s-D(s,m)}{s^{1/3}}\le c^*-\frac{c^*}{\log^2(s+e)}+\frac{1}{s^{1/3}}+\frac{\log^s 2}{s^{1/12}}\,,$$

which tends to c^* as $s\to\infty$. ☐

6. Discussion and questions

- The analysis of the paper carries through with almost no modifications in higher dimensions $\mathbb{R}^d$, proved we redefine $d(t,m)$ and $D(t,m)$ as

  $$d(t,m)=\min\{|x| : \zeta(t,x)<m\},\qquad D(t,m)\max\{|x| : \zeta(t,x)>m\}\,.$$

  At time t, the density is then at least m within the ball of radius $d(t,m)$ around 0, and less than m outside the ball of radius $D(t,m)$ around 0. In fact, the results are very slightly easier to obtain: we made life a little harder for ourselves by using asymmetric definitions of d and of D.
- I believe that Proposition 5.1 predicts the "true" front location, in that both $D(t,m)$ and $d(t,m)$ are typically at distance $o(t^{1/3})$ from $\sqrt{2}t-c^*t^{1/3}$ when t is large. This is our justification for the remark in the final paragraph of Section 1. As a first step toward proving this, one might attempt to strengthen Proposition 3.1 by replacing $2^{-5/6}$ with the constant c^* from Proposition 5.1. To prove such a result would entail, at a minimum, replacing Lemma 3.2 by a result which provided asymptotics for probabilities of the form

  $$\mathbf{P}\left\{\mathrm{Leb}\{s\in[0,t] : |B_s|<c(s^{1/3}+1)>\alpha t\right\}.$$

  Finding such asymptotics seems perhaps technical but not hopeless.

- I also expect that in fact $\sup_{t\geq 0}\sup_{x\in\mathbb{R}}\zeta(t,x)<\infty$ almost surely. One approach to proving this, which developed out of a conversation with Elie Aïdékon, would be to first show that
$$\inf\{t\geq 0:\sup_{x\in\mathbb{R}}\zeta(t,x)<1/2\}\,,$$
is almost surely finite. If all masses are at most 1/2, several of the probability tail bounds in the paper improve by a factor of 2 or more. This may then be iterated to force the greatest mass to decay exponentially quickly. Taking such decay into account in the calculations of Section 4 might lead to a proof that $\zeta(t,x)$ is bounded; I am not certain of this, but I believe it is a good line of attack.
- In the same way as the KPP equation describes the evolution of multiplicative functionals of BBM [17], it seems plausible that the model proposed in this work (or a related model) should be connected to an equation of the form
$$u_t=\frac{1}{2}u_{xx}-u(1-u)-\int_{\{y:|y-x|<1\}}u(t,y)\mathrm{d}y.$$
This equation has steady states at 0 (unstable) and 1/2 (stable), and is redolent of a family of "non-local" Fisher-KPP-type equations which was introduced [5] to model populations in which aggregation can have both a competitive advantage (safety in numbers) and disadvantage (due to competition for resources). These equations have received substantial study [2, 6, 9]; the survey [21] contains many further references, as well as perspective on the biological motivations for such study.

  If a probabilistic model for such an equation were found, it could yield new results on, e.g., the front propagation speed or temporal fluctuations of solutions to the above equation. Conversely, a glance at that literature suggests new probabilistic questions: for example, what if the effect of competition is described by a kernel κ, where $\kappa(|x-y|)$ describes the degree of competition for resources between individuals at spatial positions x and y? In our model we took $\kappa(|x-y|)=\mathbf{1}_{[|x-y|\in(0,1)]}$; a kernel which allows substantial long-range interaction might yield rather different dynamics.
- One may reasonably consider the mechanism for mass growth in our model – both children inherit the mass of the parent – nonphysical. More physically realistic (at least for amoebae) is for the children to each have half the mass of the parent. One must also then change the rules to allow for mass growth; a reasonable modification is to take
$$\overline{\zeta}(t,x)=\sum_{\{i:|X_i(t)-x|\leq 1\}}M_i(s),$$
and
$$M_i(t)=\exp\Big(\int_0^t(1-\overline{\zeta}(s,X_i(s)))\mathrm{d}s\Big)\,.$$
In other words, the mass of an individual can increase, when there is little nearby competition for resources – but the larger particles get, the harder it is for them to sustain themselves. The key point is that 1 is still a universal upper bound on the greatest mass of any particle. I believe the analyses from Sections 4 and 5 carry through essentially unchanged for this model.

  The argument from Section 3, however, breaks down, because a particle moving through an environment of constant density $m<1$ will have mass which does *not* decay exponentially, even when the loss of mass due to branching is taken into account. Instead, such a particle will (at large times) have a mass which is random and typically of order $\Theta(1-m)$.

Because of this, I only see how to obtain Proposition 3.1 in a highly weakened form, with the condition $m \geq 1$ rather than $m > 0$. (It is possible to do very slightly better, by considering a variable bound $m = m(t)$. On can then take $m(t) < 1$ if $1 - m(t)$ decays sufficiently quickly, but the pain-to-gain ratio in writing down such an argument in detail does not seem favourable.) But $m \in (0, 1)$ is the really meaningful region, and proving a genuine analogue of Proposition 3.1 for this model is the only missing step to a proof of Theorem 1.1 for the modified dynamics (unless I have failed to spot a pitfall in how the arguments Sections 4 and 5 transfer).

- In the variant just described, one intriguing possibility is that there may now be particles with mass $\Theta(1)$ at large times. If there are, they will be found near the front, since that is where they can find food. Do they really exist?
- More generally, one may take

$$M_i(t) = \exp(\int_0^t (a - b\overline{\zeta}(s, X_i(s)))\mathrm{d}s) .$$

This looks, heuristically, like some sort of spatial logistic growth [8, 14]. It may be interesting to investigate what different behaviours can occur as the parameters a and b are varied.

7. Acknowledgements

The seed for this work started to grow after a conversation with Jeremy Quastel in April, 2013. I initially thought an analysis similar to that of the current paper would have bearing on a question he asked about branching random walk with cutoff [1]. I now doubt that there is a strong connection, but still thank him for the stimulating question, and Oberwolfach for hosting the workshop where the conversation occurred. I also thank Elie Aïdékon, Henri Berestycki, Julien Berestycki, Nathanaël Berestycki and James Martin, for interesting discussions. Finally, the last strokes of this manuscript were completed at the Newton Institute, who I thank for their hospitality.

Appendix A. Probability tail bounds

We first state a Bernstein-type inequality due to Colin McDiarmid.

Theorem A.1 ([16], Theorem 2.7)**.** *Let $X_1, \ldots, X_n$ be independent with $X_k - \mathbf{E}X_k \leq 1$ for each k. Write $S_n = \sum_{k=1}^n X_k$ and fix $V \geq \mathbf{Var}\{S_n\} = \sum_{k=1}^n \mathbf{Var}\{X_i\}$. Then for any $c \geq 0$,*

$$\mathbf{P}\{S_n \geq \mathbf{E}S_n + c\} \leq e^c \cdot \left(\frac{V}{V+c}\right)^{V+c} < \left(\frac{eV}{c}\right)^c .$$

The first inequality is the heart of the theorem; the second is easy and is included to simplify an application of the theorem. The next lemma provides upper tail probability estimates for weighted geometric sums.

Lemma A.2. *Fix $\epsilon < 1/(2e^2)$ and let $(G_i, i \geq 1)$ be iid $\mathrm{Geom}(1-\epsilon)$. For any n and any non-negative real numbers $r_1, \ldots, r_n$ with $\max r_i / \sum r_i \leq 1/V < 1$, for all $\delta > 0$,*

$$\mathbf{P}\left\{\sum_{i=1}^n r_i G_i \geq (1+\delta)\sum_{i=1}^n r_i\right\} \leq e^{-2V(\delta - \epsilon e^2)} , \quad \mathbf{P}\left\{\sum_{i=1}^n r_i G_i \geq (1+\delta)\sum_{i=1}^n r_i\right\} \leq (2^{1+\delta}\epsilon^\delta)^V .$$

Proof of Lemma A.2. Let $\hat{G}_j = G_j - 1$ and $p_j = r_j / \sum_i r_i$. Then we must bound

$$\mathbf{P}\left\{\sum_{i=1}^n p_i \hat{G}_i \geq \delta\right\} ,$$

under the assumption that $\max_i p_i \le 1/V$. First note that

$$\mathbf{E}\left[\exp\left(2V \cdot \sum_{i=1}^n p_i \hat{G}_i\right)\right] = \prod_i \frac{1-\epsilon}{1-\epsilon e^{2Vp_i}} .$$

The latter product is maximized (subject to the constraint that $\max_i p_i \le 1/V$) when $p_i = 1/V$ for V values of i and $p_i = 0$ otherwise. We thus obtain

$$\mathbf{E}\left[\exp(cV \cdot \sum_{i=1}^n p_i \hat{G}_i)\right] \le \frac{(1-\epsilon)^V}{(1-\epsilon e^c)^V} < \left((1-\epsilon)\cdot e^{2\epsilon e^c}\right)^V ,$$

since $1/(1-\epsilon e^c) < e^{2\epsilon e^c}$ for $\epsilon < 1/2e^c$. For any non-negative random variable, $\mathbf{P}\{X > \delta\} \le e^{-c\delta V}\mathbf{E}e^{cVX}$; taking $c = 2$ yields

$$\begin{aligned}\mathbf{P}\left\{\sum_{i=1}^n p_i \hat{G}_i \ge \delta\right\} &\le \exp\left(-V\cdot(2\delta - \log(1-\epsilon) - 2\epsilon e^2)\right)\\ &< \exp\left(-V\cdot(2\delta - 2\epsilon e^2)\right) ,\end{aligned}$$

proving the first inequality. For the second, we take $e^c = (2\epsilon)^{-1}$ to obtain

$$\mathbf{P}\left\{\sum_{i=1}^n p_i \hat{G}_i \ge \delta\right\} \le e^{-c\delta V}\cdot\frac{(1-\epsilon)^V}{(1-\epsilon e^c)^V} = \left(2(1-\epsilon)(2\epsilon)^\delta\right)^V < 2^{(1+\delta)V}\epsilon^{\delta V} . \qquad \square$$

Appendix B. A simple sample path estimate for Brownian motion

Proof of Lemma 3.2. We may assume $y^2 < t/2$ or else the claim is trivially true (for any β such that $2e^{-2\beta} > 1$). By Brownian scaling it suffices to show the bound in the case $y = 1$, and we assume this from now on. Let $(L_t(r), t \ge 0, r \in \mathbb{R})$ be the local time process of B and let $\tau = \inf\{t : L_t(0) \ge x\}$. By the second Ray-Knight theorem [15, Theorem 2.7.1], with W a linear Brownian motion independent of B, we have

$$(L_\tau(a) + W_a^2, a \ge 0) \overset{\mathrm{d}}{=} ((W_a + \sqrt{x})^2, a \ge 0) .$$

Using this distributional identity, it follows that

$$\begin{aligned}\mathbf{P}\left\{\sup_{a\in[0,1]} L_\tau(a) \ge 4x\right\} &\le \mathbf{P}\left\{\sup_{a\in[0,1]} (L_\tau(a) + W_a^2) > 4x\right\}\\ &= \mathbf{P}\left\{\sup_{a\in[0,1]} W_a \ge \sqrt{x}\right\}\\ &= \mathbf{P}\left\{|W_1| \ge \sqrt{x}\right\}\\ &\le 2e^{-x/2} .\end{aligned}$$

By symmetry the same bound holds for $\sup_{a\in[-1,0]} L_\tau(a)$, and we obtain

$$\mathbf{P}\{\mathrm{Leb}\{s \in [0,\tau] : |B_s| \le 1\} > 8x\} = \mathbf{P}\left\{\int_{-1}^1 L_\tau(a)\mathrm{d}a > 8x\right\} \le 4e^{-x/2} .$$

On the other hand, by Lévy's theorem ([18], Theorem 2.34) we have

$$\mathbf{P}\{\tau < t\} = \mathbf{P}\left\{\inf_{s\le t} B_s \le -x\right\} \le e^{-x^2/t}.$$

Finally, if $\tau \ge t$ then $\mathrm{Leb}\{s \in [0,\tau] : |B_s| \le 1\} \le \mathrm{Leb}\{s \in [0,t] : |B_s| \le 1\}$, so for any x we obtain

$$\mathbf{P}\{\mathrm{Leb}\{s \in [0,\tau] : |B_s| \le 1\} \ge x\} \le 4e^{-x/2} + e^{-x^2/2t}.$$

Taking $x = \alpha t$ gives the claim. $\square$

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