Victor Chulaevsky

Efficient localization bounds in a continuous N-particle Anderson model with long-range interaction

Abstract We establish strong dynamical and exponential spectral localization for a class of multi-particle Anderson models in a Euclidean space with an alloy-type random potential and a sub-exponentially decaying interaction of infinite range. For the first time in the mathematical literature, the uniform decay bounds on the eigenfunction correlators at low energies are proved, in the multi-particle continuous configuration space, in the (symmetrized) norm-distance and not in the Hausdorff distance.

1 Introduction

1.1 The model

We study a multi-particle Anderson model in $\mathbb{R}^d$ with an infinite range interaction and subject to an external random potential of the so-called alloy type. The Hamiltonian $\mathbf{H}\ \Big(= \mathbf{H}^{(N)}(\omega)\Big)$ is a random Schrödinger operator of the form

$$\mathbf{H} = -\frac{1}{2}\boldsymbol{\Delta} + \mathbf{U}(\mathbf{x}) + \mathbf{V}(\omega;\mathbf{x}) \tag{1}$$

acting in $L^2\left((\mathbb{R}^d)^N\right)$. To stress the dependence on the number of particles, $N \geq 1$, while omitting a less important parameter d (= the dimension of the 1-particle configuration space), we denote $\mathscr{X} = \mathbb{R}^d$, $\mathscr{Z} = \mathbb{Z}^d \hookrightarrow \mathbb{R}^d$, and

$$\mathscr{X}^N := \left(\mathbb{R}^d\right)^N,\ \ \mathscr{Z}^N := \left(\mathbb{Z}^d\right)^N \hookrightarrow \left(\mathbb{R}^d\right)^N,\ \ N \geq 1.$$

The points $\mathbf{x} = (x_1, \dots, x_N) \in \mathscr{X}^N$ represent the positions of N distinguishable quantum particles evolving simultaneously in the physical space $\mathbb{R}^d$. In (1), $\boldsymbol{\Delta}$

V. Chulaevsky: Université de Reims, Département de Mathématiques, Moulin de la Housse, B.P. 1039, 51687 Reims Cedex, France; E-mail: victor.tchoulaevski@univ-reims.fr

stands for the Laplacian in $(\mathbb{R}^d)^N$ (or, equivalently, in $\mathbb{R}^{Nd}$). The interaction energy operator $\mathbf{U}$ acts as multiplication by a function $\mathbf{x} \mapsto \mathbf{U}(\mathbf{x})$. Finally, the potential energy $\mathbf{V}(\omega;\mathbf{x})$ (unrelated to the inter-particle interaction) is the operator of multiplication by a function

$$\mathbf{x} \mapsto V(x_1;\omega) + \cdots + V(x_N;\omega), \tag{2}$$

where $x \in \mathbb{R}^d \mapsto V(x;\omega)$ is a random external potential assumed to be of the form

$$V(x;\omega) = \sum_{a \in \mathscr{Z}} \mathscr{V}_a(\omega)\, \varphi(x-a). \tag{3}$$

Here and below $\mathscr{V}_a$, $a \in \mathscr{Z}$, are IID (independent and identically distributed) real random variables on some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ is usually referred to as a scatterer (or "bump") function.

More precise assumptions will be specified below.

1.2 The motivation and comparison with the existing results

The single-particle localization theory, describing non-interacting quantum particles evolving in a random environment, was initiated by P. W. Anderson in his seminal paper [1]. The first rigorous mathematical results on random Anderson-type Hamiltonians were obtained by Goldsheid et al. [35] (in $\mathbb{R}^1$), by Kunz and Souillard [41] (in $\mathbb{Z}^1$), and then in multi-dimensional lattice models by Fröhlich et al. [28,29], with the help of the Multi-Scale Analysis (MSA). The approach from the works [28,29] was reformulated in a series of works by von Dreifus [25], Spencer [47] and von Dreifus and Klein [26]. An alternative approach (FMM = Fractional Moment Method) was proposed by Aizenman and Molchanov [2], for the lattice models; the original technique was later substantially generalized in a series of deep works bearing a distinctive mark of Michael Aizenman's enthusiasm; cf., e.g., [3,4,5].

Martinelli and Holden [43] extended the MSA to the continuous models, i.e., to the random Hamiltonians in $L^2(\mathbb{R}^d)$.

In all the above mentioned papers, as well as in a number of other physical and mathematical works, the quantum particles were considered non-interacting; in the physical context, this is of course a conscious approximation, made already by Anderson [1] who did not hide his concerns about the possible effects of the interaction on the localization phenomena.

An outburst of new results, both in theoretical and mathematical physics, occurred in 2005-2008 (cf. [8], [36], [19,20,21,6]; some preprints appeared earlier). As usual, the physical works provided stronger statements, viz. the stability of localization phenomena under sufficiently weak interaction in physically realistic systems, with $N \sim \rho|\Lambda|$ particles in a bounded but macroscopically large domain $\Lambda \subset \mathbb{R}^d$. We stress that Λ is indeed of *finite*, albeit possibly large size. It goes without saying that for all imaginable applications the size of Λ is bounded by (or is of order of) that of our little planet, and *not actually infinite*. The first mathematical works considered a fixed number $N > 1$ of particles in an infinite configuration space: in $\mathbb{Z}^d$ (cf. [19,20,21,6]), and later in $\mathbb{R}^d$ (cf. [18], [40], [31]).

While it does not come as a big surprise that only a finite, and fixed, number of particles were allowed in the first attempts to build rigorous theory of Anderson localization in systems with nontrivial interaction, it is more surprising that the results on complete spectral and strong dynamical localization proved in [21,6] *required* the configuration space to be *actually infinite*. More precisely, localization (viz. uniform bounds on the eigenfunction correlators, or the rate of spread of an initially localized wavepacket ψ under the Hamiltonian dynamics $e^{-it\mathbf{H}}$) could not be established in arbitrarily large yet bounded domains Λ of the physical configuration space.

We would like to stress that the issue addressed in the present paper is not a mere technicality or an abstract matter (replacing one strange-looking pseudometric – the Hausdorff distance – with another, the symmetrized norm-distance). The main problem manifests itself already in a 3-particle model describing interacting quantum particles in an arbitrarily large but finite interval $\Lambda = [0,L] \subset \mathbb{R}$ (the nature of the problem does not depend upon the dimension of the physical, single-particle space). Consider the best possible situation for the scaling analysis (MPMSA or MPFMM): let the inter-particle interaction potential be compactly supported, and the marginal distribution of disorder be Lipschitz-continuous. Technically speaking, for a lattice model (in $\mathbb{Z}^1$) we mean that the random potential $V(x;\omega)$ has IID (independent and identically distributed) values, and the random variables $V(x;\omega)$ have bounded probability density. In the continuous alloy model (cf. (3)), the same assumption is made for the IID scatterers' amplitudes $\mathscr{V}(a;\omega)$. The main, quite natural question is:

"*Can one prove the complete N-particle Anderson localization in a large bounded domain Λ, for the energies in an interval $I \subset \mathbb{R}$ where it is to be expected ?*"

For the lattice model with a bounded potential, one can take $I = \mathbb{R}$, or a sufficiently large bounded interval, since the entire spectrum of the (bounded) random Hamiltonian is bounded. In the continuous model, the single-particle localization in dimension $d > 1$ can be proved with existing techniques only in a bounded interval I near the bottom of the spectrum. The fact that Anderson localization is complete in one dimension does not imply that the same must be true for $N \geq 2$ particles, since the N-particle Hamiltonian acts in the space of functions of N variables.

The first papers [21,6] on N-particle localization did *not* answer the above question, except for the particular case $N = 2$, and neither did some subsequent works (cf., e.g., [18,22,31]). What *has* been proved for $N \geq 3$ can be explained in simple terms as follows (we keep the language of the one-dimensional model). Consider an arbitrarily large interval $[-R,R]$. There exists a sufficiently large interval $[-L(R),L(R)] \supset [-R,R]$ such that the N-particle quantum states "essentially concentrated" in $[-R,R]$ are exponentially localized. However, one could not rule out existence of N-particle states "essentially supported" simultaneously by the zones near $-L(R)$ and near $+L(R)$. Such hypothetical states might be de-localized over the distance comparable to the size of the entire sample of the disordered media $[-L,L]$, although the results of [21,6] evidenced that delocalization in a "central" zone $[-R,R]$ was impossible.

Naturally, the above hypothetical situation is hardly compatible with the conventional notion of an electrical isolator: if the current can pass from one extremity of a wire to another, the wire in question qualifies as a conductor, no matter how good an isolator it might be in the middle.

For the discrete systems, the first positive answer to the above question was recently given in [23] (see a preliminary version in the preprint `arXiv:math-ph/1404.3978`). The present paper extends the techniques of [23] to the interacting models in $\mathbb{R}^d$. This extension is not automatic and, sadly, not as complete as one would expect. Some technical aspects of the eigenvalue concentration estimation compel us to restrict the localization analysis to a particular class of alloys – with the "flat tiling" condition.

The special role of the number of particles $N = 2$ is explained by an elementary geometrical fact: the Hausdorff distance in the 2-particle configuration space is equivalent to the symmetrized norm-distance, but this is no longer true for $N \geq 3$.

All this is quite opposite to the usual situation where a finite-volume analysis is only a prelude for a rigorous study of an object inspiring mathematicians – an actually infinite system. If the results of [21,6] (or their continuous-space counterparts [18,40,31]) were to be applied to the physical models, they would be valid only if our Universe were found to be infinite (and more or less uniformly disordered). Otherwise, one could not be able to rule out some tunneling processes which might result in a transfer of particles over large distances, comparable to the size of the entire system.

On the technical level, the bottleneck of prior rigorous results on the N-particle localization (starting with $N = 3$) is the eigenvalue concentration (EVC) estimate, which is analogous to, but more sophisticated than, its well-known counterpart going back to Wegner [50]. Below we show that it is rather to be qualified as an eigenvalue *comparison* estimate for stochastically strongly correlated pairs of local Hamiltonians. Despite significant differences between the techniques of [21] and [6], both approaches faced essentially the same problem, and both gave rise to the decay bounds on the eigenfunctions (EFs) and eigenfunction correlators (EFCs) expressed in terms of the so-called Hausdorff distance but not the (symmetrized) norm-distance.

Now we turn to the goals and results of the present work.

Following the approach to the multi-particle EVC bounds presented originally in [11,13] and recently extended in [16], we aim to improve the EVC estimate required for the multi-particle MSA (MPMSA) and thus achieve more efficient N-particle localization bounds for $N \geq 3$ particles in a Euclidean space $\mathbb{R}^d$, $d \geq 1$. The estimate in question is an optimized variant of its analog proved in [23]; see the comments in Sect. 2 after Theorem 2.

The main novelty of the present work is two-fold:

⧫ We give the first rigorous proof (for $N \geq 3$) of uniform decay (which we show to be at least sub-exponential) of the eigenfunction correlators with respect to the symmetrized norm-distance for a multi-particle alloy model in $\mathbb{R}^d$ or in any bounded regular sub-domain thereof. In accordance with the above discussion, the phenomenon of Anderson localization is thus firmly established in disordered systems with a fixed number of quantum particles in a physically realistic geometrical setting. For the moment, this result is proved for a particular class of alloy potentials, which we call *flat tiling* alloys.

⧫ For the first time, we prove exponential decay of the eigenfunctions in an N-particle alloy model in $\mathbb{R}^d$ with sub-exponential decay of interaction. Prior results by Fauser and Warzel [31] imply only a sub-exponential decay of eigenfunctions in such a model.

Compared to the work by Klein and Nguyen [40], who made a significant step in the scaling analysis of the continuous interactive multi-particle Anderson models, the main improvement is relaxing the condition of finite-range for the interaction potential to a fractional-exponential decay $r \mapsto \mathrm{e}^{-r^{\zeta}}$, with $\zeta > 0$ which can be arbitrarily small.

It is to be emphasized that the recent result by Fauser and Warzel [31] on exponential decay of the EFCs, for exponentially decaying interactions, remains the strongest one *among those proved in terms of Hausdorff distance* in a continuous space, hence in an *actually infinite* configuration space. Due to some well-known limitations of the Multi-Scale Analysis (single- or multi-particle), proofs of exponential strong dynamical localization are still beyond the MSA's reach. On the other hand, recall that the technique developed by Klein and Nguyen [40], based on the Quntitative Unique Continuation Principle (QUCP) (cf. [37]), made unnecessary the complete covering condition for the alloy potential, used both in [18] and [31]. This makes the class of N-particle alloy models studied in [40] the most general one, at the time of writing these lines.

It seems appropriate to attract the readers' attention to an interesting fact: while one of the most striking differences between the MSA and the FMM in the single-particle localization theory, is that the latter employs a "mono-scale" technology, appreciated both by mathematicians and physicists, in the realm of multi-particle systems both approaches – MSA and FMM – finally settle on the common ground of *multi-scale* geometrical induction.

Except for the new EVC bounds, crucial for the localization in the symmetrized norm-distance, the main strategy of the proofs in the present paper is a streamlined and improved variant of the MPMSA from [18], with important elements of the techniques developed in Refs. [32,34,39,40]. To be more precise, we do not actually make a *bootstrap*, but rather carry out two logically independent scaling analyses, analogous (but not identical) to two of the four phases of the bootstrap MSA. This simplification has however some drawbacks. Below we comment, where appropriate, on the important advantages of the full-fledged bootstrap analysis. The task of performing such analysis is beyond the scope of the present paper, but in the light of a recent preprint [17], we plan to prove in a forthcoming work that the rate of decay of the EFCs in the N-particle alloy models in $\mathbb{R}^d$ with exponentially decaying interaction admits exponential scaling *limit*:

- in the symmetrized norm-distance, under the assumption of flat tiling (cf. (8));
- in the Hausdorff distance, for a larger class of models including those studied by Klein and Nguyen [40].

1.3 Structure of the paper

After establishing the EVC bounds in Section 2, the bulk of technical work is carried out in Sections 3–4, where the fixed-energy scaling analysis is performed.

The latter prepares the ground for the energy interval (a.k.a. variable-energy) MSA estimates derived in a "soft" way from their fixed-energy counterparts which are substantially simpler to obtain. Such a derivation is presented in Section 5, where we deviate from the strategy of Ref. [40]. The derivation of strong dynamical localization from the energy-interval estimates is obtained by another "soft" and fairly general method, developed by Germinet and Klein [32,33,34] in the context of the single-particle Anderson models and adapted to the interactive models by Klein and Nguyen [39,40].

As the matter of fact, the fixed-energy stage of the MPMSA is based on the EVC bound of the form (13) which operates with a single given cube in the N-particle configurations space, and here we do not make any improvement compared to Ref. [40]. The eigenvalue *comparison* bounds given in [40, Corollary 2.3] and in Theorem 4, on the contrary, are essentially shaped by the type of metric measuring the distance between two cubes $\mathbf{\Lambda}_L^{(N)}(\mathbf{x})$ and $\mathbf{\Lambda}_L^{(N)}(\mathbf{y})$ where the Hamiltonians $\mathbf{H}_{\mathbf{\Lambda}_L^{(N)}(\mathbf{x})}(\omega)$ and $\mathbf{H}_{\mathbf{\Lambda}_L^{(N)}(\mathbf{y})}(\omega)$ are considered, along with their spectra. Ref. [40] considers the pairs of cubes separated in the Hausdorff distance (as do, in fact, Refs. [21,18]), and this is why the two-volume EVC bound in [40] is indeed a mere corollary of the one-volume, Wegner-type estimate from [40, Theorem 2.2]. We prove the crucial EV comparison estimate for the pairs of cubes of size L with centers $\mathbf{x}$ and $\mathbf{y}$ separated in the symmetrized[1] norm-distance: $\mathrm{d}_S(\mathbf{x},\mathbf{y}) \geq 4NL$. This requires, for the moment, some restrictive conditions upon the type of the scatterers forming the alloy and the regularity of the probability distributions of the scatterers' amplitudes.

On the bright side, the energy-interval scaling analysis can be conducted for the (symmetrized) norm-distant pairs of cubes, thus making the last stage of derivation of localization results much closer to that used in the conventional, single-particle theory.

In particular, the proof of exponential decay of localized eigenfunctions is obtained with a minor modification of the method going back to the works [26, 29].

Summarizing, we focus mainly on the fixed-energy multi-scale analysis of operators in *finite cubes*, aiming to obtain more efficient energy-interval bounds referring to the symmetrized norm-distance. Such bounds are much more relevant for the physical applications than their analogs in the infinitely extended configuration space.

The reader may notice that a considerable part of the text is devoted to the proof of genuine exponential decay of the localized eigenfunctions, no matter how small is the decay exponent $\zeta > 0$ of the interaction potential. The paper could have been made almost twice shorter if the detailed, if not boring, proofs of a number of auxiliary results in this part of the paper were replaced by short cross-references like *"In the same way as in the proof of sub-exponential decay of the EF correlators, ..."*.

[1] It is necessary to use the symmetrized and not the conventional norm-distance. Indeed, any symmetry generated by permutation of the coordinates $\pi : \mathbf{\Lambda}_L^{(N)}(\mathbf{x}) \to \mathbf{\Lambda}_L^{(N)}(\mathbf{y})$ gives rise to unitarily equivalent operators, hence identical spectra, due to the π-invariance of the potential energy.

1.4 Basic geometric objects and notations

Throughout this paper, we will keep fixed an integer $N^* \geq 2$ and work in Euclidean spaces of the form $(\mathbb{R}^d)^N \cong \mathbb{R}^{Nd}$, $1 \leq N \leq N^*$, and use a shorter notation $\mathscr{X}^N := (\mathbb{R}^d)^N$. A configuration of $N \geq 1$ distinguishable particles in $\mathbb{R}^d$ is identified with a vector $\mathbf{x} = (x_1, \ldots, x_N) \in (\mathbb{R}^d)^N$, where x_j is the position of the j-th particle. More generally, boldface notations are reserved for "multi-particle" objects (Hamiltonians, resolvents, cubes, etc.).

All Euclidean spaces will be endowed with the max-norm denoted by $|\cdot|$, but occasionally we use the Euclidean norm $|\cdot|_2$. We will work with Nd-dimensional cubes of integer edge length centered at lattice points $\mathbf{u} \in \mathscr{Z}^N := (\mathbb{Z}^d)^N \hookrightarrow (\mathbb{R}^d)^N = \mathscr{X}^N$ and with edges parallel to the co-ordinate axes.

The open cube of edge length $2L+1$ centered at $\mathbf{u}$ is denoted by $\boldsymbol{\Lambda}_L(\mathbf{u})$; in the max-norm it represents the open ball of radius $L + \frac{1}{2}$ centered at $\mathbf{u}$:

$$\boldsymbol{\Lambda}_L(\mathbf{u}) = \left\{\mathbf{x} \in \mathscr{X}^N : \ |\mathbf{x} - \mathbf{u}| < L + \tfrac{1}{2}\right\}. \tag{4}$$

By a slight abuse of terminology, we will say for brevity that $\boldsymbol{\Lambda}_L(\mathbf{u})$ has radius L.

The lattice counterpart for $\boldsymbol{\Lambda}_L(\mathbf{u})$ is denoted by $\mathbf{B}_L(\mathbf{u})$:

$$\mathbf{B}_L(\mathbf{u}) = \boldsymbol{\Lambda}_L(\mathbf{u}) \cap \mathscr{Z}^N, \quad \mathbf{u} \in \mathscr{Z}^N.$$

We endow the lattice $\mathscr{Z}^N$ with the graph structure: $(\mathbf{x}, \mathbf{y})$ is an edge iff $|\mathbf{x} - \mathbf{y}|_2 = 1$.

We also consider "cells" – *closed* cubes of diameter 1 centered at the lattice points $\mathbf{u} \in \mathscr{Z}^N$:

$$\mathbf{C}(\mathbf{u}) = \{\mathbf{y} \in \mathscr{X}^N : \ |\mathbf{y} - \mathbf{u}| \leq \tfrac{1}{2}\}.$$

The union of all cells $\mathbf{C}(\mathbf{u})$, $\mathbf{u} \in \mathscr{Z}^N$, covers the entire Euclidean space $\mathscr{X}^N$. Moreover, for any $\mathbf{u} \in \mathscr{Z}^N$, denoting by $\overline{\boldsymbol{A}}$ the closure of a set $\boldsymbol{A} \subset \mathscr{X}^N$, we have

$$\overline{\boldsymbol{\Lambda}_L(\mathbf{u})} = \bigcup_{\mathbf{y} \in \mathbf{B}_L(\mathbf{u})} \mathbf{C}(\mathbf{y}).$$

The cells are not necessarily pairwise disjoint, but their overlaps always have zero Lebesgue measure. Next, given a cube $\boldsymbol{\Lambda}_L(\mathbf{u})$, we denote by $\boldsymbol{\Lambda}_L^{\text{out}}(\mathbf{u})$ the (inner) 1-neighborhood of its boundary: $\boldsymbol{\Lambda}_L^{\text{out}}(\mathbf{u}) := \{\mathbf{x} \in \boldsymbol{\Lambda} : \ |\mathbf{u} - \mathbf{x}| > L - \frac{1}{2}\}$, while for the associated lattice cube $\mathbf{B}_L(\mathbf{u}) = \boldsymbol{\Lambda}_L(\mathbf{u}) \cap \mathscr{Z}^N$, we define its (internal) boundary $\partial^- \mathbf{B}_L(\mathbf{u}) := \{\mathbf{y} \in \mathbf{B}_L(\mathbf{u}) : \ |\mathbf{u} - \mathbf{y}| = L\}$ in such a way that

$$\boldsymbol{\Lambda}_L^{\text{out}}(\mathbf{u}) \subset \bigcup_{\mathbf{x} \in \partial \mathbf{B}_L(\mathbf{u})} \mathbf{C}(\mathbf{x}). \tag{5}$$

The diameters in our formulae are relative to the max-norm; the cardinality of various sets A (usually finite) will be denoted by $|A|$. In an N-particle system, we have

$$\operatorname{diam} \boldsymbol{\Lambda}_L(\mathbf{u}) = 2L + 1, \operatorname{diam} \mathbf{B}_L(\mathbf{u}) = 2L, \ \ |\mathbf{B}_L(\mathbf{u})| = (2L+1)^{Nd} \leq (3L)^{Nd}.$$

To indicate, where necessary, the value of N, we write $\boldsymbol{\Lambda}_L^{(N)}(\mathbf{u})$ and $\mathbf{B}_L^{(N)}(\mathbf{u})$.

The indicator function of a set A is denoted in general by $\mathbf{1}_A$, but for the indicators of the cells we use a shorter notation, $\chi_{\mathbf{x}} := \mathbf{1}_{\mathbf{C}(\mathbf{x})}$.

We write $\boldsymbol{\Lambda}_\ell(\mathbf{x}) \Subset \boldsymbol{\Lambda}_L(\mathbf{y})$ to indicate that $\boldsymbol{\Lambda}_\ell(\mathbf{x}) \subset \boldsymbol{\Lambda}_{L-2}(\mathbf{y})$.

Integer intervals $[a,b] \cap \mathbb{Z}$, with $a,b \in \mathbb{Z}$, will be denoted, as usual, by $[[a,b]]$. Given an integer $N \in [[1,N^*]]$, we define the (full) projection $\Pi : (x_1,\dots,x_N) \mapsto \{x_1,\dots,x_N\}$. Also, given a non-empty index subset $\mathscr{J} = (j_1,\dots,j_n) \subseteq [[1,N]]$, we define the partial $\mathscr{J}$-projection $\Pi_{\mathscr{J}} : \mathscr{X}^N \to \mathscr{X}^n$ by $\Pi_{\mathscr{J}}\mathbf{x} = (x_{j_1},\dots,x_{j_n})$.

1.5 Symmetrized norm-distance and the Hausdorff metric

A norm-distance in the N-particle configuration space is not well-adapted to the decay estimates of the eigenfunctions and of their correlators. Indeed, if the interaction potential $\mathbf{U}$ is permutation-symmetric (and the external random potential is always so), then the entire Hamiltonian $\mathbf{H}^{(N)}$ commutes with the symmetric group $\mathfrak{S}_N$ acting by permutations of the particle positions. Thus the Hilbert space $L^2(\mathscr{X}^N)$ can be decomposed into the direct sum of $\mathbf{H}^{(N)}$-invariant subspaces, including that of the symmetric functions taking identical values along any orbit of the symmetry group $\mathfrak{S}_N$. The points of such orbits can be arbitrarily distant from each other, which makes impossible any uniform decay bound.

More to the point, the physical systems are composed of indistinguishable particles, so in the framework of Bose–Einstein or Fermi–Dirac quantum statistics, the permutations of the particle positions give rise to equivalent configurations.

For these reasons, the symmetrized norm-distance in the N-particle configuration space is much more natural, even in a situation where, as in the present paper, the particles are considered distinguishable. Its formal definition is as follows:

$$\mathrm{d}_S^{(N)}(\mathbf{x},\mathbf{y}) := \min_{\pi \in \mathfrak{S}_N} |\pi(\mathbf{x}) - \mathbf{y}|,$$

where the elements of the symmetric group $\pi \in \mathfrak{S}_N$ act on $\mathbf{x} = (x_1,\dots,x_N)$ by permutations of the coordinates x_j.

Recall also the definition of the Hausdorff distance $\mathrm{d}_{\mathscr{H}}$ between two subsets X,Y of an abstract metric space $(\mathscr{M},\mathrm{d}(\cdot))$:

$$\mathrm{d}_{\mathscr{H}}(X,Y) = \max\Big[\, \sup_{x\in X} \mathrm{d}(x,Y),\ \sup_{y\in Y} \mathrm{d}(y,X) \,\Big].$$

This notion does not apply directly to the configurations $\mathbf{x} \in \mathscr{X}^N$; however, an important characteristics of $\mathbf{x} = (x_1,\dots,x_N)$ is its projection $\Pi\mathbf{x} = \{x_1,\dots,x_N\}$. In the case of indistinguishable Fermi-particles, $\Pi\mathbf{x}$ *is* the configuration. Hence one can extend formally $\mathrm{d}_{\mathscr{H}} = \mathrm{d}_{\mathscr{H}}^{(N)}$ to the pairs of configurations, keeping the same notation and setting $\mathrm{d}_{\mathscr{H}}(\mathbf{x},\mathbf{y}) := \mathrm{d}_{\mathscr{H}}(\Pi\mathbf{x},\Pi\mathbf{y})$.

It was realized in the works [6,21] that it was simpler to obtain the decay bounds on the Green functions, eigenfunctions and eigenfunction correlators with respect to the Hausdorff distance $\mathrm{d}_{\mathscr{H}}$ than in terms of the symmetrized norm-distance.

The following simple example illustrates the difference between $\mathrm{d}_{\mathrm{S}}^{(N)}$ and $\mathrm{d}_{\mathscr{H}}^{(N)}$, for $N \geq 3$. With $N = 3$, $d = 1$, let $\mathbf{a} = (0,0,R)$ and $\mathbf{b} = (0,R,R)$, $R > 0$. Then for any $L > 0$, $\mathrm{d}_{\mathrm{S}}^{(3)}(\boldsymbol{\Lambda}_L(\mathbf{a}), \boldsymbol{\Lambda}_L(\mathbf{b})) \to +\infty$ as $|R| \to \infty$, but $\mathrm{d}_{\mathscr{H}}^{(3)}(\boldsymbol{\Lambda}_L(\mathbf{a}), \boldsymbol{\Lambda}_L(\mathbf{b})) \equiv 0$.

In physical terms, $\mathbf{b}$ is obtained from $\mathbf{a}$ by transferring[2] one of the particles from 0 to a distant location R. If one has to study localization in the finite domain $[0,R]$, then the tunneling between the configurations like $\mathbf{a}$ and $\mathbf{b}$ can (or might) ruin the decay of EFs and EFCs over the distances comparable with the size of the domain.

Such a situation is impossible for $N = 2$, since $\mathrm{d}_{\mathscr{H}}^{(2)}(\cdot\,,\cdot)$ is equivalent to $\mathrm{d}_{\mathrm{S}}^{(2)}(\cdot\,,\cdot)$.

1.6 Interaction potential

We assume the following:

(U) $\mathbf{U}$ is generated by a 2-body potential $U^{(2)} : \mathbb{R}_+ \to \mathbb{R}_+$, viz.

$$\mathbf{U}(\mathbf{x}) = \sum_{1 \leq i < j \leq N} U^{(2)}(|x_i - x_j|),$$

where

$$0 \leq U^{(2)}(r) \leq C_U \mathrm{e}^{-r^{\zeta}}, \tag{6}$$

for some $\zeta > 0$, $C_U \in (0,\infty)$.

It does not make much sense to consider separately $\zeta > 1$, for the key parameters measuring the decay of the EFCs depend in fact upon the quantity $\min(1,\zeta)$.

1.7 External random potential

We assume the following conditions to be fulfilled.

(V) The external random potential is of alloy type,

$$V(x;\omega) = \sum_{a \in \mathscr{Z}} \mathscr{V}(a;\omega)\varphi_a(x-a), \tag{7}$$

where $\mathscr{V} : \mathscr{Z} \times \Omega \to \mathbb{R}$ is an IID random field on the lattice $\mathscr{Z} = \mathbb{Z}^d \hookrightarrow \mathbb{R}^d$.

The scatterer (a.k.a. bump) functions $\varphi_a \geq 0$ have the following property which we call ***flat tiling***: $\operatorname{diam}\operatorname{supp}\, \varphi_a \leq \mathrm{r}_1 < \infty$ and, for some $\mathrm{C}_\varphi \in (0,+\infty)$,

$$\sum_{a \in \mathscr{Z}} \varphi_a(x-a) \equiv \mathrm{C}_\varphi \ \text{(up to a subset of zero Lebesgue measure).} \tag{8}$$

The common marginal probability distribution of the IID scatterers' amplitudes $\mathscr{V}(\cdot;\omega)$ is a convolution $\mu = \mu_1 * \mu_2$, where μ_2 is an arbitrary probability measure with the support containing the point 0 and contained in $\mathbb{R}_+$, and μ_1 admits a probability density p_V, which is compactly supported, with supp p_V

[2] For $N \geq 3$, such a transfer process can be "partial", i.e., leaving at least one particle at each of the two distant loci 0 and R, while for $N = 2$ a similar transfer must be "complete" in one of the two directions.

$= [0, c_V]$, $c_V > 0$, and p_V is strictly positive, bounded and has bounded derivative in the open interval $(0, c_V)$:

$$\forall t \in (0, c_V) \qquad \begin{cases} 0 < p_* \le p_V(t) \le p^* < +\infty, \\ \qquad\qquad p'_V(t) \le C^* < +\infty. \end{cases} \tag{9}$$

Probably, the most natural example is where $\varphi = \mathbf{1}_{\overline{\Lambda}_{1/2}(0)}$, so that $\sum_{a \in \mathscr{Z}} \varphi(a) \equiv 1$ Lebesgue-a.e. Such a form of alloy was considered by Kotani and Simon [42], in the single-particle setting. However, flat tiling is achieved also with $\varphi = \mathbf{1}_{\overline{\Lambda}_{\ell/2}(0)}$, $\mathbb{N} \ni \ell \ge 1$. As to the scatterers' amplitudes, one can simply take the uniform probability distribution $\mathrm{Unif}([0,1])$, where $c_V = C^* = p_* = p^* = 1$. This corresponds to the convolution $\mu = \mu_1 * \mu_2$ of the uniform distribution $\mu_1 = \mathrm{Unif}([0,1])$ with the delta-measure $\mu_2 = \delta_0$.

Observe that the marginal distribution of the random potential can have an unbounded support. The *lower* unboundedness of the potential may of course pose a problem for the self-adjointness of the Hamiltonian $\mathbf{H}(\omega)$ in the entire Euclidean space, and even if the left-tail probabilities $\mathbb{P}\{\mathscr{V}(x;\omega) < t\}$ rapidly decay as $t \to -\infty$, a number of analytical arguments in our proofs would require some modifications. On the other hand, the upper unboundedness of the random amplitudes $\mathscr{V}(x;\omega)$ is fairly harmless. For example, μ_2 can be an exponential distribution.

The positivity of the random potential and the condition that $\mathrm{supp}\,\mu_2 \ni 0$ (hence, $\mathrm{supp}\,\mu \ni 0$) are assumed merely to simplify the description of the almost sure spectrum of the random operator $\mathbf{H}(\omega)$; cf. Section 1.8 below.

Note also that the admissible probability distributions μ may have probability density vanishing at the edges of the support; for example, one can take the n-th convolution power of the uniform distribution $\mathrm{Unif}([0,1])$, for any $n \ge 2$, with density $p(t) = O(t^{n-1})$ near the lower edge 0 of its support.

The assumptions (9) are used in the proof of the crucial EVC bound (cf. Lemma 2), based on the property of the random field $\mathscr{V}(\cdot, \omega)$ which we call strong regularity of the conditional mean ((SRCM); cf. (12)). As was pointed out in our earlier paper [13], a Gaussian distribution features a particularly strong form of the property (SRCM), but, again, we do not allow here for the Gaussian marginal distributions, for it would require a number of analytical adaptations to the lower-unbounded potentials. Such adaptations would not be required in the context of discrete N-particle Anderson models, where the kinetic energy operator is bounded.

For brevity, we assume $\mathrm{C}_\varphi = 1$; this is inessential for the validity of the main results.

1.8 The almost sure (a.s.) spectrum

The exact location of the almost sure spectrum $\Sigma(\mathbf{H}^N_{\mathscr{X}^N})$ of the N-particle Hamiltonian $\mathbf{H}^N(\omega)$ in the entire Euclidean space $\mathbb{R}^d$ can be easily determined with the help of the classical Weyl criterion. The flat tiling alloy is a particular case of a more general one studied by Klein and Nguyen, and the only point which prevents us from quoting their result (cf. [40, Proposition A.1]) is that they considered an interaction of finite range. A careful reading of the proof given in [40] evidences that this is a pure formality, for their argument, based on Weyl's criterion, naturally extends to interaction potentials decaying at infinity. For an interaction of finite range, there are arbitrarily large cubes in the N-particle configuration space where the interaction vanishes and all scatterers' amplitudes are as close to 0 as one pleases, so the respective finite-cube spectrum is close to that of the Laplacian. For a decaying infinite-range interaction, the interaction energy on such cubes is not vanishing, but can be made close to 0. With these considerations in mind, we come to the following characterization of the a.s. spectrum.

Proposition 1 *Under the assumptions (V) and (U), $\Sigma(\mathbf{H}^N_{\mathscr{X}^N}(\omega)) = [0,+\infty)$ with probability* 1.

1.9 Main result

Recall that we denote by ζ the decay exponent of the interaction potential (cf. (6)).

Theorem 1 *Assume the conditions* (V) *and* (U)*, and fix an integer $N^* \geq 2$. There exist $E^* > 0$, $\kappa = \kappa(\zeta, N^*) \in (0,\zeta)$, $\nu > 0$, $m > 0$ with the following properties.*

(A) *Denote $I^* = [0,E^*]$. For all $N \in [1,N^*]$ and some nonrandom constant C, for all $\mathbf{x},\mathbf{y} \in \mathscr{X}^N$ with $R := \mathrm{d}_S(\mathbf{x},\mathbf{y}) \geq 1$, and for any regular domain $\mathbf{\Lambda} \subseteq \mathscr{X}^N$ (bounded or not) such that $\mathbf{\Lambda} \supset \mathbf{\Lambda}_{R/2}(\mathbf{x}) \cup \mathbf{\Lambda}_{R/2}(\mathbf{y})$*

$$\mathbb{E}\Big[\sup_{t\in\mathbb{R}} \big\| \mathbf{1}_{\mathbf{y}} P_{I^*}\big(\mathbf{H}^{(N)}_{\mathbf{\Lambda}}\big)\, \mathrm{e}^{-it\mathbf{H}^{(N)}_{\mathbf{\Lambda}}}\, \mathbf{1}_{\mathbf{x}} \big\| \Big] \leq C\,\mathrm{e}^{-\nu(\mathrm{d}_S(\mathbf{x},\mathbf{y}))^{\kappa}}. \tag{10}$$

Here $P_{I^*}\big(\mathbf{H}^{(N)}_{\mathbf{\Lambda}}\big)$ is the spectral projection on I^* for the operator $\mathbf{H}^{(N)}_{\mathbf{\Lambda}}(\omega)$.

(B) *With probability one, all eigenfunctions $\mathbf{\Psi}_j(\omega)$ of $\mathbf{H}(\omega)$ with eigenvalues $E_j(\omega) \in I^*$ decay exponentially fast at infinity: for each $\mathbf{\Psi}_j(\omega)$ there exists an integer $r_j(\omega)$ such that for all $\mathbf{x} \in \mathscr{Z}^N$ with $|\mathbf{x}| \geq r_j(\omega)$*

$$\|\chi_{\mathbf{x}} \mathbf{\Psi}_j\| \leq \mathrm{e}^{-m|\mathbf{x}|}. \tag{11}$$

Remark 1 The main mechanism responsible for the onset of localization at low energies is of course the Lifshitz tails phenomenon. It allows one to establish localization for any nonzero amplitude of the random potential. However, introducing explicitly the disorder amplitude g in the random potential, i.e., replacing $V(\cdot;\omega)$ by $gV(\cdot;\omega)$ in the definition of the Hamiltonian $\mathbf{H}(\omega)$, one can prove localization in the energy interval $I^*_g = [0,E^*_g]$ with $E^*_g \to +\infty$ and with the decay rate of the eigenfunctions $m_g \to +\infty$ as $g \to +\infty$.

Remark 2 It is easy to see that the constant $\nu > 0$ in the bound (10) can be made equal to 1, and even arbitrarily large, by choosing a slightly smaller value $\kappa' \in (0,\kappa)$ and a suitable constant $C = C(\kappa,\nu)$. Indeed, $\nu R^{\kappa} = \big(\nu R^{\kappa-\kappa'}\big) R^{\kappa'}$, so for large $R = \mathrm{d}_S(\mathbf{x},\mathbf{y})$ the product in the parentheses becomes bigger than any fixed $\nu' > 0$, while for smaller R the required bound can be absorbed in a large pre-factor C. This quantity is much less pertinent than the exponent m in (11). As usual, the MSA allows us to prove *exponential* decay only for the EFs, but not for the EFCs.

2 EVC bounds

2.1 Strong regularity of the conditional mean (SRCM)

The key property of the probability distribution of the random scatterers in the flat tiling model, resulting in efficient EVC bounds and ultimately, in norm-bounds on the decay of EFCs, established in [16], can be formulated for a random field[3] $\mathscr{V} : \mathscr{Z} \times \Omega \to \mathbb{R}$ on a countable set $\mathscr{Z}$ and relative to some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Formally speaking, it does not presume independence or any explicit decay of correlations of $\mathscr{V}$.

Let be given a finite set $Q \subset \mathscr{Z}$. Introduce the sample, or empirical, mean $\xi_Q(\omega) := |Q|^{-1} \sum_{x \in Q} \mathscr{V}(x;\omega)$ of a finite sample $\{\mathscr{V}(x;\cdot), x \in Q\}$, and the fluctuations relative to the sample mean: $\eta_x(\omega) = \mathscr{V}(x;\omega) - \xi_Q(\omega)$, $x \in Q$. Denote by $\mathfrak{F}_Q$ the sigma-algebra generated by all the fluctuations $\{\eta_x, x \in Q\}$ and by all values $\{\mathscr{V}(y;\cdot), y \in \mathscr{Z} \setminus Q\}$.

(SRCM): *Given a random field $\mathscr{V} : \mathscr{Z} \times \Omega \to \mathbb{R}$ on a countable set $\mathscr{Z}$, there exist $C, A \in (0, +\infty)$ such that for any finite subset $Q \subset \mathscr{Z}$, any $\mathfrak{F}_Q$-measurable random variable $\mu(\cdot)$ and all $s \in (0,1]$, the following bound holds:*

$$\mathbb{P}\left\{ \xi_Q(\omega) \in [\mu(\omega), \mu(\omega)+s] \,\middle|\, \mathfrak{F}_Q \right\} \le C\,|Q|^A s. \tag{12}$$

Theorem 2 (Cf. [16]) *Under the assumption* (V), *the IID random field $\mathscr{V} : \mathscr{Z} \times \Omega \to \mathbb{R}$ satisfies the property* (SRCM).

For the reader's convenience, we summarize the proof in Appendix D. Note that Ref. [23] used a weaker analog of the bound (12), with the RHS of the form $C\,|Q|^A s^\theta$ with $\theta \in (0,1)$; the latter is insufficient for applications to the continuous N-particle models.

2.2 EVC bounds for the flat tiling alloy model

We start with the one-volume EVC bound, which is quite similar in form to the celebrated Wegner estimate [50]. The flat tiling alloy model is a particular case of a more general one, studied by Klein and Nguyen [40], so we can simply quote their result.

In fact, both Theorem 3 and Theorem 4 can be proved with the help of the condition (SRCM), but this would result in a less optimal volume dependence in Theorem 3, than in [40, Theorem 2.2].

Theorem 3 (Cf. [40, Theorem 2.2]) *Fix an interval $I^* = [0, E^*]$, $E^* > 0$, and let $\Sigma^{I^*}_{\mathbf{x},L} = \Sigma^{I^*}_{\mathbf{x},L}(\omega)$ be the random spectrum $\Sigma(\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{x})}(\omega)) \cap I^*$. Then for all $E \in I^*$, one has*

$$\mathbb{P}\left\{ \operatorname{dist}\left[\Sigma^{I^*}_{\mathbf{x},L}, E\right] \le s \right\} \le C_1(N, E^*, p_V) L^{Nd} s. \tag{13}$$

[3] Recall that in our paper, $\mathscr{V}$ is not the potential but the family of the random scatterers' amplitudes, labeled by the discrete lattice $\mathscr{Z} \subset \mathbb{R}^d$.

The estimate (13) suffices for the fixed-energy analysis[4], but the derivation of dynamical and spectral localization requires an EVC bound for pairs of local Hamiltonians (two-volume bound).

It is to be emphasized that we cannot apply the two-volume EVC bound from [40, Corollary 2.3], since this would only lead to the localization bounds relative to the Hausdorff distance. In contrast, the EVC bound from Theorem 4, despite its ostensibly non-optimal volume dependence, suits much better the main goal of the present paper: the proof of localization with bounds in the symmetrized norm-distance.

Theorem 4 *Under the assumptions (V) and (U), for any fixed N,d, E^* and the PDF F_V of the random scatterers, there exist some $C_2, A \in (0,+\infty)$ such that for any pair of $4NL$-distant cubes $\mathbf{\Lambda}_L(\mathbf{x})$, $\mathbf{\Lambda}_L(\mathbf{y})$ the following bound holds:*

$$\forall s \in (0,1] \quad \mathbb{P}\left\{ \operatorname{dist}\left[\Sigma^{I^*}_{\mathbf{x},L}, \Sigma^{I^*}_{\mathbf{y},L}\right] \le s \right\} \le C_2 L^A s. \tag{14}$$

It is this EVC estimate which makes possible the present work. Its proof relies on the following elements.

Definition 1 A cube $\mathbf{\Lambda}^{(N)}_L(\mathbf{x}) \subset (\mathbb{R}^d)^N$ is weakly separated (or weakly Q-separated) from $\mathbf{\Lambda}^{(N)}_L(\mathbf{y})$ iff there exists a bounded subset $Q \subset \mathbb{R}^d$, of diameter $R \le 2NL$, and the index subsets $\mathcal{J}_1, \mathcal{J}_2 \subset [[1,N]]$ such that $|\mathcal{J}_1| > |\mathcal{J}_2|$ (possibly, with $\mathcal{J}_2 = \varnothing$) and

$$\begin{aligned} &\left(\Pi_{\mathcal{J}_1}\mathbf{\Lambda}_L(\mathbf{x}) \cup \Pi_{\mathcal{J}_2}\mathbf{\Lambda}_L(\mathbf{y})\right) \subseteq Q, \\ &\left(\Pi_{\mathcal{J}_1^c}\mathbf{\Lambda}_L(\mathbf{x}) \cup \Pi_{\mathcal{J}_2^c}\mathbf{\Lambda}_L(\mathbf{y})\right) \cap Q = \varnothing. \end{aligned} \tag{15}$$

A pair of cubes $(\mathbf{\Lambda}_L(\mathbf{x}), \mathbf{\Lambda}_L(\mathbf{y}))$ is weakly separated if at least one of the cubes is weakly separated from the other.

In physical terms, the weak Q-separation of $\mathbf{\Lambda}_L(\mathbf{x})$ from $\mathbf{\Lambda}_L(\mathbf{y})$ means that there are more particles in Q from the configurations $\mathbf{u} \in \mathbf{\Lambda}_L(\mathbf{x})$ than from the configurations $\mathbf{v} \in \mathbf{\Lambda}_L(\mathbf{y})$. This renders the EVs of $\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{x})}$ more sensitive to the fluctuations of the random potential in Q than the EVs of $\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{y})}$. Yet, one can still have $\mathrm{d}^{(N)}_{\mathcal{H}}(\mathbf{\Lambda}_L(\mathbf{x}), \mathbf{\Lambda}_L(\mathbf{y})) = 0$, which makes impossible any *stricto sensu* stochastic decoupling of the random operators $\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{x})}(\omega)$ and $\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{y})}(\omega)$. This explains the choice of the term "weak" [separation]. In simpler terms, it is possible to have a situation where $\mathbf{\Lambda}_L(\mathbf{x})$ and $\mathbf{\Lambda}_L(\mathbf{y})$ are weakly separated, but every random value of the potential $V(z;\omega)$ affecting $\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{x})}(\omega)$ (hence its spectrum) affects also $\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{y})}(\omega)$ along with its spectrum, and vice versa. Note that previous works, starting with [19,21] (cf. also recent papers [39,40]) used a substantially stronger condition of "separation" – in the Hausdorff distance (as did in a different manner also Aizenman and Warzel [6] and recently Fauser and Warzel [31]).

Remark 3 The crucial difference between the two notions of "separation" is that the Hausdorff-separation implies the existence of a sub-sample of the random potential, generating some sub-sigma-algebra $\mathfrak{F}_{\mathbf{x},\mathbf{y},L}$ such that conditioning on $\mathfrak{F}_{\mathbf{x},\mathbf{y},L}$

[4] Notice that fast decay of the GFs implies a.s. absence of a.c. spectrum at low energies, thanks to a result by Martinelli and Scoppola [44] easily adapted to multi-particle Hamiltonians.

renders one of the random Hamiltonians $\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{x})}(\omega), \mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{y})}(\omega)$ *non-random*, while the other one still has a non-degenerate probability distribution sufficient for a suitable Wegner-type estimate[5]. In the latter sitution, one has therefore a convenient stochastic decoupling, which may or may not be available for the cubes which are only "weakly" separated, no matter how far apart they are in the symmetrized norm-distance. We stress again that the main improvement is brought up by the property (SRCM); without it, the notion of weak separation would be of no use. On the other hand, as was already said, (SRCM) is granted for the Gaussian scatterers' amplitudes, due to an elementary property of the empiric (sample) mean of Gaussian samples, usually taught in undergraduate courses of statistics, so only its extension to more general distributions requires a proof.

Lemma 1 *Any pair of N-particle cubes* $\mathbf{\Lambda}_L^{(N)}(\mathbf{x})$, $\mathbf{\Lambda}_L^{(N)}(\mathbf{y})$ *with* $\mathrm{d}_S(\mathbf{x},\mathbf{y}) > 4NL$ *is weakly separated.*

The proof is quite elementary and can be found in Ref. [11].

Now the assertion of Theorem 4 stems directly from the following result.

Lemma 2 *Let* $\mathscr{V} : \mathscr{Z} \times \Omega \to \mathbb{R}$ *be an IID random field satisfying the condition* (SRCM). *Consider two weakly separated N-particle cubes* $\mathbf{\Lambda}_L(\mathbf{x})$, $\mathbf{\Lambda}_L(\mathbf{y})$ *and the random operators* $\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{x})}(\omega)$, $\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{y})}(\omega)$. *Then the following bound holds true for their spectra* $\Sigma^{I^*}_{\mathbf{x},L}$, $\Sigma^{I^*}_{\mathbf{y},L}$:

$$\forall\, s>0 \qquad \mathbb{P}\left\{\mathrm{dist}(\Sigma^{I^*}_{\mathbf{x}}, \Sigma^{I^*}_{\mathbf{y}}) \le s\right\} \le C|\mathbf{\Lambda}_L(\mathbf{x})|\,|\mathbf{\Lambda}_L(\mathbf{y})|L^A s.$$

Proof Let Q be a set satisfying the conditions (15) for some $\mathscr{J}_1, \mathscr{J}_2 \subset [[1,N]]$ with $|\mathscr{J}_1| =: n_1 > n_2 := |\mathscr{J}_2|$. Consider the sample mean $\xi = \xi_Q$ of $\mathscr{V}$ over Q and the fluctuations $\{\eta_x, x \in Q\}$. Owing to the flat tiling[6] condition on the shape of the scatterers, the operators $\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{x})}(\omega)$, $\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{y})}(\omega)$ can be represented as follows:

$$\begin{aligned} \mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{x})}(\omega) &= n_1\xi(\omega)\mathbf{1} + \mathbf{A}(\omega),\\ \mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{y})}(\omega) &= n_2\xi(\omega)\mathbf{1} + \mathbf{B}(\omega), \end{aligned} \tag{16}$$

where the operators $\mathbf{A}(\omega)$ and $\mathbf{B}(\omega)$ are $\mathfrak{F}_Q$-measurable. Specifically, let $\mathscr{J}_1^{\mathrm{c}} = [[1,N]] \setminus \mathscr{J}_1$, $\mathscr{J}_2^{\mathrm{c}} = [[1,N]] \setminus \mathscr{J}_2$, and

$$\begin{aligned} \mathbf{A}(\omega) &= -\mathbf{\Delta} + \mathbf{U}_{\mathbf{B}_L(\mathbf{x})} + \sum_{j\in\mathscr{J}_1^{\mathrm{c}}} V(x_j;\omega) + \sum_{j\in\mathscr{J}_1} \eta_{x_j}(\omega),\\ \mathbf{B}(\omega) &= -\mathbf{\Delta} + \mathbf{U}_{\mathbf{B}_L(\mathbf{y})} + \sum_{j\in\mathscr{J}_2^{\mathrm{c}}} V(y_j;\omega) + \sum_{j\in\mathscr{J}_2} \eta_{y_j}(\omega). \end{aligned}$$

Then (16) follows from the identities

$$\begin{aligned} V(x_j;\omega) &= \xi(\omega) + \eta_{x_j}(\omega), \;\; j \in \mathscr{J}_1,\\ V(y_j;\omega) &= \xi(\omega) + \eta_{y_j}(\omega), \;\; j \in \mathscr{J}_2, \end{aligned}$$

[5] A nice analysis performed by Klein and Nguyen [40] shows that the Wegner estimate for the pairs of Hausdorff-separated cubes does not even require the complete covering condition for the scatterer functions.

[6] This is the only instance where the flat tiling is crucial for the two-volume EVC estimate. The assumption $\mathrm{C}_\varphi = 1$ made in Sect. 1.7 allows us to avoid this extra factor in the rest of the proof.

since $\Pi_{\mathscr{J}_1}\mathbf{B}_L(\mathbf{x}), \Pi_{\mathscr{J}_2}\mathbf{B}_L(\mathbf{y}) \subset Q$, $|\mathscr{J}_1| = n_1$, $|\mathscr{J}_2| = n_2$.

Let $\{\lambda_1, \dots, \lambda_{M'}\}$ and $\{\mu_1, \dots, \mu_{M''}\}$ be the sets of eigenvalues of $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}$ and of $\mathbf{H}_{\mathbf{B}_L(\mathbf{y})})$ in the interval I^*, counting multiplicity. Owing to positivity of the interaction energy, it follows (deterministically) from Weyl's law that $M' \leq c|\mathbf{B}_L(\mathbf{x})|$ and $M'' \leq c|\mathbf{B}_L(\mathbf{y})|$. By (16), these eigenvalues can be represented as follows:

$$\lambda_i(\omega) = n_1\xi(\omega) + \lambda_i^{(0)}(\omega), \;\; \mu_j(\omega) = n_2\xi(\omega) + \mu_j^{(0)}(\omega), \tag{17}$$

where the random variables $\lambda_i^{(0)}(\omega)$ and $\mu_j^{(0)}(\omega)$ are $\mathfrak{F}_Q$-measurable. Therefore,

$$\lambda_i(\omega) - \mu_j(\omega) = (n_1 - n_2)\xi(\omega) + (\lambda_i^{(0)}(\omega) - \mu_j^{(0)}(\omega)), \tag{18}$$

with $n := n_1 - n_2 \geq 1$, by assumption. Denote

$$I_s^{(ij)}(\omega) := [\theta_{ij} - t_s, \theta_{ij} + t_s], \;\; \theta_{ij}(\omega) := \frac{\mu_j^{(0)}(\omega) - \lambda_i^{(0)}(\omega)}{n}, \;\; t_s := \frac{s}{n} \leq s.$$

The random intervals $I_s^{(ij)}$ are $\mathfrak{F}_Q$-measurable (i.e., rendered nonrandom by conditioning on $\mathfrak{F}_Q$), so we can apply (SRCM) and obtain

$$\begin{aligned} \mathbb{P}\left\{\operatorname{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq s\right\} &\leq \sum_{i=1}^{M'}\sum_{j=1}^{M''} \mathbb{P}\left\{\xi \in I_s^{(ij)}(\omega)\right\} \\ &\leq c^2|\mathbf{B}_L(\mathbf{x})| \cdot |\mathbf{B}_L(\mathbf{y})| C L^A s, \end{aligned} \tag{19}$$

where the value of the constant c can be made explicit with the help of from Weyl's law; it is irrelevant for the rest of our analysis. □

The two-volume EVC estimate (14) is thus established.

It is readily seen that a more traditional, one-volume EVC bound can be proved with an analogous (indeed, simpler) argument.

Remark 4 For further use (cf. Sect. 5.1), note that, conditional on $\mathfrak{F}_Q$, the (conditional) probability distribution of $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}(\omega)$ is supported by the one-parameter family of operators $\{tn_1\mathbf{1} + \mathbf{A}, t \in \mathbb{R}\}$ (we omit the argument ω in $\mathbf{A}(\omega)$ to stress that it is rendered non-random by the conditioning), and the probability measure on the real line $\mathbb{R} \ni t$ is the conditional distribution of the sample mean $\xi = \xi_Q$, given $\mathfrak{F}_Q$. This operator family is obviously commutative, thus admits common eigenfunctions. As a result, the resolvent operators, depending upon a single random parameter $\xi(\omega)$, have the form

$$\mathbf{G}_{\mathbf{B}_L(\mathbf{x})}(E; \xi(\omega)) = \mathbf{G}_{\mathbf{B}_L(\mathbf{x})}(E - n_1\xi(\omega); 0). \tag{20}$$

A similar representation holds true for $\mathbf{G}_{\mathbf{B}_L(\mathbf{y})}(E; \xi(\omega))$:

$$\mathbf{G}_{\mathbf{B}_L(\mathbf{y})}(E; \xi(\omega)) = \mathbf{G}_{\mathbf{B}_L(\mathbf{y})}(E - n_2\xi(\omega); 0). \tag{21}$$

3 First scaling analysis. Sub-exponential decay of the GFs

With $N^* \geq 2$ fixed, we shall prove, as in Ref. [40], localization bounds for every energy E in an energy interval $I^* = [0, E^*]$ with $E^* > 0$ determined by the parameters of the model, first of all by the common marginal PDF F_V of the random scatterers' amplitudes and by the global amplitude $g > 0$ (cf. (1)). Specifically, for any given PDF F_V satisfying (V), we can guarantee that our bounds, implying exponential spectral and sub-exponential dynamical localization, hold true in an interval $I^* = I^*(F_V, g)$ of positive length; the smaller is $g > 0$, the smaller must be $E^* = E^*(g)$.

Conversely, the larger is the amplitude g in (1), the larger is $E^*(g)$ (with a fixed F_V). In fact, the latter is essentially determined by the initial length scale (ILS) bound. The starting point for the scaling analysis is, as usual, a Lifshitz tails estimate. Furthermore, for a fixed PDF F_V and $g > 0$ large enough, the large deviations estimate can be replaced with a much simpler probabilistic argument, going back to [26] and proving the ILS bound for *any continuous* F_V and $g \gg 1$. See the discussion in Sect. 3.3.

As was said, we consider the finite-volume analysis as more relevant for applications to physical models; keeping this in mind, note that the spectrum of $\mathbf{H}^{(N)}_{\Lambda^N}(\omega)$ is of course random, with the ground state energy $E_0^{(N)}(\Lambda, \omega)$ strictly positive with probability 1, but, clearly, $E_0^{(N)}(\Lambda, \omega) \to 0$ in probability, as $\Lambda \nearrow \mathbb{R}^d$. Therefore, localization bounds established even in a tiny interval $[0, E^*]$, $0 < E^* \ll 1$, make sense for all Λ large enough.

3.1 Dominated decay of the GFs

The main technical tool used here is the Geometric Resolvent Inequality (sometimes considered as an analog of the Simon-Lieb inequality), well-known in the single-particle theory and applicable to the multi-particle Anderson Hamiltonians as well, for the structure of the potential is irrelevant for this general analytic result.

Proposition 2 (Cf. [49, Lemma 2.5.4]) *Let be given two cubes* $\mathbf{\Lambda} = \mathbf{\Lambda}_\ell(\mathbf{u}) \Subset \mathbf{\Lambda}' = \mathbf{\Lambda}_L(\mathbf{v})$. *There is a real number* C^{GRI} *depending upon* $\mathrm{dist}(\mathbf{\Lambda}, \partial\mathbf{\Lambda}')$, *such that for any measurable sets* $\boldsymbol{A} \subset \mathbf{\Lambda}_{L/3}(\mathbf{u})$ *and* $\boldsymbol{B} \subset \mathbf{\Lambda}' \setminus \mathbf{\Lambda}$,

$$\|\mathbf{1}_{\boldsymbol{B}} \mathbf{G}_{\mathbf{\Lambda}}(E) \mathbf{1}_{\boldsymbol{A}}\| \leq C^{\mathrm{GRI}} \|\mathbf{1}_{\boldsymbol{B}} \mathbf{G}_{\mathbf{\Lambda}'}(E) \mathbf{1}_{\mathbf{\Lambda}^{\mathrm{out}}}\| \cdot \|\mathbf{1}_{\mathbf{\Lambda}^{\mathrm{out}}} \mathbf{G}_{\mathbf{\Lambda}}(E) \mathbf{1}_{\boldsymbol{A}}\|. \tag{22}$$

The boundary layer $\mathbf{\Lambda}^{\mathrm{out}}$ was defined in Sect. (1.4). Introduce a notation that will be often used below:

$$\left\|\mathbf{G}_{\mathbf{\Lambda}_L(\mathbf{u})}\right\|^{\curlywedge} := \left\|\mathbf{1}_{\mathbf{\Lambda}_L^{\mathrm{out}}(\mathbf{u})} \mathbf{G}_{\mathbf{\Lambda}_L(\mathbf{u})} \chi_{\mathbf{u}}\right\| \tag{23}$$

(here $\curlywedge$ symbolizes the decay from the center to the boundary of a cube). It is of course slightly abusive, for the symbol $\curlywedge$ should have been labeled by the parameters of the cube, but its meaning will always be clear from the context.

In this section, we work with the length scales $L_k = L_0 Y^k$, $L_0, Y \in \mathbb{N}^*$, $Y > 1$, $k \geq 0$.

Corollary 1 *Consider the embedded cubes* $\mathbf{\Lambda} = \mathbf{\Lambda}_{L_k}(\mathbf{x}) \Subset \widetilde{\mathbf{\Lambda}} = \mathbf{\Lambda}_{L_{k+1}}(\mathbf{u})$. *Let* $\mathbf{B} = \mathbf{\Lambda} \cap \mathscr{Z}$, $\widetilde{\mathbf{B}} = \widetilde{\mathbf{\Lambda}} \cap \mathscr{Z}$. *For any cell* $\mathbf{C}(\mathbf{y}) \subset \widetilde{\mathbf{\Lambda}}^{\text{out}}$, *one has*

$$\|\chi_{\mathbf{y}} \mathbf{G}_{\widetilde{\mathbf{\Lambda}}} \chi_{\mathbf{u}}\| \le C^{\text{GRI}} \left\|\mathbf{G}_{\mathbf{\Lambda}}\right\|^{\curlywedge} \|\chi_{\mathbf{y}} \mathbf{G}_{\widetilde{\mathbf{\Lambda}}} \mathbf{1}_{\mathbf{\Lambda}^{\text{out}}}\|, \tag{24}$$

and consequently (cf. (5)*),*

$$\begin{aligned}\|\chi_{\mathbf{y}} \mathbf{G}_{\widetilde{\mathbf{\Lambda}}} \chi_{\mathbf{u}}\| &\le \sum_{\mathbf{z}\in\partial\mathbf{B}} C^{\text{GRI}} \left\|\mathbf{G}_{\mathbf{\Lambda}}\right\|^{\curlywedge} \|\chi_{\mathbf{y}} \mathbf{G}_{\widetilde{\mathbf{\Lambda}}} \chi_{\mathbf{z}}\| \\ &\le C' L^{Nd} \left\|\mathbf{G}_{\mathbf{\Lambda}}\right\|^{\curlywedge} \max_{\mathbf{z}\in\partial\mathbf{B}} \|\chi_{\mathbf{y}} \mathbf{G}_{\widetilde{\mathbf{\Lambda}}} \chi_{\mathbf{z}}\|\end{aligned} \tag{25}$$

with $C' = C'(N,d,C^{\text{GRI}})$.

An ideal situation for the localization analysis in a cube $\mathbf{\Lambda}_{L_{k+1}}(\mathbf{x})$, in the course of the scaling step $L_k \rightsquigarrow L_{k+1}$, is where all sub-cubes $\mathbf{\Lambda}_{L_k}(\mathbf{v}) \subset \mathbf{\Lambda}_{L_{k+1}}(\mathbf{x})$ have the property $\left\|\mathbf{G}_{\mathbf{\Lambda}_{L_k}(\mathbf{v})}(E)\right\|^{\curlywedge} \ll 1$, but it is well-known, both in the single- and multi-particle MSA (cf., e.g., [26,32,21,39,40,22]) that presence of a limited number of "unsuitable" cubes is relatively harmless – under certain additional conditions which can be informally described as "absence of abnormally strong resonances". Formal statements (cf. Appendix A) and their applications to the scaling analysis require some notions introduced in Definition 2 below.

As usual, we call a generalized eigenfunction with (generalized) eigenvalue E of the operator $\mathbf{H}$ any polynomially bounded solution to the equation $\mathbf{H}\mathbf{\Psi} = E\mathbf{\Psi}$.

In the next statement, we assume the conditions (V) and (U) to be fulfilled, but the assertions are well-known to remain valid for a much larger class of Schrödinger operators; cf., e.g., [45,46,48,32,7,38,39,40].

Proposition 3 (Cf. [49, Lemma 3.3.2]) *For spectrally a.e.* $E \in \Sigma(\mathbf{H})$ *there exists a generalized eigenfunction* $\mathbf{\Psi}$, *with eigenvalue* E. *Furthermore,*

$$\|\chi_{\mathbf{x}} \mathbf{\Psi}\| \le \left\|\mathbf{G}_{\mathbf{\Lambda}_L(\mathbf{x})}(E)\right\|^{\curlywedge} \|\mathbf{1}_{\mathbf{\Lambda}_L^{\text{out}}(\mathbf{x})} \mathbf{\Psi}\|. \tag{26}$$

3.2 Induction hypothesis

Introduce the following definitions.

Definition 2 Let be given real numbers $m > 0$, $\delta \in (0,1]$, E and integers $k > 0$, $N \ge 1$. A cube $\mathbf{\Lambda}^{(N)}_{L_k}(\mathbf{u})$, as well as its lattice counterpart $\mathbf{B}^{(N)}_{L_k}(\mathbf{u})$, is called (E,δ,m)-non-singular ((E,δ,m)-NS) iff

$$C^{\text{GRI}} (3L_k)^{Nd} \left\|\mathbf{G}_{\mathbf{\Lambda}_{L_k}\mathbf{x}}(E)\right\|^{\curlywedge} \le \mathrm{e}^{-mL_k^{\delta}}. \tag{27}$$

$\mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u})$, as well as $\mathbf{B}^{(N)}_{L_{k+1}}(\mathbf{u})$, is called (E,β)-non-resonant ((E,β)-NR) iff

$$\operatorname{dist}\left(\Sigma_{\mathbf{u},L_{k+1}}, E\right) \ge 2\mathrm{e}^{-L_{k+1}^{\beta}}. \tag{28}$$

$\mathbf{\Lambda}^{(N)}_{L}(\mathbf{u})$ and $\mathbf{B}^{(N)}_{L}(\mathbf{u})$ are called E-CNR iff for all integers $\ell \in [L_k, L_{k+1} - L_k - 2]$,

$$\operatorname{dist}\left(\Sigma_{\mathbf{u},\ell}, E\right) \ge 2\mathrm{e}^{-L_{k+1}^{\beta}}. \tag{29}$$

Here $(3L)^{Nd}$ is a crude upper bound on the cardinality $|\partial^{-}\mathbf{B}_L(\mathbf{x})|$. Introducing combinatorial factors in the LHS of the bounds like (27) is not traditional, but it allows us to replace the sums appearing in the applications of the GRI with maxima, which better suits the technique of "radial descent" presented in Appendix A.

The goal of the scale induction is to prove recursively the following property for some interval $I \subset \mathbb{R}$:

$\mathsf{S}(N,k,I)$: Given integers $N^* \geq 2$, $L_0 \geq 1$, $Y \geq 2$, the integer sequence $\{L_j := L_0 Y^j,\ j \geq 0\}$, the real numbers $m^* > 0$, $\nu^* > 0$ and the finite sequences

$$m_n := m^*(1 + 4L_0^{-\delta+\beta})^{N^*-n},\ \ \nu_n := 2\nu^*(2Y^{\kappa})^{N^*-n},\ 1 \leq n \leq N^*,$$

the following property is fulfilled for all $1 \leq n \leq N$, $L \in [L_0, L_{k+1})$ and $E \in I$:

$$\mathbb{P}\left\{ \mathbf{\Lambda}_L^{(n)}(\mathbf{x}) \text{ is } (E,\delta,m_n)\text{-S} \right\} \leq \mathrm{e}^{-\nu_n L^{\kappa}}. \tag{30}$$

In fact, it will be convenient to choose the interval I N-dependent (cf. the table (33)).

3.3 Initial length scale estimate

The assumption of non-negativity of the interaction greatly simplifies the EVC analysis in the continuous multi-particle models near the bottom of the spectrum, if the disorder amplitude g is not supposed to be large; it is not required in the strong disorder regime ($g \gg 1$). The key observation here is that any non-negative interaction can only move the EVs up, thus resulting automatically in stronger ILS estimates (in any interval of the form $(-\infty, E^*]$) for the interactive model at hand than with the interaction switched off.

We cannot apply directly the ILS estimate from [18, Lemma 3.1], for the latter provides only a power-law decay of the probability of unwanted events, while we need an input for the sub-exponential MSA induction[7]. However, a direct inspection of the proof of [49, Theorem 2.2.3], on which [18, Lemma 3.1] is based, shows that, given any integer $N^* \geq 2$, for any $0 < \gamma < 1/2$ and $\varepsilon > 0$ there exist $L_0 > 0, m, \nu, E_1^* \in (0,+\infty)$ such that for all $1 \leq N \leq N^*$ and all energies $E \in [0, E_N^*]$, with

$$E_N^* := 2^{-(N-1)} E^*,\ \ N = 1, \ldots, N^*, \tag{31}$$

one has[8]

$$\mathbb{P}\left\{ \|\mathbf{1}_{\mathbf{\Lambda}_L^{(\mathrm{out})}(\mathbf{x})} \mathbf{G}_{\mathbf{\Lambda}_L(\mathbf{x})}(E) \chi_{\mathbf{x}}\| > \mathrm{e}^{-mL^{\frac{1+\gamma}{2}}} \right\} \leq \mathrm{e}^{-\nu L^{\frac{1-\gamma-\varepsilon}{2}}}$$

[7] This is the price to pay for skipping the first phases of the bootstrap in our simplified scheme.

[8] The proof of Theorem 2.2.3 in [49] is based on a very strong probabilistic (large deviations) estimate, but the final conclusion (see the last line of the multi-line equation on p.48) is volontarily made substantially weaker, to merely suit the requirements of the MSA scheme with power-law decay of unwanted probabilities.

or, equivalently,

$$\mathbb{P}\left\{\|\mathbf{1}_{\boldsymbol{\Lambda}_L^{(\mathrm{out})}(\mathbf{x})}\mathbf{G}_{\Lambda_L(x)}(E)\chi_{\mathbf{x}}\| > \mathrm{e}^{-(mL^{\varepsilon/2})L^{\frac{1+\gamma-\varepsilon}{2}}}\right\} \le \mathrm{e}^{-(\nu L^{\varepsilon/2})L^{\frac{1-\gamma-2\varepsilon}{2}}}.$$

In other words, without any assumption on the (nonzero) amplitude of the random potential, the (E,δ,m)-NS property with any fixed $\delta\in(0,3/4)$ can be proved to hold in a cube of size L_0 with probability $\le \mathrm{e}^{-\nu L_0^{\kappa}}$ for some $\kappa>0$; here $3/4$ is obtained as $\frac{1+\gamma}{2}$ with $\gamma=1/2$. For example, with $\gamma=7/18$, $\varepsilon=1/18$ we obtain

$$\mathbb{P}\left\{\|\mathbf{1}_{\boldsymbol{\Lambda}_{L_0}^{(\mathrm{out})}(\mathbf{x})}\mathbf{G}^{(1)}_{\boldsymbol{\Lambda}_{L_0}(\mathbf{x})}(E)\chi_{\mathbf{x}}\| > \mathrm{e}^{-m(L_0)L_0^{2/3}}\right\} \le \mathrm{e}^{-\nu(L_0)L_0^{1/4}},$$

where $m_0(L_0)$, $\nu_0(L_0)\to+\infty$ as $L_0\to+\infty$. In particular, we can assume, with or without the strong disorder hypothesis ($g\gg1$), that

$$\widetilde{m}\ge 5NY,\ \widetilde{\nu}\ge 55NY, \tag{32}$$

where the integer $Y>1$ defines the length scale sequence $L_k=L_0Y^k$ figuring in the induction hypothesis $\mathsf{S}(N,k,I^*)$. These estimates will be required in Sect. 5.2 (cf. (64)).

The bootstrap strategy ultimately results in stronger estimates stemming from weaker initial assumptions, but deriving such estimates requires one to go through the bootstrap steps, which takes a bit longer than a more straightforward approach summarized, e.g., in the book [49].

In the strong disorder regime, where $g\gg1$, the ILS estimate, with $\delta=1$ and $E_g^*, m^*=m_g^*\to+\infty$ as as $g\to+\infty$, can be easily proved without resorting to the large deviations theory serving as the base for the Lifshitz tails argument. Indeed, for g large, with high probability all the scatterers' amplitudes affecting the random potential in a given cube $\boldsymbol{\Lambda}_{L_0}(\mathbf{x})$ are large, so the potential energy $\ge C(g)\gg1$, thus the entire interval $[0,C(g)/2]$ is deep inside the classically forbidden energy zone. The required exponential decay of the Green functions can be derived from the Combes–Thomas estimate [24,10]; in fact, it follows from more robust mechanisms providing stronger decay bounds in the classically forbidden "under-barrier" zone (cf., e.g., [30]), well-known from standard courses of quantum mechanics.

Summarising, we come to the following

Proposition 4 *Fix any integer $N^*\ge2$. Under the assumptions (V) and (U), there exists some $\kappa>0$ with the following property: for any $g>0$, $m^*>0$, $\nu^*>0$ there exists an integer $L_0^*=L_0^*(g,m^*,\nu^*,N^*)$ and an interval $I_g^*=[0,E_g^*]$ with $E_g^*>0$ such that $S\big(N,0,I_g^*\big)$ holds true for all $E\in I_g^*$, $1\le N\le N^*$ and $L_0\ge L_0^*$. Moreover, $E_g^*\to+\infty$ as $g\to+\infty$.*

3.4 Analytic scaling step

Definition 3 A cube $\boldsymbol{\Lambda}_{L_k}^{(N)}(\mathbf{u})$ is called weakly interactive (WI) if

$$\operatorname{diam}\Pi\mathbf{u}\equiv\max_{i\ne j}|u_i-u_j|>3NL_k,$$

and strongly interactive (SI), otherwise.

Observe that the properties WI/SI are permutation-invariant, so that both the norm-distance and its symmetrized counterpart d_S can be used in the next definition.

Definition 4 A cube $\mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x})$, $k \geq 0$, is called (E,δ,m_N)-bad if it contains either a weakly interactive (E,δ,m_N)-S cube of radius L_k or a pair of $9NL_k$-distant, (E,δ,m_N)-S, strongly interactive cubes of radius L_k. Otherwise, it is called (E,δ,m_N)-good.

Allowing for at most one bad SI cube ultimately gives rise to the sub-exponential decay bounds with some (possibly small) exponent $\kappa > 0$. To achieve a better bound, one has to resort to a more elaborate scheme; this is what we are compelled to do in the course of the second scaling analysis, while here we use a slightly simpler procedure.

In Section 4, we will work with exponentially decaying Green functions; this corresponds to $\delta = 1$; for brevity, we will write there (E,m)-S instead of $(E,1,m)$-S.

For the reader's convenience, we summarize in the table below the assumptions on the key parameters used in the scale induction with $L_k = L_0 Y^k$, $k \geq 0$.

$0 < \kappa < \beta < \delta < \min\left[\zeta, \frac{3}{4}\right]$	$Y \geq \max\left[30N^*, 12^{\frac{1}{1-2\delta}}\right]$, so $\frac{1}{4}Y^{1-2\delta} \geq 3$
$E^*_N = 2^{-N+1} E^*_1$ $I^*_N = [0, E^*_N]$, $1 \leq N \leq N^*$	$\nu_N = 3\nu^* \left(2Y^{2\kappa}\right)^{N^*-N}$ $m_N = m^* \left(1 + 4L_0^{-\delta+\beta}\right)^{N^*-N}$

(33)

Remark 5 As was explained in Sect. 3.3, without any assumption on the amplitude $g > 0$ of the random potential, the exponent δ can be chosen arbitrarily close to (but smaller than) $3/4$ at the initial scale L_0; the induction requires that $\delta < \zeta$. Taking g large enough, one can achieve $\delta = 1$ at the scale L_0, but the restriction $\delta < \zeta$ due to the long-range interaction remains in force.

In Theorem 1, we claim that spectral and dynamical localization hold in a nontrivial interval $I^* = [0, E^*]$, for the sake of brevity, but in the course of the induction in the number of particles N, it is convenient to actually prove localization for the N-particle systems in the energy intervals I^*_N, where $I^*_1 \supset \cdots \supset I^*_{N^*}$. The particular form of the threshold $E^*_N = 2^{-N+1} E^*_1$ is not crucial for the proof, but we need that $E^*_{N-1} - E^*_N > 0$.

The pivot of the deterministic component of the scaling analysis is the following result, various forms of which are well-known in the single-particle theory.

Lemma 3 *Fix two integers $k \geq 0$ and $N \in [[1, N^*]]$, and suppose that a cube $\mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x})$ is (E,δ,m_N)-*good *and (E,β)-*CNR *. If L_0 is large enough, then $\mathbf{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{x})$ is (E,δ,m_N)-*NS.

For completeness, we sketch the proof in Appendix A; it requires some preliminary results (cf. Lemmas 12 and 13).

3.5 Probabilistic scaling step

3.5.1 Weakly interactive (WI) cubes

Recall that we introduced in Sect. 1.4 the full projection Π and the partial projections $\Pi_{\mathscr{J}}$, with $\mathscr{J} \subseteq [[1,N]]$.

Lemma 4 *For any weakly interactive cube $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u})$ there is a factorization $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u}) = \mathbf{\Lambda}_{L_k}^{(n')}(\mathbf{u}') \times \mathbf{\Lambda}_{L_k}^{(n'')}(\mathbf{u}'')$ with*

$$\operatorname{dist}\big(\Pi\mathbf{\Lambda}_{L_k}^{(n')}(\mathbf{u}'), \Pi\mathbf{\Lambda}_{L_k}^{(n'')}(\mathbf{u}'')\big) > L_k. \tag{34}$$

Proof Assuming $\operatorname{diam}(\Pi\mathbf{u}) > 3NL_k$, let us show that the projection $\Pi\mathbf{\Lambda}_{3L_k/2}^{(N)}(\mathbf{u})$ is not a connected subset of $\mathbb{R}^d$. Assume otherwise; then for any partition of the particle index set, $\mathscr{J} \sqcup \mathscr{J}^{\rm c} = [[1,N]]$, denoting $\mathbf{u}' = \Pi_{\mathscr{J}}\mathbf{u}$, $\mathbf{u}'' = \Pi_{\mathscr{J}^{\rm c}}\mathbf{u}$, we have $\mathrm{d}(\Pi\mathbf{u}', \Pi\mathbf{u}'') \le 2 \cdot \frac{3L_k}{2} = 3L_k$, hence $\operatorname{diam} \Pi\mathbf{u} \le (N-1) \cdot 3L < 3NL$, contrary to our hypothesis.

Thus we have $\mathrm{d}\big(\Pi\mathbf{\Lambda}_{3L_k/2}^{(n')}(\mathbf{u}'), \Pi\mathbf{\Lambda}_{3L_k/2}^{(n'')}(\mathbf{u}'')\big) > 0$, for some partition $(\mathscr{J}, \mathscr{J}^{\rm c})$, so

$$\mathrm{d}\left(\Pi\mathbf{\Lambda}_{L_k}^{(n')}(\mathbf{u}'), \Pi\mathbf{\Lambda}_{L_k}^{(n'')}(\mathbf{u}'')\right) > \tfrac{1}{2}L_k + \tfrac{1}{2}L_k = L_k,$$

as asserted. □

We will assume that one such factorization is associated with each WI cube (even if it is not unique), and call it the canonical one. For the Hamiltonian in a WI cube we have the following algebraic representation: with $\mathbf{\Lambda}' = \mathbf{\Lambda}_{L_k}^{(n')}(\mathbf{u}')$, $\mathbf{\Lambda}'' = \mathbf{\Lambda}_{L_k}^{(n'')}(\mathbf{u}'')$,

$$\begin{aligned} \mathbf{H} &= \mathbf{H}^{\rm ni} + \mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''} \\ &= \mathbf{H}_{\mathbf{\Lambda}'} \otimes \mathbf{1}^{(n'')} + \mathbf{1}^{(n')} \otimes \mathbf{H}_{\mathbf{\Lambda}''} + \mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''} \end{aligned} \tag{35}$$

where, due to the assumption (U), the interaction $\mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''}$ between the components in $\mathbf{\Lambda}'$ and $\mathbf{\Lambda}''$ obeys

$$\|\mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''}\| \le C_U N(N-1)\mathrm{e}^{-L_k^{\zeta}}. \tag{36}$$

It is the next statement which allows us to treat the interactions of infinite range. Its proof, provided in Appendix B, is based on Lemma 14 extending [18, Lemma 4.1], [21, Lemma 3] to the sub-exponentially decaying interaction potentials in $\mathbb{R}^d$.

Lemma 5 *Assume the property $\mathsf{S}(N-1,k,I^*)$. If L_0 is large enough, then for any WI cube $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u})$*

$$\mathbb{P}\left\{\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u}) \textit{ is } (E,\delta,m_N)\text{-}S\right\} \le \mathrm{e}^{-\frac{3}{2}\nu_N L_{k+1}^{\kappa}} \tag{37}$$

and therefore,

$$\mathbb{P}\left\{\mathbf{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{u}) \textit{ contains a WI } (E,\delta,m_N)\text{-}S \textit{ ball of radius } L_k\right\} \le \frac{1}{4}\mathrm{e}^{-\nu_N L_{k+1}^{\kappa}}. \tag{38}$$

3.5.2 Strongly interactive (SI) cubes

Recall that we denoted by r_1 the diameter of the support of the scatterer functions.

Lemma 6 *If two* SI *cubes* $\mathbf{\Lambda}_L^{(N)}(\mathbf{x})$, $\mathbf{\Lambda}_L^{(N)}(\mathbf{y})$ *are* $9NL$*-distant and* $L > 2\mathrm{r}_1$*, then*

$$\Pi\mathbf{\Lambda}_{L+\mathrm{r}_1}^{(N)}(\mathbf{x}) \cap \Pi\mathbf{\Lambda}_{L+\mathrm{r}_1}^{(N)}(\mathbf{y}) = \varnothing \tag{39}$$

and, consequently, the random operators $\mathbf{H}_{\mathbf{\Lambda}_L^{(N)}(\mathbf{x})}(\omega)$, $\mathbf{H}_{\mathbf{\Lambda}_L^{(N)}(\mathbf{y})}(\omega)$ *are independent.*

Proof By definition, for any SI cubes $\mathbf{\Lambda}_L^{(N)}(\mathbf{x})$, $\mathbf{\Lambda}_L^{(N)}(\mathbf{y})$ we have

$$\max_{i,j} \mathrm{d}(x_i, x_j) \le 3NL, \;\; \max_{i,j} \mathrm{d}(y_i, y_j) \le 3NL,$$

and it follows from the assumption $\mathrm{d}(\mathbf{x},\mathbf{y}) > 9NL$ that for some $i', j' \in \{1,\dots,N\}$ one has $\mathrm{d}(x_{i'}, y_{j'}) > 9NL$, thus for all $i, j \in \{1,\dots,N\}$ we have by the triangle inequality

$$\mathrm{d}(x_i, y_j) \ge \mathrm{d}(x_{i'}, y_{j'}) - \mathrm{d}(x_{i'}, x_i) - \mathrm{d}(y_{j'}, y_j) > 9NL - 2\cdot 3NL \ge 2NL + 2\mathrm{r}_1.$$

Therefore,

$$\mathrm{dist}\big(\Pi\mathbf{\Lambda}_{L+\mathrm{r}_1}(\mathbf{x}), \Pi\mathbf{\Lambda}_{L+\mathrm{r}_1}(\mathbf{y})\big) > 2(N-1)L \ge 0,$$

so $\Pi\mathbf{\Lambda}_{L+\mathrm{r}_1}^{(N)}(\mathbf{x}) \cap \Pi\mathbf{\Lambda}_{L+\mathrm{r}_1}^{(N)}(\mathbf{y}) = \varnothing$. This implies independence of the samples of the random potential affecting the operators $\mathbf{H}_{\mathbf{\Lambda}_{L+\mathrm{r}_1}^{(N)}(\mathbf{x})}(\omega)$ and $\mathbf{H}_{\mathbf{\Lambda}_{L+\mathrm{r}_1}^{(N)}(\mathbf{y})}(\omega)$. □

3.5.3 The probabilistic scale induction

Theorem 5 *Suppose that* $\mathsf{S}(N,0,I^*)$ *holds true, and for all* $k \ge 0$*, one has*

$$\mathbb{P}\left\{ \mathbf{\Lambda}_{L_{k+1}}(\mathbf{u}) \textit{ is } (E,\beta)\text{-CNR} \right\} \le \frac{1}{4}\mathrm{e}^{-\nu_N L_k^{\kappa}},$$

for some $\kappa < \beta < \delta$*. If* L_0 *is large enough, then* $\mathsf{S}(N,k,I^*)$ *holds true for all* $k \ge 0$*.*

Proof It suffices to derive $\mathsf{S}(N,k+1,I^*)$ from $\mathsf{S}(N,k,I^*)$. By Lemma 3, if $\mathbf{\Lambda}_{L_{k+1}}(\mathbf{u})$ is (E,δ,m_N)-S, then it is either (E,δ,m_N)-bad or not (E,β)-CNR. Let

$$\begin{aligned}
\mathrm{P}_i &:= \mathbb{P}\left\{ \mathbf{\Lambda}_{L_i}(\mathbf{u}) \text{ is } (E,\delta,m_N)\text{-S} \right\}, \;\; i = k, k+1, \\
\mathrm{S}_{k+1} &:= \mathbb{P}\left\{ \mathbf{\Lambda}_{L_{k+1}}(\mathbf{u}) \text{ contains a WI, } (E,\delta,m_N)\text{-S cube of radius } L_k \right\}, \\
\mathrm{Q}_{k+1} &:= \mathbb{P}\left\{ \mathbf{\Lambda}_{L_{k+1}}(\mathbf{u}) \text{ is not } (E,\beta)\text{-CNR} \right\} \le \frac{1}{4}\mathrm{e}^{-\nu_N L_k^{\kappa}}
\end{aligned} \tag{40}$$

(the last inequality is assumed, but its validity actually follows from Theorem 3). Further, an (E,δ,m_N)-bad cube $\mathbf{\Lambda}_{L_{k+1}}(\mathbf{u})$ must contain either a WI, (E,δ,m_N)-S cube of radius L_k (with probability $\mathrm{S}_{k+1} \le \frac{1}{4}\mathrm{e}^{-\nu_N L_{k+1}^{\kappa}}$ by Lemma 5), or at least one pair of $9NL_k$-distant cubes $\mathbf{\Lambda}_{L_k}(\mathbf{v}_i)$, $i=1,2$, which are SI and (E,δ,m_N)-S. By virtue of Lemma 6, the random operators $\mathbf{H}_{\mathbf{\Lambda}_{L_k}(\mathbf{v}_1)}(\omega)$, $\mathbf{H}_{\mathbf{\Lambda}_{L_k}(\mathbf{v}_2)}(\omega)$ are independent, thus such a pair inside $\mathbf{\Lambda}_{L_{k+1}}(\mathbf{u})$ exists with probability

$$\le CL_{k+1}^{2Nd}\mathrm{P}_k^2 \le \mathrm{e}^{-2\nu_N L_k^{\kappa}+C\ln L_{k+1}} \le \frac{1}{4}\mathrm{e}^{-\nu_N L_k^{\kappa}},$$

provided L_0 (hence every L_k, $k\ge 0$) is large enough. Therefore,

$$\begin{aligned}\mathrm{P}_{k+1} &\le CL_{k+1}^{2Nd}\mathrm{P}_k^2 + \mathrm{S}_{k+1} + Q_{k+1}\\ &\le \frac{1}{4}\mathrm{e}^{-\nu_N L^{\kappa}} + \frac{1}{4}\mathrm{e}^{-\nu_N L^{\kappa}} + \frac{1}{4}\mathrm{e}^{-\nu_N L^{\kappa}} < \mathrm{e}^{-\nu_N L_k^{\kappa}}.\end{aligned}$$

□

This marks the end of the fixed-energy analysis of the Green functions with the length scales $L_k = Y^k L_0$ and sub-exponential decay bounds.

4 Second scaling analysis. Exponential decay of the GFs

4.1 Induction hypothesis

Sub-exponential decay of the localized eigenfunctions can be derived from the results of the first scaling analysis with the length scales $L_k = Y^k L_0$, viz. from the fractional-exponential decay of the EF correlators. This would correspond to the general strategy of [31] where the sub-exponentially decaying interactions are considered, except for the fact that we establish the EFC decay in the symmetrized norm-distance and not in the Hausdorff distance. This is the reverse of the medal in the Fractional Moment Method: the decay analysis of the eigenfunctions is logically subordinate to that of the EF correlators. The Multi-Scale Analysis, however, is free of such a logical dependence.

In order to prove a *genuine exponential* decay of the EFs, which is the subject of this section, we have to replace the induction hypothesis $\mathsf{S}(N,k,I^*)$ with the following one:

$\widetilde{\mathsf{S}}(N,k,I^*)$: Given integers $N^*\ge 3$, $L_0\ge 1$, $\alpha\ge 2$, the integer sequence $\{L_j := (L_0)^{\alpha^j},\ j\ge 0\}$, the real numbers $E^* = 2m^* > 0$, $P^* > 0$ and the sequences

$$E_n^* = 2m_n := 2^{-n+1}E^*,\ \ P(n,k) := 2^k P^*(2\alpha)^{N^*-n},\ 1\le n\le N^*,$$

the following property is fulfilled for all $1\le n\le N$:

$$\forall\, E\in I_n^* := [0,E_n^*] \quad \mathbb{P}\left\{\mathbf{\Lambda}_{L_k}^{(n)}(\mathbf{x}) \text{ is } (E,m_n)\text{-S}\right\} \le L_k^{-P(n,k)}. \tag{41}$$

Recall that (E,m_N)-S is a shortcut for $(E,1,m_N)$-S. Observe also that the power-law decay exponent $P(n,k)$ depends not only upon n (and deteriorates as n grows) but also upon k, and grows exponentially fast as $k\to+\infty$. The latter is due to some hidden resources of the von Dreifus–Klein method [26] which remained unexploited for a long time; they have been revealed in [12] (cf. [12, Theorem]), [14] (cf. [12, Theorem 8]), and some other works. As we shall see, virtually the same argument as in [26] allows one to prove exponential decay of the EFs in finite cubes (and ultimately in the entire lattice) with probability approaching 1 at rate faster than any power law[9].

Indeed, it is not difficult to see that the decay of the RHS in (41) is equivalent to $\mathrm{e}^{-c' \ln^{1+c} L_k}$ with $c > 0$; it is faster than any power law $L \mapsto L^{-P}$.

We summarize in the table below the assumptions on the key parameters and relations between them made in this section.

$\tau > \max\left(\zeta^{-1}, 1\right)$	$\mathbb{N} \ni \alpha > 2\tau$
$0 < \beta < \min\left(\zeta, \dfrac{1}{4\alpha}\right)$	$K + 1 > 4\alpha$
$E_N^* = 2^{-N+1} E^*$	$m_N = \frac{1}{2} E_N^*,\ 2E^* = m^* \geq L_0^{-1/2}$
$P(N,k) = 2^k P^* \, (2\alpha)^{N^* - N}$	$P^* > 4N^* d\alpha$

(42)

For further use, observe that

$$\forall N = 1, \ldots, N^* \quad P(N) \geq P^* > \max\left(4Nd, 2Nd\alpha\right). \tag{43}$$

4.2 Initial length scale (ILS) estimate

The next statement is a straightforward adaptation of Proposition 4 to the case where the decay exponent (for the GFs) $\delta = 1$. As in Section 3.3, it is obtained with the help of the large deviations estimates for arbitrarily small $g > 0$, or by substantially simpler arguments for $g \gg 1$; in the latter case, one can obtain $m^* = m_g^* \to +\infty$ as $g \to +\infty$.

Proposition 5 *Under the assumption (V) and (U), there exists an interval* $I^* = [0, E^*]$*, with* $E^* > 0$*, an integer* L_0 *and a real number* $m^* > L_0^{-1/2}$ *such that* $\widetilde{\mathsf{S}}(N, 0, I^*)$ *holds true for all* $1 \leq N \leq N^*$.

Once the ILS estimates are established in the interval I^*, it is convenient to proceed with the scale induction for N-particle systems in the individual intervals $I_N^* = [0, E_N^*]$, with $E_N^* = 2^{-N+1} E_1^*$ (cf. (42)). This provides stronger bounds for the subsystems of an N-particle systems, and allows to rule out – with hight probability – the WI singular cubes in the course of the scale induction.

[9] A further development of the simple observation made in [12] gives rise to a reformulation of the bootstrap MSA, initially developed by Germinet and Klein [32], where the EF correlators admit an explicitly described *asymptotically exponential* decay bound; see the manuscript `arXiv:math-ph//1503.02529`.

4.3 Analytic scaling step

We assume that the main parameters satisfy the conditions listed in the table (42), without repeating it every time again. In particular, this concerns the exponent τ in the following definition, replacing in this section its counterpart from Section 3, and the integer K in Definition 6.

Definition 5 A cube $\mathbf{\Lambda}_L^{(N)}(\mathbf{u})$ is called weakly interactive (WI) if

$$\operatorname{diam}\Pi\mathbf{u} \equiv \max_{i\neq j}|u_i - u_j| \geq 3NL^\tau,$$

and strongly interactive (SI), otherwise.

Definition 6 A cube $\mathbf{\Lambda}_{L_{k+1}}^{(N)}(\mathbf{x})$ is called (E, m_N, K)-bad if it contains
• either a weakly interactive (E, m_N)-S cube of radius L_k, or
• a collection of $K+1$ (or more) pairwise $9NL_k^\tau$-distant, (E, m_N)-S, strongly interactive cubes of radius L_k.
Otherwise, it is called (E, m_N, K)-good.

The next (deterministic) statement is a standard result of the Multi-Scale Analysis, essentially gong back to the work [26] and later adapted to the continuous Anderson models. The nature of the potential, in particular, the presence of a nontrivial interaction, is irrelevant for the proof.

Below we assume that the readers have familiarized themselves with the arguments presented in Appendix A, where an analog of Lemma 7 (Lemma 3) is proved.

Lemma 7 *Suppose that a cube* $\mathbf{\Lambda}_{L_{k+1}}^N(\mathbf{u})$ *is* (E, m_N, K)-good *and* (E, β)-CNR . *If* L_0 *is large enough, then* $\mathbf{\Lambda}_L^N(\mathbf{u})$ *is* (E, m_N)-NS.

Proof Set $\mathbf{\Lambda} = \mathbf{\Lambda}_{L_{k+1}}^N(\mathbf{u})$, $\mathbf{B} = \mathbf{B}_{L_{k+1}-1}^N(\mathbf{u}) = \mathbf{\Lambda}_{L_{k+1}-1}^N(\mathbf{u}) \cap \mathscr{Z}^N$, $\widehat{\mathbf{B}} = \mathbf{\Lambda}^N \cap \mathscr{Z}^N$, and fix $\mathbf{y} \in \partial^- \widehat{\mathbf{B}}$. Consider the function $f_\mathbf{y} : \mathbf{B} \to \mathbb{R}_+$ defined by

$$f_\mathbf{y} : \mathbf{z} \mapsto \left\| \chi_\mathbf{z} \mathbf{G}_\mathbf{\Lambda}^{(N)}(E) \chi_\mathbf{y} \right\|.$$

By assumption, there is a collection of balls $\mathbf{B}_{L_k^\tau}(\mathbf{u}_j) \subset \mathbf{\Lambda}$, $1 \leq j \leq K'$, with $0 \leq K' \leq K$, such that any ball $\mathbf{B}_{L_k}(\mathbf{v})$ with $\mathbf{v} \in \mathbf{B} \setminus \cup_{j=1}^{K'} \mathbf{B}_{9NL_k^\tau}(\mathbf{u}_j)$ is (E, m_N)-NS. Fix such a collection. Denote $\mathscr{L}_r(\mathbf{u}) = \{\mathbf{z} \in \mathscr{Z} : \; |\mathbf{z} - \mathbf{u}| = r\}$, $r \geq 0$ and set:

$$\mathscr{S} := \left\{ \mathbf{x} \in \mathbf{B}_{L_{k+1}-L_k-1}(\mathbf{u}) : \; \mathscr{L}_{\mathrm{d}(\mathbf{u},\mathbf{x})}(\mathbf{u}) \cap \cup_{j=1}^{K'} \mathbf{B}_{9NL_k^\tau}(\mathbf{u}_j) \neq \varnothing \right\}$$

(here $\mathscr{S}$ stands for "singular"). Then any ball $\mathbf{B}_{L_k}(\mathbf{v}) \subset \mathbf{B}$ with $\mathbf{v} \in \mathbf{B} \setminus \mathscr{S}$ is (E, m_N)-NS, and $\mathscr{S}$ is covered by a family of at most K annuli with center $\mathbf{u}$ and total width $\leq K(2 \cdot 9NL_k^\tau + 1) \leq 19NKL_k^\tau$, for L_0 large enough. By Lemma 13, $f_\mathbf{y}$ is $(L_k, q, \mathscr{S})$-dominated in $\mathbf{B}$, in the sense of Definition 7, with

$$\begin{aligned} -\ln q &= m_N(1 + L_k^{-1/8})L_k - L_{k+1}^\beta - \ln(C_\mathscr{Z}^N L_{k+1}^{Nd}) \\ &\geq m_N L_k + \left(m_N L_k^{7/8} - 2L_k^{\alpha\beta}\right) \geq L_k m_N \left(1 + \tfrac{1}{2} L_k^{-1/8}\right), \end{aligned}$$

where the last inequality follows from the assumptions $\alpha\beta < 1/4$ and $m_N \geq m^* \geq L_0^{-1/2}$ listed in (42). By Lemma 12, we obtain, denoting $\mathrm{M}(f_{\mathbf{y}}, \mathbf{B}) := \max_{\mathbf{z}\in\mathbf{B}} f_{\mathbf{y}}(\mathbf{z})$,

$$f_{\mathbf{y}}(\mathbf{u}) \leq q^{\frac{(L_{k+1}-1)-19NKL_k^{\tau}-1}{L_k}} \mathrm{M}(f_{\mathbf{y}}, \mathbf{B}) \leq q^{\frac{L_{k+1}-20NKL_k^{\tau}}{L_k}} \mathrm{M}(f_{\mathbf{y}}, \mathbf{B}).$$

Thus with $\alpha > 2\tau$, $\beta < 1/4$, $m_N \geq L_0^{-1/2} \geq L_k^{-1/2}$, and L_0 is large enough, we have

$$\begin{aligned}
-\ln f_{\mathbf{y}}(\mathbf{u}) &\geq -\ln \left(\mathrm{e}^{-m_N\left(1+\frac{1}{2}L_k^{-1/8}\right)L_k} \right)^{\frac{L_{k+1}-20NKL_k^{\tau}}{L_k}} - \ln \mathrm{e}^{L_{k+1}^{\beta}} \\
&\geq m_N \left\{ \left(1+\tfrac{1}{2}L_k^{-1/8}\right) L_k \cdot \frac{L_{k+1}\left(1-20NKL_{k+1}^{-1+\frac{\tau}{\alpha}}\right)}{L_k} - \frac{L_{k+1}^{\beta}}{m_N} \right\} \\
&\geq m_N L_{k+1} \left\{ \left(1+\tfrac{1}{4}L_k^{-1/8}\right)\left(1-L_{k+1}^{-1/2}\right) - L_{k+1}^{-1+\frac{1}{2}+\frac{1}{4}} \right\} \\
&\geq L_{k+1} m_N \left(1+2L^{-1/8}\right) \geq \gamma(m_N, L) L_{k+1} + \ln\left(C_{\mathscr{X},N} L_{k+1}^{Nd}\right).
\end{aligned}$$

Therefore, $\mathbf{\Lambda}_L(\mathbf{x})$ is (E, m_N)-NS. Note that we have here the same value of m^* figuring in $m_N = m^*(1+3L_0^{-1+\beta})^{N^*-N}$, i.e., $m^* \geq L_0^{-1/2}$. This completes the proof. □

4.4 Probabilistic scaling step

4.4.1 Weakly interactive cubes

In this subsection, we make use of the flexibility of the Multi-Scale Analysis which allows one to trade the decay rate of the probabilistic estimates (making them weaker than in Section 3) for the decay rate of the GFs, making them stronger – exponential. The reader can see that the key point is the analysis of the WI cubes, or, in simpler terms, of N-particle systems decomposed into a union of 2 (or more) distant subsystems.

Lemma 8 *For any weakly interactive cube* $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u})$ *there is a factorization* $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u}) = \mathbf{\Lambda}_{L_k}^{(n')}(\mathbf{u}') \times \mathbf{\Lambda}_{L_k}^{(n'')}(\mathbf{u}'')$ *such that*

$$\operatorname{dist}\left(\Pi\mathbf{\Lambda}_{L_k}^{(n')}(\mathbf{u}'), \Pi\mathbf{\Lambda}_{L_k}^{(n'')}(\mathbf{u}'')\right) > L_k^{\tau}. \tag{44}$$

The proof repeats almost verbatim that of Lemma 4, so we omit it.

We will assume that one such factorization is associated with each WI cube (even if it is not unique), and call it the canonical one. For the Hamiltonian in a WI cube, we have the following algebraic representation: with $\mathbf{\Lambda}' = \mathbf{\Lambda}_{L_k}^{(n')}(\mathbf{u}')$, $\mathbf{\Lambda}'' = \mathbf{\Lambda}_{L_k}^{(n'')}(\mathbf{u}'')$,

$$\mathbf{H} = \mathbf{H}^{\mathrm{ni}} + \mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''} = \mathbf{H}_{\mathbf{\Lambda}'} \otimes \mathbf{1}^{(n'')} + \mathbf{1}^{(n')} \otimes \mathbf{H}_{\mathbf{\Lambda}''} + \mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''} \tag{45}$$

where, due to the assumption (U),

$$\|\mathbf{U}_{\boldsymbol{\Lambda}',\boldsymbol{\Lambda}''}\| \le C\mathrm{e}^{-L_k^{\tau\zeta}}, \quad \text{with } \tau\zeta > 1 \text{ by (42).} \tag{46}$$

Remark 6 It is the possibility to make the product $\tau\zeta > 1$ (by choosing $\tau > 1/\zeta$) which results in the exponential decay of the EFs.

Lemma 9 *Assume the property* $\mathsf{S}(N-1,k)$. *If* L_0 *is large enough, then for any WI cube* $\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u})$

$$\mathbb{P}\left\{\boldsymbol{\Lambda}^{(N)}_{L_k}(\mathbf{u}) \textit{ is } (E,m_N)\text{-S}\right\} \le L_k^{-\frac{3}{2}P(N,k+1)}, \tag{47}$$

and therefore,

$$\mathbb{P}\left\{\boldsymbol{\Lambda}^{(N)}_{L_{k+1}}(\mathbf{u}) \textit{ contains a WI } (E,m_N)\text{-S} \textit{ ball of radius } L_k\right\} \le \frac{1}{4}L_k^{-P(N,k+1)}. \tag{48}$$

See the proof in Appendix C.

4.4.2 Strongly interactive cubes

Lemma 10 *If two* SI *cubes* $\boldsymbol{\Lambda}^{(N)}_L(\mathbf{x})$, $\boldsymbol{\Lambda}^{(N)}_L(\mathbf{y})$ *are* $9NL^{\tau}$*-distant and* $L > 2\mathrm{r}_1$, *then*

$$\Pi\boldsymbol{\Lambda}^{(N)}_{L+\mathrm{r}_1}(\mathbf{x}) \cap \Pi\boldsymbol{\Lambda}^{(N)}_{L+\mathrm{r}_1}(\mathbf{y}) = \varnothing, \tag{49}$$

and consequently, the random operators $\mathbf{H}_{\boldsymbol{\Lambda}^{(N)}_L(\mathbf{x})}$ *and* $\mathbf{H}_{\boldsymbol{\Lambda}^{(N)}_L(\mathbf{y})}$ *are independent.*

The proof is similar to that of Lemma 6; notice that $\tau > 1$ (cf. (42)).

4.4.3 The scale induction

Theorem 6 *Suppose that* $\widetilde{\mathsf{S}}(N,0,I^*_N)$ *holds true for some* $N \in [[1,N^*]]$, *and for all* $k \ge 0$, *one has*

$$\forall E \in I^*_N \qquad \mathbb{P}\left\{\, \boldsymbol{\Lambda}_{L_{k+1}}(\mathbf{u}) \textit{ is } (E,\beta)\text{-CNR} \,\right\} \le \frac{1}{4}L^{-P(N,k+1)}.$$

If L_0 *is large enough, then* $\widetilde{\mathsf{S}}(N,k,I^*_N)$ *holds true for all* $k \ge 0$.

Proof It suffices to infer $\widetilde{\mathsf{S}}(N,k+1,I^*_N)$ from $\widetilde{\mathsf{S}}(N,k,I^*_N)$. By Lemma 7, if $\boldsymbol{\Lambda}_{L_{k+1}}(\mathbf{u})$ is (E,m_N)-S, then either it is not (E,β)-CNR , or it is (E,m_N,K)-bad. Let

$$\begin{aligned}
\mathrm{P}_i &:= \mathbb{P}\left\{\, \boldsymbol{\Lambda}_{L_i}(\mathbf{u}) \text{ is } (E,m_N)\text{-S}\right\}, \;\; i = k,k+1,\\
\mathrm{S}_{k+1} &:= \mathbb{P}\left\{\, \boldsymbol{\Lambda}_{L_{k+1}}(\mathbf{u}) \text{ contains a WI, } (E,m_N)\text{-S cube of radius } L_k\right\},\\
\mathrm{Q}_{k+1} &:= \mathbb{P}\left\{\, \boldsymbol{\Lambda}_{L_{k+1}}(\mathbf{u}) \text{ is not } (E,\beta)\text{-CNR } \right\} \le \frac{1}{4}L^{-P(n,k+1)}
\end{aligned} \tag{50}$$

(the last inequality is assumed, but its validity actually follows from Theorem 3). Further, by Lemma 7, if a cube $\boldsymbol{\Lambda}_{L_{k+1}}(\mathbf{u})$ is (E,β)-CNR and (E,m_N,K)-bad, then it must contain either a weakly interactive (E,m_N)-S cube of radius L_k (with probability $S_{k+1} \le \frac{1}{4} L_{k+1}^{-P(n,k+1)}$, by Lemma 5), or at least $K+1$ pairwise $9NL_k^{\tau}$-distant cubes $\boldsymbol{\Lambda}_{L_k}(\mathbf{v}_i)$, $i = 1,\dots,K+1$, which are (E, m_N)-S and SI. By Lemma 6, the random operators $\mathbf{H}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{v}_i)}(\omega)$ are independent, thus the latter event occurs with probability

$$\le C L_{k+1}^{(K+1)Nd} \mathrm{P}_k^{K+1} \le C L_{k+1}^{-(K+1)\left[\alpha^{-1}P(N,k)-Nd\right]} \le \frac{1}{4} L_{k+1}^{-2P(N,k)} = \frac{1}{4} L_{k+1}^{-P(N,k+1)}, \tag{51}$$

under the conditions $P(N,k) > 2Nd\alpha$, $K+1 \ge 4\alpha$ given in the table (42). Therefore,

$$\begin{aligned} \mathrm{P}_{k+1} &\le C L_{k+1}^{(K+1)Nd} \mathrm{P}_k^{K+1} + \mathrm{S}_{k+1} + Q_{k+1} \\ &\le \frac{1}{4} L^{-P(n,k+1)} + \frac{1}{4} L^{-P(n,k+1)} + \frac{1}{4} L^{-P(n,k+1)} < L^{-P(n,k+1)}. \end{aligned}$$

□

Remark 7 Now we can explain the choice of the parameters τ, α and K. First, $\tau > \zeta^{-1}$ is required to assess the WI cubes. Next, $\alpha > 2\tau$ is required for the analytic scaling step (see the proof of Lemma 7). Finally, we need $K+1 > 4\alpha$ for the bound (51), thus allowing for a large number of singular SI cubes (the smaller $\zeta > 0$, the larger must be the integer K).

5 Derivation of spectral and dynamical localization

The main strategy in this section is similar to that in [23]; however, in the continuous alloy model, one needs a sharper "two-volume" EV comparison estimate than in the lattice model considered in [23] (see Theorem 4).

5.1 From a fixed energy to an energy interval

Introduce the following notation which will be used in this section:

$$\mathbf{F}_{\mathbf{x}}(E) = \mathbf{F}_{\mathbf{x},L}(E) := \max_{\mathbf{z} \in \partial^{-} \mathbf{B}_L(\mathbf{x})} \left\| \chi_{\mathbf{z}} \mathbf{G}_{\boldsymbol{\Lambda}_L(\mathbf{x})} \chi_{\mathbf{x}} \right\|.$$

Theorem 7 *Fix $L \ge 1$, a pair of N-particle cubes $\boldsymbol{\Lambda}_L(\mathbf{x})$, $\boldsymbol{\Lambda}_L(\mathbf{y})$, and a bounded interval $I \subset \mathbb{R}$. Assume that the following bound holds true for some $a_L, q_L > 0$:*

$$\forall E \in I \qquad \max_{\mathbf{z} \in \{\mathbf{x},\mathbf{y}\}} \mathbb{P}\left\{ \mathbf{F}_{\mathbf{z}}(E) \ge a_L \right\} \le q_L.$$

Assume also that the EVC bound of the form (14) *holds true for the pair $\boldsymbol{\Lambda}_L(\mathbf{x})$, $\boldsymbol{\Lambda}_L(\mathbf{y})$. Then for any $b > 0$, one has*

$$\mathbb{P}\left\{ \exists E \in I : \min\left(\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E) \right) \ge a_L \right\} \le 2|I| b^{-1} q_L + C''' L^{4Nd} b. \tag{52}$$

Consequently, under the assumptions (U) and (V), the bound (52) *holds true in the interval $I = I_N^*$ for any pair of $4NL_k$-distant cubes of radius L_k, owing to Theorem 4.*

Proof Introduce the events $\mathscr{S}_{b,\mathbf{z}} := \{\omega : \mathrm{mes}\{E \in I : \mathbf{F}_{\mathbf{x}}(E) \geq a\} > b\}$, $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}$. By the Chebychev inequality[10] combined with the Fubini theorem, we have for $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}$

$$\begin{aligned} \mathbb{P}\left\{\mathscr{S}_{b,\mathbf{z}}\right\} &\leq b^{-1}\mathbb{E}\left[\int_I \mathbf{1}_{\{\mathbf{F}_{\mathbf{z}}(E)\geq a\}}\, dE\right] = b^{-1}\int_I \mathbb{E}\left[\mathbf{1}_{\{\mathbf{F}_{\mathbf{z}}(E)\geq a\}}\right] dE \\ &= b^{-1}\int_I \mathbb{P}\{\mathbf{F}_{\mathbf{z}}(E) \geq a\}\, dE \leq b^{-1}|I| q_L. \end{aligned} \tag{53}$$

For any $\omega \notin \mathscr{S}_b := \mathscr{S}_{b,\mathbf{x}} \cup \mathscr{S}_{b,\mathbf{y}}$, each of the two random sets

$$\mathscr{E}_{\mathbf{z}}(a,\omega) := \{E \in I : \mathbf{F}_{\mathbf{z}}(E,\omega) > a\},\ \mathbf{z} \in \{\mathbf{x},\mathbf{y}\},$$

has Lebesgue measure bounded by b. The norm of the resolvent is a continuous function of the energy E, on the complement to the spectrum, and the latter is discrete for any finite volume Hamiltonian, thus the set $\{E \in \mathrm{Int}(I) : \mathbf{F}_{\mathbf{z}}(E,\omega) \in (a,+\infty)\}$ is decomposed into open sub-intervals[11]; the same is true for $\mathbf{F}_{\mathbf{y}}$. Therefore,

$$\begin{aligned} &\mathscr{E}_{\mathbf{x}}(a,\omega) = \cup_{i=1}^{K'} J_{\mathbf{x},i},\ \textstyle\sum_{i=1}^{K'} |J_{\mathbf{x},i}| \leq b, \quad K' \leq +\infty, \\ &\mathscr{E}_{\mathbf{y}}(a,\omega) = \cup_{j=1}^{K''} J_{\mathbf{y},j},\ \textstyle\sum_{j=1}^{K''} |J_{\mathbf{y},j}| \leq b, \quad K'' \leq +\infty, \end{aligned} \tag{54}$$

and (again for $\omega \notin \mathscr{S}_b$ omitted from notation for brevity)

$$\begin{aligned} \mathbb{P}\left\{\exists E \in I : \min\left[\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)\right] > a\right\} &\leq \mathbb{P}\left\{\mathscr{E}_{\mathbf{x}}(a) \cap \mathscr{E}_{\mathbf{y}}(a) \neq \varnothing\right\} \\ &\leq \sum_{i=1}^{K'} \sum_{j=1}^{K''} \mathbb{P}\left\{J_{\mathbf{x},i} \cap J_{\mathbf{y},j} \neq \varnothing\right\}. \end{aligned} \tag{55}$$

Before assessing the terms in the above RHS, we make the following observations.

The intervals $J_{\mathbf{x},i} =: (E^-_{\mathbf{x},i}, E^+_{\mathbf{x},i})$ and $J_{\mathbf{y},j} =: (E^-_{\mathbf{y},j}, E^+_{\mathbf{y},j})$ may or may not be directly related to the EVs of the operators $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}$, $\mathbf{H}_{\mathbf{B}_L(\mathbf{y})}$: they are not necessarily adjacent to these EVs. In any case, we do not make such a claim, and some numerical calculations evidence that, depending on the value of $a > 0$, the intervals like $J_{\mathbf{x},i}$ can indeed be separated from the closest respective EVs. Therefore, we cannot apply Wegner-type estimates to bound the probabilities in the RHS of (55).

Instead of the Wegner-type bounds, we adapt an argument from [14, Sect. 6] and use the identities (20)–(21) established in Sect. 2.2 (cf. Remark 4) for weakly Q-separated cubes $\mathbf{B}_L(\mathbf{x})$ and $\mathbf{B}_L(\mathbf{y})$:

$$\begin{aligned} \mathbf{G}_{\mathbf{B}_L(\mathbf{x})}(E;\xi(\omega)) &= \mathbf{G}_{\mathbf{B}_L(\mathbf{x})}(E - n_1\xi(\omega);0), \\ \mathbf{G}_{\mathbf{B}_L(\mathbf{y})}(E;\xi(\omega)) &= \mathbf{G}_{\mathbf{B}_L(\mathbf{y})}(E - n_2\xi(\omega);0). \end{aligned} \tag{56}$$

---

[10] Apparently, such kind of arguments in the energy-disorder space $\mathbb{R} \times \Omega$ were first used in localization theory by Martinelli and Scoppola [44], and later by Bourgain and Kenig [9] and Elgart et al. [27].

[11] We believe that actually $K', K'' < +\infty$ in (54), but this is not crucial for (nor used in) the proof.

Equations (56) result in the following identities for the quantities $E^{\pm}_{\mathbf{x},i}(\omega)$ and $E^{\pm}_{\mathbf{y},j}(\omega)$, which, conditional on $\mathfrak{F}_Q$, also depend upon a single random parameter $\xi(\omega)$:

$$E^{\pm}_{\mathbf{x},i}(\omega) = \widehat{E}^{\pm}_{\mathbf{x},i} - n_1\xi(\omega), \;\; E^{\pm}_{\mathbf{y},j}(\omega) = \widehat{E}^{\pm}_{\mathbf{y},j} - n_2\xi(\omega),$$

where $\widehat{E}^{\pm}_{\mathbf{x},i}$, $\widehat{E}^{\pm}_{\mathbf{y},j}$ are $\mathfrak{F}_Q$-measurable. Therefore,

$$E^{\pm}_{\mathbf{x},j} - E^{\pm}_{\mathbf{y},i} = (n_1 - n_2)\xi(\omega) + \mu_{i,j}\,,$$

with $\mathfrak{F}_Q$-measurable $\mu_{i,j}$ and integers $n_1 > n_2$, hence $n_1 - n_2 \geq 1$. Denote $\varepsilon_{\mathbf{x},i} := |J_{\mathbf{x},i}|$, $\varepsilon_{\mathbf{y},i} := |J_{\mathbf{y},i}|$, then for each pair (i,j) we have[12] by virtue of Theorem 2

$$\begin{aligned}\mathbb{P}\left\{J_{\mathbf{x},i}\cap J_{\mathbf{y},i} \neq \varnothing\right\} &\leq \mathbb{P}\left\{|\xi - \mu_{ij}| \leq |n_1 - n_2|^{-1}(\varepsilon_{\mathbf{x},i} + \varepsilon_{\mathbf{y},i})\right\} \\ &\leq CL^A\left(\varepsilon_{\mathbf{x},i} + \varepsilon_{\mathbf{y},i}\right),\end{aligned} \tag{57}$$

thus

$$\mathbb{P}\left\{\exists E \in I : \min\left[\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)\right] > a\right\} \leq CL^A \sum_{i=1}^{K'}\sum_{j=1}^{K''}\left(\varepsilon_{\mathbf{x},i} + \varepsilon_{\mathbf{y},i}\right) \leq CL^A \cdot 2b.$$

□

The strategy of the above proof evidences that instead of the usual eigenvalue *concentration* bounds, extending the celebrated Wegner estimate [50], we are compelled to use eigenvalue *comparison* bounds for the pairs of local Hamiltonians which can be stochastically correlated in a very strong way. At the moment of writing these lines, this is one of the most problematic points in the decay analysis of the Green functions and eigenfunctions of N-particle models, relative to the (symmetrized) norm-distance. The partial solution to this technical problem proposed in the present paper (cf. also [23]) requires rather restrictive hypotheses on the regularity of the probability distribution of the external random potential (both in the discrete and the continuous models) and the flat tiling condition on the scatterer functions (in the continuous model). Neither of these restrictive assumptions is required for the proof of localization relative to the Hausdorff distance, as evidence the works by Klein and Nguyen [39,40].

In the next statement, the exponent $\delta \in (0,1)$ is the same as in the fixed-energy analysis carried out in Section 3).

Corollary 2 *Under the assumptions of Theorem 1, for all $N \in [2, N^*]$ there exists an interval $I^*_N = [0, E^*_N]$ with $E^*_N > 0$ such that, if L_0 or $k \geq 0$ is large enough, then, for some $C''', A', \nu^* \in (0, +\infty)$*

$$\begin{aligned}\mathbb{P}\left\{\sup_{E\in I^*_N} \min\left[\mathbf{F}_{\mathbf{x}}(E), \mathbf{F}_{\mathbf{y}}(E)\right] \geq \mathrm{e}^{-m_N L_k^{\delta}}\right\} &\leq \frac{2|I^*_N|\mathrm{e}^{-\nu_N L_k^{\kappa}}}{\mathrm{e}^{-\frac{1}{2}\nu_N L_k^{\kappa}}} + C''' L^{A'}\mathrm{e}^{-\frac{1}{2}\nu_N L_k^{\kappa}} \\ &\leq \mathrm{e}^{-\nu^* L_k^{\kappa}}.\end{aligned} \tag{58}$$

[12] For this argument, we need the RHS on (57) to be linear in ε. This explains why the analog of the EVC bound (14) used in [23] (cf. [23, Theorem 2.2]), providing the RHS of the form $CL^A s^{\theta}$ with $0 < \theta < 1$, is insufficient here and had to be improved.

The counterpart of Corollary 2 building on the results of the scaling analysis from Section 4) (cf. (41)) is as follows.

Corollary 3 *Under the assumptions of Theorem 1, for all $N \in [2, N^*]$ there exists an interval $I_N^* = [0, E_N^*]$ with $E_N^* > 0$ such that, if L_0 or $k \geq 0$ is large enough, then, for some $C''', A', c, c', c'' \in (0, +\infty)$*

$$\mathbb{P}\left\{ \sup_{E \in I_N^*} \min\left[\mathbf{F}_\mathbf{x}(E), \mathbf{F}_\mathbf{y}(E)\right] \geq \mathrm{e}^{-m_N L_k} \right\} \leq \frac{2|I_N^*| L_k^{-2c\cdot 2^k}}{L_k^{-c\cdot 2^k}} + C''' L_k^{-c'\cdot 2^k} \leq L_k^{-c''\cdot 2^k}. \tag{59}$$

5.2 Decay of the EF correlators

The derivation of the strong dynamical localization from the energy-interval bounds given in Corollary 2 can be obtained in the same way as in the works by Klein and Nguyen [39,40], where N-particle models were studied, with the help of the techniques developed earlier by Germinet and Klein [32,33,34].

Proposition 6 *Given a positive integer L, assume that the following bound holds true for a pair of disjoint balls $\mathbf{\Lambda}_L(\mathbf{x}), \mathbf{\Lambda}_L(\mathbf{y}) \subset \mathscr{X}^N$ and some positive functions u, h:*

$$\mathbb{P}\left\{ \exists E \in \mathbb{R} : \min[\mathbf{F}_\mathbf{x}(E), \mathbf{F}_\mathbf{y}(E)] > u(L) \right\} \leq h(L). \tag{60}$$

Then for some $C = C(N,d), A = A(N,d) \in (0, +\infty)$ and any regular connected domain $\mathbf{\Lambda} \supset \mathbf{\Lambda}_L(\mathbf{x}) \cup \mathbf{\Lambda}_L(\mathbf{y})$ one has

$$\mathbb{E}\left[\sup_{t \in \mathbb{R}} \left\| \mathbf{1}_\mathbf{x} P_I(\mathbf{H}_\mathbf{\Lambda}) \mathrm{e}^{-it\mathbf{H}_\mathbf{\Lambda}} \mathbf{1}_\mathbf{y} \right\| \right] \leq C L^A \big(u(L) + h(L)\big). \tag{61}$$

The proof of similar implications in [39,40] (cf. [40, Corollary 1.7], [39, Corollary 1.7]) uses a deterministic power-law bound on the growth rate of the generalized eigenfunctions of a Schrödinger operator in $\mathbb{R}^{Nd}$, with bounded potential; see [32, Lemma 2.5]. The details can be found in [39, Proof of Theorem 4.4], [32, Proof of Theorem 3.8]. The finiteness of the range of interaction is not important here.

If $\mathbf{\Lambda}$ is bounded and $\mathbf{H}_\mathbf{\Lambda}$ has compact resolvent, then the *a priori* bounds on the generalized eigenfunctions become unnecessary, and the proof easily stems from the Bessel inequality. See, e.g., the proof of [14, Theorem 7] where a discrete single-particle model was considered.

In the present paper, the situation with the derivation of strong dynamical localization is simpler than in [18,39,40] and virtually the same as in the single-particle systems, since we prove the energy-interval MSA bounds for the norm-distant and not only Hausdorff-distant pairs of cubes.

For these reasons, and for brevity, we omit the proof of Proposition 6.

Now assertion (A) of Theorem 1 for bounded domains follows from

Theorem 8 *Given $\nu > 0$, $\exists\, g_* = g_*(\nu) \in (0,\infty)$ and $C_* = C_*(\nu) \in (0,\infty)$ such that, for $g \geq g_*(\nu)$, and $1 \leq N \leq N^*$, $\forall\, \mathbf{x},\mathbf{y} \in \mathscr{Z}$ and a bounded domain $\mathbf{\Lambda} \subseteq \mathscr{X}^N$ with $\mathbf{\Lambda} \supset \mathbf{\Lambda}_{R/2}(\mathbf{x}) \cup \mathbf{\Lambda}_{R/2}(\mathbf{y})$, $R := \mathrm{d_S}(\mathbf{x},\mathbf{y})$,*

$$\Upsilon_{\mathbf{x},\mathbf{y}} := \mathbb{E}\left[\sup_{t\in\mathbb{R}} \left\|\mathbf{1_x} P_I\big(\mathbf{H_\Lambda}\big)\mathrm{e}^{-\mathrm{i}t\mathbf{H_\Lambda}}\,\mathbf{1_y}\right\|\right] \leq C_* \mathrm{e}^{-\nu(\mathrm{d_S}(\mathbf{x},\mathbf{y}))^\kappa}. \tag{62}$$

Proof Without loss of generality, it suffices to prove the assertion for the pairs of points with $R := \mathrm{d_S}(\mathbf{x},\mathbf{y}) > 4NL_0$. Indeed, an EFC correlator is always bounded by 1, so for the pairs $\mathbf{x},\mathbf{y}$ with $\mathrm{d_S}(\mathbf{x},\mathbf{y}) \leq 4NL_0$ the required decay bound can be absorbed in a sufficiently large constant C_*.

Moreover, it suffices to establish (62) only for R large enough; again, the bound for a finite number of remaining values of R can be achieved by taking C_* large enough.

Fix two points $\mathbf{x},\mathbf{y} \in \mathscr{Z}$ with $R := \mathrm{d_S}(\mathbf{x},\mathbf{y})$ and let $L = \lfloor R/(4N)\rfloor$. Notice that for $R \geq 20N$ and $N \geq 1$, one has $L \geq R/(5N)$. Consider a subset $\mathbf{\Lambda} \subset \mathscr{X}^N$, which can be the entire $\mathscr{X}^N$ or a bounded domain such that $\mathbf{B}_L^{(N)}(\mathbf{x}) \cup \mathbf{B}_L^{(N)}(\mathbf{y}) \subset \mathbf{\Lambda}$.

We have $L \geq R/(5N)$. By Corollary 2 combined with Proposition 6,

$$\Upsilon_{\mathbf{x},\mathbf{y}} \leq 4\mathrm{e}^{-\frac{m_N}{(5N)^\delta}R^\delta} + \mathrm{e}^{-\frac{\nu_N}{11(5N)^\kappa}R^\kappa} \leq 4\mathrm{e}^{-\frac{m_N}{5N}R^\kappa} + \mathrm{e}^{-\frac{\nu_N}{55N}R^\kappa}. \tag{63}$$

Given an arbitrary $\nu > 0$, choose a sufficiently large L_0, so that the initial length scale estimate $\mathsf{S}(N,0)$ is fulfilled with $m_N \geq 5N\nu$, $\nu_N \geq 55N\nu$. Then we obtain

$$\Upsilon_{\mathbf{x},\mathbf{y}} \leq 5\mathrm{e}^{-\nu R^\kappa} = 5\mathrm{e}^{-\nu(\mathrm{d}_S(\mathbf{x},\mathbf{y}))^\kappa}. \tag{64}$$

This completes the proof of Theorem 8. ☐

Virtually the same argument, combined with the Shnol–Simon type estimates (cf. [32,39,40]) provides a similar result for unbounded domains $\mathbf{\Lambda}$. We stress that we consider the finite domain estimates much more important for physical applications.

5.3 Exponential decay of eigenfunctions

In the next statement, we keep the same notations for the Hamiltonian and the cubes as before, but it can be easily seen that the result applies to a much larger class of Schrödinger operators in a Euclidean space $\mathbb{R}^D$, $D \geq 1$, with bounded measurable random potential $\mathbb{R}^D \ni x \mapsto W(x;\omega)$. In our case, $D = Nd$, x is replaced by $\mathbf{x}$, and $W(x;\omega)$ by $\mathbf{V}(\mathbf{x};\omega) + \mathbf{U}(\mathbf{x})$. The constant a figuring in Lemma 11 can be set to $4N$, owing to Theorem 4. The main argument is not new. Thanks to the bound (65) established for all pairs of cubes which are aL_k-distant in the norm-distance, and not in the Hausdorff distance (cf. [18,40]) the structure of the proof is so close to the one employed in a number of papers on the single-particle MSA, that it could have been safely omitted. We give it below precisely for illustrating the power of the norm-distance bounds.

Another reason why we give a detailed proof is that Lemma 11, along with the decay analysis of the GFs given in Section 4, provides an important complement to the results by Fauser and Warzel [31]: we show that the EFs of the N-particle model at hand, with an infinite-range interaction decaying at a sub-exponential rate $r \mapsto \mathrm{e}^{-Cr^{\zeta}}$, with arbitrarily small $\zeta > 0$, decay *exponentially* fast at infinity. Such a result is possible due to the logical independence of the decay analysis of the EFs, in the course of the MPMSA, while in the MPFMM, it is subordinate to the decay analysis of the eigenfunction correlators.

The reader familiar with Ref. [26] can see that choosing the origin $\mathbf{0} \in (\mathbb{R}^d)^N$ (which is invariant under all permutations of coordinates) as the reference point in the proof effectively puts the symmetrized and non-symmetrized, genuine norm-distance on equal footage.

Lemma 11 *Consider the random Hamiltonian $\mathbf{H}(\omega)$ and assume that for some $a \in (0,+\infty)$ and an interval $I^* \subset \mathbb{R}$, for any $k \geq 0$ and any pair of aL_k-distant cubes $\boldsymbol{\Lambda}_{L_k}(\mathbf{x})$, $\boldsymbol{\Lambda}_{L_k}(\mathbf{y})$, the following probabilistic bound holds true:*

$$\mathbb{P}\left\{\exists E \in I^* : \ \boldsymbol{\Lambda}_{L_k}(\mathbf{x}) \text{ and } \boldsymbol{\Lambda}_{L_k}(\mathbf{y}) \text{ are } (E,m_N)\text{-S}\right\} \leq L_k^{-p_k}, \tag{65}$$

where $\lim_k p_k = +\infty$. Then with probability one, every nontrivial polynomially bounded solution $\boldsymbol{\Psi}$ to the equation $\mathbf{H}(\omega)\boldsymbol{\Psi} = E\boldsymbol{\Psi}$ with $E \in I^*$ decays exponentially fast at infinity, with the decay exponent $\geq m^* > 0$. Specifically, for some $r(\boldsymbol{\Psi}) \in (0,+\infty)$ and all $\mathbf{x} \in \mathscr{Z}^N$ with $|\mathbf{x}| \geq r(\boldsymbol{\Psi})$, one has

$$\|\chi_{\mathbf{x}}\boldsymbol{\Psi}\| \leq \mathrm{e}^{-m^*|\mathbf{x}|}. \tag{66}$$

Proof Fix a polynomially bounded solution $\boldsymbol{\Psi}$ with $\|\boldsymbol{\Psi}\| > 0$; then there exists $\widehat{\mathbf{x}} \in \mathscr{Z}^N$ such that $\|\chi_{\widehat{\mathbf{x}}}\boldsymbol{\Psi}\| > 0$. Fix such a point $\widehat{\mathbf{x}}$.

Furthermore, there exists an integer $k_0 \geq 0$ such that for all $k \geq k_0$, $\boldsymbol{\Lambda}_{L_k}(\widehat{\mathbf{x}})$ is (E,m_N)-S. Assume otherwise, then there are arbitrarily large cubes $\boldsymbol{\Lambda}_{L_k}(\widehat{\mathbf{x}})$ such that

$$\|\chi_{\widehat{\mathbf{x}}}\Psi\| \leq C_1 L_k^{C_2} \mathrm{e}^{-m_N L_k} \underset{L_k \to \infty}{\longrightarrow} 0,$$

which contradicts our assumption that $\|\chi_{\widehat{\mathbf{x}}}\Psi\| > 0$.

We fix k_0 and work only with $k \geq k_0$. Denote $\mathbb{A}_k := \boldsymbol{\Lambda}_{2L_{k+2}}(\mathbf{0}) \setminus \boldsymbol{\Lambda}_{aL_k}(\mathbf{0})$ and introduce the events of the form

$$\mathscr{T}_k(\boldsymbol{\Lambda}) := \{\exists E \in I^* : \boldsymbol{\Lambda} \text{ contains two } aNL_k\text{-distant } (E,m_N)\text{-S cubes of radius } L_k\ \},$$

where $\boldsymbol{\Lambda} \subset \mathscr{X}^N$. By the assumed property (65),

$$\begin{aligned}\mathbb{P}\left\{\mathscr{T}_k(\boldsymbol{\Lambda}_{2L_{k+2}}(\mathbf{0}))\right\} &\leq |\boldsymbol{\Lambda}_{2L_{k+2}}(\mathbf{0})|^2 L_k^{-p_k} \leq CL_k^{-p_k+P^*2Nd\alpha^2} \\ &\leq C' L_k^{-p_k/2}.\end{aligned}$$

The last RHS is summable in $k \geq k_0$, so by the Borel–Cantelli lemma, there is a subset $\widetilde{\Omega} \subset \Omega$ with $\mathbb{P}\left\{\widetilde{\Omega}\right\} = 1$ such that for any $\omega \in \widetilde{\Omega}$ there exists $k_1 \geq k_0$ such that for all $k \geq k_1$, the event $\mathscr{T}_k(\boldsymbol{\Lambda}_{2L_{k+2}}(\mathbf{0}))$ does not occur. Since $\mathbb{A}_k \subset \boldsymbol{\Lambda}_{2L_{k+2}}(\mathbf{0})$, and for all $k \geq k_1 \geq k_0$, the cube $\boldsymbol{\Lambda}_{L_k}(\mathbf{0})$ is (E,m_N)-S, all cubes $\boldsymbol{\Lambda}_{L_k}(\mathbf{y}) \subset \mathbb{A}$ (with $k \geq k_1$) are (E,m_N)-NS.

Fix $\omega \in \widetilde{\Omega}$. Now the argument becomes deterministic.

Fix any $\mathbf{x}$ with $|\mathbf{x}| > aNL_{k_1}$, and let $k = k(|\mathbf{x}|) \in \mathbb{N}$ be such that $|\mathbf{x}| \in (2L_{k+1}, 2L_{k+2}]$. Consider the cube $\boldsymbol{\Lambda}_{|\mathbf{x}|-aNL_k}(\mathbf{x}) \subset \mathbb{A}_k$. It follows from the choice of k_1 ($\leq k$) that all cubes of radius L_k inside $\boldsymbol{\Lambda}_{|\mathbf{x}|-aNL_k}(\mathbf{x})$ are (E,m_N)-NS. Therefore, by Proposition 3, using the (E,m_N)-NS property of all cubes of radius L_k inside $\boldsymbol{\Lambda}_{|\mathbf{x}|-aNL_k}(\mathbf{x})$ and taking into account that $|\mathbf{x}| > L_{k+1} = L_k^{\alpha}$, we obtain:

$$\begin{aligned}
-\ln\|\chi_{\mathbf{x}}\boldsymbol{\Psi}\| &\geq \gamma(m_N,L_k)(|\mathbf{x}|-aNL_k) - C_3\ln L_k \\
&\geq m_N(1+L_k^{-1/8})|\mathbf{x}|\left(1-aNL_k^{1-\alpha} - CL_k^{-\alpha}\ln L_k\right) \\
&\geq m_N|\mathbf{x}|\left(1+L_k^{-1/8}\right)\left(1-C'NL_k^{-1}\right) \quad (\text{since } \alpha > 2) \\
&\geq m_N|\mathbf{x}| \geq m^*|\mathbf{x}|.
\end{aligned}$$

In other words, there exists $r(\boldsymbol{\Psi}) < \infty$ such that for all $\mathbf{x} \in \boldsymbol{\mathscr{Z}}^N$ with $|\mathbf{x}| \geq r(\boldsymbol{\Psi})$,

$$\|\chi_{\mathbf{x}}\boldsymbol{\Psi}\| \leq \mathrm{e}^{-m^*|\mathbf{x}|}. \tag{67}$$

Proof of assertion (B) of Theorem 1. By Proposition 3, for spectrally a.e. $E \in \mathbb{R}$ there exists a generalized eigenfunction $\boldsymbol{\Psi}$ with generalized eigenvalue E. By Lemma 11, every generalized eigenfunction $\boldsymbol{\Psi}$ with eigenvalue in I^* is square-summable, hence the spectrum of $\mathbf{H}(\omega)$ in I^* is pure point, and there is a countable family of L^2-eigenfunctions $\boldsymbol{\Psi}_j(\omega)$ of $\mathbf{H}(\omega)$ with eigenvalues in I^*. Now the claim follows from (67). ◻

Appendix A Proof of Lemma 3

In this section, we consider abstract finite connected graphs $\mathscr{G}$, without cyclic edges $\langle x,x\rangle$, endowed with the graph distance $\mathrm{d} = \mathrm{d}_{\mathscr{G}}$; recall that $\mathrm{d}_{\mathscr{G}}(x,y)$ is the length of the shortest path from x to y over the graph's edges. Given a function $f : \mathscr{G} \to \mathbb{R}$ and a subset $A \subset \mathscr{G}$, we denote $\mathrm{M}(f,A) := \max_{x\in A} f(x)$. The general results will be used in the situation where $\mathscr{G} \subset \boldsymbol{\mathscr{Z}}^N$, with the graph structure inherited from the lattice $\boldsymbol{\mathscr{Z}}^N$.

Note that we abandon here the boldface notations, relative to the multi-particle model at hand, since we operate in this section with fairly general graphs.

Denote by $\mathrm{B}_L(u)$ the ball $\{x \in \mathscr{G} : d(u,x) \leq L\}$, and by $\mathscr{L}_L(u)$ the spherical layer $\{x \in \mathscr{G} : d(u,x) = L\}$. Whenever $\mathscr{G}$ is a subgraph of a larger graph (e.g., of $\mathscr{Z}^N$), the inclusion $\mathscr{G} \supset \mathrm{B}_L(u)$ might become ambiguous; for the purposes of the main application of Lemma 12, the following convention suffices : by saying that $\mathscr{G} \supset \mathrm{B}_L(u)$, we also mean that there really are some points $x \in \mathscr{G}$ at distance L from u. Alternatively, one can restrict the analysis to the finite subgraphs $\mathscr{G} \subset \mathscr{Z}^N$ and denote $\mathrm{B}_L(u) = \{x \in \mathscr{Z}^N : d(u,x) \leq L\}$.

Introduce the following notions.

Definition 7 (1) Let be given two integers $L \geq \ell \geq 1$, a real number $q \in (0,1)$, a finite connected graph $\mathscr{G} \supset \mathrm{B}_L(u)$, and a function $f : \mathrm{B}_L(u) \to \mathbb{R}_+$. A point $x \in \mathrm{B}_{L-\ell}(u)$ is called (ℓ,q)-regular for the function f if

$$f(x) \leq q\,\mathrm{M}(f,\mathrm{B}_\ell(x)). \tag{68}$$

The set of all regular points for f is denoted by $\mathscr{R}_f$.
(2) A spherical layer $\mathscr{L}_r(u)$ is called regular if $\mathscr{L}_r(u) \subset \mathscr{R}_f$.

(3) For $x \in \mathrm{B}_{L-\ell}(u)$, set

$$r(x) := \begin{cases} \min\{r \geq \mathrm{d}(u,x) : \mathscr{L}_r \subset \mathscr{R}_f\}, & \text{if there is } \mathscr{L}_r \subset \mathscr{R}_f \text{ with } r \geq \mathrm{d}(u,x), \\ +\infty, & \text{otherwise,} \end{cases}$$

and $R_f(x) = r(x) + \ell$.

(4) Given a subset $\mathscr{S} \subset \mathrm{B}_L(u)$, the function f is called $(\ell, q, \mathscr{S})$-dominated in $\mathrm{B}_L(u)$ if $\mathrm{B}_L(u) \setminus \mathscr{S} \subset \mathscr{R}_f$, and for any $x \in \mathrm{B}_{L-\ell}(u)$ with $R_f(x) < +\infty$, one has

$$f(x) \leq q\mathrm{M}\big(f, \mathrm{B}_{R_f(x)}(u)\big).$$

The key feature of the $(\ell, q, \mathscr{S})$-dominated functions is the following result, which is a variant of [22, Theorem 2.4.1] and [14, Lemma 9], obviously inspired by Lemma 4.1 from the work by von Dreifus and Klein [26,].

Lemma 12 *Let a function* $f : \mathscr{G} \to \mathbb{R}_+$ *be* $(\ell, q, \mathscr{S})$*-dominated in a ball* $\mathrm{B} = \mathrm{B}_L(u)$, *with* $L \geq \ell \geq 1$, $q \in (0,1)$. *Assume that the set* $\mathscr{S}$ *is covered by a union* $\mathscr{A}$ *of concentric annuli* $\mathscr{A}_j := \mathrm{B}_{b_j}(u) \setminus \mathrm{B}_{a_j-1}(u)$ *with*[13] $b_j \leq a_{j+1} - 2$ *(so the consecutive annuli are disjoint and non-adjacent) and* $\mathrm{w}(\mathscr{A}) := \sum_j (b_j - a_j + 1) \leq L - \ell$. *Then*

$$f(u) \leq q^{\frac{L-\ell-\mathrm{w}(\mathscr{A})}{\ell}} \mathrm{M}\big(f, \mathrm{B}_L(u)\big).$$

Proof It follows from the hypothesis that

$$\frac{L - \mathrm{w}(\mathscr{A})}{\ell} \geq \left\lfloor \frac{L - \mathrm{w}(\mathscr{A})}{\ell} \right\rfloor =: n+1,\ n \geq 0.$$

Define recursively a finite sequence of integers $\{r_n > r_{n-1} > \cdots > r_0\}$:

$$\begin{aligned} r_n &= \max\left[r \leq L - \ell : \mathscr{L}_r \cap \mathscr{S} = \varnothing\right], \\ r_j &= \max\left[r \leq r_{j+1} - \ell : \mathscr{L}_r \cap \mathscr{S} = \varnothing\right],\ \ j = n-1, \ldots, 0. \end{aligned} \tag{69}$$

It is convenient to introduce also, formally, $r_{n+1} = L$, although the regularity property does not apply to the points in $\mathscr{L}_{r_{n+1}} = \mathscr{L}_L$. Note that one can indeed construct recursively in (69) $n+1$ integers $r_j \geq 0$, since $L - \mathrm{w}(\mathscr{A}) - (n+1)\ell \geq 0$, and we have

$$L - r_0 = \sum_{j=0}^{n} (r_{j+1} - r_j) \leq (n+1)\ell + \mathrm{w}(\mathscr{A}),$$

so $r_0 \geq L - \mathrm{w}(\mathscr{A}) - (n+1)\ell \geq 0$. Note that the non-adjacency of the annuli $\mathscr{A}_{j-1}$, $\mathscr{A}_j$ implies that there is $r_j \in [b_{j-1}, a_j - 1]$. (We assumed the non-adjacency since it can always be achieved by merging adjacent annuli.)

Introduce the non-decreasing non-negative function

$$F : r \mapsto \mathrm{M}(f, \mathrm{B}_r(u)),\ \ r \in \{0, 1, \ldots, L\}.$$

[13] Note that $\mathrm{w}(\mathscr{A}) := \sum_j (b_j - a_j + 1)$ is not the geometrical width of the annulus $\mathscr{A}_j$, but rather the cardinality of the radial section across it. Example: $\mathbb{Z}^1 \supset \mathscr{G} = [0,7]$, $u = 0$, $\mathscr{A}$ contains a single annulus with $a_1 = 1$, $b_1 = 2$. Here $\mathrm{w}(\mathscr{A}) = 2$, while the radial section across the annulus (coinciding with the annulus, in this 1-dimensional example) has length 1.

For all $0 \le j \le n$, one has $r_j + \ell \le L$, and $\mathscr{L}_{r_j}$ is regular by construction, so

$$F(r_n) = \mathrm{M}(f, \mathrm{B}_{r_n}(u)) \le q\mathrm{M}(f, \mathrm{B}_{r_n+\ell}(u)) \le q\mathrm{M}(f, \mathrm{B}),$$

and for all $0 \le j \le n-1$, similarly,

$$F(r_j) = \mathrm{M}(f, \mathrm{B}_{r_j}(u)) \le q\mathrm{M}(f, \mathrm{B}_{r_j+\ell}(u)) \le qF(r_{j+1}).$$

Now the induction in $j = n, \ldots, 0$ proves the claim:

$$\begin{aligned} f(u) &\le \mathrm{M}(f, \mathrm{B}_{r_0}(u)) = F(r_0) \le q^{n+1}\mathrm{M}(f, \mathrm{B}) \\ &\le q^{\left\lfloor \frac{L - \mathrm{w}(\mathscr{A})}{\ell} \right\rfloor} \mathrm{M}(f, \mathrm{B}) \le q^{\frac{L - \mathrm{w}(\mathscr{A}) - \ell}{\ell}} \mathrm{M}(f, \mathrm{B}). \end{aligned}$$

□

The relevance of the notion of dominated decay is explained by the next result following immediately from the GRI, by a two-fold application thereof; the main idea of such an argument is well-known and goes back to [26, Lemma 4.2].

Lemma 13 *Suppose that for some integer $L > \ell > 1$ and $\mathbf{u} \in (\mathbb{R}^d)^N$ the cube $\boldsymbol{\Lambda}_L(\mathbf{u})$ is (E, β)-CNR. Let $\boldsymbol{\Lambda}' \supset \boldsymbol{\Lambda}_{L+1}(\mathbf{u})$, $\mathbf{y} \in \boldsymbol{\Lambda}' \setminus \boldsymbol{\Lambda}_L(\mathbf{u})$. Consider the lattice cubes $\mathbf{B}_L(\mathbf{u}) \subset \mathbf{B}_{L+1}(\mathbf{u})$, and the function $f : \mathbf{B}_L(\mathbf{u}) \to \mathbb{R}_+$ given by*

$$f : \mathbf{x} \mapsto \|\mathbf{1_y} \mathbf{G}_{\boldsymbol{\Lambda}'}(E) \mathbf{1_x}\|.$$

Let $\mathscr{S} \subset \mathbf{B}_{L-\ell}(\mathbf{u})$ be a (possibly empty) set such that any cube $\mathbf{B}_\ell(\mathbf{x}) \subset \mathbf{B}_{L-\ell}(\mathbf{u}) \setminus \mathscr{S}$ is (E, δ, m)-NS. If $0 < \beta < \delta \le 1$ and

$$m\ell^\delta > 2L^\beta > L^\beta + \ln |\mathbf{B}_L(\mathbf{u})|,$$

then f is $(\ell, q, \mathscr{S})$-dominated in $\mathbf{B}_L(\mathbf{u})$, with

$$q = \mathrm{e}^{-m'\ell^\delta}, \;\; m' := m - 2\ell^{-\delta} L^\beta > 0.$$

One can see that the definition of the "norm" $\left\| \cdot \right\|^{\curlywedge}$ in Eqn. (23) is indeed well-adapted to the dominated decay bounds; all combinatorial factors are hidden in the factor $q < 1$, where they are suppressed by the small values of the respective Green functions between the center and the boundary of the cube involved.

Now Lemma 3 can be proved essentially in the same way as [22, Theorem 2.4.1], with the help of Lemma 12. The role of the graph $\mathscr{G}$ is played, of course, by the scatterers' lattice $\mathscr{Z}^N$ labeling the centers of the unit cells. Technically, the length step $\ell + 1$ used in [22] becomes here ℓ (cf. Lemma 13), due to the form of the GRI typical for the continuous models.

Proof of Lemma 3. Let $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^{(N)}(\mathbf{u}, L_{k+1})$. By hypothesis, either $\boldsymbol{\Lambda}$ contains no (E, δ, m_N)-S ball of radius L_k, or there is a ball $\mathbf{B}^{(N)}(\mathbf{w}, L_k) \subset \mathbf{B}$ such that any ball $\mathbf{B}^{(N)}(\mathbf{v}, L_k)$ with $\mathbf{v} \in \mathbf{B} \setminus \mathbf{B}^{(N)}(\mathbf{w}, 9NL_k)$ is (E, δ, m_N)-NS. Bearing in mind Lemma 12, denote by $\mathscr{S}$ the union of all spherical layers $\mathscr{L}_r(\mathbf{u})$ such that $\mathscr{L}_r(\mathbf{u}) \cap \mathbf{B}^{(N)}(\mathbf{w}, 9NL_k) \neq \varnothing$. It follows from the relation $\beta < \delta$ (cf. (33)) that, for L_0 or m_N large enough,

$$m_N - 2L_k^{-\delta} L^\beta \ge m_N \left(1 - 2m_N^{-1} L_k^{-\delta+\beta} Y^{2\beta}\right) \ge \tfrac{3}{4} m_N > 0. \tag{70}$$

Thus, by Lemma 13, the function $f : \mathbf{x} \in \mathbf{B} \mapsto \left|\mathbf{G}_{\mathbf{B}}^{(N)}(\mathbf{u},\mathbf{x};E)\right|$ is $(L_k, q, \mathscr{S})$-dominated in $\mathbf{B}$, in the sense of Definition 7, with $q \le \mathrm{e}^{-\frac{3}{4}m_N L_k^{\delta}}$.

Applying Lemma 12, we can write, with the convention $-\ln 0 = +\infty$, that

$$
\begin{aligned}
-\ln f(\mathbf{x}) &\ge -\ln\left\{\mathrm{e}^{L^{\beta}} \exp\left[-\frac{3m_N}{4} L_k^{\delta} \cdot \frac{(L-(2\cdot 9NL_k+1)L_k - L_k}{L_k}\right]\right\} \\
&\ge L^{\delta} m_N \frac{3}{4} \cdot \frac{(Y-20N)}{Y^{2\delta}} - L^{\beta}
\end{aligned}
$$

thus by virtue of the conditions listed in the table (33) (viz. $\frac{1}{4}Y^{1-2\delta} \ge 3$, $m_N \ge 1$, $\beta < \delta$, $Y \ge 30N^* \ge 30N$), one obtains by a straightforward calculation that

$$
-\ln f(\mathbf{x}) \ge 2m_N L^{\delta} \ge m_N L^{\delta} + \ln(3^{Nd} L^{Nd}),
$$

provided L_0 is large enough. □

Appendix B Proof of Lemma 5

We will need the following result.

Lemma 14 *Fix $\beta, \delta \in (0,1]$, $m^* \ge 1$, $E \in \mathbb{R}$ and an integer $k \ge 0$. Consider a* WI *cube $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u})$ with the canonical factorization $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u}) = \mathbf{\Lambda}' \times \mathbf{\Lambda}'' \equiv \mathbf{\Lambda}_{L_k}^{(n')}(\mathbf{u}') \times \mathbf{\Lambda}_{L_k}^{(n'')}(\mathbf{u}'')$. Let $E \in I_N^*$ and suppose that $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u})$ is (E,β)-NR and fulfills the following two conditions:*

$$
\forall \lambda' \in \Sigma_{I_{N-1}^*}\left(\mathbf{H}_{\mathbf{\Lambda}'}^{(n')}\right) \quad \mathbf{\Lambda}'' \text{ is } (E-\lambda', \delta, m_{n'})\text{-NS}, \tag{71}
$$

$$
\forall \lambda'' \in \Sigma_{I_{N-1}^*}\left(\mathbf{H}_{\mathbf{\Lambda}''}^{(n'')}\right) \quad \mathbf{\Lambda}' \text{ is } (E-\lambda'', \delta, m_{n''})\text{NS}. \tag{72}
$$

If L_0 is large enough then $\mathbf{B}_{L_k}^{(N)}(\mathbf{u})$ is (E,δ,m_N)-NS.

Proof The operator $\mathbf{H}_{\mathbf{\Lambda}''}$ (as well as $\mathbf{H}_{\mathbf{\Lambda}'}$) has compact resolvent, so $E'_a \uparrow +\infty$ as $a \to +\infty$. Recall that we assume the EVs of the operators appearing in our arguments to be numbered in increasing order, counting multiplicity; without loss of generality, we can also assume that $a = 0, 1, \ldots$. We have the following identities:

$$
\begin{aligned}
\mathbf{G}_{\mathbf{\Lambda}_{L_k}(\mathbf{u})}(E) &= \sum_a \mathbf{P}'_{\mathbf{\Psi}'_a} \otimes \mathbf{G}_{\mathbf{\Lambda}''}(E - E'_a) &(73)\\
&= \sum_a \mathbf{G}_{\mathbf{\Lambda}'}(E - E''_a) \otimes \mathbf{P}''_{\mathbf{\Psi}''_a}, &(74)
\end{aligned}
$$

where $\mathbf{P}_{\mathbf{\Psi}}$ stand for the rank-one spectral projections onto the respective EFs $\mathbf{\Psi}$. By the second resolvent identity, for any energy E which is not in the spectra of $\mathbf{H}_{\mathbf{\Lambda}}^{\mathrm{ni}} = \mathbf{H}_{\mathbf{\Lambda}'}^{(n')} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{H}_{\mathbf{\Lambda}''}^{(n'')}$ or $\mathbf{H}_{\mathbf{\Lambda}}$, we have for their resolvents, $\mathbf{G}_{\mathbf{\Lambda}}^{\mathrm{ni}}(E)$ and $\mathbf{G}_{\mathbf{\Lambda}}(E)$, that $\mathbf{G}_{\mathbf{\Lambda}} = \mathbf{G}_{\mathbf{\Lambda}}^{\mathrm{ni}} - \mathbf{G}_{\mathbf{\Lambda}}^{\mathrm{ni}} \mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''} \mathbf{G}_{\mathbf{\Lambda}}$, thus

$$
\begin{aligned}
\|\chi_{\mathbf{y}} \mathbf{G} \chi_{\mathbf{y}}\| &\le \|\chi_{\mathbf{y}} \mathbf{G}^{\mathrm{ni}} \chi_{\mathbf{x}}\| + \|\chi_{\mathbf{y}} \mathbf{G}^{\mathrm{ni}} \mathbf{U} \mathbf{G} \chi_{\mathbf{x}}\| \\
&\le \|\chi_{\mathbf{y}} \mathbf{G}^{\mathrm{ni}} \chi_{\mathbf{x}}\| + \|\mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''}\| \|\mathbf{G}_{\mathbf{\Lambda}}^{\mathrm{ni}}\| \|\mathbf{G}_{\mathbf{\Lambda}}\|.
\end{aligned}
$$

We start with the last term in the RHS. Since $\mathbf{\Lambda}$ is weakly interactive, we have by inequality (36) (cf. also Lemma 4), with $\beta < \zeta$ and large L_0,

$$\|\mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''}\| \leq C\mathrm{e}^{-L_k^{\zeta}} < \mathrm{e}^{-L_k^{\beta}}.$$

The assumed (E,β)-NR property gives $\|\mathbf{G}_{\mathbf{\Lambda}}\| \leq \frac{1}{2}\mathrm{e}^{L^{\beta}}$, since $\mathrm{dist}(E,\Sigma_{\mathbf{\Lambda}}) \geq 2\mathrm{e}^{-L^{\beta}}$. The min-max principle implies for the spectrum $\Sigma^{\mathrm{ni}}_{\mathbf{\Lambda}}$ of $\mathbf{H}^{\mathrm{ni}}_{\mathbf{\Lambda}}$

$$\mathrm{dist}(E,\Sigma^{\mathrm{ni}}_{\mathbf{\Lambda}}) \geq 2\mathrm{e}^{-L^{\beta}} - \|\mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''}\| \geq \mathrm{e}^{-L^{\beta}}, \tag{75}$$

so $\|\mathbf{G}^{\mathrm{ni}}_{\mathbf{\Lambda}}\| \leq \mathrm{e}^{L^{\beta}}$. Hence with $\delta < \zeta$, $L_k \leq L < L_{k+1} = YL_k$ and L_0 large enough,

$$\|\mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''}\|\|\mathbf{G}^{\mathrm{ni}}_{\mathbf{\Lambda}}\|\|\mathbf{G}_{\mathbf{\Lambda}}\| \leq C\mathrm{e}^{-L_k^{\zeta}+2L^{\beta}} \leq \frac{1}{4}\mathrm{e}^{-2m_N L^{\delta}}.$$

It remains to assess the GF of the non-interacting Hamiltonian. If $\mathbf{y} = (\mathbf{y}',\mathbf{y}'') \in \partial^{-}\mathbf{\Lambda}^{(N)}_{L_k}(\mathbf{u})$, then either $|\mathbf{y}'-\mathbf{u}'| = L_k$, in which case we shall use (74), or $|\mathbf{y}''-\mathbf{u}''| = L_k$, and then we use instead the representation (73). For brevity, we consider in detail only the former case. Denote

$$a' = a'(n') = \max\{a : E'_a \leq E^*_{n'}\}$$

(here $n' \leq N-1$, so $E^*_{n'} \geq E^*_{N-1}$) and let $3\eta = E^*_{N-1} - E^*_N \,(>0)$. First, let $a > a'$, and consider the eigenvalue of the form $\lambda' = E'_a$. By construction of a', for any $E \in I^*_N = [0, E^*_N]$ we have (cf. (42))

$$E - \lambda' = E - E'_a < E^*_N - E^*_{n'} < E^*_N - E^*_{N-1} = -E^*_N = -2m^*_N,$$

thus $\mathrm{dist}(E,\lambda') > 2m^*_N$. Therefore, applying the Combes-Thomas estimate (cf. [24,49]) combined with Weyl's law, we obtain that, for L_0 large enough,

$$\begin{aligned}\sum_{a>a'} \left\|\mathbf{P}'_{\mathbf{\Psi}'_a} \otimes \mathbf{G}_{\mathbf{\Lambda}''}(E-E'_a)\right\|^{\curlywedge} &\leq C'''\eta^{-1}\sum_{j=0}^{+\infty} L^C \left(E_* + 2m^*_N + j\right)^{C''} \mathrm{e}^{-(2m^*_N+j)L} \\ &\leq \frac{1}{4}\mathrm{e}^{-\frac{3}{2}m_N L^{\delta}}.\end{aligned}$$

It also follows from Weyl's law that $\mathrm{card}\{a : E''_a \leq E_* + 2m^*_N\} \leq L^{C'}$.

Further, for any $\lambda' = E'_a \in \Sigma_{I^*_{n'}}$, with $a \leq a(\eta)$, we have by assumption

$$\left\|\mathbf{G}_{\mathbf{\Lambda}''}(E-E'_a)\right\|^{\curlywedge} \leq \mathrm{e}^{-m_{N-1}L^{\delta}} \leq \mathrm{e}^{-2m_N L^{\delta}}.$$

Therefore,

$$\begin{aligned}\sum_a \left\|\chi_{\mathbf{y}}\mathbf{P}'_{\mathbf{\Psi}'_a} \otimes \mathbf{G}_{\mathbf{\Lambda}''}(E-E'_a)\chi_{\mathbf{y}}\right\| &\leq \Big(\sum_{a\leq a(\eta)} + \sum_{a>a(\eta)}\Big)\Big(\mathbf{P}'_{\mathbf{\Psi}'_a} \otimes \mathbf{G}_{\mathbf{\Lambda}''}(E-E'_a)\Big) \\ &\leq L_k^{C'}\,\mathrm{e}^{-2m_N L^{\delta}} + \frac{1}{4}\mathrm{e}^{-\frac{3}{2}m_N L^{\delta}} \leq \frac{1}{2}\mathrm{e}^{-\frac{3}{2}m_N L^{\delta}}.\end{aligned} \tag{76}$$

Similarly, for $\mathbf{y} \in \partial^- \mathbf{B}_{L_k}(\mathbf{u})$ with $|\mathbf{y}'' - \mathbf{u}''| = L_k$, we obtain with the identity (73)

$$\sum_a \left\| \chi_{\mathbf{y}} \mathbf{G}_{\boldsymbol{\Lambda}'}(E - E''_a) \otimes \mathbf{P}''_{\boldsymbol{\Psi}''_a} \chi_{\mathbf{y}} \right\| \le \frac{1}{2} \mathrm{e}^{-\frac{3}{2} m_N L^\delta}. \tag{77}$$

Taking the sum over all $\mathbf{y} \in \partial^- \mathbf{B}_{L_k}(\mathbf{u})$, falling into one of the two categories (76)–(77), we obtain

$$\left\| \mathbf{G}_{\boldsymbol{\Lambda}_L(\mathbf{u})}(E) \right\|^{\curlywedge} \le \mathrm{Const}\, L^{Nd} \mathrm{e}^{-\frac{3}{2} m_N L^\delta} \le \mathrm{e}^{-m_N L^\delta},$$

for L_0 large enough; this proves the claim. □

Proof of Lemma 5. Denote by $\mathscr{S}$ the event in the LHS of (37). Let $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^{(N)}(\mathbf{u}, L_k)$ and consider the canonical factorization $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}' \times \boldsymbol{\Lambda}''$. We have

$$\begin{aligned} \mathbb{P}\{\mathscr{S}\} < \mathbb{P}\{\, \boldsymbol{\Lambda} \text{ is not } (E,\beta)\text{-NR}\} \\ + \mathbb{P}\{\, \boldsymbol{\Lambda} \text{ is } (E,\beta)\text{-NR and } (E,\delta,m_N)\text{-S}\,\}. \end{aligned} \tag{78}$$

By Theorem 3, the first term in the RHS is bounded by $\mathrm{e}^{-L_{k+1}^\beta} < \frac{1}{3} \mathrm{e}^{-\frac{3}{2} \nu_N L_{k+1}^\kappa}$, since $\kappa < \beta$, so we focus on the second summand in the RHS of (78).

Let $\Sigma' = \Sigma\big(\mathbf{H}^{(n')}_{\mathbf{B}'}\big) \cap I^*_{n'}$, $\Sigma'' = \Sigma\big(\mathbf{H}^{(n'')}_{\mathbf{B}''}\big) \cap I^*_{n''}$, and consider the events

$$\begin{aligned} \mathscr{S}' &= \{\omega : \exists \lambda' \in \Sigma',\, \mathbf{B}'' \text{ is } (E - \lambda', \delta, m_{n'}) - \mathrm{NS}\}, \\ \mathscr{S}'' &= \{\omega : \exists \lambda'' \in \Sigma'',\, \mathbf{B}' \text{ is } (E - \lambda'', \delta, m_{n''}) - \mathrm{NS}\}, \end{aligned}$$

for all $E \in I^*_{N-1}$. Since $\boldsymbol{\Lambda}$ is WI, we have that $\Pi\boldsymbol{\Lambda}' \cap \Pi\boldsymbol{\Lambda}'' = \varnothing$, hence $\mathbf{H}_{\boldsymbol{\Lambda}''}(\omega)$ is independent of the sigma-algebra $\mathfrak{F}'$ generated by the random scatterers affecting $\boldsymbol{\Lambda}'$, while $\mathbf{H}_{\boldsymbol{\Lambda}'}(\omega)$ is $\mathfrak{F}'$-measurable, and so are all the EVs $\lambda' \in \Sigma'$.

Further, by non-negativity of $\mathbf{H}'$, if $E \le E^*_{N-1}$, then $E - \lambda' \le E^*_{N-1}$ for all $\lambda' \in \Sigma'$.

Replacing the quantity $E - \lambda'$, rendered nonrandom by conditioning on $\mathfrak{F}'$, with a new nonrandom parameter E', we have, by induction on $1 \le n \le N-1$, and with $\nu_{n''} \ge \nu_{N-1}$,

$$\begin{aligned} \mathbb{P}\big\{\mathscr{S}'\big\} = \mathbb{E}\big[\mathbb{P}\big\{\mathscr{S}' \,|\, \mathfrak{F}''\big\}\,\big] &\le \sup_{E' \le E^*_{N-1}} \mathbb{P}\big\{\boldsymbol{\Lambda}'' \text{ is } (E', m)\text{-S}\big\} \\ &\le |\boldsymbol{\Lambda}''|\, \mathrm{e}^{-\nu_{n''} L_k^\kappa} \le |\boldsymbol{\Lambda}''|\, \mathrm{e}^{-2\nu_N L_k^\kappa} \le \frac{1}{3} \mathrm{e}^{-\frac{3}{2} \nu_N L_k^\kappa}. \end{aligned} \tag{79}$$

Similarly,

$$\mathbb{P}\big\{\mathscr{S}''\big\} \le \frac{1}{3} \mathrm{e}^{-\frac{3}{2} \nu_N L_k^\kappa}. \tag{80}$$

Collecting (78)–(80), the assertion (37) follows.

For the second assertion (38), it suffices to apply a polynomial bound on the number of cubes of size L_k with centers on the lattice $\mathscr{Z}^N$ in a cube of radius L_{k+1}. □

Appendix C Proof of Lemma 9

The main argument in this section is very close to that in Appendix B, but several key elements here are different, e.g., the definition of the lengths scales, the notion of a WI cube, and most importantly, the quantitative form of the probabilistic bound for singular cubes, as well as the relations between the key parameters. This is why we need a separate proof for an analog of Lemma 9.

Lemma 15 *Fix $\beta \in (0,1]$, $m^* \geq 1$ and $E \in I_N^*$. Suppose that a* WI *cube $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u})$ with the canonical factorization $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u}) = \mathbf{\Lambda}_{L_k}^{(n')}(\mathbf{u}') \times \mathbf{\Lambda}_{L_k}^{(n'')}(\mathbf{u}'') = \mathbf{\Lambda}' \times \mathbf{\Lambda}''$ is (E,β)-NR and satisfies the following two conditions:*

$$\forall \lambda' \in I_{n'}^* \cap \Sigma\left(\mathbf{H}_{\mathbf{\Lambda}'}^{(n')}\right) \qquad \mathbf{\Lambda}'' \text{ is } (E-\lambda', m_{n'})\text{-NS}, \tag{81}$$

$$\forall \lambda'' \in I_{n''}^* \cap \Sigma\left(\mathbf{H}_{\mathbf{\Lambda}''}^{(n'')}\right) \qquad \mathbf{\Lambda}' \text{ is } (E-\lambda'', m_{n''})\text{-NS}. \tag{82}$$

If L_0 is large enough then $\mathbf{\Lambda}_{L_k}^{(N)}(\mathbf{u})$ is (E, m_N)-NS.

Proof Let $\mathbf{B} = \mathbf{\Lambda} \cap \mathscr{X}^N$. We start as in Lemma 14 and make use of the identities (71)–(72). Applying again the second resolvent identity, $\mathbf{G}_{\mathbf{\Lambda}} = \mathbf{G}_{\mathbf{\Lambda}}^{\rm ni} - \mathbf{G}_{\mathbf{\Lambda}}^{\rm ni}\mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''}\mathbf{G}_{\mathbf{\Lambda}}$, we obtain for $\mathbf{y} \in \partial^-\mathbf{B}$

$$\begin{aligned}\|\chi_{\mathbf{y}}\mathbf{G}\chi_{\mathbf{u}}\| &\leq \|\chi_{\mathbf{y}}\mathbf{G}^{\rm ni}\chi_{\mathbf{u}}\| + \|\chi_{\mathbf{y}}\mathbf{G}^{\rm ni}\mathbf{U}\mathbf{G}\chi_{\mathbf{u}}\| \\ &\leq \|\chi_{\mathbf{y}}\mathbf{G}^{\rm ni}\chi_{\mathbf{u}}\| + \|\mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''}\|\|\mathbf{G}_{\mathbf{\Lambda}}^{\rm ni}\|\|\mathbf{G}_{\mathbf{\Lambda}}\|.\end{aligned}$$

Consider the last term in the RHS. Since $\mathbf{\Lambda}$ is weakly interactive, we have by inequality (46) (cf. also Lemma 8), with $\tau\zeta > 1$ by (42),

$$\|\mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''}\| \leq C\mathrm{e}^{-cL_k^{\tau\zeta}} < \mathrm{e}^{-\widetilde{m}L_k},$$

where $\widetilde{m} > 0$ can be made arbitrarily large[14], provided L_0 is large enough. Below we assume that $\widetilde{m} > 2m_N$.

The assumed (E,β)-NR property gives $\|\mathbf{G}_{\mathbf{\Lambda}}\| \leq \frac{1}{2}\mathrm{e}^{L_k^\beta}$, since $\mathrm{dist}(E,\Sigma_{\mathbf{\Lambda}}) \geq 2\mathrm{e}^{-L_k^\beta}$, thus by the min-max principle for the spectrum $\Sigma_{\mathbf{\Lambda}}^{\rm ni}$ of $\mathbf{H}_{\mathbf{\Lambda}}^{\rm ni}$

$$\mathrm{dist}(E,\Sigma_{\mathbf{\Lambda}}^{\rm ni}) \geq 2\mathrm{e}^{-L_k^\beta} - \|\mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''}\| \geq \mathrm{e}^{-L_k^\beta}, \tag{83}$$

so $\|\mathbf{G}_{\mathbf{\Lambda}}^{\rm ni}\| \leq \mathrm{e}^{L_k^\beta}$. Finally, with $\widetilde{m} > 3m_N$,

$$\|\mathbf{U}_{\mathbf{\Lambda}',\mathbf{\Lambda}''}\|\|\mathbf{G}_{\mathbf{\Lambda}}^{\rm ni}\|\|\mathbf{G}_{\mathbf{\Lambda}}\| \leq C\mathrm{e}^{-\widetilde{m}L_k + 2L_k^\beta} \leq \frac{1}{2}\mathrm{e}^{-m_N L_k}.$$

---

[14] Making $\widetilde{m}$ large is not necessary in the context of the Lifshitz tails analysis, with the random potential of small amplitude, but it becomes useful for the proof of localization in the strong disorder regime.

It remains to assess the GF of $\mathbf{H}^{\mathrm{ni}}$. If $\mathbf{y}=(\mathbf{y}',\mathbf{y}'')\in\partial^-\mathbf{B}$, then either $|\mathbf{y}'-\mathbf{u}'|=L_k$, in which case we shall use (74), or $|\mathbf{y}''-\mathbf{u}''|=L_k$, and then we use instead the representation (73). For brevity, we consider in detail only the former case.

Let $4\eta=E^*_{N-1}-E^*_N\equiv E^*_N$ (cf. (42)), $a'=a'(n')=\max\{a:\,E'_a\le E^*_{n'}\}$. By the Weyl law, $\operatorname{card}\{a:\,E''_a\le E^*_N+4\eta\}\le L_k^C$, for some $C=C(d,N)<+\infty$. The Combes-Thomas estimate (cf. [24,49]) implies that

$$\sum_{a>a(\eta)}\left\|\mathbf{P}'_{\boldsymbol{\Psi}'_a}\otimes\mathbf{G}_{\boldsymbol{\Lambda}''}(E-E'_a)\right\|^{\curlywedge}\le\sum_{j=1}^{+\infty}L_k^C(E_*+3\eta+j)^{C'}\mathrm{e}^{-(4\eta+j)L_k}\le\frac{1}{2}\mathrm{e}^{-3\eta L_k}.$$

By assumption, for all $a\le a'$,

$$\left\|\mathbf{G}_{\boldsymbol{\Lambda}''}(E-E'_a)\right\|^{\curlywedge}\le\mathrm{e}^{-m_{N-1}L_k}\le\mathrm{e}^{-2m_NL_k}.$$

Recalling $3\eta=\frac{3}{4}(E^*_{N-1}-E^*_N)=\frac{3}{4}E^*_N=\frac{3}{2}m^*_N$, we conclude that

$$\begin{aligned}\sum_a\left\|\chi_{\mathbf{y}}\mathbf{P}'_{\boldsymbol{\Psi}'_a}\otimes\mathbf{G}_{\boldsymbol{\Lambda}''}(E-E'_a)\chi_{\mathbf{y}}\right\|&\le\left(\sum_{a\le a(\eta)}+\sum_{a>a(\eta)}\right)\mathbf{P}'_{\boldsymbol{\Psi}'_a}\otimes\mathbf{G}_{\boldsymbol{\Lambda}''}(E-E'_a)\\&\le L_k^{C'}\mathrm{e}^{-2m_NL_k}+\frac{1}{2}\mathrm{e}^{-\frac{3}{2}m^*_NL_k}\le\mathrm{e}^{-\frac{3}{2}m_NL_k}.\end{aligned}\tag{84}$$

Similarly,

$$\sum_a\left\|\chi_{\mathbf{y}}\mathbf{G}_{\boldsymbol{\Lambda}'}(E-E''_a)\otimes\mathbf{P}''_{\boldsymbol{\Psi}''_a}\chi_{\mathbf{y}}\right\|\le\mathrm{e}^{-\frac{3}{2}m_NL_k}.\tag{85}$$

Taking the sum over all $\mathbf{y}\in\partial^-\mathbf{B}$, falling into one of the two categories (84)–(85), we obtain for L_0 large enough

$$\left\|\mathbf{G}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{u})}(E)\right\|^{\curlywedge}\le\mathrm{Const}\,L_k^{Nd}\mathrm{e}^{-\frac{3}{2}m_NL_k}\le\mathrm{e}^{-m_NL_k},$$

which proves the claim. □

Proof of Lemma 9. Denote by $\mathscr{S}$ the event in the LHS of (47). Let $\boldsymbol{\Lambda}=\boldsymbol{\Lambda}^{(N)}(\mathbf{u},L_k)$ and consider the canonical factorization $\boldsymbol{\Lambda}=\boldsymbol{\Lambda}'\times\boldsymbol{\Lambda}''$. We have

$$\mathbb{P}\{\mathscr{S}\}<\mathbb{P}\{\,\boldsymbol{\Lambda}\text{ is not }(E,\beta)\text{-NR}\}+\mathbb{P}\{\,\boldsymbol{\Lambda}\text{ is }(E,\beta)\text{-NR and }(E,m_N)\text{-S}\}\,.\tag{86}$$

By Theorem 3, the first term in the RHS of (86) is bounded by $\mathrm{e}^{-L_{k+1}^{\beta}}$, so we focus on the second summand.

Let $\Sigma'=\Sigma\big(\mathbf{H}^{(n')}_{\boldsymbol{\Lambda}'}\big)\cap I^*$, $\Sigma''=\Sigma\big(\mathbf{H}^{(n'')}_{\boldsymbol{\Lambda}''}\big)\cap I^*$, and consider the events

$$\begin{aligned}\mathscr{S}'&=\{\omega:\,\exists\lambda'\in\Sigma',\ \boldsymbol{\Lambda}''\text{ is }(E-\lambda',m_{n'})-\mathrm{NS}\},\\\mathscr{S}''&=\{\omega:\,\exists\lambda''\in\Sigma'',\ \boldsymbol{\Lambda}'\text{ is }(E-\lambda'',m_{n''})-\mathrm{NS}\}.\end{aligned}$$

Notice that, although the spectra $\Sigma',\Sigma''\subset I^*$ are random, their cardinalities are bounded by those for the respective Laplacians, with the potential energy $\mathbf{V}+\mathbf{U}$ switched off, owing to the positivity of the latter. These cardinalities are polynomially bounded in L_k, by the Weyl law.

Since $\boldsymbol{\Lambda}$ is WI, we have that $\Pi\boldsymbol{\Lambda}' \cap \Pi\boldsymbol{\Lambda}'' = \varnothing$, hence $\mathbf{H}_{\boldsymbol{\Lambda}''}(\omega)$ is independent of the sigma-algebra $\mathfrak{F}'$ generated by the random scatterers affecting $\boldsymbol{\Lambda}'$, while $\mathbf{H}_{\boldsymbol{\Lambda}'}(\omega)$ is $\mathfrak{F}'$-measurable, and so are all the EVs $\lambda' \in \Sigma'$.

Further, by non-negativity of $\mathbf{H}'$, if $E \le E^*_{N-1}$, then $E - \lambda' \le E^*_{N-1}$ for all $\lambda' \in \Sigma'$.

Replacing the quantity $E - \lambda'$, rendered nonrandom by conditioning on $\mathfrak{F}'$, with a new nonrandom parameter $E' \le E^*$, we have by induction in $1 \le n \le N-1$

$$\begin{aligned} \mathbb{P}\left\{\mathscr{S}'\right\} &= \mathbb{E}\left[\mathbb{P}\left\{\mathscr{S}' \,|\, \mathfrak{F}''\right\}\right] \le \sup_{E' \le E^*} \mathbb{P}\left\{\boldsymbol{\Lambda}'' \text{ is } (E', m)\text{-S}\right\} \\ &\le C|\boldsymbol{\Lambda}''|\, L_k^{-P(N-1,k)} \le C'\, L_k^{-4\alpha P(N,k)+Nd} \le \frac{1}{3} L_{k+1}^{-4P(N,k)+Nd\alpha^{-1}}. \end{aligned} \tag{87}$$

Using the definition of $P(N,k)$ in (42), we have

$$4P(N,k) = 4 \cdot 2^k P^* (2\alpha)^{N^*-N} = 2P(N,k+1),$$

so

$$4P(N,k) - Nd\alpha^{-1} = 2P(N,k+1) - \frac{1}{2}Nd > \frac{3}{2}P(N,k+1),$$

since $P(N,k+1) \ge P^* > 4Nd$ (cf. (42)). Thus

$$\mathbb{P}\left\{\mathscr{S}'\right\} \le \frac{1}{3} L_{k+1}^{-\frac{3}{2}P(N,k+1)} \tag{88}$$

and, similarly,

$$\mathbb{P}\left\{\mathscr{S}''\right\} \le \frac{1}{3} L_k^{-\frac{3}{2}P(N,k)}. \tag{89}$$

Collecting (86)–(89), the assertion (37) follows.

For the second assertion (38), it suffices to apply a polynomial bound CL_{k+1}^{Nd} on the number of cubes of size L_k with centers on the lattice $\mathscr{Z}^N$ in a cube of radius L_{k+1}. □

Appendix D Proof of Theorem 2

In this section, N does not denote the number of particles.

We begin with a simple auxiliary result on IID random variables $X_1, \dots, X_N$ with uniform distribution Unif$([0,\ell])$, $\ell > 0$. As in Sect. 2.1, we denote by ξ the sample mean and by $\eta_i = X_i - \xi$ the fluctuations relative to ξ. The sample space is identified with the cube $[0,\ell]^N$ where we consider the Lebesgue sigma-algebra of measurable subsets and the normalized (viz. probability) Lebesgue measure $\mathbb{P}$. Further, introduce the sigma-algebra $\mathfrak{F}_\eta$ generated by the fluctuations or, equivalently, by the variables $Y_i = \eta_i - \eta_N = X_i - X_N$, $1 \le i \le N-1$. Fixing the values of all Y_i, we obtain a segment $\mathscr{X}(Y)$. It may have zero length, in which case we endow it with the trivial probability measure; otherwise $\mathbb{P}$ induces on $\mathscr{X}(Y)$ the conditional measure with constant density relative to the Lebesgue measure inherited from the ambient space $\mathbb{R}^N$; here $\mathscr{X}(Y) \subset \mathbb{R}^N$ are considered as Riemannian sub-manifolds of $\mathbb{R}^N$.

Let $X_* = \min_i X_i$, $X^* = \max_i X_i$. The difference $X^* - X_*$ is constant on each element $X(Y)$, and the length $\mathfrak{l}(Y)$ of the segment $\mathscr{X}(Y)$ satisfies $\mathfrak{l}(Y) = \ell - X_*(X) + X^*(X)$. It is readily seen that any variable $N^{1/2} X_i\big|_{\mathscr{X}(Y)}$ can serve as a normalized length parameter on $\mathscr{X}(Y)$.

Lemma 16 *Assume that the IID random variables X_i are uniformly distributed in the interval $[0,\ell]$. Then for all $t \in (0, \ell/2]$ one has*

$$\mathbb{P}\{\mathfrak{l}(Y) \le t\} \le N\ell^{-2}t^2.$$

Proof Given a segment $\mathscr{X}(Y)$ and a point $X \in \mathscr{X}(Y)$, one can move all X_i in X down, without leaving $\mathscr{X}(Y)$, as long as $X_* > 0$. Similarly, one can move all X_i in X up and remain in $\mathscr{X}(Y)$, as long as $X^* > 0$. For $t \in [0, \ell/2]$, $(\ell - X_i < t)$ implies $(X_i > t)$, thus denoting $A_{ij}(t) = \{X_i < t\} \cap \{\ell - X_j < t\}$, we have $A_{ii}(t) = \varnothing$, for any i. Therefore,

$$\left\{X : \max[X_*(X),\, \ell - X^*(X)] < t\right\} \subset \bigcup_{i \ne j} A_{ij}(t).$$

Since X_i are IID and admit the density bounded by ℓ^{-1}, for all $i \ne j$ we have

$$\mathbb{P}\left\{A_{ij}(t)\right\} = \mathbb{P}\{X_i < t\} \cdot \mathbb{P}\left\{\ell - X_j < t\right\} \le \ell^{-2}t^2,$$

thus

$$\begin{aligned}\mathbb{P}\{\mathfrak{l}(Y) < t\} &= \mathbb{P}\left\{N^{1/2}(\ell - X^* + X_*) < t\right\} \\ &\le \sum_{i \ne j} \mathbb{P}\left\{A_{ij}(tN^{-1/2})\right\} \le N\ell^{-2}t^2.\end{aligned} \tag{90}$$

□

It is convenient now to denote by $\widetilde{\mathscr{X}}(Y)$ the affine line in $\mathbb{R}^N$ containing $\mathscr{X}(Y)$, and identify it with $\mathbb{R}$ when necessary.

Theorem 9 *Let be given IID random variables $X_1, \ldots, X_N$ with $X_i \sim \mathrm{Unif}([0,\ell])$ and a measurable function $\lambda : Y \mapsto \lambda(Y)$. In each interval $\mathscr{X}(Y) \subset \widetilde{\mathscr{X}}(Y) \cong \mathbb{R}$, introduce the sub-interval $I_s(Y) = [\lambda(Y), \lambda(Y) + s] \cap \widetilde{\mathscr{X}}(Y)$. For any $s \in (0,1]$,*

$$\mathbb{P}\{\xi(\omega) \in I_s(Y)\} \le 3N^3\ell^{-1}s. \tag{91}$$

Proof The function ξ cannot serve as a length parameter on $\mathscr{X}(Y)$, since its gradient $(N^{-1}, \ldots, N^{-1})$ has Euclidean norm $N^{-1/2}$, so it is convenient to introduce a rescaled sample mean $\widetilde{\xi} = \sqrt{N}\xi$ and rescaled intervals $\widetilde{I}_s(Y) = [\widetilde{\lambda}(Y), \widetilde{\lambda}(Y) + \sqrt{N}s]$ of length $|\widetilde{I}_s| = \sqrt{N}|I_s|$; here $\widetilde{\lambda}(Y) := \sqrt{N}\lambda(Y)$.

Set $\mathfrak{l}(\omega) := |\mathscr{X}(Y)|$, then we have

$$\begin{aligned}\mathbb{P}\{\xi \in I_s(\eta)\} &= \mathbb{P}\left\{\widetilde{\xi} \in \widetilde{I}_s(\eta)\right\} = \mathbb{E}\left[\mathbb{P}\left\{\widetilde{\xi} \in \widetilde{I}_s(\eta) \,\middle|\, \mathfrak{F}_\eta\right\}\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\mathfrak{l}(\omega) < s}\mathbb{P}\left\{\widetilde{\xi} \in \widetilde{I}_s(\eta) \,\middle|\, \mathfrak{F}_\eta\right\}\right] + \mathbb{E}\left[\mathbf{1}_{\mathfrak{l}(\omega) \ge s}\mathbb{P}\left\{\widetilde{\xi} \in \widetilde{I}_\varepsilon(\eta) \,\middle|\, \mathfrak{F}_\eta\right\}\right] \\ &\le \mathbb{P}\left\{\mathfrak{l}(\omega) < \sqrt{N}s\right\} + \mathbb{E}\left[\mathbf{1}_{\mathfrak{l}(\omega) \ge \sqrt{N}s}\mathbb{P}\left\{\widetilde{\xi} \in \widetilde{I}_\varepsilon(\eta) \,\middle|\, \mathfrak{F}_\eta\right\}\right],\end{aligned} \tag{92}$$

where, by virtue of (90),

$$\mathbb{P}\left\{\mathfrak{l}(\omega) < s\sqrt{N}\right\} \le N\ell^{-2}(s\sqrt{N})^2 = N^2\ell^{-2}s^2, \tag{93}$$

yielding

$$\sup_{s>0} \frac{\mathbb{P}\left\{\mathfrak{l}(\omega) < s\sqrt{N}\right\}}{s^2} \le \frac{N^2}{\ell^2}. \tag{94}$$

The second summand in the RHS of (92) can be assessed as follows:

$$\mathbb{E}\left[\mathbf{1}_{\mathfrak{l}\ge s}\mathbb{P}\left\{\widetilde{\xi} \in \widetilde{I}_s(\eta)\,|\,\mathfrak{F}_\eta\right\}\right] \le \mathbb{E}\left[\mathbf{1}_{\mathfrak{l}\ge\sqrt{N}s}\frac{s\sqrt{N}}{\mathfrak{l}}\right] = s\sqrt{N}\int_{s\sqrt{N}}^{\ell\sqrt{N}} r^{-1}\,dF_{\mathfrak{l}}(r). \tag{95}$$

Using integration by parts for the Stiltjes integral and (94), we obtain

$$\begin{aligned}\int_{s\sqrt{N}}^{\ell\sqrt{N}} r^{-1}\,dF_{\mathfrak{l}}(r) &= \frac{F(r)}{r}\Big|_{s\sqrt{N}}^{\ell\sqrt{N}} + \int_{s\sqrt{N}}^{\ell\sqrt{N}} r^{-2}F_{\mathfrak{l}}(r)\,dr \\ &\le \frac{1}{\ell\sqrt{N}} + \ell\sqrt{N}\sup_{r>0}\frac{F_{\mathfrak{l}}(r)}{r^2} \le \frac{1}{\ell\sqrt{N}} + \frac{\ell\sqrt{N}\cdot N^2}{\ell^2} \le \frac{2N^{5/2}}{\ell}.\end{aligned} \tag{96}$$

Collecting (93), (95), (96) and taking into account that $s/\ell \le 1$, the assertion follows:

$$\mathbb{P}\{\xi \in I_s(\eta)\} \le \frac{N^2}{\ell^2}s^2 + \frac{2N^{5/2}}{\ell}sN^{1/2} \le \frac{3N^3}{\ell}s. \tag{97}$$

□

Theorem 10 *Assume that the common probability distribution of the IID random variables V_j, $j = 1,\dots,N$, with PDF F_V, satisfies the following conditions:*

(i) the probability distribution is absolutely continuous, and one has

$$dF_V(v) = \rho(v)\,dv, \ \operatorname{supp}\rho = [a, a+\ell']; \tag{98}$$

(ii) the probability density $\rho(\cdot)$ has bounded logarithmic derivative on $(a, a+\ell')$:

$$\left\|(\ln\rho)'\,\mathbf{1}_{(a,a+\ell')}\right\|_\infty \le C'_\rho < +\infty. \tag{99}$$

Then there exists a constant $C = C(F_V,\ell') < \infty$ such that for any $s \in (0, N^{-2})$ and any $\mathfrak{F}_\eta$-measurable random variable λ, setting $I_s(\omega) := [\lambda(\omega), \lambda(\omega)+s]$, one has the following bound:

$$\mathbb{P}\{\xi_N(\omega) \in I_s(\omega)\} \le CN^3 s. \tag{100}$$

Proof Without loss of generality, it suffices to prove the claim for $\operatorname{supp}\rho = [0,1]$, which we assume below. Introduce a partition of the sample space $[0,1]^N$ into the cubes $\mathbf{J_k}$, induced by the decomposition $[0,1] = \sqcup_k J_k$: $\mathbf{J_k} = J_{k_1}\times\cdots\times J_{k_N}$, $\mathbf{k} = (k_1,\dots,k_N)$, where $J_k = \left[a_k, a_k + N^{-1}\right]$, $a_k = (k-1)N^{-1}$, $k = 1,\dots,N$. Next, introduce in $\mathbf{J_k}$:

- the uniform probability distribution $\widetilde{\mathbb{P}}_{\mathbf{k}}$, i.e., the normalized measure with constant density $\widetilde{\mathbf{p}}_{\mathbf{k}}$ w.r.t. the Lebesgue measure;

– the probability distribution $\widetilde{\mathbb{P}}_{\mathbf{k}}$ induced by $\mathbb{P}$, conditional on $\{\mathbf{X} \in \mathbf{J_k}\}$, i.e., the normalized measure on $\mathbf{J_k}$ with density

$$\mathbf{p_k}(\mathbf{x}) = Z_{\mathbf{k}}^{-1}\mathbf{p}(\mathbf{x}), \;\; Z_{\mathbf{k}} = \int_{\mathbf{J_k}} \mathbf{p}(\mathbf{y})\,d\mathbf{y}.$$

Let $\mathbf{k} = (k_1, \dots, k_N)$ and $\mathbf{a_k} = (a_{k_1}, \dots, a_{k_N})$. By (99), $\ln \mathbf{p}(\mathbf{x})$ is well-defined in $\mathbf{J_k}$ and satisfies

$$|\ln \mathbf{p}(\mathbf{x}) - \ln \mathbf{p}(\mathbf{a_k})| \leq \sum_{i=1}^{N} |\ln \rho(x_i) - \ln \rho(a_{k_i})| \leq N C'_p N^{-1} = \mathrm{O}(1),$$

thus

$$\forall \mathbf{x} \in \mathbf{J_K} \quad \frac{\mathbf{p}(\mathbf{x})}{\mathbf{p}(\mathbf{a_k})} = \mathrm{O}(1), \quad \text{so } C^{-1} < \frac{\mathbf{p_k}(\mathbf{x})}{\widetilde{\mathbf{p}}(\mathbf{x})} = C,$$

and for any event $\mathscr{A}$, we have

$$\mathbb{P}_{\mathbf{k}}\{\mathscr{A}\} \leq \mathrm{Const}\, \widetilde{\mathbb{P}}_{\mathbf{k}}\{\mathscr{A}\}. \tag{101}$$

The measure $\widetilde{\mathbb{P}}_{\mathbf{k}}$ has constant density on $\mathbf{J_k}$, thus Theorem 9 applies, with $\ell = N^{-1}$, and it follows from (101) and (91) that

$$\begin{aligned} \mathbb{P}\{\xi \in I_s(\eta)\} &= \sum_{\mathbf{k} \in \mathbf{K}} \mathbb{P}\{\mathbf{J_k}\}\, \mathbb{E}\left[\mathbb{P}_{\mathbf{k}}\left\{\xi \in I_s(\eta) \,\middle|\, \mathfrak{F}_\eta\right\}\right] \\ &\leq \sup_{\mathbf{k}} \mathbb{P}_{\mathbf{k}}\{\xi \in I_s(\eta)\} \leq C(F_V) N^3 s. \end{aligned}$$

□

Acknowledgements I thank the Isaac Newton Institute, Cambridge, UK, and the organisers of the program "*Periodic and ergodic spectral problems*" (2015) for the support and opportunity to work for six months in a stimulating atmosphere of the Institute, where the present work was completed.

References

1. Anderson, P. W., Absence of diffusion in certain random lattices. *Phys. Rev.*, **109**, 1492–1505 (1958)
2. Aizenman, M., Molchanov, S., Localization at large disorder and at extreme energies: an elementary derivation. *Commun. Math. Phys.*, **157**, 245–278 (1993)
3. Aizenman, M., Localization at weak disorder: Some elementary bounds. *Rev. Math. Phys.* **6**, 1163–1182 (1994)
4. Aizenman, M., Shenker, J. H., Fridrich, R. M., Hundertmark, D., Finite-volume fractional-moment criteria for Anderson localization. *Commun. Math. Phys.* **224**, 219–253 (2001)

5. Aizenman, M., Elgart, A., Naboko, S., Schenker, J.H., Stoltz, G., Moment analysis for localization in random Schrödinger operators. *Invent. Math.* **163**, 343–413 (2006)
6. Aizenman, M., Warzel, S., Localization bounds for multiparticle systems. Commun. Math. Phys. **290**, 903–934 (2009)
7. Berezanskii, Ju. M., *Expansions in eigenfunctions of self-adjoint operators*. Providence, RI: Am. Mat. Soc., 1968.
8. Basko, D.M., Aleiner, I.L., Altshuler, B.L., Metal–insulator transition in a weakly interacting many-electron system with localized single-particle states. *Ann. Physics* **321**, 1126–1205 (2006)
9. Bourgain, J., Kenig, C.E., On localization in the continuous Anderson-Bernoulli model in higher dimension. *Invent. Math.* **161**, 389–426 (2005)
10. Barbaroux, J.M., Combes, J.M., Hislop, P.D., Localization near band edges for random Schrödinger operators. *Helv. Phys. Acta* **70**, 16–43 (1997)
11. Chulaevsky, V., A remark on charge transfer processes in multi-particle systems. arXiv:math-ph/1005.3387 (2010)
12. Chulaevsky, V., Direct scaling analysis of localization in single-particle quantum systems on graphs with diagonal disorder. *Math. Phys. Anal. Geom.* **15**, 361–399 (2012)
13. Chulaevsky, V., On resonances in disordered multi-particle systems. *C. R. Acad. Sci. Paris*, Ser. I **350**, 81–85 (2011)
14. Chulaevsky, V., From fixed-energy localization analysis to dynamical localization: An elementary path. *J. Stat. Phys.* **154**, 1391–1429 (2014)
15. Chulaevsky, V.: On partial charge transfer processes in multiparticle systems on graphs. *J. Oper.* **2014**, 373754 (2014)
16. Chulaevsky, V., Optimized estimates of the regularity of the conditional distribuiton of the sample mean. arXiv:math-ph/1408.4365 (2014)
17. Chulaevsky, V., Exponential scaling limit of the single-particle Anderson model via adaptive feedback scaling. arXiv:math-ph/1503.02529 (2015)
18. Chulaevsky, V., Boutet de Monvel, A., Suhov, Y., Dynamical localization for a multi-particle model with an alloy-type external random potential. *Nonlinearity* **24**, no. 5, 1451–1472 (2011)

19. Chulaevsky, V., Suhov, Y., Wegner bounds for a two-particle tight binding model. *Commun. Math. Phys.* **283**, 479–489 (2008)

20. Chulaevsky, V., Suhov, Y., Eigenfunctions in a two-particle Anderson tight binding model. *Commun. Math. Phys.* **289**, 701–723 (2009)

21. Chulaevsky, V., Suhov, Y., Multi-particle Anderson localisation: induction on the number of particles. *Math. Phys. Anal. Geom.* **12**, 117–139 (2009)

22. Chulaevsky, V., Suhov, Y., Multi-scale Analysis for Random Quantum Systems with Interaction. Progress in Mathematical Physics. Birkhäuser, Boston (2013)

23. Chulaevsky, V., Suhov, Y., *Efficient Anderson localization bounds for large multi-particle systems*. J. Spec. Theory (*to appear*).

24. Combes, J.-M., Thomas, L., Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators. *Commun. Math. Phys.* **34**, 251–270 (1973)

25. von Dreifus, H., On the effect of randomness in feromagnetic models and Schrödinger operators. PhD thesis, New York University, 1987

26. von Dreifus, H., Klein, A., A new proof of localization in the Anderson tight-binding model. *Commun. Math. Phys.* **124**, 285–299 (1989)

27. Elgart, A., Tautenhahn, M., Veselić, I., Anderson localization for a class of models with a sign-indefinite single-site potential via fractional moment method. *Ann. Henri Poincaré* **12**, no. 8, 1571–1599 (2010)

28. Fröhlich, J., Spencer, T., Absence of diffusion in the Anderson tight-binding model for large disorder or low energy. *Commun. Math. Phys.* **88**, 151–184, (1983)

29. Fröhlich, J., Martinelli, F., Scoppola, E., Spencer, T., Constructive proof of localization in the Anderson tight-binding model. *Commun. Math. Phys.* **101**, 21–46 (1985)

30. Fischer, W., Leschke, H. and Müller, P., *Spectral localization by Gaussian random potentials in multi-dimensional continuous space. J. Stat. Phys.*, **101** no. 5/6, 935–985 (2000)

31. Fauser, M., Warzel, S., Multiparticle localization for disordered systems on continuous space via the fractional moment method. arXiv:math-ph/1304.6913 (2014)

32. Germinet, F., Klein, A., Bootstrap multi-scale analysis and localization in random media. *Commun. Math. Phys.* **222**, 415–448 (2001)

33. Germinet, F., Klein, A., New characterization of the region of complete localization for random Schrödinger operators. *J. Stat. Phys.*, **122**, 73–94 (2006)

34. Germinet, F., Klein, A., A comprehensive proof of localization for continuous Anderson models with singular random potentials. *J. Eur. Math. Soc.* **15**, 55–143 (2013)

35. Goldsheid, I.Y., Molchanov S.A., Pastur, L.A.., A pure point spectrum of the one-dimensional Schrödinger operator. *Funct. Anal. Appl.* **11**, 1–10 (1977)

36. Gornyi, I.V., Mirlin, A.D., Polyakov, D.G., Interacting electrons in disordered wires, Anderson localization and low-temperature transport. *Phys. Rev. Lett.* **95**, 206603 (2005)

37. Klein, A., Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for non-ergodic random Schrödinger operators. *Commun. Math. Phys.* **323**, 1329–1246 (2013)

38. Klein, A., Koines, A., Seifert, M., Generalized eifenfunctions for waves in inhomogeneous media. *J. Funct. Anal.* **190**, 255–291 (2002)

39. Klein, A., Nguyen, S. T., Bootstrap multiscale analysis for the multi-particle Anderson model. *J. Stat. Phys.* **151**, no. 5, 938–973 (2013)

40. Klein, A., Nguyen, S. T., Bootstrap multiscale analysis for the multi-particle continuous Anderson Hamiltonians. arXiv:math-ph/1311.4220 (2013)

41. Kunz, H., Souillard, B., Sur le spectre des opérateurs aux différences finies aléatoires. *Commun. Math. Phys.* **78**, 201–246 (1980)

42. Kotani, S., Simon, B., Localization in general one-dimensional random systems. II. Continuum Schrödinger operators. *Comm. Math. Phys.* **112**, 103–119 (1987)

43. Martinelli, F., Holden, H., On absence of diffusion near the bottom of the spectrum for a random Schrödinger operator on $L^2(\mathbb{R}^d)$. *Commun. Math. Phys.* **93**, 197–217 (1984)

44. Martinelli, F., Scoppola, E., Remark on the absence of absolutely continuous spectrum for d-dimensional Schrödinger oerators with random potential for large disorder or low energy. *Commun. Math. Phys.* **97**, 465–471 (1985)

45. Shnol, I.: On the behaviour of the Schrödinger equation. (in Russian) *Mat. Sbornik* **42**, 273–286 (1957)
46. Simon, B., Schrödinger semigroups. *Bull. Am. Math. Soc.* **7**, 447-526 (1982)
47. Spencer, T., Localization for random and quasi-periodic potentials. J. Stat. Phys. **51**, 1009–1019 (1988)
48. Stolz, G., Expansions in generalized eigenfunction of Schrödinger operators with singular potentials. In *Oper. Theory: Advances and Applications*, vol.48. Birkhäuser, Basel, 1990, 353–372.
49. Stollmann, P., *Caught by disorder.* Progress in Mathematical Physics, **20**, Birkhäuser Boston Inc., Boston, MA (2001)
50. Wegner, F., Bounds on the density of states of disordered systems. *Z. Phys.* **B44**, 9–15 (1981)