Mixed spectral types for one frequency discrete quasi-periodic Schrödinger operator

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Abstract

We consider a family of one frequency discrete analytic quasi-periodic Schrödinger operators which appear in [Bjer]. We show that this family provides an example of coexistence of absolutely continuous and point spectrum for some parameters as well as coexistence of absolutely continuous and singular continuous spectrum for some other parameters.

Keywords: quasi-periodic Schrödinger operators, mixed spectral types, Lyapunov exponent, almost reducibility.

1 Introduction

In [Bjer], Bjerklöv considers the following discrete quasi-periodic Schrödinger operator on $l^2(\mathbb{Z})$

$$(H_{K,\theta,\omega}u)_n = -u_{n+1} - u_{n-1} + V(\theta + n\omega)u_n, \quad n \in \mathbb{Z} \tag{1.1}$$

where

$$V(\theta) = exp\Big(Kf(\theta+\omega)\Big) + exp\Big(-Kf(\theta)\Big) \tag{1.2}$$

$\theta, \omega \in \mathbb{T}^b$, $f : \mathbb{T}^b \to \mathbb{R}$, is assumed to be a non-constant real-analytic function with zero mean, $\int_{\mathbb{T}^b} f(\theta)\mathrm{d}\theta = 0$ and $K \in \mathbb{R}$ is any constant. Consider the Lyapunov exponent $L(E)$ (see next section). In this explicit example, Bjerklöv shows that for large K we have a situation with mixed dynamics: zero Lyapunov exponent in a region close to $E = 0$ and positive for larger E.

In this paper, we are going to show that for one frequency case, in Bjerklöv's example (1.1), mixed dynamics actually lead to mixed spectra: for some parameters (θ, ω), $H_{K,\theta,\omega}$ has mixed absolutely continuous and point spectrum, and for some other (θ, ω), $H_{K,\theta,\omega}$ has mixed absolutely continuous and singular continuous spectrum.

Without loss of generality, we assume that $\|f\|_{C^1(\mathbb{T})} = 1$ and f has analytic extension to the strip $|Imz| < h$, where $h >> K$ (e.g., f can be taken as any entire function). It

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follows from [Bjer] that $\min\{E \in \sigma(H_{K,\theta,\omega})\} = 0$ for any K, θ, ω. And also it is not hard to show that $\max\{E \in \sigma(H_{K,\theta,\omega})\} \asymp e^{K\|f\|_\infty}$. For any $\epsilon > 0$, denote $I_{\epsilon,K} = [\epsilon, 4e^{K\|f\|_\infty}]$. We have $I_{\epsilon,K} \bigcap \sigma(H_{K,\theta,\omega}) \neq \emptyset$. We say the frequency $\omega \in \mathbb{T}$ satisfies the Diophantine Condition (denote by $\omega \in DC(\kappa, \tau)$) if

$$\|\omega \cdot n\| \geq \frac{\kappa}{|n|^\tau}, \quad \forall n \in \mathbb{Z}\backslash\{0\}$$

for some $\kappa > 0, \tau > 0$. And denote $DC = \bigcup_\kappa DC(\kappa, \tau)$ [1] for some fixed $\tau > 1$. It is well known that DC has full Lebesgue measure in any box.

The main results are as follows.

Theorem 1 *Let V be given as in (1.2). Fix $\omega_0 \in DC(\kappa, \tau)$. For any $\epsilon > 0$, there are $K = K(\omega_0, \epsilon, f) > 0$, $\delta = \delta(\omega_0, \epsilon, K) > 0$, and for any $\omega \in B_\delta(\omega_0) := \{\omega \in \mathbb{T} : |\omega - \omega_0| < \delta\}$, there is $0 < \epsilon_0 = \epsilon_0(\omega, K, h, \|f\|_h) < \epsilon$ such that*

(a) *for a.e. $\omega \in B_\delta(\omega_0)$ and a.e. $\theta \in \mathbb{T}$, $H_{K,\theta,\omega}$ has pure point spectrum in $I_{\epsilon,K}$ with exponentially decaying eigenvectors and has purely absolutely continuous spectrum in $[0, \epsilon_0]$.*

(b) *for a.e. $\omega \in B_\delta(\omega_0)$, there is a dense G_δ set of θ, such that $H_{K,\theta,\omega}$ has purely singular continuous spectrum in $I_{\epsilon,K}$ and has purely absolutely continuous spectrum in $[0, \epsilon_0]$.*

(c) *for ω in a dense subset of $B_\delta(\omega_0)$ and for any θ, $H_{K,\theta,\omega}$ has purely singular continuous spectrum in $I_{\epsilon,K}$ and has purely absolutely continuous spectrum in $[0, \epsilon_0]$.*

Previously, Bourgain [Bo] constructed quasi-periodic operator with two frequencies which has coexistence of absolutely continuous and point spectrum. While mixed spectra are expected to occur for generic one-frequency operators, such examples for the discrete case have been considered difficult to construct explicitly. Recently Bjerklöv and Krikorian [BK] announced an example of this nature. For continuous model, Fedotov and Klopp [FK] showed coexistence of absolutely continuous and singular spectrum for a family of quasi-periodic operators and also gave a criterion for the existence of absolutely continuous and singular spectrum in the semi-classical regime.

Here we give a short proof which shows that the operator (1.1) with potential (1.2) has mixed spectral types. The mixed nature of spectrum follows from coexistence of positive Lyapunov exponent and zero Lyapunov exponent which was obtained in [Bjer] and a combination of several recent results on localization, reducibility and continuity .

[1] Here $\|\cdot\|$ means the distance to the closest integer. If

$$\|\omega \cdot n\| \geq \frac{\kappa}{|n|(\log(|n|+1))^2}, \quad \forall n \in \mathbb{Z}\backslash\{0\}$$

for some $\kappa > 0$, we say ω satisfies the Strong Diophantine Condition(denote by $\omega \in SDC(\kappa)$). Denote $SDC = \bigcup_\kappa SDC(\kappa)$ which also has full Lebesgue measure.

2 Singular spectrum in the positive Lyapunov exponent region

Denote

$$A(\theta,E)=\begin{pmatrix} V(\theta)-E & -1\\ 1 & 0\end{pmatrix},\ \theta\in\mathbb{T},\quad A^n(\theta,E)=\prod_{k=n-1}^{0}A(\theta+k\omega,E),n>0$$

The Lyapunov exponent as usual (see [CFKS]) is defined by

$$L(E)=\lim_{n\to\infty}\int_{\mathbb{T}}\frac{1}{n}\log\parallel A^n(\theta,E)\parallel \mathrm{d}\theta\geq 0$$

In the following, we would like to fix f and consider Lyapunov exponent $L(E,\omega,K)$ as function of energy E, frequency ω, and parameter K. In [Bjer], Bjerklöv proved that:

Theorem 2 ([Bjer]) *Assume that V is as in (1.2), and that $\omega\in DC(\kappa,\tau)$. Then for any $\epsilon>0$ there is a $K_0=K_0(\epsilon,f,\kappa,\tau)>0$ and $c=c(f)>0$ such that for all $K>K_0$, we have*

$$L(E,\omega,K)\geq cK,\quad \textit{for all}\quad E\notin[0,\epsilon].$$

The proof is based on Large Deviation Theorem (LDT)-Avalanche Principle(AvP) scheme developed by Bourgain, Goldstein, Schlag [BG, GS1]. Due to some technical reasons, the largeness of K depends on the Diophantine Conditions of ω in this theorem, which means the positivity is not uniform for all ω. However, we can get the following local non-perturbative positivity. Bourgain and Jitomirskaya showed that Lyapunov exponent is jointly continuous in (ω,E) at any irrational frequency (Theorem 1, [BJ]). The following result is obvious:

Proposition 1 *Fix any $\epsilon>0$ and $\omega_0\in DC(\kappa,\tau)$, let $K_0=K_0(\epsilon,f,\omega_0)>0$ be given as in Theorem 2, then for any $K>K_0$, there is $\delta=\delta(\omega_0,\epsilon,K)>0$, such that for any $\omega\in B_\delta(\omega_0)$, $L(E,\omega,K)>0$ on $I_{\epsilon,K}$, where the lower bound only depends on ω_0,ϵ,K,f and is uniform in E and ω.*

The absence of a.c. spectrum on $I_{\epsilon,K}$ is therefore obvious due to Kotani theory. What we want to claim is the pure point spectrum or purely singular continuous spectrum in this region.

Anderson Localization (part (a)) Let $\Omega=SDC\bigcap B_\delta(\omega_0)$, which is a full measure subset of $B_\delta(\omega_0)$. Notice that the positivity of $L(E,\omega,K)$ is uniform for $E\in I_{\epsilon,K}$ and $\omega\in\Omega$. Then according to the non-perturbative localization result in [BG], for any $\theta\in\mathbb{T}$, a.e. $\omega\in\Omega$, $H_{K,\theta,\omega}$ exhibits A.L. in $I_{\epsilon,K}$. Thus by Fubini's theorem, $H_{K,f,\theta,\omega}$ has A.L. for a.e. $\omega\in\Omega$ and a.e. $\theta\in\mathbb{T}$.

Purely s.c. spectrum (part (b)) Let Ω be the same as in the previous part. Goldstein and Schlag [GS2] show that for a.e. $\omega\in\Omega$, the intersection $\sigma(H_{K,f,\theta,\omega})\bigcap I_{\epsilon,K}$ is a Cantor set. Then according to a theorem of Gordon [G2], nowhere dense structure of the spectrum implies the absence of point spectrum for a dense G_δ set of θ (see Theorem 6 in [G2]). Therefore, for a.e. $\omega\in\Omega$, there is a dense G_δ set of θ such that $H_{K,\theta,\omega}$ has purely singular continuous spectrum in $I_{\epsilon,K}$.

Purely s.c. spectrum (part (c)) Absence of point spectrum in this part is based on rational approximation. More precisely, denote by

$$\beta(\omega) := \limsup_n \frac{\log q_{n+1}}{q_n}$$

where $\frac{p_n}{q_n}$ is the n^{th} rational approximation of ω. Notice that

$$\sup_n \sup_{(\theta,\omega)\in\mathbb{T}^2,\ E\in I_{\epsilon,K}} \frac{1}{|n|}\log\|A^n(\theta,E)\| \leq 10K.$$

Then by standard Gordon type argument (see e.g. [G1, CFKS]), if $\beta(\omega) > 40K$, then for any θ, $H_{K,\theta,\omega}$ does not have any point spectrum. Combine with the positivity of Lyapunov exponent in $I_{\epsilon,K}$, the proof for purely s.c. spectrum of part (c) is completed. Notice that for any $\beta_0 \in [0,\infty]$, the level set $\Omega_{\beta_0} := \{\omega : \beta(\omega) = \beta_0\}$ is a dense set. For later purpose, we would like to pick the dense subset $\Omega_{\beta_0} \bigcap B_\delta(\omega_0)$ with $40K < \beta_0 < h/2$.

3 Absolutely continuous spectrum near the bottom

Next we are going to show that for any $\omega \in \mathbb{T}$ with finite $\beta(\omega)$, if the energy E is sufficiently small (depends on ω), then the Schrödinger cocycle is almost reducible. This will imply purely a.c. spectrum near the bottom of the spectrum for any phase. To complete the proof of the main theorem, we first pick some ω near ω_0 and some θ which give us point spectrum or singular continuous spectrum as in the previous part. Then for these pairs of (ω,θ), we apply the almost reducibility result to get the coexistence of two types of spectrum.

The key step to find purely a.c. spectrum near the bottom is the following reducibility result at $E = 0$, which generalizes Lemma 5.1 in [Bjer] to the case $0 < \beta(\omega) < \infty$.

Proposition 2 *For any frequency ω with $\beta(\omega) < \infty$, if $h > 2\beta$, then there exists analytic transformation $C : \mathbb{T} \to SL(2,\mathbb{R})$ such that*

$$C(\theta+\omega)A(\theta,0)C(\theta)^{-1} = A_0,$$

where

$$A_0 = \begin{pmatrix} 1 & \hat{k} \\ 0 & 1 \end{pmatrix}, \quad \hat{k} \in \mathbb{R}$$

Remark 3.1 *If ω is Diophantine or $\beta = 0$, the proposition has been proved in Lemma 5.1 [Bjer]. If $\beta > 0$, we can still find such a transformation C provided h is large. The only loss is the decrease of the width of the analytic strip. Also the analytic norm of the transformation and the constant could be very large. Actually, C has an analytic extension to the strip $|Imz| < h - 2\beta$, with $\|C\|_{h-2\beta} \sim e^{K\|f\|_h}$. We also have $|\hat{k}| \sim e^{K\|f\|_h}$*

Proof: Recall the main steps in the proof of Lemma 5.1 [Bjer], if there are $g, h : \mathbb{T} \to \mathbb{R}$ satisfying the following equations

$$\begin{aligned} g(\theta+\omega) - g(\theta) &= f(\theta+\omega) && (3.1)\\ k(\theta) = -e^{-Kg(\theta-\omega)-Kg(\theta)} \quad &, \quad \hat{k} = \int_{\mathbb{T}} k(\theta)\mathrm{d}\theta \\ h(\theta+\omega) - h(\theta) &= \hat{k} - k(\theta) && (3.2) \end{aligned}$$

then set

$$C(\theta) = \begin{pmatrix} 1 & h(\theta) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & \exp\left(-Kg(\theta-\omega)\right) \\ -\exp\left(Kg(\theta-\omega)\right) & \exp\left(Kg(\theta)\right) \end{pmatrix} \in SL(2,\mathbb{R})$$

Direct computation shows that

$$C(\theta+\omega)A(\theta,0)C(\theta)^{-1} = \begin{pmatrix} 1 & \hat{k} \\ 0 & 1 \end{pmatrix}$$

which is the desired form.

For real analytic f with zero average, if ω is Diophantine, equations (3.1),(3.2) always have real analytic solutions g, h, which is the case in [Bjer].

If $\beta > 0$, recall for f analytic in the strip $|Imz| < h$, the Fourier coefficients of f satisfy $|\hat{f}_k| \le \|f\|_h e^{-h|k|}$, therefore, from Fourier series expansion, equation (3.1) has an analytic solution g in the strip $|Imz| < h - \beta$ provided $h > \beta$. From the definition of k, $k(\theta)$ also has analytic extension to the strip $|Imz| < h - \beta$ with $\|k\|_{h-\beta} \sim e^{K\|g\|_{h-\beta}} \sim e^{K\|f\|_h}$. Then for the same reason, equation (3.2) also has an analytic solution h in the strip $|Imz| < h - 2\beta$ provided $h - \beta > \beta$. ∎

Then it is easy to see that by applying C to $A(\theta, E)$, we have

$$\begin{aligned} C(\theta+\omega)A(\theta,E)C(\theta)^{-1} &= A_0 + C(\theta+\omega)\begin{pmatrix} -E & 0 \\ 0 & 0 \end{pmatrix} C(\theta)^{-1} \\ &= A_0 + EF(\theta) && (3.3)\\ &:= G(\theta,E) \in SL(2,\mathbb{R}) && (3.4) \end{aligned}$$

where

$$F(\theta) = C(\theta+\omega)\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} C(\theta)^{-1}.$$

From the proof of Proposition 2, we see that F has analytic extension to the strip $|Imz| < h - 2\beta$ and the largeness of $\|A_0\|, \|F\|_{h-2\beta}$ depend on $\omega, K, \|f\|_h$.

For Diophantine frequency, as Bjerklöv mentioned in Remark 2 in [Bjer], one can show purely absolutely continuous spectrum for sufficient small E based on KAM approach as in [E]. Here since we also need to deal with Liouvillean frequency, we want to prove all cases together with the almost reducibility concept.

We say the skew product system (ω, A) is *almost reducible* if there exist $\eta > 0$ and a sequence of analytic maps $B^{(n)} : \mathbb{T} \to PSL(2; \mathbb{R})$, admitting holomorphic extensions to the common strip $|Imz| < \eta$ such that $B^{(n)}(z+\omega)A(z)B^{(n)}(z)^{-1}$ converges to a constant uniformly in $|Imz| < \eta$. We need the following result about almost reducibility:

Proposition 3 (Corollary 1.2, [A1]) *Any one-frequency analytic quasi-periodic $SL(2, \mathbb{R})$ cocycle close to constant is analytically almost reducible.*

Proof of purely a.c. spectrum near the bottom. According to Proposition 3, $(\omega, A_0+EF(\theta))$ is almost reducible for small E. More precisely, consider G in (3.4) with form (3.3). There exists $\epsilon_0 = \epsilon_0(\omega, \|A_0\|, h, \|F\|_{h-2\beta}) < \epsilon$ such that for $0 < E < \epsilon_0$, $(\omega, G(\theta, E))$ is almost reducible. (Such a quantitative version can be found in [HY, YZ].) Therefore, $(\omega, A(\theta, E))$ is also almost reducible for $0 < E < \epsilon_0$. As a corollary of almost reducibility [A1, A2], we have that for any θ, $H_{K,f,\theta,\omega}$ has purely absolutely continuous spectrum in $[0, \epsilon_0]$. ∎

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