L^2-REDUCIBILITY AND LOCALIZATION FOR QUASIPERIODIC OPERATORS

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Abstract. We give a simple argument that if a quasiperiodic multi-frequency Schrödinger cocycle is reducible to a constant rotation for almost all energies with respect to the density of states measure, then the spectrum of the dual operator is purely point for Lebesgue almost all values of the ergodic parameter θ. The result holds in the L^2 setting provided, in addition, that the conjugation preserves the fibered rotation number. Corollaries include localization for (long-range) 1D analytic potentials with dual ac spectrum and Diophantine frequency as well as a new result on multidimensional localization.

1. Introduction

The relation between spectral properties of discrete quasiperiodic Schrödinger operators and their duals, known as Aubry duality, has a long history starting from [2]. For nice enough v, sufficiently regular normalizable solutions of $H\psi = E\psi$ with

$$(H(x)\psi)_n = \psi_{n+1} + \psi_{n-1} + v(x + n\alpha)\psi_n \tag{1.1}$$

yield Bloch wave solutions of $\widetilde{H}\psi = E\psi$ for

$$(\widetilde{H}(\theta)\psi)_m = \sum_{m'\in\mathbb{Z}} \hat{v}_{m'}\psi_{m-m'} + 2\cos 2\pi(\alpha m + \theta)\psi_m. \tag{1.2}$$

where

$$v(x) = \sum_{k\in\mathbb{Z}} \hat{v}_k e^{2\pi i k x}.$$

Conversely, nice enough Bloch waves yield normalizable eigenfunctions. There was an expectation stemming from [2] that this should translate into the duality between pure point and absolutely continuous spectra for H and $\widetilde{H}$. The fact that such correspondence in the direction "absolutely continuous spectrum for H implies pure point spectrum for $\widetilde{H}$" could be false even for the nicest of v was first understood in [27] which was a surprise at the time. Y. Last showed that for the almost Mathieu operator H_λ defined by (1.1) with $v(x) = 2\lambda\cos 2\pi x$ and Liouville α there is absolutely continuous component in the spectrum of H for $\lambda < 1$, while $\widetilde{H}$ has purely singular continuous spectrum. By now it is known that for $\lambda < 1$ the spectrum of H is purely absolutely continuous for all x, α, while for the corresponding $\widetilde{H}$ there is an interesting phase diagram of singular continuous and pure point phases depending on the arithmetic properties of both α and x [9, 24]. It should be noted that the almost Mathieu family $\{H_\lambda\}_{\lambda>0}$ is self-dual with $\widetilde{H}_\lambda = \lambda H_{1/\lambda}$.

A more refined notion than absolutely continuous spectrum can be given in terms of reducibility of the corresponding Schrödinger cocycle, see [17]. Reducibility, for instance, does imply that the generalized eigenfunctions behave as Bloch waves.

The duality has been explored at many levels, from physics motivated gauge invariance background [29], to operator-theoretic [18], to quantitative [8]. The main recent thrust, however, has been in establishing absolute continuity of the spectrum of H [12], reducibility [30], or more subtle property called almost reducibility [8], by proving Anderson localization (or almost localization) for the dual operator $\widetilde{H}$. This was motivated by the development of non-perturbative localization methods, starting with [23], that led to duality-based reducibility conclusions without the use of KAM and thus allowing for much larger ranges of parameters.

However, recent celebrated results such as [7, 5] provide methods to establish non-perturbative reducibility directly and independently of localization for the dual model. Thus, a natural question arises: does reducibility of the Schrödinger cocycles imply localization for $\widetilde{H}(\theta)$? Since nice enough reducibility directly yields nicely decaying eigenfunctions for the dual model, the main question is that of completeness. This has been a stumbling block in corresponding arguments, e.g. [32]. The question is non-trivial even for the almost Mathieu family, where the conjectured regime of localization for a.e. x has been $e^{\beta} < \lambda$ where β is the upper rate of exponential growth of denominators of continued fractions approximants to α [22], yet direct localization arguments have stumbled upon technical difficulties for $\lambda < e^{3/2\beta}$, see [28, 6]. At the same time, in the recent preprint [9], the Almost Reducibility Conjecture, recently resolved by Avila (see [3] where the $\beta > 0$ case that is needed for [9] is presented), is combined with the technique from [5, 20, 32] to establish reducibility for the dual of the entire $e^{\beta} < \lambda$ region, thus leading to the above question of completeness of the dual solutions. The authors of [9] use certain quantitative information on the model, in particular, the existence of a large collection of eigenfunctions with prescribed rate of decay to obtain their completeness, which, ultimately, implies arithmetic phase transitions for the almost Mathieu operator.

In the present paper, we obtain an elementary proof of complete localization for the dual model under the assumption of L^2-reducibility of the Schrödinger cocycle for $H(x)$ for almost all energies with respect to the density of states measure (see Theorem 3.1). We do not make any further assumptions on $H(x)$ or it's spectra, however, we require the L^2 conjugation to preserve the rotation number of the Schrödinger cocycle (which is always possible if the conjugation is continuous).

Our result implies a simple proof of localization as a corollary of dual reducibility for the almost Mathieu operator throughout the entire $e^{\beta} < \lambda$ region (first proved in [9]). Also, if the spectrum of $H(x)$ is purely absolutely continuous, v is analytic, and α is Diophantine, the assumptions of Theorem 3.1 automatically hold due to [5], which gives a new interpretation of duality in the direction from absolutely continuous spectrum to localization, as originally envisioned. The argument applies to the multi-frequency case, thus allowing some conclusions on multidimensional localization (see Theorem 3.4). It has further implications to the quasiperiodic XY spin chain, to be explored in a forthcoming paper [21].

2. Preliminaries: Schrödinger cocycles, rotation number and duality

Let

$$(H(x)\psi)_n = \psi_{n+1} + \psi_{n-1} + v(x + n\alpha)\psi_n,$$

where $n \in \mathbb{Z}$, $x = (x_1, \dots, x_d), \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{T}^d$, and α is incommensurate in the sense that $\{1, \alpha_1, \dots, \alpha_d\}$ are linearly independent over $\mathbb{Q}$. The family $\{H(x)\}_{x\in\mathbb{T}^d}$ is called a d-frequency one-dimensional quasiperiodic operator family. We will identify $\mathbb{T}^d$ with $[0, 1)^d$, and continuous functions on $\mathbb{T}^d$ with continuous $\mathbb{Z}^d$-periodic functions on $\mathbb{R}^d$. The eigenvalue equation

$$\psi_{n+1} + \psi_{n-1} + v(x + n\alpha)\psi_n = E\psi_n \tag{2.1}$$

can be written in the following form involving transfer matrices,

$$\begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} = \left(\prod_{j=n-1}^{0} S_{v,E}(x + j\alpha) \right) \begin{pmatrix} \psi_0 \\ \psi_{-1} \end{pmatrix},$$

where

$$S_{v,E}(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix},$$

and the pair $(\alpha, S_{v,E})$ is a Schrödinger cocycle understood as a map $(\alpha, S_{v,E}) : \mathbb{T}^d \times \mathbb{C}^2 \to \mathbb{T}^d \times \mathbb{C}^2$ given by $(\alpha, S_{v,E}) : (x, w) \mapsto (x + \alpha, S_{v,E}(x) \cdot w)$. Replacing $S_{v,E}$ with $A \in \mathrm{SL}(2, \mathbb{R})$ gives a definition of an $\mathrm{SL}(2, \mathbb{R})$-cocycle.

Suppose that A is an $\mathrm{SL}(2, \mathbb{R})$-cocycle homotopic to the identity (for example, a Schrödinger cocycle). Then there exist continuous functions $w\colon \mathbb{T}^d \times \mathbb{T} \to \mathbb{R}$ and $u\colon \mathbb{T}^d \times \mathbb{T} \to \mathbb{R}^+$ such that

$$A(x) \begin{pmatrix} \cos 2\pi y \\ \sin 2\pi y \end{pmatrix} = u(x, y) \begin{pmatrix} \cos 2\pi(y + w(x, y)) \\ \sin 2\pi(y + w(x, y)) \end{pmatrix}.$$

Let μ be a probability measure invariant under transformation of $\mathbb{T}^d \times \mathbb{T}$ defined by $(x, y) \mapsto (x + \alpha, y + w(x, y))$ such that it projects to Lebesgue measure over the first coordinate. Then the *fibered rotation number* of (α, A) is defined as

$$\rho(\alpha, A) = \left(\int_{\mathbb{T}^d \times \mathbb{T}} w \, d\mu \right) \mod \mathbb{Z}.$$

If $A = S_{v,E}$, then we will denote its rotation number by $\rho(E)$, ignoring the dependence on α and v.

If $H(\cdot)$ is a quasiperiodic operator family, and $E_{H(x)}(\Delta)$ are the corresponding spectral projections, then the density of states measure is defined as

$$N(\Delta) = \int_{\mathbb{T}^d} (E_{H(x)}(\Delta)\delta_0, \delta_0) \, dx,$$

$\Delta \subset \mathbb{R}$ is a Borel set. The distribution function of this measure $N(E) = N((-\infty, E)) = N((-\infty, E])$ is called the *integrated density of states* (IDS). A remarkable relation (see, for example, [25, 10] for one-frequency case and [16] for the general case) is that for Schrödinger cocycles we have $N(E) = 1 - 2\rho(E)$, and ρ is a continuous function mapping Σ onto $[0, 1/2]$, where $\Sigma = \sigma(H(x))$ is the spectrum which is known to be independent of x. The following holds [25, 16, 19].

Proposition 2.1. *Suppose that $E \notin \Sigma$. Then $2\rho(E) \in \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z} + \ldots + \alpha_d\mathbb{Z} + \mathbb{Z}$.*

Aubry duality is a relation between spectral properties of $H(x)$ and the dual Hamiltonian (1.2). We will formulate it in the multi-frequency form, with the dual operator (1.2) now being an operator in $l^2(\mathbb{Z}^d)$:

$$(\widetilde{H}(\theta)\psi)_m = \sum_{m' \in \mathbb{Z}^d} \hat{v}_{m'}\psi_{m-m'} + 2\cos 2\pi(\alpha \cdot m + \theta),$$

where

$$v(x) = \sum_{k\in\mathbb{Z}} \hat{v}_k e^{2\pi i k\cdot x}.$$

Denote the corresponding direct integral spaces (for H and $\widetilde{H}$ respectively) by

$$\mathfrak{H} := \int_{\mathbb{T}^d}^{\oplus} l^2(\mathbb{Z})\, dx, \quad \widetilde{\mathfrak{H}} = \int_{\mathbb{T}}^{\oplus} l^2(\mathbb{Z}^d)\, d\theta.$$

Consider the unitary operator $\mathcal{U}\colon \mathfrak{H} \to \widetilde{\mathfrak{H}}$ defined on vector functions $\Psi = \Psi(x,n)$ as

$$(\mathcal{U}\Psi)(\theta, m) = \hat{\Psi}(m, \theta + \alpha\cdot m),$$

where $\hat{\Psi}$ denotes the Fourier transform over $x \in \mathbb{T}^d \to m \in \mathbb{Z}^d$ combined with the inverse Fourier transform $n \in \mathbb{Z} \to \theta \in \mathbb{T}$. Let also

$$\mathcal{H} := \int_{\mathbb{T}^d}^{\oplus} H(x)\, dx, \quad \widetilde{\mathcal{H}} := \int_{\mathbb{T}}^{\oplus} \widetilde{H}(\theta)\, d\theta.$$

Aubry duality can be formulated as the following equality of direct integrals.

$$\mathcal{U}\mathcal{H}\mathcal{U}^{-1} = \widetilde{\mathcal{H}}. \tag{2.2}$$

It is well known (see, for example, [18]) that the spectra of $H(x)$ and $\widetilde{H}(\theta)$ coincide for all x, θ. We denote them both by Σ. Moreover, the IDS of the families H and $\widetilde{H}$ also coincide.[1]

A continuous cocycle (α, A) is called *L^2-reducible* if there exists a matrix function $B \in L^2(\mathbb{T}^d; \mathrm{SL}(2,\mathbb{R}))$ and a (constant) matrix $A_\star \in \mathrm{SL}(2,\mathbb{R})$ such that

$$B(x+\alpha)A(x)B(x)^{-1} = A_\star, \quad \text{for a. e.} \quad x \in \mathbb{T}^d. \tag{2.3}$$

We will call a cocycle A *L^2-degree* 0 *reducible*, if (2.3) holds with (possibly complex) B with $|\det B(x)| = 1$ and

$$A_\star = \begin{pmatrix} e^{2\pi i\rho(E)} & 0 \\ 0 & e^{-2\pi i\rho(E)} \end{pmatrix}. \tag{2.4}$$

3. Obtaining localization by duality

Our main result is as follows.

Theorem 3.1. *Suppose that $H(x)$ is a continuous quasiperiodic operator family such that for almost all $E \in \Sigma$ with respect to the density of states measure the cocycle $S_{v,E}$ is L^2-degree* 0 *reducible. Then the spectra of dual Hamiltonians $\widetilde{H}(\theta)$ are purely point for almost all $\theta \in \mathbb{T}$.*

The following fact is standard and shows that the notion of degree 0 reducibility is needed only in discontinuous setting.

Proposition 3.2. *Suppose* (α, A) *is a continuously reducible cocycle such that* (2.3) *holds with $A_\star$ being a rotation matrix. Then it is L^2 (and continuously) degree 0 reducible.*

[1]While the proof in [18] is claimed for a more specific one-frequency family, it applies without any changes in the mentioned generality.

Proof. Assume that (2.3) holds with B continuous and homotopic to a constant matrix. Since such conjugations B preserve rotation numbers, the matrix $A_\star$ will be a rotation by the rotation number of (α, A), and (2.4) can be obtained by diagonalizing $A_\star$ over $\mathbb{C}$ (the diagonalization is another conjugation with a constant complex matrix B).

In general, any continuous map from $B\colon \mathbb{T}^d \to \mathrm{SL}(2,\mathbb{R})$ is homotopic to a rotation $R_{n\cdot x}$ for some $n \in \mathbb{Z}^d$. If $B\colon \mathbb{T}^d \to \mathrm{SL}(2,\mathbb{R})$ satisfies (2.3) and is homotopic to R_{nx}, then $R_{-n\cdot x}B(x)$ is homotopic to a constant and also satisfies (2.3). ∎

Corollary 3.3. *Suppose that v is real analytic and $d = 1$, α is Diophantine, and that the spectrum of the original operator* (1.1) *is purely absolutely continuous for almost all x. Then the spectrum of the dual operator* (1.2) *is purely point for almost all θ, and the eigenfunctions decay exponentially.*

Proof. From [5, Theorem 1.2], it follows that for Lebesgue almost all $E \in \Sigma$ the cocycle $(\alpha, S_{v,E})$ is rotations reducible. If α is Diophantine, it will also be (analytically) reducible to a constant rotation (see, for example, [30]). Hence, we fall under the assumptions of Theorem 3.1. ∎

We can also formulate the following (and now elementary) result on multi-dimensional localization.

Theorem 3.4. *If α is Diophantine, then there exists $\lambda_0(\alpha, v)$ such that the operator*

$$(\widetilde{H}(\theta)\psi)_m = \sum_{m' \in \mathbb{Z}^d} \hat{v}_{m'}\psi_{m-m'} + 2\lambda \cos 2\pi(\alpha \cdot m + \theta)$$

has purely point spectrum for $\lambda > \lambda_0(\alpha, v)$ and almost all $\theta \in \mathbb{T}$.

Proof. By Eliasson's perturbative reducibility theorem (see [17, 1]), the cocycle $(\alpha, S_{\lambda^{-1}v,E})$ is continuously (even analytically) reducible for $\lambda > \lambda_0(\alpha, v)$ for almost all values of the rotation number. Hence, it satisfies the hypothesis of Theorem 3.1. ∎

Remark 3.5. We are not aware of other multidimensional localization results with purely arithmetic conditions on the frequency. In [13], Theorem 3.4 was proved for general C^2 potentials with two critical points (rather than for $\cos x$). Long range operators were also considered. However, the result was obtained for an unspecified full measure set of frequencies. In the book [11], a possibility of alternative proof was mentioned (for general analytic potentials), with a part of the proof only requiring the Diophantine condition on α.

Remark 3.6. For every v and for almost every E among those for which the Lyapunov exponent of the Schrödinger cocycle $(\alpha, S_{v,E})$ is zero, this cocycle can be L^2-conjugated to a cocycle of rotations $R_{\phi(x)}$, see [26, 14], or [7, Theorem 2.1] with a reference to [31]. Further conjugation to a constant rotation, such as in the assumption of Theorem 3.1, requires solving a cohomological equation $\varphi(x+\alpha) - \varphi(x) = \phi(x) \bmod \mathbb{Z}$.

In the remaining two sections we prove Theorem 3.1.

4. Covariant operator families with many eigenvectors

In this section, we make the following simple observation: if, for a covariant operator family $\{H(\omega)\}$, we have found a single eigenvector of $H(\omega)$ for each ω, and the eigenvectors for different ω on the same trajectory are essentially different (that is, not obtained from each other by translation), then, for almost every ω, the eigenvectors of $H(\omega)$ obtained by such translations form a complete set.

Let $(\Omega, d\omega)$ be a measurable space with a Borel probability measure, equipped with a family $\{T^n, n \in \mathbb{Z}^d\}$ of measure-preserving one-to-one transformations of Ω such that $T^{m+n} = T^m T^n$. Let $\{H(\omega), \omega \in \Omega\}$ be a weakly measurable operator family of bounded operators in $l^2(\mathbb{Z}^d)$, with the property $H(T^n\omega) = T_n H(\omega) T_{-n}$, where T_n is the translation in $\mathbb{Z}^d$ by the vector n. We call $\{H(\omega)\}$ a *covariant operator family with many eigenvectors* if

(1) there exists a Borel measurable function $E\colon \Omega \to \mathbb{R}$ such that for almost every $\omega \in \Omega$ $E(T^n\omega) = E(T^m\omega)$ if and only if $m = n$. [2]
(2) There exists a function $u\colon \Omega \to l^2(\mathbb{Z}^d)$, *not necessarily measurable*, such that for almost every $\omega \in \Omega$ we have $H(\omega)u(\omega) = E(\omega)u(\omega)$ and $\|u(\omega)\|_{l^2(\mathbb{Z}^d)} = 1$.

Theorem 4.1. *Suppose that $\{H(\omega)\}$ is a covariant operator family in $\mathbb{Z}^d$ with many eigenvectors. Then, for almost every ω, the operator $H(\omega)$ has purely point spectrum with eigenvalues $E(T^{-n}\omega)$ and eigenvectors $T_n u(T^{-n}\omega)$.*

Proof. Let $E_k(\omega) := E(T^{-k}\omega)$, and $P_k(\omega)$ be the spectral projection of $H(\omega)$ onto the eigenspace corresponding to $E_k(\omega)$. Both these functions are weakly measurable in ω. For almost all ω, we have for all $k \in \mathbb{Z}^d$

$$H(T^k\omega)T_k u(\omega) = T_k H(\omega)u(\omega) = E(\omega)T_k u(\omega) = E_k(T^k\omega)T_k u(\omega).$$

Hence, $T_k u(\omega)$ belongs to the range of $P_k(T^k\omega)$, and, for each basis vector $\delta_l \in l^2(\mathbb{Z}^d)$, we have

$$(P_k(T^k\omega)\delta_l, \delta_l) \geqslant |(T_k u(\omega), \delta_l)|^2. \tag{4.1}$$

Let $P(\omega) := \sum_{k\in\mathbb{Z}^d} P_k(\omega)$. By assumption, all $E_k(\omega)$ are different (for almost all ω), and hence $P(\omega)$ is a projection. We have

$$\int_\Omega (P(\omega)\delta_l, \delta_l)\, d\omega = \int_\Omega \sum_{k\in\mathbb{Z}^d} (P_k(\omega)\delta_l, \delta_l)\, d\omega = \tag{4.2}$$
$$= \int_\Omega \sum_{k\in\mathbb{Z}^d} (P_k(T^k\omega)\delta_l, \delta_l)\, d\omega \geqslant \int_\Omega \sum_{k\in\mathbb{Z}^d} |(T_k u(\omega), \delta_l)|^2\, d\omega = 1.$$

Since the left hand side is bounded by 1, the last inequality must be an equality, and hence, for any l, the inequality (4.1) must also be an equality for almost every ω. By passing to the countable intersection of these sets of ω, we ultimately obtain a full measure set for which $P(\omega) = I$, and hence the eigenfunctions from the statement of the theorem indeed form a complete set. ∎

Remark 4.2. The left hand side of (4.2) is the "partial density of states measure" of Ω in the sense that it is the density of states obtained only from a part $P(\omega)$ of the spectral measure. The fact that this measure is still equal to 1 implies that, for almost every ω, that part was actually the full spectral measure, which is equivalent to completeness of the eigenfunctions. The result also implies simplicity of the point spectrum and that the function $u(\cdot)$ can actually be replaced by a measurable function with the same properties (because spectral projections are measurable

[2] In other words, E should be one-to-one on almost all trajectories. However, the wording "almost all trajectories" is ambiguous as there is no natural measure on the set of trajectories.

functions, and, due to simplicity, one can choose measurable branches of eigenfunctions; see [18] for details).

5. Proof of Theorem 3.1

Let Θ_r^+ be the set of all $\theta \in [0, 1/2]$ such that $\rho(E) = \theta$ for some $E \in \Sigma$ with $S_{v,E}$ being L^2-degree 0 reducible, and $2\theta \notin \alpha_1\mathbb{Z} + \ldots + \alpha_d\mathbb{Z} + \mathbb{Z}$. By the assumptions, this set has full Lebesgue measure in $[0, 1/2]$. Moreover, for $\theta \in \Theta_r^+$ there exists exactly one E such that $\rho(E) = \theta$, because ρ is nonincreasing and can be constant only on intervals contained in $\mathbb{R} \setminus \Sigma$, on which the rotation number must be rationally dependent with α, see Proposition 2.1. Let us denote this inverse function by $E(\theta)$, initially defined on Θ_r^+, extend it evenly onto $-\Theta_r^+$, and then 1-periodically onto $\mathbb{R}$. Let us now consider

$$f(x,\theta) = \begin{cases} B_{11}^{-1}(x, E(\theta))/\|B_{11}^{-1}(\cdot, E(\theta))\|_{L^2(\mathbb{T})}, & \theta \in [0, 1/2] \\ B_{12}^{-1}(x, E(-\theta))/\|B_{12}^{-1}(\cdot, E(-\theta))\|_{L^2(\mathbb{T})}, & \theta \in [-1/2, 0), \end{cases}$$

and then also extend $f(x, \theta)$ 1-periodically over θ. Denote the domain in θ of both functions by Θ_r which is a similar (even, and then periodic) extension of Θ_r^+. For $\theta \in \Theta_r$, the function f generates Bloch solutions

$$\psi_n(x, \theta) = e^{2\pi i n\theta} f(x + n\alpha, \theta),$$

which, for almost all x, are formal solutions of the eigenvalue equation $(H(x)\psi)_n = E(\theta)\psi_n$. It is also easy to see that, if $u(\theta)_m = \hat{f}(m, \theta)$ is the Fourier transform of f in the variable x, then

$$(\widetilde{H}(\theta)u(\theta))_m = E(\theta)u(\theta)_m.$$

Finally, $E(\theta + k\alpha) = E(\theta + l\alpha)$ with $k \neq l$ can only happen if $2\theta \in \mathbb{Z} + \mathbb{Z}\alpha_1 + \ldots + \mathbb{Z}\alpha_d$, and so, the function $E(\cdot)$ satisfies the assumptions of Theorem 4.1, from which the result follows. ■

Remark 5.1. For almost all θ, we, in addition, can choose $f(x, -\theta) = \overline{f(x, \theta)}$. This is a standard argument (see, e.g., [30], [6], or [4] in the discontinuous setting). Let

$$F(x,\theta) = \begin{pmatrix} f(x,\theta) & \overline{f(x,\theta)} \\ e^{-2\pi i\theta} f(x-\alpha,\theta) & e^{2\pi i\theta}\overline{f(x-\alpha,\theta)} \end{pmatrix}.$$

From the equation, it is easy to see that $S_{v,E}F(x, \theta) = F(x + \alpha, \theta)A_\star$, and hence $d(x, \theta) = \det F(x, \theta)$ is a function of x invariant under $x \mapsto x + \alpha$. Therefore, it is almost surely a constant function of x. If $d(x, \theta) \neq 0$, then we can simply choose $B(x) = F(x, \theta)^{-1}$. Suppose that $d(x, \theta) = 0$. Then we have

$$f(x,\theta) = \varphi(x,\theta)\overline{f(x,\theta)}, \quad e^{2\pi i\theta} f(x+\alpha,\theta) = \varphi(x,\theta)e^{-2\pi i\theta}\overline{f(x+\alpha,\theta)}.$$

for some function φ with $|\varphi(x, \theta)| = 1$ and $\varphi(x + \alpha, \theta) = e^{-4\pi i\theta}\varphi(x, \theta)$. That is only possible if 2θ is a rational multiple of α modulo $\mathbb{Z}$, which excludes at most countable set of θs.

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