A UNIFORM AREA LAW FOR THE ENTANGLEMENT OF EIGENSTATES IN THE DISORDERED XY CHAIN

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ABSTRACT. We consider the isotropic or anisotropic XY spin chain in the presence of a transversal random magnetic field, with parameters given by random variables. It is shown that eigenfunction correlator localization of the corresponding effective one-particle Hamiltonian implies a uniform area law bound in expectation for the bipartite entanglement entropy of all eigenstates of the XY chain, i.e. a form of many-body localization at all energies. Here entanglement with respect to arbitrary connected subchains of the chain can be considered. Applications where the required eigenfunction correlator bounds are known include the isotropic XY chain in random field as well as the anisotropic chain in strong or strongly disordered random field.

1. INTRODUCTION

1.1. **Motivation: Towards Many-Body Localization.** We are interested here in a better understanding of mathematical characterizations of many-body localization (MBL) in interacting quantum systems. This phenomenon has recently received strong attention in theoretical physics and quantum information theory, see e.g. [7, 8, 18, 29, 30, 32, 33, 39, 40] and references therein. It is generally described as the absence of *thermalization* or *self-equilibration* in a quantum many-body system, often due to the presence of disorder. For a detailed discussion of the current understanding of thermalization and many-body localization in the physics literature see also the recent survey [14], including the extensive list of references provided there. It is made clear in these works that there is no complete consensus yet on what physically constitutes thermalization and MBL, thus leaving a multitude of questions for further investigation and clarification.

We will focus here on a number of criteria which by now are well accepted to be sufficient for MBL and on relatively simple models where these can be proven rigorously. Generally, and quite roughly, these criteria come in the form of properties which say that an interacting system, in suitable regimes, behaves similarly to a non-interacting system. They can be given in the form of properties of stationary eigenstates, such as rapid decay of correlations and small entanglement between subsystems, or refer to the many-body dynamics of the system, such as the absence of information transport within the system.

Let us stress here that the term MBL should generally be reserved for properties which hold uniformly in the number of particles in the system (e.g. in the sense that relevant constants are bounded in the particle number). In this sense, many-body localization is to be distinguished from few-body localization, such as the known rigorous results for the N-particle Anderson model [10, 11, 2, 23], which do not yet allow uniform control in the number of electrons. In particular, results expected by physicists for the many-body Anderson model,

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such as MBL at low electron density or weak interaction strength, e.g. [3, 15], can not yet be shown rigorously.

Rigorous mathematical results on localization properties of disordered many-body systems are so far essentially restricted to two models: Disordered harmonic oscillator systems [27, 28] and the XY spin chain in random field [24, 16, 36, 31]. These models are equivalent to free Boson systems and free Fermion systems, respectively, and thus can be studied in terms of an effective one-particle Hamiltonian. As a consequence, it is possible to deduce results on MBL from known localization properties of one-particle Hamiltonians such as the Anderson model. Of course, a long term goal must be to develop methods which allow to go beyond these simple models. Some progress towards understanding MBL for larger classes of spin systems has been made in [19], which, however, still needs to make an unproven assumption to limit the amount of level attraction in the system. Mathematically, describing the phenomenon of many-body localization and fully proving it for important classes of physical examples is a wide open field (even wider than for physics).

Our more modest goal is here is to contribute to a more complete understanding of MBL properties of the two models indicated above, harmonic oscillators and the XY chain.

For oscillator systems the reduction to an effective one-particle Hamiltonian is rather straight forward. In particular, the reduction works in any dimension and does not affect locality properties of the system. This has allowed to verify a quite complete list of MBL properties for disordered oscillator systems, including a zero-velocity Lieb-Robinson bound on information transport as well as exponential decay of correlations and area-law-type entanglement bounds for ground and thermal states [27, 28] (in each case requiring disorder averaging).

The reduction of the XY model via the Jordan-Wigner transform to a free Fermion system is only possible for a one-dimensional chain of spins and, as an additional difficulty, introduces non-locality. Therefore rigorously known MBL properties for the XY model are more limited. A first contribution was made in [24] which established exponential decay of certain ground state correlations for the isotropic XY chain in random exterior field, using localization properties of the Green function of the effective Hamiltonian, in this case the Anderson model, proven via multiscale analysis. More general results have recently been obtained by Sims and Warzel [36]. Using localization of eigenfunction correlators of the underlying one-particle Hamiltonian, these authors find exponential decay of stationary as well as time-dependent correlations for larger classes of states, including excited states, for systems such as the XY chain which can be mapped to free Fermions.

Localization of eigenfunction correlators is also the key property of the effective Hamiltonian used in [16] to show a zero-velocity Lieb-Robinson bound for the XY chain in random field, again after disorder averaging. An argument in [16], valid for a very general class of quantum spin systems shows that this implies exponential decay of ground state correlations up to a logarithmic correction in the size of the ground state gap (the more specific arguments used in [36] show that no such correction is required for the random XY chain). This in turn, by another general observation, leads to an area law for the ground state entanglement [4, 5]. Note, however, that [4, 5] consider deterministic systems and that some further analysis would be required to show that the relation "exponential decay of correlations implies area law" carries over to the disorder averaged quantities.

Here we prove an area law for the disorder averaged entanglement of *all* eigenstates of the XY chain in random field. In fact, we state this as a conditional result: If the corresponding

effective one-particle Hamiltonian has localized eigenfunction correlators, then the many-body eigenstates satisfy an area law, uniform in energy (i.e. in the labeling of eigenstates), see Section 1.2 for exact statements. Applications of this result, see Section 1.3, include the isotropic XY chain but also some anisotropic cases. To our knowledge this is the first result on many-body localization for eigenstates of a disordered spin chain which goes beyond the ground state.

Our result is closely related to the recent proof by Pastur and Slavin [31] of an area law for the ground state of disordered free Fermion systems. In fact, by mapping the XY chain to a free Fermion system via the Jordan-Wigner transform, we use the argument from [31] as one of the key ingredients to our proof. We extend this argument to general eigenstates and, to include the anisotropic case, also consider more general Fermion systems which do not conserve the particle number, leading to random block operators as effective Hamiltonians. It may also be worth noting that we consider entanglement with respect to arbitrary subchains Λ_1 of the chain $\Lambda = \{1, \ldots, n\}$ in Theorem 1.1 below (and not just left or right ends of Λ). This requires to work with a suitably chosen local version of the Jordan-Wigner transform, introduced in Section 3.3 below.

1.2. **The Main Result.** We consider the anisotropic XY spin chain in transversal magnetic field, given by the self-adjoint Hamiltonian

$$(1) \qquad H := -\sum_{j=1}^{n-1} \mu_j[(1 + \gamma_j)\sigma_j^X \sigma_{j+1}^X + (1 - \gamma_j)\sigma_j^Y \sigma_{j+1}^Y] - \sum_{j=1}^{n} \nu_j \sigma_j^Z$$

in $\mathcal{H} = \bigotimes_{j \in \Lambda} \mathbb{C}^2$, where $\Lambda = \{1, 2, \ldots, n\}$ for an arbitrary positive integer n. By σ_j^X, σ_j^Y and σ_j^Z we denote the standard Pauli matrices acting on the j-th component of the tensor product.

The parameters μ_j, γ_j and ν_j describe the interaction strength, anisotropy and field strength, respectively, and we will think of them as the first n components of sequences of real-valued random variables indexed by $j \in \mathbb{N}$. While stronger assumptions will be needed in applications, for our main result we only will assume their uniform boundedness, i.e. that there exists $C < \infty$ such that

$$(2) \qquad \sup_{j \in \mathbb{N}} (|\mu_j| + |\gamma_j| + |\nu_j|) \leq C \quad \text{almost surely.}$$

We will also assume that the disorder guarantees that

$$(3) \qquad H \text{ almost surely has simple spectrum,}$$

in the sense that all its eigenvalues are non-degenerate. In the applications discussed in Section 1.3 below this will follow from an argument provided in Appendix A.

As known since the work of Lieb, Schultz and Mattis [26], the XY chain Hamiltonian H can be mapped to a free Fermion system on $\mathcal{H}$ by the Jordan-Wigner transform. As a result,

the explicit diagonalization of H reduces to the diagonalization of an effective one-particle Hamiltonian which can be written in form of the symmetric 2×2-block Jacobi matrix

$$
(4) \qquad M := \begin{pmatrix} -\nu_1 \sigma^Z & \mu_1 S(\gamma_1) & & \\ \mu_1 S(\gamma_1)^t & -\nu_2 \sigma^Z & \ddots & \\ & \ddots & \ddots & \mu_{n-1} S(\gamma_{n-1}) \\ & & \mu_{n-1} S(\gamma_{n-1})^t & -\nu_n \sigma^Z \end{pmatrix},
$$

where $\sigma^Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix and

$$
(5) \qquad S(\gamma) = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}.
$$

For some more details on the reduction of H to M see Section 2 below.

Our main result says that a suitable form of localization of the effective one-particle Hamiltonian M implies a uniform area law for the bipartite entanglement entropy of all eigenstates of the n-spin Hamiltonian H. To introduce the latter, consider an arbitrary subinterval $\Lambda_1 = \{r, \dots, r + \ell - 1\}$ of Λ and the bipartite decomposition $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where

$$
(6) \qquad \mathcal{H}_1 = \bigotimes_{j \in \Lambda_1} \mathbb{C}^2 \quad \text{and} \quad \mathcal{H}_2 = \bigotimes_{j \in \Lambda \setminus \Lambda_1} \mathbb{C}^2.
$$

For a pure state ρ in $\mathcal{H}$ the entanglement entropy $\mathcal{E}(\rho)$ with respect to the subsystem Λ_1 is given by the von Neumann entropy of the reduced state $\rho_1 = \mathrm{Tr}_{\mathcal{H}_2} \rho$,

$$
(7) \qquad \mathcal{E}(\rho) = \mathcal{S}(\rho_1) := - \mathrm{Tr}\, g(\rho_1),
$$

where

$$
(8) \qquad g(x) = \begin{cases} x \log x, & 0 < x \leq 1, \\ 0, & x = 0. \end{cases}
$$

We choose log to denote the natural logarithm, as opposed to the dyadic logarithm used in this context in the information theory literature. The distinction is irrelevant for our work, as we will not keep track of universal constants.

With $\mathbb{E}(\cdot)$ denoting the disorder average, we will prove

Theorem 1.1. *Assume (2) and (3), as well as the existence of $\beta > 2$ and $C < \infty$ such that*

$$
(9) \qquad \mathbb{E} \left(\sup_{|g| \leq 1} \| g(M)_{jk} \| \right) \leq \frac{C}{1 + |j - k|^\beta}
$$

for all $n \in \mathbb{N}$ and $1 \leq j, k \leq n$.

Then there exists $\tilde{C} < \infty$ such that

$$
(10) \qquad \mathbb{E} \left(\sup_\psi \mathcal{E}(\rho_\psi) \right) \leq \tilde{C}
$$

for all n, r and ℓ with $1 \leq r \leq r + \ell - 1 \leq n$. In (10) the supremum is taken over all normalized eigenfunctions ψ of H and $\rho_\psi = |\psi\rangle\langle\psi|$ is the orthogonal rank-one projection onto $\mathbb{C}\psi$.

In (9), $g : \mathbb{R} \to \mathbb{C}$ may be any function such that $|g(x)| \le 1$ uniformly, and $g(M)$ is defined via the functional calculus for symmetric matrices. Using the block matrix notation from (4) above, the matrix elements $g(M)_{jk}$ are understood as 2×2-matrices. While any norm $\|\cdot\|$ on the 2×2-matrices could be used in (9), we will work with the Euclidean matrix norm for the sake of definiteness.

Bounds of the form (9) are generally referred to as *localization of eigenfunction correlators*. Their use in single particle localization theory originates from [1], see also [37] for an introductory survey.

In particular, considering the functions $g_t(x) = e^{-itx}$, $t \in \mathbb{R}$, the bound (9) includes dynamical localization, uniform in time, for solutions of the Schrödinger equation $\psi'(t) = -iM\psi(t)$. We note that requiring only power law decay for some $\beta > 2$ in (9) is a relatively weak form of eigencorrelator localization. In applications where (9) can be verified, one usually has much faster decay in $|j - k|$, see Section 1.3 below.

The "area law" (10), giving an upper bound for the entanglement proportional to the surface area of the subsystem Λ_1 (as opposed to the elementary bound in terms of its volume), is uniform not only in the size of the system Λ and subsystem Λ_1, but also applies uniformly to *all* eigenstates of H. It thus yields a form of many-body localization for H at *arbitrary energy*.

Note that by a general result due to Hastings [17], an area law holds for the ground state of one-dimensional spin systems with a uniform (in volume) ground state energy gap. Not only is our work not limited to the ground state, but we also do not need any explicit assumption of the size of spectral gaps. Instead, we exploit single particle localization of the effective Hamiltonian M as the mechanism to prove an area law. Of course, bounds on level statistics can be considered as being an implicit part of the underlying single particle theory. A price we have to pay for not having deterministic gaps between energy levels is the need to include disorder averages in (9) and (10).

1.3. **Applications.** Here we discuss some concrete examples, i.e. assumptions on the random variables μ_j, γ_j and ν_j which guarantee (2), (3) and, most importantly, one-particle localization of M in the form (9).

We will focus on the case of random magnetic field, where the corresponding one-particle localization properties are best understood, i.e. we choose $\mu_j = \mu \in \mathbb{R} \setminus \{0\}$ and $\gamma_j = \gamma \in \mathbb{R}$ to be constant, while the ν_j are i.i.d. random variables with compactly supported distribution ρ. Thus (2) holds. Assuming, moreover, that ρ is absolutely continuous $d\rho(\nu) = h(\nu)d\nu$, it follows from Proposition A.1 in Appendix A that (3) is satisfied as well.

The localization property (9) is known for the following cases.

(i) Let $\gamma = 0$, so that

$$
(11) \qquad H = -\mu \sum_j (\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y) - \sum_j \nu_j \sigma_j^Z
$$

is the isotropic XY chain in random field. Then all 2×2-matrix-entries of M are diagonal and thus M decomposes into $A \oplus (-A)$, where

$$
(12) \qquad A = \begin{pmatrix} -\nu_1 & \mu & & \\ \mu & \ddots & \ddots & \\ & \ddots & \ddots & \mu \\ & & \mu & -\nu_n \end{pmatrix}
$$

is the Anderson model on the finite interval Λ. If the density h is bounded and compactly supported, then it is known that, for some $C < \infty$ and $\eta > 0$,

$$
(13) \qquad \mathbb{E}\left(\sup_{|g|\leq 1} |g(A)_{jk}|\right) \leq Ce^{-\eta|j-k|},
$$

see e.g. [37]. This readily implies (9) with exponential decay in $|j - k|$.

(ii) For the anisotropic case $\gamma \neq 0$, which does not reduce to the Anderson model, localization properties of one-particle Hamiltonians given by block Jacobi matrices of the form (4) have been studied more recently.

A result by Elgart, Shamis and Sodin [13] covers the large disorder case: If the i.i.d. random variables $\nu_j^{(0)}$, $j \in \mathbb{N}$, have bounded compactly supported density, and if $\nu_j = \lambda\nu_j^{(0)}$ for $\lambda > 0$ sufficiently large, then (9) holds with exponential decay in $|j - k|$. This is done in [13] through an adaptation of the fractional moments method to a class of random block operators which includes our model (4).

If the magnetic field is strong enough to create a spectral gap for M around energy $E = 0$ (uniform in the volume n and the disorder), then we can apply a result from [9]: Suppose that for some $C > |\mu|$ it holds that either $\nu_j \geq C$ for all $j \in \mathbb{N}$ or $\nu_j \leq -C$ for all $j \in \mathbb{N}$, resulting in the spectral gap $(-(C - |\mu|), C - |\mu|)$ for M. Then for every $\xi < 1$ there is $C = C(\xi)$ and $\eta = \eta(\xi) > 0$ such that

$$
(14) \qquad \mathbb{E}\left(\sup_{|g|\leq 1} \|g(M)_{jk}\|\right) \leq Ce^{-\eta|j-k|^\xi}.
$$

This is essentially what is shown in the proof of Theorem 7.2 of [9], while the results there are only stated in terms of dynamical localization for M, i.e. for the functions $g_t(x) = e^{-itx}$ (but the argument covers general $|g| \leq 1$).

As discussed in [9], it remains an open problem if a bound such as (14) (or at least (9)) can be proven for M without the assumption of a zero-energy gap. A problem arises from the fact that the transfer matrix group (Fürstenberg group) associated with M at $E = 0$ is not irreducible, so that many of the available dynamical systems tools for proving one-dimensional localization don't apply.

1.4. **Notes on Content and Related Work.** The rest of this paper is devoted to the proof of Theorem 1.1. As in most works on the XY chain, the starting point is the representation of the XY Hamiltonian H, via the Jordan-Wigner transform, as a quadratic form in a set of Fermionic operators, see Section 2. A key tool is that eigenstates of the latter satisfy Wick's rule (sometimes also referred to as being *quasi free* or *gaussian*) and thus are fully determined by their correlation matrices with respect to the Jordan-Wigner Fermionic operators.

More specifically, the crucial formula for determining the bipartite entanglement entropy of eigenstates is the trace identity (38) in Lemma 3.1 below. We will use it for the case where the state ρ in (38) is the reduction of an eigenstate of H to the subsystem $\mathcal{H}_1$ (which will ultimately be justified in Lemma 3.7). Thus the left hand side of (38) becomes the entanglement entropy of an eigenstate. We thus need to calculate the correlation matrix on the right hand side of (38), which will be accomplished in Lemma 3.2 and Theorem 3.4: Lemma 3.2 provides the central connection between the many-body Hamiltonian H and the effective one-particle Hamiltonian M, saying that the correlation matrices of eigenstates of H are given by spectral projections of M. Moreover, the proof of Theorem 3.4 shows

that correlation matrices of reduced states are given by $2\ell \times 2\ell$-restrictions of these spectral projections.

Equipped with these tools we can then complete the proof of the area law (10) in Section 4. This is accomplished by using an extension of an argument from [31] to reduce the claim to the eigencorrelator localization bound (9).

The trace identity (38) for Fermion systems seems to originate from [38], see also [25] for a more detailed presentation and [12] for a survey of subsequent results. In these works (38) was used as a tool in the study of the ground state entanglement for the XY chain in *constant* magnetic field,

$$(15) \qquad -\sum_{j=1}^{n-1}[(1+\gamma)\sigma_j^X\sigma_{j+1}^X + (1-\gamma)\sigma_j^Y\sigma_{j+1}^Y] - \nu\sum_{j=1}^{n}\sigma_j^Z.$$

In particular, the dimension reduction from 2^n to $2n$ accomplished by (38) (or 2^ℓ to 2ℓ for the subsystem) allowed numerical predictions for the limit of large subsystems Λ_1 (and in the thermodynamic limit of an infinite chain Λ). These were later proven rigorously, see [20] for most complete results as well as references therein. Generally, it was found that the ground state entanglement for the anisotropic chain remains bounded in the size ℓ of the subsystem, i.e. satisfies an area law. Phase transitions with divergent entanglement appear for $\nu = 2$ and for the isotropic chain $\gamma = 0$. In particular, [21] shows that in the latter case the ground state entanglement grows as $\log \ell$.

Our work here differs in several significant respects from all previous works on entanglement in the XY chain and the related Fermion systems: We consider general eigenstates of the XY chain and not only ground states here, we work in finite volume (proving bounds which hold uniformly in the volume) rather than in the thermodynamic limit of an infinite chain, and, finally, use a modification of the Jordan-Wigner transform to be able to consider entanglement with respect to more general subsystems. Partly for these reasons, but also to make a number of tools from the physics literature accessible to a broader audience, we include a self-contained presentation of relevant parts of the theory of finite Fermion systems here. This includes thorough discussions of correlation matrices, Bogoliubov transforms and Wick's rule. Some of this is contained in Section 3, with more background collected in Appendix B.

2. Reduction of H to M via Jordan-Wigner

The importance of the XY chain as a model in the theory of quantum spin systems goes back to the work of Lieb, Schultz and Mattis [26], where it was shown that the XY chain with constant coefficients (and initially without magnetic field) is an exactly solvable model. Their argument proceeds by using the Jordan-Wigner transform to reduce the XY chain to a free Fermion system. For the last half century this has turned the XY chain into a canonical toy model for quantum spin systems, frequently used as a first example to illustrate new concepts.

It was understood that the methods of Lieb, Schultz and Mattis can be extended to include magnetic fields and to allow for variable coefficients μ_j, γ_j and η_j. In the latter case the model is not exactly solvable, rather the diagonalization of H can be reduced to the diagonalization of the effective Hamiltonian M.

Below we briefly summarize this reduction. A similar account with somewhat more detail (but different sign conventions) is provided in Section 3.1 of [16].

The local lowering and raising operators are

$$(16) \qquad a := \frac{1}{2}(\sigma^X - i\sigma^Y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } a^* = \frac{1}{2}(\sigma^X + i\sigma^Y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

whose actions on the j-th spin in $\mathcal{H}$ will be denoted by a_j and a_j^*.

The *Jordan-Wigner transform* refers to the operators

$$(17) \qquad c_1 := a_1, \quad c_j := \sigma_1^Z \dots \sigma_{j-1}^Z a_j, \quad j = 2, \dots, n,$$

which satisfy the *canonical anti-commutation relations* (CAR)

$$(18) \qquad \{c_j, c_k^*\} = \delta_{jk}\, \mathbb{1}, \quad \{c_j, c_k\} = \{c_j^*, c_k^*\} = 0 \quad \text{for all } j, k = 1, \dots, n.$$

Following the more general convention used in Appendix B below, we will refer to

$$(19) \qquad \mathcal{C} = (c_1, \dots, c_n, c_1^*, \dots, c_n^*)^t$$

as the *Jordan-Wigner Fermionic system*.

The XY chain H can be expressed in terms of the Jordan-Wigner operators as

$$(20) \qquad H = \mathcal{C}^* \tilde{M} \mathcal{C}.$$

Here the right hand side is interpreted in the sense of matrix multiplication of the column $\mathcal{C}$, the row $\mathcal{C}^* = (c_1^*, \dots, c_n^*, c_1, \dots, c_n)$, and the scalar $2n \times 2n$-matrix

$$(21) \qquad \tilde{M} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix},$$

where

$$(22) \quad A = \begin{pmatrix} -\nu_1 & \mu_1 & & \\ \mu_1 & \ddots & \ddots & \\ & \ddots & \ddots & \mu_{n-1} \\ & & \mu_{n-1} & -\nu_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \gamma_1\mu_1 & & & \\ -\gamma_1\mu_1 & \ddots & & \ddots & \\ & \ddots & & \ddots & \gamma_{n-1}\mu_{n-1} \\ & & -\gamma_{n-1}\mu_{n-1} & & 0 \end{pmatrix}.$$

We see $A^* = A^t = A$ and $B^* = B^t = -B$, and thus $\tilde{M}^* = \tilde{M}^t = \tilde{M}$. Note that $\tilde{M}$ and M, given by (4), coincide up to a simple change of basis: If P is the permutation matrix which maps the canonical basis vectors $e_1, \dots, e_{2n}$ of $\mathbb{C}^{2n}$ to $e_1, e_{n+1}, e_2, e_{n+2}, \dots, e_n, e_{2n}$, then

$$(23) \qquad M = P^t \tilde{M} P.$$

Depending on the context, both versions of the effective Hamiltonian will be useful here. The form M correctly reflects locality properties with respect to the physical configuration space Λ, while the block structure of $\tilde{M}$ is more suitable for the discussion of related Fermionic systems, as we will see now.

Transforming with the unitary $\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, we see that $\tilde{M}$ is unitarily equivalent to $-\tilde{M}$, and thus, in particular, has spectrum symmetric to zero. It can be diagonalized by an orthogonal Bogoliubov matrix W (see Appendix B for a definition): Let

$$(24) \qquad W := \frac{1}{2}\begin{pmatrix} V+U & V-U \\ V-U & V+U \end{pmatrix},$$

where U and V are real orthogonal matrices associated with the singular value decomposition of $S := A + B$ via

$$(25) \qquad USV^t = \Lambda := \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}.$$

Here $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are the singular values of S, i.e. the eigenvalues of $(S^*S)^{\frac{1}{2}}$, counted with multiplicity. Calculations show that W is Bogoliubov, i.e. orthogonal and satisfies (100), and

$$(26) \qquad W \tilde{M} W^t = \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix}.$$

Let

$$(27) \qquad \mathcal{B} := W\mathcal{C}.$$

By Lemma B.3 this is a Fermionic system and

$$(28) \qquad \begin{aligned} H &= \mathcal{C}^* \tilde{M} \mathcal{C} = \mathcal{B}^* W \tilde{M} W^t \mathcal{B} = \mathcal{B}^* \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} \mathcal{B} \\ &= \sum_{j=1}^{n} \lambda_j (b_j^* b_j - b_j b_j^*) = 2 \sum_{j=1}^{n} \lambda_j b_j^* b_j - E_0 \, \mathbb{1}, \end{aligned}$$

where $E_0 = \sum_{j=1}^{n} \lambda_j$. Thus H has been written in the form of a free Fermion system. Let Ω be the vacuum vector of the b_j and

$$(29) \qquad \psi_\alpha = (b_1^*)^{\alpha_1} \ldots (b_n^*)^{\alpha_n} \Omega, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{0,1\}^n$$

the orthonormal basis of $\mathcal{H}$ associated with $\mathcal{B}$, see Appendix B. The ψ_α form a complete set of eigenvectors for H with corresponding eigenvalues $2 \sum_{j:\alpha_j=1} \lambda_j - E_0$, so that the spectrum of H is

$$(30) \qquad \sigma(H) = \left\{ \sum_{j:\alpha_j=1} \lambda_j - \sum_{j:\alpha_j=0} \lambda_j : \alpha \in \{0,1\}^n \right\}.$$

3. Entanglement Entropy of Eigenstates

3.1. **Correlation matrices and Wick's rule.**
The goal of this section is to show that the entanglement of the eigenstates $\rho_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$ of H with respect to the bipartite decomposition $\mathcal{H}_1 \otimes \mathcal{H}_2$ given by (6) can be expressed in terms of *correlation matrices* of size $2\ell \times 2\ell$ and that these correlation matrices are restrictions of suitable spectral projections for the one-particle Hamiltonian $\tilde{M}$. This provides the crucial connection between properties of $\tilde{M}$ and properties of H which will be exploited in Section 4 to prove Theorem 1.1.

The expected value of an observable $A \in \mathcal{B}(\mathcal{H})$ in the mixed state $\rho \in \mathcal{B}(\mathcal{H})$ (i.e. $\rho \geq 0$ with $\operatorname{Tr} \rho = 1$) is given by

$$(31) \qquad \langle A \rangle_\rho := \operatorname{Tr} A\rho.$$

Let $\mathcal{D} = (d_1, \ldots, d_n, d_1^*, \ldots, d_n^*)^t$ be a general Fermionic system of $\mathcal{H}$ in the sense of Appendix B. The correlation matrix of the state ρ with respect to $\mathcal{D}$ is defined to be the $2n \times 2n$ matrix

$$(32) \qquad \Gamma_\rho^{\mathcal{D}} := \langle \mathcal{D}\mathcal{D}^* \rangle_\rho,$$

with the row $\mathcal{D}^* = (d_1^*, \ldots, d_n^*, d_1, \ldots, d_n)$ and $\langle \mathcal{D}\mathcal{D}^* \rangle_\rho$ to be understood in the sense of taking expectations of each of the operator-valued entries of the $2n \times 2n$-matrix $\mathcal{D}\mathcal{D}^*$.

Using the CAR, cyclicity of the trace, and $\operatorname{tr} A^* = \overline{\operatorname{tr} A}$, one checks that correlation matrices are generally of the form

$$(33) \qquad \Gamma = \frac{1}{2}\left(\mathbb{1}_{2n} + \begin{pmatrix} X & Y \\ -\overline{Y} & -\overline{X} \end{pmatrix}\right),$$

with $n \times n$ matrices X and Y such that $X^* = X$ and $Y^* = -\overline{Y}$. In particular, correlation matrices, as defined here, are self-adjoint. We mention that in the physics literature frequently the *Majorana Fermions* $d_j + d_j^*$, $-i(d_j - d_j^*)$, $j = 1,\ldots,n$, are used to define correlation matrices, which thus become skew-adjoint.

For $f : \Lambda \to \mathbb{C}$ define

$$(34) \qquad d(f) = \sum_{j=1}^{n} \overline{f}_j d_j$$

and denote its adjoint by $d^*(f) = \sum_j f_j d_j^*$. More generally, let

$$(35) \qquad D(f,g) := d(f) + d^*(g).$$

We say that the state ρ satisfies *Wick's rule* (or, more briefly, is 'Wick') with respect to $\mathcal{D}$ if

$$(36) \qquad \left\langle \prod_{j=1}^{m} D_j \right\rangle_\rho = \begin{cases} 0, & \text{if } m \text{ is odd;} \\ \sum_{k=2}^{m} (-1)^k \langle D_k D_1 \rangle_\rho \left\langle \prod_{\substack{j=2 \\ j \neq k}}^{m} D_j \right\rangle_\rho, & \text{if } m \text{ is even.} \end{cases}$$

Here D_j is short for $D(f_j,g_j)$ and a pair $f_j, g_j : \Lambda \to \mathbb{C}$. Also note that here and throughout this paper operator products as on the left hand side of (36) are to be read from the right to left, i.e. $\prod_{j=1}^{m} D_j = D_m \ldots D_1$.

For the case of even m, an iterative application of (36) shows that expectations $\left\langle \prod_{j=1}^{m} D_j \right\rangle_\rho$ can be written as a sum of products of terms of the form $\langle D_r D_s \rangle_\rho$. We note that the resulting expression is known as the *Pfaffian* (denoted by pf) of the skew-adjoint $m \times m$-matrix $D^{(\rho,m)}$ with entries $D_{s,r}^{(\rho,m)} = \langle D_r D_s \rangle_\rho$ for $1 \leq s < r \leq n$ and appropriately extended by antisymmetry. The Pfaffian of an odd-dimensional skew-adjoint matrix is generally set to zero, so that (36) can be restated as

$$(37) \qquad \left\langle \prod_{j=1}^{m} D_j \right\rangle_\rho = \operatorname{pf}\left(D^{(\rho,m)}\right).$$

As the expectations $\langle D_r D_s \rangle_\rho$ are linear combinations of elements of the correlation matrix $\Gamma_\rho^{\mathcal{D}}$, we conclude that $\left\langle \prod_{j=1}^{m} D_j \right\rangle_\rho$ is uniquely determined by the correlation matrix. By Lemma B.2 this determines $\langle A \rangle_\rho$ for *all* $A \in \mathcal{B}(\mathcal{H})$, which in turn characterizes ρ. This yields the first claim of the following Lemma:

Lemma 3.1. *If a state ρ is Wick with respect to $\mathcal{D}$, then ρ is uniquely determined by the correlation matrix $\Gamma_\rho^{\mathcal{D}}$. If, in addition, $\Gamma_\rho^{\mathcal{D}}$ is real, then the von Neumann entropy of ρ is given by*

$$(38) \qquad \mathcal{S}(\rho) = -\operatorname{Tr} \rho \log \rho = -\operatorname{tr} \Gamma_\rho^{\mathcal{D}} \log \Gamma_\rho^{\mathcal{D}}.$$

Note here that Tr and tr denote the traces in $\mathcal{B}(\mathcal{H})$ and $\mathbb{C}^{2n \times 2n}$, respectively. The second claim is restated as Proposition B.7 in Appendix B, where a proof is provided.

3.2. **Correlation matrices of eigenstates.** Next we calculate the correlation matrix of the eigenstates $\rho_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$ of H given by (29) with respect to the Jordan-Wigner Fermionic system $\mathcal{C}$.

Lemma 3.2. *Assume that $\tilde{M}$ has simple spectrum. Then, for each $\alpha \in \{0,1\}^n$, the correlation matrix of ρ_α with respect to the Jordan-Wigner Fermionic system $\mathcal{C}$ in (19) is given by*

$$(39) \qquad \Gamma^{\mathcal{C}}_{\rho_\alpha} = \chi_{\Delta_\alpha}(\tilde{M}),$$

i.e. the spectral projection for $\tilde{M}$ onto the set

$$(40) \qquad \Delta_\alpha := \{\lambda_j : \alpha_j = 0\} \cup \{-\lambda_j : \alpha_j = 1\}.$$

Here $\{\lambda_j, -\lambda_j : j = 1, 2, \ldots, n\}$ are the eigenvalues of $\tilde{M}$.

Note that, in particular, $\rho_0 = |\Omega\rangle\langle\Omega|$ is the ground state projection for H and $\Gamma^{\mathcal{C}}_{\rho_0} = \chi_{\{\lambda_1,\ldots,\lambda_n\}}(\tilde{M}) = \chi_{(0,\infty)}(\tilde{M})$. The difference between $\Gamma^{\mathcal{C}}_{\rho_0}$ and $\Gamma^{\mathcal{C}}_{\rho_\alpha}$ is that λ_j is replaced by $-\lambda_j$ for each site j in which a "particle is created" by b_j^* in (29).

Proof. The first step in the proof is to show that

$$(41) \qquad \Gamma^{\mathcal{B}}_{\rho_\alpha} = \begin{pmatrix} \delta_{\alpha,0} & 0 \\ 0 & \delta_{\alpha,1} \end{pmatrix},$$

where $\mathcal{B}$ is the Fermionic system (27),

$$(42) \qquad \delta_{\alpha,j} := \mathrm{diag}\{\delta_{\alpha_k,j}, k = 1, 2, \ldots, n\}, \quad j \in \{0,1\},$$

and $\delta_{\alpha_k,j}$ are the Kronecker delta functions. First note that, using the definition (29) and the anti-commutation properties of the b_j, one has

$$(43) \qquad b_j^* \psi_\alpha = \begin{cases} 0 & \text{if } \alpha_j = 1, \\ \pm\psi_{\alpha+e_j} & \text{if } \alpha_j = 0, \end{cases}$$

as well as

$$(44) \qquad b_j \psi_\alpha = \begin{cases} 0 & \text{if } \alpha_j = 0, \\ \pm\psi_{\alpha-e_j} & \text{if } \alpha_j = 1. \end{cases}$$

For the elements in the upper left block of $\Gamma^{\mathcal{B}}_{\rho_\alpha}$ we get that

$$(45) \qquad \begin{aligned} \mathrm{Tr}\, b_j b_k^* \rho_\alpha &= \langle\psi_\alpha, b_j b_k^* \psi_\alpha\rangle = \langle b_j^* \psi_\alpha, b_k^* \psi_\alpha\rangle \\ &= \begin{cases} 0 & \text{if } \alpha_j = 1 \text{ or } \alpha_k = 1, \\ \langle\pm\psi_{\alpha+e_j}, \pm\psi_{\alpha+e_k}\rangle = \delta_{jk} & \text{if } \alpha_j = \alpha_k = 0, \end{cases} \end{aligned}$$

We conclude that the upper left block of $\Gamma^{\mathcal{B}}_{\rho_\alpha}$ is $\delta_{\alpha,0}$. Similarly, we find

$$(46) \qquad \begin{aligned} \mathrm{Tr}\, b_j b_k \rho_\alpha &= \langle\psi_\alpha, b_j b_k \psi_\alpha\rangle = \langle b_j^* \psi_\alpha, b_k \psi_\alpha\rangle \\ &= \begin{cases} 0 & \text{if } \alpha_j = 1 \text{ or } \alpha_k = 0, \\ \pm\langle\psi_{\alpha+e_j}, \psi_{\alpha-e_k}\rangle = 0 & \text{if } \alpha_j = 0 \text{ and } \alpha_k = 1, \end{cases} \end{aligned}$$

so that the upper right block of $\Gamma^{\mathcal{B}}_{\rho_\alpha}$ vanishes.

The general structure (33) of correlation matrices determines the remaining entries of $\Gamma^{\mathcal{B}}_{\rho_\alpha}$ and completes the proof of (41).

We now prove (39). As $\mathcal{C} = W^t \mathcal{B}$, by Lemma B.3,

$$(47) \qquad \Gamma^{\mathcal{C}}_{\rho_\alpha} = W^t \Gamma^{\mathcal{B}}_{\rho_\alpha} W = W^t \begin{pmatrix} \delta_{\alpha,0} & 0 \\ 0 & \delta_{\alpha,1} \end{pmatrix} W.$$

Now, since the eigenvalues of M are simple, i.e. $-\lambda_n < \ldots < -\lambda_1 < 0 < \lambda_1 < \ldots < \lambda_n$, $\begin{pmatrix} \delta_{\alpha,0} & 0 \\ 0 & \delta_{\alpha,1} \end{pmatrix}$ is the spectral projection for $\begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix}$ onto Δ_α. Therefore (39) follows from (26). $\qquad\square$

Lemma 3.3. *The eigenstates ρ_α are Wick with respect to the Jordan-Wigner Fermionic system $\mathcal{C}$.*

Proof. Since $\mathcal{B}$ is a Bogoliubov transformation of $\mathcal{C}$, by Lemma B.4(b) it suffices to prove that ρ_α is Wick with respect to $\mathcal{B}$. Let Ω_c and $\psi^{(c)}_\alpha = (c_1^*)^{\alpha_1} \ldots (c_n^*)^{\alpha_n} \Omega_c$ be the vacuum and eigenstates associated with $\mathcal{C}$. By Lemma B.4(a) we need to prove that $\rho^{(c)}_\alpha := |\psi^{(c)}_\alpha\rangle\langle\psi^{(c)}_\alpha|$ is Wick with respect to $\mathcal{C}$ for all α.

This can be done by finding $\rho^{(c)}_\alpha$ explicitly. From the definition (17) of the c_j we see that $c_j \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes n} = 0$ for all j. This gives that, up to a phase, $\Omega_c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes n}$ is the vacuum for $\mathcal{C}$, and therefore

$$(48) \qquad \rho^{(c)}_0 = |\Omega_c\rangle\langle\Omega_c| = \bigotimes_{j=1}^{n} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \prod_{j=1}^{n} c_j c_j^*.$$

Since for $1 \leq k \leq n$ we have

$$(49) \qquad c_k^* \left(\prod_{j=1}^{n} c_j c_j^* \right) c_k = \left(\prod_{\substack{j=1 \\ j \neq k}}^{n} c_j c_j^* \right) c_k^* c_k c_k^* c_k = \left(\prod_{\substack{j=1 \\ j \neq k}}^{n} c_j c_j^* \right) c_k^* c_k,$$

it is easy to see that

$$(50) \qquad \begin{aligned} \rho^{(c)}_\alpha &= \prod_{j=1}^{n} (c_{n-j}^*)^{\alpha_{n-j}} |\Omega_c\rangle\langle\Omega_c| \prod_{j=1}^{n} (c_j)^{\alpha_j} = \prod_{\substack{j=1 \\ \alpha_j=0}}^{n} c_j c_j^* \prod_{\substack{j=1 \\ \alpha_j=1}}^{n} c_j^* c_j \\ &= \bigotimes_{j=1}^{n} \begin{pmatrix} \delta_{\alpha_j,1} & 0 \\ 0 & \delta_{\alpha_j,0} \end{pmatrix}. \end{aligned}$$

Thus, as a product of diagonal states, $\rho^{(c)}_\alpha$ is Wick with respect to $\mathcal{C}$ by Proposition B.6. $\qquad\square$

By Lemma 3.1 we may conclude that $\operatorname{Tr} \rho_\alpha \log \rho_\alpha = \operatorname{tr} \Gamma^{\mathcal{C}}_{\rho_\alpha} \log \Gamma^{\mathcal{C}}_{\rho_\alpha}$. This, however, is a trivial fact as ρ_α is a pure state in $\mathcal{H}$ and $\Gamma^{\mathcal{C}}_{\rho_\alpha}$ an orthogonal projection in $\mathbb{C}^{2n}$. The actual use of Lemmas 3.2 and 3.3 is that they provide a first step in establishing a similar trace identity for the *reduced* states (with non-vanishing entropy).

3.3. **Entanglement of reduced states.** Restrictions of $\Gamma_{\rho_\alpha}^{\mathcal{C}}$, which we have identified in Lemma 3.2 with spectral projections for $\tilde{M}$, can be used to express the entanglement of the states ρ_α. More precisely, let

(51) $\qquad\qquad \Gamma_{\alpha,1}$ be the restriction of $\Gamma_{\rho_\alpha}^{\mathcal{C}}$ to span $\{e_j, e_{j+n} : j \in \Lambda_1\}$,

i.e. the $2\ell \times 2\ell$-submatrix of $\Gamma_{\rho_\alpha}^{\mathcal{C}}$ obtained by deleting the k-th rows and columns for $k \in \{j, j+n : j \in \Lambda \setminus \Lambda_1\}$.

Theorem 3.4. *The entanglement entropy of ρ_α with respect to the bipartite decomposition $\mathcal{H}_1 \otimes \mathcal{H}_2$ is given by*

(52) $$\mathcal{E}(\rho_\alpha) = -\operatorname{tr} \Gamma_{\alpha,1} \log \Gamma_{\alpha,1}.$$

The proof of this theorem fills the rest of this section. First we want to express the correlation matrix of the reduced states $\rho_{\alpha,1}$ in $\mathcal{H}_1$ as restrictions of the correlation matrices $\Gamma_{\rho_\alpha}^{\mathcal{C}}$. The question arises as to what Fermionic system in $\mathcal{H}_1$ to use here. If $\Lambda_1 = \{1, \ldots, \ell\}$, a left end of Λ, then one may simply use the restriction of the first ℓ of the Jordan-Wigner Fermionic operators c_j to $\mathcal{H}_1$, as these operators are local in Λ_1 by definition (17).

For general $\Lambda_1 = \{r, \ldots, r+\ell-1\}$ we need to be slightly more careful and introduce *local Jordan-Wigner operators* $\{c_j^{(1)}, \ j \in \Lambda_1\}$ on $\mathcal{H}_1$ as

(53) $\qquad\qquad c_r^{(1)} := a_r, \quad c_j^{(1)} := \sigma_r^Z \ldots \sigma_{j-1}^{(1)} a_j, \ j = r+1, \ldots, r+\ell-1.$

As for the Jordan-Wigner Fermionic operators c_j, but now on Λ_1 rather than Λ, one sees that

(54) $$\mathcal{C}_1 := (c_r^{(1)}, \ldots, c_{r+\ell-1}^{(1)}, (c_r^{(1)})^*, \ldots, (c_{r+\ell-1}^{(1)})^*)^t$$

is a Fermionic system in $\mathcal{H}_1$. Let

(55) $$\tilde{c}_j := \mathbb{1}^{\otimes(r-1)} \otimes c_j^{(1)} \otimes \mathbb{1}^{\otimes(n-\ell-r+1)}, \quad j \in \Lambda_1,$$

be the extensions of the operators $c_j^{(1)}$ to $\mathcal{H}$. The $\tilde{c}_j$ are local observables for the subsystem Λ_1 and differ from the c_j by products of 'left-end' Pauli matrices,

(56) $$c_j = \sigma_1^Z \sigma_2^Z \ldots \sigma_{r-1}^Z \tilde{c}_j = \tilde{c}_j \sigma_1^Z \sigma_2^Z \ldots \sigma_{r-1}^Z, \quad j \in \Lambda_1.$$

To describe the difference between c_j and $\tilde{c}_j$ further, for $f : \Lambda_1 \to \mathbb{C}$ we set

(57) $$\tilde{c}_r(f) = \sum_{j \in \Lambda_1} \bar{f}_j \tilde{c}_j, \quad c_r(f) = \sum_{j \in \Lambda_1} \overline{f}_j c_j,$$

and denote their adjoints by $\tilde{c}_r^*(f)$ and $c_r^*(f)$. For pairs $f, g : \Lambda_1 \to \mathbb{C}$ we also write

(58) $$\tilde{C}_r(f, g) = \tilde{c}_r(f) + \tilde{c}_r^*(g), \quad C_r(f, g) = c_r(f) + c_r^*(g).$$

From (56) and basic properties of the Pauli matrices, in particular $(\sigma_1^Z \ldots \sigma_{r-1}^Z)^2 = \mathbb{1}$, we easily get

Lemma 3.5. *Let $f, g, f_1, g_1, f_2, g_2 : \Lambda_1 \to \mathbb{C}$, then*
(a) $\tilde{C}_r(f, g) = \sigma_1^Z \sigma_2^Z \ldots \sigma_{r-1}^Z C_r(f, g)$,
(b) $\tilde{C}_r(f_1, g_1) \tilde{C}_r(f_2, g_2) = C_r(f_1, g_1) C_r(f_2, g_2)$

With this we reach our first goal:

Lemma 3.6. *The correlation matrix of the reduced state* $\rho_{\alpha,1} = \mathrm{Tr}_{\mathcal{H}_2} \rho_\alpha$ *with respect to the local Jordan-Wigner Fermionic system* $\mathcal{C}_1$ *on* $\mathcal{H}_1$ *is given by*

$$(59) \qquad\qquad\qquad \Gamma^{\mathcal{C}_1}_{\rho_{\alpha,1}} = \Gamma_{\alpha,1}.$$

Proof. That the upper right blocks of the two matrices in (59) coincide is seen as follows: For $j, k \in \Lambda_1$,

$$
\begin{aligned}
(60) \qquad \langle c_j^{(1)} c_k^{(1)} \rangle_{\rho_{\alpha,1}} &= \mathrm{Tr}\left(c_j^{(1)} c_k^{(1)} \rho_{\alpha,1} \right) = \mathrm{Tr}\left(c_j^{(1)} c_k^{(1)} \, \mathrm{Tr}_{\mathcal{H}_2} \rho_\alpha \right) \\
&= \mathrm{Tr}\left(\mathrm{Tr}_{\mathcal{H}_2} \tilde{c}_j \tilde{c}_k \rho_\alpha \right) = \mathrm{Tr}\left(c_j c_k \rho_\alpha \right) = \langle c_j c_k \rangle_{\rho_\alpha}.
\end{aligned}
$$

Here we have used in the third step that

$$(61) \qquad c_j^{(1)} c_k^{(1)} \, \mathrm{Tr}_{\mathcal{H}_2} \rho_\alpha = \mathrm{Tr}_{\mathcal{H}_2}\left(\mathbb{1} \otimes c_j^{(1)} c_k^{(1)} \otimes \mathbb{1} \right) \rho_\alpha = \mathrm{Tr}_{\mathcal{H}_2} \tilde{c}_j \tilde{c}_k \rho_\alpha$$

and in the fourth step that $\tilde{c}_j \tilde{c}_k = c_j c_k$ by Lemma 3.5(b).

Identity of the other three blocks follows in the same way when replacing $c_j^{(1)}$ and/or $c_k^{(1)}$ by $(c_j^{(1)})^*$ and/or $(c_k^{(1)})^*$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

In order to conclude that $\rho_{\alpha,1}$ can be determined from the correlation matrix $\Gamma^{\mathcal{C}_1}_{\rho_{\alpha,1}}$, we prove that Wick's rule holds:

Lemma 3.7. *The reduced states* $\rho_{\alpha,1}$ *are Wick with respect to* $\mathcal{C}_1$.

Proof. We need to prove Wick's rule (36) for $\rho_{\alpha,1}$-expectations of the operators

$$(62) \qquad C^{(1)}(f,g) = c^{(1)}(f) + (c^{(1)}(g))^*, \quad c^{(1)}(f) = \sum_{j \in \Lambda_1} \bar{f}_j c_j^{(1)}$$

on $\mathcal{H}_1$, where $f, g : \Lambda_1 \to \mathbb{C}$.

This is equivalent to proving Wick's rule for ρ_α-expectations of the operators $\tilde{C}_r(f,g)$ defined earlier in (58) because of the following: For any positive integer m and functions $f_j, g_j : \Lambda_1 \to \mathbb{C}$ for $1 \leq j \leq m$ we have,

$$
\begin{aligned}
(63) \qquad \left\langle \prod_{j=1}^m C^{(1)}(f_j, g_j) \right\rangle_{\rho_{\alpha,1}} &= \mathrm{Tr}\left(\prod_{j=1}^m C^{(1)}(f_j, g_j) \, \mathrm{Tr}_{\mathcal{H}_2} \rho_\alpha \right) \\
&= \mathrm{Tr}\left(\prod_{j=1}^m \tilde{C}_r(f_j, g_j) \rho_\alpha \right) = \left\langle \prod_{j=1}^m \tilde{C}_r(f_j, g_j) \right\rangle_{\rho_\alpha},
\end{aligned}
$$

which uses the general fact $\mathrm{Tr}_{\mathcal{H}_2}(B \otimes I)A = B \, \mathrm{Tr}_{\mathcal{H}_2} A$ for $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H}_1)$ in the second step.

Case 1: If m is even, then using Lemma 3.5 we get that

$$(64) \qquad\qquad \prod_{j=1}^m \tilde{C}_r(f_j, g_j) = \prod_{j=1}^m C_r(f_j, g_j).$$

Note that the operators $C_r(f_j, g_j)$ are of the form $C(\tilde{f}_j, \tilde{g}_j)$ where $\tilde{f}_j, \tilde{g}_j$ are the extensions of f_j, g_j by zeros to Λ. Thus we can use that ρ_α is Wick with respect to $\mathcal{C}$, i.e. Lemma 3.3:

$$(65) \qquad \left\langle \prod_{j=1}^m C^{(1)}(f_j, g_j) \right\rangle_{\rho_{\alpha,1}} = \left\langle \prod_{j=1}^m C_r(f_j, g_j) \right\rangle_{\rho_\alpha} = \mathrm{pf}\left(C_r^{(\rho_\alpha, m)} \right),$$

where $C_r^{(\rho_\alpha, m)}$ is the $m \times m$ anti-symmetric matrix with entries

$$(66) \qquad \left[C_r^{(\rho_\alpha, m)} \right]_{j,k} = \langle C_r(f_k, g_k) C_r(f_j, g_j) \rangle_{\rho_\alpha}$$

for $1 \le j < k \le n$ and properly extended by antisymmetry. Arguing as in (63) and (64), but now with only two factors,

$$(67) \qquad \langle C_r(f_j, g_j) C_r(f_k, g_k) \rangle_{\rho_\alpha} = \langle \widetilde{C}_r(f_j, g_j) \widetilde{C}_r(f_k, g_k) \rangle_{\rho_\alpha} = \langle C^{(1)}(f_j, g_j) C^{(1)}(f_k, g_k) \rangle_{\rho_{\alpha,1}}.$$

Combined with (65) this shows that Wick's rule is satisfied if m is even.

   Case 2: If m is odd, then by (63), (64) and Lemma 3.5 we get

$$(68) \qquad \left\langle \prod_{j=1}^{m} C^{(1)}(f_j, g_j) \right\rangle_{\rho_{\alpha,1}} = \left\langle \tilde{C}_r(f_m, g_m) \prod_{j=1}^{m-1} \tilde{C}_r(f_j, g_j) \right\rangle_{\rho_\alpha}$$

$$= \left\langle \tilde{C}_r(f_m, g_m) \prod_{j=1}^{m-1} C_r(f_j, g_j) \right\rangle_{\rho_\alpha}$$

$$= \left\langle \left(\sigma_1^Z \sigma_2^Z \dots \sigma_{r-1}^Z \right) C_r(f_m, g_m) \prod_{j=1}^{m-1} C_r(f_j, g_j) \right\rangle_{\rho_\alpha}$$

$$= \left\langle \left(\sigma_1^Z \sigma_2^Z \dots \sigma_{r-1}^Z \right) \prod_{j=1}^{m} C_r(f_j, g_j) \right\rangle_{\rho_\alpha}.$$

Now note that

$$(69) \qquad \sigma_1^Z \sigma_2^Z \dots \sigma_{r-1}^Z = \prod_{j=1}^{r-1} (c_j + c_j^*)(c_j - c_j^*).$$

Thus $\sigma_1^Z \sigma_2^Z \dots \sigma_{r-1}^Z$ is an even product of operators $C(\hat{f}_j, \hat{g}_j)$ with suitable choices of $\hat{f}_j$ and $\hat{g}_j$. Since ρ_α is Wick with respect to $\mathcal{C}$, we conclude that the right hand side of (68) is zero. $\qquad \square$

   We can now easily complete the *proof of Theorem 3.4*: Combining Lemma 3.7 and Lemma 3.1 gives

$$(70) \qquad \mathcal{E}(\rho_\alpha) = \mathcal{S}(\rho_{\alpha,1}) = -\operatorname{tr} \Gamma_{\rho_{\alpha,1}}^{\mathcal{C}_1} \log \Gamma_{\rho_{\alpha,1}}^{\mathcal{C}_1}.$$

By Lemma 3.6, $\Gamma_{\rho_{\alpha,1}}^{\mathcal{C}_1} = \Gamma_{\alpha,1}$, which gives (52). Theorem 3.4 is proven.

4. An Area Law for the Eigenstates

   In this section we *prove* Theorem 1.1, our main result.

   Note first that, by the non-degeneracy assumption (3), it holds with probability one that all eigenprojectors ρ_ψ of H are of the form $\rho_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$. Therefore (10) is equivalent to

$$(71) \qquad \mathbb{E}\left(\max_\alpha \mathcal{E}(\rho_\alpha) \right) \le \tilde{C} < \infty$$

uniformly in n, r and ℓ, meaning we can work with the Fermion basis given by (29).

For fixed α, set $\Gamma_1 := \Gamma_{\alpha,1}$ for the restriction (51) of $\Gamma := \Gamma^{\mathcal{C}}_{\rho_\alpha} = \chi_{\Delta_\alpha}(\tilde{M})$. Γ_1 is a correlation matrix (by Lemma 3.6) and real-valued (see (47), where W is real). Thus, by the argument in the proof of Proposition B.7, Γ_1 has eigenvalues ξ_j, $1 - \xi_j$, $j = 1, \ldots, \ell$, with $0 \leq \xi_j \leq 1/2$.

The following is a calculation essentially taken from [31], extended to the more general type of correlation matrices needed here to cover quadratic Fermion Hamiltonians of the form (20).

By Theorem 3.4 we have

$$
(72) \qquad
\begin{aligned}
\mathcal{E}(\rho_\alpha) &= -\operatorname{tr}\Gamma_1 \log \Gamma_1 \\
&= -\sum_{j=1}^{\ell} \left(\xi_j \log \xi_j + (1 - \xi_j) \log(1 - \xi_j) \right) \\
&\leq 2\log 2 \sum_{j=1}^{\ell} \sqrt{\xi_j(1 - \xi_j)} \\
&= \log 2 \operatorname{tr}(\Gamma_1(\mathbb{1} - \Gamma_1))^{1/2},
\end{aligned}
$$

where we have used the elementary inequality

$$
(73) \qquad -x\log x - (1 - x)\log(1 - x) \leq 2\log 2\sqrt{x(1 - x)}, \quad \text{for } 0 < x < 1.
$$

The Peierls-Bogoliubov inequality, see Section 8.3 in [35], says that

$$
(74) \qquad \operatorname{tr} f(A) \geq \sum_{j=1}^{m} f(A_{jj})
$$

for any convex function f and $m \times m$ hermitian matrix A. Using this with $f(x) = -\sqrt{x}$ as well as the elementary inequality $\sqrt{x} + \sqrt{y} \leq \sqrt{2}\sqrt{x+y}$, we may further bound (72) by

$$
(75) \qquad
\begin{aligned}
&\leq \log 2 \sum_{j=1}^{2\ell} \left((\Gamma_1(\mathbb{1} - \Gamma_1))_{jj} \right)^{1/2} \\
&\leq \sqrt{2}\log 2 \sum_{j=1}^{\ell} \left((\Gamma_1(\mathbb{1} - \Gamma_1))_{jj} + (\Gamma_1(\mathbb{1} - \Gamma_1))_{\ell+j,\ell+j} \right)^{1/2}.
\end{aligned}
$$

So far we have worked with the spectral projections $\Gamma = \chi_{\Delta_\alpha}(\tilde{M})$ of $\tilde{M}$ and its restrictions Γ_1. It will now be convenient to proceed by working with spectral projections of M, related to $\tilde{M}$ by the change of basis (23). Thus we let

$$
(76) \qquad \hat{\Gamma} = \chi_{\Delta_\alpha}(M) = P^t \chi_{\Delta_\alpha}(\tilde{M})P = P^t \Gamma P.
$$

which, in agreement with the structure of (4), we think of as an $n \times n$-matrix with 2×2-matrix-valued entries

$$
(77) \qquad \hat{\Gamma}_{jk} := \begin{pmatrix} \Gamma_{jk} & \Gamma_{j,k+n} \\ \Gamma_{j+n,k} & \Gamma_{j+n,k+n} \end{pmatrix}, \quad j,k = 1, \ldots, n.
$$

This allows to write $(\Gamma_1(\mathbb{1} - \Gamma_1))_{jj} + (\Gamma_1(\mathbb{1} - \Gamma_1))_{\ell+j,\ell+j} = \operatorname{tr}(\hat{\Gamma}_1(\mathbb{1} - \hat{\Gamma}_1))_{jj}$, where $\hat{\Gamma}_1$ is the restriction of $\hat{\Gamma}$ to its 2×2-matrix-valued entries $\hat{\Gamma}_{jk}$, $j,k \in \Lambda_1$. Thus the calculations in

(72) and (75) summarize as

$$(78) \qquad \mathcal{E}(\rho_\alpha) \leq \sqrt{2} \log 2 \sum_{j \in \Lambda_1} \left(\text{tr}(\hat{\Gamma}_1 (\mathbb{1} - \hat{\Gamma}_1))_{jj} \right)^{1/2}.$$

Now, since $\hat{\Gamma}$ is an orthogonal projection, we use $\hat{\Gamma}^2 = \hat{\Gamma}$ with block matrix multiplication to get,

$$(79) \qquad \hat{\Gamma}_{jj} = \hat{\Gamma}_{jj}^2 + \sum_{\substack{k \in \Lambda_1 \\ j \neq k}} \hat{\Gamma}_{jk} \hat{\Gamma}_{kj} + \sum_{k \in \Lambda \backslash \Lambda_1} \hat{\Gamma}_{jk} \hat{\Gamma}_{kj}.$$

Then, for $j \in \Lambda_1$,

$$(80) \qquad \left(\hat{\Gamma}_1 (\mathbb{1} - \hat{\Gamma}_1) \right)_{jj} = \hat{\Gamma}_{jj} (\mathbb{1} - \hat{\Gamma}_{jj}) - \sum_{\substack{k \in \Lambda_1 \\ k \neq j}} \hat{\Gamma}_{jk} \hat{\Gamma}_{kj} = \sum_{k \in \Lambda \backslash \Lambda_1} \hat{\Gamma}_{jk} \hat{\Gamma}_{kj} = \sum_{k \in \Lambda \backslash \Lambda_1} \hat{\Gamma}_{jk} (\hat{\Gamma}_{jk})^t,$$

where symmetry of $\hat{\Gamma}$ was used. Inserting this into (78), using $\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$ as well as $\text{tr} \hat{\Gamma}_{jk} (\hat{\Gamma}_{jk})^t \leq 2 \|\hat{\Gamma}_{jk}\|^2$, we find

$$(81) \qquad \mathcal{E}(\rho_\alpha) \leq 2 \log 2 \sum_{\mathbf{j} \in \Lambda_1} \sum_{\mathbf{k} \in \Lambda \backslash \Lambda_1} \|\hat{\Gamma}_{jk}\|.$$

Maximizing over α and averaging gives

$$(82) \qquad \mathbb{E} \left(\sup_\alpha \mathcal{E}(\rho_\alpha) \right) \leq 2 \log 2 \sum_{\mathbf{j} \in \Lambda_1} \sum_{\mathbf{k} \in \Lambda \backslash \Lambda_1} \mathbb{E} \left(\sup_\alpha \left\| [\chi_{\Delta_\alpha}(M)]_{jk} \right\| \right)$$

Now, by assumption (9),

$$(83) \qquad \mathbb{E} \left(\sup_\alpha \left\| [\chi_{\Delta_\alpha}(M)]_{jk} \right\| \right) \leq \mathbb{E} \left(\sup_{|g| \leq 1} \left\| [g(M)]_{jk} \right\| \right) \leq \frac{C}{1 + |j - k|^\beta}$$

for some $\beta > 2$. We have

$$(84) \qquad
\begin{aligned}
\sum_{j \in \Lambda_1} \sum_{k \in \Lambda \backslash \Lambda_1} \frac{1}{1 + |j - k|^\beta} &\leq \sum_{j=1}^{\ell} \sum_{k \in \mathbb{Z} \backslash \{1, \ldots, \ell\}} \frac{1}{1 + |j - k|^\beta} \\
&= 2 \sum_{j=1}^{\ell} \sum_{k=\ell+1}^{\infty} \frac{1}{1 + (k - j)^\beta} \\
&\leq 2 \sum_{j=1}^{\ell} \sum_{k=\ell+1}^{\infty} \frac{1}{\sqrt{1 + 2(\ell - j)^\beta}} \frac{1}{\sqrt{1 + 2(k - \ell)^\beta}} \\
&\leq 2 \left(\sum_{j=0}^{\infty} \frac{1}{\sqrt{1 + 2j^\beta}} \right)^2 < \infty.
\end{aligned}$$

This gives the uniform boundedness of (82) in n, r and ℓ, and thus completes the proof of Theorem 1.1.

APPENDIX A. NON-DEGENERACY OF EIGENVALUES

Here we will prove that the non-degeneracy assumption (3) holds under the condition that the random variables ν_j, $j \in \mathbb{N}$, are i.i.d. with absolutely continuous distribution and that μ_j, γ_j, $j \in \mathbb{N}$, are independent of the ν_j. In particular, this covers the applications discussed in Section 1.3. In this case, almost sure non-degeneracy of the eigenvalues of H will follow from Proposition A.1 below.

Towards this, let A is a hermitian $n \times n$-matrix, B an anti-hermitian $n \times n$-matrix, $A(\nu) = A + \mathrm{diag}(\nu)$ for $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n$ and

$$(85) \qquad M = M(\nu) = \begin{pmatrix} A(\nu) & B \\ -B & -A(\nu) \end{pmatrix}.$$

As M and $-M$ are unitarily equivalent, M has n pairs $\pm\lambda_j$, $j = 1, \ldots, n$, of eigenvalues, where we may choose $0 \leq \lambda_1 \leq \ldots \leq \lambda_n$. As discussed in Section 2, the 2^n eigenvalues of the corresponding many-body Hamiltonian $H = \mathcal{C}^* M \mathcal{C}$ are

$$(86) \qquad E_\alpha := \sum_{j:\alpha_j=1} \lambda_j - \sum_{j:\alpha_j=0} \lambda_j$$

for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n$.

Proposition A.1. *For Lebesgue-almost every $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n$ the 2^n eigenvalues $\{E_\alpha : \alpha \in \{0,1\}^n\}$ of H are pairwise distinct.*

Remark A.2. Pairwise distinctness of the many-body eigenvalues E_α readily implies pairwise distinctness of the one-body eigenvalues, i.e. Proposition A.1 yields that $0 < \lambda_1 < \ldots < \lambda_n$ and, in particular, invertibility of M for Lebesgue-almost every $\nu \in \mathbb{R}^n$.

The elementary proof of Proposition A.1 proceeds in two steps: First we show the existence of *one* $\nu' \in \mathbb{R}^n$ such that the 2^n eigenvalues of $H(\nu')$ are pairwise distinct. Then we use analytic perturbation theory to show that this property extends to $H(\nu)$ for Lebesgue-a.e. $\nu \in \mathbb{R}^n$.

Step 1: Consider the functions $f_\beta(x) := \sum_{j=1}^n \beta_j x_j$ on $\mathbb{R}^n$, $\beta \in \{-1,1\}^n$. There exists a nullset $N \subset \mathbb{R}^n$ such that the numbers $f_\beta(x)$, $\beta \in \{-1,1\}^n$, are pairwise distinct for all $x \in \mathbb{R}^n \setminus N$ (first observe that for each pair $\beta \neq \tilde{\beta}$ there is a nullset $N_{\beta\tilde{\beta}}$ such that $f_\beta(x) \neq f_{\tilde{\beta}}(x)$ for all $x \in \mathbb{R}^n \setminus N_{\beta\tilde{\beta}}$, then take the union of these nullsets).

Fix $\nu'' \in \mathbb{R}^n \setminus N$ with $0 < \nu_1'' < \ldots < \nu_n''$ and let $\delta := \min_{\beta\neq\tilde{\beta}} |f_\beta(\nu'') - f_{\tilde{\beta}}(\nu'')|$. Let $D := \left\| \begin{pmatrix} A & B \\ -B & -A \end{pmatrix} \right\|$ and $\nu' := \frac{4nD+1}{\delta}\nu''$, so that

$$(87) \qquad \min_{\beta\neq\tilde{\beta}} |f_\beta(\nu') - f_{\tilde{\beta}}(\nu')| = 4nD + 1.$$

Suitable choices of $\beta, \tilde{\beta}$ show that $|\nu_k'| > 2D$ and $|\nu_k' - \nu_\ell'| > 2D$ for all $k \neq \ell$. Treating $M' := M(\nu')$ as a perturbation of $\mathrm{diag}(\nu_1', \ldots, \nu_n', -\nu_1', \ldots, -\nu_n')$, we see that each of the $2n$ intervals

$$(88) \qquad [\nu_j' - D, \nu_j' + D], \ [-\nu_j' - D, -\nu_j' + D], \quad j = 1, \ldots, n,$$

contains exactly one eigenvalue of M'. Denote these eigenvalues by $\pm\lambda'_j$, $j = 1, \ldots, n$. For the corresponding eigenvalues $E'_\beta := f_\beta(\lambda'_1, \ldots, \lambda'_n)$, $\beta \in \{1, -1\}^n$, of $H' = \mathcal{C}^* M' \mathcal{C}$ we get

$$
\begin{aligned}
(89) \qquad |E'_\beta - E'_{\tilde{\beta}}| &= \left| \sum_{j=1}^n (\beta_j - \tilde{\beta}_j) \lambda'_j \right| \\
&= \left| \sum_{j=1}^n (\beta_j - \tilde{\beta}_j)(\lambda'_j - \nu'_j) + f_\beta(\nu') - f_{\tilde{\beta}}(\nu') \right| \\
&\geq |f_\beta(\nu') - f_{\tilde{\beta}}(\nu')| - 2 \sum_{j=1}^n |\lambda'_j - \nu'_j| \\
&\geq |f_\beta(\nu') - f_{\tilde{\beta}}(\nu')| - 2nD \; > \; 0
\end{aligned}
$$

if $\beta \neq \tilde{\beta}$. Thus $H' = H(\nu')$ has non-degenerate spectrum.

Step 2: We now apply analytic perturbation theory iteratively to each of the parameters $\nu_1, \ldots, \nu_n$ in $H(\nu)$.

Fix $\nu'_2, \ldots, \nu'_n$, the last $n - 1$ components of ν' found above. Then

$$
(90) \qquad H(\nu_1, \nu'_2, \ldots, \nu'_n) = A_1 + \nu_1 B_1
$$

for selfadjoint A_1 and B_1. Thus by Theorem II.6.1 of [22] there are real analytic functions f_k, $k = 1, \ldots, 2^n$, such that $\{f_k(\nu_1) : k = 1, \ldots, 2^n\}$ are the eigenvalues of $H(\nu_1, \nu'_2, \ldots, \nu'_n)$ for each $\nu_1 \in \mathbb{R}$. As $H(\nu')$ has non-degenerate spectrum, the numbers $\{f_k(\nu'_1) : k = 1, \ldots, 2^n\}$ are pairwise distinct. Analyticity of the functions f_k implies the existence of a nullset $N^{(1)} \subset \mathbb{R}$ (in fact a countable set) such that the numbers $\{f_k(\nu_1) : k = 1, \ldots, 2^n\}$ are pairwise distinct for each $\nu_1 \in \mathbb{R} \setminus N^{(1)}$ (each pair of functions f_k and $f_{\tilde{k}}$, $k \neq \tilde{k}$, coincides at no more than countable many points).

For any such ν_1, we can now use analyticity in ν_2 (given that $H(\nu_1, \nu_2, \nu'_3, \ldots, \nu'_n) = A_2 + \nu_2 B_2$) to get the existence of a nullset $N^{(2)}(\nu_1)$ in $\mathbb{R}$ such that $H(\nu_1, \nu_2, \nu'_3, \ldots, \nu'_n)$ has pairwise distinct eigenvalues for all $\nu_2 \in \mathbb{R} \setminus N^{(2)}(\nu_1)$. It follows that the eigenvalues of $H(\nu_1, \nu_2, \nu'_3, \ldots, \nu'_n)$ are non-degenerate for all ν_1, ν_2 with $\nu_1 \in \mathbb{R} \setminus N^{(1)}$, $\nu_2 \in \mathbb{R} \setminus N^{(2)}(\nu_1)$, and thus for Lebesgue-a.e. $(\nu_1, \nu_2) \in \mathbb{R}^2$.

Iteration of this argument leads to Proposition A.1.

APPENDIX B. FINITE FERMIONIC SYSTEMS AND BOGOLIUBOV TRANSFORMATIONS

In a Hilbert space $\mathcal{H}$ of dimension $\dim \mathcal{H} = 2^n$, we call

$$
(91) \qquad \mathcal{D} = (d_1, \ldots, d_n, d_1^*, \ldots, d_n^*)^t
$$

a *Fermionic system* if the operators $d_j \in \mathcal{B}(\mathcal{H})$ and their adjoints satisfy the canonical commutation relations (CAR)

$$
(92) \qquad \{d_j, d_k^*\} = \delta_{jk} \mathbb{1}, \quad \{d_j, d_k\} = \{d_j^*, d_k^*\} = 0 \quad \text{for all } j, k = 1, \ldots, n.
$$

The intersection of the kernels of the d_j is one-dimensional, i.e. they contain an essentially unique normalized vector Ω, referred to as the vacuum vector, from which an orthonormal basis of $\mathcal{H}$ is found as

$$
(93) \qquad \phi_\alpha = (d_1^*)^{\alpha_1} \ldots (d_n^*)^{\alpha_n} \Omega, \quad \alpha \in \{0, 1\}^n,
$$

see for example [34].

It is easy to see that the operators $\{d_1, \ldots, d_n, d_1^*, \ldots, d_n^*\}$ in a Fermionic system are linearly independent (consider a general linear combination of these operators and calculate its anti-commutators with all d_j and d_j^*). Next we state and prove two other basic properties of finite Fermionic systems.

Lemma B.1. *Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be 2^n-dimensional Hilbert spaces and $\mathcal{D} = (d_1, \ldots, d_n, d_1^*, \ldots, d_n^*)^t$ be a Fermionic system in $\mathcal{H}$. Then*

$$\tag{94} \tilde{\mathcal{D}} = (\tilde{d}_1, \ldots, \tilde{d}_n, \tilde{d}_1^*, \ldots, \tilde{d}_n^*)^t$$

is a Fermionic system in $\tilde{\mathcal{H}}$ if and only if there exists a unitary operator $U : \mathcal{H} \to \tilde{\mathcal{H}}$ such that $U^* \tilde{d}_j U = d_j$ for all $1 \leq j \leq n$. The unitary operator U is characterized by $U\phi_\alpha = \tilde{\phi}_\alpha$ for all $\alpha \in \{0,1\}^n$, where $\{\phi_\alpha\}$ and $\{\tilde{\phi}_\alpha\}$ are the ONBs associated with $\mathcal{D}$ and $\tilde{\mathcal{D}}$ through (93).

Proof. Let $\tilde{\Omega}$ and Ω be the vacuum states of $\tilde{\mathcal{D}}$ and $\mathcal{D}$, respectively. For the basis vectors $\tilde{\phi}_\alpha = (\tilde{d}_1^*)^{\alpha_1} \ldots (\tilde{d}_n^*)^{\alpha_n} \tilde{\Omega}$ and $\phi_\alpha = (d_1^*)^{\alpha_1} \ldots (d_n^*)^{\alpha_n} \Omega$, let $U : \mathcal{H} \to \tilde{\mathcal{H}}$ the unitary determined by $U\phi_\alpha = \tilde{\phi}_\alpha$ for all α. Then for any j we have

$$\tag{95} \tilde{d}_j^* U\phi_\alpha = \tilde{d}_j^* \tilde{\phi}_\alpha = \begin{cases} 0, & \text{if } \alpha_j = 1; \\ (-1)^{\sum_{k=1}^{j} \alpha_k} \tilde{\phi}_{\alpha+e_j} = (-1)^{\sum_{k=1}^{j} \alpha_k} U\phi_{\alpha+e_j}, & \text{if } \alpha_j = 0. \end{cases}$$

$$= U d_j^* \phi_\alpha.$$

Thus $d_j^* = U^* \tilde{d}_j^* U$ and $U^* \tilde{d}_j U = d_j$ for all $1 \leq j \leq n$.

The converse follows easily using

$$\tag{96} \left\{ \tilde{d}_j^{\#_1}, \tilde{d}_k^{\#_2} \right\} = U \left\{ d_j^{\#_1}, d_k^{\#_2} \right\} U^* \quad \text{for any } 1 \leq j, k \leq n,$$

where $\#_1$ and $\#_2$ stand for $*$ or nothing. $\qquad\square$

Lemma B.2. *Let $\mathcal{D}$ be a Fermionic system in $\mathcal{H}$ and $\mathcal{A}$ be the $\star$-algebra generated by the components of $\mathcal{D}$. Then $\mathcal{A} = \mathcal{B}(\mathcal{H})$.*

Proof. With ϕ_α as in (93) we need to show that for every pair $\alpha, \beta \in \{0,1\}^n$ there exists an operator $A_{\alpha,\beta} \in \mathcal{A}$, such that $A_{\alpha,\beta}\phi_\alpha = \phi_\beta$ and $A_{\alpha,\beta}\phi_{\tilde{\alpha}} = 0$ for $\tilde{\alpha} \neq \alpha$. This operator is explicitly given by

$$\tag{97} A_{\alpha,\beta} := \left(\prod_{j=1}^{n} (d_j d_j^*)^{1-\beta_j} \right) \left(\prod_{j=0}^{n-1} (d_{n-j}^*)^{\beta_{n-j}} \right) \left(\prod_{j=1}^{n} d_j^{\alpha_j} \right).$$

Now we have, using the CAR,

$$\tag{98} A_{\alpha,\beta}\phi_\alpha = \left(\prod_{j=1}^{n} (d_j d_j^*)^{1-\beta_j} \right) \left(\prod_{j=0}^{n-1} (d_{n-j}^*)^{\beta_{n-j}} \right) d_n^{\alpha_n} \ldots d_1^{\alpha_1} (d_1^*)^{\alpha_1} \ldots (d_n^*)^{\alpha_n} \Omega$$

$$= \left(\prod_{j=1}^{n} (d_j d_j^*)^{1-\beta_j} \right) \left(\prod_{j=0}^{n-1} (d_{n-j}^*)^{\beta_{n-j}} \right) \Omega = \left(\prod_{j=1}^{n} (d_j d_j^*)^{1-\beta_j} \right) \phi_\beta = \phi_\beta,$$

where we used $d_j d_j^* d_k^* \Omega = d_k^* \Omega$ for $j \neq k$ in the last step. Finally, we will show that $A_{\alpha,\beta} \phi_{\tilde{\alpha}} = 0$ for $\tilde{\alpha} \neq \alpha$. We have

$$(99) \qquad A_{\alpha,\beta} \phi_{\tilde{\alpha}} = \left(\prod_{j=1}^{n} (d_j d_j^*)^{1-\beta_j} \right) \left(\prod_{j=0}^{n-1} (d_{n-j}^*)^{\beta_{n-j}} \right) \left(\prod_{j=1}^{n} d_j^{\alpha_j} \right) \left(\prod_{j=0}^{n-1} (d_{n-j}^*)^{\tilde{\alpha}_{n-j}} \right) \Omega.$$

Since $\tilde{\alpha} \neq \alpha$ there exists j_0 such that either $(\alpha_{j_0}, \tilde{\alpha}_{j_0}) = (1, 0)$ or $(\alpha_{j_0}, \tilde{\alpha}_{j_0}) = (0, 1)$. In the first case d_{j_0} will commute (up to signs) with all the terms to its right and thus annihilate Ω. In the second case $d_{j_0}^*$ will appear twice in (99) and commutes (again up to signs) with all the terms in between its two appearances. As $(d_{j_0}^*)^2 = 0$, we again get $A_{\alpha,\beta} \phi_{\tilde{\alpha}} = 0$. $\qquad \square$

A matrix $W \in \mathbb{C}^{2n \times 2n}$ is called a *Bogoliubov matrix* if W is unitary and

$$(100) \qquad \overline{W} J W^* = J, \text{ where } J = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

The reason for using this terminology is that, for the finite Fermionic systems considered here, Bogoliubov matrices implement Bogoliubov transformations:

Lemma B.3. *Let $\mathcal{D}$ be a Fermionic system in $\mathcal{H}$ and $W \in \mathbb{C}^{2n \times 2n}$. Then*

$$(101) \qquad \tilde{\mathcal{D}} := W\mathcal{D}$$

is a Fermionic system in $\mathcal{H}$ if and only if W is a Bogoliubov matrix. In this case, the correlation matrices, as defined in (32) above, are related by

$$(102) \qquad \Gamma_\rho^{\tilde{\mathcal{D}}} = W \Gamma_\rho^{\mathcal{D}} W^*$$

for all states ρ in $\mathcal{H}$.

Proof. Note first that (102) follows from a simple linearity argument. Let W be a Bogoliubov matrix and $\mathcal{D}$ a Fermionic system. The latter means

$$(103) \qquad \mathcal{D}\mathcal{D}^* + J (\mathcal{D}\mathcal{D}^*)^t J = \mathbb{1}_{2n}.$$

Note that, given unitarity, the condition (100) is equivalent to $J\overline{W} = WJ$ and $W^t J = JW^*$. Thus

$$(104) \qquad \tilde{\mathcal{D}}\tilde{\mathcal{D}}^* + J(\tilde{\mathcal{D}}\tilde{\mathcal{D}}^*)^t J = W \left(\mathcal{D}\mathcal{D}^* + J(\mathcal{D}\mathcal{D}^*)^t J \right) W^* = W \mathbb{1}_{2n} W^* = \mathbb{1}_{2n},$$

so $\tilde{\mathcal{D}}$ is a Fermionic system.

Conversely, assume that $\mathcal{D}$ and $\tilde{\mathcal{D}} = W\mathcal{D}$ are both Fermionic and consider the decomposition of W into $n \times n$ blocks,

$$(105) \qquad W = \begin{pmatrix} K & L \\ M & N \end{pmatrix}.$$

From the transformation (101) we get that for $j = 1, 2, \ldots, n$,

$$(106) \qquad \tilde{d}_j = \sum_{k=1}^{n} (K_{jk} d_k + L_{jk} d_k^*), \quad \tilde{d}_j^* = \sum_{k=1}^{n} (M_{jk} d_k + N_{jk} d_k^*).$$

By taking the adjoint of the left hand side of (106) and comparing it with the right hand side, using that the elements of a Fermionic system are linearly independent, we get

$$(107) \qquad N_{jk} = \overline{K_{jk}}, \quad M_{jk} = \overline{L_{jk}}, \text{ for all } j, k = 1, 2, \ldots, n,$$

meaning that

(108) $$M = \overline{L} \text{ and } N = \overline{K}.$$

Next, we will prove that W is a unitary matrix, that is the rows are an ONB of $\mathbb{C}^{2n}$. Using (106) and (108), a calculation starting with $\{\tilde{d}_j, \tilde{d}_j^*\} = \mathbb{1}$ proves that the rows of W are unit vectors. Next, for $j \neq k$, two calculations starting from $\{\tilde{d}_j, \tilde{d}_k^*\} = 0$ and $\{\tilde{d}_j, \tilde{d}_k\} = 0$ show that any two rows of W are orthogonal. Thus W is a unitary matrix, which means that $WW^* = \mathbb{1}$. In terms of the blocks of W this means that

(109) $$\begin{aligned} KK^* + LL^* = \mathbb{1}, \quad KL^t + LK^t = 0, \\ \overline{L}K^* + \overline{K}L^* = 0, \quad \overline{L}L^t + \overline{K}K^t = \mathbb{1}. \end{aligned}$$

One checks that this is equivalent to (100). □

In the situation above we say that $\tilde{\mathcal{D}}$ is a Bogoliubov transformation of $\mathcal{D}$. One can easily see that being related by a Bogoliubov transformation is an equivalence relation between Fermionic systems.

As in Section 3.1 we say that a state ρ on $\mathcal{H}$ is Wick with respect to a Fermionic system $\mathcal{D}$ if expectations of products of operators $D(f_j, g_j)$ satisfy (36).

Lemma B.4. *Let $\mathcal{D}$ and $\tilde{\mathcal{D}}$ be Fermionic systems on $\mathcal{H}$ and assume that ρ is Wick with respect to $\mathcal{D}$.*

(a) Let U be the unitary relating $\mathcal{D}$ and $\tilde{\mathcal{D}}$ as in Lemma B.1. Then $U\rho U^*$ is Wick with respect to $\tilde{\mathcal{D}}$.

(b) If $\tilde{\mathcal{D}}$ is a Bogoliubov transformation of $\mathcal{D}$, then ρ is Wick with respect to $\tilde{\mathcal{D}}$.

Proof. (a) For any positive integer m,

(110) $$\left\langle \prod_{j=1}^m \tilde{D}_j \right\rangle_{U\rho U^*} = \left\langle U^* \left(\prod_{j=1}^m \tilde{D}_j \right) U \right\rangle_\rho = \left\langle \prod_{j=1}^m D_j \right\rangle_\rho.$$

From this it is straightforward to check that $U\rho U^*$ is Wick with respect to $\tilde{\mathcal{D}}$.

(b) Since the $\tilde{d}_j$ and $\tilde{d}_j^*$ are linear combinations of the d_j and d_j^*, one can see that, for given $f, g : \{1, \ldots, n\} \to \mathbb{C}$,

(111) $$\tilde{D}(f, g) = D(h, r)$$

for some $h, r : \{1, 2, \ldots, n\} \to \mathbb{C}$. With this the claim follows easily from the definition of Wick's rule. □

The following result states the relation between correlation matrices and Wick's rule.

Lemma B.5. *Let ρ and $\tilde{\rho}$ be states in 2^n-dimensional Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively, satisfying Wick's rule with respect to the Fermionic systems $\mathcal{D}$ and $\tilde{\mathcal{D}}$, respectively. If*

(112) $$\Gamma_\rho^{\mathcal{D}} = \Gamma_{\tilde{\rho}}^{\tilde{\mathcal{D}}},$$

then ρ and $\tilde{\rho}$ are unitary equivalent.

Proof. Using notations as in Section 3.1 (for both $\mathcal{D}$ and $\tilde{\mathcal{D}}$), equality of the correlation matrices gives the equality of the correlations

(113) $$\langle D_j D_k \rangle_\rho = \langle \tilde{D}_j \tilde{D}_k \rangle_{\tilde{\rho}} \text{ for all } 1 \leq j < k \leq n.$$

Thus for any positive integer m,

(114)
$$D^{(\rho,m)} = \tilde{D}^{(\tilde{\rho},m)}.$$

Now Wick's rule implies

(115)
$$\left\langle \prod_{j=1}^m D_j \right\rangle_\rho = \operatorname{pf}\left(D^{(\rho,m)}\right) = \operatorname{pf}\left(\tilde{D}^{(\tilde{\rho},m)}\right) = \left\langle \prod_{j=1}^m \tilde{D}_j \right\rangle_{\tilde{\rho}} = \left\langle \prod_{j=1}^m U^* D_j U \right\rangle_{\tilde{\rho}}$$
$$= \left\langle \prod_{j=1}^m D_j \right\rangle_{U\tilde{\rho}U^*},$$

where we used Lemma B.1 which provides a unitary U such that $\tilde{d}_j = U^* d_j U$ for all $1 \le j \le n$. By Lemma B.2 this implies $\langle A \rangle_\rho = \langle A \rangle_{U\tilde{\rho}U^*}$ and thus

(116)
$$\rho = U\tilde{\rho}U^*$$

$\square$

Our final goal in this Appendix will be to prove the trace identity (38). We will first prove this identity as well as Wick's rule for diagonal product states with respect to the Jordan-Wigner Fermionic operators $\mathcal{C}$ given in (19). The general result (38) will then be reduced to this special case.

Proposition B.6. *Let $\rho^{(diag)} \in \mathcal{B}(\bigotimes_{j=1}^n \mathbb{C}^2)$ be the product state given by*

(117)
$$\rho^{(diag)} = \bigotimes_{j=1}^n \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix},$$

where $\eta_j \in [0,1]$ for $1 \le j \le n$. Then $\rho^{(diag)}$ is Wick with respect to the Jordan-Wigner Fermionic system $\mathcal{C}$. Also,

(118)
$$\Gamma^{\mathcal{C}}_{\rho^{(diag)}} = \operatorname{diag}\{1 - \eta_1, \ldots, 1 - \eta_n, \eta_1, \ldots, \eta_n\}$$

and

(119)
$$\operatorname{Tr} \rho^{(diag)} \log \rho^{(diag)} = \operatorname{tr} \Gamma^{\mathcal{C}}_{\rho^{(diag)}} \log \Gamma^{\mathcal{C}}_{\rho^{(diag)}}.$$

The strategy of the following proof is essentially the 'classical' argument for the proof of Wick's rule for thermal states in free Fermion systems, e.g. [6].

Proof. We first note that the formulas (118) and (119) follow from explicit calculations. For the rest of the proof we will drop the superscript $(diag)$. For odd values of m in formula (36), Wick's rule follows from proving

(120)
$$\left\langle \prod_{j=1}^m c_{r_j}^{\#r_j} \right\rangle_\rho = 0.$$

We will assume that the product of $c^\#$'s is not zero, otherwise there is nothing to prove. Since

(121)
$$aa^*a = a, \quad a^*aa^* = a^*, \quad \text{and} \quad a^\# \sigma^Z = \pm a^\#$$

it is easy to see that

$$(122) \qquad \prod_{j=1}^{m} c_{r_j}^{\# r_j} = \pm \bigotimes_{j=1}^{n} A_j,$$

where $A_j \in \{a_j, a_j^*, a_j a_j^*, a_j^* a_j, \sigma_j^Z, \mathbb{1}\}$, and since m is odd there exists $j_0 \in \{1, 2, \ldots, n\}$ such that $A_{j_0} \in \{a_{j_0}, a_{j_0}^*\}$. Then

$$(123) \qquad \operatorname{Tr} \prod_{j=1}^{m} c_{r_j}^{\# r_j} \rho = \pm \prod_{j=1}^{n} \left(\operatorname{tr} A_j \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix} \right),$$

which vanishes because

$$(124) \qquad \operatorname{tr} a^{\#} \begin{pmatrix} \eta_{j_0} & 0 \\ 0 & 1 - \eta_{j_0} \end{pmatrix} = 0.$$

The proof for even m is more involved. First, assume that $\eta_j \notin \{0, 1\}$ for all $1 \leq j \leq n$. Note that

$$(125) \qquad c_k \rho = \frac{\eta_k}{1 - \eta_k} \rho c_k,$$

because

$$(126) \qquad \sigma^Z \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix} = \begin{pmatrix} \eta_j & 0 \\ 0 & -(1 - \eta_j) \end{pmatrix} = \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix} \sigma^Z,$$

$$(127) \qquad a \begin{pmatrix} \eta_k & 0 \\ 0 & 1 - \eta_k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \eta_k & 0 \end{pmatrix} = \frac{\eta_k}{1 - \eta_k} \begin{pmatrix} \eta_k & 0 \\ 0 & 1 - \eta_k \end{pmatrix} a.$$

Also,

$$(128) \qquad c(f) \rho = \sum_{k=1}^{n} \overline{f_k} c_k \rho = \rho \sum_{k=1}^{n} \overline{f_k} \frac{\eta_k}{1 - \eta_k} c_k = \rho \, c(D_\xi f),$$

where

$$(129) \qquad D_\xi := \operatorname{diag} \{\xi_j : j = 1, 2, \ldots, n\}, \quad \xi_j := \frac{\eta_j}{1 - \eta_j}.$$

Similarly,

$$(130) \qquad c^*(g) \rho = \rho \, c^*(D_\xi^{-1} g).$$

The CAR imply

$$(131) \qquad \{c(f), c(g)\} = 0, \quad \{c(f), c^*(g)\} = \langle f, g \rangle_{\ell^2}.$$

With $C_j := C(f_j, g_j)$, $j = 1, \ldots, n$, write

$$(132) \qquad \left\langle \prod_{j=1}^{m} C_j \right\rangle_\rho = \left\langle \left(\prod_{j=2}^{m} C_j \right) c(f_1) \right\rangle_\rho + \left\langle \left(\prod_{j=2}^{m} C_j \right) c^*(g_1) \right\rangle_\rho$$

For the first term we calculate, using (128) and cyclicity,

$$(133) \qquad \left\langle \left(\prod_{j=2}^{m} C_j \right) c(f_1) \right\rangle_\rho = \left\langle c(D_\xi f_1) \prod_{j=2}^{m} C_j \right\rangle_\rho = \left\langle c(D_\xi f_1) C_m \prod_{j=2}^{m-1} C_j \right\rangle_\rho$$

$$= \left\langle \{c(D_\xi f_1), C_m\} \prod_{j=2}^{m-1} C_j \right\rangle_\rho - \left\langle C_m c(D_\xi f_1) \prod_{j=2}^{m-1} C_j \right\rangle_\rho$$

$$= \langle D_\xi f_1, g_m \rangle_{\ell^2} \left\langle \prod_{j=2}^{m-1} C_j \right\rangle_\rho - \left\langle C_m c(D_\xi f_1) \prod_{j=2}^{m-1} C_j \right\rangle_\rho,$$

where also (131) was used. We proceed by applying this argument iteratively to the second term in (133), commuting $c(D_\xi f)$ with each of the C_j, and eventually conclude that (133) coincides with

$$(134) \qquad \sum_{k=2}^{m} (-1)^k \langle D_\xi f_1, g_k \rangle_{\ell^2} \left\langle \prod_{\substack{j=2 \\ j \neq k}}^{m} C_j \right\rangle_\rho - \left\langle \left(\prod_{j=2}^{m} C_j \right) c(D_\xi f_1) \right\rangle_\rho$$

Defining $\tilde{f}_1 := (\mathbb{1} + D_\xi) f_1$, the outcome of this calculation can be rewritten as

$$(135) \qquad \left\langle \left(\prod_{j=1}^{m} C_j \right) c(\tilde{f}_1) \right\rangle_\rho = \sum_{k=2}^{m} (-1)^k \langle D_\xi (\mathbb{1} + D_\xi)^{-1} \tilde{f}_1, g_k \rangle_{\ell^2} \left\langle \prod_{\substack{j=2 \\ j \neq k}}^{m} C_j \right\rangle_\rho$$

Now

$$(136) \qquad \langle D_\xi (\mathbb{1} + D_\xi)^{-1} \tilde{f}_1, g_k \rangle_{\ell^2} = \langle D_\eta \tilde{f}_1, g_k \rangle, \quad \text{where} \quad D_\eta := \text{diag}\{\eta_j, j = 1, 2, \ldots, n\}$$

and, as we can explicitly check that $\langle c_j^* c_j \rangle_\rho = \eta_j$, the latter coincides with

$$(137) \qquad \left\langle C_k \, c(\tilde{f}_1) \right\rangle_\rho = \langle c^*(g_k) c(\tilde{f}_1) \rangle_\rho$$

$$= \sum_{j=1}^{n} \overline{\tilde{f}_{1j}} g_{kj} \langle c_j^* c_j \rangle_\rho = \langle D_\eta \tilde{f}_1, g_k \rangle.$$

Thus equation (135) becomes

$$(138) \qquad \left\langle \left(\prod_{j=1}^{m} C_j \right) c(\tilde{f}_1) \right\rangle_\rho = \sum_{k=2}^{m} (-1)^k \langle C_k \, c(\tilde{f}_1) \rangle_\rho \left\langle \prod_{\substack{j=2 \\ j \neq k}}^{m} C_j \right\rangle_\rho$$

By applying similar steps (which we omit) to the second term of (132), and introducing $\tilde{g}_1 = (\mathbb{1} + D_\xi^{-1}) g_1$ in the process, we get

$$(139) \qquad \left\langle \left(\prod_{j=1}^{m} C_j \right) c^*(\tilde{g}_1) \right\rangle_\rho = \sum_{k=2}^{m} (-1)^k \langle C_k \, c^*(\tilde{g}_1) \rangle_\rho \left\langle \prod_{\substack{j=2 \\ j \neq k}}^{m} C_j \right\rangle_\rho$$

By substituting the results (138) and (139) into (132), we obtain Wick's rule

$$(140) \qquad \left\langle \prod_{j=1}^{m} C_j \right\rangle_\rho = \sum_{j=2}^{m} (-1)^k \langle C_k C_1 \rangle_\rho \left\langle \prod_{\substack{j=2 \\ j \neq k}}^{m} C_j \right\rangle_\rho$$

for the case of even m (after renaming $\tilde{f}_j$ and $\tilde{g}_j$ as f_j and g_j).

Finally, in the general case where $\eta_j \in [0,1]$ for all $j = 1, 2, \ldots, n$, there exists a sequence

$$
(141) \qquad \rho_n = \bigotimes_{j=1}^{n} \begin{pmatrix} \eta_j^{(n)} & 0 \\ 0 & 1 - \eta_j^{(n)} \end{pmatrix}
$$

where $\eta_j^{(n)} \notin \{0,1\} \to \eta_j$ as $n \to \infty$ and thus $\rho_n \to \rho$. Now the fact that ρ_n is Wick with respect to $\mathcal{C}$ carries over to the limit. $\qquad \square$

We can now prove the following fundamental relation, which was stated earlier as the second part of Lemma 3.1.

Proposition B.7. *Let $\mathcal{D}$ be a Fermionic system in $\mathcal{H}$ and ρ a state in $\mathcal{H}$ which is Wick with respect to $\mathcal{D}$ and such that the correlation matrix $\Gamma_\rho^{\mathcal{D}}$ is real. Then*

$$
(142) \qquad \operatorname{Tr} \rho \log \rho = \operatorname{tr} \Gamma_\rho^{\mathcal{D}} \log \Gamma_\rho^{\mathcal{D}}.
$$

Proof. The real-valued correlation matrix $\Gamma_\rho^{\mathcal{D}}$ is diagonalizable by an orthogonal Bogoliubov matrix W (using the general form (33) of a correlation matrix for the real case, this follows by the same argument as discussed in Section 2 for diagonalizing $\tilde{M}$).

Conjugation with $\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ shows that $\Gamma_\rho^{\mathcal{D}}$ is unitarily equivalent to $\mathbb{1} - \Gamma_\rho^{\mathcal{D}}$, so that its spectrum is symmetric to $1/2$ and thus

$$
(143) \qquad W^t \Gamma_\rho^{\mathcal{D}} W = \operatorname{diag}(1 - \eta_1, \ldots, 1 - \eta_n, \eta_1, \ldots, \eta_n) =: D_\eta
$$

for real numbers η_j. In fact we have $\eta_j \in [0,1]$: For the Fermionic system $\tilde{\mathcal{D}} = W\mathcal{D}$, we get by (102) that $\Gamma_\rho^{\tilde{\mathcal{D}}} = D_\eta$ and thus $\eta_j = \operatorname{Tr} \tilde{d}_j^* \tilde{d}_j \rho \in [0,1]$ because $\tilde{d}_j^* \tilde{d}_j$ is an orthogonal projection.

Consider the product state $\rho^{(diag)}$ of Proposition B.6 with this choice of the numbers η_j. An explicit calculation shows $\Gamma_{\rho^{(diag)}}^{\mathcal{C}} = D_\eta$, which is therefore equal to $\Gamma_\rho^{\tilde{\mathcal{D}}}$. Also, $\rho^{(diag)}$ is Wick with respect to $\mathcal{C}$ by Proposition B.6 and ρ is Wick with respect to $\tilde{\mathcal{D}}$ by assumption and Lemma B.4(b). Thus, using Lemma B.5, we get that ρ and $\rho^{(diag)}$ are unitarily equivalent. As we have also shown that $\Gamma_\rho^{\mathcal{D}}$ is unitarily equivalent to $\Gamma_{\rho^{(diag)}}^{\mathcal{C}}$, (142) now follows from (119). $\qquad \square$

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