Lower bound for the number of critical points of minimal spectral k-partitions for k large.

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Abstract

In a recent paper with Thomas Hoffmann-Ostenhof, we proved that the number of critical points ν_k in the boundary set of a k-minimal partition tends to $+\infty$ as $k \to +\infty$. In this note, we show that ν_k increases linearly with k as suggested by a hexagonal conjecture about the asymptotic behavior of the energy of these minimal partitions. As the original proof by Pleijel, this involves Faber-Krahn's inequality and Weyl's formula, but this time, due to the magnetic characterization of the minimal partitions, we have to establish a Weyl's formula for Aharonov-Bohm operator controlled with respect to a k-dependent number of poles.

1 Introduction

We consider the Dirichlet Laplacian in a bounded regular domain $\Omega \subset \mathbb{R}^2$. In [10] we have analyzed the elations between the nodal domains of the real-valued eigenfunctions of this Laplacian and the partitions of Ω by k disjoint open sets D_i which are minimal in the sense that the maximum over the D_i's of the ground state energy (or smallest eigenvalue) of the Dirichlet realization of the Laplacian in D_i is minimal. We denote by $(\lambda_j(\Omega))_{j\in\mathbb{N}}$ the non decreasing sequence of its eigenvalues and by ϕ_j some associated orthonormal basis of real-valued eigenfunctions. The groundstate ϕ_1 can be chosen to be strictly positive in Ω, but the other eigenfunctions ϕ_j ($j > 1$) must have non empty zeroset in Ω. By the zero-set of a real-valued continuous function u on $\overline{\Omega}$, we mean $N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}}$ and call the components of $\Omega \setminus N(u)$ the nodal domains of u. The number of nodal domains of u is called $\mu(u)$. These $\mu(u)$ nodal domains define a k-partition of Ω, with $k = \mu(u)$.

We recall that the Courant nodal Theorem [6] says that, for $k \geq 1$, and if $E(\lambda_k)$ denotes the eigenspace associated with λ_k , then, for all real-valued $u \in E(\lambda_k) \setminus \{0\}$, $\mu(u) \leq k$.

A theorem due to Pleijel [15] in 1956 says that this cannot be true when the dimension (here we consider the $2D$-case) is larger than one. In the next section, we describe the link of these results with the question of spectral minimal partitions which were introduced by Helffer–Hoffmann-Ostenhof–Terracini [10].

2 Minimal spectral partitions

We now introduce for $k \in \mathbb{N}$ ($k \geq 1$), the notion of k-partition. We call k-**partition** of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of mutually disjoint sets in Ω. We denote by $\mathfrak{O}_k(\Omega)$ the set of open connected partitions of Ω. We now introduce the notion of energy of the partition $\mathcal{D}$ by

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i)\,. \tag{2.1}$$

Then we define for any k the minimal energy in Ω by

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathfrak{O}_k} \Lambda(\mathcal{D}). \tag{2.2}$$

and call $\mathcal{D} \in \mathfrak{O}_k$ a minimal k-partition if $\mathfrak{L}_k = \Lambda(\mathcal{D})$. We associate with a partition its **boundary set**:

$$N(\mathcal{D}) = \overline{\cup_i \left(\partial D_i \cap \Omega\right)}\,. \tag{2.3}$$

The properties of the boundary of a minimal partition are quite close to the properties of nodal sets can be described in the following way:

(i) Except for finitely many distinct $X_i \in \Omega \cap N$ in the neighborhood of which N is the union of $\nu_i = \nu(X_i)$ smooth curves ($\nu_i \geq 3$) with one end at X_i, N is locally diffeomorphic to a regular curve.

(ii) $\partial\Omega \cap N$ consists of a (possibly empty) finite set of points Y_i. Moreover N is near Y_i the union of ρ_i distinct smooth half-curves which hit Y_i.

(iii) N has the **equal angle meeting property**[1]

The X_i are called the critical points and define the set $X(N)$. A particular role is played by $X^{odd}(N)$ corresponding to the critical points for which ν_i is odd.

It has been proved by Conti-Terracini-Verzini (existence) and Helffer–Hoffmann-Ostenhof–Terracini (regularity) (see [10] and references therein) that for any k, there exists a minimal regular k-partition, and moreover that any minimal k-partition has a regular representative[2].

In a recent paper with Thomas Hoffmann-Ostenhof [9], we proved that the number of odd critical points of a minimal k-partition $\mathcal{D}_k$

$$\nu_k := \# X^{odd}(N(\mathcal{D}_k)) \tag{2.4}$$

tends to $+\infty$ as $k \to +\infty$.

In this note, we will show that it increases linearly with k as suggested by the hexagonal conjecture as discussed in [2, 4, 3, 9]. This conjecture says that

$$A(\Omega) \lim_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = \lambda(\mathrm{Hexa}_1)\,, \tag{2.5}$$

where Hexa_1 denotes the regular hexagon of area 1 and $A(\Omega)$ denotes the area of Ω.
Behind this conjecture, there is the idea that k-minimal partitions will look (except at the boundary where one can imagine that pentagons will appear) as the intersection with Ω of a tiling by hexagons of area $\frac{1}{k}A(\Omega)$.
The proof presented here gives not only a better result but is at the end simpler, although based on the deep magnetic characterization of minimal partitions of [8] which will be recalled in the next section.

3 Aharonov-Bohm operators and magnetic characterization.

Let us recall some definitions about the Aharonov-Bohm Hamiltonian in an open set Ω (for short $\mathbf{AB}X$-Hamiltonian) with a singularity at $X \in \Omega$ as considered in [11, 1]. We denote by $X = (x_0, y_0)$ the coordinates of the pole and consider the magnetic potential with flux at X: $\Phi = \pi$, defined in

[1]The half curves meet with equal angle at each critical point of N and also at the boundary together with the tangent to the boundary.

[2]possibly after a modification of the open sets of the partition by capacity 0 subsets.

$\dot{\Omega}_X = \Omega \setminus \{X\}$:

$$\mathbf{A}^X(x,y) = (A_1^X(x,y), A_2^X(x,y)) = \frac{1}{2}\left(-\frac{y-y_0}{r^2}, \frac{x-x_0}{r^2}\right). \qquad (3.1)$$

The $\mathbf{AB}X$-Hamiltonian is defined by considering the Friedrichs extension starting from $C_0^\infty(\dot{\Omega}_X)$ and the associated differential operator is

$$-\Delta_{\mathbf{A}^X} := (D_x - A_1^X)^2 + (D_y - A_2^X)^2 \text{ with } D_x = -i\partial_x \text{ and } D_y = -i\partial_y. \quad (3.2)$$

Let K_X be the antilinear operator $K_X = e^{i\theta_X}\,\Gamma$, with $(x-x_0)+i(y-y_0) = \sqrt{|x-x_0|^2+|y-y_0|^2}\,e^{i\theta_X}$, θ_X such that $d\theta_X = 2\mathbf{A}^X$, and where Γ is the complex conjugation operator $\Gamma u = \bar{u}$. A function u is called K_X-real, if $K_X u = u$. The operator $-\Delta_{\mathbf{A}^X}$ is preserving the K_X-real functions and we can consider a basis of K_X-real eigenfunctions. Hence we only analyze the restriction of the $\mathbf{AB}X$-Hamiltonian to the K_X-real space $L^2_{K_X}$ where

$$L^2_{K_X}(\dot{\Omega}_X) = \{u \in L^2(\dot{\Omega}_X)\;,\; K_X\, u = u\}\,.$$

This construction can be extended to the case of a configuration with ℓ distinct points $X_1, \dots, X_\ell$ (putting a flux π at each of these points). We just take as magnetic potential

$$\mathbf{A}^X = \sum_{j=1}^{\ell} \mathbf{A}^{X_j}\,, \text{ where } X = (X_1, \dots, X_\ell)\,.$$

We can also construct the antilinear operator K_X, where θ_X is replaced by a multivalued-function ϕ_X such that $d\phi_X = 2\mathbf{A}^X$. We can then consider the real subspace of the K_X-real functions in $L^2_{K_X}(\dot{\Omega}_X)$. It was shown in [11] and [1] that the K_X-real eigenfunctions have a regular nodal set (like the eigenfunctions of the Dirichlet Laplacian) with the exception that at each singular point X_j ($j = 1, \dots, \ell$) an odd number of half-lines meet.
The next theorem which is the most interesting part of the magnetic characterization of the minimal partitions given in [8] will play a basic role in the proof of our main theorem.

Theorem 3.1 *[Helffer–Hoffmann-Ostenhof]*
Let Ω be simply connected. If $\mathcal{D}$ is a k-minimal partition of Ω, then, by choosing $(X_1, \dots, X_\ell) = X^{odd}(N(\mathcal{D}))$, $\mathcal{D}$ is the nodal partition of some k-th K_X-real eigenfunction of the Aharonov-Bohm Laplacian associated with $\dot{\Omega}_X$.

4 Analysis of the critical sets in the large limit case

We can now state our main theorem, which improves (2.4) as proved in [9].

Theorem 4.1 (Main theorem)
Let $(\mathcal{D}_k)_{k\in\mathbb{N}}$ be a sequence of regular minimal k-partitions. Then there exists $c_0 > 0$ and k_0 such that for $k \geq k_0$,

$$\nu_k := \# X^{odd}(N(\mathcal{D}_k)) \geq c_0 k\,.$$

Proof
The proof is inspired by the proof of Pleijel's theorem, with the particularity that the operator, which is now the Aharonov-Bohm operator will depend on k. Hence the known Weyl asymptotics for the Aharonov-Bohm operators [13] can not be used here.
For each $\mathcal{D}_k$, we consider the corresponding Aharonov-Bohm operator as constructed in Theorem 3.1.
We come back to the proof of the lower bound of the Weyl's formula but we will make a partition in squares depending on $\lambda = \mathfrak{L}_k$.
We introduce a square Q_p of size $t/\sqrt{\lambda}$ with $t \geq 1$ which will be chosen large enough (independently of k) and will be determined later. Having in mind the standard proof of the Weyl's formula (see for example [7]), we recall the following proposition

Proposition 4.2
If $\mathcal{D}$ is a partition of Ω, then

$$\sum_i n(\lambda, D_i) \leq n(\lambda, \Omega)\,. \tag{4.1}$$

Here $n(\lambda, \Omega)$ is the counting function of the eigenvalues $< \lambda$ of $H(\Omega)$.
This proposition is actually present in the proofs of the asymptotics of the counting function. We will apply this proposition in the case of Aharonov-Bohm operators $H := -\Delta_{\mathbf{A}^X}$ restricted to K_X-real L^2 spaces. $H(D)$ means the Dirichlet realization (obtained via the Friedrichs extension theorem) of $-\Delta_{\mathbf{A}^X}$ in an open set $D \subset \Omega$.

Remark 4.3 *Note that if no pole belongs to D (a pole on ∂D is permitted) and if D is simply connected, then $H(D)$ is unitary equivalent (the magnetic potential can be gauged away) to the Dirichlet Laplacian in D. We refer to [1, 12, 14] for a careful analysis of the domains of the involved operators.*

We now consider a maximal partition of Ω with squares Q_p of size $t/\sqrt{\lambda}$ with the additional rule that the squares should not contain the odd critical points of $\mathcal{D}_k$.
The area $A(\Omega_{k,t,\lambda})$, where $\Omega_{k,t,\lambda}$ is the union of these squares, satisfies

$$A(\Omega_{k,t,\lambda}) \geq A(\Omega) - \ell t^2/\lambda - C(t,\Omega)\frac{1}{\sqrt{k}}\,.$$

The second term on the right hand side estimates from above the area of the squares containing a critical point and the last term takes account of the effect of the boundary.
Note that this lower bound of $A(\Omega_{k,t,\lambda})$ leads to the estimate of the cardinal of the squares using $\#\{Q_p\} = A(\Omega_{k,t,\lambda})\frac{\lambda}{t^2}$. In each of the squares, because (as recalled in Remark 4.3) the magnetic Laplacian is isospectral to the usual Laplacian, we have (after a dilation argument) :

$$n(\lambda, Q_p) = n\left(t,(0,1)^2\right) .$$

Hence we need to find a lower bound of $n(t) := n(t,(0,1)^2)$, the number of eigenvalues less than t^2 for the standard Dirichlet Laplacian in the fixed unit square.
We know, that for any $\epsilon > 0$ there exists t such that

$$n(t) \geq (1-\epsilon)\frac{1}{4\pi}t^2 . \tag{4.2}$$

This leads, using Proposition 4.2 for $H = -\Delta_{\mathbf{A}^X}$ (remember that X is given by the magnetic characterization of $\mathcal{D}_k$) and applying (4.2) in each square, to the lower bound as $k \to +\infty$,

$$k = n(\mathfrak{L}_k, \Omega) \geq \left(\frac{1}{4\pi}(1-\epsilon)t^2\right)\left(A(\Omega) - \ell t^2/\mathfrak{L}_k + o(1)\right)\left(\frac{\mathfrak{L}_k}{t^2}\right) \tag{4.3}$$

Let us recall from [10] the following consequence of Faber-Krahn's inequality

$$A(\Omega)\frac{\mathfrak{L}_k(\Omega)}{k} \geq \pi\mathbf{j}^2 , \tag{4.4}$$

where $\mathbf{j} \sim 2.405$ is the first zero of the first Bessel function.

Dividing (4.3) by k and using (4.4), we get, as $k \to +\infty$

$$1 \geq \frac{\mathbf{j}^2}{4}(1-\epsilon)(1-\frac{\ell}{k}t^2\pi^{-1}\mathbf{j}^{-2})(1+o(1)).$$

If we assume that the number ℓ of critical points satisfies

$$\ell \leq \alpha k \,, \text{ for some } \alpha > 0 \,,$$

we get

$$1 \geq \frac{\mathbf{j}^2}{4}(1-\epsilon)(1-\alpha t^2\pi^{-1}\mathbf{j}^{-2})(1+o(1)). \tag{4.5}$$

We see that if ϵ is small enough (this determines $t = t(\epsilon)$) and αt^2 is small enough such that

$$\frac{\mathbf{j}^2}{4}(1-\epsilon)(1-\alpha t^2\pi^{-1}\mathbf{j}^{-2}) > 1$$

(this gives the condition on α), we will get a contradiction for k large.

As recalled in [9], Euler's formula implies that for a minimal k-partition $\mathcal{D}$ of a simply connected domain Ω the cardinal of $X^{odd}(N(\mathcal{D}))$ satisfies

$$\# X^{odd}(N(\mathcal{D})) \leq 2k - 4\,. \tag{4.6}$$

This estimate seems optimal and is compatible with the hexagonal conjecture, which, for critical points, will read

Conjecture 4.4

$$\lim_{k \to +\infty} \frac{\# X^{odd}(N(\mathcal{D}_k))}{k} = 2\,. \tag{4.7}$$

5 Explicit lower bounds

Looking at the proof of the main theorem, the contradiction is obtained if (4.5) is satisfied. Using the universal lower bound for $n(t)$ (see for example [15]), we have, if $t \geq 2$

$$n(t) > \frac{1}{4\pi} t^2 - \frac{2}{\pi^2} t + \frac{1}{\pi^2}\,. \tag{5.1}$$

We look for $t = t(\epsilon) \geq 2$ such that

$$\frac{1}{4\pi} t^2 - \frac{2}{\pi^2} t + \frac{1}{\pi^2} \geq (1-\epsilon) \frac{1}{4\pi} t^2\,,$$

which leads to the condition

$$\epsilon \frac{1}{4\pi} t^2 - \frac{2}{\pi^2} t + \frac{1}{\pi^2} \geq 0\,. \tag{5.2}$$

We can choose $t(\epsilon) = \max(2, \frac{8}{\epsilon\pi})$. We then get a condition on α through (4.5). For some admissible ϵ, i.e satisfying:

$$\frac{\mathbf{j}^2}{4}(1-\epsilon) > 1\,,$$

the proof works if $\alpha < c_0(\epsilon)$, with $c_0(\epsilon)$ solution of

$$1 = \frac{\mathbf{j}^2}{4}(1-\epsilon)(1 - c_0(\epsilon) t(\epsilon)^2 \pi^{-1} \mathbf{j}^{-2})\,. \tag{5.3}$$

Hence the c_0 announced in the theorem can be chosen as

$$c_0 := \sup_{\epsilon \in (0, 1 - 4/\mathbf{j}^2)} c_0(\epsilon)\,,$$

It remains to determine this sup. Note that $1 - 4/\mathbf{j}^2 \sim 0,36$. Hence we can assume $t(\epsilon) = \frac{8}{\epsilon\pi}$ and get for $c_0(\epsilon)$ the equation

$$c_0(\epsilon) = \epsilon^2 2^{-6}\pi^3\mathbf{j}^2 \left(1 - \frac{4}{\mathbf{j}^2(1-\epsilon)}\right) . \tag{5.4}$$

But $c_0(\epsilon)$ being 0 at the ends of the interval $(0, 1-4/\mathbf{j}^2))$, the maximum is obtained inside by looking at the zero of the derivative with respect to ϵ. We get

$$\epsilon_{max} = (1-\mathbf{j}^{-2}) - \sqrt{(1-\mathbf{j}^{-2})^2 - (1-4\mathbf{j}^{-2})} = (1-\mathbf{j}^{-2}) - \mathbf{j}^{-2}\sqrt{1+2\mathbf{j}^2} . \tag{5.5}$$

and

$$c_0 = 2^{-6}\mathbf{j}^{-2}\pi^3 \left((\mathbf{j}^4 + 10\mathbf{j}^2 - 2) - 2(2\mathbf{j}^2+1)\sqrt{1+2\mathbf{j}^2}\right) . \tag{5.6}$$

Numerics with $\mathbf{j}$ replaced by its approximation gives $c_0 \sim 0.014$. This is extremely small and very far from from the conjectured value 2 !

Remark 5.1 *One can actually in* (5.6) *replace* $\mathbf{j}^2$ *by* $\frac{A(\Omega)}{\pi} \liminf \frac{\mathfrak{L}_k}{k}$. *The constant* $\mathbf{j}^2$ *appears indeed only through* (4.4). *Because of the monotonicity of* c_0 *as a function of* $\mathbf{j}^2$ *which results of the definition of* c_0 *as a sup. any improvement of a lower bound for* $\frac{A(\Omega)}{\pi} \liminf \frac{\mathfrak{L}_k}{k}$ *will lead to a corresponding improvment of* c_0.

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