# ON THE PHASE-SPACE DISTRIBUTION OF BLOCH EIGENMODES FOR PERIODIC POINT SCATTERERS

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Abstract. Consider the 3-dimensional Laplacian with a potential described by point scatterers placed on the integer lattice. We prove that for Floquet-Bloch modes with fixed quasi-momentum satisfying a certain Diophantine condition, there is a subsequence of eigenvalues of positive density whose eigenfunctions exhibit equidistribution in position space and localisation in momentum space. This result complements the result of Ueberschär and Kurlberg [15] who show momentum localisation for zero quasi-momentum in 2-dimensions, and is the first result in this direction in 3-dimensions.

## 1. Introduction

The phase space distribution of quantum eigenfunctions for large energies remains in general an unsolved problem - specifically, one would like to know whether the eigenfunctions of a given system exhibit equidistribution or some degree of localisation (or indeed both). We are motivated by the physical problem concerning propagation through a cubic crystal lattice of scatterers. It is well known that when considering a scattering problem in which the wavelength is much larger than the radius of the scatterer, we can replace the scattering potential with a Dirac $\delta$ point potential. This approach is perhaps most famously used in the 1-dimensional Kronig-Penney model [5] which considers the one dimensional Schrödinger equation with a Dirac comb potential. Periodic problems of this sort can be tackled with Floquet-Bloch theory which allows us to reduce a periodic problem in $\mathbb{R}^d$ to a family of quasiperiodic problems on $\mathbb{T}^d$ parametrised by their Bloch vector or quasimomentum $k \in \mathbb{T}^d$.

For zero quasimomentum the problem of limiting phase space distributions has been studied in two dimensions by Rudnick and Ueberschär [9], and Ueberschär and Kurlberg [15, 6], who showed that almost all eigenfunctions equidistribute in position space for all tori, and that almost all eigenfunctions *either* equidistribute or localise in momentum space dependent on the diophantine properties of the ratio of side lengths. These results were partially generalised to three dimensions by Yesha [16, 17] who showed that for the cubic torus that *all* eigenfunctions equidistribute in position space, and that almost all eigenfunctions equidistribute in phase space. In this paper we aim to generalise the results on the cubic torus to include nonzero quasimomentum.

Problems of this type have been studied extensively in the Quantum Chaos literature since Šeba [14] who considered a rectangular billiard with a point scatterer at

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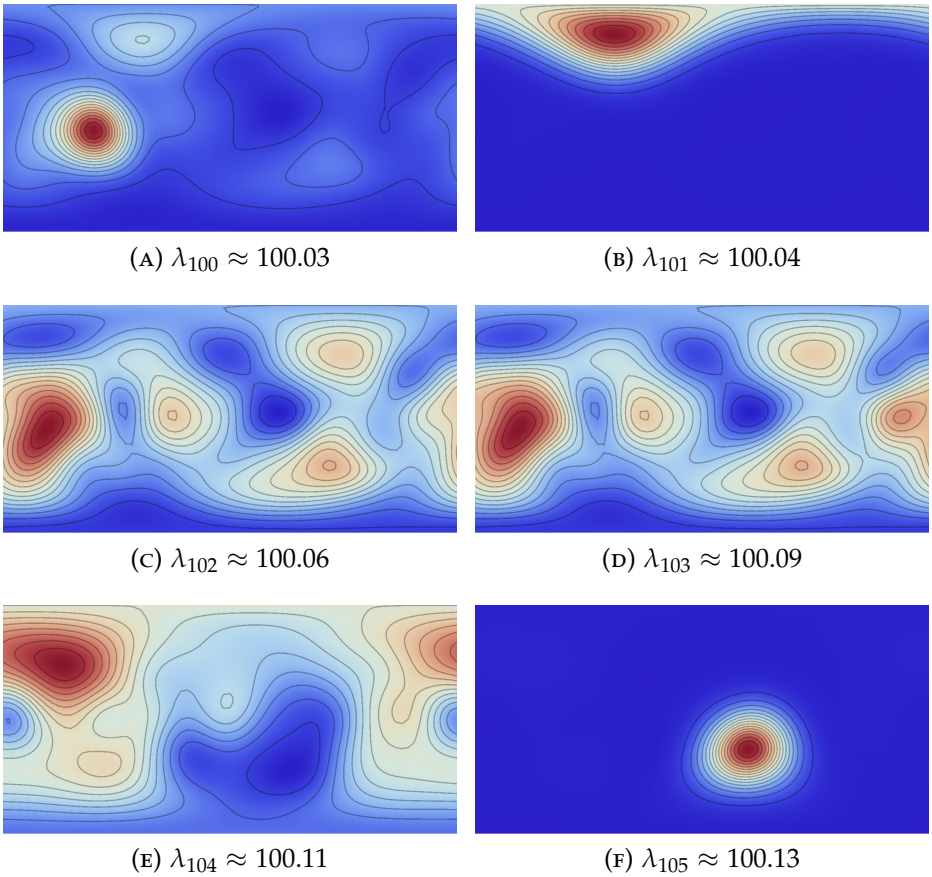
some given point. The Šeba billiard was constructed as an example of an intermediate system, meaning one that is classically integrable (the point scatterer affects only a zero measure set of trajectories) yet exhibits properties typical of chaotic systems [3, 10, 11, 12], this is interesting in view of Shnirelman's theorem [2, 13, 18] which states that for classically ergodic systems a density one subsequence of eigenfunctions equidistributes in phase space, yet when the classical dynamics is integrable eigenfunctions tend to localise or scar.

We study the Laplacian on $\mathbb{R}^3$ with potential described by point scatterers placed on $2\pi\mathbb{Z}^3$ which is described by the formal operator

$$-\Delta + c \sum_{j \in 2\pi\mathbb{Z}^3} \delta_{j+x_0}. \tag{1}$$

This operator is unitarily equivalent via a gauge transformation to a direct integral over quasimomenta $k$. That is, we can instead consider a related quasiperiodic problem on the torus which is then realised via Von Neumann self-adjoint extension theory [1]. We first show that almost all of the eigenfunctions of this operator equidistribute in position space. We then prove that there is a positive density sequence of eigenfunctions which do not equidistribute in momentum space, specifically we can find a subsequence that partially localises in a given direction.

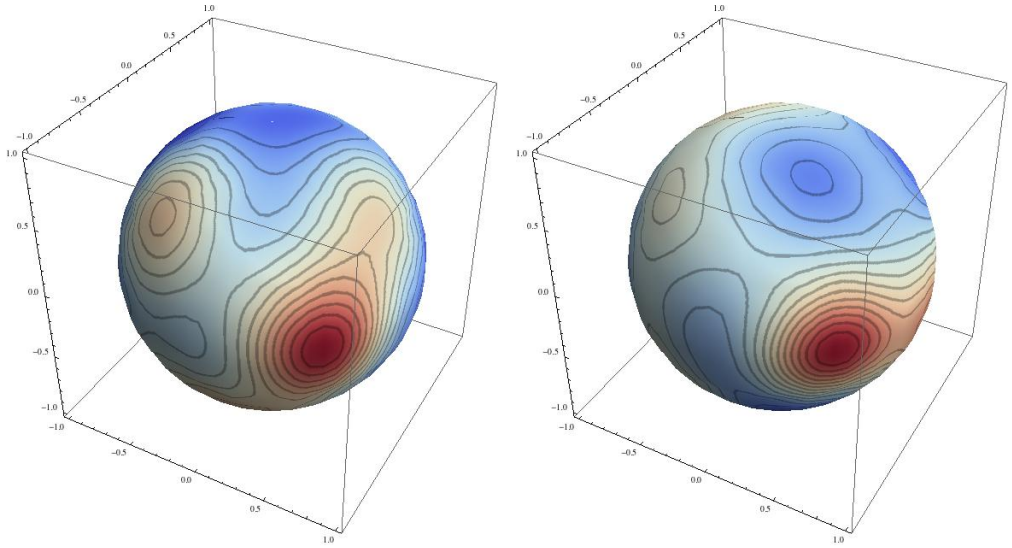


(A) $\lambda_{100} \approx 100.03$ (B) $\lambda_{101} \approx 100.04$

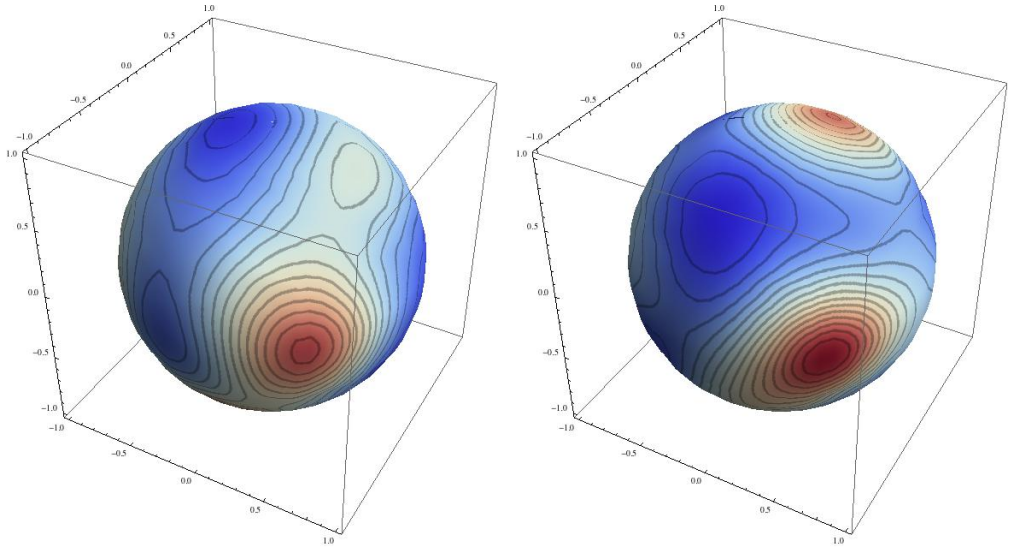
(C) $\lambda_{102} \approx 100.06$ (D) $\lambda_{103} \approx 100.09$

(E) $\lambda_{104} \approx 100.11$ (F) $\lambda_{105} \approx 100.13$

FIGURE 1. A sequence of six consecutive eigenfunction density plots on the plane $(\theta, \phi)$ showing the distribution of momentum directions. We use fixed quasimomentum $k = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{5}})$.



(A) $\lambda_{14322} \approx 203.630$ (B) $\lambda_{23985} \approx 292.147$



(C) $\lambda_{45414} \approx 454.925$ (D) $\lambda_{65109} \approx 583.445$

FIGURE 2. A collection of non-consecutive eigenfunctions in momentum space with eigenvalue $\lambda$ showing partial localisation in the fixed direction $(1,-1,0)$. We again use fixed quasimomentum $k = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{5}})$.

## 2. SETUP

Consider the positive operator $-\Delta_k$ on $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z}^3)$ defined by

$$\Delta_k = \left(\frac{\partial}{\partial x} + ik_1\right)^2 + \left(\frac{\partial}{\partial y} + ik_2\right)^2 + \left(\frac{\partial}{\partial z} + ik_3\right)^2. \tag{2}$$

The eigenfunctions of this operator are the complex exponentials

$$\frac{1}{(2\pi)^{3/2}} e^{i\langle \xi, x\rangle} \tag{3}$$

with eigenvalue $|\xi + k|^2$, $\xi \in \mathbb{Z}^3$. We will write $\mathcal{N} = \{n_k \,|\, k \in \mathbb{N}\}$ to denote the ordered sequence of these eigenvalues. Equivalently, we could consider the standard Laplacian on $\mathbb{T}^3$ on functions that satisfy the quasiperiodic boundary conditions $\psi(x+\gamma) = e^{i\langle\gamma,k\rangle}\psi(x)$ for $\gamma$ in $2\pi\mathbb{Z}^3$. In this case the eigenfunctions are proportional to the exponentials $e^{i\langle\xi+k,x\rangle}$ and again have corresponding eigenvalue $|\xi+k|^2$ - it turns out that the first formulation is more convenient in our case. It is worth noting that this operator occurs naturally when considering the Laplacian on $\mathbb{R}^3$ with some periodic potential. It is known that provided $V(x+\gamma) = V(x)$ for all $\gamma \in 2\pi\mathbb{Z}^3$ then the operator on $\mathbb{R}^3$ given by $-\Delta + V(x)$ has a direct integral decomposition into operators on $\mathbb{T}^3$ of the form $-\Delta_k + V(x)$. Full details of this procedure for a general operator can be found in [8]. We consider the perturbation of the operator $-\Delta_k$ by a $\delta$ potential at a given point $x_0 \in \mathbb{T}^3$. We realise the perturbed operator

$$H_k = -\Delta_k + \delta_{x_0} \tag{4}$$

via self-adjoint extension theory. Details of this calculation can be found in e.g. [**?**]. The idea is that if we restrict our operator to functions vanishing at the point $x_0$, it should act like $-\Delta_k$. This operator is then symmetric but not self-adjoint, so we extend the domain of functions in such a way that self-adjointness is regained. If we define the restricted Laplacian, $-\Delta_0 := -\Delta\,|_{\mathcal{D}_0}$ with

$$\mathcal{D}_0 := C^\infty(\mathbb{T}^3/\{x_0\}), \tag{5}$$

then the deficiency indices are $(1,1)$ and the deficiency elements are the Green's functions, $G_{\pm i}(x, x_0)$, where we define $G_\lambda$ by

$$G_\lambda(x,x_0) := (\Delta_k + \lambda)^{-1}\delta(x-x_0) \overset{L^2}{=} -\frac{1}{8\pi^3}\sum_{\xi\in\mathbb{Z}^3}\frac{e^{i\langle\xi,\,x-x_0\rangle}}{|\xi+k|^2-\lambda}. \tag{6}$$

There therefore exists a 1-parameter family of self-adjoint extensions parametrised by $\phi$ which we denote by $\Delta_{k,\phi}$. The domains of these operators consist of functions $f$ such that

$$f(x) = C\big(\cos(\phi/2)\frac{1}{4\pi|x-x_0|} + \sin(\phi/2)\big) + o(1) \tag{7}$$

as $x \to x_0$. The domain of $\Delta_{k,\phi}$ can be written

$$\mathcal{D}_\phi = \Big\{g + cG_i(\cdot,x_0) + ce^{i\phi}G_{-i}(\cdot,x_0) \mid g \in \mathcal{D}_0, c \in \mathbb{C}, \phi \in (-\pi,\pi)\Big\}, \tag{8}$$

and the action of $\Delta_{k,\phi}$ is given by

$$-\Delta_{k,\phi}f = -\Delta_k g + ciG_i(\cdot,x_0) - ce^{i\phi}iG_{-i}(\cdot,x_0). \tag{9}$$

The new perturbed eigenvalues are given by solutions of the equation

$$\sum_{\xi\in\mathbb{Z}^3}\left(\frac{1}{|\xi+k|^2-\lambda} - \frac{|\xi+k|^2}{|\xi+k|^4+1}\right) = c_0\tan(\phi/2), \tag{10}$$

where

$$c_0 = \sum_{\xi\in\mathbb{Z}^3}\frac{1}{|\xi+k|^4+1}. \tag{11}$$

The set of perturbed eigenvalues will be denoted by $\Lambda$.

## 3. Statement of Results

We state the main results as two separate theorems, the first concerning pure position observables, the second concerning full phase space observables. To deal with phase space we first need to define quantisation. We follow the approach used in [17]. Consider a classical symbol $a \in C^\infty(S^*\mathbb{T}^3)$, where $S^*\mathbb{T}^3 \simeq \mathbb{T}^3 \times S^2$. We define the quantisation $\mathrm{Op}(a)$ by

$$(\mathrm{Op}(a)f)(x) = \sum_{\xi \in \mathbb{Z}^3} \mathrm{e}^{\mathrm{i}\langle \xi, x\rangle} a(x, \overline{\xi + k}) \hat{f}(\xi), \tag{12}$$

where we use the notation $\overline{\xi} = \frac{\xi}{|\xi|}$. We can then expand $a$ in functions $\mathrm{e}_{\zeta,l,m}(x,\xi) = Y_{l,m}(\overline{\xi})\mathrm{e}^{\mathrm{i}\langle \zeta, x\rangle}$, where $Y_{l,m}(\xi)$ is the (normalised) spherical harmonic of degree $l$ and order $m$. Specifically we consider some finite polynomial $P$ defined by

$$P(x,\xi) = \sum_{|\zeta| \le N_1} \sum_{l \le N_2} \sum_{|m| \le l} c_{\zeta,l,m} \mathrm{e}_{\zeta,l,m}(x,\xi), \tag{13}$$

and claim that for all $a \in C^\infty(S^*\mathbb{T}^3)$ there exist $N_1$ and $N_2$ such that for all $(x,\xi) \in S^*\mathbb{T}^3$ and multi-indices $\alpha$ with $|\alpha| < 2$ we have

$$|\partial_x^\alpha (a(x,\xi) - P(x,\xi))| < \epsilon. \tag{14}$$

In light of this it suffices to prove our theorem only for these finite polynomials (see [17] for details). We are now able to state the main results. Let $\Lambda$ denote the sequence of perturbed eigenvalues.

**Definition 3.1.** A vector $k \in \mathbb{R}^d$ is said to be Diophantine of type $\kappa$ if there exists a constant $C$ such that for all $m \in \mathbb{R}^d, q \in \mathbb{N}$ we have

$$\max_j \left| k_j - \frac{m_j}{q} \right| > \frac{C}{q^\kappa}. \tag{15}$$

The first theorem concerns position space equidistribution and is proved in Section 5.

**Theorem 3.1.** *Fix $\phi \in (-\pi, \pi)$. Assume the components of $(1,k)$ are linearly independent over $\mathbb{Q}$. Then, there is a density one subset $\Lambda' \subset \Lambda$ such that for all observables $a \in C^\infty(\mathbb{T}^3)$ we have*

$$\lim_{\lambda \to \infty} \langle a(x) g_\lambda(x), g_\lambda(x)\rangle = \frac{1}{8\pi^3} \int_{\mathbb{T}^3} a(x)\, dx \tag{16}$$

*with $\lambda \in \Lambda'$.*

The second theorem concerns simultaneous equidistribution in position space and partial localisation in momentum space and is proved in Section 6.

**Theorem 3.2.** *Fix $\phi \in (-\pi, \pi)$. Let $k$ be diophantine of type $\kappa \in [4/3, 2)$ and assume the components of $(1,k)$ are linearly independent over $\mathbb{Q}$. Then, for all $\epsilon > 0$ there is a subset $\Lambda_\epsilon \subset \Lambda$ of density at least $1 - \epsilon$ such that for all subsequences $(\lambda_n)_{n \in \mathbb{N}}$ there exists a further subsequence $(\lambda_{n_j})_{j \in \mathbb{N}}$ such that for all observables $a \in C^\infty(S^*\mathbb{T}^3)$ we have*

$$\lim_{j \to \infty} \langle \mathrm{Op}(a(x,\xi)) g_{\lambda_{n_j}}(x), g_{\lambda_{n_j}}(x)\rangle = \frac{1}{\mathrm{vol}(S^*\mathbb{T}^3)} \int_{S^*\mathbb{T}^3} a(x,\xi)\, dx\, d\mu(u) \tag{17}$$

*where $\mu$ has a positive proportion of its mass supported on a finite number of points.*

## 4. Truncation

In order to consider only finite sums we define a truncated Green's function. Define $A(\lambda, L)$ by

$$A(\lambda, L) = \{\xi \in \mathbb{Z}^3 : ||\xi + k|^2 - \lambda| < L\}, \tag{18}$$

we then define the truncated Green's function by

$$G_{\lambda,L}(x, x_0) = -\frac{1}{8\pi^3} \sum_{\xi \in A(\lambda,L)} \frac{e^{i\langle \xi, x - x_0\rangle}}{|\xi + k|^2 - \lambda}. \tag{19}$$

We want to show that for $L = \lambda^{-\delta}$ for some $\delta$ this truncation is a good approximation for large $\lambda$. We first need a lower bound on the full Green's function. Define

$$\mathcal{N}(x) = \{n \in \mathcal{N} \mid n \leq x\}. \tag{20}$$

If the components of $(1, k)$ are linearly independent over $\mathbb{Q}$ then we know the asymptotic behaviour of $\mathcal{N}(x)$ to be

$$N(x) = \#\mathcal{N}(x) = \frac{4}{3}\pi x^{3/2} + O(x^{\theta}). \tag{21}$$

It is conjectured that $\theta = \frac{1}{2} + \epsilon$ for all $\epsilon$, and for $k = 0$ the current best explicit bound due to Heath-Brown [4] gives $\theta = \frac{21}{32} + \epsilon$ for all $\epsilon > 0$. For our purposes it is required that $\theta < 1$, in fact we will show in the Appendix that we have $\theta < \frac{3}{4} + \epsilon$ independent of $k$.

**Lemma 4.1.** *Let the components of $(1, k)$ be linearly independent over $\mathbb{Q}$. Then, there is a density one subsequence of eigenvalues $\lambda \in \Lambda$ such that*

$$\|G_\lambda\| \gg \lambda^{1/2-\epsilon}. \tag{22}$$

*Proof.* We have from (21) that

$$\frac{1}{N(x)} \sum_{n_k \leq x} (n_k - n_{k-1}) \asymp \frac{x}{N(x)} \asymp x^{-1/2}. \tag{23}$$

Thus, since $n_k - n_{k-1} > 0$ we must have that for a subsequence of density one that

$$n_{k+1} - n_k \ll n_{k+1}^{-1/2+\epsilon}. \tag{24}$$

Let $n_{k+1} > \lambda > n_k$ and we see

$$\|G_\lambda\|^2 \gg \sum_{n \in \mathcal{N}} \frac{1}{(n - \lambda)^2} > \frac{1}{(n_{k+1} - \lambda)^2} > \frac{1}{(n_{k+1} - n_k)^2} \gg n_{k+1}^{1-\epsilon} > \lambda^{1-\epsilon}. \tag{25}$$

□

**Lemma 4.2.** *Let $L = \lambda^{-\delta}$, then $\|g_{\lambda,L} - g_\lambda\| \to 0$ as $\lambda \to \infty$.*

*Proof.* First we see that

$$\|g_{\lambda,L} - g_\lambda\| = \left\| \frac{G_\lambda}{\|G_\lambda\|} - \frac{G_{\lambda,L}}{\|G_{\lambda,L}\|} \right\| \tag{26}$$

$$= \left\| \frac{G_\lambda}{\|G_\lambda\|} - \frac{G_{\lambda,L}}{\|G_\lambda\|} + \frac{G_{\lambda,L}}{\|G_\lambda\|} - \frac{G_{\lambda,L}}{\|G_{\lambda,L}\|} \right\| \tag{27}$$

$$\leq \frac{\|G_\lambda - G_{\lambda,L}\|}{\|G_\lambda\|} + \|G_{\lambda,L}\| \left| \frac{1}{\|G_\lambda\|} - \frac{1}{\|G_{\lambda,L}\|} \right| \tag{28}$$

$$\leq 2\frac{\|G_\lambda - G_{\lambda,L}\|}{\|G_\lambda\|}. \tag{29}$$

Then we have

$$\|G_\lambda - G_{\lambda,L}\|^2 \ll \sum_{||\xi+k|^2-\lambda|>L} \frac{1}{(|\xi+k|^2-\lambda)^2}. \tag{30}$$

We evaluate the lattice sum via Abel summation, which tells us that for a smooth function $f$ we have

$$\sum_{n_A<|\xi+k|^2<n_B} f(|\xi+k|^2) = N(n_B)f(n_B) - N(n_A)f(n_{A+1}) - \int_{n_{A+1}}^{n_B} f'(t)N(t)\,dt. \tag{31}$$

Integrating by parts we see

$$\sum_{n_A<|\xi+k|^2<n_B} f(|\xi+k|^2) = 2\pi \int_{n_{A+1}}^{n_B} f(t)t^{1/2}\,dt \tag{32}$$

$$+ O(n_B^\theta f(n_B) - n_A^\theta f(n_{A+1})) + O(\int_{n_{A+1}}^{n_B} |f'(t)|t^\theta\,dt). \tag{33}$$

Applying this to $f(n) = \frac{1}{(n-\lambda)^2}$ with $n_A = n_0$ and $n_B < \lambda - L < n_{B+1}$ we see

$$\sum_{n<\lambda-L} \frac{1}{(n-\lambda)^2} = 2\pi \int_{n_1}^{n_B} \frac{n^{1/2}}{(n-\lambda)^2}\,dn + O\left(\frac{n_B^\theta}{(n_B-\lambda)^2}\right) + O\left(\int_{n_1}^{n_B} \frac{n^\theta}{(\lambda-n)^3}\,dn\right). \tag{34}$$

We can bound the integral by

$$\int_{n_1}^{n_B} \frac{n^{1/2}}{(n-\lambda)^2}\,dn \leq \lambda^{1/2} \int_{n_1}^{n_B} \frac{1}{(n-\lambda)^2}\,dn \tag{35}$$

$$\leq \frac{\lambda^{1/2}}{L} \leq \frac{\lambda^\theta}{L^2}. \tag{36}$$

Similarly we see

$$\frac{n_B^\theta}{(n_B-\lambda)^2} \leq \frac{\lambda^\theta}{L^2}, \tag{37}$$

and also

$$\int_{n_1}^{n_B} \frac{n^\theta}{(\lambda-n)^3} \ll \frac{\lambda^\theta}{L^2}. \tag{38}$$

Now repeating this procedure with $n_A < \lambda + L < n_{A+1}$ and $n_B = \infty$ we obtain

(39)
$$\sum_{n>\lambda-L} \frac{1}{(n-\lambda)^2} = 2\pi \int_{n_{A+1}}^{\infty} \frac{n^{1/2}}{(n-\lambda)^2}\, dn + O\left(\frac{n_A^\theta}{(n_A-\lambda)^2}\right) + O\left(\int_{n_A}^{\infty} \frac{n^\theta}{(\lambda-n)^3}\, dn\right).$$

For the first integral we write

$$\begin{aligned}\int_{n_{A+1}}^{\infty} \frac{n^{1/2}}{(n-\lambda)^2}\, dn &= \int_{n_{A+1}-\lambda}^{\infty} \frac{(s+\lambda)^{1/2}}{s^2}\, ds \\ &\leq \int_L^{\lambda} \frac{(s+\lambda)^{1/2}}{s^2}\, ds + \int_{\lambda}^{\infty} \frac{(s+\lambda)^{1/2}}{s^2}\, ds \\ &\ll \frac{\lambda^{1/2}}{L} \ll \frac{\lambda^\theta}{L^2}. \end{aligned} \tag{40}$$

For the second term we have immediately

$$\frac{n_A^\theta}{(n_A-\lambda)^2} \ll \frac{\lambda^\theta}{L^2}. \tag{41}$$

For the third term we see

$$\begin{aligned}\int_{n_A}^{\infty} \frac{n^\theta}{(\lambda-n)^3}\, dn &= \int_{n_{A+1}-\lambda}^{\infty} \frac{(s+\lambda)^\theta}{s^3}\, ds \\ &\ll \int_{n_{A+1}-\lambda}^{\infty} \frac{s+\lambda}{s^3}\, ds \ll \frac{1}{\lambda}. \end{aligned} \tag{42}$$

Putting all of this together we see

$$\|G_\lambda - G_{\lambda,L}\|^2 \ll \frac{\lambda^\theta}{L^2}, \tag{43}$$

and hence that for the normalised Green's functions

$$\|g_{\lambda,L} - g_\lambda\| \ll \frac{\lambda^{-\frac{1-\theta}{2}}}{L} = \lambda^{-(1-\theta)/2+\epsilon+\delta} \tag{44}$$

which tends to 0 for all $\delta < \frac{1-\theta}{2} - \epsilon$. □

**Corollary 4.1.** *Define $g_{\lambda,L}$ as above with $L = \lambda^{-\delta}$ and $0 < \delta < \frac{1-\theta}{2} - \epsilon$ then*

$$|\langle \mathrm{Op}(e_{\zeta,l,m}) g_{\lambda,L}, g_{\lambda,L}\rangle - \langle \mathrm{Op}(e_{\zeta,l,m}) g_\lambda, g_\lambda\rangle| \to 0. \tag{45}$$

*Proof.* We have

$$|\langle \mathrm{Op}(e_{\zeta,l,m}) g_{\lambda,L}, g_{\lambda,L}\rangle - \langle \mathrm{Op}(e_{\zeta,l,m}) g_\lambda, g_\lambda\rangle| \tag{46}$$
$$\leq |\langle \mathrm{Op}(e_{\zeta,l,m}) g_{\lambda,L}, g_{\lambda,L} - g_\lambda\rangle| + |\langle \mathrm{Op}(e_{\zeta,l,m})(g_\lambda - g_{\lambda,L}), g_\lambda\rangle|. \tag{47}$$

Taking each term and using Cauchy-Schwarz gives

$$|\langle \mathrm{Op}(e_{\zeta,l,m}) g_{\lambda,L}, g_{\lambda,L}\rangle - \langle \mathrm{Op}(e_{\zeta,l,m}) g_\lambda, g_\lambda\rangle| \leq \|\mathrm{Op}(e_{\zeta,l,m})\|_\infty \|g_\lambda - g_{\lambda,L}\| \to 0. \tag{48}$$

□

## 5. Equidistribution in Position Space

The following proposition is key to the proof.

**Proposition 5.1.** *Fix $\zeta \neq 0$ and let $L = \lambda^{-\delta}$ for some $\delta > 0$. Let the components of $(1, k)$ be linearly independent over $\mathbb{Q}$. Then, for a subsequence of $\lambda$ of density one we have*

$$\langle \mathrm{Op}(\mathrm{e}_{\zeta,l,m}) g_{\lambda,L}, g_{\lambda,L} \rangle \to 0. \tag{49}$$

In order to prove this we first need a lemma.

**Lemma 5.1.** *Let the components of $(k, 1)$ be linearly independent over $\mathbb{Q}$ and fix $\zeta \in \mathbb{Z}^3$. Then, there exists some $\epsilon > 0$ such that for all $\xi \in \mathbb{Z}^3$ we have*

$$|2\langle \xi + k, \zeta \rangle + |\zeta|^2| > \epsilon. \tag{50}$$

*Proof.* First write $\xi = a\zeta_\parallel + b\zeta_\perp$, and $\zeta = c\zeta_\parallel$ where $\zeta_\parallel$ and $\zeta_\perp$ are primitive lattice vectors and $a, b, c \in \mathbb{Z}$. Then we assume for contradiction that $|2\langle \xi + k, \zeta \rangle + |\zeta|^2| < \epsilon$. We see then that

$$-\frac{c}{2} - \frac{\epsilon}{2c|\zeta_\parallel|^2} - \frac{\langle k, \zeta_\parallel \rangle}{|\zeta_\parallel|^2} < a < -\frac{c}{2} + \frac{\epsilon}{2c|\zeta_\parallel|^2} - \frac{\langle k, \zeta_\parallel \rangle}{|\zeta_\parallel|^2}. \tag{51}$$

Thus we see that unless the quantity

$$-\frac{c}{2} - \frac{\langle k, \zeta_\parallel \rangle}{|\zeta_\parallel|^2} \tag{52}$$

is an integer, we can always find $\epsilon$ small enough such that this inequality cannot be satisfied by integer $a$. Since the components of $(k, 1)$ are linearly independent over $\mathbb{Q}$, we have for all $m \in \frac{1}{2}\mathbb{Z}$ that

$$\langle k, \zeta_\parallel \rangle - m|\zeta_\parallel|^2 \neq 0, \tag{53}$$

so $\frac{\langle k, \zeta_\parallel \rangle}{|\zeta_\parallel|^2} \notin \frac{1}{2}\mathbb{Z}$, and we conclude that

$$|2\langle \xi + k, \zeta \rangle + |\zeta|^2| > \epsilon. \tag{54}$$

□

*Proof of Proposition 5.1.* First write

$$\begin{aligned}
&|\langle \mathrm{Op}(\mathrm{e}_{\zeta,l,m}) G_{\lambda,L}, G_{\lambda,L} \rangle| \\
&= \frac{1}{64\pi^6 \|G_\lambda\|^2} |\langle \sum_{\xi \in A(\lambda,L)} \frac{\mathrm{e}^{\mathrm{i}\langle \xi, x - x_0 \rangle}}{|\xi + k|^2 - \lambda} \mathrm{e}^{\mathrm{i}\langle \zeta, x \rangle} Y_{l,m}(\overline{\xi + k}), \sum_{\eta \in A(\lambda,L)} \frac{\mathrm{e}^{\mathrm{i}\langle \eta, x - x_0 \rangle}}{|\eta + k|^2 - \lambda} \rangle| \qquad (55) \\
&= \frac{1}{64\pi^6 \|G_\lambda\|^2} \Big| \int_{\tilde{\mathbb{T}}^3} \sum_{\xi, \eta \in A(\lambda,L)} \frac{\mathrm{e}^{\mathrm{i}\langle \eta - \xi, x - x_0 \rangle}}{(|\xi + k|^2 - \lambda)(|\eta + k|^2 - \lambda)} \mathrm{e}^{-\mathrm{i}\langle \zeta, x \rangle} Y^*_{l,m}(\overline{\xi + k})\, dx \Big| \qquad (56)
\end{aligned}$$

Note then by Lemma 5.1 that for $\xi \in A(\lambda, L)$,

$$||\xi + \zeta + k|^2 - \lambda| = ||\xi + k|^2 - \lambda + 2\langle \xi + k, \zeta \rangle + |\zeta|^2| \gg \epsilon \tag{57}$$

so $\xi + \zeta \notin A(\lambda, L)$ for $\lambda$ sufficiently large. Thus the integral in (56) vanishes. □

We are now able to show equidistribution for position space observables.

*Proof of Theorem 3.1.* Let $a \in C^\infty(\mathbb{T}^3)$, then the operator $\mathrm{Op}(a)$ is just given by multiplication by $a$. We consider $a$ to be some finite polynomial

$$a(x) = \sum_{|\zeta|<N} \hat{a}(\zeta)\, \mathrm{e}^{\mathrm{i}\langle \zeta, x\rangle}, \tag{58}$$

and see from Proposition 5.1 that along some density one subsequence of $\lambda$,

$$\begin{aligned} \langle a(x) g_{\lambda,L}, g_{\lambda,L}\rangle &\to \langle \hat{a}(0) g_{\lambda,L}, g_{\lambda,L}\rangle \\ &= \left(\int_{\mathbb{T}^3} a(y)\, \frac{dy}{8\pi^3}\right)\left(\int_{\mathbb{T}^3} |g_{\lambda,L}(x)|^2\, \frac{dx}{8\pi^3}\right) \\ &= \int_{\mathbb{T}^3} a(y)\, \frac{dy}{8\pi^3}. \end{aligned} \tag{59}$$

The result then follows from Corollary 4.1. □

## 6. Localisation in Momentum Space

Throughout this section we will assume $k$ is diophantine of type $\kappa < 2$. Let $a$ be defined by

$$a(x,\xi) = \sum_{|\zeta|\le N_1, l\le N_2, |m|\le l} \hat{a}(\zeta,l,m) \mathrm{e}_{\zeta,l,m}(x,\xi) \tag{60}$$

where $\hat{a}(\zeta,l,m)$ is given by

$$\hat{a}(\zeta,l,m) = \frac{1}{8\pi^3}\int_{S^2}\int_{\mathbb{T}^3} a(x,\xi)\mathrm{e}^{-\mathrm{i}\langle x,\zeta\rangle} Y^*_{l,m}(\bar{\xi})\, dx d\sigma(\xi). \tag{61}$$

We thus have that

$$\begin{aligned} \langle \mathrm{Op}(a) g_{\lambda,L}, g_{\lambda,L}\rangle &\sim \langle \sum_{l,m} \hat{a}(0,l,m)\, \mathrm{Op}(\mathrm{e}_{0,l,m}) g_{\lambda,L}, g_{\lambda,L}\rangle \\ &= \|G_\lambda\|^{-2} \frac{1}{16\pi^6} \sum_{l,m} \sum_{\xi\in A(\lambda,L)} \hat{a}(0,l,m) \frac{Y_{l,m}(\overline{\xi+k})}{(|\xi+k|^2-\lambda)^2} \\ &\asymp \sum_{\xi\in A(\lambda,L)} \int_{S^*\mathbb{T}^3} a(x,\eta)\, dx\, \frac{\delta(\bar{\eta}-\bar{\xi})}{(|\xi+k|^2-\lambda)^2} d\sigma(\eta). \end{aligned} \tag{62}$$

Thus the component of the spectral measure for each fixed $|\xi+k|^2 = m$ on $\mathbb{T}^3 \times S^2$ consists of Leb $\times \delta_{\overline{\xi+k}}$. The full (unnormalised) spectral measure is thus a weighted sum of a growing number of $\delta$ masses that become dense on $S^2$. We aim to show that for a positive density subsequence of $\lambda$, the tails of this sum can be bounded uniformly in $\lambda$ such that a positive proportion of its density will be supported on a finite number of points.

**Lemma 6.1.** *We have that* $\#\{n_i \in \mathcal{N}(T) : n_{i+1} - n_i > G/\sqrt{n_{i+1}}\} < T^{3/2}/G$.

*Proof.* We see that

$$\sum_{n_i\le T} \sqrt{n_{i+1}}(n_{i+1}-n_i) \le \sum_{n_i\le T} (n_{i+1}^{3/2} - n_i^{3/2}) \tag{63}$$

$$\ll T^{3/2}. \tag{64}$$

Thus by Chebyshev's inequality we see

$$\#\{n_i \leq T : s_i = n_{i+1} - n_i > G/\sqrt{n_{i+1}}\} < T^{3/2}/G. \tag{65}$$

□

**Lemma 6.2.** *Given $D > 0$, $E \geq 1$,*

$$\#\{n \in \mathcal{N}(T) : |\mathcal{N}(T) \cap [n - \tfrac{D}{\sqrt{n}}, n + \tfrac{D}{\sqrt{n}}]| > E + 1\} \ll \frac{D(2T)^{3/2}}{E}. \tag{66}$$

*Proof.* We have that

$$\sum_{n \in \mathcal{N}(T)} (|\mathcal{N}(T) \cap [n - \tfrac{D}{\sqrt{n}}, n + \tfrac{D}{\sqrt{n}}]| - 1) \tag{67}$$

$$= \#\{n, m \in \mathcal{N}(T) : m \neq n, \sqrt{n}|n - m| \leq D\} \tag{68}$$

$$\ll \#\{n, m \in \mathcal{N}(2T) \backslash \mathcal{N}(T) : m \neq n, \sqrt{T}|n - m| \leq D\}. \tag{69}$$

Since we assumed $k$ was diophantine, by [7] Theorem 1.6 we have

$$\#\{n, m \in \mathcal{N}(2T) \backslash \mathcal{N}(T) : m \neq n, \sqrt{T}|n - m| \leq D\} \sim D(2T)^{3/2}. \tag{70}$$

Again by Chebyshev's inequality we conclude

$$\#\{n \in \mathcal{N}(T) : |\mathcal{N}(T) \cap [n - \tfrac{D}{\sqrt{n}}, n + \tfrac{D}{\sqrt{n}}] > E + 1\} \ll \frac{D(2T)^{3/2}}{E}. \tag{71}$$

□

**Lemma 6.3.** *For all $A > 1$*

$$\sum_{\substack{n,m \in \mathcal{N}(x) \\ \sqrt{m}|n-m| > A}} \frac{1}{m(n-m)^2} \ll \frac{x^{3/2}}{A^{1/3}}. \tag{72}$$

*Proof.* We first define

$$M(k) := |\{n \in \mathcal{N} : n^{3/2} \in [k, k+1]\}|. \tag{73}$$

Then we deduce an $L^2$ bound on $M(k)$ by

$$\sum_{k \leq T} M(k)^2 = \sum_{k \leq T} |\{m, n \in \mathcal{N} : m^{3/2}, n^{3/2} \in [k, k+1]\}| \tag{74}$$

$$\leq |\{m, n \in \mathcal{N} : m^{3/2}, n^{3/2} \leq T + 1, m^{3/2} - n^{3/2} \in [-1, 1]\}| \tag{75}$$

which again by Theorem 1.6 in [7] gives us

$$\sum_{k \leq T} M(k)^2 \ll T. \tag{76}$$

Note that we can write

$$\sqrt{m}|n - m| = \frac{\sqrt{m}}{\sqrt{m} + \sqrt{n}}(\sqrt{m}|n - m| + \sqrt{n}|n - m|) \geq \frac{\sqrt{m}}{\sqrt{m} + \sqrt{n}}|n^{3/2} - m^{3/2}|,$$

and also that $\sqrt{m}|n-m| < |n^{3/2}-m^{3/2}|$. Hence, we can bound the sum in (72) by

$$\sum_{\substack{n,m\in\mathcal{N}(x)\\ \sqrt{m}|n-m|>A}} \frac{1}{m(n-m)^2} \ll \sum_{\substack{n,m\in\mathcal{N}(x)\\ |n^{3/2}-m^{3/2}|>A}} \frac{(1+\sqrt{\frac{n}{m}})^2}{(n^{3/2}-m^{3/2})^2} \tag{77}$$

$$= \sum_{k=A}^{x^{3/2}} \sum_{\substack{n,m\in\mathcal{N}(x)\\ |n^{3/2}-m^{3/2}|\in[k,k+1]}} \frac{(1+\sqrt{\frac{n}{m}})^2}{(n^{3/2}-m^{3/2})^2}.$$

Now, when $m > n$ we can immediately conclude

$$\sum_{k=A}^{x^{3/2}} \sum_{\substack{n,m\in\mathcal{N}(x)\\ |n^{3/2}-m^{3/2}|\in[k,k+1]}} \frac{(1+\sqrt{\frac{n}{m}})^2}{(n^{3/2}-m^{3/2})^2} < 4\sum_{k=A}^{x^{3/2}} \frac{1}{k^2}|\{m,n\in\mathcal{N}(x) : (n^{3/2}-m^{3/2})\in[k,k+1]\}|$$

$$\leq 4\sum_{k=A}^{x^{3/2}} \frac{1}{k^2} \sum_{m\leq x^{3/2}} M(m)(M(m+k)+M(m+k+1)) \tag{78}$$

$$\ll x^{3/2}\sum_{k=A}^{x^{3/2}} \frac{1}{k^2} \ll \frac{x^{3/2}}{A}.$$

When $m < n$, we see that

$$n^{3/2}-m^{3/2}\in[k,k+1] \implies \left(\frac{n}{m}\right)^{3/2} \leq 1+\frac{k+1}{m^{3/2}}. \tag{79}$$

We know that $m$ is bounded away from zero, say $m > C$, then we must have

$$\left(\frac{n}{m}\right)^{1/2} \leq C^{-1/2}(k+1+C^{3/2})^{1/3}. \tag{80}$$

Repeating the previous argument is this regime yields

$$\sum_{k=A}^{x^{3/2}} \sum_{\substack{n,m\in\mathcal{N}(x)\\ |n^{3/2}-m^{3/2}|\in[k,k+1]}} \frac{(1+\sqrt{\frac{n}{m}})^2}{(n^{3/2}-m^{3/2})^2} \ll \sum_{k=A}^{x^{3/2}} \frac{1}{k^{4/3}}|\{m,n\in\mathcal{N}(x) : (n^{3/2}-m^{3/2})\in[k,k+1]\}|$$

$$\leq \sum_{k=A}^{x^{3/2}} \frac{1}{k^{4/3}} \sum_{m\leq x^{3/2}} M(m)(M(m+k)+M(m+k+1)) \tag{81}$$

$$\ll x^{3/2}\sum_{k=A}^{x^{3/2}} \frac{1}{k^{4/3}} \ll \frac{x^{3/2}}{A^{1/3}}.$$

□

We are now ready to prove the second main theorem.

*Proof of Theorem 3.2.* Define $\mathcal{N}'$ as follows, first remove all points $m$ whose nearest left neighbour is further than $G/\sqrt{m}$, by Lemma 6.1 we are left with a subsequence of density at least $1-1/G$. Now choose $D$ and fix $E$ large enough such that

$$|\{m\in\mathcal{N}(T) : |\mathcal{N}(T)\cap[m-\tfrac{D}{\sqrt{m}},m+\tfrac{D}{\sqrt{m}}]>E+1\}| \leq \frac{T^{3/2}}{G} \tag{82}$$

which is possible by Lemma 6.2. Removing these points leaves us with a subsequence of density at least $1 - 2/G$. Finally, by Lemma 6.3, and Chebyshev's inequality we choose $F$ large enough such that

$$|\{m \in \mathcal{N}(T) : \sum_{\substack{n \in \mathcal{N}(T) \\ \sqrt{m}|n-m|>D}} \frac{1}{(n-m)^2} > F\, m\}| \leq \frac{T^{3/2}}{G}. \tag{83}$$

Removing these points leaves us with a subsequence of density at least $1 - 3/G$. Thus if we consider pure momentum observables and for $m \in \mathcal{N}'$ denote by $\mu_m$ the delta measure on the point corresponding to the direction $\xi + k$ with $|\xi + k|^2 = m$, we see that the unnormalised measure associated to $G_{\lambda_m}$ is

$$\sum_{n \in \mathcal{N}} \frac{\mu_n}{(n-\lambda_m)^2} = \frac{\mu_m}{(m-\lambda_m)^2} + \sum_{\substack{n \in \mathcal{N} \\ 0<|n-m|<\frac{D}{\sqrt{m}}}} \frac{\mu_n}{(n-\lambda_m)^2} + \sum_{\substack{n \in \mathcal{N} \\ |n-m|>\frac{D}{\sqrt{m}}}} \frac{\mu_n}{(n-\lambda_m)^2}. \tag{84}$$

We know that the first term is $\gg m/G$, the second sum has at most $E$ terms, and the third is bounded above by $F\, m$. Thus the normalised measure will have a positive proportion of its mass on a finite number of points. The theorem then follows from compactness of $S^*\mathbb{T}^3$ and by setting $\epsilon = 3/G$ and defining $\Lambda_\epsilon$ by $\lambda_m \in \Lambda_\epsilon \iff m \in \mathcal{N}'$. □

## Appendix A.

**Proposition A.1.** *Let $S(R) = \#\{|\xi + k| < R \mid \xi \in \mathbb{Z}^3\}$ denote the number of shifted lattice points inside a ball of radius $R$. Then we have that*

$$S(R) = \frac{4}{3}\pi R^3 + O(R^{3/2+\epsilon}). \tag{85}$$

*Proof.* We bound the quantity $S(R)$ above and below by sums over the indicator function of a shifted ball convolved with some smooth bump function with smoothing parameter $\delta$. We can then employ Poisson summation and tune $\delta$ in such a way that the error terms vanish. Let $B_k(R)$ denote the ball of radius $R$ centred at $k$, and write $\psi_\delta(x) = \delta^{-3}\psi(x/\delta)$ where $\psi$ is some smooth function with compact support in $B_0(1)$ normalised such that $\hat{\psi}(0) = 1$. Define $S_\delta(R)$ to be the smoothed sum

$$S_\delta(R) = \sum_{x\in\mathbb{Z}^3} \chi_{B_k(R)} * \psi_\delta(x). \tag{86}$$

Note that we have $S(R-\delta) \leq S_\delta(R) \leq S(R+\delta)$. By Poisson summation we see

$$\sum_{x\in\mathbb{Z}^3} \chi_{B_k(R)} * \psi_\delta(x) = \sum_{\xi\in\mathbb{Z}^3} \hat{\chi}_{B_k(R)}(\xi)\hat{\psi}_\delta(\xi). \tag{87}$$

Computing the term $\xi = 0$ yields

$$\int_{\mathbb{R}^3} \chi_{B_k(R)}(x)\,dx = \frac{4}{3}\pi R^3. \tag{88}$$

For $\xi \neq 0$, the Fourier coefficients $\hat{\chi}_{B_k(R)}(\xi)$ are given by

$$\hat{\chi}_{B_k(R)}(\xi) = \mathrm{e}^{-2\pi\mathrm{i}\langle k,\xi\rangle}\frac{1}{2\pi^2|\xi|^3}\left(\sin(2\pi R|\xi|) - 2\pi R|\xi|\cos(2\pi R|\xi|)\right). \tag{89}$$

We also have that

$$\begin{aligned}\int_{\mathbb{R}^3} \delta^{-3}\psi(x/\delta)\mathrm{e}^{-2\pi\mathrm{i}\langle x,\xi\rangle}\,dx &= \int_{\mathbb{R}^3} \delta^{-3}\psi(x/\delta)(4\pi^2|\xi|^2)^{-1}(-\Delta)\mathrm{e}^{-2\pi\mathrm{i}\langle x,\xi\rangle}\,dx \\ &= (4\pi^2|\xi|^2)^{-1}\int_{\mathbb{R}^3} \delta^{-3}\mathrm{e}^{-2\pi\mathrm{i}\langle x,\xi\rangle}(-\Delta)\psi(x/\delta)\,dx \\ &= (4\pi^2|\xi|^2\delta^2)^{-1}\int_{\mathbb{R}^3} \mathrm{e}^{-2\pi\mathrm{i}\delta\langle y,\xi\rangle}(-\Delta)\psi(y)\,dy.\end{aligned} \tag{90}$$

We thus have that $|\hat{\psi}_\delta(\xi)| \ll \min\{1, (\delta|\xi|)^{-2}\} \leq (\delta|\xi|)^{-(1+\epsilon)}$. Plugging these asymptotics in to the sum gives

$$\begin{aligned}\sum_{x\in\mathbb{Z}^3\setminus\{0\}} \chi_{B_k(R)} * \psi_\delta(x) &\ll R\delta^{-(1+\epsilon)} \sum_{\xi\in\mathbb{Z}^3\setminus\{0\}} |\xi|^{-(3+\epsilon)} \\ &\ll R\delta^{-(1+\epsilon)}.\end{aligned} \tag{91}$$

We thus have that

$$\begin{aligned}S(R) \leq S_\delta(R+\delta) &= \frac{4}{3}\pi(R+\delta)^3 + O(R\delta^{-(1+\epsilon)}) \\ &= \frac{4}{3}\pi R^3 + O(R^2\delta + R\delta^{-(1+\epsilon)}),\end{aligned} \tag{92}$$

and similarly that

$$S(R) \geq S_\delta(R-\delta) = \frac{4}{3}\pi R^3 + O(R^2\delta + R\delta^{-(1+\epsilon)}). \tag{93}$$

Setting $\delta = R^{-1/2}$ yields the result. □

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