# BALLISTIC TRANSPORT FOR THE SCHRÖDINGER OPERATOR WITH LIMIT-PERIODIC OR QUASI-PERIODIC POTENTIAL IN DIMENSION TWO

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Abstract. We prove the existence of ballistic transport for the Schrödinger operator with limit-periodic or quasi-periodic potential in dimension two. This is done under certain regularity assumptions on the potential which have been used in prior work to establish the existence of an absolutely continuous component and other spectral properties. The latter include detailed information on the structure of generalized eigenvalues and eigenfunctions. These allow to establish the crucial ballistic lower bound through integration by parts on an appropriate extension of a Cantor set in momentum space, as well as through stationary phase arguments.

## 1. Introduction

1.1. **Prior results on ballistic transport.** A rough qualitative correspondence of spectral and dynamical properties of Schrödinger operators $H = -\Delta + V$, either discrete in $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ or continuous in $\mathcal{H} = L^2(\mathbb{R}^d)$, is given by the RAGE theorem, e.g. [29]: It says that solutions $\Psi(\cdot, t) = e^{-iHt}\Psi_0$ of the time-dependent Schrödinger equation are 'bound states' if the spectral measure $\mu_{\Psi_0}$ of the initial state $\Psi_0$ is pure point, while $\Psi(\cdot, t)$ is a 'scattering state' if $\mu_{\Psi_0}$ is (abolutely) continuous. However, knowing the spectral type is not sufficient to quantify transport properties more precisely, for example in terms of diffusion exponents $\beta$. The latter, if they exist, characterize how time-averaged moments

$$\langle\langle X^m_{\Psi_0}\rangle\rangle_T := \frac{2}{T}\int_0^\infty \exp\left(-\frac{2t}{T}\right)\langle\Psi(\cdot,t), X^m\Psi(\cdot,t)\rangle_{\mathcal{H}}\, dt \tag{1.1}$$

of the position operator $X$ grow as a power $T^{m\beta}$ of time $T$, where $(Xu)(x) = |x|u(x)$. The special cases $\beta = 1$, $\beta = 1/2$ and $\beta = 0$ are interpreted as ballistic transport, diffusive transport, and dynamical localization, respectively.

Restricting to the most frequently considered case of the second moment $m = 2$, the ballistic upper bound $\langle\Psi(\cdot,t), X^2\Psi(\cdot,t)\rangle_{\mathcal{H}} \le C_1(\Psi_0)T^2 + C_2(\Psi_0)$ and, thus, also

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its averaged version

$$\langle\langle X^2_{\Psi_0}\rangle\rangle_T \leq C_1(\Psi_0)T^2 + C_2(\Psi_0) \tag{1.2}$$

are known to hold for general relatively bounded potentials $V$ and large classes of initial states $\Psi_0$ [28]. As most authors, we will work here with the Abel mean used in (1.1), but mention that the existence of a ballistic upper bound can be used to show that Abel means and Cesaro means $T^{-1}\int_0^T \ldots\, dt$ lead to the same diffusion exponents.

In the late 1980s and 1990s methods were developed which led to more concrete bounds on diffusion exponents by also taking fractal dimensions of the associated spectral measures into account. In particular, again for the special case of the second moment, the Guarneri-Combes theorem [14, 15, 5, 24] says that

$$\langle\langle X^2_{\Psi_0}\rangle\rangle_T \geq C_{\Psi_0} T^{2\alpha/d}. \tag{1.3}$$

for initial states $\Psi_0$ with uniformly $\alpha$-Hölder continuous spectral measure (and satisfying an additional energy bound in the continuum case [5]). In dimension $d = 1$ this provides the equivalence of absolutely continuous spectral measures ($\alpha = 1$) with ballistic transport, as the bounds (1.2) and (1.3) combine to $\langle\langle X^2\rangle\rangle_T \sim T^2$. In particular, this means that in cases where the spectra of one-dimensional Schrödinger operators with limit or quasi-periodic potentials were found to have an a.c. component, e.g. [2, 4, 10, 11, 25, 26, 27], one also gets ballistic transport.

One can not conclude ballistic transport from the existence of a.c. spectrum in dimension $d \geq 2$. In fact, examples of Schrödinger operators with absolutely continuous spectrum, but slower than ballistic transport have been found: A two-dimensional 'jelly-roll' example with a.c. spectrum and diffusive transport is discussed in [23], while [3] provides examples of separable potentials in dimension $d \geq 3$ with a.c. spectrum and sub-diffusive transport.

In general, growth properties of generalized eigenfunctions have to be used in addition to spectral information for a more complete characterization of the dynamics. General relations between eigenfunction growth and spectral type as well as dynamics were found in [23]. A series of works studied one-dimensional models with $\alpha < 1$ and related the dynamics to transfer matrix bounds, e.g. [6, 7, 8, 9, 13, 17, 31]. In particular, these methods can establish lower transport bounds in models with sub-ballistic transport, such as the Fibonacci Hamiltonian and the random dimer model.

Much less has been done for $d \geq 2$. Ballistic lower bounds and thus the existence of waves propagating at non-zero velocity are known only for $V = 0$, where this is classical, e.g. [29], and for periodic potentials [1]. Scattering theoretic methods show that this extends to potentials of sufficiently rapid decay, or sufficiently rapidly decaying perturbations of periodic potentials. However, to our knowledge there are no prior results on ballistic lower bounds for multidimensional Schrödinger operators with bounded potentials which are not asymptotically periodic. Providing two such results in dimension $d = 2$, one for a class of limit-periodic potentials and one for a class of quasi-periodic potentials, is our main goal here.

For both of these examples, the existence of an absolutely continuous component in the spectrum has been shown in earlier works [20, 21, 22]. Essentially, what we do here is to show that the properties of generalized eigenfunctions which were obtained in these works can be used to also conclude ballistic transport.

1.2. **Main Results.** We study the initial value problem:

$$i\frac{\partial \Psi}{\partial t} = H\Psi, \quad \Psi(\vec{x},0) = \Psi_0(\vec{x}) \tag{1.4}$$

for the Schrödinger operator

$$H = -\Delta + V(\vec{x}) \tag{1.5}$$

in two dimensions, $\vec{x} \in \mathbb{R}^2$. For the potential $V(\vec{x})$ we consider two cases, limit-periodic potentials and quasi-periodic potentials.

In the *limit-periodic case*, we assume that the potential can be written as

$$V(\vec{x}) = \sum_{r=1}^{\infty} V_r(\vec{x}), \tag{1.6}$$

where $\{V_r\}_{r=1}^{\infty}$ is a family of periodic potentials with doubling periods. More precisely, $V_r$ has orthogonal periods $2^{r-1}\vec{d_1},\ 2^{r-1}\vec{d_2}$. Without loss of generality, we assume that $\vec{d_1} = (d_1, 0)$, $\vec{d_2} = (0, d_2)$ and $\int_{Q_r} V_r(\vec{x})d\vec{x} = 0$, where $Q_r = [0, 2^{r-1}d_1] \times [0, 2^{r-1}d_2]$ is the elementary cell of periods corresponding to $V_r$. We also assume that all $V_r$ are real trigonometric polynomials with the lengths growing at most linearly in the period. Namely, there exists a positive number $R_0 < \infty$ such that each potential admits the Fourier representation

$$V_r(\vec{x}) = \sum_{q\in\mathbb{Z}^2\setminus\{0\},\ 2^{-r+1}|q|<R_0} v_{r,q} e^{i\langle 2^{-r+1}\tilde{q},\vec{x}\rangle}, \quad \tilde{q} = 2\pi\left(\frac{q_1}{d_1}, \frac{q_2}{d_2}\right), \tag{1.7}$$

$\langle\cdot,\cdot\rangle$ being the canonical scalar product and $|\cdot|$ the corresponding norm in $\mathbb{R}^2$. We assume that the series (1.6) converges super-exponentially fast:

$$\sum_q |v_{r,q}| < \hat{C}\exp(-2^{\eta r}) \tag{1.8}$$

for some $\eta > \eta_0 > 3\cdot 10^4$ uniform in $r$. Without loss of generality we can set $\hat{C} = 1$.

In the *quasi-periodic case*, we assume that $V$ is real and can be written in the form

$$V(\vec{x}) = \sum_{\mathbf{s}_1,\mathbf{s}_2\in\mathbb{Z}^2,\,\mathbf{s}_1+\alpha\mathbf{s}_2\in\mathcal{S}_Q} V_{\mathbf{s}_1,\mathbf{s}_2} e^{2\pi i\langle \mathbf{s}_1+\alpha\mathbf{s}_2,\vec{x}\rangle}, \tag{1.9}$$

where $\alpha$ is an irrational number and $\mathcal{S}_Q = \mathcal{S}_Q(\alpha)$ a finite set depending on a positive integer $Q$. To simplify the construction we will assume that $\alpha$ and $\mathcal{S}_Q$ are not degenerate in some sense. More precisely, we impose the following conditions:

**C1** $0 < \alpha < 1$ is irrational and its irrationality measure $\mu$ is finite: $\mu < \infty$ (in other words, this means that $\alpha$ is not a Liouville number). Note also that $\mu \geq 2$ for any irrational number $\alpha$.

**C2** There are $N_0, N_1 > 0$ such that if $|n_1| + |n_2| + |n_3| > N_1$ then

$$n_1 + \alpha n_2 + \alpha^2 n_3 = 0 \quad \text{or} \quad |n_1 + \alpha n_2 + \alpha^2 n_3| > (|n_1| + |n_2| + |n_3|)^{-N_0}. \tag{1.10}$$

Note that **C2** (and also **C1**) is automatically satisfied for a quadratic irrational $\alpha$: If a triple $(n_1', n_2', n_3')$ exists such that

$$n_1' + \alpha n_2' + \alpha^2 n_3' = 0, \tag{1.11}$$

then this triple is unique up to trivial multiplication (otherwise $\alpha$ is rational). If $n_1+\alpha n_2+\alpha^2 n_3 \neq 0$ for some other triple $(n_1, n_2, n_3)$, then (1.10) holds automatically, since otherwise $\mu = \infty$. **C2** allows to estimate from below the angle between two non-colinear vectors $\mathbf{s}_1 + \alpha\mathbf{s}_2$ and $\mathbf{s}_1' + \alpha\mathbf{s}_2'$ (with $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_1', \mathbf{s}_2' \in \mathbb{Z}^2$) by a negative power of $|\mathbf{s}_1 + \alpha\mathbf{s}_2| + |\mathbf{s}_1' + \alpha\mathbf{s}_2'|$.

We further require that, given $\alpha$ satisfying **C1** and **C2**, there exists a positive integer $Q$ and a finite set $\mathcal{S}_Q = \mathcal{S}_Q(\alpha) \subset \mathbb{R}^2$ with the following properties:

**C3** If $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{Z}^2$ are such that $\mathbf{s}_1 + \alpha\mathbf{s}_2 \in \mathcal{S}_Q$ then $|\mathbf{s}_1| + |\mathbf{s}_2| \leq Q$.

**C4** If $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_1', \mathbf{s}_2' \in \mathbb{Z}^2$ are such that $\mathbf{s}_1 + \alpha\mathbf{s}_2$, $\mathbf{s}_1' + \alpha\mathbf{s}_2' \in \mathcal{S}_Q$ and $\mathbf{s}_1 + \alpha\mathbf{s}_2 = c_*(\mathbf{s}_1' + \alpha\mathbf{s}_2')$, then $c_*$ is rational. This means that if there are several vectors in $\mathcal{S}_Q$ with the same direction then they form a subset of a periodic one-dimensional lattice. As $\alpha$ is irrational, the generating vector of this one-dimensional lattice is also of the form $\mathbf{s}_1 + \alpha\mathbf{s}_2$, $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{Z}^2$. Thus, without loss of generality, we will assume that $\mathcal{S}_Q$ contains generating vectors $\mathbf{s}_1+\alpha\mathbf{s}_2$ of all present directions as well as all their integer multipliers $n(\mathbf{s}_1 + \alpha\mathbf{s}_2)$, $n \in \mathbb{Z}$, such that $|n|(|\mathbf{s}_1| + |\mathbf{s}_2|) \leq Q$ (to satisfy **C3**).

In particular, the set $\mathcal{S}_Q$ is symmetric with respect to $\mathbf{0}$ and $\mathbf{0} \in \mathcal{S}_Q$. We will assume though that $V_{\mathbf{0},\mathbf{0}} = 0$. It is shown in [21] that the period of every one-dimensional sublattice in $\mathcal{S}_Q$ is not smaller than $CQ^{-\mu}$.

A basic example of a quasi-periodic potential satisfying the above assumptions and not being periodic in any direction is

$$V(x_1, x_2) = \lambda_1 \cos(2\pi x_1)+\lambda_2 \cos(2\pi x_2)+\lambda_3 \cos(2\pi(\alpha x_1+x_2))+\lambda_4 \cos(2\pi(x_1+\alpha x_2)),$$

for arbitrary nonzero coupling constants $\lambda_1, ..., \lambda_4$ and Liouville number $\alpha$ satisfying **C2**, in particular for a quadratic irrational $\alpha$[1].

Now we consider (1.4), $V$ being limit-periodic or quasi-periodic with assumptions as above. Clearly, the ballistic upper bound of [28] applies and we have (1.2), for example for $\Psi_0 \in C_0^2$, the $C^2$-functions of compact support.

In our two main results we prove that under the above assumptions, both in the limit-periodic and quasi-periodic case, one also has corresponding *ballistic lower bounds*:

**Theorem 1.1.** *There is an infinite-dimensional projector $E_\infty$ in $L^2(\mathbb{R}^2)$ such that for any $\Psi_0 \in E_\infty \mathcal{C}_0^\infty$, $\Psi_0 \neq 0$, the solution $\Psi(\vec{x}, t)$ of (1.4) satisfies the estimate*

$$\frac{2}{T}\int_0^\infty e^{-2t/T}\left\|X\Psi(\cdot, t)\right\|^2_{L^2(\mathbb{R}^2)}dt > c_1(\Psi_0)T^2, \quad c_1 > 0, \tag{1.12}$$

*when $T > T_0(\Psi_0)$.*

[1]Note, however, that the separable $V(x_1, x_2) = \cos(2\pi x_1) + \cos(2\pi x_2) + \cos(2\pi\alpha x_1) + \cos(2\pi\alpha x_2)$ does not satisfy **C4**.

**Theorem 1.2.** *If $\Psi_0 \in \mathcal{C}_0^\infty$ with $E_\infty \Psi_0 \neq 0$, where $E_\infty$ is as in Theorem 1.1, then the solution $\Psi(\vec{x}, t)$ of (1.4) satisfies the estimate*

$$\frac{2}{T}\int_0^\infty e^{-2t/T} \left\| X\Psi(\cdot, t)\right\|^2_{L^2(\mathbb{R}^2)} dt > c_1(\Psi_0)T^2, \quad c_1 > 0, \tag{1.13}$$

*when $T > T_0(\Psi_0)$.*

The constant $c_1$ is defined explicitly in (3.19) and (4.1).

The operator $E_\infty$ is a spectral projector of $H$. As will be seen from its explicit construction in Section 2.1.2, it is close to the operator of multiplication by the characteristic function of an extensive set in Fourier space.

Note the subtle but important difference in the choice of initial conditions in Theorems 1.1 and 1.2. The latter shows that suitable $C_0^\infty$-initial states lead to ballistic transport, while Theorem 1.1 requires to project onto the range of $E_\infty$. In fact, we believe that $E_\infty \Psi_0 \neq 0$ for any $0 \neq \Psi_0 \in \mathcal{C}_0^\infty$ and thus Theorem 1.2 holds for any such $\Psi_0$, but at the moment we do not have a proof.

As already remarked in Section 1.1, both results remain true if the Abel means are replaced by Cesaro means.

The proofs for the limit-periodic case (1.6) and the quasi-periodic case (1.9) are analogous. For the sake of definiteness, we present all the details for the limit-periodic case only. Where necessary, we provide some comments about the quasi-periodic setting. Also, some statements in the paper [20] on the limit-period case were presented in a form not very convenient for what we need here and thus need some technical adjustments, while the corresponding results for the quasi-periodic case in [22] are more directly applicable. This provides another reason for mostly focusing on the limit-periodic case here.

In Section 2 we start by recalling results on the spectral properties of the operator $H$ which were obtained in [20] for the limit-periodic case and in [22] for the quasi-periodic case. Some of these results will also be adjusted and refined to make them more suitable for the proof of our main results. In particular, we recall how generalized eigenvalues $\lambda_\infty(\vec{k})$ and generalized eigenfunctions $\Psi_\infty(\vec{k}, \vec{x})$ of $H$ can be constructed for momenta $\vec{k}$ in an asymptotically large Cantor-type subset $\mathcal{G}_\infty$ of $\mathbb{R}^2$. We also provide a useful construction which allows to extend the functions $\lambda_\infty$ and $\Psi_\infty(\cdot, \vec{x})$ to smooth functions of $\vec{k} \in \mathbb{R}^2$. Theorem 1.1 is proven in Section 3 and Theorem 1.2 is proven in Section 4. Having the technical background from [20] and [22] available, these proofs use rather elementary analysis methods, such as integration by parts and stationary phase estimates. In Section 5 we collect several appendices which provide technical details for some of the arguments from earlier sections.

## 2. Spectral Properties of the Operator $H$

Our proof of Theorems 1.1 and 1.2 is based on the results and properties of two-dimensional limit and quasi-periodic Schrödinger operators derived in the papers [20] and [22]. While those works derived, in particular, the existence of an absolutely continuous component of the spectrum, we will show here how the bounds obtained can be used and, in part, improved, to also conclude ballistic transport. In this section we give a thorough discussion of the results and methods from [20] and [22], mostly focusing on the limit-periodic case. In particular, we give a detailed construction of the spectral projection $E_\infty$ used in our main theorems.

### 2.1. **The Case of a Limit-Periodic Potential.**

2.1.1. *Prior results.* To describe $E_\infty$, we recall the spectral properties of $H$, obtained in [20]:

(1) The spectrum of the operator (1.5), (1.6) contains a semiaxis. A proof of an analogous result by different means can be found in the paper [30]. In [30], the authors consider the operator $H = (-\Delta)^l + V$, $8l > d + 3$, $d \neq 1 (\mathrm{mod}\ 4)$. This obviously includes our case $l = 1$, $d = 2$. However, there is an additional rather strong restriction on the potential $V(\vec{x})$ in [30], which we don't have here: In [30] all the period lattices of the potentials $V_r$ need to have a nonzero vector $\gamma$ in common, i.e., $V(\vec{x})$ is periodic in direction $\gamma$.

(2) There are generalized eigenfunctions $\Psi_\infty(\vec{k}, \vec{x})$, corresponding to the semiaxis, which are close to plane waves: for every $\vec{k}$ in an extensive subset $\mathcal{G}_\infty$ of $\mathbb{R}^2$, there is a solution $\Psi_\infty(\vec{k}, \vec{x})$ of the equation $H\Psi_\infty = \lambda_\infty \Psi_\infty$ which can be described by the formula:

$$\Psi_\infty(\vec{k}, \vec{x}) = e^{i\langle \vec{k}, \vec{x}\rangle} \left(1 + u_\infty(\vec{k}, \vec{x})\right), \tag{2.1}$$

$$\|u_\infty\|_{L^\infty(\mathbb{R}^2)} =_{|\vec{k}|\to\infty} O(|\vec{k}|^{-\gamma_1}), \quad \gamma_1 > 0, \tag{2.2}$$

where $u_\infty(\vec{k}, \vec{x})$ is a limit-periodic function, as the potential. The eigenvalue $\lambda_\infty(\vec{k})$ corresponding to $\Psi_\infty(\vec{k}, \vec{x})$ is close to $|\vec{k}|^2$:

$$\lambda_\infty(\vec{k}) =_{|\vec{k}|\to\infty} |\vec{k}|^2 + O(|\vec{k}|^{-\gamma_2}), \quad \gamma_2 > 0. \tag{2.3}$$

The "non-resonant" set $\mathcal{G}_\infty$ of the vectors $\vec{k}$, for which (2.1) – (2.3) hold, is an extensive Cantor type set: $\mathcal{G}_\infty = \cap_{n=1}^\infty \mathcal{G}_n$, where $\{\mathcal{G}_n\}_{n=1}^\infty$ is a decreasing sequence of sets in $\mathbb{R}^2$. Each $\mathcal{G}_n$ has a finite number of holes in each bounded region. More and more holes appears when $n$ increases, however holes added at each step are of smaller and smaller size. The set $\mathcal{G}_\infty$ satisfies the estimate:

$$\frac{|(\mathcal{G}_\infty \cap \mathbf{B_R})|}{|\mathbf{B_R}|} =_{R\to\infty} 1 + O(R^{-\gamma_3}), \quad \gamma_3 > 0, \tag{2.4}$$

where $\mathbf{B_R}$ is the disk of radius $R$ centered at the origin, $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^2$.

(3) The set $\mathcal{D}_\infty(\lambda)$, defined as a level (isoenergetic) set for $\lambda_\infty(\vec{k})$,

$$\mathcal{D}_\infty(\lambda) = \left\{\vec{k} \in \mathcal{G}_\infty : \lambda_\infty(\vec{k}) = \lambda\right\},$$

is proven to be a slightly distorted circle with an infinite number of holes. It can be described by the formula:

$$\mathcal{D}_\infty(\lambda) = \{\vec{k} : \vec{k} = \varkappa_\infty(\lambda, \vec{\nu})\vec{\nu},\ \vec{\nu} \in \mathcal{B}_\infty(\lambda)\}, \tag{2.5}$$

where $\mathcal{B}_\infty(\lambda)$ is a subset of the unit circle $S_1$. The set $\mathcal{B}_\infty(\lambda)$ can be interpreted as the set of possible directions of propagation for the almost plane waves (2.1). The set $\mathcal{B}_\infty(\lambda)$ has a Cantor type structure and an asymptotically full measure on $S_1$ as $\lambda \to \infty$:

$$L\left(\mathcal{B}_\infty(\lambda)\right) =_{\lambda\to\infty} 2\pi + O\left(\lambda^{-\gamma_3/2}\right), \tag{2.6}$$

here and below $L(\cdot)$ is Lebesgue measure on $S_1$. The value $\varkappa_\infty(\lambda,\vec{\nu})$ in (2.5) is the "radius" of $\mathcal{D}_\infty(\lambda)$ in a direction $\vec{\nu}$. The function $\varkappa_\infty(\lambda,\vec{\nu})-\lambda^{1/2}$ describes the deviation of $\mathcal{D}_\infty(\lambda)$ from the perfect circle of radius $\lambda^{1/2}$. It is proven that the deviation is asymptotically small, uniformly in $\vec{\nu}\in\mathcal{B}_\infty(\lambda)$:

$$\varkappa_\infty(\lambda,\vec{\nu}) =_{\lambda\to\infty} \lambda^{1/2} + O\left(\lambda^{-\gamma_4}\right), \quad \gamma_4>0. \tag{2.7}$$

(4) Absolute continuity of the branch of the spectrum (the semiaxis) corresponding to $\Psi_\infty(\vec{k},\vec{x})$ is proven, see details below.

2.1.2. *Description of methods:* To prove the above results in [20], the authors considered the sequence of operators:

$$H_0=-\Delta, \qquad H_n=H_0+\sum_{r=1}^{M_n}V_r, \quad n\geq 1,\ M_n\to\infty \text{ as } n\to\infty.$$

Obviously, $\|H-H_n\|\to 0$ as $n\to\infty$, where $\|\cdot\|$ is the norm in the class of bounded operators. Clearly,

$$H_n=H_{n-1}+W_n, \quad W_n=\sum_{r=M_{n-1}+1}^{M_n}V_r. \tag{2.8}$$

Each operator $H_n$, $n\geq 1$, is considered as a perturbation of the previous operator $H_{n-1}$. Every operator $H_n$ is periodic, however the periods go to infinity as $n\to\infty$. It is shown that there is a $\lambda_*$, $\lambda_*=\lambda_*(V)$, such that the semiaxis $[\lambda_*,\infty)$ is contained in the spectra of **all** operators $H_n$. For every operator $H_n$ there is a set of eigenfunctions (corresponding to the semiaxis) being close to plane waves: for every $\vec{k}$ in an extensive subset $\mathcal{G}_n$ of $\mathbb{R}^2$, there is a solution $\Psi_n(\vec{k},\vec{x})$ of the differential equation $H_n\Psi_n=\lambda_n\Psi_n$, which can be described by the formula:

$$\Psi_n(\vec{k},\vec{x})=e^{i\langle\vec{k},\vec{x}\rangle}\left(1+u_n(\vec{k},\vec{x})\right), \quad \|u_n\|_{L^\infty(\mathbb{R}^2)} \underset{|\vec{k}|\to\infty}{=} O(|\vec{k}|^{-\gamma_1}), \quad \gamma_1>0, \tag{2.9}$$

where $u_n(\vec{k},\cdot)$ has periods $2^{M_n-1}\vec{d}_1, 2^{M_n-1}\vec{d}_2$. The corresponding eigenvalue $\lambda_n(\vec{k})$ is close to $|\vec{k}|^2$:

$$\lambda_n(\vec{k}) =_{|\vec{k}|\to\infty} |\vec{k}|^2+O\left(|\vec{k}|^{-\gamma_2}\right), \quad \gamma_2>0. \tag{2.10}$$

The non-resonant set $\mathcal{G}_n$ for which (2.10) holds, is proven to be extensive in $\mathbb{R}^2$:

$$\frac{|\mathcal{G}_n\cap\mathbf{B_R}|}{|\mathbf{B_R}|} =_{R\to\infty} 1+O(R^{-\gamma_3}). \tag{2.11}$$

Estimates (2.9) – (2.11) are uniform in $n$. The set $\mathcal{D}_n(\lambda)$ is defined as the level (isoenergetic) set for the non-resonant eigenvalue $\lambda_n(\vec{k})$:

$$\mathcal{D}_n(\lambda)=\left\{\vec{k}\in\mathcal{G}_n:\lambda_n(\vec{k})=\lambda\right\}.$$

This set is proven to be a slightly distorted circle with a finite number of holes. The set $\mathcal{D}_n(\lambda)$ can be described by the formula:

$$\mathcal{D}_n(\lambda)=\{\vec{k}:\vec{k}=\varkappa_n(\lambda,\vec{\nu})\vec{\nu},\ \vec{\nu}\in\mathcal{B}_n(\lambda)\}, \tag{2.12}$$

where $\mathcal{B}_n(\lambda)$ is a subset of the unit circle $S_1$. The set $\mathcal{B}_n(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves (2.9). It is shown that $\{\mathcal{B}_n(\lambda)\}_{n=1}^{\infty}$ is a decreasing sequence of sets, since on each step more and more directions are excluded. Each $\mathcal{B}_n(\lambda)$ has an asymptotically full measure on $S_1$ as $\lambda\to\infty$:

$$L\left(\mathcal{B}_n(\lambda)\right)=_{\lambda\to\infty}2\pi+O\left(\lambda^{-\gamma_3/2}\right), \tag{2.13}$$

the estimate being uniform in $n$. The set $\mathcal{B}_n$ has only a finite number of holes, however their number is growing with $n$. More and more holes of a smaller and smaller size are added at each step. The value $\varkappa_n(\lambda,\vec{\nu})-\lambda^{1/2}$ gives the deviation of $\mathcal{D}_n(\lambda)$ from the perfect circle of radius $\lambda^{1/2}$ in direction $\vec{\nu}$. It is proven that the deviation is asymptotically small uniformly in $n$:

$$\varkappa_n(\lambda,\vec{\nu})=\lambda^{1/2}+O\left(\lambda^{-\gamma_4}\right),\qquad \frac{\partial\varkappa_n(\lambda,\vec{\nu})}{\partial\varphi}=O\left(\lambda^{-\gamma_5}\right),\quad \gamma_4,\gamma_5>0, \tag{2.14}$$

$\varphi$ being an angle variable $\vec{\nu}=(\cos\varphi,\sin\varphi)$.

On each step more and more points are excluded from the non-resonant sets $\mathcal{G}_n$ and, thus, $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets. The set $\mathcal{G}_\infty$ is defined as the limit set: $\mathcal{G}_\infty=\cap_{n=1}^{\infty}\mathcal{G}_n$. It has an infinite number of holes in each bounded region, but nevertheless satisfies the relation (2.4). For every $\vec{k}\in\mathcal{G}_\infty$ and every $n$, there is a generalized eigenfunction of $H_n$ of the type (2.9). It is proven that the sequence of $\Psi_n(\vec{k},\vec{x})$ has a limit in $L^\infty(\mathbb{R}^2)$ as $n\to\infty$, when $\vec{k}\in\mathcal{G}_\infty$. The function $\Psi_\infty(\vec{k},\vec{x})=\lim_{n\to\infty}\Psi_n(\vec{k},\vec{x})$ is a generalized eigenfunction of $H$. It can be written in the form (2.1)–(2.2). Naturally, the corresponding eigenvalue $\lambda_\infty(\vec{k})$ is the limit of $\lambda_n(\vec{k})$ as $n\to\infty$.

We consider the limit $\mathcal{B}_\infty(\lambda)$ of $\mathcal{B}_n(\lambda)$:

$$\mathcal{B}_\infty(\lambda)=\bigcap_{n=1}^{\infty}\mathcal{B}_n(\lambda),\quad \mathcal{B}_n\subset\mathcal{B}_{n-1}.$$

This set has a Cantor type structure on the unit circle. It is proven that $\mathcal{B}_\infty(\lambda)$ has asymptotically full measure on the unit circle (see (2.6)). We prove that the sequence $\varkappa_n(\lambda,\vec{\nu})$, $n=1,2,...,$, describing the isoenergetic curves $\mathcal{D}_n$, quickly converges as $n\to\infty$. Hence, $\mathcal{D}_\infty(\lambda)$ can be described as the limit of $\mathcal{D}_n(\lambda)$ in the sense (2.5), where $\varkappa_\infty(\lambda,\vec{\nu})=\lim_{n\to\infty}\varkappa_n(\lambda,\vec{\nu})$ for every $\vec{\nu}\in\mathcal{B}_\infty(\lambda)$. It is shown that the derivatives of the functions $\varkappa_n(\lambda,\vec{\nu})$ (with respect to the angle variable $\varphi$ on the unit circle) have a limit as $n \to \infty$ for every $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$. We denote this limit by $\frac{\partial \varkappa_\infty(\lambda,\vec{\nu})}{\partial \varphi}$. Using (2.14) we prove that

$$\frac{\partial \varkappa_\infty(\lambda,\vec{\nu})}{\partial \varphi} = O\left(\lambda^{-\gamma_5}\right). \tag{2.15}$$

Thus, the limit curve $\mathcal{D}_\infty(\lambda)$ has a tangent vector in spite of its Cantor type structure, the tangent vector being the limit of the corresponding tangent vectors for $\mathcal{D}_n(\lambda)$ as $n \to \infty$. The curve $\mathcal{D}_\infty(\lambda)$ takes the form of a slightly distorted circle with an infinite number of holes.

Absolute continuity of the branch of the spectrum $[\lambda_*(V), \infty)$, corresponding to the functions $\Psi_\infty(\vec{k}, \vec{x})$, $\vec{k} \in \mathcal{G}_\infty$, follows from continuity properties of the level curves $\mathcal{D}_\infty(\lambda)$ with respect to $\lambda$, and from convergence of spectral projections corresponding to $\Psi_n(\vec{k}, \vec{x})$, $\vec{k} \in \mathcal{G}_\infty$, to spectral projections of $H$ in the strong sense and uniformly in $\lambda$ with $\lambda > \lambda_*(V)$.

Let $\mathcal{G}_n'$ be a bounded Lebesgue measurable subset of $\mathcal{G}_n$. We consider the spectral projection $E_n\left(\mathcal{G}_n'\right)$ of $H^{(n)}$, corresponding to functions $\Psi_n(\vec{k}, \vec{x})$, $\vec{k} \in \mathcal{G}_n'$. By [12], $E_n\left(\mathcal{G}_n'\right) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ can be presented by the formula:

$$E_n\left(\mathcal{G}_n'\right) F = \frac{1}{4\pi^2} \int_{\mathcal{G}_n'} \big(F, \Psi_n(\vec{k})\big) \Psi_n(\vec{k})\, d\vec{k} \tag{2.16}$$

for any $F \in \mathcal{C}_c(\mathbb{R}^2)$, the continuous, compactly supported functions on $\mathbb{R}^2$. Here and below $\left(\cdot, \cdot\right)$ is the canonical scalar product in $L^2(\mathbb{R}^2)$, i.e.,

$$\big(F, \Psi_n(\vec{k})\big) = \int_{\mathbb{R}^2} F(x) \overline{\Psi_n(\vec{k}, \vec{x})}\, d\vec{x}.$$

The above formula can be rewritten in the form

$$E_n\left(\mathcal{G}_n'\right) = S_n\left(\mathcal{G}_n'\right) T_n\left(\mathcal{G}_n'\right), \tag{2.17}$$

$$T_n : \mathcal{C}_c(\mathbb{R}^2) \to L^2\left(\mathcal{G}_n'\right), \qquad S_n : L^\infty\left(\mathcal{G}_n'\right) \to L^2(\mathbb{R}^2),$$

$$(T_n F)(\vec{k}) = \frac{1}{2\pi} \big(F, \Psi_n(\vec{k})\big) \ \text{ for any } F \in \mathcal{C}_c(\mathbb{R}^2), \tag{2.18}$$

$T_n F$ being in $L^\infty\left(\mathcal{G}_n'\right)$, and

$$(S_n f)(\vec{x}) = \frac{1}{2\pi} \int_{\mathcal{G}_n'} f(\vec{k}) \Psi_n(\vec{k}, \vec{x})\, d\vec{k} \ \text{ for any } f \in L^\infty\left(\mathcal{G}_n'\right). \tag{2.19}$$

By [12],

$$\|T_n F\|_{L^2(\mathcal{G}_n')} \leq \|F\|_{L^2(\mathbb{R}^2)} \tag{2.20}$$

and

$$\|S_n f\|_{L^2(\mathbb{R}^2)} \leq \|f\|_{L^2(\mathcal{G}_n')}. \tag{2.21}$$

Hence, $T_n$ and $S_n$ can be extended by continuity from $\mathcal{C}_c(\mathbb{R}^2)$ and $L^\infty\left(\mathcal{G}_n'\right)$ to $L^2(\mathbb{R}^2)$ and $L^2\left(\mathcal{G}_n'\right)$, respectively. Thus, the operator $E_n\left(\mathcal{G}_n'\right)$ is described by (2.17) in the whole space $L^2(\mathbb{R}^2)$.

Let

$$\mathcal{G}_{n,\lambda} = \{\vec{k} \in \mathcal{G}_n : \lambda_n(\vec{k}) < \lambda\}. \tag{2.22}$$

This set is Lebesgue measurable, since $\mathcal{G}_n$ is open and $\lambda_n(\vec{k})$ is continuous on $\mathcal{G}_n$.

Let

$$\mathcal{G}_{\infty,\lambda}=\left\{\vec{k}\in\mathcal{G}_\infty:\lambda_\infty(\vec{k})<\lambda\right\}. \tag{2.23}$$

The function $\lambda_\infty(\vec{k})$ is a Lebesgue measurable function, since it is a limit of a sequence of measurable functions. Hence, the set $\mathcal{G}_{\infty,\lambda}$ is measurable. It is shown in [20] that the measure of the symmetric difference of the two sets $\mathcal{G}_{\infty,\lambda}$ and $\mathcal{G}_{n,\lambda}$ converges to zero as $n\to\infty$, uniformly in $\lambda$ in every bounded interval:

$$\lim_{n\to\infty}|\mathcal{G}_{\infty,\lambda}\,\Delta\,\mathcal{G}_{n,\lambda}|=0.$$

Next, we consider the sequence of operators $T_n(\mathcal{G}_{\infty,\lambda})$ which are given by (2.18) and act from $L^2(\mathbb{R}^2)$ to $L^2(\mathcal{G}_{\infty,\lambda})$. It is proven in [20] that the sequence $T_n(\mathcal{G}_{\infty,\lambda})$ has a strong limit $T_\infty(\mathcal{G}_{\infty,\lambda})$. The operator $T_\infty(\mathcal{G}_{\infty,\lambda})$ satisfies $\|T_\infty\|\le 1$ and can be described by the formula $(T_\infty F)(\vec{k})=\frac{1}{2\pi}\big(F,\Psi_\infty(\vec{k})\big)$ for any $F\in\mathcal{C}_c(\mathbb{R}^2)$. The convergence of $T_n(\mathcal{G}_{\infty,\lambda})F$ to $T_\infty(\mathcal{G}_{\infty,\lambda})F$ is uniform in $\lambda$ for every $F\in L^2(\mathbb{R}^2)$. We also consider the sequence of operators $S_n(\mathcal{G}_{\infty,\lambda})$ which are given by (2.19) with $\mathcal{G}_n'=\mathcal{G}_{\infty,\lambda}$:

$$S_n(\mathcal{G}_{\infty,\lambda}):\ L^2(\mathcal{G}_{\infty,\lambda})\to L^2(\mathbb{R}^2). \tag{2.24}$$

It is proven in [20] that the sequence of operators $S_n(\mathcal{G}_{\infty,\lambda})$ has a strong limit $S_\infty(\mathcal{G}_{\infty,\lambda})$. In fact, a slight modification of the proof (see Appendix 1 below) gives convergence in operator norm sense. The estimate

$$\|S_\infty(\mathcal{G}_{\infty,\lambda})-S_0(\mathcal{G}_{\infty,\lambda})\|<c\lambda_*^{-\gamma_6},\quad \gamma_6>0, \tag{2.25}$$

holds for $\lambda>\lambda_*$.

The operator $S_\infty(\mathcal{G}_{\infty,\lambda})$ satisfies $\|S_\infty\|=1$ and can be described by the formula

$$(S_\infty f)(\vec{x})=\frac{1}{2\pi}\int_{\mathcal{G}_{\infty,\lambda}}f(\vec{k})\Psi_\infty(\vec{k},\vec{x})\,d\vec{k} \tag{2.26}$$

for any $f\in L^\infty(\mathcal{G}_{\infty,\lambda})$. The convergence of $S_n(\mathcal{G}_{\infty,\lambda})f$ to $S_\infty(\mathcal{G}_{\infty,\lambda})f$ is uniform in $\lambda$ for every $f\in L^2(\mathcal{G}_\infty)$.

The spectral projections $E_n(\mathcal{G}_{\infty,\lambda})$ converge in norm to $E_\infty(\mathcal{G}_{\infty,\lambda})$ in $L^2(\mathbb{R}^2)$ as $n$ tends to infinity, since $T_n=S_n^*$. The operator $E_\infty(\mathcal{G}_{\infty,\lambda})$ is a spectral projection of $H$. It can be represented in the form $E_\infty(\mathcal{G}_{\infty,\lambda})=S_\infty(\mathcal{G}_{\infty,\lambda})T_\infty(\mathcal{G}_{\infty,\lambda})$, where $S_\infty(\mathcal{G}_{\infty,\lambda})$ and $T_\infty(\mathcal{G}_{\infty,\lambda})$ are limits in norm of $S_n(\mathcal{G}_{\infty,\lambda})$ and $T_n(\mathcal{G}_{\infty,\lambda})$, respectively. For any $F\in\mathcal{C}_c(\mathbb{R}^2)$, we show

$$E_\infty\left(\mathcal{G}_{\infty,\lambda}\right)F=\frac{1}{4\pi^2}\int_{\mathcal{G}_{\infty,\lambda}}\big(F,\Psi_\infty(\vec{k})\big)\Psi_\infty(\vec{k})\,d\vec{k}, \tag{2.27}$$

$$HE_\infty\left(\mathcal{G}_{\infty,\lambda}\right)F=\frac{1}{4\pi^2}\int_{\mathcal{G}_{\infty,\lambda}}\lambda_\infty(\vec{k})\big(F,\Psi_\infty(\vec{k})\big)\Psi_\infty(\vec{k})\,d\vec{k}. \tag{2.28}$$

Absolute continuity of the branch of the spectrum corresponding to the functions $\Psi_\infty(\vec{k})$ follows from properties of $E_\infty\left(\mathcal{G}_{\infty,\lambda}\right)$.

The projections $E_\infty(\mathcal{G}_{\infty,\lambda})$ have a strong limit $E_\infty(\mathcal{G}_\infty)$ as $\lambda$ goes to infinity. Hence, the operator $E_\infty(\mathcal{G}_\infty)$ is a projection. The projections $E_\infty(\mathcal{G}_{\infty,\lambda})$, $\lambda\in\mathbb{R}$, and $E_\infty(\mathcal{G}_\infty)$ reduce the operator $H$. The family of projections $E_\infty(\mathcal{G}_\infty,\lambda)$ is the resolution of the identity of the operator $HE_\infty(\mathcal{G}_\infty)$ acting in $E_\infty(\mathcal{G}_\infty)L^2(\mathbb{R}^2)$. Further we denote $E_\infty(\mathcal{G}_\infty)$ just by $E_\infty$.

In what follows, we may need to increase the parameter $\lambda_*$ in a controlled way. We denote the new $\lambda_*$ by $\lambda_{**}$. This means a change of the set $\mathcal{G}_\infty$: $\mathcal{G}_{\infty,\lambda_{**}} = \mathcal{G}_{\infty,\lambda_*} \setminus B_{k_{**}}$, where here and below $k_{**} = \lambda_{**}^{1/2}$. However, for any fixed value of $\lambda_{**}$ the projector $E_\infty(\mathcal{G}_\infty)$ corresponds to a sufficiently rich branch of the absolutely continuous spectrum covering the half-line $[\lambda_{**}, \infty)$.

2.1.3. *Extension of $\lambda_\infty(\vec{k})$ from $\mathcal{G}_\infty$ to $\mathbb{R}^2$.* First, we extend the function $\lambda_\infty(\vec{k})$ from $\mathcal{G}_\infty$ to $\mathbb{R}^2$, the result being a $C^M(\mathbb{R}^2)$ function. Note that the extended function is not an eigenvalue outside of $\mathcal{G}_\infty$.

Indeed, let $M$ be a natural number (in fact, we will need $M = 7$ later). First, following [20], we represent $\lambda_\infty(\vec{k}) - k^2$, $k := |\vec{k}|$, $\vec{k} \in \mathcal{G}_\infty$, in the form:

$$\lambda_\infty(\vec{k}) - k^2 = \lambda_1(\vec{k}) - k^2 + \sum_{n=1}^{\infty} \left(\lambda_{n+1}(\vec{k}) - \lambda_n(\vec{k})\right).$$

By Theorem 2.6 in [20], with $D^m := \partial_1^{m_1}\partial_2^{m_2}$ we obtain

$$\left| D^m \left(\lambda_1(\vec{k}) - k^2\right)\right| < Ck^{-\gamma_2+\gamma_0|m|} \tag{2.29}$$

when $\vec{k}$ is in the $k^{-\gamma_0}$-neighborhood of $\mathcal{G}_1 \supset \mathcal{G}_\infty$ and the constant depends only on $V$ and $m$. Moreover, by Theorem 3.8 and e.t.c. in [20],

$$\left|\lambda_{n+1}(\vec{k}) - \lambda_n(\vec{k})\right| < e^{-k^{\eta s_n}}, \tag{2.30}$$

for any $n \geq 1$, where $s_n = 2^{n-1}s_1$, $s_1$ being chosen sufficiently small with $0 < s_1 < 10^{-4}$. The value of $s_1$ is chosen at the beginning of the iteration procedure and, eventually, $\lambda_*(V)$ and the constants in the estimates depend on $s_1$. Estimate (2.30) is valid in the $(\epsilon_n e^{-1-\delta_0})$-neighborhood of each $\vec{k} \in \mathcal{G}_n \supset \mathcal{G}_\infty$, where $\epsilon_n = e^{-\frac{1}{4}k^{\eta s_n}}$ and $\delta_0 > 0$. The constant $\frac{1}{4}$ in the definition of $\epsilon_n$, see [20], is chosen at random. Instead of $\frac{1}{4}$, one can take any fraction $\frac{1}{M+1}$, $M \geq 1$. This will lead, generally speaking, to an increase of $\lambda_*(V)$, when $M > 3$. We will denote the new $\lambda_*(V)$ by $\lambda_{**}(V, M)$. Further we use the notation $\epsilon_n = e^{-\frac{1}{M+1}k^{\eta s_n}}$ and assume $k^2 > \lambda_{**}(V, M)$. Then we can rewrite (2.30) as

$$\left|\lambda_{n+1}(\vec{k}) - \lambda_n(\vec{k})\right| < \epsilon_n^{M+1} \tag{2.31}$$

in the $(\epsilon_n k^{-1-\delta_0})$-neighborhood of any $\vec{k} \in \mathcal{G}_n$. Using analyticity of $\lambda_{n+1}(\vec{k})$ and $\lambda_n(\vec{k})$ in the complex $(\epsilon_n k^{-1-\delta_0})$-neighborhood of any $\vec{k} \in \mathcal{G}_n$, we obtain (see [20])

$$\left|D^m \left(\lambda_{n+1}(\vec{k}) - \lambda_n(\vec{k})\right)\right| < \epsilon_n^{M+1-|m|}k^{(1+\delta_0)|m|} \tag{2.32}$$

in $\mathcal{G}_n$ for all $m$. Next, let $\eta_1(\vec{k})$ be a function in $C^\infty$ with support in the (real) $k^{-\gamma_0}$-neighborhood of $\mathcal{G}_1$, satisfying $\eta_1 = 1$ on $\mathcal{G}_1$ and $\left|D^m\eta_1(\vec{k})\right| < k^{\gamma_0|m|}$. This is possible since we can take a convolution of the characteristic function of the $\frac{1}{2}k^{-\gamma_0}$-neighborhood of $\mathcal{G}_1$ with $\omega(2k^{\gamma_0}\vec{k})$, where $\omega$ is a smooth cut-off function with support in the unit disc centered at the origin. Similarly, let $\eta_n(\vec{k})$, $n \geq 2$, be a $C^\infty$ function with support in the $(\epsilon_n k^{-1-\delta_0})$-neighborhood of $\mathcal{G}_n$, satisfying $\eta_n = 1$ on $\mathcal{G}_n$ and

$$\left| D^m \eta_n(\vec{k}) \right| \leq \left( \epsilon_n k^{-1-\delta_0} \right)^{-|m|}. \tag{2.33}$$

In the estimate (2.29), $\gamma_2 = 2 - 30s_1 - 20\delta_0$, $\gamma_0 = 1 + 16s_1 + 11\delta_0$. However, (2.29) can be improved when $|m| < k^{s_1/2}$, see Lemma 2.5 in [20]. In this case, one can take $\gamma_0 = 3s_1 + 2\delta_0$. Choose $s_1$ small enough so that $2\gamma_0 M < \gamma_2$, i.e.,

$$2(3s_1 + 2\delta_0)M < 2 - 30s_1 - 20\delta_0$$

and for sufficiently large $k$, $M < k^{s_1/2}$ and so (2.29) holds with $\gamma_0 = 3s_1 + 2\delta_0$.

Next, we extend $\lambda_\infty(\vec{k}) - k^2$ from $\mathcal{G}_\infty$ to $\mathbb{R}^2$ using the formula

$$\lambda_\infty(\vec{k}) - k^2 = (\lambda_1(\vec{k}) - k^2)\eta_1(\vec{k}) + \sum_{n=1}^{\infty} \left( \lambda_{n+1}(\vec{k}) - \lambda_n(\vec{k}) \right) \eta_{n+1}(\vec{k}). \tag{2.34}$$

It follows from (2.31) and (2.32) that the series converges in $C^M(\mathbb{R}^2)$. Moreover, the next lemma follows from (2.31)–(2.33).

**Lemma 2.1.** *For every natural number $M$, there exists $\lambda_{**}(V, M) > 0$ such that the function $\lambda_\infty(\vec{k}) - k^2$ can be extended, as a $\mathcal{C}^M$ function, from $\mathcal{G}_{\infty,\lambda_{**}}$ to $\mathbb{R}$ and it satisfies*

$$\left| D^m \left( \lambda_\infty(\vec{k}) - k^2 \right) \right| < C_M k^{-\gamma_2 + \gamma_0 |m|}, \tag{2.35}$$

*for any $m \in \mathbb{N}_0^2$ with $|m| \leq M$, where $-\gamma_2 + 2\gamma_0 M < 0$.*

**Remark 1.** *For our needs $M = 7$ is sufficient and in what follows we assume that the corresponding $s_1$ and $\lambda_{**}$ are chosen for $M = 7$.*

2.1.4. *Extension of $\Psi_\infty(\vec{k}, \vec{x})$ from $\mathcal{G}_\infty$ to $\mathbb{R}^2$.* We extend $\Psi_\infty(\vec{k}, \vec{x})$ by a formula analogous to (2.34):

$$\Psi_\infty(\vec{k}, \vec{x}) - e^{i\langle \vec{k}, \vec{x} \rangle} = \left( \Psi_1(\vec{k}, \vec{x}) - e^{i\langle \vec{k}, \vec{x} \rangle} \right) \eta_1(\vec{k}) + \sum_{n=1}^{\infty} \left( \Psi_{n+1}(\vec{k}, \vec{x}) - \Psi_n(\vec{k}, \vec{x}) \right) \eta_{n+1}(\vec{k}). \tag{2.36}$$

The series converges by (5.5). Using the last formula and (2.26), we define $S_\infty(\widetilde{\mathcal{G}}_\infty)$ for any $\widetilde{\mathcal{G}}_\infty \supset \mathcal{G}_\infty$:

$$\left( S_\infty(\widetilde{\mathcal{G}}_\infty) f \right)(\vec{x}) := \frac{1}{2\pi} \int_{\widetilde{\mathcal{G}}_\infty} f(\vec{k}) \Psi_\infty(\vec{k}, \vec{x})\, d\vec{k}. \tag{2.37}$$

It is easy to see that

$$S_\infty(\widetilde{\mathcal{G}}_\infty) = S_0(\widetilde{\mathcal{G}}_\infty) + \sum_{n=0}^{\infty} \big( S_{n+1}(\widetilde{\mathcal{G}}_\infty) - S_n(\widetilde{\mathcal{G}}_\infty) \big) \eta_{n+1}, \tag{2.38}$$

where $S_0(\widetilde{\mathcal{G}}_\infty)$ is defined by

$$S_0(\widetilde{\mathcal{G}}_\infty) f = \frac{1}{2\pi} \int_{\widetilde{\mathcal{G}}_\infty} f(\vec{k}) e^{-i\langle \vec{k}, \vec{x} \rangle} d\vec{k},$$

$\eta_{n+1}$ is multiplication by $\eta_{n+1}(\vec{k})$ and $S_n(\widetilde{\mathcal{G}}_\infty)$ is given by (2.19) with $\mathcal{G}_n'$ being the intersection of $\widetilde{\mathcal{G}}_\infty$ with the $(\epsilon_n k^{-1-\delta_0})$-neighborhood of $\mathcal{G}_n$ for $n \geq 2$ and the $k^{-\gamma_0}$-neighborhood of $\mathcal{G}_1$ for $n = 1$.

Similarly to (2.25), we show that

$$\|S_\infty(\widetilde{\mathcal{G}}_\infty) - S_0(\widetilde{\mathcal{G}}_\infty)\| < c(V)\lambda_{**}^{-\gamma_6}. \tag{2.39}$$

In what follows we assume that $\lambda_{**}$ is chosen so that, in particular, $c(V)\lambda_{**}^{-\gamma_6} \leq 1/2$. Thus we have

$$\|S_\infty(\widetilde{\mathcal{G}}_\infty)\| \leq 2. \tag{2.40}$$

Similarly, with $T_0$ the Fourier transform,

$$\begin{aligned}(T_\infty F)(\vec{k}) &:= \frac{1}{2\pi}(F(\cdot), \Psi_\infty(\vec{k}, \cdot)) \\ &= (T_0 F)(\vec{k}) + \sum_{n=0}^{\infty}\big((T_{n+1} - T_n)F\big)(\vec{k})\eta_{n+1}(\vec{k}).\end{aligned} \tag{2.41}$$

.

We need one more auxiliary result.

**Lemma 2.2.** *For any given $L \in \mathbb{N}$ there exists $\lambda_{**}(V, L)$ such that for any $F \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, the function $T_\infty F$ as defined above is in $\mathcal{C}^L(\mathbb{R}^2)$. Moreover, if $0 \leq j \leq L$ and $m \in \mathbb{N}_0^2$, $|m| \leq L$, then*

$$\left||\vec{k}|^j D^m (T_\infty F)(\vec{k})\right| < C(L, F), \tag{2.42}$$

*for all $\vec{k} \in \mathbb{R}^2$.*

A proof of this lemma is given in Appendix 2 below.

**Remark 2.** *In fact, for our needs $L = 6$ is sufficient and in what follows we assume that the corresponding $\lambda_{**}$ is chosen for $L = 6$.*

2.2. **The Case of a Quasi-periodic Potential.** The main results in the case of quasi-periodic potential [22] are completely analogous to those for limit-periodic potential in Section 2.1.1, the only difference being that $u_\infty$ in (2.1) is quasi-periodic, i.e., has a representation analogous to that for the potential, but not necessarily a trigonometric polynomial. The operators $H_n$ in the approximation procedure are, naturally, quite different from (2.8). However, the rest of Section 2.2.2 is completely analogous for both types of potentials, the quasi-periodic case being even somewhat simpler, since convergence of the sequence $S_n$ in norm, proven in Appendix 1 for the limit-periodic case, is already proven in [21], [22] for the quasi-periodic potential. Extension of $\lambda_\infty(\vec{k})$ and $E_\infty(\vec{k})$ to $\mathbb{R}^2$ (Section 2.1.3) are also completely similar in both cases. Note only that in the quasi-periodic case Lemma 2.1 holds with $\gamma_2 = 2 - 88\mu\delta$, $\gamma_0 = (40\mu + 1)\delta$, $\delta > 0$, by Theorem 3.3, Corollary 3.4 and Lemma 3.5 in [22] and $\epsilon_n^{M+1} = k^{-\frac{\beta}{10}k^{r_{n-1}-r_{n-2}}}$ (see (2.31)), here $\beta$ is a positive constant, $r_n$ is an increasing sequence going to infinity as $n \to \infty$, see Corollaries 5.4, 6.4 in [22].

## 3. Proof of Theorem 1.1

Let $\mathcal{S}$ be the class of functions in $T_\infty \mathcal{C}_0^\infty(\mathbb{R}^2)$, see (2.41). As shown in Lemma 2.2, if $\widehat{\Psi}_0 \in \mathcal{S}$, then

$$\big||\vec{k}|^j D^m(\widehat{\Psi}_0)(\vec{k})| < C(j,m,\widehat{\Psi}_0) \tag{3.1}$$

for any $\vec{k} \in \mathbb{R}^2$ when $j \le 6$ and $|m| \le 4$.

Let $\widehat{\Psi}_0 \in \mathcal{S}$ and

$$\Psi(\vec{x},t) := \frac{1}{2\pi}\int_{\mathcal{G}_\infty} \Psi_\infty(\vec{k},\vec{x}) e^{-it\lambda_\infty(\vec{k})}\widehat{\Psi}_0(\vec{k})\, d\vec{k}, \tag{3.2}$$

then this function solves the initial value problem (1.4), where

$$\Psi_0(\vec{x}) = \frac{1}{2\pi}\int_{\mathcal{G}_\infty} \Psi_\infty(\vec{k},\vec{x})\widehat{\Psi}_0(\vec{k})\, d\vec{k} \tag{3.3}$$

and $\Psi_0(\vec{x}) \in S_\infty \mathcal{S} = E_\infty \mathcal{C}_0^\infty$. Obviously, $S_\infty \mathcal{S}$ is dense in $E_\infty L^2(\mathbb{R}^2)$.

The first step of the proof is replacing $\mathcal{G}_\infty$ by a small neighborhood $\tilde{\mathcal{G}}_\infty$ and to estimate the resulting errors in the integrals. This is an important step, since $\mathcal{G}_\infty$ is a closed Cantor-type set, while $\tilde{\mathcal{G}}_\infty$ is an open set. The second step is integrating by parts in an integral over $\tilde{\mathcal{G}}_\infty$ with the purpose of obtaining (1.12), the fact that $\tilde{\mathcal{G}}_\infty$ is open being used for handling boundary terms. All further considerations are essentially identical for the limit-periodic and quasi-periodic cases. The notations are mostly identical, in situations where they are different we consider the limit-periodic case.

To get the lower bound (1.12), we first note that

$$\|X\Psi\|^2_{L^2(\mathbb{R}^2)} \ge \|X\Psi\|^2_{L^2(B_R)} \ge \frac{1}{2}\|Xw\|^2_{L^2(B_R)} - \|X(\Psi - w)\|^2_{L^2(B_R)},$$

where $B_R$ is the open disc with radius $R$ centered at the origin, $R = c_0 T$, $c_0$ to be chosen later, and $w(\vec{x},t)$ is an approximation of $\Psi$ when $\mathcal{G}_\infty$ is replaced by its small neighborhood $\tilde{\mathcal{G}}_\infty$. Namely,

$$w(\vec{x},t) := \frac{1}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty} \Psi_\infty(\vec{k},\vec{x}) e^{-it\lambda_\infty(\vec{k})}\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\, d\vec{k}, \tag{3.4}$$

$\eta_\delta$ being a smooth cut-off function with support in a $\delta$-neighborhood $\widetilde{\mathcal{G}}_\infty$ of $\mathcal{G}_\infty$ and $\eta_\delta = 1$ on $\mathcal{G}_\infty$. The parameter $\delta$ $(0 < \delta < 1)$ will be chosen later to be sufficiently small and depend only on $\widehat{\Psi}_0$. We take $\eta_\delta$ to be a convolution of a function $\omega(\vec{k}/2\delta)$ with the characteristic function of the $\delta/2$-neighborhood of $\mathcal{G}_\infty$, where $\omega(\vec{k})$ is a nonnegative $\mathcal{C}_0^\infty(\mathbb{R}^2)$-function with a support in the unit ball centered at zero and integral one. Then, $\eta_\delta \in \mathcal{C}_0^\infty(\mathbb{R}^2)$,

$$0 \le \eta_\delta \le 1,\ \eta_\delta(\vec{k}) = 1 \text{ when } \vec{k} \in \mathcal{G}_\infty,\ \eta_\delta(\vec{k}) = 0 \text{ when } \vec{k} \notin \widetilde{\mathcal{G}}_\infty,\ \|D^m\eta_\delta\|_{L^\infty} < C_m\delta^{-|m|}. \tag{3.5}$$

To prove (1.12), we will show that there exist a positive constant $c_1$ and constants $c_2$ and $c_3$ such that

$$\frac{2}{T}\int_0^\infty e^{-2t/T}\big\|Xw(\cdot,t)\big\|^2_{L^2(B_R)} dt \ge 6c_1T^2 - c_2T - c_3, \tag{3.6}$$

as long as $c_0$ in the definition of $R$ exceeds a certain value depending only on $\widehat{\Psi}_0$. In formula (3.6), the constant $c_1 = c_1(\widehat{\Psi}_0)$ depends on $\widehat{\Psi}_0$, but not $\delta$ or $c_0$, while the constants $c_2 = c_2(\widehat{\Psi}_0, \delta)$ and $c_3 = c_3(\widehat{\Psi}_0, \delta)$ depend on $\widehat{\Psi}_0$ and $\delta$, but not $c_0$.

We also prove that

$$\frac{2}{T}\int_0^\infty e^{-2t/T}\big\|X(\Psi - w)(\cdot,t)\big\|^2_{L^2(B_R)}dt \le \gamma(\delta,\widehat{\Psi}_0)c_0^2T^2, \tag{3.7}$$

$\gamma(\delta,\widehat{\Psi}_0) = o(1)$ as $\delta \to 0$ uniformly in $c_0$.

*Proof of* (3.7). Since $\eta_\delta = 1$ on $\mathcal{G}_\infty$,

$$\Psi(\vec{x},t) - w(\vec{x},t) = -\frac{1}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty\setminus\mathcal{G}_\infty}\Psi_\infty(\vec{k},\vec{x})e^{-it\lambda_\infty(\vec{k})}\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\,d\vec{k} =: f(\vec{x},t).$$

Since $\|X\| \le R$, it suffices to show that

$$\|f(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \le \gamma(\delta,\widehat{\Psi}_0). \tag{3.8}$$

Note that $f = S_\infty(\widetilde{\mathcal{G}}_\infty)g_1$, where $g_1 = e^{-it\lambda_\infty(\vec{k})}\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\chi(\widetilde{\mathcal{G}}_\infty\setminus\mathcal{G}_\infty)$ and $S_\infty$ is defined by (2.37). Now, the estimate (2.40) and Lebesgue's Dominated Convergence Theorem complete the proof, where Lemma 2.2 is used to show that $\widehat{\Psi}_0(\vec{k})$ decays sufficiently fast at infinity.

■

*Proof of* (3.6). Let

$$v(\vec{x},t) := \frac{1}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty}\Psi_\infty(\vec{k},\vec{x})e^{-it\lambda_\infty(\vec{k})}\nabla\Big(\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\Big)\,d\vec{k}. \tag{3.9}$$

Then, using integration by parts and then (2.1), we get

$$\begin{aligned} v(\vec{x},t) &= -\frac{1}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty}\Big[\nabla_{\vec{k}}\Psi_\infty(\vec{k},\vec{x}) - it\Psi_\infty(\vec{k},\vec{x})\nabla\lambda_\infty(\vec{k})\Big]e^{-it\lambda_\infty(\vec{k})}\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\,d\vec{k} \\ &= -\frac{i}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty}\Big[\vec{x} - t\nabla\lambda_\infty(\vec{k})\Big]\Psi_\infty(\vec{k},\vec{x})e^{-it\lambda_\infty(\vec{k})}\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\,d\vec{k} \\ &\quad - \frac{1}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty}e^{i\langle\vec{k},\vec{x}\rangle}\nabla_{\vec{k}}u_\infty(\vec{k},\vec{x})e^{-it\lambda_\infty(\vec{k})}\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\,d\vec{k}, \end{aligned}$$

where the boundary term is vanishing due to $\eta_\delta$ and the fast decay of $\widehat{\Psi}_0$. In short, $v = -iXw + it\phi - \phi_s$, where

$$\begin{aligned} \phi(\vec{x},t) &:= \frac{1}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty}\nabla\lambda_\infty(\vec{k})\Psi_\infty(\vec{k},\vec{x})e^{-it\lambda_\infty(\vec{k})}\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\,d\vec{k} \\ \phi_s(\vec{x},t) &:= \frac{1}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty}e^{i\langle\vec{k},\vec{x}\rangle}\nabla_{\vec{k}}u_\infty(\vec{k},\vec{x})e^{-it\lambda_\infty(\vec{k})}\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\,d\vec{k}, \end{aligned} \tag{3.10}$$

and, therefore, $\|Xw\|^2_{L^2(B_R)} > \frac{t^2}{3}\|\phi\|^2_{L^2(B_R)} - \|v\|^2_{L^2(B_R)} - \|\phi_s\|^2_{L^2(B_R)}$. Integrating the last inequality with respect to $t$, we obtain:

$$\frac{2}{T}\int_0^\infty e^{-2t/T}\|Xw\|^2_{L^2(B_R)}\,dt$$

$$\geq \frac{1}{3}\cdot\frac{2}{T}\int_0^\infty t^2 e^{-2t/T}\|\phi\|^2_{L^2(B_R)}\,dt - \frac{2}{T}\int_0^\infty e^{-2t/T}\|v\|^2_{L^2(B_R)}\,dt$$
$$-\frac{2}{T}\int_0^\infty e^{-2t/T}\|\phi_s\|^2\,dt =: \frac{1}{3}I_1 - I_2 - I_3. \tag{3.11}$$

Now we show that:

$$I_1 \geq 18(c_1T^2 - c_2T), \tag{3.12}$$
$$I_2 \leq c\delta^{-2}\|\widehat{\Psi}_0\|^2_{W_2^1(\mathbb{R}^2)}, \tag{3.13}$$
$$I_3 \leq c(V)\|\widehat{\Psi}_0\|^2_{L^2(\mathbb{R}^2)}. \tag{3.14}$$

Let us prove (3.13) first. From (3.9), we see that $v = S_\infty(\widetilde{\mathcal{G}}_\infty)g_2$, where $g_2 = e^{-it\lambda_\infty(\vec{k})}\nabla\big(\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\big)$, and, therefore, by (2.40), we get

$$\begin{aligned}\|v\|_{L^2(\mathbb{R}^2)} &\leq 2\|\nabla\Big(\widehat{\Psi}_0\eta_\delta\Big)\|_{L^2(\mathbb{R}^2)}\\ &\leq 2\Big(\|\nabla\widehat{\Psi}_0\|_{L^2(\mathbb{R}^2)} + \|\widehat{\Psi}_0\|_{L^2(\mathbb{R}^2)}\|\nabla\eta_\delta\|_{L^\infty(\mathbb{R}^2)}\Big)\\ &\leq c\delta^{-1}\|\widehat{\Psi}_0\|_{W_2^1(\mathbb{R}^2)}\end{aligned}$$

Now (3.13) is obvious.

Estimate (3.14) can be obtained in the same way as (2.25) or (2.39) with $\nabla_{\vec{k}}u_\infty$ instead of $u_\infty$ (see Appendix 3 for details).

Finally, we show the estimate (3.12). Substituting (2.1) into (3.10) yields

$$\begin{aligned}\phi(\vec{x},t) &= \frac{1}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty}\nabla\lambda_\infty(\vec{k})e^{i\langle\vec{k},\vec{x}\rangle}e^{-it\lambda_\infty(\vec{k})}\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\,d\vec{k}\\ &\quad+\frac{1}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty}\nabla\lambda_\infty(\vec{k})e^{i\langle\vec{k},\vec{x}\rangle}u_\infty(\vec{k},\vec{x})e^{-it\lambda_\infty(\vec{k})}\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\,d\vec{k}\\ &=: \widetilde{\phi}(\vec{x},t) + \widetilde{\phi}_s(\vec{x},t).\end{aligned}$$

We use

$$\|\phi\|^2_{L^2(B_R)} \geq \frac{1}{2}\|\widetilde{\phi}\|^2_{L^2(B_R)} - \|\widetilde{\phi}_s\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{2}\|\widetilde{\phi}\|^2_{L^2(\mathbb{R}^2)} - \frac{1}{2}\|\widetilde{\phi}\|^2_{L^2(\mathbb{R}^2\setminus B_R)} - \|\widetilde{\phi}_s\|^2_{L^2(\mathbb{R}^2)}.$$

Thus,

$$\begin{aligned}\frac{2}{T}\int_0^\infty t^2e^{-2t/T}\|\phi\|^2_{L^2(B_R)}\,dt &= \frac{2}{T}\int_0^\infty t^2e^{-2t/T}\left(\frac{1}{2}\|\widetilde{\phi}\|^2_{L^2(\mathbb{R}^2)} - \|\widetilde{\phi}_s\|^2_{L^2(\mathbb{R}^2)}\right)dt\\ -\frac{1}{T}\int_0^\infty t^2e^{-2t/T}\|\widetilde{\phi}\|^2_{L^2(\mathbb{R}^2\setminus B_R)}\,dt &=: R_1 - R_2.\end{aligned} \tag{3.15}$$

To get a lower bound for $R_1$, we notice that

$$\frac{1}{2}\|\widetilde{\phi}\|^2_{L^2(\mathbb{R}^2)} - \|\widetilde{\phi}_s\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{2}\|S_0(\widetilde{\mathcal{G}}_\infty)g_3\|^2_{L^2(\mathbb{R}^2)} - \|(S_\infty(\widetilde{\mathcal{G}}_\infty) - S_0(\widetilde{\mathcal{G}}_\infty))g_3\|^2_{L^2(\mathbb{R}^2)} \tag{3.16}$$

where $g_3(\vec{k}) := \nabla\lambda_\infty(\vec{k})\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})$. Now, using (2.39) with $c(V)\lambda_{**}^{-\gamma_6} \le 1/4$ and noticing that $S_0$ is just the Fourier transform, we get

$$\begin{aligned}&\frac{1}{2}\|\widetilde{\phi}\|^2_{L^2(\mathbb{R}^2)} - \|\widetilde{\phi}_s\|^2_{L^2(\mathbb{R}^2)} \ge (\frac{1}{2}-\frac{1}{4})\|g_3\|^2_{L^2(\widetilde{\mathcal{G}}_\infty)}\\ &= \frac{1}{4}\int_{\widetilde{\mathcal{G}}_\infty} |\nabla\lambda_\infty(\vec{k})|^2|\widehat{\Psi}_0(\vec{k})|^2\eta_\delta(\vec{k})^2\,d\vec{k} \ge \frac{1}{4}\int_{\mathcal{G}_\infty}|\vec{k}|^2|\widehat{\Psi}_0(\vec{k})|^2\,d\vec{k}.\end{aligned} \tag{3.17}$$

Here we also used that on $\mathcal{G}_\infty$ we have $\eta_\delta = 1$ and $|\nabla\lambda_\infty| \ge |\vec{k}|$.

The bound (3.17) immediately implies the main estimate of the paper:

$$R_1 \ge \frac{1}{8}T^2\int_{\mathcal{G}_\infty}|\vec{k}|^2|\widehat{\Psi}_0(\vec{k})|^2\,d\vec{k} =: 20c_1T^2, \tag{3.18}$$

$$c_1 = c_1(\Psi_0) := \frac{1}{160}\int_{\mathcal{G}_\infty}|\vec{k}|^2|\widehat{\Psi}_0(\vec{k})|^2\,d\vec{k}. \tag{3.19}$$

For $R_2$, let us introduce a new variable $\vec{z} := \vec{x}/t$ and consider

$$\widetilde{\phi}(\vec{z}t,t) = \frac{1}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty} e^{it\left(\langle\vec{k},\vec{z}\rangle - \lambda_\infty(\vec{k})\right)}g_3(\vec{k})\,d\vec{k}. \tag{3.20}$$

We use the method of stationary phase and integration by parts. Considering (2.34) and Lemma 2.1, we conclude that the equation for a stationary point

$$\vec{z} - \nabla\lambda_\infty(\vec{k}) = 0$$

has a unique solution $\vec{k}_0(z) := \vec{k}_0$ and

$$\vec{k}_0 = \frac{1}{2}\vec{z} + O(|\vec{z}|^{-\gamma_5}), \quad \gamma_5 > 0.$$

Let $\eta$ be a smooth cut-off function satisfying

$$\eta(\vec{k}) = \begin{cases} 0, & \left|\vec{k}-\vec{k}_0\right| \le 1 \\ 1, & \left|\vec{k}-\vec{k}_0\right| \ge 2\end{cases}.$$

Then,

$$\widetilde{\phi}(\vec{z}t,t) = \frac{1}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty\cap\{\vec{k}\,:\,|\vec{k}-\vec{k}_0|<2\}} e^{it\left(\langle\vec{k},\vec{z}\rangle-\lambda_\infty(\vec{k})\right)}g_3(\vec{k})\big(1-\eta(\vec{k})\big)\,d\vec{k} \tag{3.21}$$

$$+\frac{1}{2\pi}\int_{\widetilde{\mathcal{G}}_\infty\cap\{\vec{k}:\,|\vec{k}-\vec{k}_0|>1\}} e^{it\left(\langle\vec{k},\vec{z}\rangle-\lambda_\infty(\vec{k})\right)}g_3(\vec{k})\eta(\vec{k})\,d\vec{k} \tag{3.22}$$

$$=: \widetilde{\phi}_1(\vec{z}t,t) + \widetilde{\phi}_2(\vec{z}t,t). \tag{3.23}$$

To estimate $\widetilde{\phi}_1(\vec{z}t,t)$, we first note that $g_3(1-\eta) \in \mathcal{C}_0^4(\mathbb{R}^2)$ and $\langle\vec{k},\vec{z}\rangle - \lambda_\infty(\vec{k}) \in \mathcal{C}^7(\mathbb{R}^2)$, the estimate (2.35) holding for $|m| \le 7$ with $-\gamma_2 + 7\gamma_0 < 0$. Therefore, applying Theorem 7.7.5 in [16] yields:

$$\widetilde{\phi}_1(\vec{z}t,t) = \frac{1}{2i}e^{it\left(\langle\vec{k}_0,\vec{z}\rangle-\lambda_\infty(\vec{k}_0)\right)}\left(1+O(|\vec{z}|^{-\gamma_5})\right)g_3(\vec{k}_0)t^{-1} + \epsilon(g_3)t^{-2} \tag{3.24}$$

for $|z|^2 > \lambda_*$ and 0 otherwise. Here

$$|\epsilon(g_3)| \leq c \sum_{|m|\leq 4} \sup_{|\vec{k}-\vec{k}_0|<2} |D^m g_3(\vec{k})| \leq c \left\| |\vec{k}|^3 \widehat{\Psi}_0(\vec{k}) \right\|_{\mathcal{C}^4(\mathbb{R}^2)} \delta^{-4} |\vec{z}|^{-2},$$

Next, we consider $\widetilde{\phi}_2(\vec{z}t, t)$. There is no stationary point. Integrating by parts twice, we obtain

$$|\widetilde{\phi}_2(\vec{z}t, t)| \leq C(\widehat{\Psi}_0)(\delta t)^{-2}(1 + |\vec{z}|)^{-2}, \tag{3.25}$$

where $C(\widehat{\Psi}_0)$ is a combination of integrals of the type $\int |\vec{k}|^j |D^m \widehat{\Psi}_0(\vec{k})| d\vec{k}$, $0 \leq j \leq 3$, $0 \leq |m| \leq 2$.

Now, we consider $\|\widetilde{\phi}(\vec{x}, t)\|^2_{L^2(\mathbb{R}^2 \setminus B_R)}$. Using the estimates (3.24) and (3.25), we obtain

$$\|\widetilde{\phi}(\vec{x}, t)\|^2_{L^2(\mathbb{R}^2 \setminus B_R)} = t^2 \|\widetilde{\phi}(\vec{z}t, t)\|^2_{L^2(\mathbb{R}^2 \setminus B_{R/t})} \leq \int_{\mathbb{R}^2 \setminus B_{c_0 T/t}} |g_3(\vec{k}_0(\vec{z}))|^2 \, d\vec{z} + O(t^{-1})$$

as $t \to \infty$, the constant in $O(t^{-1})$ depending on $\delta$ and $\widehat{\Psi}_0$. Next, substituting the above estimate into the formula for $R_2$ (see (3.15)) and changing the variables $s = t/T$, we obtain:

$$R_2 \leq T^2 \int_0^\infty s^2 e^{-2s} \int_{\mathbb{R}^2 \setminus B_{c_0/s}} |g_3(\vec{k}_0(\vec{z}))|^2 \, d\vec{z} \, ds + O(T). \tag{3.26}$$

By Lebesgue's Dominated Convergence Theorem, the integral on (3.26) goes to zero when $c_0 \to \infty$ uniformly in $\delta$. We choose $c_0$ large enough to ensure that

$$R_2 \leq c_1 T^2 + cT,$$

the constant $c_1$ being defined by (3.19). Notice that the choice of $c_0$ depends on $\widehat{\Psi}_0$, but not $\delta$. Considering the last estimate together with (3.18), we obtain (3.12).

∎

*Proof of* (1.12). After $c_0$ is fixed as above we choose a sufficiently small $\delta = \delta(c_0, \widehat{\Psi}_0)$ so that the constant $\gamma c_0^2$ from (3.7) is smaller than $c_1$. Thus, we obtain:

$$\frac{2}{T} \int_0^\infty e^{-2t/T} \left\| X \Psi(\cdot, t) \right\|^2_{L^2(\mathbb{R}^2)} dt > 2c_1(\Psi_0) T^2 - c_2(\Psi_0) T - c_3(\Psi_0), \quad c_1 > 0. \tag{3.27}$$

Taking $T$ sufficiently large, we obtain (1.12) for any non-zero $\Psi_0 \in E_\infty \mathcal{C}_0^\infty$.

∎

## 4. Proof of Theorem 1.2

Now we prove Theorem 1.2. Let $\Psi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ then by Lemma 2.2, $\widehat{\Psi}_0 \in \mathcal{C}^L$ decays faster than any polynomials of degree at most $L$, where

$$\widehat{\Psi}_0(\vec{k}) = (T_\infty \Psi_0)(\vec{k}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \overline{\Psi_\infty(\vec{k}, \vec{x})} \Psi_0(\vec{x}) \, d\vec{x}. \tag{4.1}$$

We denote

$$\Psi_{0,\mathrm{ac}} := E_\infty(\mathcal{G}_\infty)\Psi_0 = \frac{1}{2\pi} \int_{\mathcal{G}_\infty} \Psi_\infty(\vec{k}, \vec{x}) \widehat{\Psi}_0(\vec{k}) \, d\vec{k}$$

and

$$\Psi_{0,\mathrm{s}} := \Psi_0 - \Psi_{0,\mathrm{ac}}.$$

We notice that $\Psi_{0,\mathrm{s}} \perp E_\infty L^2(\mathbb{R}^2)$ and $\|\Psi_{0,\mathrm{s}}\|_{L^2(\mathbb{R}^2)} \le \|\Psi_0\|_{L^2(\mathbb{R}^2)}$. Assume that $\Psi_{0,\mathrm{ac}}$ is not identically zero. We put

$$\Psi(\vec{x},t) = \Psi_{\mathrm{ac}}(\vec{x},t) + \Psi_{\mathrm{s}}(\vec{x},t) := e^{-itH}\Psi_{0,\mathrm{ac}} + e^{-itH}\Psi_{0,\mathrm{s}}.$$

As in the proof of Theorem 1.1, we use

$$\|X\Psi\|_{L^2(\mathbb{R}^2)} \ge \|X\Psi\|_{L^2(B_R)}$$

and approximate $\Psi_{\mathrm{ac}}$ by $w$ defined as in (3.4). Next, we rewrite

$$\|X(\Psi_{\mathrm{s}} + w)\|^2_{L^2(B_R)} = \|X\Psi_{\mathrm{s}}\|^2_{L^2(B_R)} + \|Xw\|^2_{L^2(B_R)} + 2\Re(X\Psi_{\mathrm{s}}, Xw)_{L^2(B_R)}.$$

Let us note that $(X\Psi_{\mathrm{s}}, Xw)_{L^2(B_R)} = (\Psi_{\mathrm{s}}, X^2 w)_{L^2(B_R)}$ and consider its integral over $t$:

$$\hat{I} := \frac{2}{T}\int_0^\infty e^{-2t/T} |(\Psi_{\mathrm{s}}, X^2 w)_{L^2(B_R)}|\, dt. \tag{4.2}$$

Considering (3.6), we see that it is enough to show that $\hat{I}$ is small compared with the r.h.s. of (3.6). We achieve this by proving that $X^2 w$ is orthogonal to $\Psi_{\mathrm{s}}$ up to minor terms. Indeed, by (3.4) and (2.1),

$$\begin{aligned}(X^2 w)(\vec{x},t) &= \frac{1}{2\pi}\int_{\tilde{\mathcal{G}}_\infty} |\vec{x}|^2 \Psi_\infty(\vec{k},\vec{x}) e^{-it\lambda_\infty(\vec{k})} \widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\, d\vec{k} \\ &= -\frac{1}{2\pi}\int_{\tilde{\mathcal{G}}_\infty} (\Delta_{\vec{k}} e^{i\langle \vec{k},\vec{x}\rangle})(1 + u_\infty(\vec{k},\vec{x})) e^{-it\lambda_\infty(\vec{k})} \widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})\, d\vec{k}.\end{aligned}$$

Using $g_4(\vec{k},\vec{x}) := (1 + u_\infty(\vec{k},\vec{x}))\widehat{\Psi}_0(\vec{k})\eta_\delta(\vec{k})$ and applying integration by parts as above, we obtain

$$\begin{aligned}(X^2 w)(\vec{x},t) = {} & t^2 \frac{1}{2\pi}\int_{\tilde{\mathcal{G}}_\infty} e^{i\langle \vec{k},\vec{x}\rangle} \left|\nabla\lambda_\infty(\vec{k})\right|^2 e^{-it\lambda_\infty(\vec{k})} g_4(\vec{k},\vec{x})\, d\vec{k} \\ & - \frac{1}{2\pi}\int_{\tilde{\mathcal{G}}_\infty} e^{i\langle \vec{k},\vec{x}\rangle} e^{-it\lambda_\infty(\vec{k})} \Delta_{\vec{k}} g_4(\vec{k},\vec{x})\, d\vec{k} \\ & + t\frac{i}{\pi}\int_{\tilde{\mathcal{G}}_\infty} e^{i\langle \vec{k},\vec{x}\rangle} e^{-it\lambda_\infty(\vec{k})} \left\langle \nabla\lambda_\infty(\vec{k}), \nabla_{\vec{k}} g_4(\vec{k},\vec{x})\right\rangle\, d\vec{k} \\ & + t\frac{i}{2\pi}\int_{\tilde{\mathcal{G}}_\infty} e^{i\langle \vec{k},\vec{x}\rangle} \left(\Delta\lambda_\infty(\vec{k})\right) e^{-it\lambda_\infty(\vec{k})} g_4(\vec{k},\vec{x})\, d\vec{k}.\end{aligned}$$

The last three integrals can be estimated as in the proof of (3.6) (see (3.13),(3.14)), and the corresponding contribution to $\hat{I}$ is bounded by a linear function of $T$ for every fixed $\delta > 0$. For the first integral, we have

$$\begin{aligned}& t^2 \frac{1}{2\pi}\int_{\tilde{\mathcal{G}}_\infty} e^{i\langle \vec{k},\vec{x}\rangle} \left|\nabla\lambda_\infty(\vec{k})\right|^2 e^{-it\lambda_\infty(\vec{k})} g_4(\vec{k},\vec{x})\, d\vec{k} \\ & = t^2 \frac{1}{2\pi}\int_{\mathcal{G}_\infty} \Psi_\infty(\vec{k},\vec{x}) \left|\nabla\lambda_\infty(\vec{k})\right|^2 e^{-it\lambda_\infty(\vec{k})} \widehat{\Psi}_0(\vec{k})\, d\vec{k}\end{aligned}$$

$$
\begin{aligned}
&+ t^2 \frac{1}{2\pi} \int_{\tilde{\mathcal{G}}_\infty \setminus \mathcal{G}_\infty} \Psi_\infty(\vec{k}, \vec{x}) \left| \nabla \lambda_\infty(\vec{k}) \right|^2 e^{-it\lambda_\infty(\vec{k})} \widehat{\Psi}_0(\vec{k}) \eta_\delta(\vec{k}) \, d\vec{k} \\
&=: t^2 (J_1 + J_2).
\end{aligned}
$$

Obviously, $\|J_2\|_{L^2(\mathbb{R}^2)} = o(1)$ as $\delta \to 0$ uniformly in $t$ (cf. (3.8)) and its contribution to $\hat{I}$ is bounded by $\gamma T^2$, where $\gamma(\delta, \Psi_0) \to 0$ as $\delta \to 0$ . To estimate the contribution from $J_1$ we notice that $J_1 = E_\infty(\mathcal{G}_\infty) J_1$ and, thus, we arrive at the main point of the proof:

$$
(\Psi_s, J_1)_{L^2(B_R)} = -(\Psi_s, J_1)_{L^2(\mathbb{R}^2 \setminus B_R)}.
$$

It remains to estimate

$$
\hat{I}_1 = \frac{2}{T} \int_0^\infty t^2 e^{-2t/T} |(\Psi_s, J_1)_{L^2(\mathbb{R}^2 \setminus B_R)}| dt. \tag{4.3}
$$

It is easy to see that

$$
\begin{aligned}
\hat{I}_1 &\leq \frac{2}{T} \int_0^\infty t^2 e^{-2t/T} (\epsilon \|\Psi_s\|^2_{L^2(\mathbb{R}^2 \setminus B_R)} + \frac{1}{4\epsilon} \|J_1\|^2_{L^2(\mathbb{R}^2 \setminus B_R)}) dt \\
&\leq \epsilon C(\Psi_0) T^2 + \frac{1}{\epsilon T} \int_0^\infty t^2 e^{-2t/T} (\|J_1 + J_2\|^2_{L^2(\mathbb{R}^2 \setminus B_R)} + \|J_2\|^2_{L^2(\mathbb{R}^2 \setminus B_R)}) dt.
\end{aligned} \tag{4.4}
$$

The estimate for the integral with $J_1 + J_2$ is similar to the estimate for $R_2$ (see (3.26)), while the estimate for the integral with $J_2$ repeats the proof for (3.8). Thus, (4.4) is bounded by

$$
\epsilon C(\Psi_0) T^2 + \frac{1}{2\epsilon} \left( T^2 \hat{\gamma}(c_0, \Psi_0) + C(\Psi_0, \delta) T + T^2 \gamma(\delta, \Psi_0) \right),
$$

where $\hat{\gamma}(c_0, \Psi_0) \to 0$ as $c_0 \to \infty$ and $\gamma(\delta, \Psi_0) \to 0$ as $\delta \to 0$. Now, one chooses small $\epsilon$, then large $c_0$, small $\delta$ and large $T_0$ to prove (4.2).

## 5. Appendices

Here we provide detailed proofs of some of the facts which were used in Sections 2 and 3 above.

**Remark 3.** *Using the a priori estimates (see* [20]*) for the solutions* $\Psi_n$ *from* (2.9) *and their Fourier coefficients defined by*

$$
\Psi_n(\vec{k}, \vec{x}) = e^{i\langle \vec{k}, \vec{x} \rangle} \left( 1 + u_n(\vec{k}, \vec{x}) \right), \tag{5.1}
$$

$$
u_n(\vec{k}, \vec{x}) = \sum_{r \in \mathbb{Z}^2} C_r^{(n)}(\vec{k}) e^{i\langle \vec{p}_r^{(n)}, \vec{x} \rangle}, \tag{5.2}
$$

$\vec{p}_r^{(n)}$ *being vectors of the dual lattice corresponding to* $W_n$*, and repeating the arguments which led to Lemma 2.1, one can obtain that the extended coefficients are sufficiently smooth and satisfy estimates of the type* (2.29)*,* (2.32)*,* (2.35)*. We omit the details.*

### 5.1. **Appendix 1.**

**Lemma 5.1.** *The sequence of operators $S_n(\mathcal{G}_{\infty,\lambda})$ given by (2.24) has a limit $S_\infty(\mathcal{G}_{\infty,\lambda})$ in the class of bounded operators. The convergence of $S_n(\mathcal{G}_{\infty,\lambda})$ to $S_\infty(\mathcal{G}_{\infty,\lambda})$ is uniform in $\lambda$ and estimate* (2.25) *holds.*

*Proof.* It suffices to prove that $S_n(\mathcal{G}_{\infty,\lambda})f$ is a Cauchy sequence. Given $Q_n$ is the cell of periods of the operator $H^{(n)}$, the function $\Psi_n(\vec{k},x)$ is quasi-periodic in $Q_n$. It can be represented as a combination of plane waves (5.1), (5.2). The Fourier transform of $\widehat{\Psi}_n$ is a combination of $\delta$-functions:

$$\widehat{\Psi}_n(\vec{k},\vec{\xi}) = \sum_{r\in\mathbb{Z}^2} C_r^{(n)}(\vec{k})\delta\big(\vec{\xi}+\vec{k}+\vec{p}_r(0)/\tilde{N}_n\big).$$

From this, we compute easily the Fourier transform of $S_nf$

$$(\widehat{S_nf})(\vec{\xi}) = \frac{1}{2\pi}\sum_{r\in\mathbb{Z}^2} C_r^{(n)}\big(-\vec{\xi}-\vec{p}_r(0)/\tilde{N}_n\big)f\big(-\vec{\xi}-\vec{p}_r(0)/\tilde{N}_n\big)\chi\big(\mathcal{G}_{\infty,\lambda},-\vec{\xi}-\vec{p}_r(0)/\tilde{N}_n\big),$$

where $\chi(\mathcal{G}_{\infty,\lambda},\cdot)$ is the characteristic function on $\mathcal{G}_{\infty,\lambda}$. Since $\mathcal{G}_{\infty,\lambda}$ is bounded, the series contains only a finite number of non-zero terms for every $\vec{\xi}$. By Parseval's identity, triangle inequality and a parallel shift of the variable,

$$\|S_nf\|_{L^2(\mathbb{R}^2)} = \|\widehat{S_nf}\|_{L^2(\mathbb{R}^2)}$$

$$\leq \frac{1}{2\pi}\sum_{r\in\mathbb{Z}^2}\Big\|C_r^{(n)}\big(-\vec{\xi}-\vec{p}_r(0)/\tilde{N}_n\big)f\big(-\vec{\xi}-\vec{p}_r(0)/\tilde{N}_n\big)\chi\big(\mathcal{G}_{\infty,\lambda},-\vec{\xi}-\vec{p}_r(0)/\tilde{N}_n\big)\Big\|_{L^2(\mathbb{R}^2)}$$

$$= \frac{1}{2\pi}\sum_{r\in\mathbb{Z}^2}\|C_r^{(n)}(\vec{k})f(\vec{k})\|_{L^2(\mathcal{G}_{\infty,\lambda})}.$$

Assume first that the support of $f$ belongs to a ring $R_{k,2k}$ for some $k$ such that $k^2 > \lambda_*(V)$. Then, the last inequality yields:

$$\|S_nf\|_{L^2(\mathbb{R}^2)} \leq \frac{1}{2\pi}\|f\|_{L^2(R_{k,2k})}\sum_{r\in\mathbb{Z}^2}\|C_r^{(n)}\|_{L^\infty(R_{k,2k})}. \tag{5.3}$$

By (5.2), Fourier coefficients $C_r^{(n)}(\vec{k})$ can be estimated as follows:

$$p_r^4(0)|C_r^{(n)}(\vec{k})| \leq 2\pi\|\Psi_n(\vec{k},\cdot)\exp\Big(-i\langle\vec{k},\cdot\rangle\Big)\|_{W_2^4(Q_n)}|Q_n|^{-1/2}\tilde{N}_n^4$$

$$\leq 16\pi|\vec{k}|^4\|\Psi_n(\vec{k},\cdot)\|_{W_2^4(Q_n)}|Q_n|^{-1/2}\tilde{N}_n^4.$$

Considering that $\sum_{r\neq 0}p_r^{-4}(0) < c$, we obtain:

$$\sum_{r\in\mathbb{Z}^2}\|C_r^{(n)}\|_{L^\infty(R_{k,2k})} < c\sup_{\vec{k}\in R_{k,2k}}\left(|\vec{k}|^4\|\Psi_n(\vec{k},\cdot)\|_{W_2^4(Q_n)}|Q_n|^{-1/2}\tilde{N}_n^4\right).$$

Using (5.3), we arrive at

$$\|S_nf\|_{L^2(\mathbb{R}^2)} < ck^4\|f\|_{L^2(R_{k,2k})}\sup_{\vec{k}\in R_{k,2k}}\left(|Q_n|^{-1/2}\tilde{N}_n^4\sup_{\vec{k}\in R_{k,2k}}\|\Psi_n(\vec{k},\cdot)\|_{W_2^4(Q_n)}\right).$$

Similarly,

$$\|(S_{n+1}-S_n)f\|_{L^2(\mathbb{R}^2)} <$$
$$ck^4\|f\|_{L^2(R_{k,2k})}\sup_{\vec{k}\in R_{k,2k}}\left(|Q_{n+1}|^{-1/2}\tilde{N}_{n+1}^4\sup_{\vec{k}\in R_{k,2k}}\|\big(\Psi_{n+1}(\vec{k},\cdot)-\Psi_n(\vec{k},\cdot)\big)\|_{W_2^4(Q_{n+1})}\right).$$

It is proven in [23] (Section 5.2) that

$$\|\Psi_{n+1}(\vec{k},\cdot)-\Psi_n(\vec{k},\cdot)\|_{L^2(Q_{n+1})} < c\epsilon_n^3|Q_{n+1}|^{1/2}, \quad n\geq 1, \quad \text{when} \quad \vec{k}\in R_{k,2k}. \tag{5.4}$$

Applying the equation for eigenfunctions twice, we arrive to:

$$\|\Psi_{n+1}(\vec{k},\cdot)-\Psi_n(\vec{k},\cdot)\|_{W_2^4(Q_{n+1})} < ck^4\epsilon_n^3|Q_{n+1}|^{1/2}, \quad n\geq 1, \quad \text{when} \quad \vec{k}\in R_{k,2k}. \tag{5.5}$$

Using the last estimate, we obtain

$$\|(S_n-S_{n+1})f\|_{L^2(\mathbb{R}^2)} \leq ck^8\|f\|_{L^2(R_{k,2k})}\sup_{\vec{k}\in R_{k,2k}}\left(\tilde{N}_n^4\epsilon_n^3\right). \tag{5.6}$$

when the support of $f$ is in $R_{k,2k}$. Considering that $\epsilon_n$ decays super-exponentially with $n$ (see the formula above (2.31) and the estimate $\tilde{N}_n \approx k^{s_n}$, we conclude that

$$\|(S_n-S_{n+1})f\|_{L^2(\mathbb{R}^2)} \leq c\|f\|_{L^2(R_{k,2k})}\epsilon_n^2(k), \tag{5.7}$$

i.e., $S_nf$ is a Cauchy sequence in $L^2(\mathbb{R}^2)$ for every $f\in L^2\left(R_{k,2k}\right)$.

If $f\in L^2(\mathcal{G}_{\infty,\lambda})$, then we can express it as a sum of functions $f_k$ such that $f_k$ has support in $R_{k,2k}$. Summing up estimates (5.7) over all $k$ and using the Cauchy-Schwatz inequality on the right, we easily see that:

$$\|(S_n-S_{n+1})f\|_{L^2(\mathbb{R}^2)} \leq c\|f\|_{L^2(\mathcal{G}_{\infty,\lambda})}\epsilon_n(k_*), \quad n\geq 1, \tag{5.8}$$

i.e., $S_n$ is a Cauchy sequence in the space of bounded operators. We denote the limit of $S_n(\mathcal{G}_{\infty,\lambda})f$ by $S_\infty(\mathcal{G}_{\infty,\lambda})f$. By Theorem 2.3 in [23]

$$\|(S_0-S_1)\|_{L^2(\mathbb{R}^2)} < \lambda_*^{-\gamma_6}. \tag{5.9}$$

Estimate (2.25) easily follows. ∎

5.2. **Appendix 2 (Proof of Lemma 2.2).** Next, using (5.1), (5.2) and integrating by parts $j$ times in (2.18), we obtain:

$$\begin{aligned}\left|(T_nF)(\vec{k})\right| \quad \leq \quad & \left(\sum_{r:|\vec{k}+\vec{p}_r^{(n)}|\geq|\vec{k}/4|}\frac{|C_r^{(n)}(\vec{k})|}{|\vec{k}+\vec{p}_r^{(n)}|^j}\right)\|F\|_{W_j^1(\mathbb{R}^2)} \\ & + \left(\sum_{r:|\vec{k}+\vec{p}_r^{(n)}|<|\vec{k}/4|}|C_r^{(n)}(\vec{k})|\right)\|F\|_{L^1(\mathbb{R}^2)}.\end{aligned} \tag{5.10}$$

Noting that

$$\sum_r |C_r^{(n)}(\vec{k})| < 2, \tag{5.11}$$

we can estimate the first term in the right hand side of (5.10), i.e.,

$$|\vec{k}|^j \left( \sum_{r:|\vec{k}+\vec{p}_r^{(n)}|\geq|\vec{k}/4|} \frac{|C_r^{(n)}(\vec{k})|}{|\vec{k}+\vec{p}_r^{(n)}|^j} \right) \|F\|_{W_j^1(\mathbb{R}^2)} \leq 2^{2j+2}\|F\|_{W_j^1(\mathbb{R}^2)}. \tag{5.12}$$

Next, since $H\Psi_n = \lambda_n \Psi_n$, we have

$$\left( H\Psi_n, \frac{e^{i\langle \vec{k}+\vec{p}_r^{(n)},\cdot\rangle}}{|Q_n|} \right) = \lambda_n(\vec{k}) C_r^{(n)}(\vec{k}). \tag{5.13}$$

Note that

$$\left( H\Psi_n, \frac{e^{i\langle \vec{k}+\vec{p}_r^{(n)},\cdot\rangle}}{|Q_n|} \right) = \left( \Psi_n, \frac{|\vec{k}+\vec{p}_r^{(n)}|^2 e^{i\langle \vec{k}+\vec{p}_r^{(n)},\cdot\rangle}}{|Q_n|} \right) + \left( \Psi_n, W_n \frac{e^{i\langle \vec{k}+\vec{p}_r^{(n)},\cdot\rangle}}{|Q_n|} \right).$$

Since the length of $V_r$ grows at most linearly with period, i.e., if $|\vec{p}_r^{(n)} - \vec{p}_{r'}^{(n)}| \geq R_0$, then $(W_n)_{r-r'} = 0$, we get

$$\left( \Psi_n, W_n \frac{e^{i\langle \vec{k}+\vec{p}_r^{(n)},\cdot\rangle}}{|Q_n|} \right) = \sum_{r':|\vec{p}_{r'}^{(n)}-\vec{p}_r^{(n)}|<R_0} \left( \Psi_n, (W_n)_{r-r'} \frac{e^{i\langle \vec{k}+\vec{p}_{r'}^{(n)},\cdot\rangle}}{|Q_n|} \right).$$

Hence,

$$C_r^{(n)}(\vec{k}) = (\lambda_n(\vec{k}) - |\vec{k}+\vec{p}_r^{(n)}|^2)^{-1} \sum_{r':|\vec{p}_{r'}^{(n)}-\vec{p}_r^{(n)}|<R_0} (W_n)_{r-r'} C_{r'}^{(n)}(\vec{k}). \tag{5.14}$$

and therefore

$$\begin{aligned} \sum_{r:|\vec{k}+\vec{p}_r^{(n)}|<|\vec{k}/2|} |C_r^{(n)}(\vec{k})| &\leq \sum_{r:|\vec{k}+\vec{p}_r^{(n)}|<|\vec{k}/2|} \sum_{r':|\vec{p}_{r'}^{(n)}-\vec{p}_r^{(n)}|<R_0} \frac{|(W_n)_{r-r'}||C_{r'}^{(n)}(\vec{k})|}{|\lambda_n(\vec{k}) - |\vec{k}+\vec{p}_r^{(n)}|^2|} \\ &\leq \frac{C}{|\vec{k}|^2} \sum_{r,r'} |(W_n)_{r-r'}||C_{r'}^{(n)}(\vec{k})| \leq \frac{C(V)}{|\vec{k}|^2}. \end{aligned}$$

By a recursive argument, while $j < k_{**}/(4R_0)$, we obtain

$$\sum_{r:|\vec{k}+\vec{p}_r^{(n)}|<|\vec{k}/4|} |C_r^{(n)}(\vec{k})| \leq \frac{C(j,V)}{|\vec{k}|^j}. \tag{5.15}$$

Using (5.12) and (5.15) in (5.10), we obtain

$$\left| |\vec{k}|^j (T_n F)(\vec{k}) \right| \leq C(j,V,F).$$

The case $|m| > 0$ can be covered using integration by parts. To obtain the decay rate for $D^m(C_r^{(n)}(\vec{k})\eta_n(\vec{k}))$ one differentiates the recursive version of (5.14) and uses a priori estimates for $\sum_r |D^m(C_r^{(n)}(\vec{k})\eta_n(\vec{k}))|$ (see [20] and (5.1), (5.2)).

5.3. **Appendix 3 (Proof of** (3.14)**).** We consider the limit-periodic case. A proof for the quasi-periodic case is analogous. Indeed, we can write $u_\infty(\vec{k},\vec{x})$ as follows:

$$u_\infty(\vec{k},\vec{x}) = \sum_{n=1}^{\infty} \widetilde{u}_n(\vec{k},\vec{x}),$$

where $\widetilde{u}_n(\vec{k},\vec{x}) = \sum_{r\in\mathbb{Z}^2} \widetilde{C}_r^{(n)} e^{i\langle \vec{p}_r^{(n)},\vec{x}\rangle}$, $\vec{p}_r^{(n)}$ are vectors of the dual lattice corresponding to $W_n$. We obtain:

$$\begin{aligned}
&\|\phi_s\|^2_{L^2(B_R)} \le \|\phi_s\|^2_{L^2(\mathbb{R}^2)} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\widetilde{\mathcal{G}}_\infty}\int_{\widetilde{\mathcal{G}}_\infty} e^{i\langle \vec{k}-\vec{\xi},\vec{x}\rangle}\langle \nabla_{\vec{k}}\widetilde{u}_\infty(\vec{k},\vec{x}),\nabla_{\vec{\xi}}\widetilde{u}_\infty(\vec{\xi},\vec{x})\rangle e^{-it\left(\lambda_\infty(\vec{k})-\lambda_\infty(\vec{\xi})\right)} \\
&\qquad\qquad \widehat{\Psi}_0(\vec{k})\overline{\widehat{\Psi}_0(\vec{\xi})}\eta_\delta(\vec{k})\eta_\delta(\vec{\xi})\,d\vec{k}\,d\vec{\xi}\,d\vec{x} \\
&= \frac{1}{(2\pi)^2}\sum_{n,m\in\mathbb{N}}\sum_{r,q\in\mathbb{Z}^2}\int_{\mathbb{R}^2}\int_{\widetilde{\mathcal{G}}_\infty}\int_{\widetilde{\mathcal{G}}_\infty} e^{i\langle \vec{k}-\vec{\xi}+\vec{p}_r^{(n)}-\vec{p}_q^{(m)},\vec{x}\rangle}\langle \nabla\widetilde{C}_r^{(n)}(\vec{k}),\nabla\widetilde{C}_q^{(m)}(\vec{\xi})\rangle \\
&\qquad\qquad e^{-it\left(\lambda_\infty(\vec{k})-\lambda_\infty(\vec{\xi})\right)}\widehat{\Psi}_0(\vec{k})\overline{\widehat{\Psi}_0(\vec{\xi})}\eta_\delta(\vec{k})\eta_\delta(\vec{\xi})\,d\vec{k}\,d\vec{\xi}\,d\vec{x} \\
&= \sum_{n,m\in\mathbb{N}}\sum_{r,q\in\mathbb{Z}^2}\int_{\widetilde{\mathcal{G}}_\infty}\int_{\widetilde{\mathcal{G}}_\infty}\delta(\vec{k}-\vec{\xi}+\vec{p}_r^{(n)}-\vec{p}_q^{(m)})\langle \nabla\widetilde{C}_r^{(n)}(\vec{k}),\nabla\widetilde{C}_q^{(m)}(\vec{\xi})\rangle \\
&\qquad\qquad e^{-it\left(\lambda_\infty(\vec{k})-\lambda_\infty(\vec{\xi})\right)}\widehat{\Psi}_0(\vec{k})\overline{\widehat{\Psi}_0(\vec{\xi})}\eta_\delta(\vec{k})\eta_\delta(\vec{\xi})\,d\vec{k}\,d\vec{\xi} \\
&= \sum_{n,m\in\mathbb{N}}\sum_{r,q\in\mathbb{Z}^2}\int_{\widetilde{\mathcal{G}}_\infty\cap(\widetilde{\mathcal{G}}_\infty-\vec{p}_r^{(n)}+\vec{p}_q^{(m)})}\langle \nabla\widetilde{C}_r^{(n)}(\vec{k}),\nabla\widetilde{C}_q^{(m)}(\vec{k}+\vec{p}_r^{(n)}-\vec{p}_q^{(m)})\rangle \\
&\quad e^{-it\left(\lambda_\infty(\vec{k})-\lambda_\infty(\vec{k}+\vec{p}_r^{(n)}-\vec{p}_q^{(m)})\right)}\widehat{\Psi}_0(\vec{k})\overline{\widehat{\Psi}_0(\vec{k}+\vec{p}_r^{(n)}-\vec{p}_q^{(m)})}\eta_\delta(\vec{k})\eta_\delta(\vec{k}+\vec{p}_r^{(n)}-\vec{p}_q^{(m)})\,d\vec{k} \\
&\le \frac{1}{2}\sum_{n,m\in\mathbb{N}}\sum_{r,q\in\mathbb{Z}^2}\int_{\widetilde{\mathcal{G}}_\infty\cap(\widetilde{\mathcal{G}}_\infty-\vec{p}_r^{(n)}+\vec{p}_q^{(m)})}|\nabla\widetilde{C}_r^{(n)}(\vec{k})|\;|\nabla\widetilde{C}_q^{(m)}(\vec{k}+\vec{p}_r^{(n)}-\vec{p}_q^{(m)})|\;|\widehat{\Psi}_0(\vec{k})|^2\,d\vec{k} \\
&+ \frac{1}{2}\sum_{n,m\in\mathbb{N}}\sum_{r,q\in\mathbb{Z}^2}\int_{\widetilde{\mathcal{G}}_\infty\cap(\widetilde{\mathcal{G}}_\infty-\vec{p}_r^{(n)}+\vec{p}_q^{(m)})}|\nabla\widetilde{C}_r^{(n)}(\vec{k})|\;|\nabla\widetilde{C}_q^{(m)}(\vec{k}+\vec{p}_r^{(n)}-\vec{p}_q^{(m)})| \\
&\qquad\qquad |\widehat{\Psi}_0(\vec{k}+\vec{p}_r^{(n)}-\vec{p}_q^{(m)})|^2\,d\vec{k} \\
&= \int_{\widetilde{\mathcal{G}}_\infty}\left[\sum_{n,m\in\mathbb{N}}\sum_{r,q\in\mathbb{Z}^2}|\nabla\widetilde{C}_r^{(n)}(\vec{k})|\;|\nabla\widetilde{C}_q^{(m)}(\vec{k}+\vec{p}_r^{(n)}-\vec{p}_q^{(m)})|\chi_{\widetilde{\mathcal{G}}_\infty}(\vec{k}+\vec{p}_r^{(n)}-\vec{p}_q^{(m)})\right]|\widehat{\Psi}_0(\vec{k})|^2\,d\vec{k} \\
&\le c_4(V)\|\Psi_0\|^2_{L^2(\mathbb{R}^2)},
\end{aligned}$$

since $\sum_{n,m\in\mathbb{N}}\sum_{r,q\in\mathbb{Z}^2}|\nabla\widetilde{C}_r^{(n)}(\vec{k})|\;|\nabla\widetilde{C}_q^{(m)}(\vec{k}+\vec{p}_r^{(n)}-\vec{p}_q^{(m)})|\chi_{\widetilde{\mathcal{G}}_\infty}(\vec{k}+\vec{p}_r^{(n)}-\vec{p}_q^{(m)})$ is bounded uniformly in $\vec{k}\in\widetilde{\mathcal{G}}_\infty$, say, it is bounded by $c_4(V)$, see [20].

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