Two-scale series expansions for travelling wave packets in one-dimensional periodic media

Kirill D. Cherednichenko[1]

[1]Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK

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Abstract

Starting from the wave equation for a medium with material properties that vary periodically, we study a system of recurrence relations that describe propagation of wave packets that oscillate on the microscale (*i.e.* on lengths of the order of the period of the medium) and vary slowly on the macroscale (*i.e.* on lengths that contain a large number of periods). The resulting equations contain a version of the geometric optics and the overall energy transport description for periodic media.

1 Introduction

In the present article we study wave propagation through periodic composites. In particular, we are interested in wave-packet-like solutions $u = u(x, t, \varepsilon)$ (*i.e.* $u(\cdot, t, \varepsilon) \in L^2(\mathbb{R}^d)$ for all (t, ε)) to the wave equation

$$u_{tt} - \varepsilon^2 \mathrm{div}_x A(x/\varepsilon)\nabla_x u = 0, \tag{1}$$

where A is a uniformly positive-definite 1-periodic smooth matrix function. We will assume that $A^\top = A$ everywhere. Our aim is to develop a general asymptotic theory for wave packets that oscillate on the scale ε of the period in the coefficients of (1) with an amplitude that "varies slowly", *i.e.* whose gradient is of the order $O(1)$ as $\varepsilon \to 0$.

Our formal asymptotic expansion (see (3), (7)) is a generalisation of the series adopted by Allaire, Palombaro and Rauch in [1], [2], [3], where solutions to the Cauchy problem for (1) with "specially prepared" initial conditions are analysed in the regimes $t = O(1)$, $t = O(\varepsilon^{-1})$ as $\varepsilon \to 0$. We do not make any assumptions about the initial data, except for those that are required by the standard existence theory.

Our asymptotic procedure results in the eikonal equation (11), which is similar to the equation (35) in [2] but involves in addition the dependence of its solution on the "quasimomentum" $\varkappa$ in the integral representation (3). We consider general solutions to (11) which provide a general form of the amplitude function $u^{(0)}$ satisfying the transport equation (22). Our analysis of the "Gelfand transform" $\hat{u}$ in (3) is supplemented by the use of the method of stationary phase in order to provide the general asymptotic form (see Section 4) for a rapidly oscillating wavetrain with a slowly varying amplitude.

We conclude our analysis by the derivation of the propagation properties of the quasimomentum $\varkappa$, the "local wavenumber" k and the "energy" $\int |u|^2$, see Sections 5, 6. We show that these take place with the "group velocity" Ω' calculated at the given "macroscopic location" x, see (11).

2 One-dimensional formulation

In the present paper we focus on the (1+1)-version (*i.e.* $d = 1$) of the equation (1), where A is a scalar, for which we use the letter a:

$$u_{tt} - \varepsilon^2 \big(a(\cdot/\varepsilon)u_x\big)_x = 0. \tag{2}$$

Here a is a positive bounded scalar function with a bounded inverse.

Note first that $u = u^\varepsilon$ can be written in the form

$$u^\varepsilon(x,t) = \int_{Q'} \hat{u}\Big(x, \frac{x}{\varepsilon}, t, \varkappa, \varepsilon\Big) \exp\Big(\mathrm{i}\varkappa \frac{x}{\varepsilon}\Big) d\varkappa, \tag{3}$$

where $\hat{u}$ is the scaled Gelfand transform (see [6], [4]) of the function u with respect to the spatial variable x, written in the two-scale form:

$$\hat{u}(x, y, t, \varkappa, \varepsilon) = \varepsilon^d \sum_{n\in\mathbb{Z}} u(x + \varepsilon n, t) \exp\big[-\mathrm{i}(y + n)\big]\varkappa, \qquad \varkappa \in Q' := [0, 2\pi)^d. \tag{4}$$

The equation (2) implies that for each $\varkappa \in Q'$ the function $\hat{u}(x, x/\varepsilon, t, \varkappa, \varepsilon)$, which is ε-periodic in x, satisfies the equation

$$\hat{u}_{tt} - \Big(\varepsilon\frac{d}{dx} + \mathrm{i}\varkappa\Big) a(x/\varepsilon) \Big(\varepsilon\frac{d}{dx} + \mathrm{i}\varkappa\Big)\hat{u} = 0.$$

Introducing the "slow time" $\varepsilon t = \tau$ results in an equation for the function $\tilde{u}(\tau) := \hat{u}(\tau/\varepsilon)$:

$$\varepsilon^2 \tilde{u}_{\tau\tau} - \Big(\varepsilon\frac{d}{dx} + \mathrm{i}\varkappa\Big) a(x/\varepsilon) \Big(\varepsilon\frac{d}{dx} + \mathrm{i}\varkappa\Big)\tilde{u} = 0, \tag{5}$$

which we analyse in the next section.

Remark 2.1. *For each value of $\varkappa \in Q'$, both real and imaginary parts of the "elementary Floquet solution"*

$$\tilde{u}(x, t, \varkappa, \varepsilon) := \hat{u}\Big(x, \frac{x}{\varepsilon}, t, \varkappa, \varepsilon\Big) \exp\Big(\mathrm{i}\varkappa \frac{x}{\varepsilon}\Big) \tag{6}$$

is a solution to the original equation (2).

3 Asymptotic expansion

We are looking to determine a more specific form of the solution $\hat{u}$ to (5), as a (formal) asymptotic expansion in powers of the small parameter ε:

$$\tilde{u}(x, \tau, \varkappa, \varepsilon) = \exp\Big(-\frac{\mathrm{i}}{\varepsilon}\Phi(x, \tau, \varkappa, \varepsilon)\Big) \sum_{n=0}^{\infty} \varepsilon^n \mathcal{U}^{(n)}\Big(x, \frac{x}{\varepsilon}, \tau, \varkappa\Big) \tag{7}$$

Remark 3.1. *By representing $\hat{u}$ in the form (7) we make an implicit "slow time" assumption, namely, the phase velocity $\Phi_x^{-1}\Phi_\tau$ is much smaller than ε^{-1} as $\varepsilon \to 0$. In other words, the "typical relaxation time" of the solution u is much larger than the typical time it takes for the solution to travel across the ε-periodicity cell.*

Next, we write the phase function Φ as a power expansion with respect to the parameter ε :

$$\Phi(x, \tau, \varkappa, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \varphi^{(n)}(x, \tau, \varkappa). \tag{8}$$

In the following we always assume that the terms $\varphi_x^{(n)}$, $n = 0, 1, 2, ...$, in the power expansion for "local wavenumber" are uniformly bounded below in absolute value, for $(x, t, \varkappa) \in \mathbb{R} \times \mathbb{R} \times Q'$. Under this assumption the requirement of Remark 3.1 is satisfied, and in fact for small values of ε the phase velocity $\Phi_x^{-1}\Phi_\tau$ is of the order $Q(1)$.

3.1 Eikonal equation for the phase function

Substituting (7)–(8) into the equation (5) yields a system of recurrence relations for the amplitude $\mathcal{U}^{(n)} = \mathcal{U}^{(n)}(\varepsilon y, y, \tau, \varkappa)$ and the phase $\varphi^{(n)} = \varphi^{(n)}(\varepsilon y, \tau, \varkappa)$ coefficients in the series (7)–(8). Here $\varepsilon y = x$ is the "slow" variable and $y = x/\varepsilon$ is the "fast" variable in the spatial behaviour of the solution u to the equation (2).

In particular, at the order ε^0 we obtain

$$-\bigl(\varphi_t^{(0)}\bigr)^2 \mathcal{U}^{(0)} - \left(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi_x^{(0)}\right) a(y) \left(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi_x^{(0)}\right) \mathcal{U}^{(0)} = 0, \tag{9}$$

i.e. on the microscale, in the vicinity of the point x at time τ, the leading-order amplitude $\mathcal{U}^{(0)}$ behaves as an eigenfunction of the differential operator

$$-\left(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi_x^{(0)}\right) a(y) \left(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi_x^{(0)}\right),$$

corresponding to the value $\varkappa$ of the "quasimomentum".

Hence we write

$$\mathcal{U}^{(0)}(x, y, \tau, \varkappa) = u^{(0)}(x, \tau, \varkappa) U^{(0)}\Bigl(y, \Omega\bigl(\varkappa - \varphi_x^{(0)}\bigr)\Bigr), \tag{10}$$

and the function $\varphi^{(0)}$ solves the equation

$$\varphi_\tau^{(0)} = \pm\Omega\bigl(\varkappa - \varphi_x^{(0)}\bigr). \tag{11}$$

where $U^{(0)} = U^{(0)}\bigl(y, \Omega)$ is a normalised eigenfunction in (9) corresponding to the eigenvalue $\Omega^2 = \Omega^2\bigl(\varkappa - \varphi_x^{(0)}\bigr)$, *i.e.* one has

$$-\left(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi_x^{(0)}\right) a(y) \left(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi_x^{(0)}\right) U^{(0)} = \Omega^2\, U^{(0)}, \qquad \int_Q \bigl|U^{(0)}(y, \Omega)\bigr|^2 dy = 1. \tag{12}$$

In what follows we choose Ω to be the positive root of the eigenvalue Ω^2 in (12). Also, in all formulae containing "±" or "∓", the upper sign corresponds to the choice of "+" in (11) and the lower sign corresponds to the choice of "−".

Differentiating (11) with respect to τ and x in turn, we obtain

$$\varphi^{(0)}_{\tau\tau} = \mp\varphi^{(0)}_{x\tau}\Omega'$$

and

$$\varphi^{(0)}_{\tau x} = \pm\Big[\Omega\big(\varkappa - \varphi^{(0)}_x\big)\Big]_x =: \pm\Omega_x,$$

respectively. Henceforth, the values of Ω and its derivatives are taken at the point $\varkappa - \varphi^{(0)}(x,\tau,\varkappa)$. Combining these two equations yields

$$\varphi^{(0)}_{\tau\tau} = -\Omega'\Omega_x = -\frac{1}{2}\big[\Omega^2\big]_x. \tag{13}$$

We will use this result to simplify the equation (21) in the next section.

3.2 Transport equation for the overall amplitude

Further, collecting the terms of order ε^1 yields

$$-\big(\varphi^{(0)}_t\big)^2\mathcal{U}^{(1)} - \Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big)a(y)\Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big)\mathcal{U}^{(1)}$$
$$= \Big(\mathrm{i}\varphi^{(1)}_{\tau\tau} + 2\varphi^{(0)}_\tau\varphi^{(1)}_\tau\Big)\mathcal{U}^{(0)} + 2\mathrm{i}\varphi^{(0)}_\tau\mathcal{U}^{(0)}_\tau - \mathrm{i}\varphi^{(0)}_{xx}a(y)\mathcal{U}^{(0)}$$
$$+\Big\{a(y)\Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big) + \Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big)a(y)\Big\}\mathcal{U}^{(0)}_x$$
$$-\mathrm{i}\varphi^{(1)}_x\Big\{a(y)\Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big) + \Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big)a(y)\Big\}\mathcal{U}^{(0)}$$

which we re-write using the representation (10), as follows:

$$-\big(\varphi^{(0)}_t\big)^2\mathcal{U}^{(1)} - \Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big)a(y)\Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big)\mathcal{U}^{(1)}$$
$$= \Big(\mathrm{i}\varphi^{(1)}_{\tau\tau} + 2\varphi^{(0)}_\tau\varphi^{(1)}_\tau\Big)u^{(0)}U^{(0)} + 2\mathrm{i}\varphi^{(0)}_\tau u^{(0)}_\tau U^{(0)} - 2\mathrm{i}\varphi^{(0)}_\tau u^{(0)}U^{(0)}_\Omega\Omega'\varphi^{(0)}_{x\tau}$$
$$-\mathrm{i}\varphi^{(0)}_{xx}u^{(0)}a(y)U^{(0)} + \Big\{a(y)\Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big) + \Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big)a(y)\Big\}u^{(0)}_x U^{(0)}$$
$$+\Big\{a(y)\Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big) + \Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big)a(y)\Big\}u^{(0)}U^{(0)}_\Omega\Omega'\varphi^{(0)}_{xx}$$
$$-\mathrm{i}\varphi^{(1)}_x\Big\{a(y)\Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big) + \Big(\frac{d}{dy} + \mathrm{i}\varkappa - \mathrm{i}\varphi^{(0)}_x\Big)a(y)\Big\}u^{(0)}U^{(0)}. \tag{14}$$

We treat (14) as an equation for $\mathcal{U}^{(1)}$, hence we write the condition of solvability of (14). To this end we multiply (14) by the complex conjugate of the function $U^{(0)}$ found at the previous step and integrate the result with respect to the variable $y \in Q = [0,1)^d$. Using the eigenfunction equation (12) yields

$$\Big(\mathrm{i}\varphi^{(0)}_{\tau\tau} + 2\varphi^{(0)}_\tau\varphi^{(1)}_\tau\Big)u^{(0)} + 2\mathrm{i}\varphi^{(0)}_\tau u^{(0)}_\tau - 2\mathrm{i}\varphi^{(0)}_\tau u^{(0)}\Omega'\varphi^{(0)}_{x\tau}\int_Q U^{(0)}_\Omega\overline{U^{(0)}} - \mathrm{i}\varphi^{(0)}_{xx}\int_Q a(y)\big|U^{(0)}\big|^2$$

$$+u_x^{(0)}\int_Q\left\{a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)+\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)a(y)\right\}U^{(0)}\overline{U^{(0)}}$$

$$-u^{(0)}\Omega'\varphi_{xx}^{(0)}\int_Q\left\{a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)+\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)a(y)\right\}U_\Omega^{(0)}\overline{U^0}$$

$$-\mathrm{i}\varphi_x^{(1)}u^{(0)}\int_Q\left\{a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)+\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)a(y)\right\}U^{(0)}\overline{U^{(0)}}=0. \qquad (15)$$

Lemma 3.2. *The expression*

$$g_1(x,y,\tau,\varkappa):=-\mathrm{i}\int_Q U_\Omega^{(0)}\overline{U^{(0)}}$$

is real-valued.

Proof. Notice that

$$0=\frac{d}{d\Omega}\int_Q U^{(0)}\overline{U^{(0)}}=\int_Q U_\Omega^{(0)}\overline{U^{(0)}}+\int_Q U^{(0)}\overline{U_\Omega^{(0)}}$$

$$=\int_Q U_\Omega^{(0)}\overline{U^{(0)}}+\overline{\int_Q \overline{U^{(0)}}U_\Omega^{(0)}}=2\Re\int_Q U_\Omega^{(0)}\overline{U^{(0)}}.$$

□

Considering the real part of (15) results in an equation for $\varphi_t^{(1)}$:

$$2\varphi_\tau^{(0)}\varphi_\tau^{(1)}+2\varphi_\tau^{(0)}\Omega'\varphi_{x\tau}^{(0)}g_1$$

$$-\Omega'\varphi_{xx}^{(0)}\Re\int_Q\left\{a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)+\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)a(y)\right\}U_\Omega^{(0)}\overline{U^0}$$

$$+2\varphi_x^{(1)}\Im\int_Q a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)U^{(0)}\overline{U^{(0)}}=0, \qquad (16)$$

while taking the imaginary part of (15) yields an equation for $u^{(0)}$:

$$\varphi_{\tau\tau}^{(0)}u^{(0)}+2\varphi_\tau^{(0)}u_\tau^{(0)}-\varphi_{xx}^{(0)}\int_Q a(y)|U^{(0)}|^2+2u_x^{(0)}\Im\int_Q a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)U^{(0)}\overline{U^{(0)}}$$

$$-\Omega'\varphi_{xx}^{(0)}u^{(0)}\Im\int_Q\left\{a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)+\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)a(y)\right\}U_\Omega^{(0)}\overline{U^0}. \qquad (17)$$

Notice that

$$\left(\Im\int_Q a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)U^{(0)}\overline{U^{(0)}}\right)_x=-\varphi_{xx}^{(0)}\int_Q a(y)|U^{(0)}|^2$$

$$-\Omega'\varphi_{xx}^{(0)}\Im\int_Q a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)U_\Omega^{(0)}\overline{U^{(0)}}-\Omega'\varphi_{xx}^{(0)}\Im\int_Q a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)U^{(0)}\overline{U_\Omega^{(0)}},$$

which is the sum of the third and fifth terms in (17). Hence, we obtain

$$\varphi_{\tau\tau}^{(0)}u^{(0)}+2\varphi_\tau^{(0)}u_\tau^{(0)}+2u_x^{(0)}\Im\int_Q a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)U^{(0)}\overline{U^{(0)}}$$

$$+u^{(0)}\left(\Im\int_Q a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)U^{(0)}\overline{U^{(0)}}\right)_x=0,$$

or, after multiplication by $u^{(0)}$ and using the product rule,

$$\left[\left(u^{(0)}\right)^2\varphi_\tau^{(0)}\right]_\tau+\left[\left(u^{(0)}\right)^2\Im\int_Q a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)U^{(0)}\overline{U^{(0)}}\right]_x=0, \tag{18}$$

which is a transport equation for $\left(u^{(0)}\right)^2$. Finally, we use the following statement.

Lemma 3.3. *The formula*

$$\Im\int_Q a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)U^{(0)}\overline{U^{(0)}}=\frac{1}{2}\left(\Omega^2\right)', \tag{19}$$

holds, where the right-hand side is evaluated at $\varkappa-\varphi_x^{(0)}(x,\tau,\varkappa)$.

Proof. Differentiating with respect to $\varkappa$ the eigenvalue equation (12), we obtain

$$-\mathrm{i}\left(1-\varphi_{x\varkappa}^{(0)}\right)\left\{a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)+\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)a(y)\right\}U^{(0)}$$

$$-\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)\frac{d}{d\varkappa}U^{(0)}=\left(1-\varphi_{x\varkappa}^{(0)}\right)\left(\Omega^2\right)'U^{(0)}+\Omega^2\frac{d}{d\varkappa}U^{(0)}.$$

Multiplying both sides of the last equation by $\overline{U^{(0)}}$, integrating by parts in the last term on the left-hand side and using once again the eigenvalue equation (12) yields

$$-\mathrm{i}\left(1-\varphi_{x\varkappa}^{(0)}\right)\int_Q\left\{a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)+\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)a(y)\right\}U^{(0)}\overline{U^{(0)}}=\left(1-\varphi_{x\varkappa}^{(0)}\right)\left(\Omega^2\right)'.$$

Finally, we obtain (19) by noticing that

$$\int_Q\left\{a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)+\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)a(y)\right\}U^{(0)}\overline{U^{(0)}}$$

$$=2\mathrm{i}\Im\int_Q a(y)\left(\frac{d}{dy}+\mathrm{i}\varkappa-\mathrm{i}\varphi_x^{(0)}\right)U^{(0)}\overline{U^{(0)}}.$$

□

Combining (18) and Lemma 3.3 we obtain

$$\left[\left(u^{(0)}\right)^2\varphi_\tau^{(0)}\right]_\tau+\frac{1}{2}\left[\left(u^{(0)}\right)^2\left(\Omega^2\right)'\right]_x=0. \tag{20}$$

Using the product rule we re-write (20) as

$$\left(\varphi_{\tau\tau}^{(0)}+\Omega'\Omega_x\right)\left(u^{(0)}\right)^2+\left[\left(u^{(0)}\right)^2\right]_\tau\varphi_\tau^{(0)}+\left[\pm\left(u^{(0)}\right)^2\Omega'\right]_x(\pm\Omega)=0, \tag{21}$$

where the first term vanishes in view of (13), and hence

$$\left[\left(u^{(0)}\right)^2\right]_\tau\varphi_\tau^{(0)}+\left[\pm\left(u^{(0)}\right)^2\Omega'\right]_x(\pm\Omega)=0.$$

Finally, using (11) results in

$$\left[\left(u^{(0)}\right)^2\right]_\tau+\left[\pm\left(u^{(0)}\right)^2\Omega'\right]_x=0, \tag{22}$$

which is the transport equation for the modulating function $u^{(0)}=u^{(0)}(x,\tau,\varkappa)$.

3.3 Formula for the leading-order term in (3)

Summarising, the leading-order term in (3) is given by

$$\int\limits_{Q'} u^{(0)}(x,\varepsilon t,\varkappa)\,U^{(0)}\Bigl(\frac{x}{\varepsilon},\Omega\bigl(x,\varkappa-\varphi_x^{(0)}(x,\varepsilon t,\varkappa)\bigr)\Bigr)\exp\Bigl[-\frac{\mathrm{i}}{\varepsilon}\theta(x,\varepsilon t,\varkappa)\Bigr]d\varkappa, \tag{23}$$

where $u^{(0)}$ is the "amplitude" of the "asymptotic elementary solution", given by the equation (20), and

$$\theta(x,\tau,\varkappa):=\varphi^{(0)}(x,\tau,\varkappa)-\varkappa x,\qquad \tau=t\varepsilon, \tag{24}$$

is the "phase" of the solution. Recall that for all values of Ω the eigenfunction $U^{(0)}(\cdot,\Omega)$ is Q-periodic. The expression under the integral in (23) is the leading-order term, with respect to the small parameter ε, of the elementary solution (6) for each value of $\varkappa$.

4 Asymptotic analysis of the leading-order term as $\varepsilon\to 0$

The main contribution to the integral (23) is given by the neighbourhoods of the points $\varkappa=\hat\varkappa$ for which the phase function θ is "stationary", *i.e.*

$$\theta_\varkappa(x,\tau,\varkappa)=0,\quad \tau=\varepsilon t,$$

which in view of (24) is written as

$$\varphi_\varkappa^{(0)}(x,\tau,\varkappa)=x. \tag{25}$$

Using the standard formulae (see *e.g.* [5, Section 2.9]) of the method of stationary phase, in combination with the argument of [7, Appendix C], we write, as $\varepsilon\to 0$:

$$u^\varepsilon(x,t)\sim u^{(0)}(x,\varepsilon t,\hat\varkappa)\Bigl\langle U^{(0)}\Bigl(\cdot,\hat\varkappa,\Omega\bigl(\hat\varkappa-\varphi_x^{(0)}(x,\varepsilon t,\hat\varkappa)\bigr)\Bigr)\Bigr\rangle$$
$$\times\sqrt{\frac{2\pi}{\varphi_{\varkappa\varkappa}^{(0)}(x,\varepsilon t,\hat\varkappa)}}\exp\Bigl[-\frac{\mathrm{i}}{\varepsilon}\Bigl\{\varphi^{(0)}(x,\varepsilon t,\hat\varkappa)+\frac{\pi}{4}\mathrm{sgn}\bigl(\varphi_{\varkappa\varkappa}^{(0)}(x,\varepsilon t,\hat\varkappa)\bigr)\Bigr\}\Bigr], \tag{26}$$

where we substitute $\hat\varkappa=\hat\varkappa(x,\varepsilon t)$ from (25) and "sgn" stands for the sign function.

5 Propagation of the local values of the quasimomentum $\varkappa$ and wavenumber k

In what follows we continue using the notation $\tau=\varepsilon t$ whenever applicable. Note that the equation (11) can be written as

$$\omega=\mp\Omega(k). \tag{27}$$

where we denote by

$$\omega=\omega(x,\tau,\varkappa):=-\theta_\tau(x,\tau,\varkappa)=-\varphi_\tau^{(0)}(x,\tau,\varkappa), \tag{28}$$

$$k=k(x,\tau,\varkappa):=-\theta_x(x,\tau,\varkappa)=\varkappa-\varphi_x^{(0)}(x,\tau,\varkappa) \tag{29}$$

the local values of "frequency" and "wavenumber" in a nonuniform wavetrain, in particular, in a "wave packet" such as (26), *cf.* [8]. We assume that for all $x, \tau, \varkappa$ the "ampltude function" $u^{(0)}$ does not include any phase of the function $\hat{u}$ by requiring that

$$u^{(0)}(x, \tau, \varkappa) = \big|u^{(0)}(x, \tau, \varkappa)\big|.$$

This requirement is met by including the expression for the corresponding phase into the function $\varphi^{(0)}$.

Differentiating the equation (27) with respect to $\varkappa$ yields

$$\omega_\varkappa = \mp k_\varkappa \Omega'. \tag{30}$$

For the function $\hat{\varkappa} = \hat{\varkappa}(x, \tau)$ describing the stationary value of $\varkappa$ in (25), one has, by differentiating (25) with respect to x,

$$\varphi^{(0)}_{\varkappa x} + \varphi^{(0)}_{\varkappa\varkappa} \hat{\varkappa}_x = 1,$$

or, equivalently, by additionally using (29),

$$\varphi^{(0)}_{\varkappa\varkappa} \hat{\varkappa}_x = k_\varkappa. \tag{31}$$

Further, differentiating (25) with respect to τ for fixed x, we write

$$\varphi^{(0)}_{\varkappa\tau} + \varphi^{(0)}_{\varkappa\varkappa} \hat{\varkappa}_\tau = 0,$$

from where, using the definition (28), we obtain

$$\omega_\varkappa - \varphi^{(0)}_{\varkappa\varkappa} \hat{\varkappa}_\tau = 0. \tag{32}$$

Finally, combining (32), (30) and (31) yields

$$\varphi^{(0)}_{\varkappa\varkappa} \hat{\varkappa}_\tau = \omega_\varkappa = \mp k_\varkappa \Omega' = \mp \varphi^{(0)}_{\varkappa\varkappa} \hat{\varkappa}_x \Omega',$$

and hence

$$\hat{\varkappa}_\tau \pm \hat{\varkappa}_x \Omega' = 0, \tag{33}$$

assuming that $\varphi^{(0)}_{\varkappa\varkappa}$ does not vanish in the domain of $\varphi^{(0)}$. A version of the transport equation

$$k_\tau \pm k_x \Omega' = 0, \tag{34}$$

for the local wave number described in [8] also holds for the quantity $\hat{k}$ given by (*cf.* (29))

$$\hat{k}(x, \tau) := k\big(x, \tau, \hat{\varkappa}(x, \tau)\big) = \hat{\varkappa}(x, \tau) - \varphi^{(0)}_x\big(x, \tau, \hat{\varkappa}(x, \tau)\big),$$

namely

$$\hat{k}_\tau \pm \hat{k}_x \Omega' = 0.$$

This is obtained immediately by differentiating the equation (*cf.* (27))

$$\hat{\omega} = \mp \Omega(\hat{k}), \qquad \hat{\omega} := \omega\big(x, \tau, \hat{\varkappa}(x, \tau)\big)$$

with respect to x, and by noting first that

$$\omega_x = -\theta_{\tau x} = k_\tau = \varphi^{(0)}_{x\tau} = \pm \varphi^{(0)}_{xx} \Omega' \tag{35}$$

in view of (28), (29), (11), and second that

$$\omega_\varkappa = -\varphi^{(0)}_{\varkappa\tau} = \mp\big(1 - \varphi^{(0)}_{x\varkappa}\big)\Omega'$$

in view of (11).

The equations (33) and (34) are interpreted in the sense that the local quasimomentum $\hat{\varkappa}$ and the local wavenumber $\hat{k}$ propagate at each point (x, τ) with the "group velocity" $\Omega'(\hat{k})$. As we shall see in the next section, the quantity $\Omega'(\hat{k})$ describes the speed of propagation of the wave energy in the wave-train: the amount of the energy carried between two points moving with group velocities remains unchanged with time.

6 Transport of amplitude

First we note that the points x that have a fixed value of $\varkappa$ are transported with the group velocity $\Omega'(\hat{k})$. Indeed, differentiating (25) with respect to τ for fixed $\varkappa$ we write

$$\varphi^{(0)}_{\varkappa x} x_\tau + \varphi^{(0)}_{\varkappa\tau} = x_\tau.$$

At the same time, differentiating (11) with respect to $\varkappa$ we obtain

$$\varphi^{(0)}_{\tau\varkappa} = \pm\big(1 - \varphi^{(0)}_{x\varkappa}\big)\Omega'.$$

Combining the above two equalities yields

$$\big(1 - \varphi^{(0)}_{x\varkappa}\big)x_\tau = \pm\big(1 - \varphi^{(0)}_{x\varkappa}\big)\Omega',$$

where Ω' is evaluated at the point $\varkappa - \varphi^{(0)}_x\big(x(\tau,\varkappa),\tau,\varkappa\big)$, and hence (assuming that $\varphi^{(0)}_{x\varkappa} \neq 1$)

$$x_\tau(\tau, \varkappa) = \pm\Omega', \tag{36}$$

as claimed.

Now, consider the integral of the modulus of the solution u^ε squared, between any two points $x_1 = x_1(\tau)$, $x_2 = x_2(\tau)$ moving with group velocities corresponding to the values $\varkappa_1$ $\varkappa_2$ of the quasimomentum (and hence have the local values of the quasimomentum $\hat{\varkappa}(x_1, \tau) = \varkappa_1$, $\hat{\varkappa}(x_2, \tau) = \varkappa_2$ constant in time):

$$\begin{aligned} Q(\tau) &:= \int_{x_1(\tau)}^{x_2(\tau)} \big|u^\varepsilon(x)\big|^2 dx \\ &\sim 2\pi \int_{x_1(\tau)}^{x_2(\tau)} \big(u^{(0)}(x,\tau,\hat{\varkappa})\big)^2 \Big\langle U^{(0)}\big(\cdot, \hat{\varkappa} - \varphi^{(0)}_x(x,\tau,\hat{\varkappa})\big)\Big\rangle^2 \frac{dx}{\varphi^{(0)}_{\varkappa\varkappa}(x,\tau,\hat{\varkappa})}, \end{aligned} \tag{37}$$

as $\varepsilon \to 0$.

Making the change of the variable from x to $\varkappa$ according to the equation (25) results in

$$Q(\tau) \sim 2\pi \int_{\varkappa_1}^{\varkappa_2} \Big(u^{(0)}\big(x(\tau,\varkappa),\tau,\varkappa\big)\Big)^2 F\big(\tilde{k}(\tau,\varkappa)\big)\Big(1 - \varphi^{(0)}_{\varkappa x}\big(x(\tau,\varkappa),\tau,\varkappa\big)\Big)^{-1} d\varkappa, \tag{38}$$

where

$$F(\tilde{k}) := \Big\langle U^{(0)}(\cdot, \tilde{k})\Big\rangle^2, \qquad \tilde{k}(\tau,\varkappa) := \varkappa - \varphi^{(0)}_x\big(x(\tau,\varkappa),\tau,\varkappa\big).$$

We claim that the energy (38) is constant in time τ. Indeed, for the derivative of the expression under the integral in (38) with respect to τ, one has

$$\mathcal{D} := \frac{\partial}{\partial\tau}\biggl\{\Bigl(u^{(0)}\bigl(x(\tau,\varkappa),\tau,\varkappa\bigr)\Bigr)^2 F\bigl(\tilde{k}(\tau,\varkappa)\bigr)\Bigl(1-\varphi^{(0)}_{\varkappa x}\bigl(x(\tau,\varkappa),\tau,\varkappa\bigr)\Bigr)^{-1}\biggr\}$$

$$= \biggl\{\Bigl[\bigl(u^{(0)}\bigr)^2\Bigr]_x x_\tau + \Bigl[\bigl(u^{(0)}\bigr)^2\Bigr]_\tau\biggr\}F\bigl(\tilde{k}\bigr)\Bigl(1-\varphi^{(0)}_{\varkappa x}\bigl(x(\tau,\varkappa),\tau,\varkappa\bigr)\Bigr)^{-1}$$

$$+\bigl(u^{(0)}\bigr)^2 F\bigl(\tilde{k}\bigr)\Bigl(1-\varphi^{(0)}_{\varkappa x}\bigl(x(\tau,\varkappa),\tau,\varkappa\bigr)\Bigr)^{-2}\bigl(\varphi^{(0)}_{\varkappa xx}x_\tau+\varphi^{(0)}_{\varkappa x\tau}\bigr)$$

$$+\bigl(u^{(0)}\bigr)^2\Bigl(1-\varphi^{(0)}_{\varkappa x}\bigl(x(\tau,\varkappa),\tau,\varkappa\bigr)\Bigr)^{-1}F'\bigl(\tilde{k}\bigr)\bigl(\tilde{k}_x x_\tau+\tilde{k}_\tau\bigr).$$

The last term in the above expression vanishes, due to the fact that

$$\tilde{k}_x x_\tau + \tilde{k}_\tau = -\varphi^{(0)}_{xx}x_\tau - \varphi^{(0)}_{x\tau} = 0 \tag{39}$$

holds, by virtue of (36) and (11), *cf.* (35). Further, notice that[1]

$$\varphi^{(0)}_{\varkappa xx}x_\tau + \varphi^{(0)}_{\varkappa x\tau} = \mp\varphi^{(0)}_{xx}(\Omega')_\varkappa = \mp\bigl(1-\varphi^{(0)}_{x\varkappa}\bigr)\varphi^{(0)}_{xx}\Omega'' = \pm\bigl(1-\varphi^{(0)}_{x\varkappa}\bigr)(\Omega')_x,$$

in view of

$$\varphi^{(0)}_{\tau x\varkappa} = \pm\bigl(-\varphi^{(0)}_{xx}\bigr)(\Omega')_\varkappa \pm \bigl(-\varphi^{(0)}_{xx\varkappa}\bigr)\Omega',$$

which, in turn, is obtained by differentiating the last equality in (35) with respect to $\varkappa$, *cf.* (39).

Combining the above observations and the equation (36) yields

$$\mathcal{D} = \biggl\{\Bigl[\pm\bigl(u^{(0)}\bigr)^2\Omega'\Bigr]_x + \Bigl[\bigl(u^{(0)}\bigr)^2\Bigr]_\tau\biggr\}\Bigl(1-\varphi^{(0)}_{\varkappa x}\bigl(x(\tau,\varkappa),\tau,\varkappa\bigr)\Bigr)^{-1},$$

which vanishes thanks to the transport equation (22) for the function $u^{(0)}$. This concludes the proof of the fact that (38) is constant.

The above argument implies, in particular, that

$$\mathcal{E}_\tau + \mathcal{F}_x = 0,$$

where $\mathcal{E}$ is the energy density, given for each (x,τ) by the expression under the integral in (37), and

$$\mathcal{F} = \pm\Omega'\bigl(\hat{k}(x,\tau)\bigr)\mathcal{E}, \qquad \hat{k} := \hat{\varkappa}(x,\tau) - \varphi^{(0)}\bigl(x,\tau,\hat{\varkappa}(x,\tau)\bigr)$$

is the density of the "energy flux". In other words, propagation of the energy of the wave packet takes place with the group velocity corresponding to the local value $\hat{\varkappa}$ of the quasimomentum.

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[1] As before, expressions Ω', Ω'' are evaluated at $\varkappa - \varphi^{(0)}_x(x,\tau,\varkappa)$.

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