

# AN OVERVIEW OF PERIODIC ELLIPTIC OPERATORS

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*Dedicated to the memory of mathematicians and dear friends  
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ABSTRACT. The article surveys the main techniques and results of the spectral theory of periodic operators arising in mathematical physics and other areas. Close attention is paid to studying analytic properties of Bloch and Fermi varieties, which influence significantly most properties of such operators.

The approaches described are applicable not only to the standard model example of Schrödinger operator with periodic electric potential  $-\Delta + V(x)$ , but to a wide variety of elliptic periodic equations and systems, equations on graphs,  $\bar{\partial}$ -operator, and other operators on abelian coverings of compact bases.

Many important applications are mentioned. However, due to the size restrictions, they are not dealt with in details.

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## INTRODUCTION

Elliptic PDEs with periodic coefficients, notably the stationary Schrödinger operator  $-\Delta + V(x)$  with a periodic potential  $V$ , have

been intensively studied in physics and mathematics literature for close to a century, due to their crucial role in the solid state theory (e.g., see books and surveys [13, 26, 162, 162, 239, 295, 296]). In spite of that, some important questions have remained unresolved. The interest in periodic elliptic (and sometimes parabolic) equations and systems received a strong boost in the last decades, due to new applications in areas such as photonic crystals and other metamaterials [172], fluid dynamics [51, 291], carbon nanostructures [158], inverse scattering method of solving integrable systems [221, 222], and lately topological insulators [25]. When the author was preparing his introductory lectures at the Newton Institute in January 2015 [177], he discovered that there was no comprehensive source devoted to the theory of periodic elliptic PDEs and their applications. There are several books and surveys completely or partially devoted to this topic (e.g., [8, 24, 37, 85, 120, 125, 156, 170, 171, 239, 258, 275, 278]), but none of them collects all useful techniques (e.g., the analytic geometry of Bloch and Fermi varieties) and is up to date. Thus, the idea was born to expand the lectures to this survey that would contain the main techniques and results of the spectral theory of periodic operators arising in mathematical physics and other areas. These are applicable not only to the standard model example of Schrödinger operator with periodic electric potential  $-\Delta + V(x)$ , but to a wide variety of elliptic periodic equations and systems, equations on graphs,  $\bar{\partial}$ -operator, and other operators on abelian coverings of compact bases. Close attention is paid to studying analytic properties of Bloch and Fermi varieties, which influence significantly most properties of such operators.

There are many important applications of what is discussed. However, the author realized at this stage that even a large survey cannot do justice to such applications, and even to many details of techniques employed. Thus, several sections (especially toward the end of the text) contain mostly hints and pointers to the literature. The task of more detailed and comprehensive discussion is postponed till the third iteration of my lectures, which is a monograph in preparation. The same applies to the bibliography of this survey, which is necessarily extensive, but far from being comprehensive, and many references are given through secondary sources (books and surveys).

The author is indebted to many colleagues, co-authors, and former and current students for their publications, discussions, and encouragement, which have made possible over the years my work on this subject. The list of these people is so long, that I have decided, with apologies to many, to postpone it to the monograph planned. In here, I

express gratitude to the people who influenced my work in this area the most, M. Birman, L. Ehrenpreis, S. Krein, V. Palamodov, B. Simon, L. Zelenko, and V. Zhikov. I take full responsibility for all errors that are inevitably present in this text.

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## 1. 1D - A BRIEF SKETCH

In this section, we will survey briefly what is commonly called Floquet (or Floquet-Lyapunov) theory [112, 196, 197], the main tool in studying periodic linear ODEs and systems of ODEs. Its basics can be found in many ODE textbooks, e.g. in [9, 10, 45, 46]. There are also nice books dedicated to (mostly spectral theory of) the periodic ODEs of the second order, see e.g. [12, 85, 198], [275, Ch. XXI] and [239, Section XIII.16] (see also [84] for the case of singular potentials). I want to attract the reader's attention to the not sufficiently well known amazing treatise [287], which contains an enormous amount of information concerning periodic ODEs and systems of ODEs, including Hamiltonian and canonical systems, parametric resonance, stability domains, various applications, etc. Here, we will touch upon a few basics things only.

**1.1. Euler's theorem.** Consider a linear system of ODEs

$$(1.1) \quad \frac{dx}{dt} = Cx, t \in \mathbb{R}, x \in \mathbb{C}^n,$$

with a constant  $n \times n$  matrix  $C$ . Then all solutions look as follows:

$$x(t) = e^{Ct}x_0,$$

where  $x_0$  is the initial value of  $x(t)$ . One thus obtains the following **Euler's theorem** [95], which can be found in any ODE textbook:

**Theorem 1.1.** *All solutions of (1.1) are linear combinations of the exponential-polynomial solutions of the form*

$$(1.2) \quad x(t) = e^{i\lambda t} \sum_{j \in \mathbb{Z}, j \geq 0} p_j t^j,$$

where  $i\lambda$  is an eigenvalue of  $C$  and the sum is finite.

If  $C$  has no Jordan blocks, then only  $j = 0$  is present in (1.2).

Floquet-Lyapunov theory, sketched below, generalizes this result to the case of systems with periodic coefficients.

**1.2. Floquet-Lyapunov theory.** Let us consider a linear system of ODEs

$$(1.3) \quad \frac{dx}{dt} = A(t)x, t \in \mathbb{R}, x \in \mathbb{C}^n,$$

with a 1-periodic  $n \times n$  matrix function  $A(t)$ . As promised before, we will not dwell here on imposing weakest conditions on the entries of the matrix [287, Ch. II], assuming that they are continuous. Let  $X(t)$  be the  $n \times n$  matrix **fundamental solution** of (1.3). In other words, if

$$e_j = (0, \dots, 0, \underbrace{1}_{j\text{th entry}}, 0, \dots, 0)$$

is the standard basis of  $\mathbb{C}^n$  and  $x_j(t)$  - the solution of (1.3) such that  $x_j(0) = e_j$ , then  $X(t)$  has each  $x_j(t)$  as the  $j$ th column. One thinks of  $X(t)$  as the operator that shifts by time  $t$  along the trajectories. In particular, when the value of  $t$  coincides with the period (i.e., is equal to 1 under our assumption), we introduce the following

**Definition 1.2.** The matrix

$$(1.4) \quad M := X(1)$$

of the shift by the period 1 along the trajectories of the system is said to be the **monodromy matrix** of the equation (1.3).

The main result of the Floquet theory is the following

**Theorem 1.3. Floquet theorem**

*There exists a 1-periodic matrix function  $P(t)$  and a constant matrix  $C$ , such that*

$$(1.5) \quad X(t) = P(t)e^{Ct}.$$

*Remark 1.4.* Notice that

$$(1.6) \quad M = X(1) = e^C.$$

**Corollary 1.5.** *Any solution of (1.3) is a linear combination of **Floquet solutions***

$$(1.7) \quad x(t) = e^{ikt} \sum_{j \in \mathbb{Z}^+} p_j(t)t^j,$$

where the sum is finite, coefficients  $p_j(t)$  are 1-periodic, and  $e^{ik}$  are the eigenvalues of the monodromy matrix  $M$ .

When the monodromy matrix does not have Jordan blocks, only the term with  $j = 0$  is present. In this case, the solution is  $x(t) = e^{ikt}p(t)$  with a 1-periodic function  $p(t)$  and is sometimes called a **Bloch solution**.

**Definition 1.6.** The eigenvalues  $z := e^{ik}$  of the monodromy matrix  $M$  are called **Floquet multipliers** and numbers  $k$  - **quasimomenta** or **crystal momenta** (the latter names coming from solid state physics [13]).

*Remark 1.7.* One notices that the value of a quasimomentum  $k$  is defined only modulo  $2\pi\mathbb{Z}$ -shifts<sup>1</sup>.

In fact, the following, stronger than Theorem 1.3, result holds:

**Theorem 1.8. *Lyapunov reduction theorem.*** *There exists a periodic invertible matrix-function  $B(t)$  such that the substitution  $x(t) = B(t)y(t)$  reduces the system (1.3) to the one with a constant matrix  $C$ :*

$$\frac{dy}{dt} = Cy(t).$$

**1.3. Hill operator.** Studying higher order ODEs with periodic coefficients can be reduced to the case of first order systems. It is, however, useful to consider the special case of the so called **Hill operator**:

$$(1.8) \quad H = -\frac{d^2u}{dx^2} + V(x)$$

with a “nice” real 1-periodic potential  $V$  on  $\mathbb{R}$ . Again, we will overkill assuming continuity of  $V$ , although much weaker conditions suffice (see, e.g., [84, 85, 198, 239, 287]). The domain of the operator will be, by definition, the Sobolev space  $H^2(\mathbb{R})$ . Defined this way, the operator is self-adjoint [85]<sup>2</sup>.

**1.3.1. Monodromy, discriminant, and such.** We will be interested in the spectral problem for the Hill operator:

$$(1.9) \quad Hu = \lambda u.$$

In accordance to the general Floquet theory approach, we consider the fundamental system of two (analytic in  $\lambda$ ) solutions  $\phi(x, \lambda), \psi(x, \lambda)$ :

$$(1.10) \quad \phi(0, \lambda) = \psi'_x(0, \lambda) = 1, \phi'_x(0, \lambda) = \psi(0, \lambda) = 0.$$

This enables us to consider the **monodromy matrix**

$$(1.11) \quad M(\lambda) := \begin{pmatrix} \phi(1, \lambda) & \psi(1, \lambda) \\ \phi'_x(1, \lambda) & \psi'_x(1, \lambda) \end{pmatrix}.$$

<sup>1</sup>See the discussion of dual lattices and quasimomenta in Section 2.

<sup>2</sup>It is not hard to show that if one defines the operator on the space  $C_0^\infty(\mathbb{R})$ , it will be essentially self-adjoint, with the only self-adjoint extension described above.

The determinant  $\det M(\lambda)$  is the Wronskian of this fundamental system of solutions, evaluated at  $t = 1$ . Since the Wronskian is constant in time and equal to 1 at  $t = 0$ , we conclude that

$$(1.12) \quad \det M(\lambda) = 1.$$

Thus, since we are interested in the eigenvalues of the monodromy matrix, all the pertinent information is contained in the trace of  $M$ :

$$(1.13) \quad \Delta(\lambda) := \operatorname{Tr} M(\lambda) = \phi(1, \lambda) + \psi'_x(1, \lambda),$$

which is called the **discriminant**, or the **Lyapunov function** of the Hill operator (1.8).

Thus, all Floquet multipliers  $z$ , being the eigenvalues of  $M$ , can be found from the secular equation

$$(1.14) \quad z^2 - \Delta(\lambda)z + 1 = 0.$$

Its roots provide Floquet multipliers and quasimomenta for a given  $\lambda$ :

$$(1.15) \quad e^{ik} = z = 0.5 \left( \Delta(\lambda) \pm \sqrt{\Delta(\lambda)^2 - 4} \right).$$

It is now easy to come up with the following result (a compilation of various statements from standard books, e.g. [85, 287]):

**Theorem 1.9.**

- (1) If  $|\Delta(\lambda)| < 2$ , then
  - the Floquet multipliers are complex, distinct, of the absolute value 1, and complex conjugate to each other.
  - In particular, the quasimomentum is real and if  $k$  is a quasi-momentum, then  $-k$  is<sup>3</sup>.
  - All solutions are bounded.
  - The **Lyapunov exponent**<sup>4</sup> is equal to zero.
- (2) If  $|\Delta(\lambda)| = 2$ , then
  - The two Floquet multipliers coincide and are equal to 1 or  $-1$ .
  - Quasimomentum is either  $k = 0$ , or  $k = \pi \pmod{2\pi}$ .
  - All solutions are bounded or polynomially bounded.
  - The Lyapunov exponent is equal to zero.
- (3) If  $|\Delta(\lambda)| > 2$ , then

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<sup>3</sup>This observation also follows from the fact that, due to the potential being real and for real  $\lambda$ , complex conjugate of a solution is a solution as well. In physics terms, this is a reflection of the **time reversibility** of the dynamical system with the Hill operator as the Hamiltonian.

<sup>4</sup>The Lyapunov exponent characterises the maximal possible rate of the exponential growth of solutions.

- the Floquet multipliers are real, distinct, and reciprocal to each other.
- Quasimomenta are complex.
- Solutions grow exponentially.
- The Lyapunov exponent is positive and equal to  $|\Im k|$ .

Let us look at the case of the free operator (i.e., with zero potential). The straightforward calculation shows then that the free discriminant is  $\Delta_0(\lambda) = 2 \cos(\sqrt{\lambda})$ . In particular, it is an entire function of exponential order  $1/2$ , i.e.

$$|\Delta_0(\lambda)| \leq C e^{C|\lambda|^{1/2}}.$$

Its graph looks as shown in Fig. 1. The properties of the discriminant

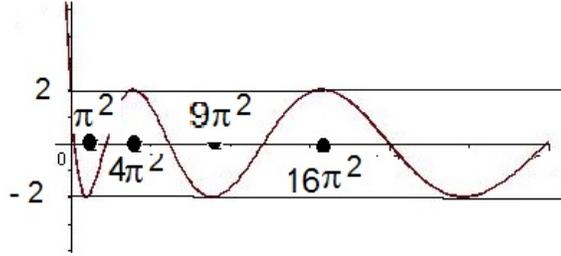


FIGURE 1. The graph of the free discriminant.

in presence of a periodic potential have also been studied thoroughly (see, e.g., [85, 198]). As in the free case, it is an entire function of exponential order  $1/2$ :

$$|\Delta(\lambda)| \leq C e^{C|\lambda|^{1/2}}.$$

The Figure 2 gives an idea of its behavior.

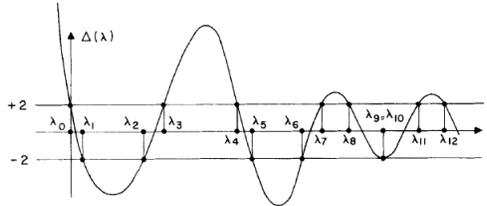


FIGURE 2. Discriminant for a periodic Hill equation.

Notice that the extrema of the free discriminant are located exactly on the boundary of the horizontal strip  $|\Delta| \leq 2$ . They get shifted in the periodic case (without any new ones being created), but cannot

move into the interior of this strip (see detailed considerations, e.g., in [85]). In particular, the pieces of the function  $\Delta(\lambda)$  between its consecutive hits of the lines  $\Delta = \pm 2$ , are **monotonic**. We will provide a simple explanation of this monotonicity a little bit later. The reader will see later on that absence of any analog of this monotonicity in higher dimensions is responsible for significant changes in some spectral properties in comparison with 1D.

**Definition 1.10.** The segments where  $|\Delta(\lambda)| \leq 2$  (shown in red in the figure below) are called the **stability zones** or **spectral bands**. The segments between the bands are called **instability zones** or **spectral gaps**.

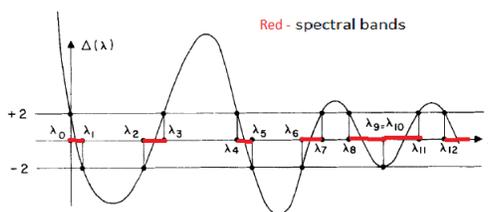


FIGURE 3. Stability zones (spectral bands), shown in red, are separated by the instability zones (spectral gaps)

*Remark 1.11.* Thus, we conclude that the spectral bands are characterized by the quasimomentum being real and Lyapunov exponent being equal to zero. Correspondingly, the spectral gaps are characterized by the complex quasimomentum and positive Lyapunov exponent.

1.3.2. *Spectrum of the Hill operator.* So far, the name “spectral band” was not explicitly related to the spectrum of the Hill operator  $H = -d^2/dx^2 + V(x)$ . It is not hard to show (which will be explained later on in the higher dimensions), that the spectrum  $\sigma(H)$  indeed coincides with the union of all spectral bands. E.g., existence of bounded generalized solutions inside the spectral bands allows one to show that the bands do belong to the spectrum. Indeed, cutting off these solutions at infinity provides approximate eigenfunctions. On the other hand, one can show that inside the gaps there is no spectrum.

We now collect some important features of the spectrum:

**Theorem 1.12.**

(1) *There exists a sequence of real numbers*

$$(1.16) \quad a_1 < b_1 \leq b_2 < a_2 \leq a_3 < \dots \mapsto \infty$$

(ends of the spectral bands) such that the spectrum  $\sigma(H)$  coincides with the union of all spectral bands  $I_j$

$$(1.17) \quad \sigma(H) = \bigcup_{j \in \mathbb{Z}^+} I_j,$$

where

$$(1.18) \quad I_1 = [a_1, b_1], I_2 = [b_2, a_2], I_3 = [a_3, b_s], \dots$$

- (2) The spectral bands have finite lengths and **do not overlap**, but might touch.
- (3) **Generically w.r.t. the potential, all gaps are present (open)**, i.e. there are no equalities in (1.16), so the bands do not touch [256].
- (4) **Finite gap (finite zone) potentials**, i.e. those that lead to finitely many open gaps only, are very special and can be all described [81, 82, 221, 223].
- (5) There are no open gaps if and only if the potential is constant (the famous **Borg's theorem** [35]).
- (6) The rate of the **gaps' sizes decay** determines smoothness of the potential [84, 199, 200]. For instance,  $C^\infty$  potentials correspond to the gap size decaying faster than algebraically.
- (7) There are **isospectral potentials**. The sets of isospectral finite-zone potentials form tori [203, 204, 223].

1.3.3. *Dispersion relation.* We will introduce now the important notion of the dispersion relation for the Hill operator. It is rarely discussed in mathematics literature devoted to periodic ODEs, but it is central in solid state physics and crucial for the higher dimension considerations.

The dispersion relation describes the spectral parameter  $\lambda$  as a (multiple valued) function of the crystal momentum  $k$ :

**Definition 1.13.** The **dispersion relation** (or the **Bloch variety**  $B_H$ ) of the Hill operator  $H$  is the subset of  $\mathbb{R}_k \times \mathbb{R}_\lambda$  defined as follows:

$$(1.19) \quad B_H := \{(k, \lambda) \in \mathbb{R}^2 \mid Hu = \lambda u \text{ has a } \neq 0 \text{ Bloch-Floquet solution } u(x) = e^{ikx}p(x) \text{ with the quasi-momentum } k\}$$

The **complex dispersion relation** (or **complex Bloch variety**  $B_{H,\mathbb{C}}$ ) is defined analogously, only allowing both  $k$  and  $\lambda$  to be complex.

One immediately obtains the following statement:

**Theorem 1.14.**

- (1) The dispersion relation is  $2\pi$ -periodic with respect to  $k$ .

- (2) *The real dispersion relation is even with respect to  $k$ . The complex one is invariant with respect to the mapping  $(k, \lambda) \mapsto (-k, \bar{\lambda})$ . (Indeed, due to the potential being real, the complex conjugate of a Floquet solution is also a Floquet solution.)*
- (3) *The spectrum of the operator  $H$  coincides with the projection of its (real) dispersion relation onto the spectral axis.*
- (4) *If the branches of the multiple valued function  $k \in \mathbb{R} \mapsto \lambda \in \mathbb{R}$  are labeled in increasing order ( $\lambda_1(k) \leq \lambda_2(k) \leq \dots$ ), then the range of the  $j$ th branch is the  $j$ th spectral band of  $H$ .*
- (5) *In terms of the discriminant, the dispersion relation (both in the real and complex incarnations) is described as the set of all (correspondingly real or complex) solutions  $(k, \lambda)$  of the equation*

$$(1.20) \quad \Delta(\lambda) - 2 \cos k = 0.$$

*In other words, it is the graph of the multiple-valued function*

$$(1.21) \quad \lambda = \Delta^{-1}(2 \cos k).$$

Formula (1.21) is less useful than (1.20), due to its multiple valued (and branching) nature. On the other hand, the function  $F(\lambda, k) := \Delta(\lambda) - 2 \cos k$  is an entire function in  $\mathbb{C}^2$  of a finite exponential order (equal to 1 in the Hill's case). In particular, one can make the following observation, which gains prominence in PDE situation:

**Proposition 1.15.** *The dispersion relation of the Hill operator is a principal (i.e., of co-dimension one) analytic subset in  $\mathbb{C}^2$ .*

Let us look at the case of the free operator. Then the dispersion relation boils down to  $\cos \sqrt{\lambda} = \cos k$ , or to the union of infinitely many parabolic branches

$$(1.22) \quad \lambda = (k + 2n\pi)^2, n \in \mathbb{Z},$$

see Fig. 4. Due to the periodicity with respect to  $k$ , it is customary to draw the dispersion relation over the interval  $[-\pi, \pi]$  (later on, this interval will acquire the name of a **Brillouin zone**) only. In fact, due to the evenness, one can, without losing any information, restrict to  $[0, \pi]$  (**reduced Brillouin zone**) only (see Fig. 5).

One can see that the seeming complexity of the dispersion relation even in the free case, is an illusion, since one observes a single period view of a periodic sequence of parabolas.

Let us see how the picture gets perturbed when a small periodic potential  $V(x)$  is turned on (Fig. 6). You see that under this perturbation some gaps are opening. However, it seems that over  $[0, \pi]$  each branch is monotonic. Let us see why this is indeed true:

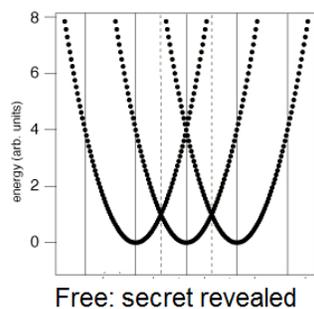


FIGURE 4. The dispersion relation (over the whole quasi-momentum line) of the free operator.

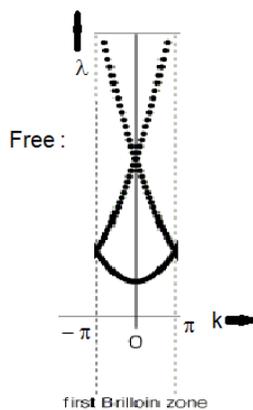


FIGURE 5. The dispersion relation of the free operator over one period.

**Lemma 1.16.**

- (1) Each branch  $\lambda_j(k)$  of the dispersion relation is monotonic over  $[0, \pi]$ .
- (2) In particular, the spectral band edges occur only at  $k = 0$  and  $k = \pi$ , i.e. for periodic or anti-periodic problems for the Hill operator.

*Proof.* Indeed if this were not true, taking into account the evenness with respect to  $k$ , one would have situation like in Fig. 7. If we pick the value of  $\lambda$  at the level shown, we will find four values of  $k$  where this level is reached. This means that the equation  $Hu = \lambda u$  has four independent solutions. The ODE being of the second order, this is impossible.  $\square$

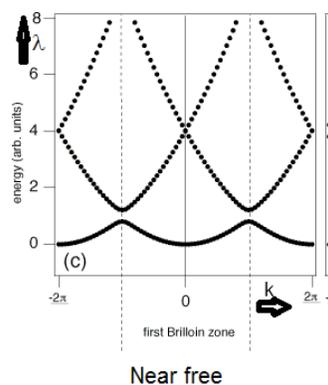


FIGURE 6. The dispersion relation in presence of a potential.

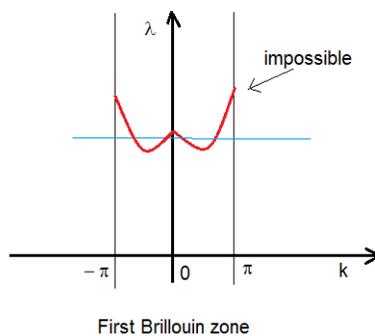


FIGURE 7. Non-monotonicity is impossible.

The same argument shows impossibility of band overlaps, as in Fig. 8:

Thus, one concludes:

**Lemma 1.17.** *The spectral bands of the Hill operator do not overlap (although they might touch).*

*Remark 1.18.* Notice that the conclusions about monotonicity, band edges occurring at  $k = 0$  and  $k = \pi$  only, and absence of band overlap were drawn from the fact that the second order ODE cannot have more than two independent solutions. One can wonder, whether this is just an artifact of our proof, or something deeper. For instance, this counting cannot be used for PDEs. And indeed, we will see that for PDEs all these claims are in general incorrect.

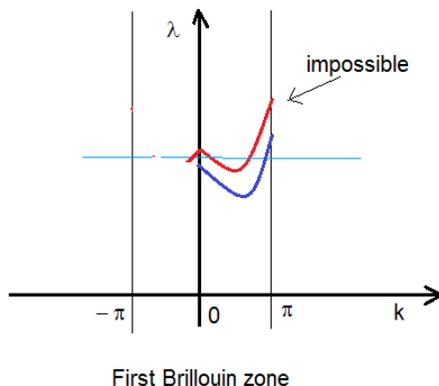


FIGURE 8. No band overlap.

We formulate now without proofs some additional spectral properties:

**Theorem 1.19.**

- (1) *The pure point and singular continuous spectra of the Hill operator are empty. Thus, the spectrum is absolutely continuous [85, 171, 239]<sup>5</sup>.*
- (2) *The dispersion relation, as an analytic set, is irreducible (modulo the  $2\pi$  shifts of the quasi-momentum). In other words, any open part of the dispersion relation uniquely determines the whole of it. To put it still differently, the smooth part of  $B_H$  is connected [166] (see also [16]).*
- (3) *The Bloch variety (which is analytic, as we know), is generically not algebraic (e.g., [203, 204]).*

## 2. LATTICES IN $\mathbb{R}^n$

Switching to the multi-periodic case, we need first to go through some basics concerning lattices.

**Definition 2.1.** A (Bravais) **lattice**  $\Gamma$  in  $\mathbb{R}^n$  is the set of all integer linear combinations of  $n$  linearly independent vectors  $a_1, \dots, a_n$  (see Fig. 9:

$$(2.1) \quad \Gamma = \left\{ \gamma \in \mathbb{R}^n \mid \gamma = \sum_{j=1}^n \gamma_j a_j, \gamma_j \in \mathbb{Z} \right\}.$$

We will identify  $\Gamma$  with the corresponding group of shifts of  $\mathbb{R}^n$ .

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<sup>5</sup>We will discuss this at length for the PDE case.

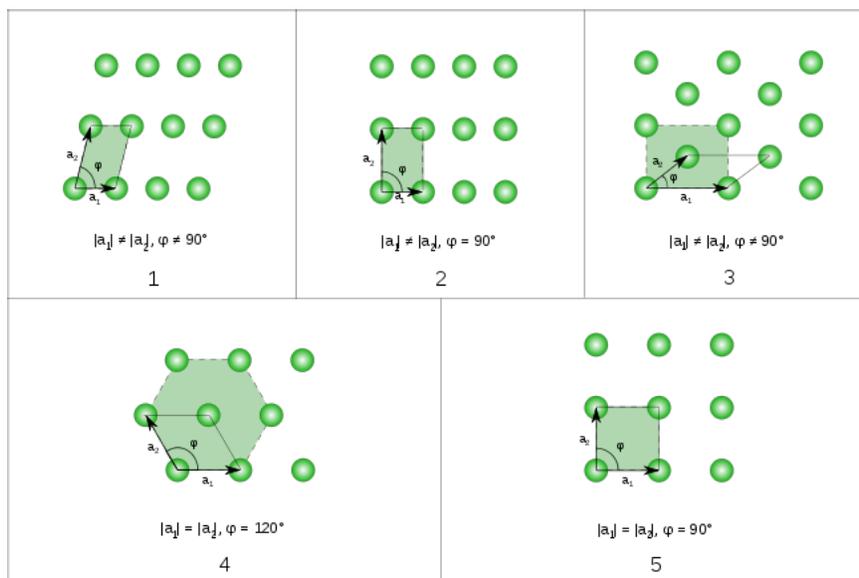


FIGURE 9. Lattices and fundamental domains (shaded).

**Definition 2.2.** The **dual (reciprocal) lattice** to  $\Gamma$ ,  $\Gamma^*$  in  $(\mathbb{R}^n)^*$  is defined as follows:

$$(2.2) \quad \Gamma^* = \{k \in (\mathbb{R}^n)^* \mid \langle k, \gamma \rangle \in 2\pi\mathbb{Z} \text{ for any } \gamma \in \Gamma\}.$$

The original lattice  $\Gamma$  is sometimes called the **real lattice**.

In particular, when  $\Gamma = \mathbb{Z}^n$  and the duality is coming from the standard Euclidean scalar product, then  $\Gamma^* = 2\pi\mathbb{Z}^n$ .

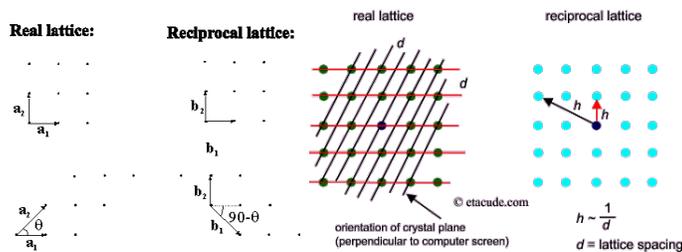


FIGURE 10. Real and reciprocal lattices.

**Definition 2.3.** We fix a **fundamental domain  $W$  (Wigner-Seitz cell)** of (the real lattice)  $\Gamma$  in  $\mathbb{R}^n$ , i.e. a domain such that its  $\Gamma$ -shifts may intersect only along their boundaries and cover the whole space

(See Fig. 9, 11)<sup>6</sup>. We will work only with connected fundamental domains.

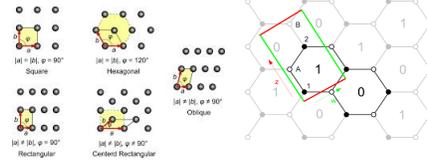


FIGURE 11. More lattices and fundamental domains.

We will also fix a fundamental domain  $\mathcal{B}$  of (the reciprocal lattice)  $\Gamma^*$  in  $(\mathbb{R}^n)^*$ , which we will call the **Brillouin zone**. One of the options (standard in physics) is to choose the set of all points such that the origin is the closest point to them in  $\Gamma^*$  (Fig. 12).

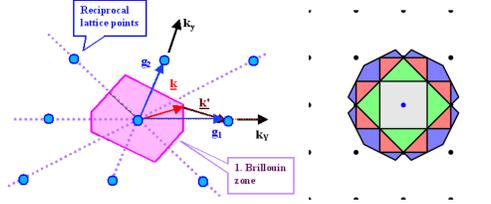


FIGURE 12. Brillouin zones.

The particular choice of a lattice does not matter in many, and does matter in some results, but for the sake of simplicity, we will assume, unless noted otherwise, that

$$(2.3) \quad \Gamma = \mathbb{Z}^n, \Gamma^* = 2\pi\mathbb{Z}^n, W = [0, 1]^n, \mathcal{B} = [-\pi, \pi]^n.$$

We also introduce two **tori** that correspond to the two lattices

$$(2.4) \quad \mathbb{T} := \mathbb{R}^n / \Gamma \text{ and } \mathbb{T}^* := (\mathbb{R}^n)^* / \Gamma^*,$$

equipped with normalized Haar measures on both. E.g., under our standard choice (2.3), the measures are correspondingly  $dx$  and  $(2\pi)^{-n}dk$ .

Note that under our assumption that  $\Gamma = \mathbb{Z}$ , the torus  $\mathbb{T}^*$  can be considered as the unit torus in  $\mathbb{C}^n$ :

$$(2.5) \quad \mathbb{T}^* = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1| = \dots = |z_n| = 1\}.$$

As usual,  $\Gamma$  - ( $\Gamma^*$  -) periodic functions on  $\mathbb{R}^n$  (on  $(\mathbb{R}^n)^*$ ) can be identified with functions on the torus  $\mathbb{T}$  ( $\mathbb{T}^*$ ).

<sup>6</sup>A fundamental domain is clearly not defined uniquely.

**Fourier series (FS)** identify  $L^2(\mathbb{T})$  with the  $l^2$  space on  $\Gamma^*$ :

$$(2.6) \quad f(x) \mapsto \{f_k := \int_{\mathbb{T}} f(x) e^{-ik \cdot x} dx\}_{k \in \Gamma^*}$$

Analogously with  $L^2(\mathbb{T}^*)$  and  $l^2$  on  $\Gamma$ .

One can consider (as it will be extremely useful for us) vector-valued Fourier series. Namely, let  $\mathcal{H}$  be a Hilbert space and  $L^2(\mathbb{T}, \mathcal{H})$ ,  $l^2(\Gamma^*, \mathcal{H})$  be the spaces of  $\mathcal{H}$ -valued  $L^2$ - functions on  $\mathbb{T}$  and  $\Gamma^*$  correspondingly and the Fourier Series expansion is defined as in (2.6). The following theorem restates standard results concerning Fourier series:

**Theorem 2.4.**

- *The FS expansion is (up to a constant factor) an isometry of  $L^2(\mathbb{T}, \mathcal{H})$  onto  $l^2(\Gamma^*, \mathcal{H})$ .*
- *A function  $\in L^2(\mathbb{T}, \mathcal{H})$  is infinitely differentiable if and only if its Fourier coefficients  $f_k$  decay faster than any power of  $k$ .*
- *A function  $\in L^2(\mathbb{T}, \mathcal{H})$  allows analytic continuation into a complex neighborhood of  $\mathbb{T}$  (see (2.5)) if and only if its Fourier coefficients  $f_k$  decay exponentially.*

### 3. PERIODIC OPERATORS

From now on, we will be interested in studying linear elliptic PDEs with periodic coefficients. Our main “test” example is the Schrödinger operator in  $L_2(\mathbb{R}^n)$ :

$$(3.1) \quad H = -\Delta + V(x),$$

with a “sufficiently nice” real electric potential  $V$ , periodic with respect to the group  $\Gamma = \mathbb{Z}^n$  (and thus the dual lattice is  $\Gamma^* = 2\pi\mathbb{Z}^n$ ). The domain of  $H$  is the Sobolev space  $H^2(\mathbb{R}^n)$ . The reader in most cases will not be misled thinking just of this operator.

In fact, many other periodic operators of mathematical physics arise in applications and need to be studied. The techniques described in this survey work (sometimes with some caveats) for them as well. We will from time to time address some of those, but so far we just briefly describe some examples worthy of studying (and being studied):

- Although we assumed above a “nice” (continuous or even smooth) potential  $V(x)$ , this condition is a severe overkill (e.g.,  $L_{2,loc}$  usually suffices). However, the main issues already arise in the smooth case. So, unless specified otherwise, **coefficients in all operators will be assumed smooth** (albeit in practical applications they often are even discontinuous).

- The Schrödinger operator (3.1) is self-adjoint, however **many of the techniques and results do not require self-adjointness**. For instance, when we will discuss absolute continuity of the spectrum, the question does not make sense in non-self-adjoint case. However, we will see that the equivalent statement, absence of the pure point spectrum, can be stated and holds in the non-self-adjoint setting as well.
- More general Schrödinger operators are also of interest and present much more difficulties in studying them. These are, first of all, the **magnetic Schrödinger operator**

$$-\Delta + iA(x) \cdot \nabla + V(x)$$

with periodic magnetic and electric potentials  $A(x)$  and  $V(x)$ .

Even more difficulties one encounters in **presence of periodic metric**:

$$-\nabla \cdot g(x) \nabla + iA(x) \cdot \nabla + V(x),$$

where  $g(x)$  is a periodic positive definite matrix-function.

- Periodic elliptic **operators of higher than second order** are also of some interest. Here, one should beware that such operators, unlike the 2nd order ones, might not obey even the weakest uniqueness of continuation laws, which influences the validity of some of the results (e.g., absence of the pure point spectrum).
- Periodic elliptic **systems** (including overdetermined ones) can also be considered, first of all the **Maxwell operator** (in its 2nd order incarnation):

$$(3.2) \quad \nabla \times \varepsilon^{-1}(x) \nabla \times,$$

where  $\varepsilon(x)$  is a periodic positively definite scalar function (or tensor) and  $\nabla \times$  is the *curl* operator.

Here one encounters an additional difficulty, since this operator is not elliptic by itself, but only as a **member of an elliptic complex** of operators.

- The natural mapping  $\mathbb{R}^n \mapsto \mathbb{T}$  is a normal **abelian covering** of the (compact) torus  $\mathbb{T}$ . The techniques that we will consider, as well as many results, apply to **periodic operators on coverings**

$$M \mapsto N,$$

subject to the following three conditions:

- (1) The base  $N$  is compact.

- (2) The deck group  $\Gamma$  of the covering is finitely generated and virtually abelian<sup>7</sup>.
- (3) The operator on  $M$  is “elliptic” in the sense that being pushed down to  $N$ , it is a Fredholm operator in appropriate spaces.

In particular, **periodic operators on abelian coverings of compact Riemannian manifolds, analytic manifolds, and even graphs**, succumb gladly to the theory.

- **Elliptic periodic boundary value problems** are important in various applications (e.g., waveguides and photonic crystals). Here periodicity is imposed on the shape of the domain, coefficients of the operator, and boundary conditions. Some issues easily resolved for operators in the whole space become much harder in this new setting.
- Ellipticity condition can be weakened to **hypoellipticity, e.g. parabolicity**, although the known results are much weaker here (see Section 9.1 and [171, and references therein]).

**Non-hypoelliptic periodic equations**, e.g. important time periodic **hyperbolic or non-stationary Schrödinger** equations **require different techniques**, due to lack of Fredholmity (see, e.g. [284–286] and Section 9.1).

- Some, but not all, studies carry over to periodic elliptic **pseudo-differential operators** ( $\Psi$ DOs) (see [171] and [261]).

#### 4. FLOQUET TRANSFORM AND DIRECT INTEGRAL DECOMPOSITION

Our main tools in 1D were the spectral analysis of the monodromy operator and the Lyapunov reduction theorem. Both of them rely upon the propagator along the solutions of the Cauchy problem. This raises some hopes that an analog of the 1D Floquet theory **might** work for time-periodic PDEs of evolution type (parabolic, hyperbolic, non-stationary Schrödinger), where such propagators are nicely defined. And indeed, **sometimes** this does happen, although, surprisingly (and shamefully) some basic issues remain still unresolved (see some details in Section 9.1). However, for non-evolution (e.g., elliptic) periodic PDEs using such propagators is all but impossible 🙄. One thus has to resort to a different technique, which we outline below.

**4.1. Floquet transform.** As we have recalled above, Fourier series transform functions of  $\gamma \in \Gamma$  (even with values in a Hilbert space) into

---

<sup>7</sup>I.e., contains a finite index abelian subgroup.

functions of  $z \in \mathbb{T}^*$ :

$$\{f_\gamma\}_{\gamma \in \Gamma} \mapsto \widehat{f}(z) := \sum_{\gamma \in \Gamma} f_\gamma z^\gamma = \sum_{\gamma \in \Gamma} f_\gamma e^{ik \cdot \gamma},$$

with standard  $L^2$  isometry and Paley-Wiener type theorems being preserved. Here  $k \in \mathbb{R}^n$  and  $z$  belongs to the unit torus  $\mathbb{T}^*$ .

Let now  $f(x)$  be a function in  $L^2_{loc}(\mathbb{R}^n)$ . We cover  $\mathbb{R}^n$  by the shifted copies of the fundamental domain  $W$ :

$$(4.1) \quad \mathbb{R}^n = \bigcup_{\gamma \in \Gamma} (W + \gamma),$$

cut  $f$  into the corresponding pieces  $f|_{W-\gamma}$  and then shift them all back to the original  $W$ :

$$(4.2) \quad f_\gamma(x) := f|_{W-\gamma}(x - \gamma).$$

If we restrict here  $x$  to being in the Wigner-Seitz cell  $W$ , we get a function on  $\Gamma$  with values in the Hilbert space  $L_2(W)$ :

$$(4.3) \quad \gamma \in \Gamma \mapsto f_\gamma \in L^2(W).$$

Now summing the (vector-valued) Fourier series, we arrive to our main tool:

**Definition 4.1.** The **Floquet transform**<sup>8</sup>  $\mathcal{U}_\Gamma$  acts as follows:

$$(4.4) \quad f(x) \mapsto \mathcal{U}_\Gamma f(x, k) := \sum_{\gamma \in \Gamma} f_\gamma(x) e^{ik \cdot \gamma} = \sum_{\gamma \in \Gamma} f_\gamma(x) z^\gamma.$$

In other words, this is just the sum of the Fourier series, whose coefficients are the pieces of  $f$  over shifted copies of the fundamental domain.

Here, as before,  $z = (z_1, \dots, z_n) := (e^{ik_1}, \dots, e^{ik_n})$  is the Floquet multiplier corresponding to the crystal momentum  $k = (k_1, \dots, k_n)$ . We will also abuse notations, writing

$$\mathcal{U}_\Gamma f(x, z) = \sum_{\gamma \in \Gamma} f_\gamma(x) z^\gamma$$

---

<sup>8</sup>Neither the name ‘‘Floquet transform’’ is commonly accepted, nor Floquet introduced this transform. Various other names have been suggested: Bloch transform, Gelfand transform, Zak transform, and probably some others. Here we see an instance of the so called **Arnold principle**: *If a notion bears a personal name, then this name is not the name of the discoverer* [11, as well as everywhere on the Web]. (See also the **Berry Principle**: *the Arnold Principle is applicable to itself*.) However, it is handy in writing to use a name, rather than pointing one’s finger to a formula. Thus, the author chose Floquet, since this transform is very natural for Floquet theory. Hopefully, no one will object to it.

It will be convenient to extend  $\mathcal{U}_\Gamma f(x, k)$  from  $x \in W$  to the whole  $\mathbb{R}^n$  just by removing the restriction to  $W - \gamma$  in (4.2), which results in

$$(4.5) \quad f(x) \mapsto \mathcal{U}_\Gamma f(x, k) := \sum_{\gamma \in \Gamma} f(x - \gamma) e^{ik \cdot \gamma} = \sum_{\gamma \in \Gamma} f(x - \gamma) z^\gamma.$$

If this is done, then one observes the following useful properties:

$$(4.6) \quad \mathcal{U}_\Gamma f(x + \gamma, k) = \mathcal{U}_\Gamma f(x, k) e^{ik \cdot \gamma}, \gamma \in \Gamma,$$

$$(4.7) \quad \mathcal{U}_\Gamma f(x + \gamma, z) = \mathcal{U}_\Gamma f(x, k) z^\gamma, \gamma \in \Gamma,$$

$$(4.8) \quad \mathcal{U}_\Gamma f(x, k + k') = \mathcal{U}_\Gamma f(x, k), k' \in \Gamma^*.$$

The first two equations, (4.6) and (4.7), say that, being considered on the whole space  $\mathbb{R}^n$ ,  $\mathcal{U}_\Gamma f(x, k)$  is  $\Gamma$ -**automorphic** with respect to  $\Gamma$  with the character  $e^{ik \cdot \gamma}$  (this is also called **cyclic property, quasi-periodicity, or Floquet property**). The last one, (4.8) claims  $\Gamma^*$ -**periodicity** with respect to  $k$ .

One can get easily some inversion formulas for the Floquet transform. Indeed, it is enough to find the Fourier coefficients of  $\mathcal{U}_\Gamma f(x, k)$  with respect to  $k$  and place them where they belong, i.e. on the shifted copies of the fundamental domain. This leads to the following inversion formula:

$$(4.9) \quad f(x) = \int_{\mathbb{T}^*} \mathcal{U}_\Gamma f(x + \gamma, k) e^{ik \cdot \gamma} dk, x \in W - \gamma, \text{ for any } \gamma \in \Gamma.$$

On the other hand, if we consider the Fourier coefficients  $f_\gamma$  as the shifted copies of  $f$  in the whole space  $\mathbb{R}^n$ , then just the zero's coefficient recovers the whole function:

$$(4.10) \quad f(x) = \int_{\mathbb{T}^*} \mathcal{U}_\Gamma f(x, k) dk, x \in \mathbb{R}^n.$$

If one compares the two inversion formulas, one might have a feeling that something is wrong. However, a brief consideration, which uses (4.6) shows that the formulas are indeed equivalent.

**4.2. Plancherel and Paley-Wiener type theorems.** Now, application of Theorem 2.4 to  $\mathcal{H} = L_2(W)$  leads to the following range theorems for the Floquet transform:

**Theorem 4.2.**

- $\mathcal{U}_\Gamma$  is isometry from  $L^2(\mathbb{R}^n)$  onto  $L^2(\mathbb{T}^*, L^2(W))$ .
- $\mathcal{U}_\Gamma f(\cdot, k)$  is infinitely differentiable on  $\mathbb{T}^*$  as a function with values in  $L_2(W)$  iff the norms  $\|f\|_{W+\gamma}$  decay when  $\|\gamma\| \mapsto \infty$  faster than any power of  $\|\gamma\|$ .

- $\mathcal{U}_\Gamma f(\cdot, k)$  is analytic in a complex neighborhood of  $\mathbb{T}^*$  as a function with values in  $L_2(W)$ , iff the norms  $\|f\|_{W+\gamma}$  decay exponentially fast when  $\|\gamma\| \mapsto \infty$ .

Dealing with differential operators, we will be also interested in the behavior of Sobolev spaces  $H^s(\mathbb{R}^n)$  under the Floquet transform. This seems to be a piece of cake, just replacing everywhere  $L_2$  with  $H^s$ . Well, not so fast:

**Proposition 4.3.**  $\mathcal{U}_\Gamma$  is an isometry from  $H^s(\mathbb{R}^n)$  onto a proper subspace of  $L^2(\mathbb{T}^*, H^s(W))$ .

Oops! what is wrong with Sobolev spaces? The answer is not hard to figure out. Indeed, if one takes an arbitrary function  $F(x, k)$  in  $L^2(\mathbb{T}^*, H^s(W))$ , its alleged pre-image under the Floquet transform can be, as we have figured out before, recovered by taking Fourier coefficients of  $F$  and distributing them to the appropriate shifts of the fundamental domain  $W$ . The resulting function is, by construction, in  $H^s$  inside of each of the shifted copies of  $W$ . Moreover, the sum of squares of their  $H^s$  norms is finite. The problem is that there is nothing enforcing the smoothness across the boundary between two adjacent cells. This explains why there is no surjectivity of the Floquet transform in this setting.

Fortunately, it is not hard to fix this and describe the range of the transform explicitly. Clearly some boundary conditions are needed to glue the pieces together. And indeed these are easy to establish for instance when  $s = 2$  (or any  $s \in \mathbb{Z}^+$ ). It is, however, easier to avoid this and to adopt a much more telling (and applicable for any  $s \geq 0$ ) view. The clue on how to do this comes from the automorphic property (4.6, 4.7). We introduce the following

**Definition 4.4.** For any  $k \in \mathbb{C}^n$ , the space  $H_k^s(W)$  as the space of restrictions to the fundamental domain  $W$  of  $H_{loc}^s$ -function on  $\mathbb{R}^n$  that are automorphic with the given  $k$ :

$$(4.11) \quad H_k^s(W) := \{f|_W \mid f \in H_{loc}^s(\mathbb{R}^n) \text{ s.t. } f \text{ satisfies (4.6)}\}$$

Another useful object is a  $k$ -dependent smooth linear bundle over  $\mathbb{T}$ . Let us fix  $k \in \mathbb{C}^n$  and consider first the trivial linear bundle  $E := \mathbb{R}^n \times \mathbb{C}$  over  $\mathbb{R}^n$ . We define a ( $k$ -dependent)  $\Gamma$ -action on  $E$  as follows:

$$(4.12) \quad \text{For any } \gamma \in \Gamma \text{ and } (x, c) \in E, \tau_k(\gamma)(x, c) := (x + \gamma, e^{ik \cdot x} c).$$

**Definition 4.5.** We denote by  $E_k$  the linear bundle over the torus  $\mathbb{T}$  obtained by factoring out the action  $\tau_k$  of  $\Gamma$  on the bundle  $E$ .

Now the following claims are immediate to obtain:

**Proposition 4.6.** *Let  $s \geq 0$ . Then, for any  $k$ ,*

- (1)  $H_k^s$  is a closed subspace in  $H^s(W)$ .
- (2)  $H_k^0 = L_{2,k} = L_2(W)$ .
- (3)  $H_{k+k'}^s = H_k^s$  for any  $k' \in \Gamma^*$ .
- (4) A function  $u$  belongs to  $H_k^s$  if and only if it can be represented as
 

(4.13)  $u(x) = e^{ik \cdot x} v(x)$  with  $v(x)$  being  $\Gamma$ -periodic, i.e.  $v \in H^s(\mathbb{T})$ .
- (5) Moreover,  $\mathcal{H}^s := \bigcup_k H^s(k)$  is an analytic Banach vector subbundle of the trivial bundle  $\mathbb{C}^n \times H^s(W)$  over  $\mathbb{C}^n$ .
- (6) There is a natural correspondence between  $\Gamma$ -automorphic functions on  $\mathbb{R}^n$  with a quasi-momentum  $k$  and sections of the linear bundle  $E_k$  over  $\mathbb{T}$ .
- (7) Under this correspondence, elements of  $H_k^s$  correspond to  $H^s$ -sections of the bundle  $E_k$ .
- (8) Due to periodicity, the operator  $H$  preserves the  $\Gamma$ -automorphy. In particular,  $H$  defines an elliptic operator  $H(k)$  on sections of the bundle  $E_k$ .

*Remark 4.7.*

- In the particular case when  $H$  is the Laplacian  $-\Delta$ , the operators  $-\Delta(k)$  are called **twisted Laplacians**. We can call the operators  $H(k)$  **twisted Schrödinger operators**, or (more common) **Bloch Hamiltonians**.
- The advantage of dealing with the family of operators  $H(k)$  rather than the single operator  $H$  is that each  $H(k)$  is an elliptic operator in sections of a bundle over a **compact** manifold  $\mathbb{T}$  and thus has purely discrete spectrum, unlike the operator  $H$ , whose spectrum is continuous.
- For the operator  $H = -\Delta + V(x)$ , the corresponding  $H(k)$  acts by the same differential expression on  $W$  with the domain  $H_k^2$ , i.e. with the cyclic boundary conditions. In this view, the domain of the operator analytically rotates with  $k$ , while the differential expression of  $H$  stays the same. One of its advantages is that the domain  $H_k^s$  of the operator is  $\Gamma^*$ -periodic with respect to  $k$ , so it can be considered as depending on the Floquet multiplier  $z = (e^{ik_1}, \dots, e^{ik_n})$  only.
- Alternatively, according to (4.13), the operator  $H(k)$  can be considered as acting, for all  $k$ , on periodic functions on  $\mathbb{T}$ . This is achieved by commuting with an exponent:

$$(4.14) \quad e^{-ik \cdot x} \circ (-\Delta + V(x)) \circ e^{ik \cdot x}.$$

This leads to the following differential expression for  $H(k)$

$$(4.15) \quad -\Delta - 2ik \cdot \nabla + k^2 + V(x)$$

acting on the torus  $\mathbb{T}$ , i.e., on  $\Gamma$ -periodic functions. The advantage of this representation is the explicit polynomial dependence of the differential expression on the quasi-momentum and the domain being fixed. On the other hand, explicit periodicity in  $k$  is lost.

There are advantages and disadvantages of both representations. It is thus recommended to choose them judiciously, depending on the problem.

- The latter representation can be generalized to any periodic differential operator  $L(x, D)$ , where  $D := -\frac{1}{i} \frac{\partial}{\partial x}$ :

$$(4.16) \quad L(k) = L(x, D + k).$$

We can now formulate the exact range theorem for the Floquet transform in Sobolev spaces:

**Theorem 4.8.**

- $\mathcal{U}_\Gamma$  is isometry from  $H^s(\mathbb{R}^n)$  onto the space  $L^2(\mathbb{T}^*, \mathcal{H}^s)$  of the  $L^2$ -sections of the bundle  $\mathcal{H}^s$ .
- The section  $\mathcal{U}_\Gamma f(\cdot, k)$  is infinitely differentiable on  $\mathbb{T}^*$  iff the norms  $\|f\|_{H^s(W+\gamma)}$  decay when  $\|\gamma\| \mapsto \infty$  faster than any power of  $\|\gamma\|$ .
- The section  $\mathcal{U}_\Gamma f(\cdot, k)$  is analytic in a complex neighborhood of  $\mathbb{T}^*$  iff the norms  $\|f\|_{H^s(W+\gamma)}$  decay exponentially fast when  $\|\gamma\| \mapsto \infty$ .

**4.3. Direct integral decomposition.** We can summarize the results of previous sections as the following **direct integral decompositions** of functional spaces and of the operator  $H$ :

$$(4.17) \quad L^2(\mathbb{R}^n) = \int_{\mathbb{T}^*}^{\oplus} L^2(W), \quad H^s(\mathbb{R}^n) = \int_{\mathbb{T}^*}^{\oplus} H_k^s, \quad H = \int_{\mathbb{T}^*}^{\oplus} H(k).$$

We will not get into any deeper discussions of the direct integral technique (see, e.g. [55, 75, 113, 220, 239, 245]), since in our situations this is usually not required.

## 5. DISPERSION RELATION AND ALL THAT

As there is in general no analog of the 1D discriminant in higher dimensions, the dispersion relation takes the lead. Its definition and

many properties are analogous to the ones in 1D, although some important distinctions do arise.

### 5.1. Dispersion relation = Bloch variety.

**Definition 5.1.** The (real) **dispersion relation** (or the (real) **Bloch variety**  $B_H$ ) of the periodic Schrödinger operator<sup>9</sup>  $H$  is the subset of  $\mathbb{R}_k^n \times \mathbb{R}_\lambda$  defined as follows:

$$(5.1) \quad B_H := \{(k, \lambda) \in \mathbb{R}^{n+1} \mid Hu = \lambda u \text{ has a } \neq 0 \text{ Bloch-Floquet solution } u(x) = e^{ik \cdot x} p(x) \text{ with the quasi-momentum } k\}$$

The **complex dispersion relation** (or **complex Bloch variety**) is defined analogously, only allowing both  $k$  and  $\lambda$  to be complex, i.e.  $B_{H, \mathbb{C}} \subset \mathbb{C}^{n+1}$ . As we will see later on, the ability of considering the *complex* Bloch variety turns out to be very important.

**5.2. Dispersion relation and the spectrum.** Let us assume that  $k \in \mathbb{R}^n$ . The operator  $H(k)$  being a self-adjoint elliptic operator in sections of a linear bundle on the torus, implies

**Lemma 5.2.** *For any  $k \in \mathbb{R}^n$ , the operator  $H(k)$  is bounded below and has a discrete (real) spectrum*

$$(5.2) \quad \sigma(H(k)) = \{\lambda_j(k) \mid \lambda_1(k) \leq \lambda_2(k) \leq \dots \leq \lambda_d(k), \dots \mapsto \infty\}.$$

**Definition 5.3.** The function  $k \mapsto \lambda_j(k)$  is called the  *$j$ th band function*.

*Remark 5.4.* Notice that for complex quasimomenta  $k$ , the spectrum is also discrete [5, pp.180-190], although labeling them becomes an issue.

The following statements are routine, due to the direct integral expansion (4.17) of the operator, perturbation theory [157], and some standard properties of analytic Fredholm operator-functions (e.g., [171, 282, 292]):

### Theorem 5.5.

- (1) *The band functions are continuous and piece-wise analytic.*
- (2) *The graph of the multiple-valued mapping*

$$(5.3) \quad k \in \mathbb{R}^n \mapsto \sigma(H(k))$$

*coincides with the dispersion relation (Bloch variety)  $B_H$  of  $H$ .*

---

<sup>9</sup>The definition stays the same for other periodic operators,

- (3) *The latter claim also holds in the complex case, i.e. the graph of the multiple-valued mapping*

$$(5.4) \quad k \in \mathbb{C}^n \mapsto \sigma(H(k))$$

*coincides with the complex dispersion relation (complex Bloch variety) of  $H$ .*

- (4) *The dispersion relation is  $\Gamma^*$ -periodic with respect to  $k$  and thus it is sufficient (and common) to consider it only over the Brillouin zone  $\mathcal{B}$ .*
- (5) *The dispersion relation is symmetric (even) with respect to the mapping  $k \mapsto -k$ . (This fails if the potential is not real.)*
- (6) *The spectrum  $\sigma(H)$  is the range of the (real) dispersion relation, i.e.*

$$(5.5) \quad \sigma(H) = \bigcup_{k \in \mathcal{B}} \sigma(H(k))$$

$$(5.6) \quad = \{\lambda \in \mathbb{R} \mid \exists k \in \mathbb{R}^n, \text{ such that } \lambda \in \sigma(H(k))\}$$

$$(5.7) \quad = \{\lambda \in \mathbb{R} \mid \exists k \in \mathbb{R}^n, j \in \mathbb{Z}^+, \text{ such that } \lambda = \lambda_j(k)\}.$$

The last statement of the theorem is a very general claim about spectra of direct integrals of operators (see, e.g., [171, 239]).

The following remark is often very useful:

**Lemma 5.6.** *Let  $S$  be a dense subset of the Brillouin zone  $\mathcal{B}$ . Then*

$$(5.8) \quad \sigma(H) = \overline{\bigcup_{k \in S} \sigma(H(k))}.$$

In other words, if for some reason one wants to avoid some exceptional values of  $k$  (e.g.,  $k = 0$  causes troubles for the Maxwell operator), one can avoid them by choosing an appropriate dense subset of quasi-momenta.

We can now define the spectral bands of the operator  $H$ :

**Definition 5.7.** The segment

$$(5.9) \quad I_j := \text{range}(\lambda_j) = \{\lambda \in \mathbb{R} \mid \exists k \in \mathbb{R}^n \text{ such that } \lambda = \lambda_j(k)\}$$

is the  $j$ th band of the spectrum  $\sigma(H)$ .

Due to  $\Gamma^*$ -periodicity with respect to  $k$ , one can also write

$$(5.10) \quad I_j = \{\lambda \in \mathbb{R} \mid \exists k \in \mathcal{B} \text{ such that } \lambda = \lambda_j(k)\}.$$

**Corollary 5.8.**

- (1) *Each band  $I_j$  is a finite closed interval, both of whose endpoints tend to infinity when  $j \rightarrow \infty$ .*

(2) *The bands cover the whole spectrum:*

$$(5.11) \quad \sigma(H) = \bigcup_{j \in \mathbb{Z}^+} I_j.$$

(3) *The **bands can**<sup>10</sup> **overlap**.*

As it was in 1D, it is easy to check that for the free operator  $H = -\Delta$ ,  $B_H$  is the union of the paraboloid  $\lambda = k^2$  and its  $\Gamma^*$ -shifts. One can deduce from this (which requires sometimes highly non-trivial work involving number theory (e.g., [232, 235, 258])) the following:

**Theorem 5.9.** *Consecutive spectral bands of the free operator overlap, and thus the spectrum of the free operator has no gaps. Moreover, the length of the overlap tends to infinity when one goes up along the spectrum.*

This is in a stark contrast with 1D, where the bands do not overlap (touching in the worst case).

In Fig. (13) one finds an example of the dispersion relation for a non-zero potential:

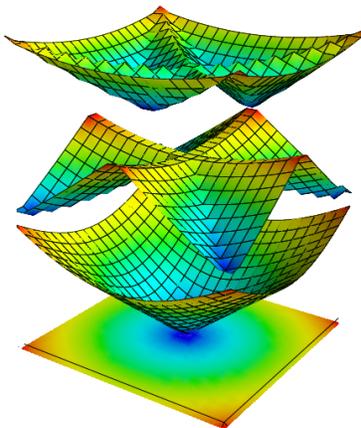


FIGURE 13. The dispersion relation of a non-free operator, drawn over a Brillouin zone.

**5.3. Born-Karman approximation.** A very useful way of obtaining the dispersion relation (and thus the spectrum) of a periodic operator is due to Born and Von Karman (see, e.g., [13]).

We will assume, as before, that  $\Gamma = \mathbb{Z}^n$  (although the approach works for any lattice).

<sup>10</sup>And, as we will see, usually do.

**Theorem 5.10.** *Consider for an integer  $N$  the cube  $Q_N := [-N, N]^n \in \mathbb{R}^n$  and define the operator  $H^{(N)}$  acting as the differential expression of  $H$  on  $Q_N$  with periodic boundary conditions (i.e., on the cube  $Q_N$  folded onto a large torus). Then*

$$(5.12) \quad \sigma(H) = \lim_{N \rightarrow \infty} \sigma(H^{(N)}) = \overline{\bigcup_{N \in \mathbb{Z}^+} \sigma(H^{(N)})},$$

where the limit is taken with respect to the Hausdorff distance.

Indeed, let us denote by  $\mathbb{QB}$  the subset of the Brillouin zone consisting of all quasi-momenta with components that are rational multiples of  $\pi$ . Then, it is a rather straightforward exercise to see that

$$\bigcup_{N \in \mathbb{Z}^+} \sigma(H^{(N)}) = \bigcup_{k \in \mathbb{QB}} \sigma(H(k)).$$

Now, using Lemma 5.6, one proves Theorem 5.10.

*Remark 5.11.* Another important observation is that this theorem is more than just the claim about Hausdorff convergence of spectra, or density of states convergence. Indeed, such claims hold also for other “reasonable” choices of boundary conditions (e.g., Dirichlet), or other choices of large domains instead of large cubes assembled from the fundamental unit cubes [263]. In those cases, however, one can often see **spurious spectra** appearing outside  $\sigma(H)$ , but disappearing in the limit. In the Born-Karman case, though, this does not happen, since  $\sigma(H^{(N)}) \subset \sigma(H)$  for any  $N$ .

**5.4. Analytic properties of the dispersion relation  $B_H$ .** We will explore now analytic properties of the dispersion relation of the operator  $H$ . Consider the complex Bloch variety

$$(5.13) \quad B_{H, \mathbb{C}} \subset \mathbb{C}^{n+1}.$$

**Theorem 5.12.** (*[168, Lemma 8 of §4], [171, Theorem 4.4.2], see also [165]*)

- (1) *There exists an entire function  $f(k, \lambda)$  on  $\mathbb{C}^{n+1}$ , such that*
- (a)

$$(5.14) \quad |f(k, \lambda)| \leq C_p e^{(|k|+|\lambda|)^p}$$

for any  $p > n$ . (Similar statement holds for more general elliptic periodic operators, with the exponent  $p$  depending on the dimension and order of the operator [171].)

- (b)

$$(5.15) \quad B_{H, \mathbb{C}} = \{(k, \lambda) \in \mathbb{C}^{n+1} \mid f(k, \lambda) = 0\}.$$

(2) If  $p > n + 1$ , one can make sure that  $f$  is  $\Gamma^*$ -periodic w.r.t.  $k$ . In particular,  $B_{H,\mathbb{C}}$  is a  $\Gamma^*$ -periodic complex analytic sub-variety of  $\mathbb{C}^{n+1}$  of co-dimension 1.

An easier statement, without estimates on growth of the function  $f$ , can be obtained from the analytic Fredholm theory (e.g., [171, 282, 292]). The proof of theorem as stated uses the theory of **regularized determinants in Shatten-von Neumann classes** [83, 127, 257].

**5.5. Floquet variety.** It is natural to reduce  $B_H$  with respect to its  $\Gamma^*$ -periodicity (i.e., using Floquet multipliers  $z = e^{ik}$  instead of crystal momenta  $k$ ). We thus introduce the following (not commonly adopted) notion:

**Definition 5.13.** The **Floquet variety**  $\mathcal{F}_H$  of  $H$  is defined as follows:  
(5.16)

$$\mathcal{F}_H := \{(z, \lambda) \in (\mathbb{C}^*)^n \times \mathbb{C} \mid z = e^{ik} = (e^{ik_1}, \dots, e^{ik_n}), \text{ where } (k, \lambda) \in B_{H,\mathbb{C}}\},$$

where  $\mathbb{C}^*$  is the **punctured complex plane**  $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ .

In other words,  $(z, \lambda) \in \mathcal{F}_H$ , iff there exists a non-zero Floquet solution of  $Hu = \lambda u$  with the Floquet multiplier  $z$ .

**5.6. Non-algebraicity.** As we have seen already, in the free case,  $B_H$  is the union of  $\Gamma^*$ -shifts of a single paraboloid. Thus, we get

**Proposition 5.14.** *In the free case,*

- (1) *All irreducible components of the Bloch variety  $B_H$  are algebraic (shifted paraboloids). In particular, the Bloch variety  $B_H$  is irreducible modulo  $\Gamma^*$ -shifts.*
- (2) *The Floquet variety  $\mathcal{F}_H$  is irreducible.*

It is thus interesting to ask what to expect in terms of algebraicity and irreducibility in the non-free periodic case.

In general, the irreducible components of neither Bloch, nor Floquet variety are algebraic. There exist, however, examples of **non-selfadjoint periodic operators  $H$  with algebraic components of  $B_H$**  (see [170, Theorem 11] and [171, Theorem 4.1.2]).

**In discrete problems, the Floquet varieties are algebraic** [24, 169], since the corresponding operators  $H(z)$  are finite Laurent series with respect to  $z$ :

$$\sum_{j \in \mathbb{Z}^n} H_j z^j,$$

where the sum is finite and  $H_j$  are matrices of a finite size.

A more interesting (and as we will see later, important) question is about irreducibility of the Bloch variety in higher dimensions, which is addressed in the next sub-section.

**5.7. Irreducibility.** It is conjectured that the analog of the statement (2) of Proposition 5.14 holds in higher dimensions:

**Conjecture 5.15.** *The Bloch variety of the self-adjoint periodic Schrödinger operator  $H = -\Delta + V(x)$  is irreducible modulo  $\Gamma^*$ .*

Proving this conjecture seems to be extremely hard. It has been proven in dimension 2 by a tour de force in [165].

A much weaker conjecture formulated below is already non-trivial:

**Conjecture 5.16.** *The Bloch variety  $B_H$  for a periodic elliptic **second order** operator  $H$  with sufficiently “decent” coefficients does not have any flat components  $\lambda = \text{const}$ .*

We will see later on, how important is the resolution of this conjecture for the spectral theory. That is why it has attracted attention of many mathematicians and has been proven in many cases (albeit still not in its full generality), see Section 6.3.

We provide here an easy to complete sketch of the proof of the following well known result, due to L. Thomas [272], although he did not formulate the statement in this form (see also [239]):

**Theorem 5.17.** *[272] Let  $V \in L_\infty(\mathbb{R}^n)$  be periodic. Then the Bloch variety of the Schrödinger operator  $-\Delta + V(x)$  has no flat components  $\lambda = \lambda_0$ . (Notice that it is not assumed that the potential  $V$  is real.)*

*Proof.* Suppose that this is incorrect, i.e. there exists  $\lambda_0 \in \mathbb{C}$  such that the equation  $-\Delta(k)u + Vu = \lambda_0 u$  has a non-trivial 1-periodic solution  $u$  for any  $k \in \mathbb{C}^n$ . Absorbing the spectral parameter into the potential, we can assume that  $\lambda_0 = 0$ . Thus, we conclude that

$$\Delta(k)u = Vu \text{ has a non-trivial solution } u \text{ for any } k \in \mathbb{C}^n.$$

Let  $C = \|V\|_{L_\infty}$ . Then  $\|Vu\|_{L_2(\mathbb{T})} \leq C\|u\|_{L_2(\mathbb{T})}$  and thus

$$\|\Delta(k)u\|_{L_2(\mathbb{T})} \leq C\|u\|_{L_2(\mathbb{T})}$$

for any  $k$  (with a non-trivial  $u$  depending on  $k$ ).

Let us now pick quasimomentum with a large imaginary part and judiciously chosen real part. Namely, assuming that  $\Gamma = \mathbb{Z}^n$  (analogous consideration works for any lattice), let

$$k = (0.5 + i2a, 0, \dots, 0), \quad a \in \mathbb{R}.$$

Expansion of the periodic function  $u$  into Fourier series easily leads to the estimate

$$\|\Delta(k)u\|_{L_2(\mathbb{T})} \geq |a|\|u\|_{L_2(\mathbb{T})}.$$

If  $a > C$ , we get a contradiction, unless  $\|u\| = 0$ , which we assumed does not happen.  $\square$

The assumption of boundedness of the potential is an overkill (see, e.g. [239]).

In fact, a stronger statement holds:

**Theorem 5.18.** *Under the same conditions as in Theorem 5.17, for any  $\lambda_0 \in \mathbb{R}$ , the level set of the dispersion relation:*

$$(5.17) \quad F_{\lambda_0} = \{k \in \mathbb{R}^n \mid H(k)u = \lambda_0 u \text{ has a non-zero solution} \}$$

*has measure zero in  $\mathbb{R}^n$ .*

We will see later that this leads to the absolute continuity of the spectrum of the periodic Schrödinger operators, which was the main goal in [272]. Due to this relation, the statements of Theorems 5.17 and 5.18 have been extended in the last three decades to a large variety of scalar and matrix periodic operators of mathematical physics (Pauli, Dirac, Magnetic Schrödinger, Maxwell, see Section 6.3 and references therein), although the proof of the whole Conjecture 5.16 remains elusive.

We need first the following auxiliary statement:

**Lemma 5.19.** *Let  $a(k)$  be a real analytic function on  $\mathbb{R}^n$ . If the measure of the zero set of  $a$  is positive, then the function vanishes identically.*

The author has tried to find in various real analytic geometry sources this result stated explicitly, but has failed so far. It can be derived from the Weierstrass Preparatory Theorem (e.g., [136]). It is also a direct corollary of the Lojasiewicz's result on existence of Whitney stratification of real analytic sets [195], as well as of the desingularization theorems by Bierstone and Milman [27, 28], which seems to be quite an overkill in both cases.

Now, Theorem 5.18 becomes a simple corollary of Theorem 5.17. Indeed, the level set of dispersion relation, is the set of real zeros of the real analytic function  $f(k, \lambda_0)$  on  $\mathbb{R}^n$  (see Theorem 5.12 for the definition of the function  $f$ ). Thus, if this set has a positive measure, then, according to Lemma 5.19, the function vanishes identically, and thus  $f(k, \lambda_0) = 0$  for all  $k \in \mathbb{C}^n$ . This is impossible according to Theorem 5.17.

One should notice that for periodic elliptic operators of orders higher than 2 neither of these conjectures holds true. E.g., there are examples of such operators of the 4th order whose Bloch variety is reducible, and moreover, has a flat irreducible component [171, pp. 135–136].

Let us address another issue of irreducibility. We have seen that in the free case any non-zero  $\Gamma^*$ -shift of an irreducible component  $\Sigma$  of  $B_H$  produces another component, different from  $\Sigma$ . In other words, irreducible components are not invariant with respect to any non-zero  $\Gamma^*$ -shifts. It seems that in presence of a generic periodic potential the situation is different.

**Conjecture 5.20.** *For a generic potential, unless an irreducible component  $\Sigma \subset B_H$  is algebraic, there exists a non-zero  $\Gamma^*$ -shift that leaves it invariant.*

If this conjecture is correct, then perturbation of the free operator by a periodic potential not only deforms the paraboloids, but also connects them together.

**5.8. Extrema.** Since, according to (5.5) – (5.7) the edges of the spectrum occur at extrema of the band functions, studying these extrema is an important task.

**5.8.1. Location.** In  $1D$ , as we have seen, the extrema of the dispersion relation  $\lambda(k)$  occur only at  $k = 0$  and  $k = \pi$  in the first Brillouin zone, which was due to the monotonicity of band functions on the reduced Brillouin zone.

Let us try to look at the points  $k = 0$  and  $k = \pi$  in a different light. The reciprocal lattice  $\Gamma^*$  is invariant with respect to the symmetry group generated by its shifts and “time reversion”  $k \mapsto -k$ . One discovers then that the points  $k = 0, \pi$  are exactly the fixed points of some of these symmetries inside of the reduced Brillouin zone.

In higher dimensions, the symmetry group can be larger. If one looks at the free case, one discovers that critical points of these overlapping paraboloids do occur at some symmetry points as well. Most computations for periodic Schrödinger and Maxwell operators also showed this effect. This has led to the following

**formerly popular belief:** *For  $-\Delta + V(x)$ , the extrema of dispersion relation must be attained at fixed points  $k \in \mathcal{B}$  of some symmetries (probably at the highest symmetry points). Thus, computing the dispersion only around the symmetry points gives the correct spectrum as a set.*

No monotonicity (or any other) reason for this claim exists. Some numerical evidence against it have been around, but not widely known and/or believed. **This claim was discussed and disproved** analytically in [97, 137]. It is also discussed in [137] why in practice the extrema are indeed often located at the symmetry points.

### 5.8.2. *Generic structure of spectral edges.*

After discussing the possible locations of the extrema, we move to the probably more important question of their structure. The following complications can (and do) occur at a spectral edge value  $c$ :

- A:** The extremal value  $c$  is attained by more than one band function.
- B:** The extremum  $c$  of a single band function  $\lambda_j$  is non-isolated (i.e., a “Mexican hat” type picture occurs).
- C:** The extremum is isolated, but degenerate (i.e., the Hessian of  $\lambda_j(k)$  at the extremum point is degenerate).

Although all of the above can occur, the general belief is that **generically** they do not.

**Conjecture 5.21.** *Generically (with respect to the potentials and other free parameters of the operator), the extrema of band functions*

- (1) *are attained by a single band;*
- (2) *are isolated;*
- (3) *are non-degenerate, i.e. have non-degenerate Hessians.*

In other words, one conjectures that generically near a spectral edge the dispersion relation looks like a single parabolic shape, i.e. resembles the dispersion relation at the bottom of the spectrum of  $-\Delta$ . As we will see later (Section 7), then various analogs of the properties of the Laplacian would be applicable to a generic periodic elliptic operator.

Why would one conjecture this? Well, existence of a degenerate extremum of the dispersion relation is an analytic equality type restriction. Then it is natural to think that it should either hold almost never, or for (almost) all operators. It is hard to believe that such a restriction for all periodic potentials exists.

The common idea is that generically, the dispersion relation probably behaves like the spectrum of a “generic” family of self-adjoint matrices. This has been conjectured in various forms by several authors (e.g., [16, 48]), but has never been proven (or even formulated) in any rigorous form.

Although the validity of the conditions (1)-(3) of the Conjecture 5.21 is often assumed in mathematics and physics literature (it is involved

with the definition of effective masses in solid state physics, homogenization, Green's function asymptotics, Liouville type theorems, Anderson localization, etc., see Section 7)), the conjecture remains largely unproven.

Let us mention what is known in this regard.

The following well known result by Kirsch and Simon [161] establishes the conjecture (in a stronger form) at the bottom of the spectrum of a Schrödinger operator with periodic electric potential:

**Theorem 5.22.** [161] *Let*

$$H = -\Delta + V(x)$$

*be a periodic Schrödinger operator in  $\mathbb{R}^n$ . Then the bottom of the spectrum of  $H$  is attained by the non-degenerate minimum at  $k = 0$  of the lowest eigenfunction  $\lambda_1$  only.*

Regretfully, the arguments of [161] are not applicable in the case of an internal spectral edge. J. Ralston [164]:

**Theorem 5.23.** *The claim (1) of Conjecture 5.21 holds for a generic Schrödinger operator with periodic electric potential.*

The conjecture has not been established even in the discrete situation, except for the following partial result:

**Proposition 5.24.** [78] *Let  $H$  be the periodic Laplace-Beltrami type operator<sup>11</sup> on a  $\mathbb{Z}^2$ -periodic graph  $\Gamma$ , having just two vertices (atoms) per a fundamental domain. Then the set of parameters (vertex and edge weights) for which the dispersion relation has a degenerate extremum is a (semi-)algebraic subset of co-dimension 1 in the space of all parameters.*

In principle, if the conjecture were proven for the discrete case, one could attempt the following bootstrap procedure to carry the result over to the continuous case (such a procedure was used for a different purpose in [137]):

- A.** prove the result for discrete periodic graphs
- $\Rightarrow$  **B.** carry it over to periodic quantum graphs, using the known relations between the two (e.g., [24])
- $\Rightarrow$  **C.** fatten the graph (e.g., [24, 173]) to extend to the full dimension.

Here the main difficulty arguably lies in the first step A.

An interesting example of a discrete  $\mathbb{Z}^2$ -periodic problem with degenerate (non-isolated) band edges was constructed by Filonov [105].

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<sup>11</sup>See [24, 44] for the explicit form of such operators.

A direct approach to the continuous case, skipping the discrete one, would be desirable.

Allowing change of the lattice of periods to a sub-lattice, possibility of removing degenerate edges by perturbation of potential was shown in  $2D$  in [110]. It seems that isolated nature of extrema was recently established in  $2D$  by Filonov and Kachkovskii [106].

**5.9. Dirac cones.** When two branches of the dispersion relation meet, they often form a conical junction point, called a Dirac cone or sometimes a “diabolic point.” The former name reflects the fact that the conical structure resembles the one for the dispersion of a  $2D$  massless Dirac equation.

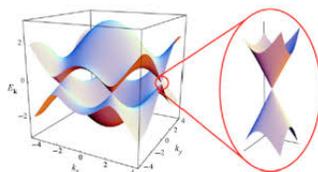


FIGURE 14. Dirac cones for the tight-binding model of graphene.

Such conical structures are usually unstable under perturbation of parameters of the operator (the potential). However, it has been noticed that if the structure has honeycomb lattice symmetry, preserved under perturbation, this protects the cone from splitting. Thus, in particular, such a cone mandatorily arises in the dispersion relation of the famous **graphene**, which explains its amazing electric properties (see, e.g., [158]).

The observation of the mandatory appearance of Dirac cones in honeycomb-symmetric structures was first made (way before graphene

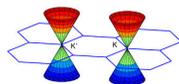


FIGURE 15. Dirac cones for the honeycomb structure.

came into play) for the tight binding model of the discrete honeycomb lattice [281], which provides an approximate picture of the graphene’s first two dispersion bands. An infinite-band quantum graph model, which has an infinite-dimensional freedom of choosing honeycomb-symmetric potentials was considered in [182], where detailed structure

of the spectrum and dispersion relation, including in particular presence of Dirac cones, was described<sup>12</sup>. For 2D Schrödinger operators with honeycomb-symmetric potentials, existence and stability of Dirac cones was established in [23, 98, 133]. The Schrödinger operator with honeycomb lattice of point scatterers was considered in [190].

One can say, however, that the story does not end here, since presence of stable Dirac cones (including very interesting non-isotropic ones) has been observed for more general 2D (**graphyne**) structures, which lack honeycomb symmetry. See, e.g. [89] for physics discussion and [76, 77] for quantum graph studies of a simplest graphyne. So far, there is no complete understanding of this phenomenon (analogous to the one provided in [23, 98, 133] for the honeycomb case).

**5.10. Fermi surfaces.** The notion of the Fermi surface is among the central ones in the solid state theory, as one can see from any solid state textbook, e.g. [13, 40, 162, 296]. There are plenty of books and databases dedicated to Fermi surfaces of various crystals, e.g. [56, 57, 99, 138, 295–297]. The reason is that many physical properties of, e.g., metals depend upon the geometry of its Fermi surface. We will try to show that hard questions about the analytic geometry of Fermi surfaces are intimately related to some basic spectral and other properties of the relevant operator. Answering these questions is usually extremely hard (see, e.g., the book [125], and a shorter survey [124], devoted solely to studying the surface for a discrete periodic Schrödinger operator on  $\mathbb{Z}^2$ ).

To define a Fermi surface, let us consider the (real or complex) dispersion relation  $B_H$  as the graph of the multiple valued function

$$k \in \mathbb{C}^n \mapsto \sigma(H(k)).$$

**Definition 5.25.** The **Fermi surface**  $F_{\lambda,H}$  of  $H$  at a scalar value  $\lambda$  is the  $\lambda$ -level set of the dispersion relation. I.e.,

**Complex Fermi surface is:**

$$\begin{aligned} F_{\lambda,H,\mathbb{C}} &:= \{k \in \mathbb{C}^n \mid (k, \lambda) \in B_{H,\mathbb{C}}\} \\ &= \{k \in \mathbb{C}^n \mid H(k)u = \lambda u \text{ has a non-zero solution}\}. \end{aligned}$$

**Real Fermi surface for a real  $\lambda$  is:**

$$\begin{aligned} F_{\lambda,H} &:= \{k \in \mathbb{R}^n \mid (k, \lambda) \in B_H\} \\ &= \{k \in \mathbb{R}^n \mid H(k)u = \lambda u \text{ has a non-zero solution}\}. \end{aligned}$$

An obvious statement is:

---

<sup>12</sup>The article [182] also contained the detailed spectral analysis of quantum graph models of all possible carbon nanotubes.

**Lemma 5.26.** *The Fermi surface is  $\Gamma^*$ -periodic.*

The following result follows from its analog for the Bloch variety  $B_H$ :

**Theorem 5.27.**  *$F_{\lambda,H}$  is the zero set of an entire function  $f(k)$  of the same exponential order as for  $B_H$  (see Theorem 5.12).*

The following reformulation of Theorem 5.18 is useful:

**Theorem 5.28.** *At any energy level  $\lambda \in \mathbb{R}$ , the real Fermi surface either has measure zero in  $\mathbb{R}^n$ , or it coincides with the whole  $\mathbb{R}^n$  (and thus the complex Fermi surface at this level is  $\mathbb{C}^n$ ).*

*Remark 5.29.*

- (1) In physics, the name Fermi surface is used only for a specific real level of the energy  $\lambda$ , the so called **Fermi level**  $\lambda_F$  [13, 40, 162, 296]. It is convenient for us, though, to consider Fermi surfaces at arbitrary levels.
- (2) The definition of the Fermi surface seems to be rather innocuous. Just to impress on you its possible complexity, Fig. 16 presents the Fermi surface of Niobium (Nb). One can find more

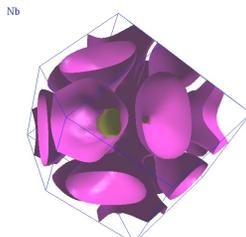


FIGURE 16. The Fermi surface of Niobium (Nb)

(often more complex) pictures of Fermi surfaces at URL [99].

**Conjecture 5.30.** *The Fermi surface for  $H = -\Delta + V(x)$  is irreducible (modulo  $\Gamma^*$ ) for  $\lambda \in \mathbb{R}$ , except for a discrete set of values of  $\lambda$ .*

This conjecture has been proven for discrete Schrödinger on  $\mathbb{Z}^2$  [124, 125]. In the case of continuous Schrödinger operator with periodic electric potential in dimensions  $n = 2, 3$  and separable potential  $V(x) = \sum V_j(x_j)$ , as well as in 3D for potential  $U(x_1, x_2) + V(x_3)$ , where the axis  $x_j$  are oriented along the basis vectors of  $\Gamma$  [185].

As we will see later, the “esoteric” question of irreducibility of the Fermi surface is closely related to the question of absence of impurity eigenvalues embedded into the continuous spectrum.

5.11. **Bloch bundles.** Let  $S \subset \sigma(H)$  be a subset consisting of  $m$  spectral bands and surrounded by spectral gaps. We will call such a subset a **composite band** (see Fig. 5.11). Let us surround  $S$  with a

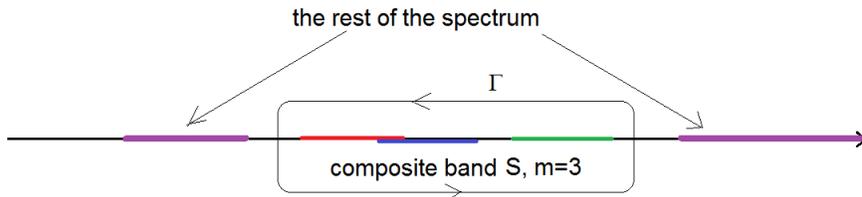


FIGURE 17. A composite band composed of three spectral bands.

contour  $\Gamma$  (Fig. 5.11) and introduce the corresponding  $m$ -dimensional spectral projector for  $H(k)$ :

$$P(k) := \frac{1}{2\pi i} \oint_{\Gamma} (\zeta - H(k))^{-1} d\zeta.$$

Projector  $P(k)$  depends analytically on  $k$  in a complex neighborhood of  $\mathbb{T}^*$  in  $\mathbb{C}^n$ . Its range forms the so called **Bloch bundle** over this neighborhood, which corresponds to the composite band  $S$ . Sections of this bundle form the invariant subspace for  $H$  that corresponds to  $S \subset \sigma(H)$ .

This bundle over the torus  $\mathbb{T}^*$  can be topologically non-trivial (e.g., in presence of magnetic potential [273] or in the case of topological insulators [25]).

As we have seen, the bundle is defined and analytic in a complex neighborhood of the torus. We are sometimes interested in its analytic triviality. The question arises whether there are additional analytic obstructions to this. The following result is a simple incarnation of the Oka's principle [132, 136]:

**Theorem 5.31.** (e.g., [176]) *Topological triviality of the Bloch bundle over  $\mathbb{T}^*$  is equivalent to its analytic triviality in a complex neighborhood.*

Indeed, there is a neighborhood of the torus that is the product of 1D complex domains. Then it is a Stein manifold, and the above result is a part of the famous Grauert's theorem [131] for such manifolds.

A variety of results on sufficient conditions of triviality of Bloch bundles are known, see e.g. [176, 228] and references therein (see also [292] for the survey of techniques of Banach bundles).

**5.12. Analyticity in the space of parameters.** An often very useful thing to do is to add to the spectral parameter  $\lambda$  and quasimomenta  $k$  an infinite-dimensional space  $P$  of parameters of the periodic operator (i.e., an appropriate Banach space of periodic potentials). Then the Fermi and Bloch varieties can be defined in this extended space. When the space of parameters is defined properly, it is usually straightforward to prove the following:

**Meta-theorem** *Bloch and Fermi surfaces are analytic subsets in the enlarged space  $\mathbb{C}^{n+1} \times P$  (respectively  $\mathbb{C}^n \times P$ ).*

**5.13. Inverse problems.** Inverse spectral problems are known to arise frequently and have been studied a lot for periodic ODEs (e.g., [1, 82, 189, 203, 221] and references therein).

The periodic PDEs in this regard are much trickier and there are not that many results available. We look here at a few examples.

5.13.1. *Borg's theorem.* The famous **Borg's uniqueness theorem** for the Hill operator  $-\frac{d^2}{dx^2} + q(x)$  with periodic potential  $q$  says:

**Theorem 5.32.** [35] *TFAE:*

- (1) *The potential is constant.*
- (2) *There are no spectral gaps.*
- (3) [16] *There exists an entire function, whose graph  $\lambda = f(k)$  belongs to the dispersion relation.*

The equivalence 1.  $\Leftrightarrow$  2. fails miserably in dimensions  $n > 1$ ! In fact, as we will see later, for any sufficiently small bounded periodic potential  $V(x)$  in  $\mathbb{R}^n$  with  $n > 1$ , the Schrödinger operator  $-\Delta + V(x)$  has no gaps in the spectrum. Moreover, when the gaps do appear, there are only finitely many of them, which is clearly not enough to recover the potential whatsoever. Spectral gaps in higher dimensions do not provide nearly as much information as in the ODE case. However, the equivalence 1.  $\Leftrightarrow$  3. probably still holds:

**Conjecture 5.33.** *TFAE:*

- (1) *The potential is constant.*
- (2) *There exists an entire function, whose graph  $\lambda = f(k)$  belongs to the dispersion relation.*

In  $2D$ , the conjecture was proven by Knörrer and Trubowitz [165].

5.13.2. *Floquet rigidity.* Let us consider the Schrödinger operator  $H = -\Delta + V(x)$  with the potential periodic with respect to a lattice  $\Gamma$  and the corresponding Bloch operators  $H(k)$ , which have, as we know, discrete spectra.

**Definition 5.34.**

- The **periodic spectrum** is defined as follows:

$$(5.18) \quad \sigma_0(H) := \sigma(H(0)) = \text{spectrum of } H \text{ on } \Gamma\text{-periodic functions.}$$

- The  **$k$ -spectrum** is defined as follows:

$$(5.19) \quad \begin{aligned} & \sigma_k(H) := \sigma(H(k)) \\ & = \text{spectrum on Bloch functions with quasimomentum } k. \end{aligned}$$

Clearly, the union over all  $k$  of the graphs of these spectra forms the Bloch variety for  $H$ .

**Definition 5.35.**

- The  $\Gamma$ -periodic potentials  $V$  and  $W$  are **isospectral**, if

$$(5.20) \quad \sigma_0(-\Delta + V) = \sigma_0(-\Delta + W).$$

- The  $\Gamma$ -periodic potentials  $V$  and  $W$  are **Floquet isospectral**, if

$$(5.21) \quad \sigma_k(-\Delta + V) = \sigma_k(-\Delta + W), \forall k \in \mathbb{R}^n.$$

One is interested in understanding when two potentials are isospectral, or even Floquet isospectral. These questions have been understood well in  $1D$  (see, e.g., [203]), while for  $n \geq 2$  problems are far from being resolved.

There are simple examples of isospectral potentials. For instance, one easily observes

**Lemma 5.36.**

- (1) Potentials  $V(x)$  and  $W(x) = V(\pm x + a)$ , for any  $a \in \mathbb{R}^n$ , are Floquet isospectral.
- (2) If the lattice  $\Gamma$  is invariant with respect to an orthogonal transformation  $O$  of  $\mathbb{R}^n$ . Then the potentials  $V(x)$  and  $V(Ox)$  are isospectral.

Imposing the following condition on the lattice  $\Gamma$ , one eliminates the second option in the lemma above:

**Definition 5.37.** A lattice  $\Gamma$  has a **simple length spectrum**, if

$$(5.22) \quad \text{If } \gamma_1, \gamma_2 \in \Gamma \text{ and } |\gamma_1| = |\gamma_2|, \text{ then } \gamma_2 = \pm\gamma_1.$$

One other option is to use the well known results on  $1D$  isospectrality. Indeed, let  $v(s)$  and  $w(s)$  be smooth 1-periodic functions of one variable and  $\delta$  be a vector such that

$$(5.23) \quad \{\delta \cdot \gamma \mid \gamma \in \Gamma\} = \mathbb{Z}.$$

It is easy to establish [93, 94] that **if 1D potentials  $v$  and  $w$  are isospectral, then the “ridge”  $n$ -dimensional potentials  $v(\delta \cdot x)$  and  $w(\delta \cdot x)$  are isospectral.** One wonders whether there are any other options for isospectrality (Floquet isospectrality)<sup>13</sup>.

Many significant results on multi-dimensional isospectrality were obtained by Eskin-Ralston-Trubowitz in [93, 94] (see also an overview in [92]), mostly under conditions of analyticity of the potentials and the simplicity of the length spectrum (5.22). See also preceding and following works of various authors with additional results and related discussions in [90, 91, 93, 128–130, 134, 135, 152–154, 276].

We present below a few results of [93, 94]<sup>14</sup>.

**Theorem 5.38.** *If the length spectrum of  $\Gamma$  is simple and  $\Gamma$ -periodic potentials  $V$  and  $W$  are analytic, then*

- *If they are isospectral, then they are Floquet isospectral.*
- *Let  $k_0 \in \mathbb{R}^n$  be such that  $\cos(2\pi k_0 \cdot \gamma) \neq 0$  for all  $\gamma \in \Gamma$ . If  $\sigma_{k_0}(-\Delta + V) = \sigma_{k_0}(-\Delta + W)$ , then  $V$  and  $W$  are Floquet isospectral.*

This result shows that (under appropriate conditions) the whole Floquet spectrum (i.e., the whole Bloch variety) has an overdetermined information, and the fiber of  $B_H$  over an appropriate quasimomentum  $k_0$  carries as much information as the whole  $B_H$ .

It is natural to look in the “horizontal,” i.e. along quasimomenta, direction. In particular,

**does the knowledge of a single band function  $\lambda_j(k)$  determine the whole dispersion relation (Bloch variety)?**

In fact, the Conjecture 5.15 (proven in 1D and 2D) implies, if correct, that *even any open part of a single branch  $\lambda_j(k)$  uniquely determines the dispersion relation.*

Let now  $V$  be a  $C^\infty$   $\Gamma$ -periodic potential with the Fourier series expansion

$$(5.24) \quad V(x) = \sum_{\delta \in \Gamma^*} a_\delta e^{i\delta \cdot x}.$$

<sup>13</sup>One can easily achieve Floquet isospectrality by allowing complex potentials [171, Theorem 4.1.2].

<sup>14</sup>The sophisticated techniques used in [93, 94] to prove the results involved, in particular, the spectral invariants extracted from the heat and wave kernel asymptotics.

**Definition 5.39.** Let  $\gamma \in \Gamma$ . The **reduced potential**  $V_\gamma(x)$  is defined as follows:

$$(5.25) \quad V_\gamma(x) = \int_0^1 V(x + s\gamma) ds = \sum_{\delta \cdot \gamma = 0} a_\delta e^{i\delta \cdot x}.$$

Another important observation in [92–94] concerns the relations between the Floquet spectra of the original and reduced potentials:

**Theorem 5.40.** *For any  $\gamma \in \Gamma$ , the Floquet spectrum  $\sigma_k(-\Delta + V)$  determines uniquely  $\sigma_k(-\Delta + V_\gamma)$ .*

Let now  $\delta \in \Gamma^*$  be such that there exists  $\gamma_0 \in \Gamma$  for which  $\delta \cdot \gamma_0 = 2\pi$ . Then one can introduce the 1D **directional potential**

$$(5.26) \quad V_\delta(s) := \sum_{n=-\infty}^{\infty} a_{n\delta} e^{ins}$$

**Theorem 5.41.** *The Floquet spectrum  $\sigma_k(-\Delta + V)$  determines uniquely the Floquet spectra the 1D spectral problem*

$$(5.27) \quad \left( = |\delta|^2 \frac{d^2}{ds^2} + V_\delta(s) \right) u(s) = \lambda u(s)$$

for any directional potential  $V_\delta$ .

The isospectrality of potentials  $V(\pm x + a)$  suggests a question whether under some conditions one can prove that, modulo shifts, the number of isospectrality classes is finite. A variety of important results of this type can be found in [94].

## 6. SPECTRAL STRUCTURE OF PERIODIC ELLIPTIC OPERATORS

As we have seen, both in 1D and in higher dimensions, periodic elliptic operators have band-gap structure of their spectra. There are, however, significant differences between ODE and PDE cases. We make the comparison in the table below:

	$n = 1$	$n > 1$
band overlaps	none	frequent
gaps	generically all open	many gapless potentials
no gaps	constant potential	all small potentials
free operator bands	touch to cover $\mathbb{R}^+$	overlap to cover $\mathbb{R}^+$

The rest of this section is devoted to discussing various spectral properties of periodic operators, such as the band-gap structure of the spectrum, the qualitative structure of the spectrum, eigenfunction expansions, etc.

**6.1. Spectral gaps.** We start with considerations of spectral gaps: their existence, number, and ways of creating them.

6.1.1. *Existence of gaps.* As we have discussed already, according to the Borg's theorem, in 1D every non-constant periodic potential creates spectral gaps. This is manifestly incorrect higher dimensions:

**Theorem 6.1.** *If the  $L_\infty(\mathbb{R}^n)$ -norm of a periodic potential  $V(x)$  is sufficiently small, the Schrödinger operator  $H = -\Delta + V(x)$  has no spectral gaps.*

This is an immediate consequence of Theorem 5.9, if one considers  $H$  as a perturbation of the free operator  $-\Delta$ .

6.1.2. *Maximal abelian coverings and Sunada's no gap conjecture.* Consider the standard covering  $\mathbb{R}^n \mapsto \mathbb{T} (= \mathbb{R}^n/\mathbb{Z}^n)$ . Its deck group is  $\mathbb{Z}^n$ , which coincides with  $H_1(\mathbb{T}, \mathbb{Z})$ . It can also be understood as the quotient of the universal cover of the torus by the commutator subgroup of the fundamental group  $\pi_1(\mathbb{T})$ .

**Definition 6.2.** The **maximal abelian covering** of a compact manifold  $X$  is the covering  $Y \mapsto X$  with the deck group  $H_1(X, \mathbb{Z})$ , obtained as the quotient of the universal cover of  $X$  by the commutator subgroup of the fundamental group  $\pi_1(X)$ .

The following monotonicity result holds [264]:

**Theorem 6.3.** *Let  $Y \mapsto X$  be a Riemannian covering with an amenable deck group. Then*

$$\sigma(-\Delta_X) \subset \sigma(-\Delta_Y),$$

where  $\Delta_M$  denotes the Laplace-Beltrami operator on a Riemannian manifold  $M$ .

Thus, the spectrum of the Laplacian on the maximal abelian covering is the largest (and thus has the fewest gaps) among all abelian coverings of the same base. The following, still not proven, **no gap conjecture** has been formulated by T. Sunada [265, 266]:

**Conjecture 6.4.** *Let  $Y \mapsto X$  be the maximal abelian covering of a compact Riemannian manifold  $X$  of a constant negative curvature. Then the spectrum of the Laplace-Beltrami operator  $-\Delta_Y$  on  $Y$  has no gaps.*

Analogous conjecture has been formulated in the graph case, where the base graph  $X$  is assumed to be regular (i.e., degrees of all its vertices are the same). The graph version of the conjecture was proven for all regular graphs of even degree [142] and for various examples of regular graphs of odd degree [142, 167], e.g., the so called  $K_4$  graph  $X$ .

6.1.3. *Number of gaps. Bethe-Sommerfeld conjecture.* What about the possible number of spectral gaps, if they exist at all? In  $1D$ , the statement (3) of Theorem 1.12 shows that generically the number of gaps is infinite.

The situation is quite different in dimensions  $n > 1$ , as the following old **Bethe-Sommerfeld Conjecture (BSC)** [26] shows:

**Conjecture 6.5.** *When  $n > 1$ , there can only be a finite number of gaps in the spectrum of any periodic Schrödinger operator*

$$-\Delta + V(x)$$

in  $\mathbb{R}^n$ .

By now, after several decades of efforts, the conjecture has been proven in its full generality. The story started with wonderful works by Dahlberg and Trubowitz [59] and by Skriganov [258, and references therein], where some conditions on the lattice and/or dimension were imposed, and culminated in the work of Parnowski [232] (see some further generalizations in [235]), along the way absorbing contributions of other experts [20, 88, 156, 188, 214, 215, 231, 233, 234, 260, 277, 280]. The results also cover the case of magnetic potentials.

*Remark 6.6.*

- (1) **A stronger version of BSC-** for any second order periodic elliptic operator, is still not proven, except for

$$H = (-\Delta)^m + \text{lower order periodic terms.}$$

- (2) The Maxwell operator case is mostly unresolved, except in a simplest instance [280].
- (3) The validity of BSC for periodic waveguide systems has not been established and is sometimes questioned.
- (4) For periodic graphs/quantum graphs the statement of BSC does not hold [24].

6.1.4. *Gap creation.* Presence of gaps is necessary in many instances. E.g., they are responsible for properties of semi-conductors. Fortunately, such materials exist already in nature. The situation is different, for instance, when one tries to create the so called **photonic crystals**, where the operator of interest is Maxwell in a periodic medium and one needs to create such a medium with spectral gaps [149, 172]. We know by now that periodicity, although leading to the band-gap structure of the spectrum, does not guarantee existence of gaps. We thus address here briefly how spectral gaps can be “engineered”.

Let us address first the Schrödinger operator  $-\Delta + V(x)$ . We have just said, engineering gaps is not such a hot issue in this case. It also happens to be rather easy to create spectral gaps. Indeed, the following procedure can be easily made precise: creating a local potential well, one can create a bound state (eigenvalue) below the spectrum of the Laplacian. Now, repeating periodically copies of the well at large distances from each other will lead to spreading of this eigenvalue into a thin spectral band at a distance from the spectrum of the free operator, and thus creating a periodic medium with a spectral gap.

The potential well technique fails for Maxwell and some other operators, when local perturbation cannot lower the bottom of the spectrum. Another deficiency here is that the resulting gaps are rather hard to manipulate.

Another approach is creating very high contrast media, which succeeds in creating spectral gaps for periodic Maxwell operators [100–102, 104] as well as for Laplace-Beltrami operators on abelian coverings of compact Riemannian manifolds [141, 238, and references therein]. A two-scale homogenization approach has also been developed by Zhikov [294].

The high contrast approach also has its deficiencies. First of all, it is still hard to manipulate the locations and sizes of spectral gaps. Besides, the high contrast frequently requires non-physical values of material parameters.

The moral is that in higher dimensions it often is very hard to create and manipulate the spectral gaps arising due to Bragg scattering (i.e., due to the periodicity of the medium).

A different promising mechanism of **opening resonant gaps** exists. The author has learned it first from Pavlov’s work [236], although the idea probably has been around for a longer time (Arnold principle again). The idea is that spreading small identical resonators throughout the medium tends to create spectral gaps around the eigenvalues of the resonator. No periodicity is required. Regretfully, in the continuous case, this idea apparently has never been made precise. It has been implemented (as the so called **decoration procedure**), in its simplest incarnations in the graph [246] and quantum graph cases [174]. In practice, a more involved “spider decoration” procedures are desirable (see discussion in [24, 174]), which are at an initial stage of development (see, e.g., [224]). One can compare this with the more elaborate **zig-zag** decoration procedure, used to create **expander graphs** (which boils down to controlling the size of the principal spectral gap) [240].

Resonant gaps can also be used to create “slowing down light” optical media (e.g., [139, 178, 288]).

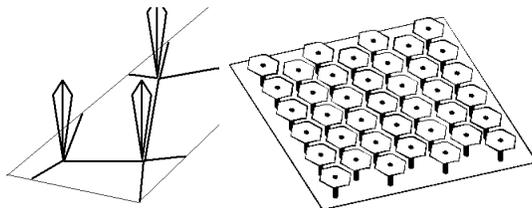


FIGURE 18. A decorated graph (left). An antenna implementing the concept(right).

**6.2. Eigenfunction expansion.** Let us recall that

$$(6.1) \quad \lambda_1(k) \leq \lambda_2(k) \leq \lambda_3(k) \leq \dots$$

is the sequence of eigenvalues of the operator  $H(k)$ , listed for  $k \in \mathbb{R}^n$  (with their multiplicity) in a non-decreasing order.

The standard perturbation theory (e.g., [157]) implies that

**each  $\lambda_j(k)$  is a continuous, piece-wise analytic function of  $k$ .**

By definition of  $\lambda_j$ , for each  $\lambda_j(k)$  the operator  $H$  has a generalized (since it is not square-summable) Bloch eigenfunction  $\psi(x)$  with the quasi-momentum  $k$  (i.e.,  $\psi(x) = e^{ik \cdot x} p(x)$  with a  $\Gamma$ -periodic function  $p(x)$ ):

$$(6.2) \quad H\psi = \lambda_j(k)\psi.$$

The question is whether the eigenfunction  $\psi$  (and thus the corresponding periodic function  $p$ ) can be chosen with a nice dependence of  $k$ . Whenever the multiplicity of the eigenvalue  $\lambda_j(k)$  stays constant near a point  $k_0$ , one can choose the Bloch eigenfunction analytically depending on  $k$ . See, e.g., [171, 292] for this standard fact. However, whenever the band functions collide and thus the multiplicity changes, one cannot extend the Bloch eigenfunction even continuously w.r.t.  $k$ . However, cutting along these thin bad sets of quasimomenta, one can establish the following

**Lemma 6.7.** [282] *There exists a null-set  $\mathcal{Z} \subset \mathbb{T}^*$  and a sequence  $\psi_j(k)$  of Bloch eigenfunctions, analytic on  $\mathbb{T}^* \setminus \mathcal{Z}$  that form an orthonormal basis in  $L_2(W)$  for  $k \in \mathbb{T}^* \setminus \mathcal{Z}$ .*

This immediately leads to the eigenfunction expansion of any function  $f(x) \in L^2(\mathbb{R}^n)$  into Bloch eigenfunctions:

**Theorem 6.8.** ([119], see also [193]) *Let  $f \in L^2(\mathbb{R}^n)$  and*

$$(6.3) \quad \mathcal{U}_\Gamma f(\cdot, k) = \sum_j f_j(k) \psi_j(\cdot, k)$$

be the expansion of its Floquet transform into the basis  $\psi_j$ . Then

$$(6.4) \quad \|f\|_{L^2}^2 = \sum_j \|f_j\|_{L^2(\mathbb{T}^*)}^2.$$

In other words,

**Theorem 6.9.** *The operator  $H$  is unitarily equivalent to the orthogonal direct sum of operators of multiplication by piece-wise analytic continuous functions  $\lambda_j(k)$  in  $L^2(\mathbb{T}^*)$ .*

**6.3. Absolutely continuous, pure point, and singular continuous parts of the spectrum.** According to Theorem 6.9, one needs to understand the spectrum of the operator  $A$  acting in  $L^2(\mathbb{T}^*)$  as multiplication by a continuous piece-wise analytic function  $a(k)$

$$(6.5) \quad (Af)(k) = a(k)f(k).$$

**Theorem 6.10.** *The following holds:*

- (1) *The singular continuous spectrum of the operator  $A$  is empty:*

$$(6.6) \quad \sigma_{sc}(A) = \emptyset.$$

- (2) *The pure point spectrum consists of all values  $\lambda$  such that the  $\lambda$ -level set of  $a(k)$  has positive measure. In particular, if there is no positive measure “flat piece”  $a = \text{const}$ , then  $\sigma_{pp}(A) = \emptyset$  and thus the whole spectrum is absolutely continuous:*

$$\sigma(A) = \sigma_{ac}(A).$$

- (3) *For each  $\lambda \in \sigma_{pp}(A)$ , the function  $a(k)$  is constant on an open set.*

We thus get the following result:

**Corollary 6.11.** *For any periodic self-adjoint elliptic operator  $H$ , one has:*

- (1) *The singular continuous spectrum is empty:  $\sigma_{sc}(A) = \emptyset$ .*
- (2) *The pure point spectrum consists of all values  $\lambda$  such that the  $\lambda$ -level set of some of  $\lambda_j(k)$  has positive measure.*
- (3) *If the pure point spectrum is non-empty, then at least one of the band functions  $\lambda_j$  is constant on an open set.*
- (4) *The pure point spectrum consists of all values  $\lambda_0$ , such the complex Bloch variety  $B_{H,\mathbb{C}}$  contains the flat component  $\lambda = \lambda_0$ .*

The last statement of this theorem follows from analyticity of the Bloch variety (Theorem 5.12), which forces any small piece of a constant branch to extend to the whole flat component. Since Theorem 5.17 prohibits such a situation for the Schrödinger operator  $-\Delta + V(x)$ , one reaches the following conclusion:

**Theorem 6.12.** [272] *Under the conditions of Theorem 5.12, the spectrum of the Schrödinger operator with periodic potential is absolutely continuous.*

As it was the case with Theorem 5.17, here the condition of boundedness of the periodic potential is a significant overkill (see, e.g., [239]). One, however, should not get over-excited about dropping conditions on the coefficients in the highest order terms, since when they are going below guaranteeing weak uniqueness of continuation property, one can get point spectrum, as it was shown by Filonov [103]. We will return to this discussion a little but further in the text.

Since the famous work [272] by L. Thomas, all proofs of the absolute continuity of the spectra of periodic elliptic operators followed the same scheme: proving absence of flat components in the dispersion relation (Bloch variety). Due to the usually hard technical work of many researchers (Birman & Suslina, Danilov, Derguzov, Filonov, Friedlander, Klopp, Kuchment & Levendorskii, Morame, Simon, Shen, Shterenberg, Sobolev, Thomas, and many others), absolute continuity of the spectrum has been proven for many (**but still not all**) second order scalar elliptic periodic operators, as well as periodic Pauli, Dirac, and in some cases Maxwell operators, see [30, 33, 61–71, 111, 114, 122, 123, 155, 163, 171, 179, 239, 247–249, 252, 259, 272, 274] and references therein. However, the following conjecture still remains unproven, even under assumption of infinite differentiability of the coefficients of the operator:

**Conjecture 6.13.** *The spectrum of any self-adjoint second order periodic elliptic scalar operator with “nice” (e.g., smooth) coefficients is absolutely continuous.*

In all works, except [114], the proof goes analogously to the one of Theorem 5.17: dominating in the Fourier domain the lower order terms by the principal part, which gets increasingly hard in presence of variable coefficients in the first order terms (magnetic potential) and it does not look feasible at all to treat variable coefficients in the principal part of the operator. The least technical approach, which beautifully avoids these domination estimates and thus allows for variable coefficients in the second order terms, was due to L. Friedlander [114]. Regretfully, it requires some symmetry condition on the operator, which seems to be superfluous, but no one has succeeded in removing it.

There are also quite a few, often even more demanding, results on absolute continuity for operators in periodic waveguides, on periodic systems of curves and surfaces, etc. [22, 72, 96, 107–109, 116, 150, 151, 170, 254, 262, 268, 269].

*Remark 6.14.* Absolute continuity of the spectrum **fails for some periodic elliptic operators of higher order.**

Indeed, the Plis' example [237] of a 4th order elliptic operator which fails unique continuation property, i.e. has a compactly supported eigenfunction, can be easily massaged to produce a periodic example, where  $\sigma_{pp} \neq \emptyset$  [171].

This relation to the uniqueness of continuation theorems does not seem accidental. As we have already mentioned, Filonov [103] constructed an example of a 2nd order periodic elliptic operator with non-empty pure point spectrum, with the leading coefficients falling just below what is needed for the weak uniqueness of continuation to hold. Another evidence of this is the following result:

**Theorem 6.15.** [171, Theorems 4.1.5 and 4.1.6] *Existence of an  $L^2$ -eigenfunction for a periodic elliptic operator is equivalent to existence of a (different) super-exponentially decaying eigenfunction, i.e. such that*

$$(6.7) \quad |u(x)| \leq Ce^{-|x|^\gamma},$$

with some  $\gamma > 1$  (depending on the order of the operator and dimension, see details in [171, proof of Theorem 4.1.6]).

Existence of such a solution must violate some “unique continuity at infinity” property. Regretfully, the only such result known, due to Froese&Herbst& M.&T. Hoffman-Ostenhof [117] and independently Meshkov [205], is not strong enough to lead to new absolute continuity theorems. E.g., it does not allow variable coefficients in the terms of the first and second orders.

Another confirmation of the relation with the uniqueness of continuation comes from equations on discrete or quantum graphs, where it does not hold. And sure enough, one can find compactly supported eigenfunctions for 2nd order elliptic periodic operators on such graphs (see [24, and references therein] and Fig. 19).

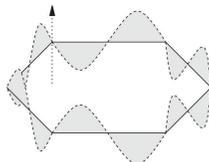


FIGURE 19. A compactly supported eigenfunction on a quantum graph.

6.3.1. *Density of states.* An important object in spectral theory and solid state physics is the so called *density of states* of a self-adjoint periodic elliptic operator  $H$ . Here is the idea: consider an “appropriate” sequence of expanding when  $N \rightarrow \infty$  domains  $V_N \subset \mathbb{R}^n$ , eventually covering the whole space. After imposing “appropriate” boundary conditions on these domains, one can consider the counting function of eigenvalues  $\lambda_j$  of the resulting operator with discrete spectrum:

$$(6.8) \quad \#\{\lambda_j < \lambda\}.$$

Now one normalizes this function by the volume of the domain  $V_N$  and takes the limit when  $n \rightarrow \infty$ :

**Definition 6.16.** The **integrated density of states (IDS)** of the operator  $H$  is

$$(6.9) \quad \rho(\lambda) := \lim_{N \rightarrow \infty} \frac{1}{|V_N|} \#\{\lambda_j < \lambda\}.$$

One certainly wonders whether the limit exists and is independent of the choice of the expanding domains  $V_N$  and boundary conditions imposed. Under appropriate conditions on those, the answer is a “yes” to both questions (see for instance [263] where this is done even in almost periodic situation and [2] for a more general discussion).

Without listing the conditions, we just acknowledge that, assuming that the lattice is transformed to  $\mathbb{Z}^n$ , this is true when one picks  $V_N$  as the sequence of cubes  $\{x \in \mathbb{R}^n \mid |x_j| \leq N, j = 1, \dots, n\}$  for  $N \in \mathbb{Z}^+$  and one imposes periodic boundary conditions on those (recall our previous discussion of Born-Karman conditions).

Then a simple calculation shows that the density of states can be described in terms of the dispersion relation  $\lambda_j(k)$  as follows:

**Theorem 6.17.**

$$(6.10) \quad \rho(\lambda) = \sum_j \mu\{k \in \mathbb{T}^* \mid \lambda_j(k) < \lambda\},$$

where  $\mu$  is the previously defined Haar measure on the torus  $\mathbb{T}^*$ .

Due to the analytic nature of the band functions we have studied, one gets

**Corollary 6.18.**

- (1) *The IDS is a piecewise analytic function of  $\lambda$ .*
- (2) *Unless a flat band function  $\lambda = \text{const}$  is present, the IDS is continuous.*
- (3) *Singularities of the IDS can arise if  $\lambda$  is a singular or critical value of the dispersion relation (**Van Hove singularities**).*

The Figure 20 illustrates such possibilities:

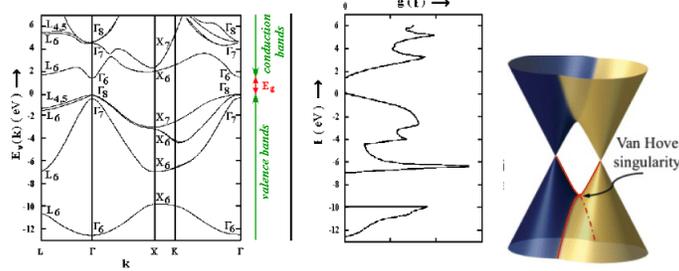


FIGURE 20. Dispersion relation (left), IDS (center), and a surface plot of the dispersion relation.

**Definition 6.19.** The **density of states** is the Radon-Nikodim derivative of the IDS with respect to the Lebesgue measure:

$$(6.11) \quad g(\lambda) := \frac{d\rho}{d\lambda}.$$

**Theorem 6.20.**

$$(6.12) \quad g(\lambda) = (2\pi)^{-n} \sum_j \int_{\lambda_j=\lambda} \frac{ds}{|\nabla_k \lambda_j|}.$$

Noticing that in (6.12) one encounters the integral of a holomorphic form over a real cycle on the Fermi surface  $F_\lambda$ , allows one to use powerful techniques of SCV (see [120]), which we are unable to address here.

**6.4. Wannier functions.** Bloch functions clearly are analogs for the periodic case of plane waves  $e^{i\xi \cdot x}$ , localized in Fourier domain. Standard Fourier transform converts the plane waves into delta functions, which are localized in the physical space. One wonders whether there is an analog of the delta function basis for the periodic situation, and if yes, whether such functions are useful. The answer is a “yes” for both questions, although new non-trivial issues arise here.

The functions in question are called **Wannier functions**, which are used frequently in numerical computations in solid state physics and photonic crystal theory [38, 39, 201], since they lead to nice discretizations, close to the tight-binding models. One expects (in analogy with the plane waves and delta functions) that they could arise as inverse Floquet transforms w.r.t. the crystal momentum, of Bloch functions.

Let us try to be more precise.

**Definition 6.21.** Suppose that we have a Bloch function  $\phi_k(x)$  that depends “sufficiently nicely” on the quasi-momentum  $k$ . The corresponding Wannier function is defined as follows:

$$(6.13) \quad w(x) := \int_{\mathbb{T}^*} \phi_k(x) dk.$$

Standard Fourier series argument shows that **smoothness of  $\phi_k$  with respect to  $k$  translates into decay of  $w(x)$** . For instance, infinite differentiability implies that the  $L_2$ -norm of  $w(x)$  on a cube decays faster than algebraically with the shift of the cube. Analogously, analyticity of  $\phi_k$  would produce exponential decay of the Wannier function.

Suppose now that one has a multiple band  $S$  consisting of  $m$  bands of the spectrum, which is separated from the rest of the spectrum by gaps. As it was discussed in Section 5.11, one obtains an  $m$ -dimensional analytic Bloch bundle over  $\mathbb{T}^*$ . If this bundle is analytically trivial, it has an analytic basis of Bloch functions:  $\phi_{k,1}, \dots, \phi_{k,m}$ . It is a simple exercise to see (e.g., [176]) that the corresponding exponentially decaying Wannier functions  $w_j = \int_{\mathbb{T}^*} \phi_{k,j} dk$  and all their  $\Gamma$ -shifts form a basis (which can also be made orthonormal under appropriate choice of Bloch functions) in the spectral subspace of the operator  $H$  that corresponds to the isolated part of the spectrum  $S$ .

This construction does not work, if the Bloch bundle is not analytically trivial. An incarnation of Oka’s principle, due to Grauert [131], shows that the only obstacle is topological: if the Bloch bundle is topologically trivial, then it is automatically analytically trivial (see also [292] for the related discussions). For a while this issue had not been understood and there was a belief that nice Wannier bases always exist, till Thouless showed [273] that in presence of magnetic terms in a periodic Schrödinger operator, the Bloch bundle can be non-trivial (such non-triviality also arises in topological insulators). A lot of effort has been concentrated on establishing sufficient conditions of the triviality, as well as on efficient finding the Wannier bases (see, e.g., [176, 201, 202, 219, 228, 229, and references therein]).

In the presence of the topological obstacle, however, no basis of even slowly decaying Wannier functions exists [176]. However, it was shown that instead of the (non-existing) analytic basis of the bundle, one can always find a “nice” Parseval frame (i.e., overdetermined system) of exponentially decaying Wannier functions [176]. Rather crude estimates on the number of extra functions needed is also provided in [176]. However, in dimensions up to 4, instead of  $m$  families of Wannier functions, one only needs  $(m + 1)$  [14].

**6.5. Impurity spectra.** Consider now the periodic Schrödinger operator  $H = -\Delta + V(x)$ . An important and frequently studied issue is the existence and properties of the **impurity spectrum** arising when a localized (compactly supported or sufficiently fast decaying) perturbation  $W(x)$  is added to the potential. The well known theorem claims that only eigenvalues of finite multiplicity can appear, leaving the otherwise absolutely continuous spectrum unchanged (see, e.g. [126, Section 18 and references therein]).

Since  $V$  is periodic and thus has band-gap structure of the spectrum, these eigenvalues have two options: to appear in the spectral gaps, or embed into the AC spectrum (**embedded eigenvalues**), see Fig.21. The general wisdom is that when the perturbation decays sufficiently

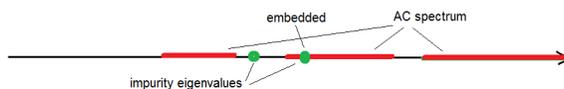


FIGURE 21. Impurity eigenvalues on a band-gap structured spectrum.

fast, embedded eigenvalues cannot arise, while if the decay is not fast enough, one can indeed have them. In the case when the background potential  $V$  is equal to zero, starting with the famous work by von Neumann and Wigner [279], many results of this nature have been obtained, see e.g. a nice survey in [86].

If, however, the background periodic potential is present, the issue is far from being resolved. The results of expected type, i.e. confirming the “general wisdom,” were obtained in the ODE case by Rofo-Beketov [241–244].

Although similar answers are expected in the higher dimension case, proving them happens to be really hard. Some results, rather weak, but instructive were obtained by Vainberg and the author [184–186]. E.g., the following theorem was proven:

**Theorem 6.22.** [184, 185] *Let  $n \leq 4$  and the perturbation  $W(x)$  decay exponentially. If for a given  $\lambda \in \sigma(H)$  the Fermi surface at that level is irreducible (modulo periodicity), then  $\lambda \notin \sigma_{pp}(H + W)$ .*

*In particular, if all Fermi surfaces are irreducible, no embedded eigenvalues can arise.*

*Remark 6.23.* Notice that irreducibility of the Fermi surface at a level does not depend on the impurity potential  $W$ .

As our previous discussion has indicated, it is very hard to establish irreducibility of the Fermi surface, although for a second order periodic

ODE it is automatic. One wonders whether the irreducibility condition of this theorem is just an artifact of the techniques used in the proof. It is possible, but does not seem to be very likely. The proof indicated that reducibility could in principle lead to a situation similar to the waveguide theory, where impurity eigenvalues can get embedded. And indeed, there are examples of a periodic fourth order ODE [230] and of second order multi-periodic quantum graph operators [250] where reducible Fermi surfaces arise and impurity eigenvalues can be embedded.

In the graph case, due to non-trivial topology, embedded eigenvalues can and do arise. However,

**Theorem 6.24.** *[186] In the graph case, embedded eigenvalues might arise, but in the case of irreducible Fermi surface, the corresponding eigenfunction must be supported “close” to the support of the perturbation. (See details in [186].)*

It is interesting to note that in the reducible case, eigenfunctions can spread and indeed have unbounded support [250].

## 7. THRESHOLD EFFECTS

The name **threshold effects** (as coined by Birman and Suslina), is used here for the features that depend upon the infinitesimal structure, or maybe just on a finite jet, of the dispersion at a spectral edge. And there are quite a few of those. A popular now days example is of graphene, where the Dirac cones lead to conductance as if governed by the massless Dirac equation [158]. We will not dwell on this particular example, but rather list a number of other important relations.

Regretfully, in order to keep the (already excessive) length of the text in check, the author can afford only brief pointers rather than any detailed discussion or formulation of the results.

**7.1. Homogenization.** Homogenization is an effective medium theory for (long) waves in a highly oscillating periodic (or even random) medium. It is probably the best known threshold effect. Indeed, finding the homogenized (effective medium) operator is, roughly, equivalent to determining the second order jet of the dispersion relation at the bottom of the spectrum (see, e.g. [6, 7, 21, 50, 52–54, 148] and references therein). It is sometimes a highly non-trivial procedure (e.g., for Maxwell operator).

While the usual homogenization leads to a homogeneous medium (which gave the name to the area), and so cannot address finite spectral gaps, more sophisticated approaches (see, e.g. [115, 140, 294]) can achieve this and more.

It is natural to ask whether there is a version of homogenization that occurs near the spectral edges of the internal gaps, where the parabolic shape of the dispersion relation might resemble the one at the bottom of the spectrum. This indeed happen to be the case, as it was shown in a series of works by Birman and Suslina [29, 31, 32, 34].

**7.2. Liouville theorems.** The classical Liouville theorem says that *Any harmonic function in  $\mathbb{R}^n$  of polynomial growth of order  $N$  is a polynomial of degree  $N$ . The dimension of this space is*

$$\binom{n+N}{N} - \binom{n+N-2}{N-2}.$$

S. T. Yau posed a problem [289] of generalizing this theorem to harmonic functions on noncompact manifolds of nonnegative curvature, which was resolved by Colding and Minicozzi [47] (there were also partial contributions from various researchers, in particular P. Li, see the references in [191, 192]). Thus, finite-dimensionality of the spaces of harmonic functions of a given polynomial growth, as well as estimates (rather than exact formulas) for their dimensions were obtained.

On the other hand, a wonderful observation was made by Avellaneda and Lin [15] and Moser and Struwe [216]: the spaces of polynomially growing solutions of periodic divergence type second order elliptic equations in  $\mathbb{R}^n$  are finite-dimensional. Moreover, their dimensions coincide with those of harmonic polynomials. Homogenization techniques were used in both cases, which restricted the consideration to the bottom of the spectrum only. In Pinchover and author's works [180, 181] a wide range generalization of this result was obtained: for an elliptic periodic operator on an abelian covering of a compact manifold (or graph) it was shown that validity of Liouville theorem at some energy level  $\lambda$  is equivalent to the corresponding Fermi surface consisting of finitely many points (i.e., essentially being at a spectral edge). Moreover, dimensions of the spaces of polynomially growing solutions were explicitly computed in terms of the lowest order non-trivial germ of the dispersion equation at the corresponding spectral edge.

**7.3. Green's function.** Another threshold effect is the behavior of the Green's function (the Schwartz kernel of the resolvent) near and at spectral edges. There are well known general resolvent exponential decay estimates with the exponential rate depending upon the distance

to the spectrum, e.g. the Combes-Thomas estimates [19, 49]. However, being applied to self-adjoint periodic elliptic operators, these estimates are not very precise. Indeed, first of all, one expects the exponential decay to be direction-dependent, while the operator estimates mentioned above would provide only an isotropic estimate. Moreover, at the bottom of the spectrum the dispersion relation of periodic non-magnetic Schrödinger operator has, as it has been mentioned before, a non-degenerate parabolic extremum [161], i.e. resembles the one for the Laplace operator. Thus, one can hope that at least near the spectrum the decay should resemble the one for the Laplacian case, i.e. involving an additional algebraically decaying factor. Moreover, at the edge of the spectrum one expects (in dimensions three and higher) some algebraic decay. And indeed, principal terms were found for the asymptotics of the Green's functions of such operators below the spectrum in [17, 18, 217] (see also a simplified derivation for the discrete case in [283]). These results confirm the expectation.

One can ask the same question near and at the edges of the internal gaps of the spectrum, as long as the dispersion relation has a non-degenerate (parabolic) extremum there. Results of this type were obtained in the recent works [159, 160, 183].

Going inside the spectral bands, it is also natural to mention here the results on the absorption principle for periodic elliptic operators, see [120–122, 218].

## 8. SOLUTIONS

In this section we provide a very brief overview, with pointers to the literature, of various results concerning solutions of homogeneous and inhomogeneous periodic elliptic equations.

**8.1. Floquet-Bloch expansions.** The Euler's theorem [95], mentioned in Section 1.1 claims that all solutions of homogeneous constant coefficient linear ODEs (or systems of those) are linear combinations of exponential-polynomial solutions. This classical theorem has an extremely non-trivial generalization to the case of constant coefficient PDEs (sometimes called **Erenpreis' Fundamental Principle**), see the books [87, 225, 226] devoted to its proof and applications. Here instead of finite linear combinations of exponential-polynomial solutions, one needs to involve the integrals over the characteristic variety of the operator, which in turn relies upon commutative algebra, algebraic geometry, and several complex variables techniques. One can ask the natural question whether there is an analog of this result for

periodic elliptic PDEs, providing expansions into Bloch-Floquet solutions. If yes, one expects it to be much harder to prove, due to the non-algebraic nature of the situation (i.e., of the Fermi surface, which replaces the characteristic variety). Nevertheless, somewhat more restrictive than in the constant coefficient case, such results have been obtained in [168, 171, 175, 187, 227]. Most of the book [171] is devoted to their proofs. Regretfully, probably due to a more restricted nature of these results, they do not seem to have much of consequence. Thus, we do not address them in any detail here.

## 8.2. Generalized eigenfunctions and Shnol'-Bloch theorems.

One of the consequences of the Floquet theory is that for periodic elliptic operators  $H$  detecting whether  $\lambda$  is in the spectrum is equivalent to existence of a non-trivial Bloch solution  $u$  with a real quasi-momentum  $k$  of the equation  $Hu = \lambda u$ . Since the solution is not square integrable, we would call it a **generalized eigenfunction**. Since one can easily come up with a generalized eigenfunction (just an exponent) of Laplace operator for arbitrary  $\lambda \in \mathbb{C}$ , it is clear that existence of a growing generalized eigenfunction does not mean being on the spectrum. However, Bloch solutions with real quasimomenta are bounded, one can ask whether presence of a bounded generalized eigenfunction detects the spectrum. This is indeed the case and the following **Bloch theorem** (probably never proven by Bloch) holds:

**Theorem 8.1.** *[171, and references therein] Existence of a non-trivial bounded solution of a periodic elliptic equation  $Hu = \lambda u$  implies existence of a Bloch solution with a real quasi-momentum, and thus  $\lambda \in \sigma(H)$ .*

Indeed, a stronger type of result is known, the so called **Shnol' Theorem** [58, 126, 171, 251, 255], which holds also in non-periodic case. Its simplest version is:

**Theorem 8.2.** *If for any  $\epsilon > 0$  there exists a non-trivial generalized eigenfunction  $u_\epsilon$  with the estimate*

$$(8.1) \quad |u_\epsilon(x)| \leq e^{\epsilon|x|},$$

*then  $\lambda \in \sigma(H)$ .*

Moreover, the following stronger version holds:

**Theorem 8.3.** *If for some  $a > 0$  there exists a non-trivial generalized eigenfunction  $u$  with the estimate*

$$(8.2) \quad |u_\epsilon(x)| \leq e^{a|x|},$$

then  $\text{dist}(\lambda, \sigma(H)) \leq C\sqrt{e^{2a} - 1}$ , where  $C$  depends only on the operator  $H$ .

The detailed (although somewhat outdated) discussion of the Shnol' type can be found in [126, Section 54].

In the periodic elliptic case one can formulate a stronger version [171, Theorem 4.3.1]:

**Theorem 8.4.** *Let  $L(x, D)$  be a scalar periodic elliptic operator with smooth coefficients in  $\mathbb{R}^n$ . If  $Lu = 0$  has a non-zero solution satisfying the inequality*

$$(8.3) \quad |f(x)| \leq Ce^a \sum |x_j|$$

for some  $a > 0$ , then it also has a Bloch solution with a quasimomentum  $k$  such that  $\Im k_j \leq a$  for all  $j = 1, \dots, n$ , that is, a Bloch solution that also satisfies (8.3).

In other words, there is a complex quasi-momentum vector, whose distance to the unit torus (and thus  $\text{dist}(\lambda, \sigma(H))$ ) can be estimated from above.

*Remark 8.5.* The Shnol' theorem stated with the point-wise estimates like in (8.3) fails in non-euclidean situation. E.g., the hyperbolic Laplacian has a bounded generalized eigenfunction for  $\lambda = 0$ , although 0 is not in the spectrum. Analogous situation occurs for operator on trees. However, analysis of the proof (e.g., in [126]) shows that in fact only an integrated ( $L_2$ ) estimate of growth is used, which implicitly incorporates the rate of the ball volume growth. Thus, there is an  $L_2$ -version that holds even when the volume of the ball grows exponentially (see [24, Section 3.2] for the graph case).

**8.3. Positive solutions.** As it was shown in [3, 4], positive solutions of a periodic elliptic second order equation in  $\mathbb{R}^n$  (or on a co-compact abelian covering) allow integral expansions into positive Bloch solutions (see also [171, Section 4.6] for the description of this result, as well as [194] for the case of nilpotent co-compact covering).

**8.4. Inhomogeneous equations.** It is well known in the theory of periodic ODEs (e.g., [12, 287]) that unique solvability of inhomogeneous equations in  $L_2$  or in the space of bounded functions is equivalent to absence of Floquet multipliers of absolute value one (equivalently, absence of Bloch solutions with real quasimomenta). A simple PDE analog of this result (as well as solvability in the spaces of exponentially decaying functions) also holds [171, Section 4.2].

## 9. MISCELLANY

**9.1. Parabolic time-periodic equations.** Some fluid dynamics problems of stability of periodic flows [290, 291] lead naturally to the task of developing an analog of Floquet theory (e.g., completeness of and expansion into Floquet solutions) for parabolic time periodic problems in Banach and Hilbert spaces:

$$(9.1) \quad \frac{dx}{dt} = A(t)x, x(0) = x_0, A(t+1) = A(t).$$

The simplest, albeit already important and non-trivial example is the heat equation

$$(9.2) \quad \frac{du}{dt} = \Delta_x u + b(x, t)u$$

with time periodic function  $b(x, t)$  in an infinite cylinder, with Dirichlet or Neumann boundary conditions. Embarrassingly enough, there is no general result guaranteeing existence of at least one Floquet solution of (9.2), even less completeness of such solution. Forget the Lyapunov theorem!

Only in one spatial dimension an impressive result of this kind for (9.2) is achieved in beautiful works by Chow, Lu, and Mallet-Paret [41, 42], where inverse scattering method is used. Nothing comparable is available in higher dimensions, unless severe restrictions are imposed on the periodic term [171, Ch. 5 and references therein]. What seems to be the problem? After all, these equations are hypoelliptic, and the general Floquet theory techniques to a large extent applies to such equations [171, Ch. 3]! The thing is that the general operator theory approach to parabolic periodic equations does not work nearly as nicely as it does for the elliptic case. Namely, as it was discovered by Miloslavskii there are extremely nice abstract periodic equations (9.1) with constant highest order terms, which have no Floquet solutions at all (see [206–213] as well as [171, Ch. 5] for the results and discussion). The positive results known ([206–213] and [171, Ch. 5 and references therein]) require very strong restrictions on periodic terms, such as  $b(x, t)$  in (9.2). The author hopes that the “pathological” non-existence of Floquet solutions does not hold for (9.2) (all known counterexamples are abstract parabolic equations). The similar difficulty is also encountered in the Floquet theory for evolution equations (9.1) with a bounded operator coefficient  $A(t)$  (see [60])

Time periodic hyperbolic and Schrödinger type equations have also been attracting a lot of attention in physics (e.g., [43, 118, 293]). However, most of the analytic theory described before fails here and other

techniques are required and are being developed. See, e.g. the survey by Yajima [286] and references therein, as well as, e.g. [144–147, 270, 271, 284, 285].

**9.2. Photonic crystals.** Photonic crystals are artificial periodic optical media that are optical analogs of semiconductors, which bring about major technological advances (see [80, 149, 172] for a nice physics introduction and bibliography, as well as [79, 172] for mathematics surveys). The main equation to study here is Maxwell operator with periodic coefficients. Floquet theory for this operator works in many regards in parallel with the elliptic equations considered in this article, with some notable analytic and numerical quirks (see, e.g. [100–102, 172, 214] for details and references).

**9.3. Waveguides.** Periodic waveguides form one of the important applications of the Floquet theory. Here one considers boundary value problems in periodically shaped domains, where the boundary conditions and the governing equations are also periodic. In some applications, such as quantum waveguides or photonic crystal waveguides, one has to deal also with problems on zero-width surface or curve systems, or where the waves rather being confined to the interior of a waveguide, leak (being evanescent) into the surrounding space. Many spectral problems here (e.g., absolute continuity of the spectrum) happen to be much harder in the waveguide situations and are not completely understood, in spite of significant progress achieved [36, 72–74, 116, 143, 171, 253, 254, 262, 267–269].

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## REFERENCES

- [1] M. J. Ablowitz and P. A. Clarkson, *Nonlinear evolution equations and inverse scattering*, Cambridge University Press, Cambridge, UK, 2001.
- [2] Toshiaki Adachi and Toshikazu Sunada, *Density of states in spectral geometry*, Comment. Math. Helv. **68** (1993), no. 3, 480–493. MR MR1236765 (94k:58149)

- [3] Shmuel Agmon, *On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds*, Methods of functional analysis and theory of elliptic equations (Naples, 1982), Liguori, Naples, 1983, pp. 19–52. MR 819005 (87b:58087)
- [4] ———, *On positive solutions of elliptic equations with periodic coefficients in  $\mathbf{R}^n$ , spectral results and extensions to elliptic operators on Riemannian manifolds*, Differential equations (Birmingham, Ala., 1983), North-Holland Math. Stud., vol. 92, North-Holland, Amsterdam, 1984, pp. 7–17. MR 799327 (87a:35060)
- [5] ———, *Lectures on elliptic boundary value problems*, AMS Chelsea Publishing, Providence, RI, 2010. MR 2589244 (2011c:35004)
- [6] Grégoire Allaire and Carlos Conca, *Bloch wave homogenization and spectral asymptotic analysis*, J. Math. Pures Appl. (9) **77** (1998), no. 2, 153–208. MR 1614641 (99d:35017)
- [7] ———, *Bloch wave homogenization and spectral asymptotic analysis*, J. Math. Pures Appl. (9) **77** (1998), no. 2, 153–208. MR 1614641 (99d:35017)
- [8] Grégoire Allaire, Carlos Conca, and Muthusamy Vanninathan, *The Bloch transform and applications*, Actes du 29ème Congrès d'Analyse Numérique: CANum'97 (Larnas, 1997), ESAIM Proc., vol. 3, Soc. Math. Appl. Indust., Paris, 1998, pp. 65–84 (electronic). MR 1642454 (2000a:35006)
- [9] V. I. Arnol'd, *Ordinary differential equations*, MIT Press, Cambridge, Mass.-London, 1978, Translated from the Russian and edited by Richard A. Silverman. MR 0508209 (58 #22707)
- [10] ———, *Geometrical methods in the theory of ordinary differential equations*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 250, Springer-Verlag, New York, 1988, Translated from the Russian by Joseph Szücs [József M. Szücs]. MR 947141 (89h:58049)
- [11] ———, *On the teaching of mathematics*, Uspekhi Mat. Nauk **53** (1998), no. 1(319), 229–234. MR 1618209 (99k:00011)
- [12] F. M. Arscott, *Periodic differential equations. An introduction to Mathieu, Lamé, and allied functions*, International Series of Monographs in Pure and Applied Mathematics, Vol. 66. A Pergamon Press Book, The Macmillan Co., New York, 1964. MR 0173798 (30 #4006)
- [13] N. W. Ashcroft and N. D. Mermin, *Solid state physics*, Holt, Rinehart and Winston, New York-London, 1976.
- [14] D. Auckly and P. Kuchment, *On parseval frames of wannier functions*, in preparation (2015).
- [15] Marco Avellaneda and Fang-Hua Lin, *Un théorème de Liouville pour des équations elliptiques à coefficients périodiques*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 5, 245–250. MR MR1010728 (90j:35072)
- [16] J. E. Avron and B. Simon, *Analytic properties of band functions*, Ann. Phys. **110** (1978), no. 1, 85–101. MR MR0475384 (57 #14992)
- [17] M. Babillot, *Théorie du renouvellement pour des chaînes semi-markoviennes transientes*, Ann. Inst. H. Poincaré Probab. Statist. **24** (1988), no. 4, 507–569. MR MR978023 (90h:60082)
- [18] Martine Babillot, *Asymptotics of Green functions on a class of solvable Lie groups*, Potential Anal. **8** (1998), no. 1, 69–100. MR MR1608646 (99g:60133)

- [19] J. M. Barbaroux, J. M. Combes, and P. D. Hislop, *Localization near band edges for random Schrödinger operators*, *Helv. Phys. Acta* **70** (1997), no. 1–2, 16–43, Papers honoring the 60th birthday of Klaus Hepp and of Walter Hunziker, Part II (Zürich, 1995). MR MR1441595 (98h:82028)
- [20] G. Barbatis and L. Parnowski, *Bethe-Sommerfeld conjecture for pseudodifferential perturbation*, *Comm. Partial Differential Equations* **34** (2009), no. 4–6, 383–418. MR 2530702 (2010d:35253)
- [21] Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou, *Asymptotic analysis for periodic structures*, *Studies in Mathematics and its Applications*, vol. 5, North-Holland Publishing Co., Amsterdam, 1978. MR MR503330 (82h:35001)
- [22] Francois Bentosela, Pierre Duclos, and Pavel Exner, *Absolute continuity in periodic thin tubes and strongly coupled leaky wires*, *Lett. Math. Phys.* **65** (2003), no. 1, 75–82. MR MR2019393 (2004k:81119)
- [23] G. Berkolaiko and A. Comech, *Symmetry and dirac points in graphene spectrum*, 2014.
- [24] Gregory Berkolaiko and Peter Kuchment, *Introduction to quantum graphs*, *Mathematical Surveys and Monographs*, vol. 186, American Mathematical Society, Providence, RI, 2013. MR 3013208
- [25] B. A. Bernevig, *Topological insulators and topological superconductors*, Princeton UNiv. Press.
- [26] Hans Bethe and Arnold Sommerfeld, *Elektronentheorie der metalle*, *Handbuch der Physik* (H. Geiger and K. Scheel, eds.), vol. 24,2, Springer, 1933, This nearly 300-page chapter was later published as a separate book: *Elektronentheorie der Metalle* (Springer, 1967), pp. 333–622.
- [27] Edward Bierstone and Pierre D. Milman, *Semianalytic and subanalytic sets*, *Inst. Hautes Études Sci. Publ. Math.* (1988), no. 67, 5–42. MR 972342 (89k:32011)
- [28] ———, *A simple constructive proof of canonical resolution of singularities*, *Effective methods in algebraic geometry* (Castiglione, 1990), *Progr. Math.*, vol. 94, Birkhäuser Boston, Boston, MA, 1991, pp. 11–30. MR 1106412 (92h:32053)
- [29] M. Sh. Birman, *On the homogenization for periodic operators in a neighborhood of an edge of an internal gap*, *Algebra i Analiz* **15** (2003), no. 4, 61–71. MR MR2068979 (2006i:35010)
- [30] M. Sh. Birman and T. A. Suslina, *A periodic magnetic Hamiltonian with a variable metric. The problem of absolute continuity*, *Algebra i Analiz*; translation in *St. Petersburg Math. J.* **11** (2000), no. 2, 203232 **11** (1999), no. 2, 1–40. MR MR1702587 (2000i:35026)
- [31] ———, *Periodic second-order differential operators. Threshold properties and averaging*, *Algebra i Analiz* **15** (2003), no. 5, 1–108. MR MR2068790 (2005k:47097)
- [32] ———, *Homogenization of a multidimensional periodic elliptic operator in a neighborhood of an edge of an inner gap*, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **318** (2004), no. Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. **36** [35], 60–74, 309. MR MR2120232 (2005j:35006)

- [33] M. Sh. Birman, T. A. Suslina, and R. G. Shterenberg, *Absolute continuity of the two-dimensional Schrödinger operator with delta potential concentrated on a periodic system of curves*, Algebra i Analiz **12** (2000), no. 6, 140–177. MR MR1816514 (2002k:35227)
- [34] Michael Birman and Tatyana Suslina, *Threshold effects near the lower edge of the spectrum for periodic differential operators of mathematical physics*, Systems, approximation, singular integral operators, and related topics (Bordeaux, 2000), Oper. Theory Adv. Appl., vol. 129, Birkhäuser, Basel, 2001, pp. 71–107. MR MR1882692 (2003f:35220)
- [35] G. Borg, *Eine umkehrung der sturm-liouwillischen eigenwertaufgabe*, Acta Math. **78** (1946), 1–96.
- [36] B. M. Brown, V. Hoang, M. Plum, and I. G. Wood, *Floquet-Bloch theory for elliptic problems with discontinuous coefficients*, Spectral theory and analysis, Oper. Theory Adv. Appl., vol. 214, Birkhäuser/Springer Basel AG, Basel, 2011, pp. 1–20. MR 2808460 (2012i:35067)
- [37] B. Malcolm Brown, Michael S. P. Eastham, and Karl Michael Schmidt, *Periodic differential operators*, Operator Theory: Advances and Applications, vol. 230, Birkhäuser/Springer Basel AG, Basel, 2013. MR 2978285
- [38] K. Busch, G. von Freymann, S. Linden, S. F. Mingaleev, L. Tkeshelashvili, and M. Wegener, *Periodic nanostructures for photonics*, Physics Reports **444** (2007), no. 3–6, 101–202.
- [39] Kurt Busch, Sergei F Mingaleev, Antonio Garcia-Martin, Matthias Schillinger, and Daniel Hermann, *The wannier function approach to photonic crystal circuits*, Journal of Physics: Condensed Matter **15** (2003), no. 30, R1233.
- [40] Joseph Callaway, *Energy band theory*, Pure and Applied Physics, vol. 16, Academic Press, New York, 1964. MR MR0162578 (28 #5776)
- [41] Shui-Nee Chow, Kening Lu, and John Mallet-Paret, *Floquet theory for parabolic differential equations*, J. Differential Equations **109** (1994), no. 1, 147–200. MR 1272403 (95c:35116)
- [42] ———, *Floquet bundles for scalar parabolic equations*, Arch. Rational Mech. Anal. **129** (1995), no. 3, 245–304. MR 1328478 (96c:35070)
- [43] Shih-I Chu, *Nonperturbative approaches to atomic and molecular multiphoton (half-collision) processes in intense laser fields*, Multiparticle quantum scattering with applications to nuclear, atomic and molecular physics (Minneapolis, MN, 1995), IMA Vol. Math. Appl., vol. 89, Springer, New York, 1997, pp. 343–387. MR 1487928 (98i:81291)
- [44] Fan R. K. Chung, *Spectral graph theory*, CBMS Regional Conference Series in Mathematics, vol. 92, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1997. MR MR1421568 (97k:58183)
- [45] Earl A. Coddington and Robert Carlson, *Linear ordinary differential equations*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997. MR 1450591 (99j:34001)
- [46] Earl A. Coddington and Norman Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955. MR 0069338 (16,1022b)

- [47] Tobias H. Colding and William P. Minicozzi, II, *Harmonic functions on manifolds*, Ann. of Math. (2) **146** (1997), no. 3, 725–747. MR MR1491451 (98m:53052)
- [48] Yves Colin de Verdière, *Sur les singularités de van Hove génériques*, Mém. Soc. Math. France (N.S.) (1991), no. 46, 99–110, Analyse globale et physique mathématique (Lyon, 1989). MR MR1125838 (92h:35160)
- [49] J. M. Combes, *Asymptotic behaviour of eigenfunctions for multiparticle schrödinger operators*, Comm. Math. Phys. **34**, 251–270.
- [50] C. Conca, R. Orive, and M. Vanninathan, *Bloch approximation in homogenization on bounded domains*, Asymptot. Anal. **41** (2005), no. 1, 71–91. MR 2124894 (2006g:35011)
- [51] C. Conca, J. Planchard, and M. Vanninathan, *Fluids and periodic structures*, RAM: Research in Applied Mathematics, vol. 38, John Wiley & Sons, Ltd., Chichester; Masson, Paris, 1995. MR 1652238 (99k:73094)
- [52] Carlos Conca, Rafael Orive, and Muthusamy Vanninathan, *Bloch approximation in homogenization and applications*, SIAM J. Math. Anal. **33** (2002), no. 5, 1166–1198 (electronic). MR 1897707 (2003d:35015)
- [53] Carlos Conca and M. Vanninathan, *Fourier approach to homogenization problems*, ESAIM Control Optim. Calc. Var. **8** (2002), 489–511, A tribute to J. L. Lions. MR 1932961 (2003j:35017)
- [54] Carlos Conca and Muthusamy Vanninathan, *Homogenization of periodic structures via Bloch decomposition*, SIAM J. Appl. Math. **57** (1997), no. 6, 1639–1659. MR 1484944 (98j:35017)
- [55] Lawrence J. Corwin and Frederick P. Greenleaf, *Representations of nilpotent Lie groups and their applications. Part I*, Cambridge Studies in Advanced Mathematics, vol. 18, Cambridge University Press, Cambridge, 1990, Basic theory and examples. MR 1070979 (92b:22007)
- [56] A. P. Cracknell, *The Fermi surfaces of metals*, Taylor & Francis, London, 1971.
- [57] A. P. Cracknell and K. S. Wong, *The Fermi surface*, Clarendon Press, Oxford, 1973.
- [58] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, study ed., Texts and Monographs in Physics, Springer-Verlag, Berlin, 1987. MR MR883643 (88g:35003)
- [59] B. E. J. Dahlberg and E. Trubowitz, *A remark on two-dimensional periodic potentials*, Comment. Math. Helv. **57** (1982), no. 1, 130–134. MR 672849 (84h:35119)
- [60] Daletskiĭ, Yu. L. and Kreĭn, M. G., *Ustoichivost reshenii differentsialnykh uravnenii v banakhovom prostranstve*, Izdat. “Nauka”, Moscow, 1970, Non-linear Analysis and its Applications Series. MR 0352638 (50 #5125)
- [61] L. I. Danilov, *On the spectrum of the Dirac operator in  $\mathbf{R}^n$  with periodic potential*, Teoret. Mat. Fiz. **85** (1990), no. 1, 41–53. MR 1083951 (92a:35119)
- [62] ———, *Resolvent estimates and the spectrum of the Dirac operator with a periodic potential*, Teoret. Mat. Fiz. **103** (1995), no. 1, 3–22. MR 1470934 (98f:35112)
- [63] ———, *On the spectrum of the two-dimensional periodic Dirac operator*, Teoret. Mat. Fiz. **118** (1999), no. 1, 3–14. MR 1702856 (2000h:35117)

- [64] ———, *Absolute continuity of the spectrum of a periodic Dirac operator*, *Differ. Uravn.* **36** (2000), no. 2, 233–240, 287. MR 1773794 (2001f:47082)
- [65] ———, *On the spectrum of the periodic Dirac operator*, *Teoret. Mat. Fiz.* **124** (2000), no. 1, 3–17. MR 1821309 (2002b:81028)
- [66] ———, *On the absolute continuity of the spectrum of a periodic Schrödinger operator*, *Mat. Zametki* **73** (2003), no. 1, 49–62. MR 1993539 (2004f:35130)
- [67] ———, *On the spectrum of the two-dimensional periodic Schrödinger operator*, *Teoret. Mat. Fiz.* **134** (2003), no. 3, 447–459. MR 2001818 (2004j:35210)
- [68] ———, *On the nonexistence of eigenvalues in the spectrum of a generalized two-dimensional periodic Dirac operator*, *Algebra i Analiz* **17** (2005), no. 3, 47–80. MR 2167843 (2006m:35261)
- [69] ———, *On absolute continuity of the spectrum of a periodic magnetic Schrödinger operator*, *J. Phys. A* **42** (2009), no. 27, 275204, 20. MR 2512122 (2010f:81090)
- [70] ———, *On absolute continuity of the spectrum of three- and four-dimensional periodic Schrödinger operators*, *J. Phys. A* **43** (2010), no. 21, 215201, 13. MR 2644606 (2011i:81056)
- [71] Leonid I. Danilov, *On absolute continuity of the spectrum of a 3D periodic magnetic Dirac operator*, *Integral Equations Operator Theory* **71** (2011), no. 4, 535–556. MR 2854864
- [72] V. I. Derguzov, *A mathematical investigation of periodic cylindrical wave guides. I, II*, *Vestnik Leningrad. Univ.* (1972), no. 13 *Mat. Meh. Astronom. Vyp.* 3, 32–40, 149; *ibid.* No. 19 *Mat. Meh. Astronom. Vyp.* 3 (1972), 14–20, 145. MR MR0340856 (49 #5606)
- [73] ———, *Linear equations with periodic coefficients and their applications to wave guide systems*, Ph.D. thesis, Leningrad, 1975, (In Russian).
- [74] ———, *On the discrete spectrum of a periodic boundary value problem connected with the study of periodic waveguides*, *Sibirsk. Mat. Zh.* **21** (1980), no. 5, 27–38, 189. MR MR592214 (82g:78023)
- [75] Jacques Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann)*, *Cahiers scientifiques, Fascicule XXV*, Gauthier-Villars, Paris, 1957. MR 0094722 (20 #1234)
- [76] Ngoc T. Do, *On the quantum graph spectra of graphyne nanotubes*, *Analysis and Mathematical Physics* (2014) **5** (2014), no. 1, 39–65.
- [77] Ngoc T. Do and P. Kuchment, *Quantum graph spectra of a graphyne structure*, *Nanoscale Systems: Mathematical Modeling, Theory and Applications* **2** (2013), no. 1, 107–123.
- [78] Ngoc T. Do, P. Kuchment, and F. Sottile, *On the generic structure of spectral edges for periodic difference operators*.
- [79] W. Dörfler, A. Lechleiter, M. Plum, G. Schneider, and C. Wieners, *Photonic crystals: Mathematical analysis and numerical approximation*, *Oberwolfach SEminars*, vol. 42.
- [80] Jonathan P. Dowling, *Photonic and acoustic band-gap bibliography*, <http://phys.lsu.edu/jdowling/pbgbib.html>.
- [81] B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, *Integrable systems. I* [*MR0842910 (87k:58112)*], *Dynamical systems, IV*, *Encyclopaedia Math. Sci.*, vol. 4, Springer, Berlin, 2001, pp. 177–332. MR 1866633

- [82] ———, *Topological and algebraic geometry methods in contemporary mathematical physics*, Classic Reviews in Mathematics and Mathematical Physics, vol. 2, Cambridge Scientific Publishers, Cambridge, 2004. MR 2187099 (2006f:37002)
- [83] Nelson Dunford and Jacob T. Schwartz, *Linear operators. Part II*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1988, Spectral theory. Selfadjoint operators in Hilbert space, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1963 original, A Wiley-Interscience Publication. MR 1009163 (90g:47001b)
- [84] P. Dzhakov and B. S. Mityagin, *Instability zones of one-dimensional periodic Schrödinger and Dirac operators*, Uspekhi Mat. Nauk **61** (2006), no. 4(370), 77–182. MR 2279044 (2007i:47054)
- [85] M. S. P. Eastham, *The spectral theory of periodic differential equations*, Scottish Academic Press Ltd., London, 1973.
- [86] Michael S. P. Eastham and Hubert Kalf, *Schrödinger-type operators with continuous spectra*, Research Notes in Mathematics, vol. 65, Pitman (Advanced Publishing Program), Boston, Mass., 1982. MR MR667015 (84i:35107)
- [87] Leon Ehrenpreis, *Fourier analysis in several complex variables*, Pure and Applied Mathematics, Vol. XVII, Wiley-Interscience Publishers A Division of John Wiley & Sons, New York-London-Sydney, 1970. MR 0285849 (44 #3066)
- [88] Daniel M. Elton, *The Bethe-Sommerfeld conjecture for the 3-dimensional periodic Landau operator*, Rev. Math. Phys. **16** (2004), no. 10, 1259–1290. MR 2114599 (2005k:81070)
- [89] Ivanovskii A. Enyanshin, A., *Graphene allotropes: stability, structural and electronic properties from df-tb calculations*, Phys. Status Solidi (B).
- [90] G. Eskin, *Inverse spectral problem for the Schrödinger equation with periodic vector potential*, Comm. Math. Phys. **125** (1989), no. 2, 263–300. MR 1016872 (91a:35052)
- [91] ———, *Inverse spectral problem for the Schrödinger operator with periodic magnetic and electric potentials*, Séminaire sur les Équations aux Dérivées Partielles, 1988–1989, École Polytech., Palaiseau, 1989, pp. Exp. No. XIII, 6. MR MR1032289 (91a:58193)
- [92] G. Eskin, J. Ralston, and E. Trubowitz, *The multidimensional inverse spectral problem with a periodic potential*, Microlocal analysis (Boulder, Colo., 1983), Contemp. Math., vol. 27, Amer. Math. Soc., Providence, RI, 1984, pp. 45–56. MR 741038 (85k:58079)
- [93] Gregory Eskin, James Ralston, and Eugene Trubowitz, *Isospectral periodic potentials on  $\mathbf{R}^n$* , Inverse problems (New York, 1983), SIAM-AMS Proc., vol. 14, Amer. Math. Soc., Providence, RI, 1984, pp. 91–96. MR MR773705
- [94] ———, *On isospectral periodic potentials in  $\mathbf{R}^n$ . II*, Comm. Pure Appl. Math. **37** (1984), no. 6, 715–753. MR 762871 (86e:35109b)
- [95] Leonard Euler, *De integratione aequationum differentialium altiorum gradum*, Miscellanea berol. **7** (1743), 193–242.
- [96] Pavel Exner and Rupert L. Frank, *Absolute continuity of the spectrum for periodically modulated leaky wires in  $\mathbb{R}^3$* , Ann. Henri Poincaré **8** (2007), no. 2, 241–263. MR 2314447 (2008a:81063)

- [97] Pavel Exner, Peter Kuchment, and Brian Winn, *On the location of spectral edges in  $\mathbb{Z}$ -periodic media*, J. Phys. A **43** (2010), no. 47, 474022, 8.
- [98] Charles L. Fefferman and Michael I. Weinstein, *Honeycomb lattice potentials and Dirac points*, J. Amer. Math. Soc. **25** (2012), no. 4, 1169–1220. MR 2947949
- [99] The Fermi Surface Database, <http://www.phys.ufl.edu/fermisurface/>.
- [100] A. Figotin and P. Kuchment, *Band-gap structure of spectra of periodic dielectric and acoustic media. II. Two-dimensional photonic crystals*, SIAM J. Appl. Math. **56** (1996), no. 6, 1561–1620. MR MR1417473 (98f:35012)
- [101] ———, *Spectral properties of classical waves in high-contrast periodic media*, SIAM J. Appl. Math. **58** (1998), no. 2, 683–702 (electronic). MR MR1617610 (99f:47061)
- [102] Alex Figotin and Peter Kuchment, *Band-gap structure of spectra of periodic dielectric and acoustic media. I. Scalar model*, SIAM J. Appl. Math. **56** (1996), no. 1, 68–88. MR MR1372891 (97g:35122)
- [103] N. Filonov, *Second-order elliptic equation of divergence form having a compactly supported solution*, J. Math. Sci. (New York) **106** (2001), no. 3, 3078–3086, Function theory and phase transitions. MR 1906035 (2003h:35047)
- [104] ———, *Gaps in the spectrum of the Maxwell operator with periodic coefficients*, Comm. Math. Phys. **240** (2003), no. 1-2, 161–170. MR 2004984 (2004i:35293)
- [105] ———, *Personal communication*, Unpublished (2015).
- [106] N. Filonov and I. Kachkovskii, *Personal communication*, Unpublished (2015).
- [107] N. Filonov and F. Klopp, *Absolute continuity of the spectrum of a Schrödinger operator with a potential which is periodic in some directions and decays in others*, Doc. Math. **9** (2004), 107–121. MR 2054982 (2005i:35045a)
- [108] ———, *Erratum to: “Absolute continuity of the spectrum of a Schrödinger operator with a potential which is periodic in some directions and decays in others” [Doc. Math. 9 (2004), 107–121]*, Doc. Math. **9** (2004), 135–136. MR MR2054984 (2005i:35045b)
- [109] ———, *Absolutely continuous spectrum for the isotropic Maxwell operator and coefficients that are periodic in some directions and decay in others*, Comm. Math. Phys. **258** (2005), no. 1, 75–85. MR 2166840 (2006f:35197)
- [110] N. Filonov, L. Parnovskii, and R. Shterenberg, *Personal communication*, Unpublished (2015).
- [111] N. Filonov and A. V. Sobolev, *Absence of the singular continuous component in the spectrum of analytic direct integrals*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **318** (2004), no. Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 36 [35], 298–307, 313. MR 2120804 (2005m:47040)
- [112] G. Floquet, *Sur les équations différentielles linéaires à coefficients périodiques*, Ann. Ecole Norm. **12** (1883), no. 2, 47–89.
- [113] Gerald B. Folland, *A course in abstract harmonic analysis*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR 1397028 (98c:43001)
- [114] L. Friedlander, *On the spectrum of a class of second order periodic elliptic differential operators*, Comm. Math. Phys. **229** (2002), no. 1, 49–55. MR MR1917673 (2003k:35179)

- [115] Leonid Friedlander, *On the density of states of periodic media in the large coupling limit*, Comm. Partial Differential Equations **27** (2002), no. 1–2, 355–380. MR MR1886963 (2003d:35194)
- [116] ———, *Absolute continuity of the spectra of periodic waveguides*, Waves in periodic and random media (South Hadley, MA, 2002), Contemp. Math., vol. 339, Amer. Math. Soc., Providence, RI, 2003, pp. 37–42. MR MR2042530 (2005a:35211)
- [117] R. Froese, I. Herbst, M. Hoffmann-Ostenhof, and T. Hoffmann-Ostenhof,  *$L^2$ -lower bounds to solutions of one-body Schrödinger equations*, Proc. Roy. Soc. Edinburgh Sect. A **95** (1983), no. 1–2, 25–38. MR MR723095 (86a:35044)
- [118] M. Gavrilá, *Atomic structure and decay in high-frequency fields, in atoms in intense laser fields*, Atoms in Intense Laser Fields, Academic Press, 1992, pp. 435–510.
- [119] I. M. Gelfand, *Expansion into eigenfunctions of an equation with periodic coefficients*, Dokl. Akad. Nauk SSSR (N.S.) **73** (1950), 1117–1120.
- [120] Christian Gérard, *Resonance theory for periodic Schrödinger operators*, Bull. Soc. Math. France **118** (1990), no. 1, 27–54. MR 1077086 (91h:35231)
- [121] ———, *A proof of the abstract limiting absorption principle by energy estimates*, J. Funct. Anal. **254** (2008), no. 11, 2707–2724. MR 2414218 (2009e:47028)
- [122] Christian Gérard and Francis Nier, *The Mourre theory for analytically fibered operators*, J. Funct. Anal. **152** (1998), no. 1, 202–219. MR MR1600082 (99b:47030)
- [123] ———, *Scattering theory for the perturbations of periodic Schrödinger operators*, J. Math. Kyoto Univ. **38** (1998), no. 4, 595–634. MR MR1669979 (2000e:47023)
- [124] D. Gieseke, H. Knörrer, and E. Trubowitz, *An overview of the geometry of algebraic Fermi curves*, Algebraic geometry: Sundance 1988, Contemp. Math., vol. 116, Amer. Math. Soc., Providence, RI, 1991, pp. 19–46. MR MR1108630 (92i:14026)
- [125] ———, *The geometry of algebraic Fermi curves*, Perspectives in Mathematics, vol. 14, Academic Press Inc., Boston, MA, 1993. MR MR1184395 (95b:14019)
- [126] I. M. Glazman, *Direct methods of qualitative spectral analysis of singular differential operators*, Translated from the Russian by the IPST staff, Israel Program for Scientific Translations, Jerusalem, 1965; Daniel Davey & Co., Inc., New York, 1966. MR 0190800 (32 #8210)
- [127] I. C. Gohberg and M. G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators*, Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18, American Mathematical Society, Providence, RI, 1969. MR MR0246142 (39 #7447)
- [128] Carolyn S. Gordon, Pierre Guerini, Thomas Kappeler, and David L. Webb, *Inverse spectral results on even dimensional tori*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 7, 2445–2501. MR 2498357 (2010d:58034)
- [129] Carolyn S. Gordon and Thomas Kappeler, *On isospectral potentials on tori*, Duke Math. J. **63** (1991), no. 1, 217–233. MR 1106944 (92f:58185)
- [130] ———, *On isospectral potentials on flat tori. II*, Comm. Partial Differential Equations **20** (1995), no. 3–4, 709–728. MR 1318086 (96a:58197)

- [131] H. Grauert, *Analytische Faserungen über holomorph-vollständigen Räumen*, Math. Ann.
- [132] Hans Grauert and Reinhold Remmert, *Theory of Stein spaces*, Classics in Mathematics, Springer-Verlag, Berlin, 2004, Translated from the German by Alan Huckleberry, Reprint of the 1979 translation. MR 2029201 (2004k:32015)
- [133] V. V. Grushin, *Application of the multiparameter theory of perturbations of Fredholm operators to Bloch functions*, Mat. Zametki **86** (2009), no. 6, 819–828. MR 2643450 (2011b:47032)
- [134] Victor Guillemin, *Spectral theory on  $S^2$ : some open questions*, Adv. in Math. **42** (1981), no. 3, 283–298. MR 642394 (83d:58070)
- [135] ———, *Inverse spectral results on two-dimensional tori*, Journal of the American Mathematical Society **3** (1990), no. 2, 375–387.
- [136] Robert C. Gunning and Hugo Rossi, *Analytic functions of several complex variables*, AMS Chelsea Publishing, Providence, RI, 2009, Reprint of the 1965 original. MR 2568219 (2010j:32001)
- [137] J. M. Harrison, P. Kuchment, A. Sobolev, and B. Winn, *On occurrence of spectral edges for periodic operators inside the Brillouin zone*, J. Phys. A **40** (2007), no. 27, 7597–7618. MR 2369966 (2008j:81039)
- [138] W. A. Harrison and M. B. Webb (eds.), *The Fermi surface*, John Wiley & Sons, New York, London, 1960.
- [139] J. E. Heebner, R. W. Boyd, and Q. Park, *Scissor solitons & other propagation effects in microresonator modified waveguides*, JOSA B **19** (2002), 722–731.
- [140] Rainer Hempel and Karsten Lienau, *Spectral properties of periodic media in the large coupling limit*, Comm. Partial Differential Equations **25** (2000), no. 7-8, 1445–1470. MR 1765136 (2001h:47074)
- [141] Rainer Hempel and Olaf Post, *Spectral gaps for periodic elliptic operators with high contrast: an overview*, Progress in analysis, Vol. I, II (Berlin, 2001), World Sci. Publ., River Edge, NJ, 2003, pp. 577–587. MR 2032728 (2004k:35281)
- [142] Yusuke Higuchi and Tomoyuki Shirai, *Some spectral and geometric properties for infinite graphs*, Discrete geometric analysis, Contemp. Math., vol. 347, Amer. Math. Soc., Providence, RI, 2004, pp. 29–56.
- [143] Vu Hoang and Maria Radosz, *Absence of bound states for waveguides in two-dimensional periodic structures*, J. Math. Phys. **55** (2014), no. 3, 033506, 20. MR 3221271
- [144] James S. Howland, *Scattering theory for Hamiltonians periodic in time*, Indiana Univ. Math. J. **28** (1979), no. 3, 471–494. MR 529679 (80e:47008)
- [145] ———, *Floquet operators with singular spectrum. I, II*, Ann. Inst. H. Poincaré Phys. Théor. **50** (1989), no. 3, 309–323, 325–334. MR 1017967 (91i:81018)
- [146] ———, *Quantum stability*, Schrödinger operators (Aarhus, 1991), Lecture Notes in Phys., vol. 403, Springer, Berlin, 1992, pp. 100–122. MR 1181243 (94b:81026)
- [147] ———, *Floquet operators with singular spectrum. III*, Ann. Inst. H. Poincaré Phys. Théor. **69** (1998), no. 2, 265–273. MR 1638899 (99i:81037)

- [148] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, 1994, Translated from the Russian by G. A. Yosifian [G. A. Iosif'yan]. MR MR1329546 (96h:35003b)
- [149] J. D. Joannopoulos, S. Johnson, R. D. Meade, and J. N. Winn, *Photonic crystals: Molding the flow of light*, 2nd ed., Princeton University Press, Princeton, N.J., 2008.
- [150] I. Kachkovskii and N. Filonov, *Absolute continuity of the spectrum of a periodic Schrödinger operator in a multidimensional cylinder*, Algebra i Analiz **21** (2009), no. 1, 133–152. MR 2553054 (2010i:35054)
- [151] ———, *Absolute continuity of the spectrum of the periodic Schrödinger operator in a layer and in a smooth cylinder*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **385** (2010), no. Kraevye Zadachi Matematicheskoi Fiziki i Smezhnye Voprosy Teorii Funktsii. 41, 69–81, 235. MR 2749370 (2011k:35032)
- [152] Thomas Kappeler, *On isospectral periodic potentials on a discrete lattice. I*, Duke Math. J. **57** (1988), no. 1, 135–150. MR 952228 (90b:39010a)
- [153] ———, *On isospectral potentials on a discrete lattice. II*, Adv. in Appl. Math. **9** (1988), no. 4, 428–438. MR 968676 (90b:39010b)
- [154] ———, *Isospectral potentials on a discrete lattice. III*, Trans. Amer. Math. Soc. **314** (1989), no. 2, 815–824. MR 961624 (90b:39010c)
- [155] Yulia Karpeshina and Young-Ran Lee, *Absolutely continuous spectrum of a polyharmonic operator with a limit periodic potential in dimension two*, Comm. Partial Differential Equations **33** (2008), no. 7-9, 1711–1728. MR 2450178 (2009g:35046)
- [156] Yulia E. Karpeshina, *Perturbation theory for the Schrödinger operator with a periodic potential*, Lecture Notes in Mathematics, vol. 1663, Springer-Verlag, Berlin, 1997. MR 1472485 (2000i:35002)
- [157] Tosio Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition. MR MR1335452 (96a:47025)
- [158] M. I. Katsnelson, *Graphene. carbon in two dimensions*, Cambridge Univ. Press, 2012.
- [159] Kuchment Peter Kha, Minh and Andrew Raich, *Green's function asymptotics near the internal edges of spectra of periodic elliptic operators: spectral gap interior*, arXiv:1508.06703, submitted.
- [160] Minh Kha, *Green's function asymptotics of periodic elliptic operators on abelian coverings of compact manifolds*, preprint.
- [161] Werner Kirsch and Barry Simon, *Comparison theorems for the gap of Schrödinger operators*, J. Funct. Anal. **75** (1987), no. 2, 396–410. MR MR916759 (89b:35127)
- [162] C. Kittel, *Introduction to solid state physics*, Wiley, New York, 1976.
- [163] Frédéric Klopp, *Absolute continuity of the spectrum of a Landau Hamiltonian perturbed by a generic periodic potential*, Math. Ann. **347** (2010), no. 3, 675–687. MR 2640047 (2011g:47106)
- [164] Frédéric Klopp and James Ralston, *Endpoints of the spectrum of periodic operators are generically simple*, Methods Appl. Anal. **7** (2000), no. 3, 459–463, Cathleen Morawetz: a great mathematician. MR 1869296 (2002i:47055)

- [165] H. Knörrer and E. Trubowitz, *A directional compactification of the complex Bloch variety*, Comment. Math. Helv. **65** (1990), no. 1, 114–149. MR 1036133 (91k:58134)
- [166] W. Kohn, *Analytic properties of Bloch waves and Wannier functions*, Phys. Rev. (2) **115** (1959), 809–821. MR 0108284 (21 #7000)
- [167] P. Kuchment, *On sunada's no gap conjecture*, unpublished.
- [168] P. A. Kuchment, *Floquet theory for partial differential equations*, Uspekhi Mat. Nauk; translation in Russian Math. Surveys, 37:4 (1982), 1–60 **37** (1982), no. 4(226), 3–52, 240. MR MR667973 (84b:35018)
- [169] ———, *On the Floquet theory of periodic difference equations*, Geometrical and algebraical aspects in several complex variables (Cetraro, 1989), Sem. Conf., vol. 8, EditEl, Rende, 1991, pp. 201–209. MR MR1222215 (94k:39001)
- [170] PA Kuchment, *Floquet theory for partial differential equations*, Russian Mathematical Surveys **37** (1982), no. 4, 1–60.
- [171] Peter Kuchment, *Floquet theory for partial differential equations*, Operator Theory: Advances and Applications, vol. 60, Birkhäuser Verlag, Basel, 1993. MR MR1232660 (94h:35002)
- [172] ———, *The mathematics of photonic crystals*, Mathematical modeling in optical science (G. Bao, L. Cowsar, and W. Masters, eds.), Frontiers Appl. Math., vol. 22, SIAM, Philadelphia, PA, 2001, pp. 207–272. MR MR1831334 (2002k:78002)
- [173] ———, *Graph models for waves in thin structures*, Waves Random Media **12** (2002), no. 4, R1–R24. MR 1937279 (2003h:35215)
- [174] ———, *Quantum graphs. II. Some spectral properties of quantum and combinatorial graphs*, J. Phys. A **38** (2005), no. 22, 4887–4900. MR 2148631 (2006a:81035)
- [175] ———, *Integral representations of solutions of periodic elliptic equations*, Probability and mathematical physics, CRM Proc. Lecture Notes, vol. 42, Amer. Math. Soc., Providence, RI, 2007, pp. 323–339. MR 2352277 (2008k:35077)
- [176] ———, *Tight frames of exponentially decaying Wannier functions*, J. Phys. A **42** (2009), no. 2, 025203, 16. MR 2525287 (2011d:81103)
- [177] ———, *Introduction to periodic operators*, video at <http://www.newton.ac.uk/event/pepw01/timetable>, 2015.
- [178] Peter Kuchment and Leonid A. Kunyansky, *Spectral properties of high contrast band-gap materials and operators on graphs*, Experiment. Math. **8** (1999), no. 1, 1–28. MR MR1685034 (2000c:78027)
- [179] Peter Kuchment and Sergei Levendorskii, *On the structure of spectra of periodic elliptic operators*, Trans. Amer. Math. Soc. **354** (2002), no. 2, 537–569. MR 1862558 (2002g:35163)
- [180] Peter Kuchment and Yehuda Pinchover, *Integral representations and Liouville theorems for solutions of periodic elliptic equations*, J. Funct. Anal. **181** (2001), no. 2, 402–446. MR MR1821702 (2001m:35067)
- [181] ———, *Liouville theorems and spectral edge behavior on abelian coverings of compact manifolds*, Trans. Amer. Math. Soc. **359** (2007), no. 12, 5777–5815. MR 2336306 (2008h:58037)
- [182] Peter Kuchment and Olaf Post, *On the spectra of carbon nano-structures*, Comm. Math. Phys. **275** (2007), no. 3, 805–826. MR 2336365 (2008j:81041)

- [183] Peter Kuchment and Andrew Raich, *Green's function asymptotics near the internal edges of spectra of periodic elliptic operators. Spectral edge case*, Math. Nachr. **285** (2012), no. 14-15, 1880–1894. MR 2988010
- [184] Peter Kuchment and Boris Vainberg, *On embedded eigenvalues of perturbed periodic Schrödinger operators*, Spectral and scattering theory (Newark, DE, 1997), Plenum, New York, 1998, pp. 67–75. MR MR1625271 (99e:35032)
- [185] ———, *On absence of embedded eigenvalues for Schrödinger operators with perturbed periodic potentials*, Comm. Partial Differential Equations **25** (2000), no. 9-10, 1809–1826. MR 1778781 (2001i:47074)
- [186] ———, *On the structure of eigenfunctions corresponding to embedded eigenvalues of locally perturbed periodic graph operators*, Comm. Math. Phys. **268** (2006), no. 3, 673–686. MR 2259210 (2007h:39034)
- [187] Peter Kuchment and Leonid Zelenko, *On the floquet representations of exponentially increasing solutions of elliptic equations with periodic coefficients*, Soviet Math. Dokl.
- [188] I. S. Lapin, *One version of the Bethe-Sommerfeld conjecture*, J. Math. Sci. (N. Y.) **117** (2003), no. 3, 4157–4166, Nonlinear problems and function theory. MR 2027452 (2004k:81114)
- [189] Peter D. Lax, *Periodic solutions of the KdV equation*, Comm. Pure Appl. Math. **28** (1975), 141–188. MR 0369963 (51 #6192)
- [190] Minjae Lee, *Conic dispersion surfaces for point scatterers on a honeycomb lattice*, 2014, arXiv:1402.5179.
- [191] Peter Li, *Curvature and function theory on Riemannian manifolds*, Surveys in differential geometry, Surv. Differ. Geom., VII, Int. Press, Somerville, MA, 2000, pp. 375–432. MR MR1919432 (2003g:53047)
- [192] ———, *Geometric analysis*, Cambridge Studies in Advanced Mathematics, vol. 134, Cambridge University Press, Cambridge, 2012. MR 2962229
- [193] V. B. Lidskii, *Eigenfunction expansions for equations with periodic coefficients*, Appendix VIII to the Russian (1961) edition of Titchmarsh' book.
- [194] Vladimir Ya. Lin and Yehuda Pinchover, *Manifolds with group actions and elliptic operators*, Mem. Amer. Math. Soc. **112** (1994), no. 540, vi+78. MR MR1230774 (95d:58119)
- [195] Stanisław Łojasiewicz, *Introduction to complex analytic geometry*, Birkhäuser Verlag, Basel, 1991, Translated from the Polish by Maciej Klimek. MR 1131081 (92g:32002)
- [196] A. M. Lyapunov, *Sur une série relative à la théorie des équations différentielles linéaires à coefficients périodiques.*, Compt. Rend. **123** (1896), no. 26, 1248–1252.
- [197] ———, *Sur une équation transcendante et les équations différentielles linéaires du second ordre à coefficients périodiques.*, Compt. Rend. **128** (1899), no. 18, 1085–1088.
- [198] Wilhelm Magnus and Stanley Winkler, *Hill's equation*, Interscience Tracts in Pure and Applied Mathematics, No. 20, Interscience Publishers John Wiley & Sons New York-London-Sydney, 1966. MR MR0197830 (33 #5991)
- [199] V. A. Marchenko and Ī. V. Ostrovskii, *Approximation of periodic potentials by finite zone potentials*, Vestnik Khar'kov. Gos. Univ. (1980), no. 205, 4–40, 139. MR 643352 (84b:34032)

- [200] V. A. Marchenko and I. V. Ostrovsky, *Corrections to the article: “Approximation of periodic by finite-zone potentials”* [*Selecta Math. Soviet.* **6** (1987), no. 2, 101–136; see MR0910538 (88f:00011)], *Selecta Math. Soviet.* **7** (1988), no. 1, 99–100, Selected translations. MR 967083 (89j:34033)
- [201] N. Marzari, I Souza, I., and D. Vanderbilt, *An introduction to maximally-localized wannier functions*, *Highlight of the Month, Psi-K Newsletter* **57** (2003), no. 4, 129–168.
- [202] N. Marzari and D. Vanderbilt, *Maximally localized generalized wannier functions for composite energy bands*, *Phys. Rev. B* **56** (1997, NUMBER = 20, PAGES = 12847–12865,).
- [203] H. P. McKean and E. Trubowitz, *Hill’s operator and hyperelliptic function theory in the presence of infinitely many branch points*, *Comm. Pure Appl. Math.* **29** (1976), no. 2, 143–226. MR 0427731 (55 #761)
- [204] H. P. McKean and P. van Moerbeke, *The spectrum of Hill’s equation*, *Invent. Math.* **30** (1975), no. 3, 217–274. MR 0397076 (53 #936)
- [205] V. Z. Meshkov, *On the possible rate of decrease at infinity of the solutions of second-order partial differential equations*, *Mat. Sb.* **182** (1991), no. 3, 364–383, (In Russian. English transl. in *Math. USSR-Sbornik*). MR MR1110071 (92d:35032)
- [206] A. I. Miloslavskii, *Floquet solutions, and the reducibility of a certain class of differential equations in a Banach space*, *Izv. Severo-Kavkaz. Naučn. Centra Vysš. Školy Ser. Estestv. Nauk.* (1975), no. 4, 82–87, 118. MR 0470395 (57 #10149)
- [207] ———, *The decay of the solutions of an abstract parabolic equation with a periodic operator coefficient*, *Izv. Severo-Kavkaz. Naučn. Centra Vysš. Školy Ser. Estestv. Nauk.* (1976), no. 2, 12–15, 116. MR 0435541 (55 #8500)
- [208] ———, *Floquet theory for abstract parabolic equations with periodic coefficients*, Ph.D. thesis, Rostov State University. Russia, 1976.
- [209] ———, *On the Floquet theory for parabolic equations*, *Funkcional. Anal. i Priložen.* **10** (1976), no. 2, 80–81. MR 0473390 (57 #13057)
- [210] ———, *On the Floquet theory for parabolic equations*, *Teor. Funktsii Funktsional. Anal. i Prilozhen.* (1979), no. 31, 98–102, 168. MR 548296 (80k:34086)
- [211] ———, *Asymptotic behavior of the solutions of abstract integro-differential equations with periodic coefficients*, *Funktsional. Anal. i Prilozhen.* **22** (1988), no. 3, 77–78. MR 961767 (89j:45007)
- [212] ———, *An abstract integro-differential equation with a periodic coefficient. I*, *Teor. Funktsii Funktsional. Anal. i Prilozhen.* (1990), no. 53, 100–108. MR 1077229 (91k:47126a)
- [213] ———, *An abstract integro-differential equation with a periodic coefficient. II*, *Teor. Funktsii Funktsional. Anal. i Prilozhen.* (1990), no. 54, 57–68. MR 1080726 (91k:47126b)
- [214] Abderemane Morame, *The absolute continuity of the spectrum of Maxwell operator in a periodic media*, *J. Math. Phys.* **41** (2000), no. 10, 7099–7108. MR 1781426 (2001h:82103)
- [215] Sergey Morozov, Leonid Parnovski, and Irina Pchelintseva, *Lower bound on the density of states for periodic Schrödinger operators*, *Operator theory and its applications*, *Amer. Math. Soc. Transl. Ser. 2*, vol. 231, Amer. Math. Soc., Providence, RI, 2010, pp. 161–171. MR 2757529 (2012e:35179)

- [216] Jürgen Moser and Michael Struwe, *On a Liouville-type theorem for linear and nonlinear elliptic differential equations on a torus*, Bol. Soc. Brasil. Mat. (N.S.) **23** (1992), no. 1–2, 1–20. MR MR1203171 (94b:35051)
- [217] Minoru Murata, *Asymptotics of Green functions and Martin boundaries for elliptic operators with periodic coefficients*, J. Differential Equations **195** (2003), no. 1, 82–118. MR 2019244 (2004k:35067)
- [218] ———, *Asymptotics of Green functions and the limiting absorption principle for elliptic operators with periodic coefficients*, J. Math. Kyoto Univ. **46** (2006), no. 4, 713–754. MR 2320348 (2008d:35002)
- [219] G. Nenciu, *Existence of the exponentially localised wannier functions*, Comm. Math. Phys. **91** (1983, NUMBER = 1, PAGES = 81–85,).
- [220] Ole A. Nielsen, *Direct integral theory*, Lecture Notes in Pure and Applied Mathematics, vol. 61, Marcel Dekker, Inc., New York, 1980. MR 591683 (82e:46081)
- [221] S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of solitons*, Contemporary Soviet Mathematics, Consultants Bureau [Plenum], New York, 1984, The inverse scattering method, Translated from the Russian. MR 779467 (86k:35142)
- [222] S. P. Novikov, *Two-dimensional Schrödinger operators in periodic fields*, Current problems in mathematics, Vol. 23; (English translation in Journal of Mathematical Sciences, **28** (1985), No.1 1–20.), Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1983, pp. 3–32. MR 734312 (85i:81020)
- [223] ———, *Solitons and geometry*, Lezioni Fermiane. [Fermi Lectures], Published for the Scuola Normale Superiore, Pisa, 1994. MR 1272686 (95i:58091)
- [224] Beng-Seong Ong, *Spectral problems of optical waveguides and quantum graphs*, Ph.D. thesis, Texas A&M University, 2006.
- [225] V. P. Palamodov, *Lineinye differentsialnye operatory s postoyaannymi koef-fitsientami*, Izdat. “Nauka”, Moscow, 1967. MR 0243193 (39 #4517)
- [226] ———, *Linear differential operators with constant coefficients*, Translated from the Russian by A. A. Brown. Die Grundlehren der mathematischen Wissenschaften, Band 168, Springer-Verlag, New York-Berlin, 1970. MR 0264197 (41 #8793)
- [227] Victor P. Palamodov, *Harmonic synthesis of solutions of elliptic equation with periodic coefficients*, Ann. Inst. Fourier (Grenoble) **43** (1993), no. 3, 751–768. MR 1242614 (95f:35037)
- [228] Gianluca Panati, *Triviality of Bloch and Bloch-Dirac bundles*, Ann. Henri Poincaré **8** (2007), no. 5, 995–1011. MR 2342883 (2008d:81074)
- [229] Gianluca Panati and Adriano Pisante, *Bloch bundles, Marzari-Vanderbilt functional and maximally localized Wannier functions*, Comm. Math. Phys. **322** (2013), no. 3, 835–875. MR 3079333
- [230] Vassilis Papanicolaou, *Private communication*.
- [231] L. Parnowski and N. Sidorova, *Critical dimensions for counting lattice points in Euclidean annuli*, Math. Model. Nat. Phenom. **5** (2010), no. 4, 293–316. MR 2662460 (2011e:11162)
- [232] Leonid Parnowski, *Bethe-Sommerfeld conjecture*, Ann. Henri Poincaré **9** (2008), no. 3, 457–508. MR 2419769 (2009f:81068)

- [233] Leonid Parnovski and Roman Shterenberg, *Asymptotic expansion of the integrated density of states of a two-dimensional periodic Schrödinger operator*, Invent. Math. **176** (2009), no. 2, 275–323. MR 2495765 (2010d:35048)
- [234] Leonid Parnovski and Alexander V. Sobolev, *On the Bethe-Sommerfeld conjecture*, Journées “Équations aux Dérivées Partielles” (La Chapelle sur Erdre, 2000), Univ. Nantes, Nantes, 2000, pp. Exp. No. XVII, 13. MR 1775693 (2002i:35137)
- [235] ———, *Bethe-Sommerfeld conjecture for periodic operators with strong perturbations*, Invent. Math. **181** (2010), no. 3, 467–540. MR 2660451 (2011f:35242)
- [236] B. S. Pavlov, *The theory of extensions, and explicitly solvable models*, Uspekhi Mat. Nauk **42** (1987), no. 6(258), 99–131, 247. MR MR933997 (89b:47009)
- [237] A. Pliš, *Non-uniqueness in Cauchy’s problem for differential equations of elliptic type*, J. Math. Mech. **9** (1960), 557–562. MR MR0121568 (22 #12305)
- [238] Olaf Post, *Periodic manifolds with spectral gaps*, J. Differential Equations **187** (2003), no. 1, 23–45. MR MR1946544 (2003m:58047)
- [239] Michael Reed and Barry Simon, *Methods of modern mathematical physics. I–4. Functional analysis*, Academic Press, New York, 1972. MR MR0493419 (58 #12429a)
- [240] Omer Reingold, Salil Vadhan, and Avi Wigderson, *Entropy waves, the zig-zag graph product, and new constant-degree expanders*, Ann. of Math. (2) **155** (2002), no. 1, 157–187. MR 1888797 (2003c:05145)
- [241] F. S. Rofo-Beketov, *On the spectrum of non-selfadjoint differential operators with periodic coefficients*, Dokl. Akad. Nauk SSSR **152** (1963), 1312–1315. MR MR0157033 (28 #274)
- [242] ———, *A finiteness test for the number of discrete levels which can be introduced into the gaps of the continuous spectrum by perturbations of a periodic potential*, Dokl. Akad. Nauk SSSR **156** (1964), 515–518. MR MR0160967 (28 #4176)
- [243] ———, *A perturbation of a Hill’s operator, that has a first moment and a non-zero integral, introduces a single discrete level into each of the distant spectral lacunae*, Mathematical physics and functional analysis, No. 4 (Russian), Fiz.-Tehn. Inst. Nizkih Temperatur, Akad. Nauk Ukrain. SSR, Kharkov, 1973, pp. 158–159, 163. MR MR0477257 (57 #16798)
- [244] ———, *Spectrum perturbations, the knezer-type constants and the effective mass of zones-type potentials*, Constructive Theory of Functions 1984 (Sofia), 1984, Proceedings of the International Conference on Constructive Function Theory, Varna 1984, pp. 757–766.
- [245] Bruno Scarpellini, *Stability, instability, and direct integrals*, Chapman & Hall/CRC Research Notes in Mathematics, vol. 402, Chapman & Hall/CRC, Boca Raton, FL, 1999. MR MR1682260 (2000j:35023)
- [246] Jeffrey H. Schenker and Michael Aizenman, *The creation of spectral gaps by graph decoration*, Lett. Math. Phys. **53** (2000), no. 3, 253–262. MR MR1808253 (2001k:47008)
- [247] Zhongwei Shen, *Absolute continuity of generalized periodic Schrödinger operators*, Harmonic analysis and boundary value problems (Fayetteville, AR, 2000), Contemp. Math., vol. 277, Amer. Math. Soc., Providence, RI, 2001, pp. 113–126. MR 1840430 (2002j:35078)

- [248] ———, *On absolute continuity of the periodic Schrödinger operators*, Internat. Math. Res. Notices (2001), no. 1, 1–31. MR 1809495 (2002a:47078)
- [249] Zhongwei Shen and Peihao Zhao, *Uniform Sobolev inequalities and absolute continuity of periodic operators*, Trans. Amer. Math. Soc. **360** (2008), no. 4, 1741–1758. MR 2366961 (2010j:35095)
- [250] Stephen P. Shipman, *Eigenfunctions of unbounded support for embedded eigenvalues of locally perturbed periodic graph operators*, Comm. Math. Phys. **332** (2014), no. 2, 605–626. MR 3257657
- [251] È. È. Shnol', *On the behavior of the eigenfunctions of Schrödinger's equation*, Mat. Sb. (N.S.) **42 (84)** (1957), no. 3, 273–286. MR 0125315 (23 #A2618)
- [252] R. G. Shterenberg, *Absolute continuity of a two-dimensional magnetic periodic Schrödinger operator with electric potential of measure derivative type*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **271** (2000), no. Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 31, 276–312, 318. MR MR1810620 (2002m:35171)
- [253] ———, *The Schrödinger operator in a periodic waveguide on a plane and quasiconformal mappings*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **295** (2003), no. Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 33, 204–243, 247. MR MR1983118 (2004e:78030)
- [254] R. G. Shterenberg and T. A. Suslina, *Absolute continuity of the spectrum of the Schrödinger operator with the potential concentrated on a periodic system of hypersurfaces*, Algebra i Analiz **13** (2001), no. 5, 197–240, English translation in St. Petersburg Math. J. 13 (2002), no. 5. MR MR1882869 (2002m:35172)
- [255] M. A. Shubin, *Weak Bloch property and weight estimates for elliptic operators*, Séminaire sur les Équations aux Dérivées Partielles, 1989–1990, École Polytech., Palaiseau, 1990, With an appendix by Shubin and J. Sjöstrand, pp. Exp. No. V, 36. MR MR1073180 (92f:58167)
- [256] Barry Simon, *On the genericity of nonvanishing instability intervals in Hill's equation*, Ann. Inst. H. Poincaré Sect. A (N.S.) **24** (1976), no. 1, 91–93. MR 0473321 (57 #12992)
- [257] ———, *Trace ideals and their applications*, second ed., Mathematical Surveys and Monographs, vol. 120, American Mathematical Society, Providence, RI, 2005. MR 2154153 (2006f:47086)
- [258] M. M. Skriganov, *Geometric and arithmetic methods in the spectral theory of multidimensional periodic operators*, Trudy Mat. Inst. Steklov. **171** (1985), 122. MR MR798454 (87h:47110)
- [259] Alexander V. Sobolev, *Absolute continuity of the periodic magnetic Schrödinger operator*, Invent. Math. **137** (1999), no. 1, 85–112. MR 1703339 (2000g:35028)
- [260] ———, *Recent results on the Bethe-Sommerfeld conjecture*, Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday, Proc. Sympos. Pure Math., vol. 76, Amer. Math. Soc., Providence, RI, 2007, pp. 383–398. MR 2310211 (2009d:35245)
- [261] ———, *The gauge transform method in the theory of periodic operators*, LECTURES AT THE INTERNATIONAL NEWTON INSTITUTE, JANUARY 2015 (2015).

- [262] Alexander V. Sobolev and Jonathan Walthoe, *Absolute continuity in periodic waveguides*, Proc. London Math. Soc. (3) **85** (2002), no. 3, 717–741. MR 1936818 (2003j:35240)
- [263] M. A. Šubin, *Spectral theory and the index of elliptic operators with almost-periodic coefficients*, Uspekhi Mat. Nauk **34** (1979), no. 2(206), 95–135. MR 535710 (81f:35090)
- [264] Toshikazu Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of Math. (2) **121** (1985), no. 1, 169–186. MR MR782558 (86h:58141)
- [265] ———, *A periodic Schrödinger operator on an abelian cover*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **37** (1990), no. 3, 575–583. MR MR1080871 (92e:58222)
- [266] ———, *Discrete geometric analysis*, Analysis on graphs and its applications, Proc. Sympos. Pure Math., vol. 77, Amer. Math. Soc., Providence, RI, 2008, pp. 51–83. MR MR2459864
- [267] T. A. Suslina, *On the absence of eigenvalues of a periodic matrix Schrödinger operator in a layer*, Russ. J. Math. Phys. **8** (2001), no. 4, 463–486. MR MR1932011 (2003k:81057)
- [268] ———, *Absolute continuity of the spectrum of the periodic Maxwell operator in a layer*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **288** (2002), no. Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 32, 232–255, 274.
- [269] T. A. Suslina and R. G. Shterenberg, *Absolute continuity of the spectrum of the magnetic Schrödinger operator with a metric in a two-dimensional periodic waveguide*, Algebra i Analiz **14** (2002), no. 2, 159–206. MR MR1925885 (2003h:35185)
- [270] V. V. Sviridov, *On the completeness of quasienergetic states*, Dokl. Akad. Nauk SSSR **274** (1984), no. 6, 1366–1367. MR 740452 (85i:35040)
- [271] ———, *Some problems of the theory of differential equations with periodic coefficients*, Ph.D. thesis, 1986.
- [272] Lawrence E. Thomas, *Time dependent approach to scattering from impurities in a crystal*, Comm. Math. Phys. **33** (1973), 335–343. MR MR0334766 (48 #13084)
- [273] D. J. Thouless, *Wannier functions for magnetic sub-bands*, J. Phys. C: Solid State Phys.
- [274] M. Tikhomirov and N. Filonov, *Absolute continuity of an “even” periodic Schrödinger operator with nonsmooth coefficients*, Algebra i Analiz **16** (2004), no. 3, 201–210. MR 2083570 (2005f:35056)
- [275] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations. Part I*, Second Edition, Clarendon Press, Oxford, 1962. MR 0176151 (31 #426)
- [276] Pierre van Moerbeke, *About isospectral deformations of discrete Laplacians*, Global analysis (Proc. Biennial Sem. Canad. Math. Congr., Univ. Calgary, Calgary, Alta., 1978), Lecture Notes in Math., vol. 755, Springer, Berlin, 1979, pp. 313–370. MR 564908 (81g:58018)
- [277] O. A. Veliev, *Perturbation theory for the periodic multidimensional Schrödinger operator and the Bethe-Sommerfeld conjecture*, Int. J. Contemp. Math. Sci. **2** (2007), no. 1-4, 19–87. MR 2292470 (2009b:47080)

- [278] Oktay Veliev, *Multidimensional periodic Schrödinger operator*, Springer Tracts in Modern Physics, vol. 263, Springer, Cham, 2015, Perturbation theory and applications. MR 3328527
- [279] J. Von Neumann and E. P. Wigner, *über merkwürdige diskrete eigenwerte*, Z. Phys. **30** (1929), 465–467.
- [280] Mariya Vorobets, *On the Bethe-Sommerfeld conjecture for certain periodic Maxwell operators*, J. Math. Anal. Appl. **377** (2011), no. 1, 370–383. MR 2754836 (2012f:35527)
- [281] P. R. Wallace, *The band theory of graphite*, Phys. Rev. **71**.
- [282] Calvin H. Wilcox, *Theory of Bloch waves*, J. Analyse Math. **33** (1978), 146–167. MR MR516045 (82b:82068)
- [283] Wolfgang Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, vol. 138, Cambridge University Press, Cambridge, 2000.
- [284] Kenji Yajima, *Scattering theory for Schrödinger equations with potentials periodic in time*, J. Math. Soc. Japan **29** (1977), no. 4, 729–743. MR 0470525 (57 #10276)
- [285] ———, *Large time behaviors of time-periodic quantum systems*, Differential equations (Birmingham, Ala., 1983), North-Holland Math. Stud., vol. 92, North-Holland, Amsterdam, 1984, pp. 589–597. MR 799400 (86k:81042)
- [286] ———, *Quantum dynamics of time periodic systems*, Phys. A **124** (1984), no. 1-3, 613–619, Mathematical physics, VII (Boulder, Colo., 1983). MR 759208
- [287] V. A. Yakubovich and V. M. Starzhinskii, *Linear differential equations with periodic coefficients. v. 1, 2*, Halsted Press [John Wiley & Sons] New York-Toronto, Ont., 1975, Translated from Russian by D. Louvish. MR MR0364740 (51 #994)
- [288] Amnon Yariv, Yong Xu, Redinald K. Lee, and Axel Scherer, *Coupled-resonator optical waveguide: a proposal and analysis*, Opt. Lett. **24** (1999), 711–713.
- [289] Shing Tung Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.
- [290] V. I. Yudovich, *The linearization method in the stability problem for periodic flows of low viscosity fluids*, Proceedings of the VI- th winter school on mathematical programming and related problems, Drogobych 1973: Functional analysis and its applications. (In Russian) (1975), 44–113.
- [291] ———, *The linearization method in hydrodynamical stability theory*, Translations of Mathematical Monographs, vol. 74, American Mathematical Society, Providence, RI, 1989, Translated from the Russian by J. R. Schulenberger. MR 1003607 (90h:76001)
- [292] M. G. Zaïdenberg, S. G. Kreĭn, P. A. Kučment, and A. A. Pankov, *Banach bundles and linear operators*, Uspehi Mat. Nauk **30** (1975), no. 5(185), 101–157, English translation: Russian Math. Surveys **30** (1975), no. 5, 115–175.
- [293] Ya. B. Zeldovich, *Scattering and emission of a quantum system in a strong electromagnetic wave*, Soviet Physics Uspekhi.
- [294] V. V. Zhikov, *Gaps in the spectrum of some elliptic operators in divergent form with periodic coefficients*, Algebra i Analiz **16** (2004), no. 5, 34–58. MR 2106666 (2006a:35053)

- [295] J. M. Ziman, *Electrons in metals. a short guide to the Fermi surface*, Taylor & Francis, London, 1962.
- [296] ———, *Principles of the theory of solids*, Cambridge University Press, New York, 1964.
- [297] Nataliya A. Zimbovskaya, *Local geometry of the Fermi surface. and high-frequency phenomena in metals*, Springer Verlag, New York, 2001.

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