PARTITIONS AND CONSERVATIVITY

ANDREAS BLASS

Abstract. We study the partition properties enjoyed by the "next best thing to a P-point" ultrafilters introduced recently in joint work with Dobrinen and Raghavan. That work established some finite-exponent partition relations, and we now analyze the connections between these relations for different exponents and the notion of conservativity introduced much earlier by Phillips. In addition, we establish some infinite-exponent partition relations for these ultrafilters and also for sums of non-isomorphic selective ultrafilters indexed by selective ultrafilters.

1. Introduction

This paper is, in part, a sequel to an earlier joint paper with Natasha Dobrinen and Dilip Raghavan [4] in which we studied the ultrafilters created by a certain forcing construction. These ultrafilters were shown to have numerous combinatorial properties, including some weak partition properties and a classification of arbitrary functions modulo the ultrafilter.

Part of the present paper is an analysis of the implications between these combinatorial properties in general, i.e., independently from the connection with the generic ultrafilters of [4]. This analysis involves also a notion of conservativity that was introduced by Phillips [9], was studied further in [1, 2], and is connected with the model-theoretic notion of stability.

The other part of the present paper concerns infinitary partition properties of the generic ultrafilters in [4]. These partition properties lead to so-called "complete combinatorics" for the forcing notion used in [4]. A different approach to these partition properties was developed independently by Dobrinen [6].

This paper was written while the author was a visiting fellow in the "Mathematical, Foundational and Computational Aspects of the Higher Infinite" program at the Isaac Newton Institute for Mathematical Sciences.

2. Fréchet-Squared Genericity

In this section, we review the results from [4] that will play a major role in the present paper, and we take the opportunity to also fix some notation and terminology.

We deal with filters on countably infinite sets, and, in the absence of a contrary statement, filters are assumed to extend the cofinite filter. Sometimes we formulate definitions and results in terms of filters on the set ω of natural numbers, but these definitions and results can be transferred to other countably infinite sets via arbitrary bijections.

Definition 1. An ultrafilter $\mathcal{U}$ on ω is a *P-point* if, for every function f on ω, there is some $A \in \mathcal{U}$ such that the restriction $f \restriction A$ is finite-to-one or constant. $\mathcal{U}$ is *selective* if, for every function f on ω, there is some $A \in \mathcal{U}$ such that the restriction $f \restriction A$ is one-to-one or constant.

Clearly, every selective ultrafilter is a P-point. The existence of selective ultrafilters as well as the existence of non-selective P-points can be proved if the continuum hypothesis is assumed, but it is consistent with ZFC that there are no P-points.

We shall need a well-known eqivalent characterization of P-points. An ultrafilter $\mathcal{U}$ is a P-point if, given any countably many sets $A_n \in \mathcal{U}$, there is a set $B \in \mathcal{U}$ almost included in all of them, i.e., $B - A_n$ is finite for all n. The proof of the equivalence is based on making an f in the definition have value n on $A_n - A_{n-1}$ (where A_{-1} means $\varnothing$).

Selective ultrafilters enjoy remarkable partition properties and are therefore sometimes called Ramsey ultrafilters. Specifically, we have the following result, in which the part about partitions of $[\omega]^n$ is due to Kunen (published in Booth's thesis [5]) and the parts about $[\omega]^\omega$ are due to Mathias [8].

Proposition 2. *Every selective ultrafilter $\mathcal{U}$ on ω has the following partition properties.*

1. *For all $n, k \in \omega$, we have $\omega \to (\mathcal{U})^n_k$, which means that, whenever the set $[\omega]^n$ of n-element subsets of ω is partitioned into k pieces, then there is a set $H \in \mathcal{U}$ such that $[H]^n$ is included in one piece.*
2. *$\omega \overset{analytic}{\longrightarrow} (\mathcal{U})^\omega_2$, which means that, whenever the set $[\omega]^\omega$ of infinite subsets of ω is partitioned into an analytic piece and its complement, then there is a set $H \in \mathcal{U}$ such that $[H]^\omega$ is included in one piece.*
3. *Suppose the universe is obtained from some ground model by Lévy-collapsing to ω all cardinals below a Mahlo cardinal of the*

ground model. Then $\omega \xrightarrow{HOD\mathbb{R}} (\mathcal{U})^\omega_2$*, where* $HOD\mathbb{R}$ *means hereditarily ordinal definable from reals.*

In part (2) of this proposition, "analytic" refers to the topology of $[\omega]^\omega$ as a subspace of the power set $\mathcal{P}(\omega)$, which is, in turn, identified with 2^ω topologized as a product of discrete two-point spaces. Thus, two infinite subsets of ω are near each other in $[\omega]^\omega$ just in case they have a long common initial segment.

The partition relation $\omega \to (\mathcal{U})^n_k$ for any particular n and $k \geq 2$ easily implies the same relation for the same n and all k; it also implies the same relation for all smaller n. Also, $\omega \to (\mathcal{U})^2_2$ easily implies selectivity, because, given a function f on ω, we can partition the set $[\omega]^2$ of pairs $\{a, b\}$ according to whether $f(a) = f(b)$. The proposition thus implies the additional fact that the partition relation $\omega \to (\mathcal{U})^n_k$ for any one value of n implies the same for larger values of n, and this is not such an easy result.

The most natural way to produce a nonprincipal ultrafilter $\mathcal{U}$ on ω by forcing is to take, as forcing conditions, the infinite subsets of ω, ordered by inclusion (so smaller sets are stronger conditions). The intended meaning of a condition A is that it forces $A \in \mathcal{U}$. This ordering is not separative; the separative quotient identifies two infinite sets if and only if their symmetric difference is finite, so it amounts to the Boolean algebra[1] $\mathcal{P}(\omega)/\mathrm{fin}$. This separative quotient is countably closed, so the forcing adds no new reals. That makes it easy to check, by a density argument, that the generic object added by the forcing is an ultrafilter and indeed a selective ultrafilter.

Part (3) of Proposition 2 has an important consequence concerning this $\mathcal{P}(\omega)/\mathrm{fin}$ forcing. First, let us weaken that part of the proposition as follows. In the Lévy-Mahlo model (i.e., the model obtained by Lévy collapsing all cardinals below some Mahlo cardinal to ω), if $\mathcal{D} \subseteq [\omega]^\omega$ is in $HOD\mathbb{R}$, then either (a) there is an infinite $H \subseteq \omega$ such that no infinite subset of H is in $\mathcal{D}$, or (b) there is an $H \in \mathcal{U} \cap \mathcal{D}$. Note that, in both alternatives, we have weakened the conclusion in Proposition 2, part (3). In (a), we allow H to be any infinite set, not necessarily in $\mathcal{U}$. In (b), we only require H itself, not all its infinite subsets, to be in $\mathcal{D}$. This weakened form of the partition relation can be succinctly restated as follows.

Corollary 3. *In the Lévy-Mahlo model, every selective ultrafilter is* $\mathcal{P}(\omega)/\mathrm{fin}$*-generic over* $HOD\mathbb{R}$*.*

[1]When we use Boolean algebras as notions of forcing, we always mean the algebras with their zero elements removed.

Because of this corollary, one says that selectivity is *complete combinatorics* for $\mathcal{P}(\omega)/\mathrm{fin}$ forcing. Any generic ultrafilter for this forcing has the combinatorial property of selectivity, and there is no genuinely[2] additional combinatorial information that follows from $\mathcal{P}(\omega)/\mathrm{fin}$-genericity, because, at least in the Lévy-Mahlo model, selectivity by itself already ensures genericity over a large inner model.

Remark 4. The forcing notion $\mathcal{P}(\omega)/\mathrm{fin}$ is equivalent, for forcing, to the set of countably generated filters on ω (extending the cofinite filter, as usual), partially ordered by reverse inclusion. Indeed, we can associate to any infinite subset A of ω the filter $\{X \in [\omega]^\omega : A \subseteq^* X\}$, where $\subseteq^*$ means inclusion modulo finite sets. This gives a dense embedding of $\mathcal{P}(\omega)/\mathrm{fin}$ into the partially ordered set of countably generated filters.

The analog of Proposition 2 part (3) for this partial ordering of filters is false. Specifically, the image of our embedding of $\mathcal{P}(\omega)/\mathrm{fin}$ is, as mentioned above, dense, but so is the complement of this image. That is, every countably generated filter can be extended to one that does not contain a set A that is included modulo finite in all other elements of the extended filter.

Curiously, though, despite the failure of Proposition 2 part (3) for the filter poset, the weakened version used in proving Corollary 3 is true. One way to see this is to observe that this weakened version is equivalent to the corollary, which, being only about forcing, is clearly unchanged when we replace the forcing notion $\mathcal{P}(\omega)/\mathrm{fin}$ by an equivalent one.

One of the objectives of the paper [4] was to obtain, for a suitable ultrafilter that is not a P-point, results analogous to those for selective ultrafilters described above. To this end, we introduced what seems to be the simplest notion of forcing that adjoins a non-P-point ultrafilter on ω, and we studied the properties of these ultrafilters in considerable detail. In order to describe this forcing and the resulting ultrafilters, it is convenient to first introduce some notation, which will also be useful later in other contexts.

Definition 5. For a subset A of ω^2, we define its *(vertical) sections* to be the sets

$$A(x) = \{y \in \omega : \langle x, y\rangle \in A\}.$$

[2]The word "genuinely" here is intended to exclude information like the partition properties in Proposition 2, which intuitively look like more information than selectivity but in fact follow from selectivity.

For filters $\mathcal{U}$ and $\mathcal{V}_n$ on ω, we define the $\mathcal{U}$-indexed *sum* of the $\mathcal{V}_n$'s to be

$$\mathcal{U}\text{-}\sum_n \mathcal{V}_n = \{A \subseteq \omega^2 : \{n \in \omega : A(n) \in \mathcal{V}_n\} \in \mathcal{U}\}.$$

That is, a subset of the plane is large with respect to this sum if and only if almost all (with respect to $\mathcal{U}$) of its sections are large (with respect to the appropriate $\mathcal{V}_n$). It is easy to verify that this sum is always a filter, and that it is an ultrafilter if $\mathcal{U}$ and all of the $\mathcal{V}_n$'s are ultrafilters.

If all of the $\mathcal{V}_n$'s are the same filter $\mathcal{V}$, then we write $\mathcal{U} \otimes \mathcal{V}$ for the sum. If, furthermore, $\mathcal{V} = \mathcal{U}$, then we use the notation $\mathcal{U}^{\otimes 2}$.

Finally, we write $\mathfrak{Fr}$ for the filter of cofinite subsets of ω, because it is often called the Fréchet filter.

The forcing notion that formed the main subject of [4] is the Boolean algebra[3] $\mathcal{P}(\omega^2)/\mathfrak{Fr}^{\otimes 2}$. It is shown in [4] that this is a countably closed forcing notion and therefore does not add reals. As a consequence, the generic object that it adjoins is an ultrafilter $\mathcal{G}$ on ω^2 extending the filter $\mathfrak{Fr}^{\otimes 2}$. This implies that, for every $A \in \mathcal{G}$, there are infinitely many $n \in \omega$ such that A_n is infinite; indeed, this property of A is exactly what it means for A to intersect every set in $\mathfrak{Fr}^{\otimes 2}$ or, equivalently, not to be in the ideal dual to $\mathfrak{Fr}^{\otimes 2}$.

A frequently useful notion of forcing equivalent to $\mathcal{P}(\omega^2)/\mathfrak{Fr}^{\otimes 2}$ can be obtained as follows. First, replace the equivalence classes that constitute the quotient algebra $\mathcal{P}(\omega^2)/\mathfrak{Fr}^{\otimes 2}$ with all their representatives. That is, form the poset of all subsets of ω^2 that meet every set in $\mathfrak{Fr}^{\otimes 2}$, i.e., that have infinitely many infinite sections. This poset is not separative; its separative quotient is $\mathcal{P}(\omega^2)/\mathfrak{Fr}^{\otimes 2}$. Second, pass to the sub-poset consisting of those elements that have no finite sections. This is a dense sub-poset, and in fact its separative quotient is still $\mathcal{P}(\omega^2)/\mathfrak{Fr}^{\otimes 2}$, because, when we remove all finite sections from a set, we do not alter its equivalence class modulo $\mathfrak{Fr}^{\otimes 2}$. Third, pass to the even smaller poset consisting of those conditions on which the second projection $\omega^2 \to \omega$ is one-to-one. This sub-poset is still dense in our forcing, because, given any condition A, we can thin out all its sections so as to be disjoint from each other yet still infinite. Fourth, restrict to those conditions that lie entirely above the diagonal, i.e., that consist only of pairs (x, y) with $x < y$. This is again a dense subset, and it makes no difference in the separative quotient since the part of

[3] I prefer to think of quotients of Boolean algebras as being determined by filters rather than by ideals. So I use the filter notation $\mathfrak{Fr}^{\otimes 2}$ here, even though I used the standard notation $\mathcal{P}(\omega)/\mathrm{fin}$ earlier rather than my preferred $\mathcal{P}(\omega)/\mathfrak{Fr}$.

ω^2 that lies above the diagonal is a set in $\mathfrak{Fr}^{\otimes 2}$. Finally, for technical convenience, consider only those conditions A such that we never have the x-coordinate of one point in A equal to the y-coordinate of another point in A. Conditions of this sort are easily seen to be dense in our poset, so the final result is still forcing equivalent to the original $\mathcal{P}(\omega^2)/\mathfrak{Fr}^{\otimes 2}$. Summarizing this construction, we have that $\mathcal{P}(\omega^2)/\mathfrak{Fr}^{\otimes 2}$ is equivalent, as a forcing notion, to the poset defined as follows.

Definition 6. $\mathbb{P}$ is the set of those subsets A of ω^2 that satisfy

(1) A has infinitely many infinite sections and no nonempty finite sections.
(2) The sections of A are pairwise disjoint.
(3) All elements $\langle x, y\rangle$ of A have $x < y$.
(4) For any $\langle x, y\rangle$ and $\langle x', y'\rangle$ in A, we have $x \neq y'$.

We sometimes identify the above-diagonal subset of ω^2, $\{\langle x, y\rangle : x < y\}$, with the set $[\omega]^2$ of two-element subsets $\{x < y\}$ of ω. Thus, the forcing conditions in $\mathbb{P}$ can be viewed as subsets of $[\omega]^2$, and the generic $\mathcal{G}$ can be viewed as an ultrafilter on $[\omega]^2$.

An immediate consequence of the "infinitely many infinite sections" property of sets in $\mathcal{G}$ is that the projection $\pi_1 : \omega^2 \to \omega$ to the first factor is neither finite-to-one nor constant on any set in $\mathcal{G}$. Therefore, the generic filter is not a P-point.

Here is a list of additional properties of $\mathcal{P}(\omega^2)/\mathfrak{Fr}^{\otimes 2}$-generic ultrafilters[4] proved in [4, Section 3]; the references are to the propositions, theorems, and corollaries of that paper. After the list, I shall provide definitions for the concepts used in the list.

Proposition 7. *If $\mathcal{G}$ is $\mathcal{P}(\omega^2)/\mathfrak{Fr}^{\otimes 2}$-generic over V, then the following statements hold in $V[\mathcal{G}]$.*

(1) *The ultrafilter $\pi_1(\mathcal{G})$ is $\mathcal{P}(\omega)$/fin-generic over V and therefore selective. (Proposition 30)*
(2) *The projection $\pi_2 : \omega^2 \to \omega$ to the second factor is one-to-one on a set in $\mathcal{G}$. (Corollary 32)*
(3) *For any function f on ω^2, there is a set $A \in \mathcal{G}$ such that $f \restriction A$ is one of*
   - *a constant function,*
   - *π_1 followed by a one-to-one function, and*
   - *a one-to-one function. (Corollary 33)*

[4]Genericity here is over V, which I think of as the universe of all sets, so generic objects are in Boolean extensions. The results remain correct under any of the other customary ways to view forcing, for example taking V to be a countable transitive model so that generic objects exist in the ordinary, two-valued universe of sets.

(4) *$\mathcal{G}$ is a weak P-point. (Theorem 36)*
(5) *For any partition of $[\omega^2]^n$ into finitely many pieces, there is a set $H \in \mathcal{G}$ such that, for any n-type τ, all n-element subsets of $[H]^n$ that realize τ lie in the same class of the partition. Therefore, $\mathcal{G}$ is $(n, T(n))$-weakly Ramsey, where $T(n)$ is the number of n-types. (Theorem 31)*

For the sake of completeness, we explain here the notation $\pi_1(\mathcal{G})$ and the terminology "weak P-point", "n-type", "realize", and "weakly Ramsey" used in items 1, 4, and 5 of this proposition. More information about these can be found in [4]; the notion of weak P-point comes from Kunen's paper [7], where he proved (in ZFC) that weak P-points exist.

For any function $f : X \to Y$ and any filter $\mathcal{U}$ on X, its image on Y is defined as

$$f(\mathcal{U}) = \{A \subseteq Y : f^{-1}(A) \in \mathcal{U}\}.$$

This is always a filter, except that it may fail to satisfy our convention that filters on ω must contain all cofinite sets. If $\mathcal{U}$ is an ultrafilter, then so is $f(\mathcal{U})$, which extends the cofinite filter as long as f is not constant on any set in $\mathcal{U}$.

Definition 8. An ultrafilter $\mathcal{U}$ on a countably infinite set S is a *weak P-point* if, given any countable set of (nonprincipal) ultrafilters $\mathcal{W}_n$ on S, all distinct from $\mathcal{U}$, we have a set A that is in $\mathcal{U}$ but in none of the $\mathcal{W}_n$.

In terms of the topology of $\beta S - S$, this means that $\mathcal{U}$ is not in the closure of any countable set of ultrafilters distinct from $\mathcal{U}$. It is easy to verify that, as the terminology implies, all P-points are weak P-points. Kunen showed in [7] that there always exist weak P-points that are not P-points.

A sum $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ is never a weak P-point, as can be seen by taking the $\mathcal{W}_n$ in Definition 8 to be copies of the $\mathcal{V}_n$ on the vertical sections of ω^2. That is, let $i_n : \omega \to \omega^2$ be the map $y \mapsto \langle n, y\rangle$ and set $\mathcal{W}_n = i_n(\mathcal{V}_n)$.

Definition 9. Let n and t be natural numbers, let S be a countably infinite set, and let $\mathcal{U}$ be a (nonprincipal) ultrafilter on S. Then $\mathcal{U}$ is (n, t)-*weakly Ramsey* if, whenever $[S]^n$ is partitioned into finitely many pieces, there is a set $H \in \mathcal{U}$ such that $[H]^n$ meets at most t of the pieces.

If $t = 1$, this is the partition property in part (1) of Proposition 2. As t increases, the (n, t)-weak Ramsey property gets weaker. As n increases, the property gets stronger.

Another common notation for this property is $S \to [\mathcal{U}]^n_{t+1}$. The reason for the subscript $t+1$ is that this partition relation is equivalent

to saying that, whenever $[S]^n$ is partitioned into $t+1$ pieces, there is an $H \in \mathcal{U}$ such that $[H]^n$ is disjoint from at least one piece. Our definition, with an arbitrary finite number of pieces follows easily from this version with just $t+1$ pieces, by induction on the number of pieces.

Note that, in arrow notations like $S \to [\mathcal{U}]^n_{t+1}$, the square brackets are used to indicate the weak form of homogeneity, merely missing a piece, whereas round parentheses indicate the strong form, meeting only one piece. This notation is so common that one often speaks of square-bracket partition relations.

To explain n-types, it is useful to begin by considering an arbitrary element A of the notion of forcing $\mathbb{P}$ from Definition 6. Let us agree to write any n-element subset of A as $\{\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle\}$ with $b_1 < \cdots < b_n$. Recall that clause (2) in Definition 6 ensures that the b_i's are all distinct, so we are merely adopting the convention to list the n pairs $\langle a_i, b_i\rangle$ in the order of increasing second components. The n-type realized by A will be defined as all the information about the relative ordering of the a_i's and b_i's, with no information about their actual values. More precisely, we define types and realization as follows.

Definition 10. An *n-type* is a linear pre-order of the set of $2n$ formal symbols $x_1, \dots, x_n, y_1, \dots, y_n$ such that

- $y_1 < \cdots < y_n$,
- each x_i precedes the corresponding y_i, and
- if two distinct symbols are equivalent in the pre-order, then both of them are x's.

Recall that a pre-order on a set is a reflexive, transitive, binary relation $\leq$ on that set; that it is linear if every two elements of the set are ordered one way or the other; that two elements are called equivalent if each is $\leq$ the other; that identifying equivalent elements leads to a partial order (linear if the preorder was linear) on the quotient set; and that $<$ means "$\leq$ and not $\geq$."

In [4, Definition 2.9], we used a different formulation of the notion of n-type, namely a list of the x_i's and y_i with $=$ or $<$ between each consecutive pair, subject to requirements corresponding to the clauses of the present definition. We pointed out, in [4, Remark 2.11], that this list form of n-types is equivalent to the pre-order form adopted here. My main reason for now preferring the pre-order version is that is generalizes more naturally to the case of infinite sets in place of $\{a_1, \dots, a_n, b_1, \dots, b_n\}$. Nevertheless, the list form is also convenient, for example in the following definition.

Definition 11. The n-type *realized* by an n-element subset $\{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\}$ of an element of $\mathbb{P}$ is the pre-order of $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ whose list form becomes true when the x_i's and y_i's are interpreted as denoting the corresponding a_i's and b_i.

It is easy to verify, in the light of Definition 6 and our convention that $b_1 < \cdots < b_n$, that every n-element subset of an element of $\mathbb{P}$ realizes a unique type. Furthermore, every $A \in \mathbb{P}$ has n-element subsets realizing all of the n-types. This result is essentially Proposition 2.14 of [4], though it is stated and proved in somewhat greater generality there.

Notation 12. $T(n)$ denotes the number of n-types.

The preceding discussion shows that the generic ultrafilter cannot be $(n, T(n)-1)$-weakly Ramsey. Indeed, the same goes for any ultrafilter having a basis of sets from $\mathbb{P}$ and, as [4, Corollary 2.16] shows, for any ultrafilter that is not a P-point. Thus, part 5 of Proposition 7 says that $\mathcal{G}$ has the strongest weak-Ramsey properties that are possible for a non-P-point.

It will be convenient to have shorter name for the property in part (3) of Proposition 7.

Definition 13. An ultrafilter $\mathcal{U}$ on ω^2 has the *three functions property* if every function on ω^2 is, when restricted to some set in $\mathcal{U}$, either one-to-one or constant or the composition of π_1 followed by a one-to-one function.

The motivation for this terminology is that there are, up to restriction to sets in $\mathcal{U}$ and post-composition with one-to-one functions, just three functions on ω, namely the identity, π_1, and any constant function. By analogy, selectivity could be called the two-functions property.

3. Selective-Indexed Sums of Selective Ultrafilters

In this section, we describe another family of ultrafilters on ω^2 enjoying many but not all of the properties of the $\mathcal{P}(\omega^2)/\mathfrak{Fr}^{\otimes 2}$-generic (or equivalently $\mathbb{P}$-generic) ultrafilters discussed in the preceding section. These ultrafilters are the sums $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$, as in Definition 5, of selective ultrafilters, in the special case that the summands are pairwise non-isomorphic and the indexing ultrafilter $\mathcal{U}$ is also selective. To avoid having to repeatedly use the long phrase "**s**elective-**i**ndexed **s**um of **n**on-**i**somorphic **s**electives", we introduce the following acronym.

Definition 14. A *sisnis* ultrafilter is an ultrafilter on ω^2 of the form

$$\mathcal{U}\text{-}\sum_n \mathcal{V}_n$$

where $\mathcal{U}$ and all of the $\mathcal{V}_n$'s are selective ultrafilters and, for $m \neq n$, there is no permutation f of ω with $f(\mathcal{V}_m) = \mathcal{V}_n$.

The content of the definition would be unchanged if we required only $\mathcal{U}$-almost all of the $\mathcal{V}_n$'s to be selective and pairwise non-isomorphic. This is because a sum $\mathcal{U}\text{-}\sum_n \mathcal{V}_n$ is unchanged if we change the $\mathcal{V}_n$'s arbitrarily for a set of n's whose complement is in $\mathcal{U}$. For the same reason, we can assume that $\mathcal{U}$ is not isomorphic to any of the $\mathcal{V}_n$'s, since this can be arranged by altering at most one $\mathcal{V}_n$. The content would also be unchanged if we allowed f to be an arbitrary function $\omega \to \omega$ rather than a permutation. This is because, by selectivity, any $f : \omega \to \omega$ is $\mathcal{V}_m$-almost everywhere equal to either a constant function or a one-to-one function. A constant f cannot map $\mathcal{V}_m$ to a non-principal ultrafilter such as $\mathcal{V}_n$, and a one-to-one map would be $\mathcal{V}_m$-almost everywhere equal to a permutation of ω.

The next proposition summarizes information from [4, Section 2] about sisnis ultrafilters; see Lemmas 2.2 and 2.3, Proposition 2.4, and Theorem 2.17 of [4].

Proposition 15. *Every sisnis ultrafilter $\mathcal{W} = \mathcal{U}\text{-}\sum_n \mathcal{V}_n$ on ω^2 has the following properties.*

1. *The ultrafilter $\pi_1(\mathcal{W})$ is the selective ultrafilter $\mathcal{U}$.*
2. *The projection $\pi_2 : \omega^2 \to \omega$ to the second factor is one-to-one on a set in $\mathcal{W}$.*
3. *For any function f on ω^2, there is a set $A \in \mathcal{W}$ such that $f \restriction A$ is one of*
   - *a constant function,*
   - *π_1 followed by a one-to-one function, and*
   - *a one-to-one function.*
4. *$\mathcal{W}$ is not a weak P-point.*
5. *For any partition of $[\omega^2]^n$ into finitely many pieces, there is a set $H \in \mathcal{W}$ such that, for any n-type τ, all n-element subsets of $[H]^n$ that realize τ lie in the same class of the partition. Therefore, $\mathcal{W}$ is $(n, T(n))$-weakly Ramsey.*

Do not be lulled by the apparent similarity between this proposition and Proposition 7. Although the other clauses in the two propositions match, clause (4) is entirely different. $\mathbb{P}$-generic ultrafilters are weak P-points but sisnis ultrafilters are not. Indeed, as pointed out earlier, sums of ultrafilters are never weak P-points.

Of course, it follows that sisnis ultrafilters $\mathcal{W}$ are not P-points. It is easy to see this directly, because the first projection π_1 is neither finite-to-one nor constant on any set in $\mathcal{W}$.

We need one additional property of sisnis ultrafilters, the analog of a trivial property of $\mathbb{P}$-generic ultrafilters.

Lemma 16. *Every sisnis ultrafilter has a basis consisting of sets in* $\mathbb{P}$.

Proof. Consider an arbitrary sisnis ultrafilter, say $\mathcal{W} = \mathcal{U}\text{-}\sum_n \mathcal{V}_n$, and an arbitrary set $A \in \mathcal{W}$. We must find a subset of A that is in $\mathcal{W}$ and also in $\mathbb{P}$. Referring to the four clauses in the definition of $\mathbb{P}$, we see that A already satisfies half of the first clause: It has infinitely many infinite sections simply because $\mathcal{U}$ and the $\mathcal{V}_n$'s are nonprincipal. To achieve clause 2 (pairwise disjoint sections), we intersect A with a set in $\mathcal{W}$ on which π_2 is one-to-one. Such a set exists by part (2) of Proposition 15, and the resulting intersection A' satisfies clause (2) of the definition of $\mathbb{P}$.

Next, we shrink A' to an $A'' \in \mathcal{W}$ satisfying the fourth clause in the definition of $\mathbb{P}$, namely that the x-coordinates are distinct from the y-coordinates of elements of A''. This was already done during the proof of [4, Theorem 2.17], but for convenience we repeat the brief argument here. It suffices to find some $B \in \mathcal{U}$ such that, for all n, we have $B \notin \mathcal{V}_n$, for then we can take $A'' = A' \cap (B \times (\omega - B))$. Since $\mathcal{U}$ is distinct from the $\mathcal{V}_n$'s, we have, for each n, some $B_n \in \mathcal{U}$ such that $B_n \notin \mathcal{V}_n$. Because $\mathcal{U}$ is selective and therefore a P-point, it contains a set B such that $B - B_n$ is finite for every n. Then, for each n, we have that $\mathcal{V}_n$ contains $\omega - B_n$ and therefore also contains its almost-superset $\omega - B$, as required.

Finally, to achieve the remaining half of the first clause (no finite sections) and the third clause (no elements below the diagonal), we need only remove finitely many elements from some sections of A''; that will not affect whether those sections are in the ultrafilters $\mathcal{V}_n$ and so the resulting set A''' will be in $\mathcal{W}$. This completes the proof that A has a subset $A''' \in \mathcal{W} \cap \mathbb{P}$. □

4. Weak Ramsey Properties

In this section, we begin the analysis of the connections between the $(n, T(n))$-weak Ramsey properties in part (5) of Propositions 7 and 15 as well as the three-functions property expressed by part (3) in these propositions. We shall study these properties in the context of non-P-points. This context makes the weak Ramsey properties quite strong, in the sense that the next stronger such properties, $(n, T(n) - 1)$-weak Ramseyness, are impossible for non-P-points.

Throughout this section, we assume that we are dealing with a non-P-point $\mathcal{W}$. Replacing $\mathcal{W}$ by its isomorphic image under a suitable function to ω^2, we assume further that $\mathcal{W}$ is an ultrafilter on ω^2 such

that the first projection $\pi_1 : \omega^2 \to \omega$ is neither finite-to-one nor constant on any set in $\mathcal{W}$. We refer to such a $\mathcal{W}$ as a *non-P-point in standard position*.

Notice that, for such a $\mathcal{W}$, every set $A \in \mathcal{W}$ has infinitely many infinite sections, and therefore has a subset in $\mathbb{P}$. (The subset might not be in $\mathcal{W}$.)

We begin by showing that the weak homogeneity in the definition of $(n, T(n))$-weak Ramseyness necessarily arises from full homogeneity for each n-type.

Notation 17. For any set $S \subseteq \omega^2$ and any n-type τ, denote by $[S]_\tau$ the set of all those n-element subsets of S that realize the type τ.

Proposition 18. *Let $\mathcal{W}$ be an $(n, T(n))$-weakly Ramsey non-P-point in standard position, let τ be any n-type, and let the set $[\omega^2]_\tau$ be partitioned into finitely many pieces. Then there is a set $H \in \mathcal{W}$ such that $[H]_\tau$ is included in one of the pieces.*

Proof. Let $\mathcal{W}$, τ, and a partition Π of $[\omega^2]_\tau$ into, say, p pieces be as in the hypothesis of the proposition. Define a partition Π' of $[\omega^2]^n$ into $p + T(n) - 1$ pieces by letting the first p pieces be those of Π and letting the remaining $T(n) - 1$ pieces be $[\omega^2]_\sigma$ for the $T(n) - 1$ n-types σ other than τ. As $\mathcal{W}$ is $(n, T(n))$-weakly Ramsey, let $H \in \mathcal{W}$ be such that $[H]^n$ meets only $T(n)$ pieces of Π'.

Now H, being in $\mathcal{W}$, has a subset in $\mathbb{P}$, and, as we noticed right after Definition 11, such a subset contains realizers for all n-types. Therefore, $[H]^n$ meets all those pieces of our partition that have the form $[\omega^2]_\sigma$ for $\sigma \neq \tau$. That's $T(n) - 1$ pieces, so $[H]^n$ can meet at most one of the remaining pieces of Π', which are the original pieces of Π. Therefore, $[H]_\tau$ is included in that single piece of τ. □

We shall refer to the conclusion of this proposition as *τ-homogeneity* or, when we want to refer to all types τ together, as *n-type homogeneity* for $\mathcal{W}$ and for H. We note that the converse of the proposition is easy when $\mathcal{W}$ has a basis of sets from $\mathbb{P}$. For such $\mathcal{W}$, n-type homogeneity implies $(n, T(n))$-weak Ramseyness. To verify this, consider any partition of $[\omega^2]^n$ into finitely many pieces and find, for each n-type τ, a homogeneous set $H_\tau \in \mathcal{W}$ for that type. Then all subsets of $\bigcap_\tau H_\tau$ that realize n-types lie in at most $T(n)$ pieces of the original partition, namely the pieces that contain the sets $[H_\tau]_\tau$. Finally, shrink $\bigcap_\tau H_\tau$ to a set in $\mathcal{W} \cap \mathbb{P}$, so that all its n-element subsets realize n-types.

Notice that the assumption, in the preceding paragraph, that $\mathcal{W}$ has a basis consisting of sets in $\mathbb{P}$, was used only at the end of the argument, to ensure that all n-element subsets of the homogeneous set realize some

n-types. In some situations, the property that all n-element subsets realize n-types can be obtained from other hypotheses. The following lemma is a useful instance of this. It provides, for weakly Ramsey ultrafilters, some information that would be automatic for $\mathbb{P}$-generic ultrafilters and for sisnis ultrafilters, because these are generated by sets in $\mathbb{P}$.

Lemma 19. *If $\mathcal{W}$ is an $(n, T(n))$-weakly Ramsey non-P-point in standard position, then there is a set $P \in \mathcal{W}$ such that every n-element subset of P realizes an n-type.*

Proof. Partition $[\omega^2]^n$ into $T(n)+1$ pieces by making each of the $T(n)$ sets $[\omega^2]_\tau$, for n-types τ a piece, and then adding one more piece containing all the n-element sets that don't realize a type (because they have two elements with the same y-coordinate, or an element whose x-coordinate equals another element's y-coordinate, or because an element isn't above the diagonal). By hypothesis, there is a set $P \in \mathcal{W}$ that meets only $T(n)$ of these pieces. Recall that every set in $\mathcal{W}$ has a subset in $\mathbb{P}$ and every set in $\mathbb{P}$ has subsets realizing all types. So $[P]^n$ meets all of the pieces of the form $[\omega^2]_\tau$ in our partition and must therefore miss the one remaining piece, the piece consisting of n-element sets that don't realize types. □

Corollary 20. *If a non-P-point is $(n+1, T(n+1))$-weakly Ramsey, then it is also $(n, T(n))$-weakly Ramsey.*

Proof. We assume, without loss of generality, that $\mathcal{W}$ is in standard position. By Lemma 19, there is a set $A \in \mathcal{W}$ all of whose $(n+1)$-element subsets realize $(n+1)$-types. It follows immediately that every n-element subset of A realizes an n-type. This observation allows us to apply the comments immediately preceding Lemma 19 without needing the assumption that $\mathcal{W}$ has a basis of sets in $\mathbb{P}$. That assumption was needed only to ensure that the final homogeneous set can be shrunk so that all its n-element subsets realize n-types.

Thanks to those comments, it suffices to prove τ homogeneity for each n-type τ. Enlarge τ to an $(n+1)$-type τ' by appending $x_{n+1} < y_{n+1}$ after all of the x's and y's pre-ordered by τ. Notice that, if an $n+1$-element set realizes τ' then its first n elements (first in the usual ordering by y-coordinates) form an n-element set realizing τ. Given a partition Π of $[\omega^2]_\tau$, form a new partition Π' of $[\omega^2]_{\tau'}$ by putting two sets realizing τ' into the same piece of the new partition if their initial n-element subsets (realizing τ) are in the same piece of Π. By hypothesis, $\mathcal{W}$ contains a set homogeneous for Π', and it immediately follows that this set is also homogeneous for Π. (This uses the trivial

fact that every n-element set realizing τ is the initial n-element subset of some $(n+1)$-element set realizing τ'; just adjoin an $(n+1)^{\text{th}}$ element far beyond the given elements.) □

Our next goal is to establish a connection between weak Ramseyness and the three-functions property in part (3) of Propositions 7 and 15. In connection with the $(2,4)$ in the following proposition, recall that $T(2) = 4$.

Proposition 21. *If a non-P-point $\mathcal{W}$ in standard position is $(2,4)$-weakly Ramsey, then it has the three-functions property, i.e., every function on ω^2 is, on some $H \in \mathcal{W}$, either constant or one-to-one or π_1 followed by a one-to-one function.*

Proof. Let $\mathcal{W}$ be as in the hypothesis of the proposition. By Lemma 19, let $P \in \mathcal{W}$ be a set all of whose two-element subsets realize 2-types.

As a preliminary step, we show that $\pi_1(\mathcal{W})$ is selective. Let g be any function on ω; we show that is it one-to-one or constant on some set in $\pi_1(\mathcal{W})$. Let τ be the type given in list form by $x_1 < x_2 < y_1 < y_2$. Partition the set of pairs $\{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\}$ into two pieces according to whether $g(a_1) = g(a_2)$. Let $H \in \mathcal{W}$ be τ-homogeneous for this partition. Because $\mathcal{W}$ is in standard position, π_1 is not finite-to-one on any set in $\mathcal{W}$, so we can arrange that all nonempty sections of H are infinite. It easily follows that g is constant or one-to-one on $\pi_1(H) \in \pi_1(\mathcal{W})$.

Now to prove the three-functions property, let f be any function on ω^2. Partition $[\omega^2]^2$ into two pieces, the first being $\{\{x,y\} : f(x) = f(y)\}$ and the second being $\{\{x,y\} : f(x) \neq f(y)\}$.

Apply τ-type homogeneity for all four 2-types τ and intersect the resulting homogeneous sets with each other and with P. The result is a set $H \in \mathcal{W}$ such that all its 2-element subsets realize 2-types and, for each 2-type τ, either $f(a) = f(b)$ whenever $\{a,b\} \subseteq H$ realizes τ or $f(a) \neq f(b)$ whenever $\{a,b\}$ realizes τ.

Consider first the type given in list form by $x_1 = x_2 < y_1 < y_2$, i.e., the type of pairs that lie in a vertical column. Suppose all pairs $\{a,b\} \subseteq H$ realizing this type have $f(a) = f(b)$, so f is constant on vertical columns of H. Then, on H, we have $f = g \circ \pi_1$ for some function g on ω. Since $\pi_1(\mathcal{W})$ is selective, it contains a set K on which g is constant or one-to-one. Then on $H \cap \pi_1^{-1}(K) \in \mathcal{W}$, the function f is either constant (if g is constant on K) or π_1 followed by a one-to-one function (namely g). So the proposition is true in this case.

There remains the case that all pairs $\{a,b\}$ realizing the type $x_1 = x_2 < y_1 < y_2$ have $f(a) \neq f(b)$, so f is one-to-one on vertical columns

of H. We want to show that f is globally one-to-one on H. We shall do this by considering, one at a time, the three remaining 2-types, the types realized by pairs not on a vertical column, and show, in each of the three cases, that f must take different values at the two points of such a pair. In each case, we shall show this by contradiction, assuming that f takes the same value at the two points of each pair realizing the type, and deducing that f also takes the same value at two points in a vertical column.

Consider first the 2-type τ given by $x_1 < x_2 < y_1 < y_2$ and suppose, toweard a contradiction, that $f(a) = f(b)$ for some, and therefore for all pairs $\{a, b\} \in [H]_\tau$. Let τ' be the 3-type $x_1 = x_2 < x_3 < y_1 < y_2 < y_3$ and consider any 3-element subset $\{a, b, c\}$ of H realizing τ'. (Such a set exists because every set in $\mathcal{W}$ contains realizers for all types.) By inspection of the types, we see that both $\{a, c\}$ and $\{b, c\}$ realize τ. therefore we have both $f(a) = f(c)$ and $f(b) = f(c)$. But a and b are in the same vertical column, so $f(a) \neq f(b)$, and we have a contradiction.

The remaining two cases are very similar to the one just considered. When τ is $x_2 < x_1 < y_1 < y_2$, we take τ' to be $x_3 < x_1 = x_2 < y_1 < y_2 < y_3$, and when τ is $x_1 < y_1 < x_2 < y_2$, we take τ' to be $x_1 = x_2 < y_1 < y_2 < x_3 < y_3$. The rest of the argument is verbatim as in the preceding paragraph. (The general recipe for producing τ' from τ is to first change the subscript 2 in τ to 3 and then insert x_2 equal to x_1 and y_2 immediately after y_1.) □

The results proved so far in this section give the chain of implications, in which "w.r." abbreviates "weakly Ramsey" and "3f" abbreviates "the three-functions property".

$$\cdots \to (n+1, T(n+1))\text{-w.r.} \to (n, T(n))\text{-w.r.} \to \cdots \to (2, 4)\text{-w.r.} \to \text{3f}.$$

The question naturally arises whether these implications can be reversed. There is a reasonable hope for reversals, by analogy with what happens for selective ultrafilters. There, we have easy implications from the partition properties for larger exponents n to the partition properties for smaller n and from the partition property for exponent 2 to selectivity, just as in the chain above. Kunen's theorem gives reversals for the whole chain by showing that selectivity implies all the finite-exponent partition properties. Is there an analog of Kunen's theorem in the present situation, i.e., does the three-functions property (for a non-P-point) imply $(n, T(n))$-weak Ramseyness for all n? We conclude this section by showing that the answer is "no"; in the next section, though, we will show a way to correct the problem.

Theorem 22. *Assume the continuum hypothesis. There is a non-P-point in standard position that satisfies the three-functions property but is not* $(2,4)$*-weakly Ramsey.*

Proof. Because the continuum hypothesis is assumed, there is an enumeration, in an ω_1-sequence, of all the functions $\omega^2 \to \omega$; fix such an enumeration $\langle f_\alpha : \alpha < \omega_1 \rangle$.

Using the continuum hypothesis again, fix a selective ultrafilter $\mathcal{U}$ on ω.

Also, fix a symmetric, irreflexive, binary relation E on ω that makes ω a copy of the random graph. This means that, given any finite list of elements $a_0, \dots, a_{n-1} \in \omega$ and any subset $S \subseteq n$, there is some $b \in \omega$ distinct from all the a_i's such that, for all $i < n$, we have $a_i E b$ if and only if $i \in S$. In this situation, we say that b realizes the configuration[5] $\langle a_0, \dots, a_{n-1}; S \rangle$; we refer to the a_i's as the parameters of this configuration. It is easy to define such a relation E by an inductive construction, realizing all configurations, one at a time. It is well known and easy to prove, by a back-and-forth argument, that any two such relations E yield isomorphic graphs $\langle \omega, E \rangle$. So it makes sense to speak of a copy of the (rather than a) random graph. Call a subset X of ω *rich* if it has a subset $Y \subseteq X$ on which the restriction of E is a copy of the random graph.

With these preliminary items available, we are ready to construct the desired ultrafilter $\mathcal{W}$ on ω^2. It will be generated by $\aleph_1$ sets S_α indexed by the ordinals $\alpha < \omega_1$, and subject to the following requirements.

(1) $\mathcal{U}$-almost all sections of S_α are rich, i.e., $\{n \in \omega : S_\alpha(n) \text{ is rich}\} \in \mathcal{U}$.
(2) The S_α's are decreasing modulo $\mathcal{U}$, i.e., if $\alpha < \beta$ then $\{n \in \omega : S_\beta(n) \subseteq S_\alpha(n)\} \in \mathcal{U}$.
(3) The restriction of f_α to $S_{\alpha+1}$ either is one-to-one or factors through the first projection, $f_\alpha \restriction S_{\alpha+1} = g \circ \pi_1$ for some $g : \omega \to \omega$.

Before constructing the sets S_α, we verify, on the basis of these three requirements, that the S_α's together with the sets $\pi_1^{-1}(X)$ for $X \in \mathcal{U}$ generate an ultrafilter as required in the theorem.

First, we verify that the proposed generators have the finite intersection property, so they generate a filter. Consider any finitely many of the proposed generators, say $S_{\alpha_1}, \dots, S_{\alpha_k}$ and $\pi_1^{-1}(X_1), \dots, \pi_1^{-1}(X_l)$. Let α be the largest of the α_i's and notice that, by requirement (2), there is a set $Y \in \mathcal{U}$ such that $S_{\alpha_i}(n) \supseteq S_\alpha(n)$ for all $n \in Y$ and all i.

[5]The word "type" is often used instead of "configuration"; we prefer the latter here, to avoid any confusion with the "n-type" terminology.

Let Z be the interseection of Y and all of the X_j's, so $Z \in \mathcal{U}$. Then, whenever $n \in Z$, we have that the intersection of all of $S_{\alpha_1}, \ldots, S_{\alpha_k}$ and $\pi_1^{-1}(X_1), \ldots, \pi_1^{-1}(X_l)$ has the same section at n as S_α does. In particular, this section is rich, by requirement (1), and, a fortiori, infinite. This completes the verifiction that the sets S_α for $\alpha < \omega_1$ and $\pi_1^{-1}(X)$ for $X \in \mathcal{U}$ generate a filter $\mathcal{W}$ on ω^2.

The preceding argument shows something more, which we record here as a lemma for future reference.

Lemma 23. *For every set $A \in \mathcal{W}$, there is a set $Z \in \mathcal{U}$ such that $A(n)$ is rich, and therefore infinite, for all $n \in Z$.*

Next, we verify that the filter $\mathcal{W}$ is an ultrafilter. Let any subset A of ω^2 be given, and let $f : \omega^2 \to \omega$ be one-to-one on A and constant on $\omega^2 - A$. This f is f_α for some $\alpha < \omega_1$, and so it is either one-to-one or fiberwise constant on $S_{\alpha+1} \in \mathcal{W}$ by requirement (3). This means that $S_{\alpha+1}$ is (up to perhaps one point) included in A or in $\omega^2 - A$. Since A was arbitrary, this proves that $\mathcal{W}$ is an ultrafilter.

It now follows immediately from Lemma 23 that $\mathcal{W}$ is a non-P-point in standard position.

To verify the three-functions property, consider any function $\omega^2 \to \omega$. It is f_α for some $\alpha < \omega_1$, so its restriction to $S_{\alpha+1}$ either is constant or factors through π_1, by requirement (3). Since $S_{\alpha+1} \in \mathcal{W}$, we have the desired conclusion if f_α is constant on this set. So assume $f_\alpha = g \circ \pi_1$ on $S_{\alpha+1}$. Because $\mathcal{U}$ is selective, it contains a set H on which g is constant or one-to-one. Then $S_{\alpha+1} \cap \pi_1^{-1}(H)$ is a set in $\mathcal{W}$ on which f_α is constant or of the form $g \circ \pi_1$ with g one-to-one. This completes the verification of the three-functions property for $\mathcal{W}$.

To show that $\mathcal{W}$ is not (2,4)-weakly Ramsey, it suffices, by Proposition 18, to show that $\mathcal{W}$ lacks τ-type homogeneity for some 2-type τ. We use the type given by $x_1 = x_2 < y_1 < y_2$, i.e., the type of vertical pairs. These pairs are partitioned by the edge relation E; more precisely, we use the partition of $[\omega^2]_\tau$ into two pieces, one of which is $\{\{\langle a, b\rangle, \langle a, c\rangle\} \in [\omega^2]_\tau : bEc\}$. For any set $H \in \mathcal{W}$, Lemma 23 implies that $[H]_\tau$ meets both pieces of this partition. This completes the verification, on the basis of requirements (1)–(3), that $\mathcal{W}$ is as required in the theorem.

It remains to produce sets S_α satisfying the three requirements, and we shall do this by recursion on α. We begin the construction by setting $S_0 = \omega^2$; this clearly satisfies requirement (1), and the other two requirements are vacuous at this stage.

We consider next the limit stages of the construction. Let λ be a countable limit ordinal, and suppose, as an induction hypothesis, that

S_α has been defined for all $\alpha < \lambda$, satisfying requirements (1)–(3). Fix an increasing ω-sequence $0 = \alpha(0) < \alpha(1) < \dots$ with supremum λ. We shall define S_λ in such a way that $\mathcal{U}$-almost all of its sections are rich (so that requirement (1) continues to be satisfied) and so that $\{n \in \omega : S_\lambda(n) \subseteq S_{\alpha(i)}(n)\} \in \mathcal{U}$ for each $i \in \omega$. This will ensure that requirement (2) holds when $\beta = \lambda$ and α is one of the $\alpha(i)$'s, but it immediately implies the same for all $\alpha < \lambda$. Indeed, for any such α, there is an i (in fact infinitely many of them) with $\alpha < \alpha(i)$, and then, thanks to the induction hypothesis, we have that $S_\lambda(n) \subseteq S_{\alpha(i)}(n) \subseteq S_\alpha(n)$ for $\mathcal{U}$-almost all n. Note that requirement (3) is exclusively about successor S's and thus imposes no condition on S_λ.

As a preliminary normalization, we modify the sequence of sets $\langle S_{\alpha(i)} : i < \omega\rangle$, which is, by requirement (2) in the induction hypothesis, decreasing modulo $\mathcal{U}$, to make it literally decreasing. That is, we set

$$S'_i = \bigcap_{j=0}^{i} S_{\alpha(j)},$$

and we note that this has not ruined requirement (1) of the induction hypothesis; $\mathcal{U}$-almost all sections of S'_i are rich. Indeed, for $\mathcal{U}$-almost all n, we have that $S_{\alpha(i)}(n) \subseteq S_{\alpha(j)}(n)$ for all of the (finitely many) $j < i$ and so $S'_i(n) = S_{\alpha(i)}(n)$.

We also observe that, since $\alpha(0) = 0$, we have $S'_0 = S_{\alpha(0)} = S_0 = \omega^2$. In particular, all sections of S'_0 are rich.

For each $n \in \omega$, let $h(n)$ be the largest $i \leq n$ such that $S'_i(n)$ is rich. This exists because of the observation in the preceding paragraph. Then define

$$S_\lambda = \{\langle n, y\rangle \in \omega^2 : y \in S'_{h(n)}(n)\}.$$

The definition of h ensures that $S_\lambda(n)$ is rich for all $n \in \omega$, so requirement (1) is satisfied at λ. Requirement (3) says nothing about limit stages, so it remains only to verify that, for all i, $\mathcal{U}$-almost all n satisfy $S_\lambda(n) \supseteq S_{\alpha(i)}(n)$ for all i.

We remarked, immediately after defining S'_i, that it agrees with $S_{\alpha(i)}$ in $\mathcal{U}$-almost all vertical sections. So it suffices to verify that, for each i, $\mathcal{U}$-almost all n satisfy $S_\lambda \subseteq S'_i$.

So fix some $i \in \omega$. As we pointed out above, $\mathcal{U}$-almost all n have $S'_i(n)$ rich, and of course, as $\mathcal{U}$ is nonprincipal, $\mathcal{U}$-almost all n have $i \leq n$. Therefore, $\mathcal{U}$-almost all n have $h(n) \geq i$. For these n, we have, since the S' sequence is literally decreasing,

$$S_\lambda(n) = S'_{h(n}(n) \subseteq S'_i(n),$$

as desired. This completes the limit stage of our induction.

Finally, we deal with the successor stage. Suppose, as an induction hypothesis, that S_β has been defined for all $\beta \leq \alpha$ and satisfies requirements (1)–(3) in this range; our objective is to define $S_{\alpha+1}$ so as to satisfy all the new instances of requirements (1)–(3). In the case of requirement (2), it suffices to ensure that $S_{\alpha+1}(n) \subseteq S_\alpha(n)$ for $\mathcal{U}$-almost all n; the rest of requirement (2), comparing S_β with $S_{\alpha+1}$ for smaller β, is then an immediate consequence via the induction hypothesis comparing S_β with S_α.

To avoid unnecessary clutter, let us take advantage of the fact that, in this step of the construction α is fixed, so we can write simply f for f_α. We write f^n for the restriction of f to the n^{th} column in ω^2, regarded as a function on ω; that is, $f^n(y) = f(\langle n, y\rangle)$.

Our goal is thus to produce a set $S = S_{\alpha+1}$ with the following properties, corresponding to the three requirments above.

(1) $\mathcal{U}$-almost all sections $S(n)$ are rich.
(2) $S(n) \subseteq S_\alpha(n)$ for $\mathcal{U}$-almost all n.
(3) The restriction of f to S either is one-to-one or factors through π_1.

Call a number n good (for this argument) if f^n is constant on some rich subset of $S_\alpha(n)$; otherwise, call n bad. If $\mathcal{U}$-almost all n are good, then our task is easy. For each good n, choose some rich $R_n \subseteq S_\alpha(n)$ on which f^n is constant, and set

$$S = \{\langle n, y\rangle \in \omega^2 : n \text{ is good and } y \in R_n\}.$$

This has rich sections at all good n, so we have (1), while (2) holds because $R_n \subseteq S_\alpha(n)$ and (3) holds because f^n is constant on R_n.

It remains to treat the case that $\mathcal{U}$-almost all n are bad. In this case, we shall construct S one element at a time, making sure that they all map to different values under f, so that f will be one-to-one on S. The elements that we put into S will all be of the form $\langle n, y\rangle$ with n bad and $y \in S_\alpha(n)$. This will ensure that (2) holds. The sections of S at good n will be empty, but this does no harm to (1) because $\mathcal{U}$-almost no n are good. We shall also ensure that every section of S at a bad n is not only rich but a copy of the random graph. (Recall that a rich set is a superset of a copy of the random graph.) We may also assume that the nonempty sections of S_α are copies of the random graph; simply shrink each rich section to a copy of the random graph, and remove all non-rich sections.

To inductively produce S, we first make a list of conditions that should be satisfied by our construction. These conditions are represented formally by tuples of the form $\langle k_0, k_1, \ldots, k_{l-1}; Q\rangle$ where $k_0 <$

$k_1 < \cdots < k_{l-1}$ are natural numbers and $Q \subseteq l$. The meaning of this tuple is: "If the elements put into S at the ${k_0}^{\text{th}}, {k_1}^{\text{th}}, \ldots, {k_{l-1}}^{\text{th}}$ steps of the construction exist[6] and are in the same column, say they are $\langle n, y_1\rangle, \langle n, y_2\rangle, \ldots, \langle n, y_{l-1}\rangle$ for some bad n, then put into S an element $\langle n, z\rangle$ such that, for all $i < l$, we have zEy_i if and only if $i \in Q$." Thus, the tuple $\langle k_0, k_1, \ldots, k_{l-1}; Q\rangle$ requests fulfillment of one instance of the condition that $S(n)$ be a copy of the random graph. There are only countably many such tuples, so we can enumerate them as an ω-sequence. Fix such an enumeration in which, for all $q \in \omega$, the q^{th} tuple has all of its k_i's smaller than q (if necessary, repeat the vacuous tuple, where $l = 0$, numerous times).

We now explain the q^{th} step of the construction of S. Consider the q^{th} condition in our enumeration, say $\langle k_0, k_1, \ldots, k_{l-1}; Q\rangle$, and suppose that, as in the meaning of this condition explained above, the k_i'th step (which has already been done, because of the way we arranged the enumeration of conditions) put $\langle n, y_i\rangle$ into S for each $i < l$. We wish to adjoin some $\langle n, z\rangle$ to S subject to two desiderata. First, it should do what the condition $\langle k_0, k_1, \ldots, k_{l-1}; Q\rangle$ requests; zEy_i should hold when $i \in Q$ and fail when $i \in l - Q$. Second, $f(\langle n, z\rangle)$ should be different from $f(a)$ for all of the finitely many elements a already put into S during previous steps. Let us call these finitely many values $f(a)$ the forbidden values.

From now on, we work within $S_\alpha(n)$, which we recall is, with the binary relation E, a copy of the random graph. We seek an element z that has the correct configuration relative to the y_i's, as specified by $\langle k_0, k_1, \ldots, k_{l-1}; Q\rangle$, and such that $f^n(z)$ is different from the forbidden values.

For each forbidden value v, consider the set B_v obtained by removing from $(f^n)^{-1}(\{v\})$ any y_i's that happen to lie in $(f^n)^{-1}(\{v\})$. Since f^n is constant on B_v (with value v) and since n is bad, we know that B is not a copy of the random graph. So we can fix a finite subset F_v of B_v and a configuration C_v with respect to F_v that is not realized by any element of B_v. Combine all these configurations C_v, and also the configuration that $\langle k_0, k_1, \ldots, k_{l-1}; Q\rangle$ tells us to realize, into a single large but finite configuration C, relative to all the members of the F_v's and all the y_i's. Since S_α is a copy of the random graph, it contains an element z, distinct from all members of the F_v's and from the y_i's, and realizing C. Then, for each forbidden value v, we have that z realizes C_v and is therefore not in B_v. It is also not among the y_i's that were

[6]Some steps won't put any elements into S, so "exist" is not a vacuous requirement here.

removed from $(f^n)^{-1}(\{v\})$ when we defined B_v, so $z \notin (f^n)^{-1}(\{v\})$. That is, $f^n(z)$ is not the forbidden value v. Furthermore, z realizes the configuration requested by $\langle k_0, k_1, \ldots, k_{l-1}; Q\rangle$. Therefore, z satisfies all our desiderata, and we adjoin $\langle n, z\rangle$ as the next element of S. This completes the q^{th} step of the construction of $S = S_{\alpha+1}$. So we have obtained $S_{\alpha+1}$ with the required properties, and the proof of the theorem is complete. □

The theorem just proved shows that we do not have a perfect analog of Kunen's results for selective ultrafilters. The three-functions property, which is the analog of selectivity (a two-functions property) in our non-P-point situation, does not imply the strongest possible Ramsey properties. There remain at least three natural questions.

- Does the three-functions property imply weaker Ramsey properties, say (n, t)-weak Ramseyness for some $t > T(n)$?
- Do any of the $(n, T(n))$-weak Ramsey properties imply other such properties with larger n?
- Can the three-functions property be combined with some other (reasonable) property to imply weak Ramsey properties?

The second and third of these questions will be answered in the next section. As for the first, the following remark sketches a negative answer.

Remark 24. The random graph can be viewed as a random edge 2-coloring of the complete graph on $\aleph_0$ vertices. Edges of the random graph are colored red, and edges of the complete graph that are not in the random graph are colored green. There is an entirely analogous random edge $\aleph_0$-coloring of the complete graph on $\aleph_0$ vertices. It defining property is that, given any finite set F of vertices and any function f assigning to each vertex $v \in F$ one of the colors, there is a vertex $z \notin F$ whose edge to any $v \in F$ has the color $f(v)$.

The proof of Theorem 22 can be carried out essentially unchanged but with this random $\aleph_0$-coloring in place of the random graph. The result is a non-P-point $\mathcal{W}$ in standard position, enjoying the three-functions property, but with the following strong negative partition property. For the 2-type τ given by $x_1 = x_2 < y_1 < y_2$, there is a partition of $[\omega^2]_\tau$ into infinitely many pieces such that, for every $H \in \mathcal{W}$, all of the pieces meet $[H]_\tau$.

In particular, by merging some of the pieces of this partition, we can get, for any finite number t, a partition of $[\omega^2]_\tau$ into t pieces such that all pieces meet $[H]_\tau$ for all $H \in \mathcal{W}$. Combining this partition

of $[\omega^2]_\tau$ with the three other pieces $[\omega^2]_\sigma$ for 2-types $\sigma \neq \tau$, we get a counterexample showing that $\mathcal{W}$ is not $(2, t+2)$-weakly Ramsey.

5. Conservativity

In this section, we recall the notion of conservative elementary extensions, introduced in the context of models of arithmetic by Phillips [9]; we explain its connection with weak Ramsey properties; and we show that it, when combined with the three-functions property, implies $(n, T(n))$-weak Ramseyness for all $n \in \omega$.

To avoid excessive repetition, we refer the reader to [2] for some of the results that we shall need and we give only a short summary here.

We adopt the convention that, for a structure $\mathfrak{A}$, its underlying set (also called its domain or its universe) is denoted by $|\mathfrak{A}|$.

Definition 25. Let $\mathfrak{A}$ be a structure for a first-order language, and let $\mathfrak{B}$ be an elementary extension of $\mathfrak{A}$. Then $\mathfrak{B}$ is a *conservative extension* of $\mathfrak{A}$ if, whenever X is a parametrically definable subset of $\mathfrak{B}$, then $X \cap |\mathfrak{A}|$ is a parametrically definable subset of $\mathfrak{A}$.

This concept makes good sense in the context of general model theory. In fact, it can be used to characterize stable theories as those theories T such that all elementary extensions of models of T are conservative extensions. We shall, however, use only the special case where the models are elementary extensions of the standard model $\mathfrak{N}$ of full arithmetic. By full arithmetic, we mean the language that has function and relation symbols for all of the functions and relations (of arbitrary finite arity) on the set of natural numbers. $\mathfrak{N}$ is the model with underlying set ω and with all the symbols having the obvious meanings. In fact, we shall be concerned only with ultrapowers $\mathcal{U}\text{-prod}\,\mathfrak{N}$ of the standard model $\mathfrak{N}$.

Since all subsets of $\mathfrak{N}$ are definable in the language of full arithmetic, all elementary extensions of $\mathfrak{N}$ are conservative extensions.

If $\mathcal{U}$ is an ultrafilter on ω, then $\mathcal{U}\text{-prod}\,\mathfrak{N}$ is generated by a single element, namely the equivalence class modulo $\mathcal{U}$ of the identity function, $[\mathrm{id}]_\mathcal{U}$. Indeed, every element $[f]_\mathcal{U}$ of $\mathcal{U}\text{-prod}\,\mathfrak{N}$ is ${}^*f([\mathrm{id}]_\mathcal{U})$, where, as is customary in nonstandard analysis, ${}^*f$ is the function on $\mathcal{U}\text{-prod}\,\mathfrak{N}$ denoted there by the function symbol that denotes f in $\mathfrak{N}$.

If $\mathcal{U}$ is an ultrafilter on a countably infinite set S other than ω, it is still the case that $\mathcal{U}\text{-prod}\,\mathfrak{N}$ is generated by a single element; the equivalence class modulo $\mathcal{U}$ of any bijection between S and ω will do.

If $\mathcal{U}$ is an ultrafilter on S and f is a function with domain S, then f induces an elementary embedding $f_* : f(\mathcal{U})\text{-prod}\,\mathfrak{N} \to \mathcal{U}\text{-prod}\,\mathfrak{N}$,

namely the function that sends each $[g]_{f(\mathcal{U})}$ to $[g \circ f]_{\mathcal{U}}$. We sometimes identify $f(\mathcal{U})$-prod $\mathfrak{N}$ with its image under this embedding.

Definition 26. If $\mathcal{U}$-prod $\mathfrak{N}$ is a conservative extension of the image under f_* of $f(\mathcal{U})$-prod $\mathfrak{N}$, then we call f a *conservative map* on $\mathcal{U}$.

We shall be particularly interested, for reasons to be explained later, in the situation where $\mathcal{U}$ is an ultrafilter on ω^2 and f is the projection π_1 to the first cooordinate.

The key property of conservative extensions for our purposes is the following result, which is part of Theorem 3 in [2].

Proposition 27. *Suppose that $\mathfrak{B}$ and $\mathfrak{C}$ are elementary extensions of a model $\mathfrak{A}$ of full arithmetic and that $\mathfrak{B}$ is a conservative extension of $\mathfrak{A}$. Then there is, up to isomorphism, only one amalgamation of $\mathfrak{B}$ and $\mathfrak{C}$ over $\mathfrak{A}$ in which all the elements of $|\mathfrak{B}| - |\mathfrak{A}|$ are above (with respect to the nonstandard extension ${}^*<$ of the standard order on ω) all elements of $|\mathfrak{C}|$.*

The existence of such amalgamations is established by a fairly easy compactness argument; see Theorem 2(b) in [2]. The important part of Proposition 27 is the uniqueness.

The relevance of amalgamations for our purposes is the following connection with weak Ramsey properties; it is Theorem 5 of [2].

Proposition 28. *An ultrafilter $\mathcal{U}$ on ω is (n,t)-weakly Ramsey if and only if there are, up to isomorphism, at most t ways to amalgamate n copies of $\mathcal{U}$-prod $\mathfrak{N}$ with a specified ordering of the n copies of the generator $[id]_{\mathcal{U}}$.*

These propositions allow us to prove the first main result of this section.

Theorem 29. *Let $\mathcal{W}$ be a non-P-point in standard position. Assume that $\mathcal{W}$ has the three-functions property and that π_1 is a conservative map on $\mathcal{W}$. Then $\mathcal{W}$ is $(n, T(n))$-weakly Ramsey for all $n \in \omega$.*

Proof. We write $\mathcal{U}$ for the ultrafilter $\pi_1(\mathcal{W})$ on ω.

Thanks to the three-functions property, the ultrapower $\mathcal{W}$-prod $\mathfrak{N}$ has exactly three elementary submodels, namely the standard model $\mathfrak{N}$, the whole model $\mathcal{W}$-prod $\mathfrak{N}$, and the copy of $\mathcal{U}$-prod $\mathfrak{N}$ induced by the projection π_1 from $\mathcal{W}$ to $\mathcal{U}$. The copy of $\mathcal{U}$-prod $\mathfrak{N}$ is generated by $[\pi_1]_{\mathcal{W}}$. The whole model $\mathcal{W}$-prod $\mathfrak{N}$ is generated by the equivalence class, modulo $\mathcal{W}$, of any bijection $\omega^2 \to \omega$. It is also generated by $[\pi_2]_{\mathcal{W}}$ because π_2 is one-to-one on a set in $\mathcal{W}$ and therefore coincides, on a possibly smaller set in $\mathcal{W}$, with a bijection.

Temporarily, consider a single n-type τ. The notion of an n-tuple of elements of ω^2, with y-coordinates in increasing order, realizing τ can, like any relation on ω, be canonically extended to any elementary extension of $\mathfrak{N}$.

In particular, suppose we have an amalgamation of n copies of $\mathcal{W}$-prod $\mathfrak{N}$, with their generators, copies of $[\pi_2]$, in a specified order. Let the n copies of $\mathcal{W}$-prod $\mathfrak{N}$ be listed in the order of their copies of $[\pi_2]$ in the amalgamation, and write $[f]^i$ for the image, in the amalgamation, of an element $[f]$ of the i^{th} copy of $\mathcal{W}$-prod $\mathfrak{N}$. So our convention for the order of listing these copies ensures that $[\pi_2]^1 < [\pi_2]^2 < \cdots < [\pi_2]^n$ in the amalgamation. Then the elements $[\mathrm{id}]^i$, which are pairs in the amalgamation, realize τ if and only if the list form of τ is satisfied when the x_i are interpreted as $[\pi_1]^i$ and the y_i as $[\pi_2]^i$.

We now check how many amalgamations there are, of n copies of $\mathcal{W}$-prod $\mathfrak{N}$, in which the generators $[\mathrm{id}]^i$ realize τ. We build such an amalgamation by starting with the standard model $\mathfrak{N}$ and extending it in as many steps as there are inequivalent elements in the pre-order τ; we go through these elements in increasing order according to τ.

At a step corresponding to an equivalence class of x_i's in τ, we must amalgamate the model $\mathfrak{M}$ already constructed in previous steps with $\mathcal{U}$-prod $\mathfrak{N}$, identifying only the standard parts, and putting all nonstandard elements of $\mathcal{U}$-prod $\mathfrak{N}$ above all elements of $\mathfrak{M}$. Since $\mathcal{U}$-prod $\mathfrak{N}$ is, like any model of full arithmetic, a conservative extension of $\mathfrak{N}$, there is only one way, up to isomorphism, to perform this amalgamation.

At a step corresponding to a y_i in τ, we must amalgamate the model $\mathfrak{M}$ already constructed with $\mathcal{W}$-prod $\mathfrak{N}$, identifying the copy of $\mathcal{U}$-prod $\mathfrak{N}$ in this $\mathcal{W}$-prod $\mathfrak{N}$ with the copy already amalgamated into $\mathfrak{M}$ at the earlier step corresponding to x_i, and putting all elements of $|\mathcal{W}\text{-prod }\mathfrak{N}| - |\mathcal{U}\text{-prod }\mathfrak{N}|$ above all elements of $\mathfrak{M}$. Again, there is only one way to perform this amalgamation, because $\mathcal{W}$-prod $\mathfrak{N}$ is, by hypothesis, a conservative extension of the submodel identified with $\mathcal{U}$-prod $\mathfrak{N}$.

The preceding two paragraphs show that there is only one way, up to isomorphism, to amalgamate n copies of $\mathcal{W}$-prod $\mathfrak{N}$ with the generators in a specified order and realizing τ. Since this holds for each n-type τ, and since there are $T(n)$ n-types, we conclude that there are only $T(n)$ ways to amalgamate n copies of $\mathcal{W}$-prod $\mathfrak{N}$ with the generators in a specified order. By Proposition 28, it follows that $\mathcal{W}$ is $(n, T(n))$-weakly Ramsey. □

Theorem 29 shows that conservativity is a sufficient condition to add to the three-function property and produce all the weak Ramsey

properties enjoyed by sisnis ultrafilters and $\mathbb{P}$-generic ultrafilters. The question naturally arises whether conservativity is necessary for this purpose. Theorem 22 shows that some additional condition is needed, but might conservativity be excessive? Might a weaker additional condition suffice? The next theorem answers these questions negatively. It shows that conservativity is really needed.

Theorem 30. *Suppose $\mathcal{W}$ is a $(2,4)$-weakly Ramsey non-P-point in standard position. Then π_1 is a conservative map on $\mathcal{W}$.*

Proof. As before, we write $\mathcal{U}$ for $\pi_1(\mathcal{W})$. As a first step toward the proof, we analyze the definition of "π_1 is a conservative map on $\mathcal{W}$" in order to replace it with an equivalent condition of a combinatorial, rather than model-theoretic, flavor.

Notice first that, when considering a parametrically definable subset X of $\mathcal{W}$-prod $\mathfrak{N}$, we may assume without loss of generality that the only parameter used in the definition is the pair $[\mathrm{id}]_{\mathcal{W}}$. This is because any other parameter $[f]_{\mathcal{W}}$ can be defined from $[\mathrm{id}]_{\mathcal{W}}$. Furthermore, although the definition could, a priori, look like

$$X = \{z \in \mathcal{W}\text{-prod}\,\mathfrak{N} : \varphi([\mathrm{id}]_{\mathcal{W}}, z)\}$$

for an arbitrary formula φ, we may assume without loss of generality that it has the form

$$X = \{z \in \mathcal{W}\text{-prod}\,\mathfrak{N} : {}^*R([\pi_1]_{\mathcal{W}}, [\pi_2]_{\mathcal{W}}, z)\}$$

for some ternary relation R on ω. This is because we are working in full arithmetic, so any formula φ applied to a pair and a single element is equivalent, in $\mathfrak{N}$ and therefore in any elementary extension, to an atomic ternary relation.

So let us consider an arbitrary X of this form, obtained from some arbitrary ternary relation R. The intersection of X with the elementary submodel $\mathcal{U}$-prod $\mathfrak{N}$ is, taking into account the identification via $(\pi_1)_*$,

$$Y = \{[f]_{\mathcal{U}} : \mathcal{W}\text{-prod}\,\mathfrak{N} \models {}^*R([\pi_1]_{\mathcal{W}}, [\pi_2]_{\mathcal{W}}, [f \circ \pi_1]_{\mathcal{W}})\}.$$

Our goal is to prove that this Y is parametrically definable in $\mathcal{U}$-prod $\mathfrak{N}$. As in the case of X, if there is such a definition, there will be one whose only parameter is $[\mathrm{id}]_{\mathcal{U}}$ and indeed one of the form

$$Y = \{[f]_{\mathcal{U}} : \mathcal{U}\text{-prod}\,\mathfrak{N} \models {}^*S([\mathrm{id}]_{\mathcal{U}}, [f]_{\mathcal{U}})\}$$

for some binary relation S on ω.

Taking into account the definition of how relation symbols are interpreted in ultrapowers, we find that what must be proved is the following. For every ternary relation R on ω, there exists a binary relation

S on ω such that, for all functions $f : \omega \to \omega$, we have

$$\{\langle x, y\rangle \in \omega^2 : R(x, y, f(x))\} \in \mathcal{W} \iff \{x \in \omega : S(x, f(x))\} \in \mathcal{U}.$$

Now that we have a combinatorial version of the desired conclusion, we work toward deducing this version form the assumption that $\mathcal{W}$ is a (2,4)-weakly Ramsey non-P-point in standard position. In fact, we will use (2,4)-weak Ramseyness only to obtain τ-homogeneity for the 2-type τ given by $x_1 = x_2 < y_1 < y_2$.

Let an arbitrary ternary relation R be given. Associate to each pair $\langle x, y\rangle \in \omega^2$ the function $C_{\langle x,y\rangle} : \omega \to 2$ that sends any $z \in \omega$ to 1 if $R(x, y, z)$ and to 0 otherwise. Partition $[\omega^2]_\tau$ into two pieces, putting $\{\langle x, y\rangle, \langle x, y'\rangle\}$ (where $y < y'$ by our usual convention) into the first piece if $C_{\langle x,y\rangle}$ lexicographically precedes $C_{\langle x,y'\rangle}$ and into the second piece otherwise. By hypothesis, there is a set $H \in \mathcal{W}$ such that $[H]_\tau$ lies entirely in one of the two pieces.

Suppose $[H]_\tau$ is included in the first piece of our partition. (The alternative possibility, that it is included in the second piece, is handled by an entirely analogous argument.) We may assume all nonempty sections of H are infinite, since removing any finite sections only deletes a set not in $\mathcal{W}$ and thus changes none of the properties we have for H. For each $x \in \pi_1(H)$, the sequence of functions $\langle C_{\langle x,y\rangle} : y \in H(x)\rangle$ is lexicographically increasing. Any such sequence eventually stabilizes componentwise. That is, for each $z \in \omega$, there is some N_z such that $C_{\langle x,y\rangle}(z)$ is independent of y once $y \geq N_z$. To see this, argue by induction on z. For $z = 0$, the lexicographic ordering forces the values of $C_{\langle x,y\rangle}(0)$ to never decrease as y increases, so they are either all 0, or, once one of them is 1, all the later ones, for larger y, are also 1. Once the values for $z = 0$ have stabilized, the values for $z = 1$ can never decrease, so these too must stabilize. And so on; once the values for all $z < k$ have stabilized, the values for $z = k$ can no longer decrease, so they also stabilize.

Now define S by putting a pair $\langle x, z\rangle$ into S if and only if the eventual, stable value of $C_{\langle x,y\rangle}(z)$ for all sufficiently large y is 1. We claim that this S works. Let an arbitrary $f : \omega \to \omega$ be given.

Suppose first that $\{x \in \omega : S(x, f(x))\} \in \mathcal{U}$, and let B denote this set in $\mathcal{U}$. For each $x \in B$ we have, by definition of S, that $R(x, y, f(x))$ holds for all sufficiently large $y \in H(x)$. Thus, the set $\{\langle x, y\rangle \in \omega^2 : R(x, y, f(x))\}$ includes the intersection of H, $\pi_1^{-1}(B)$, and a set of the form $\{\langle x, y\rangle : y > M(x)\}$ for some function M. All three of these are in $\mathcal{W}$, the last because of standard position: π_1 is not finite-to-one on any set in $\mathcal{W}$. Therefore the intersection is in $\mathcal{W}$, as required.

The remaining case, that $\{x \in \omega : S(x, f(x))\} \notin \mathcal{U}$ is handled the same way, using, in place of R and S, their negations. □

Corollary 31. *For non-P-points, the properties of $(n, T(n))$-weak Ramseyness for different $n \geq 2$ are all equivalent.*

Proof. We already know, from Corollary 20, that these weak Ramsey properties for larger n imply the proiperties for smaller n. For the converse, we assume that the ultrafilter is in standard position; this can be arranged by applying a suitable bijection and thus entails no loss of generality. Then any of these weak Ramsey properties implies $(2, 4)$-weak Ramseyness (by Corollary 20), which in turn implies both the three-functions property (by Proposition 21) and conservativity of π_1 (by Theorem 30). These, in turn, imply $(n, T(n))$-weak Ramseyness for all n (by Theorem 29). □

Summarizing, we have, for non-P-points in standard position, the equivalence of all the $(n, T(n))$-weak Ramsey properties and the conjunction of the three-functions property with conservativity of π_1. For non-P-points not in standard position, the only change that is needed is that conservativity applies not to π_1 but to any function p that is neither finite-to-one nor constant on any set of the ultrafilter. (The three-functions property ensures that p is essentially unique.)

6. Infinitary Partition Relations and Complete Combinatorics

In the preceding sections, we have dealt only with finitary partition relations. In the present section, we turn to infinitary partition relations enjoyed by $\mathbb{P}$-generic ultrafilters and by sisnis ultrafilters. By analogy with Mathias's results for selective ultrafilters, parts (2) and (3) of Proposition 2, and thinking of $\mathbb{P}$ in our situation as being the analog of $[\omega]^\omega$ in Mathias's situation, we might hope that our ultrafilters $\mathcal{W}$ enjoy a partition relation of the following sort: Whenever $\mathbb{P}$ is partitioned into two nice pieces, then there is some $A \in \mathcal{W}$ all of whose subsets in $\mathbb{P}$ lie in the same piece. Here, "nice" could mean analytic/coanalytic, or, in the case of the Lévy-Mahlo model, it could mean $HOD\mathbb{R}$.

Unfortunately, such a partition relation is extremely false. It is possible to partition $\mathbb{P}$ into continuum many Borel pieces, all of which are dense in the forcing notion $\mathbb{P}$. To see this, we extend the notion of n-types (Definition 10) to ω-types.

Definition 32. An *ω-type* is a linear pre-order of the infinite set of formal symbols $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ such that

- $y_1 < y_2 < \dots$,
- each x_i precedes the corresponding y_i,
- each equivalence class in the pre-order consists of either a single y_i or infinitely many x_i's,
- there are infinitely many equivalence classes of x's, and
- the induced linear order of the equivalence classes has order-type ω.

The intention here is that an ω-type describes the order-relationships between the x and y coordinates of the points in an element of $\mathbb{P}$. Recall that the definition of $\mathbb{P}$ requires that, if $A \in P$, then all the points in A have distinct y-coordinates; as before, we adopt the convention of thinking of the points in A as listed in order of increasing y-coordinates.

Definition 33. The ω-type realized by an element A of $\mathbb{P}$ is the pre-order consisting of exactly those inequalities between the formal symbols x_i and y_j that hold when A is listed as $\{\langle a_i, b_i\rangle : i \in \omega\}$ in order of increasing b_i's and then each x_i is interpreted as denoting a_i and each y_j is interpreted as denoting b_j.

The definition of $\mathbb{P}$ easily implies that every $A \in \mathbb{P}$ realizes a (unique) ω-type.

Notation 34. Let $\mathbb{P}_\tau$ be the subset of $\mathbb{P}$ consisting of those elements of $\mathbb{P}$ that realize the ω-type τ.

Given any $A \in \mathbb{P}$ and any ω-type τ, it is easy to construct a subset $B \subseteq A$ in $\mathbb{P}$ (i.e., an extension of A in the forcing notion $\mathbb{P}$) that realizes τ. That is, each $\mathbb{P}_\tau$ is dense in $\mathbb{P}$. It is easy to check also that each $\mathbb{P}_\tau$ is a Borel set. So we have, as claimed, a partition of $\mathbb{P}$ into continuum many Borel sets, all of which are dense in $\mathbb{P}$.

Although this result constitutes a strong counterexample to natural partition relations for $\mathbb{P}$, it also suggests a way around the problem. Each $\mathbb{P}_\tau$ is a notion of forcing equivalent to $\mathbb{P}$, and we might hope for a partition relation satisfied by one of these notions of forcing. Recall Remark 4, where we pointed out that an infinite-exponent partition relation can hold for a notion of forcing while failing for an equivalent notion. Perhaps this happens here.

In fact, the next theorem shows that this happens for every ω-type.

Theorem 35. *Let τ be an ω-type, and let $\mathbb{P}_\tau$ be partitioned into an analytic subset and its complement.*

(1) *There is a set $H \in \mathbb{P}_\tau$ such that all its subsets in $\mathbb{P}_\tau$ lie in the same piece of the partition.*

(2) *Any sisnis ultrafilter on ω^2, contains an H such that all its subsets in $\mathbb{P}_\tau$ lie in the same piece of the partition.*
(3) *Any $\mathbb{P}$-generic ultrafilter on ω^2, contains an $H \in \mathbb{P}_\tau$ such that all its subsets in $\mathbb{P}_\tau$ lie in the same piece of the partition.*

Parts (1) and (3) were proved for a particular ω-type τ by Dobrinen in [6] using an entirely different method, based on Todorcevic's theory of topological Ramsey spaces. It is very likely that her method can be applied to arbitrary ω-types, not just the one she used in [6].

Proof. The main work in this proof is to establish part (2) of the theorem; afterward, parts (1) and (3) will follow fairly easily. Fortunately, the main work was already done in [3], specifically in proving Theorem 7 of that paper. So our first task here is just to show how (2) follows from that theorem. This argument parallels part of the proof of Theorem 2.17 in [4], which also relied on the same result from [3].

We begin by stating, in the next lemma, the result from [3]; afterward, we shall show how part (2) of the present theorem follows from it.

Lemma 36 (Theorem 7 of [3])**.** *Assume that selective ultrafilters $\mathcal{D}(s)$ have been assigned to all finite subsets s of ω, and assume that every two of these ultrafilters are either equal or not isomorphic. Let $\mathcal{X}$ be an analytic subset of $[\omega]^\omega$. Then there is a function Z assigning, to each ultrafilter $\mathcal{D}$ that occurs among the $\mathcal{D}(s)$'s, some element $Z(\mathcal{D}) \in \mathcal{D}$ such that $\mathcal{X}$ contiains all or none of the infinite sets $\{z_0 < z_1 < z_2 < \dots\} \in [\omega]^\omega$ that satisfy $z_n \in Z(\mathcal{D}(\{z_0, \dots, z_{n-1}\}))$ for all $n \in \omega$.*

We emphasize that, if the same ultrafilter $\mathcal{D}$ occurs as $\mathcal{D}(s)$ for several sets s, then a single set $Z(\mathcal{D})$ is assigned to it by Z, not a possibly different set for each occurrence.

Using this lemma, we proceed with the proof of part (2) of our theorem. Let $\mathcal{W} = \mathcal{U}\text{-}\sum_n \mathcal{V}_n$ be a sisnis ultrafilter on ω^2, so $\mathcal{U}$ and all of the $\mathcal{V}_n$ are non-isomorphic selective ultrafilters on ω. Let τ be an ω-type, and let $\mathbb{P}_\tau$ be partitioned into an analytic piece $\mathcal{Y}$ and its complement.

There is a natural bijection φ from $[\omega]^\omega$ onto $\mathbb{P}_\tau$, defined as follows. Given a set $\{z_0 < z_1 < z_2 < \dots\} \in [\omega]^\omega$, assign the value z_i to the formal variables in the i^{th} equivalence class[7] of the pre-order τ. (So z_i becomes the value of either a single y_j or infinitely many x_k's). For each $j \in \omega$, the values assigned to x_j and y_j determine a point in ω^2, and we let $\varphi(\{z_0, z_1, \dots\})$ be the set of these points. Because τ is an ω-type, this set is in $\mathbb{P}_\tau$.

[7]Here and below, our enumeration of the equivalence classes begins with 0. So the i^{th} equivalence class is the one with exactly i strict predecessors.

Let $\mathcal{X} = \varphi^{-1}(\mathcal{Y})$. Since φ is clearly continuous and since $\mathcal{Y}$ is analytic, $\mathcal{X}$ is an analytic subset of $[\omega]^\omega$, as required by the hypothesis of Lemma 36.

To apply the lemma, we must still define appropriate selective ultrafilters $\mathcal{D}(s)$; these will be chosen from among the ultrafilters $\mathcal{U}$ and $\mathcal{V}_n$ that produced the sisnis ultrafilter $\mathcal{W}$. The scheme for associating these ultrafilters to finite subsets s of ω is as follows. Suppose $s = \{z_0 < z_1 < \cdots < z_{k-1}\}$. Assign the value z_i to the formal variables in the i^{th} equivalence class of the pre-order τ, for each $i < k$. (This is exactly like the definition of φ above, except that, because s is finite, only k equivalence classes of variables get values.) Consider equivalence class number k, the first one not assigned a value here. If it is an equivalence class of x_j's, then let $\mathcal{D}(s) = \mathcal{U}$. If, on the other hand, it consists of a single y_j, then, since the corresponding x_j precedes y_j in τ, s has assigned a value v to x_j; set $\mathcal{D}(s) = \mathcal{V}_v$.

Since $\mathcal{U}$ and all the $\mathcal{V}_n$ are non-isomorphic selective ultrafilters, we have satisfied the hypotheses of Lemma 36, so we obtain a function Z as described there. It remains to chase through all the relevant definitions to see what the conclusion of Lemma 36 tells us in this situation.

That conclusion concerns sets $\{z_0 < z_1 < \dots\}$ such that each z_n is a member of $Z(\mathcal{D}(\{z_0, \dots, z_{n-1}\}))$. By our choice of $\mathcal{D}$'s this means that, when we compute φ of such a set, the values assigned to the x_j's are in $Z(\mathcal{U})$ and the value assigned to any y_j is in $Z(\mathcal{V}_v)$ where v is the value of the corresponding x_j. This means that $\varphi(\{z_0 < z_1 < \dots\})$ is a subset of

$$H = \{\langle a, b\rangle \in \omega^2 : a \in Z(\mathcal{U}) \text{ and } b \in Z(\mathcal{V}_a)\}.$$

This H is in $\mathcal{W} = \mathcal{U}\text{-}\sum_n \mathcal{V}_n$ because each $Z(\mathcal{D})$ is in the corresponding $\mathcal{D}$. Furthermore, any subset of H of type τ is $\varphi(\{z_0 < z_1 < \dots\})$ for some $\{z_0 < z_1 < \dots\}$ as in the conclusion of Lemma 36. The lemma tells us that either all or none of these sets $\{z_0 < z_1 < \dots\}$ are in $\mathcal{X}$, and, in view of our choice of $\mathcal{X}$, this means that all or none of the subsets of H of type τ are in $\mathcal{Y}$. This completes the proof of part (2) of the theorem.

We turn next to part (1). We first prove a slightly weaker version, replacing the assertion that $H \in \mathbb{P}_\tau$ with the assertion that H has infinitely many infinite vertical sections. This weaker version would be an immediate consequence of part (2) if we knew that there exists a sisnis ultrafilter, because any set in a sisnis ultrafilter has infinitely many infinite sections. The existence of a sisnis ultrafilter, which is equivalent to the existence of infinitely many non-isomorphic selective ultrafilters, is not provable in ZFC; indeed, it is not provable that

there exists even one selective ultrafilter. Nevertheless, we can still use part (2) to obtain the weakened part (1) as follows. Regardless of the existence or non-existence of sisnis ultrafilters, we can pass to a forcing extension of the universe in which the continuum hypothesis holds and there are no new reals. (It suffices to adjoin a generic subset of ω_1 with countable forcing conditions.) In the extension, there are, thanks to the continuum hypothesis, plenty of selective ultrafilters ($2^{2^{\aleph_0}}$ of them), so we have a sisnis ultrafilter and therefore have the weakened part (1) of the theorem. But this result is a statement entirely about real numbers (note in particular that the partition can be coded by a real number, as it involves only an analytic set and its complement). Since the forcing extension didn't add reals, the same result holds in the original universe, as required.

To pass from the weakened version of part (1) to the original version where H is required to be in $\mathbb{P}_\tau$, it suffices to recall that every set with infinitely many infinite sections has a subset in $\mathbb{P}$ and that $\mathbb{P}_\tau$ is dense in $\mathbb{P}$. Therefore, we can just replace the H from the weakened version with a subset in $\mathbb{P}_\tau$ to complete the proof of part (1).

Before proceeding to part (3), we explain a technical strengthening of part (1) that will be used in the proof of part (3). Suppose we are given, in addition to the ω-type τ and the partition, a subset A of ω^2 with infinitely many infinite sections. Then, in the forcing extension used in the proof of part (1), we can choose the selective ultrafilters so that the resulting sisnis ultrafilter $\mathcal{W}$ contains A. Then, when we apply part (2) with this sisnis ultrafilter, we can arrange for the homogeneous set H to be a subset of A; since both A and H are in $\mathcal{W}$, we can replace H by its intersection with A. The passage from the forcing extension to the ground model and the shrinking of H to put it into $\mathbb{P}_\tau$ preserve this arrangement. Therefore, in part (1) of the theorem, we can get the homogeneous set to be included in any prescribed A that has infinitely many infinite sections.

Finally, we prove part (3). The preceding technical improvement of part (1) applies in particular to any $A \in \mathbb{P}_\tau$. So we have that, for any partition into an analytic set and its complement, the homogeneous sets H are dense in $\mathbb{P}_\tau$, so any $\mathbb{P}_\tau$-generic ultrafilter contains such a homogeneous set. Recall that $\mathbb{P}_\tau$ is dense in $\mathbb{P}$, so genericity is the same for these two forcing notions, and we therefore have that every $\mathbb{P}$-generic ultrafilter contains a homogeneous set. (We have tacitly used the fact that forcing by $\mathbb{P}$ adds no new reals, so the pieces of the partition, being analytic or coanalytic, are the same before and after the forcing.) $\square$

Theorem 35 is the analog, in our non-P-point context, of part (2) of Proposition 2 for selective ultrafilters. We also have the following analog of part (3) of that proposition.

Theorem 37. *Suppose the universe is obtained from some ground model by Lévy-collapsing to ω all cardinals below some Mahlo cardinal of the ground model. Then the partition properties in Theorem 35 hold with $HOD\mathbb{R}$ in place of analytic.*

Proof. We can proceed as in the proof of Theorem 35 with only the following changes. Instead of citing Theorem 7 of [3], we cite Corollary 11.1, which asserts (among other things) that the Lévy-Mahlo model satisfies Theorem 7 with $HOD\mathbb{R}$ in place of analytic. Also, in the proof of part (1), it is no longer necessary to force to obtain the continuum hypothesis; the Lévy-Mahlo model satisfies the continuum hypothesis, so plenty of selective ultrafilters are available in it. □

Finally, we point out that part (3) of Theorem 37, the part about $\mathbb{P}$-generic ultrafilters admits an easy converse, which could be viewed as a sort of complete combinatorics.

Proposition 38. *Suppose the universe is obtained from some ground model by Lévy-collapsing to ω all cardinals below some Mahlo cardinal of the ground model. Suppose further that $\mathcal{W}$ is a non-P-point in standard position and that, for at least one ω-type τ, $\mathcal{W}$ has the following partition property. For any $HOD\mathbb{R}$ partition of $\mathbb{P}_\tau$ into two pieces, there is a set $H \in \mathcal{W} \cap \mathbb{P}_\tau$ such that all of its subsets in $\mathbb{P}_\tau$ lie in the same piece of the partition. Then $\mathcal{W}$ is $\mathbb{P}$-generic over $HOD\mathbb{R}$.*

Proof. Since $\mathbb{P}_\tau$ is dense in $\mathbb{P}$, it suffices to prove that $\mathcal{W}$ intersects every dense $HOD\mathbb{R}$ subset $\mathcal{D}$ of $\mathbb{P}_\tau$. Because $\mathcal{D}$ is dense, there cannot be any $H \in \mathbb{P}_\tau$ (whether in $\mathcal{W}$ or not) such that all its subsets in $\mathbb{P}_\tau$ are outside $\mathcal{D}$. By the assumed partition property of $\mathcal{W}$, we infer that there is $H \in \mathcal{W} \cap \mathbb{P}_\tau$ such that H lies in $\mathcal{D}$ (and so do all its subsets in $\mathbb{P}_\tau$, but we don't need this part of the result). So H witnesses that $\mathcal{W}$ meets $\mathcal{D}$, as required. □

Note that Proposition 38 needs to assume the partition property for only one ω-type τ. Genericity follows, and with genericity, the partition properties for all other ω-types also follow.

Proposition 38 is a partial analog, in our non-P-point context, of Corollary 3 for selective ultrafilters. A more complete analog would result from a positive answer to the following open problem.

Question 39. Suppose $\mathcal{W}$ is a non-P-point in standard position, and suppose it is $(2,4)$-weakly Ramsey (and therefore $(n, T(n))$-weakly Ramsey for all n by Corollary 31). Must it have the infinitary partition property in part (2) of Theorem 35? If, in addition, the universe is a Lévy-Mahlo model, must $\mathcal{W}$ have the corresponding partition property for $HOD\mathbb{R}$ partitions?

References

[1] Andreas Blass, "End extensions, conservative extensions, and the Rudin-Frolík ordering," *Trans. Amer. Math. Soc.* 225 (1977) 325–340.

[2] Andreas Blass, "A model-theoretic view of some special ultrafilters," in *Logic Colloquium '77,* ed. L. Pacholski and J. Paris, North-Holland Studies in Logic and Foundations of Mathematics 96 (1978) 79–90.

[3] Andreas Blass, "Selective ultrafilters and homogeneity," *Ann. Pure Appl. Logic* 38 (1988) 215–255.

[4] Andreas Blass, Natasha Dobrinen, and Dilip Raghavan, "The next best thing to a P-point," *J. Symbolic Logic* 80 (2015) 866–900.

[5] David Booth, "Ultrafilters on a countable set," *Ann. Math. Logic* 2 (1970) 1–24.

[6] Natasha Dobrinen, "High dimensional Ellentuck spaces and initial chains in the Tukey structure of non-p-points," to appear in *J. Symbolic Logic.* `http://web.cs.du.edu/dobrinen/Dobrinen_HighDimensionalEllentuckSpaces.pdf`

[7] Kenneth Kunen, "Weak P-points in $\mathbf{N}^*$," in *Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978)*, *Colloq. Math. Soc. János Bolyai* 23 (1980) 741–749.

[8] Adrian R. D. Mathias, "Happy Families," *Ann. Math. Logic* 12 (1977) 59–111.

[9] Robert G. Phillips, "Omitting types in arithmetic and conservative extensions," in *Victoria Symposium on Nonstandard Analysis,* ed. A. Hurd and P. Loeb, Springer-Verlag Lecture Notes in Math. 369 (1974) 195–202.

Mathematics Department, University of Michigan, Ann Arbor, MI 48109–1043, U.S.A.

E-mail address: `ablass@umich.edu`