

CREATURE FORCING AND TOPOLOGICAL RAMSEY SPACES

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Celebrating Alan Dow and his tours de force in set theory and topology

ABSTRACT. This article introduces a line of investigation into connections between creature forcings and topological Ramsey spaces. Three examples of pure candidates for creature forcings are shown to contain dense subsets which are actually topological Ramsey spaces. A new variant of the product tree Ramsey theorem is proved in order to obtain the pigeonhole principles for two of these examples.

1. INTRODUCTION

Connections between partition theorems and creature forcings have been known for some time. Partition theorems are used to establish various norm functions and to deduce forcing properties, for instance, properness. Conversely, creature forcings can give rise to new partition theorems, as was shown, for example, by Rosłanowski and Shelah in [13]. Todorćević pointed out to the author in 2008 that there are strong connections between creature forcings and topological Ramsey spaces deserving of a systematic investigation. The purpose of this note is to initiate this line of research and provide some tools for future investigations.

We show that the collections of pure candidates for three examples of creature forcings presented in [13] contain dense subsets which are actually topological Ramsey spaces (see Section 4). For two of these examples, the pigeonhole principles relies on a Ramsey theorem for unbounded finite products of finite sets, where exactly one of the sets in the product can be replaced with the collection of its k -sized subsets. This is proved in Theorem 3 in Section 3, building on work of DiPrisco, Llopis and Todorćević in [2]. The method of proof for Theorem 3 easily lends itself to generalizations, setting the stage for future work regarding more types of creature forcings, as well as possible density versions of Theorem 3 and variants in the vein of [15], in which Todorćević and Tyros proved the density version of Theorem 4.

There are several immediate benefits to showing that a forcing has a dense subset which is a topological Ramsey space. Most importantly, it puts at one's disposal all the strength of Ramsey theoretic machinery, including an abstract version of the Ellentuck Theorem, (see Theorem 2 below). This theorem yields infinitary partition relations for all subsets which have the property of Baire in the abstract Ellentuck topology. Second, by work of DiPrisco, Mijares, and Nieto in [3], in the presence of a supercompact cardinal, the generic ultrafilter forced by a topological Ramsey space, partially ordered by almost reduction, has complete combinatorics in over

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$L(\mathbb{R})$. Third, if an ultrafilter is forced by a topological Ramsey space, then the availability of the Ramsey machinery aids in finding the partition relations satisfied by the ultrafilter. In the examples considered here, each generic ultrafilter has a base set consisting of creatures. Fourth, having at one's disposal the Abstract Nash-Williams Theorem aids in proving canonical equivalence relations on fronts and barriers, in the vein of Pudlák and Rödl [10]. This in turn makes possible investigations of initial Rudin-Keisler and Tukey structures below these generic ultrafilters in the line of [11], [7], [8], [6], [4], and [5].

As this note merely forms the beginning of an investigation into connections between topological Ramsey spaces and creature forcings, we only include the basics of topological Ramsey spaces in Section 2 needed to understand the present work and refer the reader to Todorćević's book [14] for a more thorough background. Likewise, we do not attempt to adequately present background material on creature forcing, rather referring the reader to Rosłanowski and Shelah's book [12], and to their paper [13] for the examples included here.

On a personal note, I would like to thank Alan Dow for his inspiring and encouraging influence on my early and present mathematics. He and his work are truly exceptional. Happy Birthday, Alan!

2. BASICS OF TOPOLOGICAL RAMSEY SPACES

A brief review of topological Ramsey spaces is provided in this section for the reader's convenience. Building on prior work of Carlson and Simpson in [1], Todorćević distilled key properties of the Ellentuck space into four axioms, **A.1** - **A.4**, which guarantee that a space is a topological Ramsey space. (For further background, the reader is referred to Chapter 5 of [14].) The axioms **A.1** - **A.4** are defined for triples (\mathcal{R}, \leq, r) of objects with the following properties: \mathcal{R} is a nonempty set, \leq is a quasi-ordering on \mathcal{R} , and $r : \mathcal{R} \times \omega \rightarrow \mathcal{AR}$ is a map producing the sequence $(r_n(\cdot) = r(\cdot, n))$ of restriction maps, where \mathcal{AR} is the collection of all finite approximations to members of \mathcal{R} . For $u \in \mathcal{AR}$ and $X, Y \in \mathcal{R}$,

$$(1) \quad [u, X] = \{Y \in \mathcal{R} : Y \leq X \text{ and } (\exists n) r_n(Y) = u\}.$$

For $u \in \mathcal{AR}$, let $|u|$ denote the length of the sequence u . Thus, $|u|$ equals the integer k for which $u = r_k(u)$. For $u, v \in \mathcal{AR}$, $u \sqsubseteq v$ if and only if $u = r_m(v)$ for some $m \leq |v|$. $u \sqsubset v$ if and only if $u = r_m(v)$ for some $m < |v|$. For each $n < \omega$, $\mathcal{AR}_n = \{r_n(X) : X \in \mathcal{R}\}$.

- A.1** (1) $r_0(X) = \emptyset$ for all $X \in \mathcal{R}$.
(2) $X \neq Y$ implies $r_n(X) \neq r_n(Y)$ for some n .
(3) $r_m(X) = r_n(Y)$ implies $m = n$ and $r_k(X) = r_k(Y)$ for all $k < n$.

A.2 There is a quasi-ordering \leq_{fin} on \mathcal{AR} such that

- (1) $\{v \in \mathcal{AR} : v \leq_{\text{fin}} u\}$ is finite for all $u \in \mathcal{AR}$,
(2) $Y \leq X$ iff $(\forall n)(\exists m) r_n(Y) \leq_{\text{fin}} r_m(X)$,
(3) $\forall u, v, y \in \mathcal{AR} [y \sqsubset v \wedge v \leq_{\text{fin}} u \rightarrow \exists x \sqsubset u (y \leq_{\text{fin}} x)]$.

The number $\text{depth}_X(u)$ is the least n , if it exists, such that $u \leq_{\text{fin}} r_n(X)$. If such an n does not exist, then we write $\text{depth}_X(u) = \infty$. If $\text{depth}_X(u) = n < \infty$, then $[\text{depth}_X(u), X]$ denotes $[r_n(X), X]$.

- A.3** (1) If $\text{depth}_X(u) < \infty$ then $[u, Y] \neq \emptyset$ for all $Y \in [\text{depth}_X(u), X]$.
 (2) $Y \leq X$ and $[u, Y] \neq \emptyset$ imply that there is $Y' \in [\text{depth}_X(u), X]$ such that $\emptyset \neq [u, Y'] \subseteq [u, Y]$.

Additionally, for $n > |u|$, let $r_n[u, X]$ denote the collection of all $v \in \mathcal{AR}_n$ such that $u \sqsubset v$ and $v \leq_{\text{fin}} X$.

- A.4** If $\text{depth}_X(u) < \infty$ and if $\mathcal{O} \subseteq \mathcal{AR}_{|u|+1}$, then there is $Y \in [\text{depth}_X(u), X]$ such that $r_{|u|+1}[u, Y] \subseteq \mathcal{O}$ or $r_{|u|+1}[u, Y] \subseteq \mathcal{O}^c$.

The *Ellentuck topology* on \mathcal{R} is the topology generated by the basic open sets $[u, X]$; it refines the metric topology on \mathcal{R} , considered as a subspace of the Tychonoff cube $\mathcal{AR}^{\mathbb{N}}$. Given the Ellentuck topology on \mathcal{R} , the notions of nowhere dense, and hence of meager are defined in the natural way. We say that a subset \mathcal{X} of \mathcal{R} has the *property of Baire* iff $\mathcal{X} = \mathcal{O} \cap \mathcal{M}$ for some Ellentuck open set $\mathcal{O} \subseteq \mathcal{R}$ and Ellentuck meager set $\mathcal{M} \subseteq \mathcal{R}$.

Definition 1 ([14]). A subset \mathcal{X} of \mathcal{R} is *Ramsey* if for every $\emptyset \neq [u, X]$, there is a $Y \in [u, X]$ such that $[u, Y] \subseteq \mathcal{X}$ or $[u, Y] \cap \mathcal{X} = \emptyset$. $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey null* if for every $\emptyset \neq [u, X]$, there is a $Y \in [u, X]$ such that $[u, Y] \cap \mathcal{X} = \emptyset$.

A triple (\mathcal{R}, \leq, r) is a *topological Ramsey space* if every subset of \mathcal{R} with the property of Baire is Ramsey and if every meager subset of \mathcal{R} is Ramsey null.

The following result can be found as Theorem 5.4 in [14].

Theorem 2 (Abstract Ellentuck Theorem). *If (\mathcal{R}, \leq, r) is closed (as a subspace of $\mathcal{AR}^{\mathbb{N}}$) and satisfies axioms **A.1**, **A.2**, **A.3**, and **A.4**, then every subset of \mathcal{R} with the property of Baire is Ramsey, and every meager subset is Ramsey null; in other words, the triple (\mathcal{R}, \leq, r) forms a topological Ramsey space.*

3. A VARIANT OF THE PRODUCT TREE RAMSEY THEOREM

The main theorem of this section, Theorem 3, is a Ramsey theorem on unbounded finite products of finite sets. This is a variation of Theorem 4 below, due to Todorćević, with the strengthenings that exactly one of the entries K_p in each finite product is replaced with $[K_p]^k$ and p is allowed to vary over all numbers less than or equal to the length of the product, and the weakening that some of the chosen subsets may have cardinality one. The conclusion is what is needed to prove Axiom **A.4** for two of the examples of forcing with pure candidates in the next section; essentially, it is the pigeonhole principle for $r_k[k-1, \bar{t}]$, for \bar{t} in a particular dense subset of the creature forcing. The hypothesis of $|K_j| \geq j+1$ lends itself to our intended applications.

We point out that by minor adjustments in the proof, a true generalization of Theorem 4 can be proved, where some function R_k on all finite sequences of positive integers is found guaranteeing that each H_i may be taken of a prescribed size m_i , rather than requiring many of them to having size 1, and that $L = \omega$ (and N is some infinite subset of ω). As this is not our intended application, we leave its proof to the interested reader.

Throughout, for $p \leq n$, $[K_p]^k \times \prod_{j \in (n+1) \setminus \{p\}} K_j$ is used to denote

$$K_0 \times \cdots \times K_{p-1} \times [K_p]^k \times K_{p+1} \times \cdots \times K_n.$$

Theorem 3. *Given $k \geq 1$, a sequence of positive integers (m_0, m_1, \dots) , sets K_j , $j < \omega$ such that $|K_j| \geq j + 1$, and a coloring*

$$c : \bigcup_{n < \omega} \bigcup_{p \leq n} ([K_p]^k \times \prod_{j \in (n+1) \setminus \{p\}} K_j) \rightarrow 2,$$

there are infinite sets $L, N \subseteq \omega$ such that, enumerating L and N in increasing order, $l_0 \leq n_0 < l_1 \leq n_1 < \dots$, and there are subsets $H_j \subseteq K_j$, $j < \omega$, such that $|H_{l_i}| = m_i$ for each $i < \omega$, $|H_j| = 1$ for each $j \in \omega \setminus L$, and c is constant on

$$\bigcup_{n \in N} \bigcup_{l \in L \cap (n+1)} ([H_l]^k \times \prod_{j \in (n+1) \setminus \{l\}} H_j).$$

Theorem 3 is a variation of the following product tree Ramsey theorem, (Lemma 2.2 in [2] and Theorem 3.21 in [14]), which we now state since it will be used in the proof of Theorem 3.

Theorem 4 (DiPrisco-Llopis-Todorćevic, [2]). *There is an $R : [\omega \setminus \{0\}]^{<\omega} \rightarrow [1, \omega)$ such that for every infinite sequence $(m_i)_{i < \omega}$ of positive integers and for every coloring*

$$c : \bigcup_{n < \omega} \prod_{i \leq n} R(m_0, \dots, m_i) \rightarrow 2,$$

there exist $H_i \subseteq R(m_0, \dots, m_i)$, $|H_i| = m_i$, for $i < \omega$, such that c is constant on the product

$$\prod_{i \leq k} H_i$$

for infinitely many $n < \omega$.

The proof of Theorem 3 closely follows the proof of Theorem 4 as presented in [14]. It will follow from Corollary 8 (proved via Lemmas 5 and 6 and Theorem 7) along with a final application of Theorem 4.

The following lemma and its proof are minor modifications of Lemma 2.1 in [2] (see also Lemma 3.20 in [14]), the only difference being the use of $[H_0]^k$ in place of H_0 . We make the notational convention that for $n = 0$, $[H_0]^k \times \prod_{j=1}^n H_j$ denotes $[H_0]^k$.

Lemma 5. *For any given $k \geq 1$ and sequence $(m_j)_{j < \omega}$ of positive integers, there are numbers $S_k(m_0, \dots, m_j)$ such that for any $n < \omega$ and any coloring*

$$c : [S_k(m_0)]^k \times \prod_{j=1}^n S_k(m_0, \dots, m_j) \rightarrow 2,$$

there are sets $H_j \subseteq S_k(m_0, \dots, m_j)$, $j \leq n$, such that $|H_j| = m_j$ and c is monochromatic on $[H_0]^k \times \prod_{j=1}^n H_j$.

Proof. Let $S_k(m_0)$ be the least number r such that $r \rightarrow (m_0)_2^k$. This satisfies the lemma when $n = 0$. Now suppose that $n \geq 1$ and the numbers $S_k(m_0, \dots, m_j)$, $j < n$, have been obtained satisfying the lemma. Let N denote the number $|[S_k(m_0)]^k| \cdot S_k(m_0, m_1) \cdots S_k(m_0, \dots, m_{n-1})$, and let $S_k(m_0, \dots, m_n) = m_n \cdot 2^N$. Given a coloring $c : [S_k(m_0)]^k \times \prod_{j=1}^n S_k(m_0, \dots, m_j) \rightarrow 2$, for each $t \in [S_k(m_0)]^k \times \prod_{j=1}^{n-1} S_k(m_0, \dots, m_j)$, let c_t denote the coloring on $S_k(m_0, \dots, m_n)$ given by $c_t(x) = c(\widehat{t}x)$, for $x \in S_k(m_0, \dots, m_n)$. Let $\langle t_i : i < N \rangle$ be an enumeration of the members

of $[S_k(m_0)]^k \times \prod_{j=1}^{n-1} S_k(m_0, \dots, m_j)$, and let $K_0 = S_k(m_0, \dots, m_n)$. Given $i < N$ and K_i , take $K_{i+1} \subseteq K_i$ of cardinality $m_n \cdot 2^{N-(i+1)}$ such that c_{t_i} is constant on $\{t_i \frown x : x \in K_{i+1}\}$. By induction on $i < N$, we obtain $K_N \subseteq S_k(m_0, \dots, m_n)$ of size m_n such that for each $i < N$, c_{t_i} is constant on K_N . Let $H_n = K_N$. Now let c' be the coloring on $[S_k(m_0)]^k \times \prod_{j=1}^{n-1} S_k(m_0, \dots, m_j)$ given by $c'(t) = c(t \frown x)$, for any (every) $x \in H_n$. By the induction hypothesis, there are $H_j \subseteq S_k(m_0, \dots, m_j)$ of cardinality m_j , $j < n$, such that c' is constant on $[H_0]^k \times \prod_{j=1}^{n-1} H_j$. Then c is constant on $[H_0]^k \times \prod_{j=1}^n H_j$. \square

Remark. It follows from $k \geq 2$ that each $S_k(m_0, \dots, m_j)$ must be greater than the number $S(m_0, \dots, m_j)$ from Lemma 3.20 in [14]. The case $k = 1$ is simply a re-statement of Lemma 3.20 in [14].

Given $k \geq 1$ and $M \in [\omega]^\omega$, letting $\{m_i : i < \omega\}$ be the increasing enumeration of M , the notation $M_o \xrightarrow{k} M_e$ means that for each 2-coloring $c : \bigcup_{n < \omega} ([m_1]^k \times \prod_{j=1}^n m_{2j+1}) \rightarrow 2$, there are $H_j \subseteq m_{2j+1}$ such that $|H_j| = m_{2j}$ and c is constant on $[H_0]^k \times \prod_{j=1}^n H_j$ for infinitely many n . The following lemma and its proof are almost identical with those of Lemma 3.18 in [14], the only changes being the substitution of $\bigcup_{n < \omega} ([\omega]^k \times \omega^{n-1})$ for the domain of the function c in place of $\omega^{<\omega}$, the substitution of $[\omega]^k$ for one of the copies of ω , and an application of Lemma 5 in place of the application of Lemma 3.20 in [14]. Thus, we omit its proof.

Lemma 6. *For each $k \geq 1$, there is an infinite subset $N \subseteq \omega$ such that $M_o \xrightarrow{k} M_e$ for each $M \in [N]^\omega$.*

The next theorem is a slight generalization of Theorem 4, replacing $R(m_0)$ there with $[R_k(m_0)]^k$; its proof is almost exactly the same, merely replacing an instance of Lemma 3.18 in [14] with Lemma 6.

Theorem 7. *Given $k \geq 1$, there is a function $R_k : [\omega \setminus \{0\}]^{<\omega} \rightarrow [1, \omega)$ such that for each sequence $(m_k)_{k < \omega}$ of positive integers for each coloring*

$$c : \bigcup_{n < \omega} [R_k(m_0)]^k \times \prod_{j=1}^n R_k(m_0, \dots, m_j) \rightarrow 2,$$

there are subsets $H_j \subseteq R_k(m_0, \dots, m_j)$ such that $|H_j| = m_j$ and c is monochromatic on $[H_0]^k \times \prod_{j=1}^n H_j$, for infinitely many n .

Proof. Pick an infinite subset $N = (n_j)_{j < \omega}$ of positive integers enumerated in increasing order and satisfying Lemma 6. Set

$$R_k(m_0, \dots, m_j) = n_{2(\sum_{i=0}^j m_i)+1}.$$

Then for every infinite sequence $(m_i)_{i < \omega}$ of positive integers, if we let

$$P = \{n_{2(\sum_{i=0}^j m_i)+\varepsilon} : j \in \omega, \varepsilon < 2\},$$

we get an infinite subset P of N such that $P_o = (R_k(m_0, \dots, m_i))_{i < \omega}$, while the sequence P_e pointwise dominates our given sequence $(m_i)_{i < \omega}$. By our choice of N , it follows that $P_o \xrightarrow{k} P_e$. P_o supplies the infinitely many levels of n satisfying the theorem. \square

The following corollary forms the basis of the proof of Theorem 3 below.

Corollary 8. *Let L, N be infinite subsets of ω such that $l_0 \leq n_0 < l_1 \leq n_1 < \dots$. Let $k \geq 1$, $m_0 \geq 1$, and K_j , $j \geq l_0$, be nonempty sets with $|K_{l_0}| = R_k(m_0)$, $|K_{l_i}| \geq i$ for each $i \geq 1$, and $|K_j| = 1$ for each $j \in (l_0, \omega) \setminus L$. Then for each coloring*

$$c: \bigcup_{n \in N} ([K_{l_0}]^k \times \prod_{j \in (l_0, n]} K_j) \rightarrow 2,$$

and each $r < \omega$, there are infinite $L' \subseteq L$, $N' \subseteq N$ with $l'_0 = l_0 \leq n'_0 < l'_1 \leq n'_1 < \dots$, and there are $H_j \subseteq K_j$ such that $|H_{l_0}| = m_0$, $|H_{l'_i}| = r + i$ for each $i \geq 1$, $|H_j| = 1$ for each $j \in (l_0, \omega) \setminus L'$, and c is constant on

$$\bigcup_{n \in N'} ([H_{l_0}]^k \times \prod_{j \in (l_0, n]} H_j).$$

Proof. Let $i_0 = 0$, and let $r < \omega$ be fixed. For each $p \geq 1$, take i_p a strictly increasing sequence so that $|K_{l_{i_p}}| \geq R_k(m_0, r+1 \dots, r+p)$. For each $j \in (l_0, \omega) \setminus \{l_{i_p} : p \geq 1\}$, take $H_j \subseteq K_j$ of size one. Then the coloring c on

$$\bigcup_{n \in N} [K_{l_0}]^k \times \prod \{K_{l_{i_p}} : p \geq 1 \text{ and } l_{i_p} \leq n\} \times \prod \{H_j : j \in (l_0, n] \setminus \{l_{i_q} : q \geq 1\}\}$$

induces a coloring c' on $\bigcup_{p < \omega} [J_0]^k \times \prod_{q=1}^p J_q$, where $J_q = K_{l_{i_q}}$, as follows: For $p < \omega$ and $(X_0, x_1, \dots, x_p) \in [J_0]^k \times \prod_{q=1}^p J_q$, letting $Y_{l_0} = X_0$, $y_{l_{i_q}} = x_q$, and for each $j \in (l_0, n_{i_p}] \setminus \{l_{i_q} : q \leq p\}$ letting y_j denote the member of H_j , we define $c'(X_0, x_1, \dots, x_p) = c(Y_{l_0}, y_{l_0+1}, \dots, y_{n_{i_p}})$. Apply Theorem 7 to c' to obtain $H_{l_0} \in [K_{l_0}]^{m_0}$, subsets $H_{l_{i_p}} \in [K_{l_{i_p}}]^{r+p}$ for each $p \geq 1$, and an infinite set P such that c' is constant on $\bigcup_{p \in P} [H_{l_0}]^k \times \prod_{1 \leq q \leq p} H_{l_{i_q}}$. Then letting $N' = \{n_{i_p} : p \in P\}$, c is constant on $\bigcup_{n \in N'} [H_{l_0}]^k \times \prod_{l_0 < j \leq n} H_j$. Letting $L' = \{l_{i_p} : p \in P\}$ finishes the proof. \square

Now we are equipped to prove Theorem 3.

Proof of Theorem 3. Take l_0 least such that $|K_{l_0}| \geq R_k(R(m_0))$, and let $L_0 = N_0 = [l_0, \omega)$. For each $j < l_0$, take some $H_j \in [K_j]^1$ and let $h \upharpoonright l_0$ denote $\prod_{j < l_0} H_j$. Then c restricted to $\bigcup_{n \in N_0} (h \upharpoonright l_0) \times [K_{l_0}]^k \times \prod_{j \in (l_0, n]} K_j$ induces a 2-coloring on $\bigcup_{n \in N_0} [K_{l_0}]^k \times \prod_{j \in (l_0, n]} K_j$. By Corollary 8, there are infinite $L'_0 \subseteq L_0$ and $N'_0 \subseteq N_0$ such that $l_0 = l'_0 \leq n'_0 < l'_1 \leq n'_1 < \dots$, and there are subsets $H_j^0 \subseteq K_j$, $j \geq l_0$, such that $|H_{l_0}^0| = R(m_0)$, $|H_{l'_i}^0| = i$ for each $i \geq 1$, $|H_j^0| = 1$ for each $j \in \omega \setminus L'_0$, and c is constant on $\bigcup_{n \in N'_0} (h \upharpoonright l_0) \times [H_{l_0}^0]^k \times \prod_{j \in (l_0, n]} H_j^0$.

Let $H_{l_0} = H_{l_0}^0$, and let $n_0 = \min(N'_0)$. Then $n_0 \geq l_0$. Let $R_k^1(m)$ denote $R_k(m)$ and in general, let $R_k^{i+1}(m)$ denote $R_k(R_k^i(m))$. Fix an $l_1 \in L'_0$ such that $l_1 > n_0$ and $|H_{l_1}^0| \geq R_k^{R(m_0)}(R(m_0, m_1))$. For $j \in (l_0, l_1)$, fix some $H_j \in [H_j^0]^1$, and let $h \upharpoonright l_1$ denote $\prod_{j \in l_1 \setminus \{l_0\}} H_j$. Enumerate H_{l_0} as $\{h_{l_0}^i : i < m_0\}$. Successively apply Corollary 8 $R(m_0)$ times to obtain $L_1 \subseteq L'_0$ with $\min(L_1) = l_1$, $N_1 \subseteq N'_0$, $H_{l_1} \subseteq K_{l_1}$ of cardinality $R(m_0, m_1)$, and subsets $H_j^1 \subseteq K_j$ for $j \in [l_1, \omega)$, such that listing L_1 as $l_1 = l_1^1 < l_2^1 < \dots$ we have $|H_{l_i^1}^1| \geq i$ and satisfying the following: For each fixed $h_{l_0}^* \in H_{l_0}$, the coloring c is constant on

$$\bigcup_{n \in N_1} (h \upharpoonright l_1) \times \{h_{l_0}^*\} \times [H_{l_1}^1]^k \times \prod_{j \in (l_1, n]} H_j^1.$$

In general, suppose for $p \geq 1$, we have fixed $l_0 \leq n_0 < \dots < l_p \leq n_p$, and chosen infinite sets L_p, N_p with $l_p = \min(L_p)$ and $l_p = l_p^p \leq n_p^p < l_{p+1}^p \leq n_{p+1}^p < \dots$ and sets $H_j \subseteq K_j$ for $j \leq l_p$ and sets $H_j^p \subseteq K_j$ for $j > l_p$ such that the following hold:

- (1) for each $i \leq p$, $|H_{l_i}| = R(m_0, \dots, m_i)$,
- (2) for each $l \in l_p \setminus \{l_i : i < p\}$, $|H_l| = 1$,
- (3) for each $i > p$, $|H_{l_i^p}^p| \geq i$,
- (4) and for each $j \in (l_p, \omega) \setminus L_p$, $|H_j^p| = 1$.

Let $h \upharpoonright l_p$ denote $\prod_{j \in l_p \setminus \{l_0, \dots, l_{p-1}\}} H_j$, which is a product of singletons. By our construction so far, we have ensured that for each sequence $\bar{x} \in \prod_{i < p} H_{l_i}$, c is constant on

$$\bigcup_{n \in N_p} h \upharpoonright l_p \times \bar{x} \times [H_{l_p}]^k \times \prod_{j \in (l_p, n]} H_j^p.$$

Let $n(p) = |\prod_{i \leq p} H_{l_i}|$. Fix $n_p \in N_p$ such that $n_p \geq l_p$ and take $l_{p+1} \in L_p$ such that $l_{p+1} > n_p$ and $|H_{l_{p+1}}^p| = R_k^{n(p)}(R(m_0, \dots, m_{p+1}))$. After $n(p)$ successive applications of Corollary 8, we obtain $L_{p+1} \subseteq L_p$ and $N_{p+1} \subseteq N_p$ with $\min(L_{p+1}) = l_{p+1}$, subsets $H_j \subseteq K_j$ for $j \in (l_p, l_{p+1}]$ and sets $H_j^{p+1} \subseteq H_j^p$ for $j > l_{p+1}$ such that the following hold:

- (1) $|H_{l_{p+1}}| = R(m_0, \dots, m_{p+1})$,
- (2) for each $l \in (l_p, l_{p+1})$, $|H_l| = 1$,
- (3) for each $i > p+1$, $|H_{l_i^{p+1}}^{p+1}| \geq i$,
- (4) and for each $j \in (l_{p+1}, \omega) \setminus L_{p+1}$, $|H_j^{p+1}| = 1$;

and moreover, letting $n_{p+1} = \min(N_{p+1})$ and letting $h \upharpoonright l_{p+1} = \prod_{j \in l_{p+1} \setminus \{l_0, \dots, l_p\}} H_j$, for each $\bar{x} \in \prod_{i \leq p} H_{l_i}$, c is constant on

$$\bigcup_{n \in N_{p+1}} (h \upharpoonright l_{p+1}) \times \bar{x} \times [H_{l_{p+1}}]^k \times \prod_{j \in (l_{p+1}, n]} H_j^{p+1}.$$

Then fix an $n_{p+1} \in N_{p+1}$ such that $n_{p+1} \geq l_{p+1}$.

In this manner, we obtain $L = \{l_i : i < \omega\}$ and $N = \{n_i : i < \omega\}$ such that $l_0 \leq n_0 < l_1 \leq n_1 < l_2 \leq n_2 < \dots$, and $H_j \subseteq K_j$, $j < \omega$, such that $|H_{l_i}| = R(m_0, \dots, m_i)$ for each $i < \omega$, $|H_j| = 1$ for each $j \in \omega \setminus L$, and for each $p < \omega$, for each $\bar{x} \in \prod_{i \leq p} H_{l_i}$, c is constant on

$$\bigcup_{n \in N \cap [l_p, \omega)} (h \upharpoonright l_p) \times \bar{x} \times [H_{l_p}]^k \times \prod_{j \in (l_p, n]} H_j.$$

Defining $c'(\bar{x})$ to be this constant color induces a 2-coloring on $\bigcup_{p \in \omega} \prod_{i \leq p} H_{l_i}$. Since each $|H_{l_i}| = R(m_0, \dots, m_i)$, we may apply Theorem 4 to obtain $H_{l_i}^* \subseteq H_{l_i}$ of cardinality m_i and an infinite subset $N^* \subseteq N$ such that \bar{c} is constant on $\bigcup_{n \in N^*} \prod_{i \leq n} H_{l_i}^*$. Then letting $H_j^* = H_j$ for $j \notin L$, and letting L^* be any subset of $\{l_i : i < \omega\}$ such that $l_0^* \leq n_0^* < l_1^* \leq n_1^* < \dots$, c is constant on

$$\bigcup_{n \in N^*} \bigcup_{l \in L^* \cap (n+1)} [H_l^*]^k \times \prod_{j \in (n+1) \setminus \{l\}} H_j^*.$$

□

4. TOPOLOGICAL RAMSEY SPACES AS DENSE SUBSETS OF THREE EXAMPLES OF CREATURE FORCINGS

In [13], Roslanowski and Shelah provided four specific examples of pure candidates for creature forcings, Examples 2.10 - 2.13, to which their theory of partition theorems in [13] is applied. In this section, we show that for three of these examples, the collections of pure candidates contain dense subsets which form topological Ramsey spaces. Theorem 3 is the essence of the pigeonhole principle, axiom **A.4**, for two of these examples, and the Hales-Jewett Theorem provides **A.4** for the last example. For more background on creature forcing, the reader is referred to [12].

To show that a dense subset of a collection of pure candidates \bar{t} forms a topological Ramsey space, it suffices by the Abstract Ellentuck Theorem 2 to define a notion of k -th approximation of \bar{t} and a quasi-ordering \leq_{fin} on the collection of finite approximations (in our cases this will be a partial ordering), and then prove that the Axioms **A.1** - **A.4** hold. In each of the examples below, given a creating pair (K, Σ) , we shall form a dense subset of the pure candidates, call it $\mathcal{R}(K, \Sigma)$, partially ordered by the partial ordering inherited from the collection of all pure candidates. For each $\bar{t} = (t_0, t_1, \dots) \in \mathcal{R}(K, \Sigma)$, for $k < \omega$, we let $r_k(\bar{t}) = (t_i : i < k)$. Thus, $r_0(\bar{t})$ is the empty sequence, and $r_1(\bar{t}) = (t_0)$, which is a sequence of length one containing exactly one member of K . The partial ordering \leq_{fin} on $\mathcal{AR} = \{r_n(\bar{t}) : \bar{t} \in \mathcal{R}(K, \Sigma), n < \omega\}$ be defined by $(t_0, \dots, t_{n-1}) \leq_{\text{fin}} (s_0, \dots, s_{m-1})$ if and only if there is a strictly increasing sequence $0 = j_0 < \dots < j_n = m - 1$ such that for each $i < n$, $t_i \in \Sigma((s_l : j_i \leq l < j_{i+1}))$, (or $\Sigma^*((s_l : j_i \leq l < j_{i+1}))$, as appropriate). The basic open sets in the Ellentuck topology are then defined as $[u, \bar{t}] = \{\bar{s} \in \mathcal{R}(K, \Sigma) : \exists n (r_n(\bar{s}) = u) \text{ and } \bar{s} \leq \bar{t}\}$, for $u \in \mathcal{AR}$ and $\bar{t} \in \mathcal{R}(K, \Sigma)$, with $u \leq_{\text{fin}} \bar{t}$.

Given this set-up, it is clear that **A.1** holds. Since given $(s_0, \dots, s_{m-1}) \in \mathcal{AR}$, $\Sigma(s_0, \dots, s_{m-1})$ and $\Sigma^*(s_0, \dots, s_{m-1})$ are finite, **A.2** (1) holds. **A.2** (2) is simply the definition of the partial ordering \leq when restricted to $\mathcal{R}(K, \Sigma)$, and **A.2** (3) is straightforward to check, using the definition of \leq_{fin} . **A.3** follows from the definition of Σ (or Σ^*). Thus, showing that the pigeonhole principle **A.4** holds for these examples is the main focus of this section.

Example 2.10 in [13]. Let $\mathbf{H}_1(n) = n + 1$ for $n < \omega$ and let K_1 consist of all FP creatures t for \mathbf{H}_1 such that

- **dis** $[t] = (u, i, A) = (u^t, i^t, A^t)$, where $u \subseteq [m_{\text{dn}}^t, m_{\text{up}}^t)$, $i \in u$, $\emptyset \neq A \subseteq \mathbf{H}_1(i)$,
- **nor** $[t] = \log_2(|A|)$,
- **val** $[t] \subseteq \prod_{j \in u} \mathbf{H}_1(j)$ is such that $\{f(i) : f \in \mathbf{val}[t]\} = A$.

For $t_0, \dots, t_n \in K_1$ with $m_{\text{up}}^{t_i} = m_{\text{dn}}^{t_{i+1}}$, let $\Sigma_1^*(t_0, \dots, t_n)$ consist of all creatures $t \in K_1$ such that

$$m_{\text{dn}}^t = m_{\text{dn}}^{t_0}, \quad m_{\text{up}}^t = m_{\text{up}}^{t_n}, \quad u^t = \bigcup_{l \leq n} u^{t_l}, \quad i^t = i^{t_{l^*}}, \quad A^t \subseteq A^{t_{l^*}} \text{ for some } l^* \leq n,$$

and $\mathbf{val}[t] \subseteq \{f_0 \cup \dots \cup f_n : (f_0, \dots, f_n) \in \mathbf{val}[t_0] \times \dots \times \mathbf{val}[t_n]\}$. Roslanowski and Shelah proved that (K_1, Σ_1^*) is a tight FFCC pair with bigness and t -multiadditivity, and is gluing on every $\bar{t} \in \text{PC}_{\infty}^{\text{tt}}(K_1, \Sigma_1^*)$. The partial ordering \leq on $\text{PC}_{\infty}^{\text{tt}}(K_1, \Sigma_1^*)$ is defined by $\bar{t} \leq \bar{s}$ if and only if there is a strictly increasing sequence $(j_n)_{n < \omega}$, with $j_0 = 0$, such that each $t_n \in \Sigma_1^*(s_{j_n}, \dots, s_{j_{n+1}-1})$.

Remark. Here, \bar{t} is the stronger condition. This reversal of the partial order notation of Roslanowski and Shelah is better suited to the topological Ramsey space framework.

First define \mathcal{S}_1 to consist of those $\bar{t} \in \text{PC}_\infty^{\text{tt}}(K_1, \Sigma_1^*)$ such that for each $l < \omega$, there is a function $g^{t_l} \in \prod_{j \in u_l \setminus \{i^{t_l}\}} \mathbf{H}_1(j)$ such that for each $f \in \mathbf{val}[t_l]$, $f \upharpoonright (u_l \setminus \{i^{t_l}\}) = g^{t_l}$. In other words, $\mathbf{val}[t_l] = \{g^{t_l} \cup \{(i^{t_l}, k)\} : k \in A^{t_l}\}$. Notice that \mathcal{S}_1 is dense in $\text{PC}_\infty^{\text{tt}}(K_1, \Sigma_1^*)$. Next, we define a dense subset of \mathcal{S}_1 , which we shall prove is a topological Ramsey space.

Definition 9 (The space $\mathcal{R}(\text{PC}_\infty^{\text{tt}}(K_1, \Sigma_1^*), \leq, r)$). Let $\mathcal{R}(\text{PC}_\infty^{\text{tt}}(K_1, \Sigma_1^*))$ consist of those members $\bar{s} \in \mathcal{S}_1$ such that for each $l < \omega$, $|A^{t_l}| = l+1$. For each $k < \omega$ and $\bar{s} = (s_0, s_1, \dots) \in \mathcal{R}(\text{PC}_\infty^{\text{tt}}(K_1, \Sigma_1^*))$, the k -th restriction of \bar{s} is $r_k(\bar{s}) = (s_0, \dots, s_{k-1})$. For $(s_0, \dots, s_{k-1}), (t_0, \dots, t_{m-1}) \in \mathcal{AR}$, define $(t_0, \dots, t_{m-1}) \leq_{\text{fin}} (s_0, \dots, s_{k-1})$ if and only if there is a strictly increasing sequence $(j_n)_{n \leq m}$, with $j_0 = 0$, such that for each $n < m$, $t_n \in \Sigma_1^*(s_{j_n}, \dots, s_{j_{n+1}-1})$.

Theorem 10. $\mathcal{R}(\text{PC}_\infty^{\text{tt}}(K_1, \Sigma_1^*), \leq, r)$ is a topological Ramsey space which is dense in the partial ordering of all tight pure candidates $\text{PC}_\infty^{\text{tt}}(K_1, \Sigma_1^*)$.

Proof. First, abbreviate $\mathcal{R}(\text{PC}_\infty^{\text{tt}}(K_1, \Sigma_1^*))$ as $\mathcal{R}^{\text{tt}}(K_1, \Sigma_1^*)$. The space $\mathcal{R}^{\text{tt}}(K_1, \Sigma_1^*)$ is dense in \mathcal{S}_1 and hence is dense in $\text{PC}_\infty^{\text{tt}}(K_1, \Sigma_1^*)$. Towards showing **A.4** holds, let $k \geq 1$ be fixed and let \bar{t} be a given member of $\mathcal{R}_1^{\text{tt}}$. Let $c : r_k[k-1, \bar{t}] \rightarrow 2$ be a given coloring. We shall show that there is a $\bar{u} \in [k-1, \bar{t}]$ such that c is constant on $r_k[k-1, \bar{u}]$.

Notice that each $\bar{x} \in r_k[k-1, \bar{t}]$ is of the form $\bar{x} = (t_0, \dots, t_{k-2}, x_{k-1})$, with $x_{k-1} \in \Sigma_1^*(t_{k-1}, \dots, t_n)$ for some $n \geq k-1$, and $i^{x_{k-1}} = i^{t_l}$ and $A^{x_{k-1}} \in [A^{t_l}]^k$, for some $l \in [k-1, n]$. Thus, x_{k-1} is completely determined by the triple (n, l, A) , where $n \geq k-1$, $l \in [k-1, n]$, and $A = A^{x_{k-1}} \in [A^{t_l}]^k$. Therefore, c induces a coloring on

$$\bigcup_{k-1 \leq l \leq n} [A^{t_l}]^k \times \prod_{j \in (n+1) \setminus \{l\}} A^{t_j}.$$

Apply Theorem 3 to the sequence of sets A^{t_j} , $j \geq k-1$ to obtain infinite sets L, N and $H_j \subseteq A^{t_j}$ such that $k-1 \leq l_0 \leq n_0 < l_1 \leq n_1 < \dots$, and for each $p < \omega$, $|H_{l_p}| = k+p$, and for each $j \in \omega \setminus L$, $|H_j| = 1$; and moreover, c is constant on

$$\bigcup_{n \in N} \bigcup_{l \in L \cap (n+1)} [H_l]^k \times \prod_{j \in (n+1) \setminus \{l\}} H_j.$$

Take $\bar{s} \in \mathcal{R}_1^{\text{tt}}$ as follows: $(s_0, \dots, s_{k-2}) = r_{k-1}(\bar{t})$. s_{k-1} is given by $m_{\text{dn}}^{s_{k-1}} = m_{\text{dn}}^{t_{k-1}}$ and $m_{\text{up}}^{s_{k-1}} = m_{\text{up}}^{t_{k-1}}$, $i^{s_{k-1}} = i^{t_{k-1}}$ and $A^{s_{k-1}} = H_{l_{k-1}}$, and $g^{s_{k-1}} = \prod_{j \in u^{s_{k-1}} \setminus \{l_{k-1}\}} H_j$. In general, for $p \geq k$, take $s_p \in \Sigma_1^*(t_{n_{p-1}+1}, \dots, t_{n_p})$ such that $i^{s_p} = i^{t_{l_p}}$, $A^{s_p} = H_{l_p}$, $m_{\text{dn}}^{s_p} = m_{\text{dn}}^{t_{n_{p-1}+1}}$ and $m_{\text{up}}^{s_p} = m_{\text{up}}^{t_{n_p}}$, and $g^{s_p} = \prod_{j \in u^{s_p} \setminus \{l_p\}} H_j$. This defines $\bar{s} \leq \bar{t}$ in $\mathcal{R}_1^{\text{tt}}$ with $r_{k-1}(\bar{s}) = r_{k-1}(\bar{t})$ such that the coloring c is constant on $r_k[k-1, \bar{s}]$. Thus, **A.4** holds, and therefore, by the Abstract Ellentuck Theorem 2 and earlier remarks, $(\mathcal{R}^{\text{tt}}(K_1, \Sigma_1^*), \leq, r)$ is a topological Ramsey space. \square

Example 2.11 in [13]. Let $\mathbf{H}_2(n) = 2$ for $n < \omega$ and let K_2 consist of all FP creatures t for \mathbf{H}_2 such that

- $\emptyset \neq \mathbf{dis}[t] \subseteq [m_{\text{dn}}^t, m_{\text{up}}^t)$,

- $\emptyset \neq \mathbf{val}[t] \subseteq \mathbf{dis}[t]_2$,
- $\mathbf{nor}[t] = \log_2(|\mathbf{val}[t]|)$.

For $t_0, \dots, t_n \in K_2$ with $m_{\text{up}}^{t_l} \leq m_{\text{dn}}^{t_{l+1}}$, let $\Sigma_2(t_0, \dots, t_n)$ consist of all creatures $t \in K_2$ such that

$$m_{\text{dn}}^t = m_{\text{dn}}^{t_0}, m_{\text{up}}^t = m_{\text{up}}^{t_n}, \mathbf{dis}[t] = \mathbf{dis}[t_{l^*}], \text{ and } \mathbf{val}[t] \subseteq \mathbf{val}[t_{l^*}], \text{ for some } l^* \leq n.$$

The partial ordering \leq on $\text{PC}_\infty(K_2, \Sigma_2)$ is defined by $\bar{t} \leq \bar{s}$ if and only if there is a strictly increasing sequence $(j_n)_{n < \omega}$, $j_0 = 0$, such that each $t_n \in \Sigma_2(s_{j_n}, \dots, s_{j_{n+1}-1})$. Roslanowski and Shelah proved that (K_2, Σ_2) is a loose FFCC pair for \mathbf{H}_2 which is simple except omitting and has bigness. For this example, the tight and loose versions are the same, so we shall drop the \mathbf{tt} superscript.

Definition 11 (The space $(\mathcal{R}(\text{PC}_\infty(K_2, \Sigma_2)), \leq, r)$). Let

$$\mathcal{R}(\text{PC}_\infty(K_2, \Sigma_2)) = \{\bar{s} \in \text{PC}_\infty(K_2, \Sigma_2) : \forall l < \omega, |\mathbf{val}[t_l]| = l + 1\},$$

with its inherited partial ordering. Abbreviate this space by $\mathcal{R}(K_2, \Sigma_2)$. For each $k < \omega$ and $\bar{s} = (s_0, s_1, \dots) \in \mathcal{R}(K_2, \Sigma_2)$, the k -th restriction of \bar{s} is simply $r_k(\bar{s}) = (s_0, \dots, s_{k-1})$. For $(s_0, \dots, s_{k-1}), (t_0, \dots, t_{m-1}) \in \mathcal{AR}$, define $(t_0, \dots, t_{m-1}) \leq_{\text{fin}} (s_0, \dots, s_{k-1})$ if and only if there is a strictly increasing sequence $(j_n)_{n \leq m}$, with $j_0 = 0$, such that for each $n < m$, $t_n \in \Sigma_2(s_{j_n}, \dots, s_{j_{n+1}-1})$.

Theorem 12. $(\mathcal{R}(\text{PC}_\infty(K_2, \Sigma_2)), \leq, r)$ is a topological Ramsey space which is dense in the partial ordering of all pure candidates $\text{PC}_\infty(K_2, \Sigma_2)$.

Proof. It is clear that $\mathcal{R}(K_2, \Sigma_2)$ forms a dense subset of $\text{PC}_\infty(K_2, \Sigma_2)$. Towards proving that **A.4** holds, let $k \geq 1$ be fixed, $\bar{t} \in \mathcal{R}(K_2, \Sigma_2)$, and $c : r_k[k-1, \bar{t}] \rightarrow 2$ be a given coloring. Each $\bar{x} \in r_k[k-1, \bar{t}]$ is of the form $\bar{x} = (t_0, \dots, t_{k-2}, x_{k-1})$, with $x_{k-1} \in \Sigma_2(t_{k-1}, \dots, t_n)$ for some $n \geq k-1$, $\mathbf{dis}[x_{k-1}] = \mathbf{dis}[t_l]$ for some $l \in [k-1, n]$, and $\mathbf{val}[x_{k-1}] \subseteq \mathbf{val}[t_l]$. Hence, x_{k-1} is completely determined by the triple $(n, l, \mathbf{val}[x_{k-1}])$, so we may regard c as a coloring of triples from

$$\{(n, l, K) : k-1 \leq l \leq n, \text{ and } K \in [\mathbf{val}[t_l]]^k\}.$$

Letting $K_l = \mathbf{val}[t_l]$, we see that c induces a coloring c' on

$$\bigcup_{k-1 \leq l \leq n} [K_l]^k \times \prod \{K_j : k-1 \leq j \leq n+1, j \neq l\},$$

as follows: For $k-1 \leq l \leq n$, any $p_j \in K_j$ ($j \neq l$) and $J_l \in [K_l]^k$, define

$$c'(p_{k-1}, \dots, p_{l-1}, J_l, p_{l+1}, \dots, p_n) = c(n, l, J_l).$$

By Theorem 3, we obtain infinite sets $L = \{l_p : p \geq k-1\}$, $N = \{n_p : p \geq k-1\}$ such that $k-1 \leq l_{k-1} := \min(L) \leq n_{k-1} < l_k \leq n_k < \dots$, and subsets $H_j \subseteq K_j$ such that for each $p < \omega$, $|H_{l_p}| = k+p$, and for each $j \in \omega \setminus L$, $|H_j| = 1$, and moreover, c' is constant on

$$\bigcup_{n \in N} \bigcup_{l \in L \cap (n+1)} [H_l]^k \times \prod \{H_j : k-1 \leq j \leq n+1, j \neq l\}.$$

Take $\bar{s} \in \mathcal{R}(K_2, \Sigma_2)$ such that $(s_0, \dots, s_{k-2}) = r_{k-1}(\bar{t})$; and for each $p \geq k-1$, (letting n_{k-2} denote $m_{\text{dn}}^{t_{k-2}}$), s_p is the creature determined by $m_{\text{dn}}^{s_p} = m_{\text{dn}}^{t_{n_{p-1}+1}}$, $m_{\text{up}}^{s_p} = m_{\text{up}}^{t_{n_p}}$, $\mathbf{dis}[s_p] = \mathbf{dis}[t_{l_p}]$ and $\mathbf{val}[s_p] = H_{l_p}$. Then the coloring c is constant on $r_k[k-1, \bar{s}]$. Thus, **A.4** holds. \square

Example 2.13 in [13]. Let $N > 0$ and $\mathbf{H}_N(n) = N$ for $n < \omega$. Let K_N consist of all FP creatures t for \mathbf{H}_N such that

- $\mathbf{dis}[t] = (X_t, \varphi_t)$, where $X_t \subsetneq [m_{\text{dn}}^t, m_{\text{up}}^t)$, and $\varphi_t : X_t \rightarrow N$,
- $\mathbf{nor}[t] = m_{\text{up}}^t$,
- $\mathbf{val}[t] = \{f \in [m_{\text{dn}}^t, m_{\text{up}}^t)N : \varphi_t \subseteq f \text{ and } f \text{ is constant on } [m_{\text{dn}}^t, m_{\text{up}}^t) \setminus X_t\}$.

For $t_0, \dots, t_n \in K_2$ with $m_{\text{up}}^{t_i} = m_{\text{dn}}^{t_{i+1}}$, let $\Sigma_N(t_0, \dots, t_n)$ consist of all creatures $t \in K_N$ such that

- $m_{\text{dn}}^t = m_{\text{dn}}^{t_0}$, $m_{\text{up}}^t = m_{\text{up}}^{t_n}$, $X_{t_0} \cup \dots \cup X_{t_n} \subseteq X_t$,
- for each $l \leq n$, either $X_t \cap [m_{\text{dn}}^{t_l}, m_{\text{up}}^{t_l}) = X_{t_l}$ and $\varphi_t \upharpoonright [m_{\text{dn}}^{t_l}, m_{\text{up}}^{t_l}) = \varphi_{t_l}$,
or $[m_{\text{dn}}^{t_l}, m_{\text{up}}^{t_l}) \subsetneq X_t$ and $\varphi_t \upharpoonright [m_{\text{dn}}^{t_l}, m_{\text{up}}^{t_l}) \in \mathbf{val}[t_l]$.

The partial ordering \leq on $\text{PC}_{\infty}^{\text{tt}}(K_N, \Sigma_N)$ is defined by $\bar{t} \leq \bar{s}$ if and only if there is a strictly increasing sequence $(j_n)_{n < \omega}$, with $j_0 = 0$, such that each $t_n \in \Sigma_N(s_{j_n}, \dots, s_{j_{n+1}-1})$. Roslanowski and Shelah proved that (K_N, Σ_N) is a tight FFCC pair for \mathbf{H}_N which has the \mathbf{t} -multiadditivity and weak bigness, and is gluing for each candidate in $\text{PC}_{\infty}^{\text{tt}}(K_N, \Sigma_N)$.

We show below that this forcing itself forms a topological Ramsey space. The pigeonhole principle **A.4** will follow from the Hales-Jewett Theorem [9]. For each $k < \omega$ and $\bar{s} = (s_0, s_1, \dots) \in \text{PC}_{\infty}^{\text{tt}}(K_N, \Sigma_N)$, the k -th restriction of \bar{s} is simply $r_k(\bar{s}) = (s_0, \dots, s_{k-1})$. For $(s_0, \dots, s_{k-1}), (t_0, \dots, t_{m-1}) \in \mathcal{AR}$, define $(t_0, \dots, t_{m-1}) \leq_{\text{fin}} (s_0, \dots, s_{k-1})$ if and only if there is a strictly increasing sequence $(j_n)_{n \leq m}$, with $j_0 = 0$, such that for each $n < m$, $t_n \in \Sigma_N(s_{j_n}, \dots, s_{j_{n+1}-1})$.

Theorem 13. $(\text{PC}_{\infty}^{\text{tt}}(K_N, \Sigma_N), \leq, r)$ is a topological Ramsey space.

Proof. Let $k \geq 1$ and $\bar{t} \in \mathcal{R}^{\text{tt}}(K_N, \Sigma_N)$ be given. There is a one-to-one correspondence σ between $r_k[k-1, \bar{t}]$ and the set of finite variable words on alphabet N : For $s \in r_k[k-1, \bar{t}]$, the variable word (l_{k-1}, \dots, l_m) equals $\sigma(s)$ if and only if $s \in \Sigma_N(t_{k-1}, \dots, t_m)$ and for each $i \in [k-1, m]$, $l_i \in N$ if and only if $\varphi_s \upharpoonright [m_{\text{dn}}^{t_i}, m_{\text{up}}^{t_i}) \setminus X_{t_i} \equiv l_i$; and $l_i = v$ if and only if $X_s \cap [m_{\text{dn}}^{t_i}, m_{\text{up}}^{t_i}) = X_{t_i}$.

Given a coloring $c : r_k[k-1, \bar{t}] \rightarrow 2$, let c' color the collection of all variable words on alphabet N by $c'(\sigma(s)) = c(s)$. By the Hales-Jewett Theorem, there is an infinite sequence of variable words $(x_i)_{i < \omega}$ such that c' is constant on all variable words of the form $x_{i_0}[\lambda_0] \frown \dots \frown x_{i_n}[\lambda_n]$, where each $\lambda_j \in N \cup \{v\}$ and at least one $\lambda_j = v$.

For each $i \geq k-1$, let $l(i) = |x_i|$, the length of the word x_i . Let $m_0 = k-1+l(k-1)$, and given $i < \omega$ and m_i , let $m_{i+1} = m_i+l(i)$. Let $(l_0^i, \dots, l_{l(i)-1}^i) = x_i$. Define s_{k-1} to be the member of $\Sigma_N(t_{k-1}, \dots, t_{m_0-1})$ such that $\sigma(s) = x_0$, and in general, for $i \geq 1$ define s_{k-1+i} to be the member of $\Sigma_N(t_{m_{i-1}}, \dots, t_{m_i-1})$ such that $\sigma(s) = x_i$. Letting $\bar{s} = r_{k-1}(\bar{t}) \frown (s_{k-1}, s_k, \dots)$, it is routine to check that c is monochromatic on $r_k[k-1, \bar{s}]$. Hence, **A.4** holds. \square

5. REMARKS AND FURTHER LINES OF INQUIRY

Whenever a forcing contains a topological Ramsey space as a dense subset, this has implications for the properties of the generic extension as well as providing Ramsey theory machinery for streamlining proofs. Although this note only showed that the pure candidates for three examples of creature forcings contain dense subsets forming topological Ramsey spaces, the work here points to and lays some groundwork for several natural lines of inquiry.

One obvious line of exploration is to develop stronger versions and other variants of Theorem 3 to obtain the pigeonhole principle for the pure candidates for other creating pairs, in particular for Example 2.12 in [13]. Another is to develop this theory for the loose candidates, as we only considered tight types here. A deeper line of inquiry is to determine the implications that the existence of a topological Ramsey space dense in a collection of pure candidates for a creating pair has for the forcing notion generated by that creating pair.

Forcing with any topological Ramsey space modulo *almost reduction* (see [3]) generates a generic ultrafilter satisfying the space's version of the Abstract Nash-Williams Theorem. The topological Ramsey spaces considered here force ultrafilters on base set K , a set of creatures, which in turn generate ultrafilters on a countable set of finite functions $\mathcal{F}_{\mathbf{H}}$. Our work yields the conclusion of Theorem 4.7 of Roslanowski and Shelah in [13] for the examples considered in Section 4. Their means of proof via analogues of Galvin-Glazer methods is subsumed in the proof of the Abstract Ellentuck Theorem 2, that axioms **A.1** - **A.4** imply a space is a topological Ramsey space. It seems that further study will yield fruitful cross-pollination between these two approaches to similar problems. We are also interested in what the implications of the complete combinatorics for an ultrafilter generic for some topological Ramsey space of pure candidates over $L(\mathbb{R})$ (in the presence of a supercompact cardinal) will be for the induced ultrafilter on $\mathcal{F}_{\mathbf{H}}$.

The hope is that further investigations into connections between creature forcings and topological Ramsey spaces will lead to new Ramsey-type theorems and new topological Ramsey spaces, while adding to the collection of available machinery and streamlining at least some genres of the myriad of creature forcings.

REFERENCES

1. Timothy J. Carlson and Stephen G. Simpson, *Topological Ramsey theory*, Mathematics of Ramsey theory, volume 5 of Algorithms and Combinatorics, Springer, 1990, pp. 172–183.
2. C. A. DiPrisco, J. Llopis, and S. Todorcevic, *Parametrized partitions of products of finite sets*, *Combinatorica* **24** (2004), no. 2, 209–232.
3. Carlos DiPrisco, José Grigorio Mijares, and Jesus Nieto, *Local Ramsey theory. An abstract approach*, arXiv:1506.03488 (2015).
4. Natasha Dobrinen, *High dimensional Ellentuck spaces and initial chains in the Tukey structure of non- p -points*, *Journal of Symbolic Logic*, 27pp, To appear.
5. ———, *Infinite dimensional Ellentuck spaces and Ramsey-classification theorems*, 36 pp, Submitted.
6. Natasha Dobrinen, José Grigorio Mijares, and Timothy Trujillo, *Topological Ramsey spaces from Fraïssé classes, Ramsey-classification theorems, and initial structures in the Tukey types of p -points*, *Archive for Mathematical Logic*, special issue in honor of James E. Baumgartner (2015), 34 pp, To appear.
7. Natasha Dobrinen and Stevo Todorcevic, *A new class of Ramsey-classification Theorems and their applications in the Tukey theory of ultrafilters, Part 1*, *Transactions of the American Mathematical Society* **366** (2014), no. 3, 1659–1684.
8. ———, *A new class of Ramsey-classification Theorems and their applications in the Tukey theory of ultrafilters, Part 2*, *Transactions of the American Mathematical Society* **367** (2015), no. 7, 4627–4659.
9. A. W. Hales and R. I. Jewett, *Regularity and positional games*, *Transactions of the American Mathematical Society* **106** (1963), 222–229.
10. Pavel Pudlák and Vojtech Rödl, *Partition theorems for systems of finite subsets of integers*, *Discrete Mathematics* **39** (1982), 67–73.
11. Dilip Raghavan and Stevo Todorcevic, *Cofinal types of ultrafilters*, *Annals of Pure and Applied Logic* **163** (2012), no. 3, 185–199.

12. Andrzej Rosłanowski and Saharon Shelah, *Norms on possibilities. I. Forcing with trees and creatures*, vol. 141, American Mathematical Society, 1999.
13. ———, *Partition theorems from creatures and idempotent ultrafilters*, *Annals of Combinatorics* **17** (2013), no. 2, 353–378.
14. Stevo Todorćević, *Introduction to Ramsey Spaces*, Princeton University Press, 2010.
15. Stevo Todorćević and Konstantinos Tyros, *Subsets of products of finite sets of positive upper density*, *Journal of Combinatorial Theory, Series A* **120** (2013), 183–193.

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