INFINITE DIMENSIONAL ELLENTUCK SPACES AND RAMSEY-CLASSIFICATION THEOREMS

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Abstract. We extend the hierarchy of finite-dimensional Ellentuck spaces to infinite dimensions. Using uniform barriers B on ω as the prototype structures, we construct a class of continuum many topological Ramsey spaces $\mathcal{E}_B$ which are Ellentuck-like in nature, and form a linearly ordered hierarchy under projection. We prove new Ramsey-classification theorems for equivalence relations on fronts, and hence also on barriers, on the spaces $\mathcal{E}_B$, extending the Pudlák-Rödl Theorem for barriers on the Ellentuck space.

The inspiration for these spaces comes from continuing the iterative construction of the forcings $\mathcal{P}([\omega]^k)/\mathrm{Fin}^{\otimes k}$ to the countable transfinite. The σ-closed partial order $(\mathcal{E}_B, \subseteq^{\mathrm{Fin}^B})$ is forcing equivalent to $\mathcal{P}(B)/\mathrm{Fin}^B$, which forces a non-p-point ultrafilter $\mathcal{G}_B$. The present work forms the basis for further work classifying the Rudin-Keisler and Tukey structures for the hierarchy of the generic ultrafilters $\mathcal{G}_B$.

1. Overview

Extending work in [1], we construct a new class of topological Ramsey spaces which are structurally based on uniform barriers. The inspiration for these spaces comes from the continuation of the iterative construction of the Boolean algebras $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ ($1 \le k < \omega$) to the countable transfinite. These spaces form a natural hierarchy in complexity over the Ellentuck space and the finite-dimensional Ellentuck spaces of [1] from several viewpoints. First, whenever B, C are uniform barriers on ω with rank of C less than rank of B, then $\mathcal{E}_B$ projects to the space $\mathcal{E}_C$. Second, any restriction of the members of a space $\mathcal{E}_B$ to the extensions of some fixed finite initial segment in some member of the space yields an isomorphic copy of one of the lower-dimensional Ellentuck spaces that were used to form $\mathcal{E}_B$. This is one of the justifications of the terminology infinite-dimensional Ellentuck space. Third,

for any uniform barriers B and C, one of $\mathcal{E}_B$ and $\mathcal{E}_C$ projects to the other, modulo possibly restricting below some $X \in \mathcal{E}_B$ and some $Y \in \mathcal{E}_C$.

The three main contributions of this work are the construction of the spaces $\mathcal{E}_B$, the proof that they are indeed topological Ramsey spaces, and the Ramsey-classification Theorem canonizing equivalence relations on fronts on $\mathcal{E}_B$ in terms of projection maps to subtrees which are unique among projection maps satisfying a certain natural property. The Ramsey-classification Theorem can be seen as the natural extension of the Pudlák-Rödl Theorem in [2] to these spaces, the Pudlák-Rödl Theorem, and moreover the Ramsey-classification Theorems for the finite-dimensional Ellentuck spaces in [1], being recovered from our theorem via a projection map.

2. Introduction

The Infinite Ramsey Theorem states that given any positive integers k and l and any coloring c of the subsets of the natural numbers of size k into l many colors, there is an infinite set M of natural numbers such that c is constant on the subsets of M of size k (see [3]). Nash-Williams extended this to finite colorings of any barrier in [4]. A finite coloring of a barrier codes a partition of the Baire space into finitely many clopen sets. Galvin and Prikry extended the Nash-Williams Theorem to Borel partitions of the Baire space in [5], and Silver extended it to analytic partitions in [6], though by non-constructive methods.

Shortly thereafter, Ellentuck found the optimal Ramsey theorem for finite partitions of the Baire space. He captured the essence of the Ramsey property by equipping the Baire space with a topology finer than the metric topology, now called the Ellentuck topology. In [7], Ellentuck proved that a subset of the Baire space is Ramsey (see Definition 1) if and only if it has the property of Baire in the Ellentuck topology, thus providing a constructive proof of Silver's Theorem and moreover extending extending it to all partitions of the Baire space into finitely many sets each of which has the property of Baire in the Ellentuck topology.

Independently and around the same time, Louveau developed ultra-Ramsey methods using a topology similar to Ellentuck's but with infinite sets in a fixed Ramsey ultrafilter, and proved the analogue of Ellentuck's Theorem in the ultra-Ramsey setting (see [8]). His work provided a topological proof of Silver's Theorem a theorem of Mathias in [9] in the ultra-Ramsey setting. An ultrafilter is *Ramsey* if for any finite coloring of the pairs of natural numbers, there is a member of the ultrafilter on which the coloring is homogeneous. Ramsey ultrafilters are closely connected with the Ellentuck space. Assuming some axiom in addition to the standard axioms of set theory, for instance the Continuum Hypothesis, Martin's Axiom, or even less ($\mathfrak{p} = \mathfrak{c}$), or by the method of forcing, one can construct a Ramsey ultrafilter using the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$. Equivalently, one may use the Ellentuck space partially ordered by almost inclusion. This sets the stage for later

connections between topological Ramsey spaces and ultrafilters satisfying partition properties.

The Ellentuck space is the quintessential example of the more general notion of a topological Ramsey space (see Definition 1). Classic examples of topological Ramsey spaces include the Carlson-Simpson space in [10] of equivalence relations on the natural numbers with infinitely many equivalence classes, and the Milliken space of infinite block sequences (see [11]) which has proved fundamental for progress in certain areas of Banach space theory. Building on prior work of Carlson and Simpson, Todorcevic distilled four axioms which, when satisfied, guarantee that a space is a toplogical Ramsey space (see Section 3). This axiomatic approach to topological Ramsey spaces paved the way for recent work involving connections between ultrafilters satisfying partition properties and constructions of new topological Ramsey spaces (see [12], [13], [14], and [1]).

Similar to the connection between the Ellentuck space and Ramsey ultrafilters, each topological Ramsey space has associated ultrafilters, which are 'selective' or 'Ramsey' with respect to the space (see [15] and [16]). Using the Continuum Hypothesis or Martin's Axiom (indeed, less is actually required) or forcing, any topological Ramsey space gives rise to an associated ultrafilter satisfying some partition properties, dependent on the space.

One current research program of ours involves finding the essential Ramsey-like structures inside partial orderings which are responsible for the partition properties of the ultrafilters that they construct. The motivation is as follows. Given a partial order which constructs an ultrafilter with certain partition properties, if one can show that the partial order contains a topological Ramsey space as a dense subset, then one gains much machinery and a fine-tuned approach to investigations of the properties of the ultrafilter. To begin with, each topological Ramsey space satisfies its version of the Abstract Ellentuck Theorem and the Abstract Nash-Williams Theorem (see Theorems 2 and 4). The structure of the members of the topological Ramsey space provide insight and these theorems streamline proofs involving finding the Ramsey numbers for the associated ultrafilters. Further, the structure sets the stage for canonizing equivalence relations on fronts and barriers, yielding analogues of the Pudlák-Rödl theorem in [2]. This in turn makes it possible to find the exact Rudin-Keisler and Tukey structures below the ultrafilters.

Todorcevic was the first to notice the connection between canonical equivalence relations on fronts and the Tukey type of a Ramsey ultrafilter. Tukey reduction on ultrafilters is a generalization of Rudin-Keisler reduction and has been widely studied in [17], [18], [19], [20], [12], [13], [14], [21], and [1]. An ultrafilter $\mathcal{V}$ is *Tukey reducible* to an ultrafilter $\mathcal{U}$ if and only if there is a function $f : \mathcal{U} \to \mathcal{V}$ which sends each filter base of $\mathcal{U}$ to a filter base of $\mathcal{V}$. Since the Tukey theory of ultrafilters is not the focus of the results in this paper, we merely mention its strong motivating force in the current work

without going into detail. For more background on Tukey theory of ultra-filters, the reader is referred to our survey paper [22]. In [20], Todorcevic proved that the Tukey type of any Ramsey ultrafilter is minimal via a judicious application of the Pudlák-Rödl Theorem on the Ellentuck space. In [12] and [13], Dobrinen and Todorcevic constructed a new class of topological Ramsey spaces which are dense in the partial orderings of Laflamme in [23] for ultrafilters with weaker partition properties. By proving and applying Ramsey-classification theorems, they found the exact initial Rudin-Keisler and Tukey structures for this class of ultrafilters. In [14], Dobrinen, Mijares, and Trujillo developed a template for constructions of topological Ramsey spaces, including spaces which generate the k-arrow ultrafilters of Baumgartner and Taylor in [24]; they proved broad class of Ramsey-classification theorems and applied them to find exact Rudin-Keisler and Tukey structures of the associated ultrafilters. Each of the new topological Ramsey spaces mentioned above has only finitely many components extending each node, and thus cannot be thought of as a generalizing the Ellentuck space.

The work in this paper builds a new class of continuum many infinite dimensional Ellentuck spaces. This extends our previous work in [1] on finite dimensional Ellentuck spaces. That paper was motivated by [21], in which Blass, Dobrinen, and Raghavan initiated the study of the Tukey type of the generic ultrafilter $\mathcal{G}_2$ forced by $\mathcal{P}(\omega \times \omega)/\mathrm{Fin} \otimes \mathrm{Fin}$, but did not find the exact Tukey structure below $\mathcal{G}_2$. (Hrusak and Verner also considered this forcing in [25].) In [1], the author developed a hierarchy of spaces $\mathcal{E}_k$ extending the Ellentuck space to every finite dimension k. The space $\mathcal{E}_k$ forms dense subsets of the partial order $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ distilling the internal structure responsible for its forcing and Ramsey properties. We proved Ramsey-classification theorems for the $\mathcal{E}_k$ spaces and applied them to find that indeed, the initial Rudin-Keisler and Tukey structures below $\mathcal{G}_k$ is simply a linear order of size k (see Theorems 40 and 41 in [1]).

The inspiration for the the infinite-dimensional Ellentuck spaces comes from the continuation of the iterative construction of the Boolean algebras $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ to the countable transfinite. The underlying structure of these spaces $\mathcal{E}_B$ are based on uniform barriers on ω, which also give the rank of the space. These spaces form a natural hierarchy in complexity over the Ellentuck space and the hierarchy of finite dimensional Ellentuck spaces in [1] space from several viewpoints. First, whenever B, C are uniform barriers on ω with rank of C less than rank of B, then $\mathcal{E}_B$ projects to $\mathcal{E}_C$. Second, as seen in Fact 21, if we fix any finite initial segment of an element of a member of the space and restrict the space to those members that extend it, then we obtain an isomorphic copy of one of the lower-dimensional Ellentuck spaces that was used to form $\mathcal{E}_B$. Lastly, for any uniform barriers B and C, we can always find members $X \in \mathcal{E}_B$ and $Y \in \mathcal{E}_C$ so that one of $\mathcal{E}_B|X$ and $\mathcal{E}_Y$ projects to the other (see Fact 25).

Associated to each $\mathcal{E}_B$ is the σ-closed forcing $\mathcal{P}(B)/\mathrm{Fin}^B$, which forces a non-p-point ultrafilter $\mathcal{G}_B$ (see just below Fact 13 in Section 4). Like the

spaces $\mathcal{E}_B$, these ultrafilters form a hierarchy, in that whenever C is obtained by a projection of B, then $\mathcal{G}_C$ is Rudin-Keisler reducible to $\mathcal{G}_B$. The Ramsey-classification Theorem 45 canonizing equivalence relations on fronts on the spaces $\mathcal{E}_B$ will be applied in forthcoming work to find the Rudin-Keisler and Tukey structures below the ultrafilters $\mathcal{G}_B$. For now, we will only mention that, for infinite rank B, we have proved that there are continuum sized linear orders in both the Rudin-Keisler and Tukey structures of ultrafilters reducible to $\mathcal{G}_B$. This contrasts with all previous related works which only yielded countable initial Tukey structures.

The paper is organized as follows. Section 3 provides the necessary background on topological Ramsey spaces from Todorcevic's book [26] as well as a review of the Ellentuck space and the Pudlák-Rödl Theorem. Section 4 begins with a review of uniform barriers and then carries out the recursive construction of the σ-closed ideals Fin^B, B a uniform barrier, and the Boolean algebras $\mathcal{P}(B)/\mathrm{Fin}^B$. This section also includes a brief discussion of the ultrafilters constructed by $\mathcal{P}(B)/\mathrm{Fin}^B$ and some basic facts about their Tukey and Rudin-Keisler structures, forming the backdrop to a sequel paper.

The infinite dimensional Ellentuck spaces $\mathcal{E}_B$ are defined in Definition 18 in Section 5. These spaces are constructed recursively on the rank of the barrier B, via defining the legitimate domains (Definitions 15 and 17) so as to ensure that spaces of smaller rank embed into spaces of larger rank, and in a way that the space restricted above any fixed stem is isomorphic to a space of smaller rank (see Fact 21). In one of the main theorems of this paper, Theorem 28, we prove that for each uniform barrier B, the space $\mathcal{E}_B$ is a topological Ramsey space.

In Section 6, Theorems 33 and 34 prove the existence of canonical equivalence relations on first approximations and on 1-extensions. These are given in terms of uniform projections, which are similar in form to uniform barriers and have a notion of rank. Thus, there are continuum many canonical equivalence relations on first approximations, in contrast to the finite-dimensional Ellentuck spaces which have only finitely many such canonical equivalence relations. These theorems lay the groundwork for the Ramsey-classification theorems in the next section.

In Section 7 we prove the Ramsey-classification Theorem 45 for each $\mathcal{E}_B$. In words, it says that each equivalence relation on a front $\mathcal{F}$ on the space $\mathcal{E}_B$ is canonized by an inner Nash-Williams function which projects each member u of $\mathcal{F}$ to some subtree of u. Further, this projection function uniquely canonizes the equivalence relation among those inner functions which satisfy the property $(*)$ (see Definition 42 and Theorem 44). Though the proof of Theorem 45 follows the standard outline of the analogous theorems in [12], [13], [14], and [1], the proof methods are quite different, involving new ideas central to the structure of the spaces $\mathcal{E}_B$.

3. Basics of topological Ramsey spaces

A brief review of topological Ramsey spaces is provided in this section for the reader's convenience. Building on prior work of Carlson and Simpson in [27], Todorcevic distilled key properties of the Ellentuck space into four axioms, **A.1** - **A.4**, which guarantee that a space is a topological Ramsey space. (For further background, the reader is referred to Chapter 5 of [26].) The axioms **A.1** - **A.4** are defined for triples $(\mathcal{R}, \leq, r)$ of objects with the following properties: $\mathcal{R}$ is a nonempty set, $\leq$ is a quasi-ordering on $\mathcal{R}$, and $r : \mathcal{R} \times \omega \to \mathcal{AR}$ is a map producing the sequence $(r_n(\cdot) = r(\cdot, n))$ of restriction maps, where $\mathcal{AR}$ is the collection of all finite approximations to members of $\mathcal{R}$. For $u \in \mathcal{AR}$ and $X, Y \in \mathcal{R}$,

(1) $$[u, X] = \{Y \in \mathcal{R} : Y \leq X \text{ and } (\exists n) \ r_n(Y) = u\}.$$

For $u \in \mathcal{AR}$, let $|u|$ denote the length of the sequence u. Thus, $|u|$ equals the integer k for which $u = r_k(u)$. For $u, v \in \mathcal{AR}$, $u \sqsubseteq v$ if and only if $u = r_m(v)$ for some $m \leq |v|$. $u \sqsubset v$ if and only if $u = r_m(v)$ for some $m < |v|$. For each $n < \omega$, $\mathcal{AR}_n = \{r_n(X) : X \in \mathcal{R}\}$.

A.1 (1) $r_0(X) = \emptyset$ for all $X \in \mathcal{R}$.

     (2) $X \neq Y$ implies $r_n(X) \neq r_n(Y)$ for some n.

     (3) $r_m(X) = r_n(Y)$ implies $m = n$ and $r_k(X) = r_k(X)$ for all $k < n$.

A.2 There is a quasi-ordering $\leq_{\text{fin}}$ on $\mathcal{AR}$ such that

     (1) $\{v \in \mathcal{AR} : v \leq_{\text{fin}} u\}$ is finite for all $u \in \mathcal{AR}$,

     (2) $Y \leq X$ iff $(\forall n)(\exists m) \ r_n(Y) \leq_{\text{fin}} r_m(X)$,

     (3) $\forall u, v, y \in \mathcal{AR}[y \sqsubset v \wedge v \leq_{\text{fin}} u \to \exists x \sqsubset u \ (y \leq_{\text{fin}} x)]$.

The number $\text{depth}_X(u)$ is the least n, if it exists, such that $u \leq_{\text{fin}} r_n(X)$. If such an n does not exist, then we write $\text{depth}_X(u) = \infty$. If $\text{depth}_X(u) = n < \infty$, then $[\text{depth}_X(u), X]$ denotes $[r_n(X), X]$.

A.3 (1) If $\text{depth}_X(u) < \infty$ then $[u, Y] \neq \emptyset$ for all $Y \in [\text{depth}_X(u), X]$.

     (2) $Y \leq X$ and $[u, Y] \neq \emptyset$ imply that there is $Y' \in [\text{depth}_X(u), X]$ such that $\emptyset \neq [u, Y'] \subseteq [u, Y]$.

Additionally, for $n > |u|$, let $r_n[u, X]$ denote the collection of all $v \in \mathcal{AR}_n$ such that $u \sqsubset v$ and $v \leq_{\text{fin}} X$.

A.4 If $\text{depth}_X(u) < \infty$ and if $\mathcal{O} \subseteq \mathcal{AR}_{|u|+1}$, then there is $Y \in [\text{depth}_X(u), X]$ such that $r_{|u|+1}[u, Y] \subseteq \mathcal{O}$ or $r_{|u|+1}[u, Y] \subseteq \mathcal{O}^c$.

The *Ellentuck topology* on $\mathcal{R}$ is the topology generated by the basic open sets $[u, X]$; it refines the metric topology on $\mathcal{R}$, considered as a subspace of the Tychonoff cube $\mathcal{AR}^{\mathbb{N}}$. Given the Ellentuck topology on $\mathcal{R}$, the notions of nowhere dense, and hence of meager are defined in the natural way. We say that a subset $\mathcal{X}$ of $\mathcal{R}$ has the *property of Baire* iff $\mathcal{X} = \mathcal{O} \cap \mathcal{M}$ for some Ellentuck open set $\mathcal{O} \subseteq \mathcal{R}$ and Ellentuck meager set $\mathcal{M} \subseteq \mathcal{R}$.

Definition 1 ([26])**.** A subset $\mathcal{X}$ of $\mathcal{R}$ is *Ramsey* if for every $\emptyset \neq [u, X]$, there is a $Y \in [u, X]$ such that $[u, Y] \subseteq \mathcal{X}$ or $[u, Y] \cap \mathcal{X} = \emptyset$. $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey null* if for every $\emptyset \neq [u, X]$, there is a $Y \in [u, X]$ such that $[u, Y] \cap \mathcal{X} = \emptyset$.

A triple $(\mathcal{R}, \leq, r)$ is a *topological Ramsey space* if every subset of $\mathcal{R}$ with the property of Baire is Ramsey and if every meager subset of $\mathcal{R}$ is Ramsey null.

The following result can be found as Theorem 5.4 in [26].

Theorem 2 (Abstract Ellentuck Theorem)**.** *If $(\mathcal{R}, \leq, r)$ is closed (as a subspace of $\mathcal{AR}^{\mathbb{N}}$) and satisfies axioms* **A.1**, **A.2**, **A.3**, *and* **A.4**, *then every subset of $\mathcal{R}$ with the property of Baire is Ramsey, and every meager subset is Ramsey null; in other words, the triple $(\mathcal{R}, \leq, r)$ forms a topological Ramsey space.*

Definition 3 ([26])**.** A family $\mathcal{F} \subseteq \mathcal{AR}$ of finite approximations is

(1) *Nash-Williams* if $u \neq v \in \mathcal{F}$ implies $v \not\sqsubseteq u$;
(2) *Ramsey* if for every partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ and every $X \in \mathcal{R}$, there are $Y \leq X$ and $i \in \{0, 1\}$ such that $\mathcal{F}_i | Y = \emptyset$.

The Abstract Nash-Williams Theorem (Theorem 5.17 in [26]), which follows from the Abstract Ellentuck Theorem, is actually sufficient for the proofs in Sections 6 and 7.

Theorem 4 (Abstract Nash-Williams Theorem)**.** *Suppose $(\mathcal{R}, \leq, r)$ is a closed triple that satisfies* **A.1** *-* **A.4**. *Then every Nash-Williams family of finite approximations is Ramsey.*

Definition 5. Suppose $(\mathcal{R}, \leq, r)$ is a closed triple that satisfies **A.1** - **A.4**. Let $X \in \mathcal{R}$. A family $\mathcal{F} \subseteq \mathcal{AR}$ is a *front* on $[0, X]$ if

(1) For each $Y \in [0, X]$, there is a $u \in \mathcal{F}$ such that $u \sqsubset Y$; and
(2) $\mathcal{F}$ is Nash-Williams.

Remark. The stronger notion of *barrier* on a topological Ramsey space simply replaces Nash-Williams with *Sperner* in (2) of Definition 5: whenever $u, v \in \mathcal{F}$, $v \neq u \to v \not\leq_{\text{fin}} u$. For topological Ramsey spaces in which the quasi-order $\leq_{\text{fin}}$ is actually a partial order, each front is a barrier when restricted to some member of $\mathcal{R}$ (see Corollary 5.19 in [26]). Since $\leq_{\text{fin}}$ is a partial order for all the spaces in this article, the terms 'front' and 'barrier' may be used interchangeably with no change to our results.

This section concludes by recalling the Ellentuck space and stating the Pudlák-Rödl Theorem for canonical equivalence relations on fronts on the Ellentuck space.

Definition 6. Let $(\mathcal{E}, \leq, r)$ denote the *Ellentuck space* $([\omega]^{\omega}, \subseteq, r)$, where for each $X \in [\omega]^{\omega}$ and $n < \omega$, $r_n(X)$ denotes the set of the n least members of X.

Definition 7. A map φ on a front $\mathcal{F} \subseteq [\omega]^{<\omega}$ on the Ellentuck space is called

    (1) *inner* if for each $u \in \mathcal{F}$, $\varphi(u) \subseteq u$.
    (2) *Nash-Williams* if for all pairs $u, v \in \mathcal{F}$, $\varphi(u) \not\sqsubset \varphi(v)$.
    (3) *irreducible* if it is inner and Nash-Williams.

Theorem 8 (Pudlák/Rödl, [2])**.** *Let R be an equivalence relation on a front $\mathcal{F}$ on the Ellentuck space. Then there is an irreducible map φ and an $X \in [\omega]^\omega$ such that for all $u, v \in \mathcal{F}$ with $u, v \subseteq X$,*

$$(2) \qquad\qquad\qquad u\,R\,v \quad \longleftrightarrow \quad \varphi(u) = \varphi(v).$$

This theorem has been generalized to new classes of topological Ramsey spaces in the papers [12], [13], [14] and [1]. In Section 7, we will extend it to the infinite dimensional Ellentuck spaces, which will be defined in Section 5.

4. Uniform barriers B, the Boolean algebras $\mathcal{P}(B)/\mathrm{Fin}^B$, and the generic ultrafilters $\mathcal{G}_B$

This section provides background on uniform barriers B on ω and our recursive constructions of the σ-closed ideals Fin^B on $\mathcal{P}(B)$. These ideals are used to extend the class of Boolean algebras $\mathcal{P}([\omega]^k)/\mathrm{Fin}^{\otimes k}$ in [1] to the class of Boolean algebras $\mathcal{P}(B)/\mathrm{Fin}^B$, for every uniform barrier B of countable ordinal rank. Since each such Boolean algbebra has a σ-closed dense subset, forcing with it adds no new subsets of ω or of B. Using the Continuum Hypthothesis, Martin's Axiom, or by forcing, $\mathcal{P}(B)/\mathrm{Fin}^B$ constructs a non-p-point ultrafilter, which we shall denote $\mathcal{G}_B$, satisfying some partition properties. Investigations of the Rudin-Keisler and Tukey structures of these generic ultrafilters motiviated the construction of the topological Ramsey spaces $\mathcal{E}_B$ and their Ramsey-classification Theorems, which will be applied to classify their Rudin-Keisler and Tukey structures in forthcoming work.

For $a, b \in [\omega]^{<\omega}$, we shall use the notation $a \trianglelefteq b$ to denote that a is an initial segment of b, and $a \triangleleft b$ to denote that a is a proper initial segment of b. This will serve to distinguish the partial ordering of initial segment on $[\omega]^{<\omega}$ from the partial ordering $u \sqsubseteq v$ of initial segment for $u, v \in \mathcal{AR}$, finite approximations of members of a topological Ramsey space $\mathcal{R}$.

The following two definitions can be found in [28]. The notation is slightly modified to be more suitable for the present work.

Definition 9. A family B of finite subsets of ω is a *front* on $\bigcup B$ if

    (1) $a \not\trianglelefteq b$ whenever $a \neq b \in B$; and
    (2) $\bigcup B$ is infinite and for every infinite $M \subseteq \bigcup B$ there is an $a \in B$ such that $a \triangleleft M$.

B is a *barrier* on $\bigcup B$ if

    (1′) $a \not\subseteq b$ whenever $a \neq b \in B$; and

(2) holds.

Notation. For a barrier B and $n \in \omega$, let $B_n = \{b \in B : n = \min(b)\}$; and let $B_{\{n\}} = \{a \in [\omega]^{<\omega} : \min(a) > n \text{ and } \{n\} \cup a \in B\}$. For N an infinite subset of $\bigcup B$, $B|N = \{b \in B : b \subseteq N\}$.

Definition 10. Let $\alpha < \omega_1$ and M be an infinite subset of ω. A subset $B \subseteq [\omega]^{<\omega}$ is an α-*uniform family on* M provided that

    (a) $\alpha = 0$ implies $B = \emptyset$.

    (b) $\alpha = \beta + 1$ implies that $\emptyset \notin B$ and $B_{\{n\}}$ is β-uniform on $M \setminus (n+1)$, for all $n \in M$.

    (c) $\alpha > 0$ is a limit ordinal implies that there is an increasing sequence $\{\alpha_n\}_{n \in M}$ of ordinals converging to α such that $B_{\{n\}}$ is α_n-uniform on $M \setminus (n+1)$, for all $n \in M$.

A barrier $B \subseteq [\omega]^{<\omega}$ which is also a uniform family is called a *uniform barrier*.

The following facts show that, were we to use fronts instead of uniform barriers as bases for the structures Fin^B and $\mathcal{E}_B$, we would gain nothing more. The following results stated in Fact 11 appear as Lemmas II.3.2, II.3.3, II.3.8 (due to Galvin), Corollary II.3.10 and Lemma II.3.17, respectively, in [28].

Facts 11.   (1) *Every α-uniform family on M is a front on M.*

    (2) *If B is α-uniform on M and if N is an infinite subset of M, then $B|N$ is α-uniform on N.*

    (3) *For every family $B \subseteq [\omega]^{<\omega}$ and every infinite $M \subseteq \omega$, there is an infinite $N \subseteq M$ such that the restriction $B|N$ is either empty or contains a barrier.*

    (4) *For every family $B \subseteq [\omega]^{<\omega}$ and for every infinite $M \subseteq \omega$, there is an infinite $N \subseteq M$ such that either $B|N = \emptyset$ or $B|N$ contains a uniform barrier.*

    (5) *The lexicographical rank of an α-uniform barrier is equal to ω^α.*

Thus, for every front $B \subseteq [\omega]^{<\omega}$ there is an infinite $N \subseteq \bigcup B$ such that $B|N$ is a uniform barrier, and any further restriction $B|N'$ for $N' \in [N]^\omega$ is again a uniform barrier with the same rank as $B|N$. Since the lexicographic rank of an α-uniform barrier is ω^α, we will simply say that the barrier is *uniform of rank α*.

Examples 12. $\{\emptyset\}$ is the uniform barrier of rank 0. For each $1 \leq k < \omega$, $[\omega]^k$ is the uniform barrier of rank k. For $\omega \leq \alpha < \omega_1$ there are a myriad of uniform barriers of rank α. The most well-known is the *Schreier barrier* $S = \{b \in [\omega]^{<\omega} : |s| = \min(s) + 1\}$, which is uniform of rank ω.

For $B, C \subseteq [\omega]^{<\omega}$, write $C \lhd B$ if and only if for each $c \in C$ there is a $b \in B$ such that $c \lhd b$. The next fact is an immediate consequence of the following: Every barrier on ω is a Nash-Williams family on the Ellentuck

space, and hence is Ramsey (recall Definition 3). Given barriers B and C on ω, we may color each $c \in C$ by 0 if there is a $b \in B$ such that $b = c$, 1 if there is a $b \in B$ such that $b \lhd c$, and 2 if there is a $b \in B$ such that $b \rhd c$. Applying the Nash-Williams Theorem for this 3-coloring on C yields the following. In the case of $B|M \lhd C|M$, we shall call $B|M$ a *projection* of $C|M$.

Fact 13. *Suppose B and C are two barriers on the same infinite set $N \subseteq \omega$. Then there is an infinite $M \subseteq N$ such that one of the following hold: $B|M = C|M$, $B|M \lhd C|M$, or $B|M \rhd C|M$.*

Now we develop the hierarchy of ideals Fin^B on $\mathcal{P}(B)$, for B a uniform barrier. Recall that Fin denotes the collection of finite subsets of ω. In [1], we let $\mathrm{Fin}^{\otimes 1}$ denote Fin, and given $\mathrm{Fin}^{\otimes k}$, $k < \omega$, we defined $\mathrm{Fin}^{\otimes k+1}$ to be the collection of all $A \subseteq [\omega]^{k+1}$ such that for all but finitely many $a_0 \in \omega$, the set $\{\{a_1, \ldots, a_k\} \in [\omega]^k : a_1 > a_0 \text{ and } \{a_0, a_1, \ldots, a_k\} \in A\}$ is a member of $\mathrm{Fin}^{\otimes k}$. In the present paper, we shall use the simpler notation Fin^B, for B a uniform barrier. Then $\mathrm{Fin}^{[\omega]^k}$ denotes $\mathrm{Fin}^{\otimes k}$ from [1], for $k < \omega$. Given a uniform barrier B on ω of rank α, note that $B_{\{n\}}$ is a uniform barrier on $\omega \setminus (n+1)$ of rank less than α, for any $n < \omega$. Assuming we have defined Fin^C for each uniform barrier C of rank less than α, then we have also defined $\mathrm{Fin}^{B_{\{n\}}}$ for each $n < \omega$ (relativizing to $\omega \setminus n + 1$). Let $\mathrm{Fin}^{B_n} = \{X \subseteq B_n : \{a \setminus \{n\} : a \in X\} \in \mathrm{Fin}^{B_{\{n\}}}\}$.

Definition 14. For B a uniform barrier, define

$$(3) \qquad\qquad \mathrm{Fin}^B = \{A \subseteq B : \forall^\infty n \ (A_n \in \mathrm{Fin}^{B_n})\}.$$

For each uniform barrier B on ω, Fin^B is a σ-closed ideal; hence, $(\mathcal{P}(B)/\mathrm{Fin}^B) \setminus \{\mathbf{0}\}$ is a σ-closed partial order which can be used to construct an ultrafilter $\mathcal{G}_B$ (on the countable base set B). If B and C are uniform barriers with $C \lhd B$, the projection map $\pi_{B,C}$ from B to C induces a projection from $\mathcal{P}(B)/\mathrm{Fin}^B$ to $\mathcal{P}(C)/\mathrm{Fin}^C$, and hence induces a projection from the ultrafilter $\mathcal{G}_B$ to $\mathcal{G}_C$. It follows that $\pi_{B,C}(\mathcal{G}_B)$ is generic for $\mathcal{P}(C)/\mathrm{Fin}^C$. Thus, the generic ultrafilters form a hierarchy in the isomorphism classes of ultrafilters: If C is a projection of B, then $\mathcal{G}_C$ is Rudin-Keisler reducible to $\mathcal{G}_B$.

Let $\subseteq^{\mathrm{Fin}^B}$ denote the following partial order on $\mathcal{P}(B)$: For $X, Y \subseteq B$, $Y \subseteq^{\mathrm{Fin}^B} X$ if and only if $Y \setminus X \in \mathrm{Fin}^B$. By a routine induction on the rank of B, one shows that $\mathcal{P}(B)/\mathrm{Fin}^B$ is forcing equivalent to $(\mathcal{P}(B), \subseteq^{\mathrm{Fin}^B})$. In the next section, we construct topological Ramsey spaces $\mathcal{E}_B$ which, when considered as partially ordered by $\subseteq^{\mathrm{Fin}^B}$, form dense subsets of $(\mathcal{P}(B), \subseteq^{\mathrm{Fin}^B})$.

5. Infinite dimensional Ellentuck Spaces

This section contains the construction of the spaces $\mathcal{E}_B$, B a uniform barrier of infinite rank, and Theorem 28 proving that each one is a topological

Ramsey space. These spaces extend the hierarchy of the finite dimensional Ellentuck spaces $\mathcal{E}_k$, $1 \leq k < \omega$, in [1], $\mathcal{E}_1$ denoting the Ellentuck space here. Each new space $\mathcal{E}_B$ is modeled on the structure of some uniform barrier B on ω of infinite rank and is constructed so that $(\mathcal{E}_B, \subseteq^{\mathrm{fin}^B})$ is forcing equivalent to $\mathcal{P}(B)/\mathrm{Fin}^B$. The spaces form a hierarchy in that whenever C is a projection of B, then $\mathcal{E}_C$ is isomorphic to a projection of $\mathcal{E}_B$ (Fact 19). Similarly to the way that uniform barriers of rank α are constructed by recursion using uniform barriers of rank less than α, so too the space $\mathcal{E}_B$ is formed using spaces $\mathcal{E}_C$ for C with rank less than the rank of B. Copies of smaller dimensional Ellentuck spaces are seen inside $\mathcal{E}_B$ both as projections of $\mathcal{E}_B$ and as upward images above some fixed root via the recursive construction (Fact 21). We point out that for any two spaces $\mathcal{E}_B$ and $\mathcal{E}_C$, one is embeddable into the other if we allow for restricting below some members of the spaces (Fact 25).

Fix the following notation.

Notation. Let $\omega^{\nearrow<\omega}$ denote the collection of finite non-decreasing sequences of members of ω. For $s \in \omega^{\nearrow<\omega}$, let $\mathrm{lh}(s)$ denote the *length of s*, that is, the cardinality of the domain of the sequence s. Let $<_{\mathrm{lex}}$ denote the lexicographic ordering on $\omega^{\nearrow<\omega}$, where we also consider any proper initial segment of a sequence to be lexicographically below that sequence. Given $s, t \in \omega^{\nearrow<\omega}$, define $s \prec t$ if and only if either

   (1) $\max(s) < \max(t)$, or
   (2) $\max(s) = \max(t)$ and $s <_{\mathrm{lex}} t$.

Thus, $\prec$ well-orders $\omega^{\nearrow<\omega}$ in order-type ω, with the empty sequence $()$ as the $\prec$-minimum.

Define the function $\sigma : [\omega]^{<\omega} \to \omega^{\nearrow<\omega}$ as follows: Let $\sigma(\emptyset) = ()$, the empty sequence; and for $\{a_0, \ldots, a_n\} \in [\omega]^{<\omega} \setminus \{\emptyset\}$, let

$$(4) \qquad \sigma(a) = (a_0, a_1 - 1, a_2 - 2, \ldots, a_n - n).$$

Thus, σ is a bijection between the collection of all finite subsets of ω and the collection of all finite non-decreasing sequences of members of ω.

For $B \subseteq [\omega]^{<\omega}$, let $\hat{B}$ denote $\{a \in [\omega]^{<\omega} : \exists b \in B \, (a \trianglelefteq b)\}$. For sequences $s, t \in \omega^{\nearrow<\omega}$, we shall also use the notation $s \trianglelefteq t$ to denote that $s = t \upharpoonright m$ for some $m \leq \mathrm{lh}(t)$. Write $s \triangleleft t$ if and only if $s \trianglelefteq t$ and $s \neq t$. For $S \subseteq \omega^{\nearrow<\omega}$, let $\hat{S}$ denote $\{s \in \omega^{\nearrow<\omega} : \exists t \in S \, (s \trianglelefteq t)\}$.

Construct the related spaces of finite sequences as follows.

Definition 15 (The Sequence Spaces $(S_{\hat{B}}, \prec)$ and $(S_B, \prec)$). Let B be a uniform barrier on ω. Let

$$(5) \qquad S_{\hat{B}} = \{\sigma(a) : a \in \hat{B}\} \quad \text{and} \quad S_B = \{\sigma(b) : b \in B\}.$$

Thus, $S_{\hat{B}}$ and S_B are collections of non-decreasing finite sequences of members of ω which retain the structure of $\hat{B}$ and B, respectively. Note

that $\prec$ well-orders $S_{\hat{B}}$ and S_B in order-type ω. We now define the maximum member $\mathbb{W}_B$ of the space $\mathcal{E}_B$.

Definition 16 (The top member $\mathbb{W}_B$ of $\mathcal{E}_B$). Let B be a uniform barrier on ω. Since $(S_{\hat{B}}, \prec)$ has order-type ω, let $\nu_B : (S_{\hat{B}}, \prec) \to (\omega, <)$ denote this order isomorphism. For $s \in S_B$, let $\mathrm{lh}(s)$ denote the length of s and let

$$(6) \qquad\qquad \mathbb{W}_B(s) = \{\nu_B(s \restriction m) : 1 \le m \le \mathrm{lh}(s)\},$$

which is a member of $[\omega]^{\mathrm{lh}(s)}$. Define

$$(7) \qquad\qquad \mathbb{W}_B = \{\mathbb{W}_B(s) : s \in S_B\}.$$

Define the map $\psi_B : S_B \to \mathbb{W}_B$ by $\psi(s) = \mathbb{W}_B(s)$, for $s \in S_B$. Let ρ_B denote $\psi_B \circ \sigma$.

Remark. Note that $\mathbb{W}_B$ is a subset of $[\omega]^{<\omega}$ which is isomorphic to B via the map ρ_B. That is, ρ_B preserves the tree structure of B, in particular, lengths of nodes and lexicographic order. Further, extend ψ_B to a map from $S_{\hat{B}}$ to $\widehat{\mathbb{W}}_B$ as follows: For $s \in S_B$ and $1 \le m \le \mathrm{lh}(s)$, let $\psi_B(s \restriction m) = \{\nu_B(s \restriction i) : 1 \le i \le m\}$. Then $\prec$ on $S_{\hat{B}}$ induces a well-order on $\widehat{\mathbb{W}}_B$; so we shall abuse notation and consider $\widehat{\mathbb{W}}_B$ as also being well-ordered by $\prec$.

Next, we define the domain spaces D_B. These are the collections of sets $S \subseteq S_B$ which we allow to serve as templates, or domains, for members of $\mathcal{E}_B$, in the sense that $\mathcal{E}_B$ will be defined to be $\{\rho_B(S) : S \in D_B\}$. Since the definition is by recursion on the rank of B, we begin with $B = [\omega]^1$, presenting an equivalent definition of the Ellentuck space $\mathcal{E}$, which, in the present notation, is denoted as $\mathcal{E}_{[\omega]^1}$. The restriction map $r_k^B(S)$ is defined to give the first k members of S, and the full approximation map $\mathfrak{a}_k^B(S)$ is designed to ensure that the structure of previously formed spaces is preserved in the recursive construction.

Definition 17 (The Domain Space D_B, r_n^B, and $\mathfrak{a}_n^B$). For each uniform barrier B, we define the *domain space* D_B, the *restriction map* r_k^B and the *full approximation map* $\mathfrak{a}_k^B$. The construction is by recursion on the rank of B.

Starting with $B = [\omega]^1$, let

$$(8) \qquad\qquad D_{[\omega]^1} = \{S \subseteq S_{[\omega]^1} : |S| = \omega\}.$$

For each $S \in D_{[\omega]^1}$ and $k < \omega$, the k-th restriction and the k-th full approximation of S are the same: Letting $\{s_i : i < \omega\}$ enumerate S according to the $\prec$-well-ordering of S, define

$$(9) \qquad\qquad r_k^{[\omega]^1}(S) = \mathfrak{a}_k^{[\omega]^1}(S) = \{s_i : i < k\},$$

the k $\prec$-least members of S.

Suppose that for all uniform barriers C of rank less than α, D_C has been defined, and also $r_k^C(S)$ and $\mathfrak{a}_k^C(S)$ have been defined, for all $S \in D_C$ and $k < \omega$. Given a uniform barrier B of rank α, recalling Notation 4, for each

$n < \omega$, $B_{\{n\}}$ is a uniform barrier of some rank less than α on $[n+1, \omega)$. Let C^n denote the uniform barrier of rank less than α on ω obtained by shifting the members of $B_{\{n\}}$ down by $n+1$. That is, C^n is the collection of all sets $\{a_0 - (n+1), \ldots, a_l - (n+1)\}$, where $\{a_0, \ldots, a_l\} \in B_{\{n\}}$. Since $\mathrm{rank}(C^n) < \alpha$, D_{C^n} is already defined, along with the restriction and full approximation maps.

Given $s = (s_0, \ldots, s_l)$, let $s + (n+1)$ denote $(s_0 + n + 1, \ldots, s_l + n + 1)$. Let $v_n : S_{C^n} \to S_{B_n}$ be the map $v_n(s) = (n)^\frown(s + (n+1))$. Define

$$(10) \qquad D_{B_n} = \{v_n(S) : S \in D_{C^n}\}.$$

For $T \in D_{B_n}$ and $k < \omega$, define

$$(11) \qquad \mathfrak{a}_k^{B_n}(T) = v_n(\mathfrak{a}_k^{C^n}(v_n^{-1}(T))).$$

Let $\widehat{\mathfrak{a}_k^{B_n}(T)}$ denote $\{s \upharpoonright m : s \in \mathfrak{a}_k^{B_n}(T) \text{ and } 1 \le m \le \mathrm{lh}(s)\}$, the closure of $\mathfrak{a}_k^{B_n}(T)$ under initial segments. Given two subsets $T, T' \subseteq S_{\hat{B}_n}$, write $T \prec T'$ if and only if for each $s \in T$ and each $s' \in T'$, $s \prec s'$. For $S \subseteq S_B$ and $n < \omega$, let S_n denote $\{s \in S : s \unrhd (n)\}$.

For the sake of efficiency we shall define S_B and $\mathfrak{a}_k^B$ simultaneously, (though technically the definition of S_B does not depend on the definition of $\mathfrak{a}_k^B$). A subset $S \subseteq S_B$ is defined to be in D_B if and only if $\hat{S} \cap \omega^1$ is infinite, and enumerating $\hat{S} \cap \omega^1$ as $\{(n_0), (n_1), \ldots\}$,

   (1) For each $i < \omega$, $S_{n_i} := \{s \in S : s \unrhd (n_i)\} \in D_{B_{n_i}}$; and
   (2) Defining $\mathfrak{a}_0^B(S) = \emptyset$ and $\mathfrak{a}_1^B(S) = \mathfrak{a}^{B_{n_0}}(S_{n_0})$, given $k \ge 1$ and $\mathfrak{a}_k^B(S)$, we require that

   (a) $\mathfrak{a}_k^B(S) \prec \widehat{\mathfrak{a}_{k+1}^{B_{n_0}}(S_{n_0})} \setminus \widehat{\mathfrak{a}_k^B(S)}$, and

   (b) for each $1 \le i \le k$, $\mathfrak{a}_{k-i}^{B_{n_{i-1}}}(S_{n_{i-1}}) \prec \widehat{\mathfrak{a}_{k+1-i}^{B_{n_i}}(S_{n_i})} \setminus \widehat{\mathfrak{a}_{k-i}^{B_{n_i}}(S_{n_i})}$.
   Define $\mathfrak{a}_{k+1}^B(S) = \mathfrak{a}_k^B(S) \cup \mathfrak{a}_{k+1}^{B_{n_0}}(S_{n_0}) \cup \cdots \cup \mathfrak{a}_1^{B_{n_k}}(S_{n_k})$.

Finally, the restriction map r^B is simply defined by letting $r_k^B(S)$ be the set of the k $\prec$-least members of S.

Remark. First, notice that (2) is equivalent to requiring that

$$\mathfrak{a}_2^{B_{n_0}}(S_{n_0}) \prec \widehat{\mathfrak{a}_1^{B_{n_1}}(S_{n_1})}$$

$$\prec \widehat{\mathfrak{a}_3^{B_{n_0}}(S_{n_0})} \setminus \widehat{\mathfrak{a}_2^{B_{n_0}}(S_{n_0})} \prec \widehat{\mathfrak{a}_2^{B_{n_1}}(S_{n_1})} \setminus \widehat{\mathfrak{a}_1^{B_{n_1}}(S_{n_1})} \prec \widehat{\mathfrak{a}_1^{B_{n_2}}(S_{n_2})}$$

$$(12) \quad \prec \widehat{\mathfrak{a}_4^{B_{n_0}}(S_{n_0})} \setminus \widehat{\mathfrak{a}_3^{B_{n_0}}(S_{n_0})} \prec \widehat{\mathfrak{a}_3^{B_{n_1}}(S_{n_1})} \setminus \widehat{\mathfrak{a}_2^{B_{n_1}}(S_{n_1})} \prec \ldots$$

Second, the $k+1$-st full approximation $\mathfrak{a}_{k+1}^B(S)$ is equal to $\mathfrak{a}_{k+1}^{B_{n_0}}(S_{n_0}) \cup \mathfrak{a}_k^{B_{n_1}}(S_{n_1}) \cup \cdots \cup \mathfrak{a}_1^{B_{n_k}}(S_{n_k})$. Third, the last member of a full approximation is free to range over a tail of S. That is, given $S \in D_B$ and $k \ge 1$, letting n be such that $r_{n+1}^B(S) = \mathfrak{a}_k^B(S)$, the set $\{s \in S : r_n^B(S) \cup \{s\} = \mathfrak{a}_k^B(S')$ for

some $S' \in D_B\}$ is exactly the set $\bigcup\{S_m : m \geq \max(r_n^B(S))\}$, which is again a member of D_B.

Now we have the correct machinery to define the spaces which extend the Ellentuck space up to all countably infinite ranks.

Definition 18 (The Infinite Dimensional Ellentuck Spaces $(\mathcal{E}_B, \leq, r)$). For a uniform barrier B, let

$$(13) \qquad \mathcal{E}_B = \{\psi_B(S) : S \in D_B\}.$$

For $X, Y \in \mathcal{E}_B$, $Y \leq X$ if and only if $Y \subseteq X$. Given $X \in \mathcal{E}_B$ and $k < \omega$, letting S be the member of D_B such that $X = \psi_B(S)$, define the k-th restriction of X to be $r_k^B(X) = \psi_B(r_k^B(S))$ and the k-th full approximation of X to be $\mathfrak{a}_k^B(X) = \psi_B(\mathfrak{a}_k^B(S))$.

Fact 19. *If B and C are uniform barriers such that $C \lhd B$, then the projection map from B to C induces a projection from $\mathcal{E}_B$ to $\mathcal{E}_C$.*

This follows from noting that the map $\rho_C \circ \pi_{C,B} \circ \rho_B^{-1} : \mathcal{E}_B \to \mathcal{E}_C$ preserves that $\prec$, $\unlhd$, and $<_{\text{lex}}$ structures on the stems which are not projected, where $\pi_{C,B}$ denotes the projection map from B to C.

Example 20 (The Ellentuck space based on the Shreier barrier). Let $\mathcal{S}$ denote the *Shreier barrier*, the set $\{b \in [\omega]^{<\omega} : \text{lh}(b) = \max(b) + 1\}$. Then the first few members in the $\prec$-well-ordering of $S_{\mathcal{S}}$ are $() \prec (0) \prec (1) \prec (1,1) \prec (1,2) \prec (2) \prec (2,2) \prec (2,2,2) \prec (1,3) \prec (2,2,3) \prec (2,3) \prec (2,3,3) \prec (3) \prec (3,3) \prec (3,3,3) \prec (3,3,3,3) \prec (1,4) \prec \ldots$. The tree structure of sequence space $S_{\mathcal{S}}$ is the following.

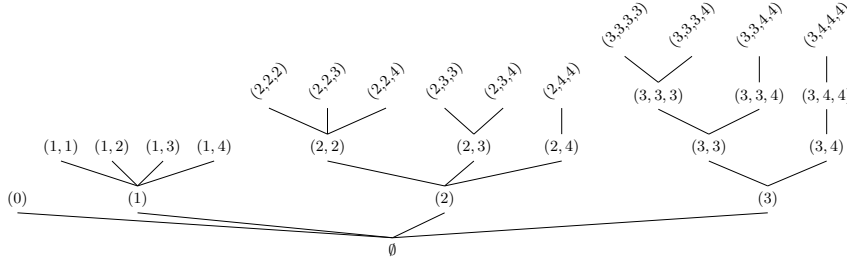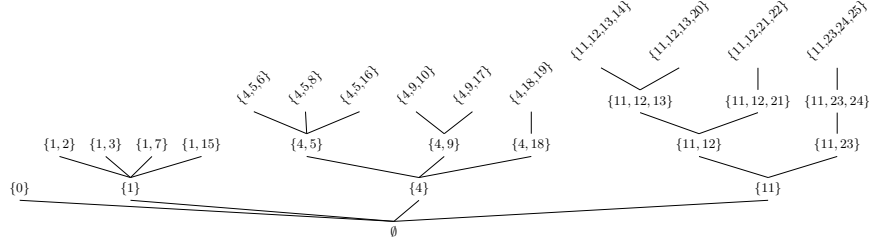
FIGURE 1. $S_{\mathcal{S}}$

The maximum member of the space $\mathcal{E}_{\mathcal{S}}$ is $\mathbb{W}_{\mathcal{S}}$.

The k-th restriction of $\mathbb{W}_{\mathcal{S}}$ consists of the k $\prec$-least members of $\mathbb{W}_{\mathcal{S}}$; for instance, $r_1^{\mathcal{S}}(\mathbb{W}_{\mathcal{S}}) = \{\{0\}\}$, $r_3^{\mathcal{S}}(\mathbb{W}_{\mathcal{S}}) = \{\{0\}, \{1,2\}, \{1,3\}\}$, and $r_5^{\mathcal{S}}(\mathbb{W}_{\mathcal{S}}) = \{\{0\}, \{1,2\}, \{1,3\}, \{1,7\}, \{4,5,8\}\}$.

The first full approximation of $\mathbb{W}_{\mathcal{S}}$ is $\mathfrak{a}_1^{\mathcal{S}}(\mathbb{W}_{\mathcal{S}}) = \{\{0\}\} = r_1^{\mathcal{S}}(\mathbb{W}_{\mathcal{S}})$. The second full approximation is $\mathfrak{a}_2^{\mathcal{S}}(\mathbb{W}_{\mathcal{S}}) = \{\{0\}, \{1,2\}\} = r_2^{\mathcal{S}}(\mathbb{W}_{\mathcal{S}})$. $\mathfrak{a}_3^{\mathcal{S}}(\mathbb{W}_{\mathcal{S}}) = \{\{0\}, \{1,2\}, \{1,3\}, \{4,5,6\}\} = r_4^{\mathcal{S}}(\mathbb{W}_{\mathcal{S}})$ and $\mathfrak{a}_4^{\mathcal{S}}(\mathbb{W}_{\mathcal{S}})$ equals the following:

$$\{\{0\}, \{1,2\}, \{1,3\}, \{4,5,6\}, \{1,7\}, \{4,5,8\}, \{4,9,10\}, \{11,12,13,14\}\},$$

FIGURE 2. $\mathbb{W}_{\mathcal{S}}$

which is equal to $r_8^{\mathcal{S}}(\mathbb{W}_{\mathcal{S}})$. We point out that for each $k < \omega$, $\min(\mathbb{W}_{\mathcal{S}}(k) \setminus r_k^{\mathcal{S}}(\mathbb{W}_{\mathcal{S}})) > \max(r_k^{\mathcal{S}}(\mathbb{W}_{\mathcal{S}}))$.

Notation. $\mathcal{AE}_k^B$ denotes the set $\{r_k^B(X) : X \in \mathcal{E}_B\}$, and $\mathcal{AE}^B$ denotes $\bigcup_{k<\omega} \mathcal{AE}_k^B$. For $X \in \mathcal{E}_B$ and $w, w' \in X$, note that $\max(w) < \max(w')$ if and only if $\psi_B^{-1}(w) \prec \psi_B^{-1}(w')$. Thus, we will often also use $\prec$ to denote the well-ordering of members of X. For $n < \omega$, $X(n)$ denotes the $w \in X$ such that $r_n(X) \cup \{w\} = r_{n+1}(X)$. Note that $X(n)$ has the n-th smallest value $\max(w)$ in X, and is the n-th least member in the $\prec$ ordering of X.

Given $a \in \hat{B} \setminus B$, let B_a denote $\{b \in B : b \rhd a\}$ and $w_a = \rho_B(a)$. For $X \in \mathcal{E}_B$ and any $w_* \in \hat{X} \setminus X$, let X_{w_*} denote $\{w \in X : w \rhd w_*\}$. Define $\mathcal{E}_{B_a} = \{X_{w_a} : X \in \mathcal{E}_B\}$. Further, let $n = \max(a)$, and let $\delta_n : [n+1, \omega) \to \omega$ be the 'shift down by $n+1$' map given by $\delta_n(k) = k - n - 1$. Letting B^a denote $\{\delta_n(b \setminus a) : b \in B_a\}$, we see that B^a is a uniform barrier on ω.

Remark. The idea behind the full approximation map is to ensure that our inductive construction yields the property that given any $a \in \hat{B} \setminus B$, the collection $\{X_{w_a} : X \in \mathcal{E}_B\}$ is isomorphic to the space $\mathcal{E}_{B^a}$. This next fact is one of the key properties of these spaces, justifying the use of terminology of infinite dimensional *Ellentuck space*.

Fact 21. *For $a \in \hat{B} \setminus B$, the space $\mathcal{E}_{B_a}$ is isomorphic to $\mathcal{E}_{B^a}$, via the map $X_{w_a} \mapsto \{\rho_{B^a} \circ \delta_n(\rho_B^{-1}(w) \setminus a) : w \in X_{w_a}\}$ for $X_{w_a} \in \mathcal{E}_{B_a}$, where $n = \max(a)$.*

In particular, for each $b \in B$, letting $a = b \setminus \{\max(b)\}$, $\mathcal{E}_{B_a}$ is isomorphic to *the* Ellentuck space. The fact is proved by fixing B and doing a straightforward induction on the rank of B_a, using the recursive definition of the domain spaces D_{B_a}.

The next fact shows that the set of all 1-extensions of a given $u \in \mathcal{AE}^B$ is also isomorphic to $\mathcal{E}_C$ for some uniform front C.

Fact 22. *Let $u \leq_{\text{fin}} X \in \mathcal{E}_B$. There is a unique $w_u \in \hat{X} \setminus X$ such that $r_{|u|+1}^B[u, X]$ is equal to the set*

$$(14) \qquad \{u \cup \{w\} \in X : w \rhd w_u \text{ and } \min(w \setminus w_u) > \max(u)\}.$$

*Let X_u denote $\{w \in X : w \rhd w_u$ and $\min(w \setminus w_u) > \max(u)\}$, and let a_u
denote $\rho_B^{-1}(w_u)$. Then $X_u = \{v(|u|) : v \in r_{|u|+1}^B[u, X]\}$ and is a member of
$\mathcal{E}_{B_{a_u}}$. In fact, $\{X_u : X \in \mathcal{E}_B$ and $u \leq_{\mathrm{fin}} X\}$ is a dense subset of $\mathcal{E}_{B_{a_u}}$.*

Proof. The proof follows by induction on $\mathrm{rank}(B)$ using the recursive defi-
nition of $r_{|u|+1}^B[u, X]$. The statement we are actually proving is that there is
a unique w_u such that X_u is the collection of last members of a full approx-
imation in $\mathcal{E}_{B_{a_u}}$ which extend $u \cap X_{w_u}$ by one element. For $\mathrm{rank}(B) < \omega$,
this fact was shown in [1]. Let B have infinite rank and suppose the fact
holds for all C with smaller rank. If $r_{|u|+1}^B(X) = \mathfrak{a}_k^B(X)$ for some k, then
X_u is exactly $\{w \in X : \min(w) > \max(u)\}$, which is a member of $\mathcal{E}_B$. In
this case, $w_u = \emptyset$ and $\{X_u : X \in \mathcal{E}_B$ and $u \leq_{\mathrm{fin}} X$ is a dense subset of $\mathcal{E}_B$.
If $r_{|u|+1}^B(X)$ is not a full approximation, then by the recursive construction
of D_B, there is some $a \in \hat{B} \setminus B$ such that the 1-extensions of u in $\mathcal{E}_B$ are
according to 1-approximations in the structure of D_{B_a}. Thus, by the in-
duction hypothesis, there is a unique $w_u \unrhd \rho_B(a)$ such that the collection
of 1-extensions of u into X in $\mathcal{E}_B$ is exactly X_u, which is a member of $\mathcal{E}_{B_a}$.
Then $a_u = a$. $\square$

We point out that for $u \leq_{\mathrm{fin}} X \in \mathcal{E}_B$, letting $S = \psi_B^{-1}(X)$ and $S_u = \psi_B^{-1}(X_u)$, then S_u is equal to $\{s \in S : s \rhd \psi_B^{-1}(w_u)$ and $\min(\psi_B(s) \setminus w_u) > \max(u)\}$ and is a member of $D_{B_{a_u}}$.

In the next Lemma, use the following notation: For $S \in D_B$, let S_n denote
$\{s \in S : S \unrhd (n)\}$; and for $X \in \mathcal{E}_B$, let X_n denote $\{w \in X : w \unrhd \rho_B(\{n\})\}$,
and note that this is equal to $\{w \in X : w \unrhd \psi_B((n))\}$. Note that for each
n, either $X_n \in \mathcal{E}_{B_n}$ or else $X_n = \emptyset$.

The following two Fusion Lemmas will be used in numerous diagonal
constructions.

Lemma 23 (Fusion). *Suppose that $S \in D_B$ and for infinitely many $n < \omega$,
S_n' is a member of D_{B_n} such that $S_n' \subseteq S_n$. Then there is an $S'' \in D_B$ such
that for each $n < \omega$ for which $S_n'' \neq \emptyset$, we have $S_n'' \subseteq S_n'$.*

*Hence, if $X \in \mathcal{E}_B$ and for infinitely many $n < \omega$, $Y_n \in \mathcal{E}_{B_n}$ with $Y_n \subseteq X_n$,
then there is a $Z \leq X$ in $\mathcal{E}_B$ such that for all $\rho_B(\{n\}) \in \hat{Z}$, $Z_n \subseteq Y_n$.*

Proof. Let $S \in D_B$ and $N \in [\omega]^\omega$, and suppose that for each $n \in N$,
S_n' is a member of D_{B_n} such that $S_n' \subseteq S_n$. The following is a mech-
anism for constructing $S'' \in D_B$ by constructing its full approximations
so that for each $(n) \in S''$, $S_n'' \subseteq S_n'$. Let n_0 be the least member of
N and define $\mathfrak{a}_2^B(S'')$ to be $\mathfrak{a}_2^{B_{n_0}}(S_{n_0}')$. Let n_1 be least in N such that
$n_1 > \max \mathfrak{a}_2^B(S'')$. Then take $\mathfrak{a}_2^B(S'')$ to be $\mathfrak{a}_2^{B_{n_0}}(S_{n_0}') \cup \mathfrak{a}_1^{B_{n_1}}(S_{n_1}')$. Now
let $m(2, 0) = \max \mathfrak{a}_2^B(S'')$. Extend $\mathfrak{a}_2^{B_{n_0}}(S_{n_0}')$ to any $\mathfrak{a}_3^{B_{n_0}}(S_{n_0}'') \subseteq S_{n_0}'$
such that the minimum of $\widehat{\mathfrak{a}_3^{B_{n_0}}(S_{n_0}'')} \setminus \widehat{\mathfrak{a}_3^{B_{n_0}}(S_{n_0}'')}$ is greater than $m(2, 0)$.
Next, let $m(2, 1)$ be the maximum of $\mathfrak{a}_3^{B_{n_0}}(S_{n_0}'')$ and extend $\mathfrak{a}_1^{B_{n_1}}(S_{n_1}'')$ to

any $\mathfrak{a}_2^{B_{n_1}}(S''_{n_1}) \subseteq S'_{n_1}$ such that the minimum of $\widehat{\mathfrak{a}_2^{B_{n_1}}(S''_{n_1})} \setminus \widehat{\mathfrak{a}_1^{B_{n_1}}(S''_{n_1})}$ is greater than $m(2,1)$. Then let n_2 be the least member of N greater than maximum of $\mathfrak{a}_2^{B_{n_1}}(S''_{n_1})$. Let $\mathfrak{a}_3^B(S'') = \mathfrak{a}_3^{B_{n_0}}(S''_{n_0}) \cup \mathfrak{a}_2^{B_{n_1}}(S''_{n_1}) \cup \mathfrak{a}_1^{B_{n_2}}(S'_{n_2})$. Continuing in this manner, one constructs $S'' = \bigcup_{i \geq 2} \mathfrak{a}_i^B(S'')$ in D_B and an infinite subset $\{n_i : i < \omega\} \subseteq N$ such that for each $i < \omega$, $S''_{n_i} \subseteq S'_{n_i}$, and for each n not equal to some n_i, $S''_n = \emptyset$. $\qquad \square$

Lemma 24 (Fusion). *Let $u \leq_{\text{fin}} X \in \mathcal{E}_B$. Let N be an infinite set of integers greater than $\max(a_u)$, and for each $n \in N$, let $Y_n \in \mathcal{E}_{B_{a_n}}$ with $Y_n \subseteq \{w \in X : w \trianglerighteq \rho_B(a_n)\}$, where a_n denotes $a_u \cup \{n\}$. Then there is a $Z \in [u, X]$ such that for each $n > \max(a_u)$, $Z_n \subseteq Y_n$.*

In particular, given $u \leq_{\text{fin}} X \in \mathcal{E}_B$ and $Y_u \in \mathcal{E}_{B_{a_u}}$ such that $Y_u \subseteq X_u$, then there is a $Z \in [u, X]$ such that $Z_u \subseteq Y_u$.

Proof. Apply Fusion Lemma 23 in $\mathcal{E}_{B_{a_u}}$ to the collection of Y_n, $n \in N$, to find a $U \in \mathcal{E}_{B_{a_u}}$ such that for each $n \in \omega$, either $U_n = \emptyset$, or else both $n \in N$ and $U_n \subseteq Y_n$, where U_n denotes $\{w \in U : w \trianglerighteq \rho_B(a_n)\}$. Then extend u into X recursively as follows. Let $k = |u|$ and let $z_k = u$. For $m \geq k$, given z_m, if $w_{z_m} \trianglerighteq w_u$, then take $z_{m+1} \in r_{m+1}[z_m, X]$ such that $z_{m+1}(m) \in U$; otherwise, take any $z_{m+1} \in r_{m+1}[z_m, X]$. Letting $Z = \bigcup_{m \geq k} z_m$ proves the first part.

The second part follows immediately from the construction method of Z in the first part. $\qquad \square$

The next fact shows that any two spaces are comparable, allowing for restrictions below some members.

Fact 25. *For any uniform barriers B and C, there are $X \in \mathcal{E}_B$ and $Y \in \mathcal{E}_C$ such that either the subspace $\mathcal{E}_B|X$ embeds into the subspace $\mathcal{E}_C|Y$, or vice versa.*

The proof is by a straightforward induction on the maximum of the ranks of B and C using the Fusion Lemma 23. This base case comes from [1], as this fact is proved there for all pairs of uniform barriers of finite rank.

This section concludes by proving the main theorem of this section, Theorem 28: Each space $(\mathcal{E}_B, \leq, r)$ is a topological Ramsey space; hence, every subset of $\mathcal{E}_B$ with the property of Baire is Ramsey. Since $\mathcal{E}_B$ is a closed subspace of $(\mathcal{AE}^B)^{\mathbb{N}}$, it suffices, by the Abstract Ellentuck Theorem 2, to show that $(\mathcal{E}_B, \leq, r)$ satisfies the axioms **A.1** - **A.4**. Leaving the routine checking of axioms **A.1** and **A.2** to the reader, we first show that **A.3** holds for $\mathcal{E}_B$ for all uniform barriers B. Then we show by induction on rank of B that **A.4** holds for $\mathcal{E}_B$, assuming that $\mathcal{E}_C$ for C of smaller rank than B have already been proved to be topological Ramsey spaces.

Fact 26. *Given $u \in \mathcal{AE}^B$, if $u \subseteq X \in \mathcal{E}_B$, then $r_{|u|+1}[u, X] \neq \emptyset$.*

Lemma 27. *For each B, the space $(\mathcal{E}_B, \leq, r)$ satisfies Axiom **A.3**.*

Proof. Given $u \in \mathcal{AE}^B$ and $X \in \mathcal{E}_B$ such that $\mathrm{depth}_X(u) = d < \infty$, and letting $Y \in [d, X]$, construct $Z \in [u, Y]$ as follows. Let $z_m = u$, where $m = |u|$. Note that $z_m \subseteq X$. Given $n \geq m$ and $z_n \sqsupseteq u$ such that $z_n \subseteq X$, by Fact 26, $r_{n+1}[z_n, X] \neq \emptyset$, so we may take $z_{n+1} \in r_{n+1}[z_n, X]$. Then $z_{n+1} \subseteq Z$, so the induction process continues. Letting $Z = \bigcup_{n \geq m} z_n$, we see that $Z \in [u, X]$; hence **A.3** (1) holds.

For **A.3** (2), suppose $Y \leq X$ and $[u, Y] \neq \emptyset$. Let $d = \mathrm{depth}_X(u)$ and let $z_d = r_d(X)$. For $n \geq d$ with z_n already chosen, if $w_{z_n} \in \hat{u}$, then choose $z_{n+1} \in r_{n+1}[z_n, X]$ such that $z(n) \in Y$. Otherwise, simply choose any $z_{n+1} \in r_{n+1}[z_n, X]$. Let $Z = \bigcup_{n \geq d} z_n$. Then $Z \in [d, X]$. Now if $Z' \in [u, Z]$, then for each $n < \omega$, there is a $w \in \hat{u}$ such that $Z'(n) \rhd w$, which implies that $Z'(n) \in Y$. Hence, $Z' \in [u, Y]$, so $[u, Z] \subseteq [u, Y]$. $\qquad\square$

Theorem 28. *For each uniform barrier B on ω, $(\mathcal{E}_B, \leq, r)$ is a topological Ramsey space.*

Proof. The proof is by induction on $\mathrm{rank}(B)$. If $\mathrm{rank}(B) = 1$, then $\mathcal{E}_B$ is the Ellentuck space; for $2 \leq k < \omega$, $\mathcal{E}_{[\omega]^k}$ is the space $\mathcal{E}_k$ which is a topological Ramsey space by Theorem 21 in [1]. Now suppose $\mathrm{rank}(B) = \alpha \geq \omega$, and for all uniform barriers C with $\mathrm{rank}(C) < \alpha$, $\mathcal{E}_C$ is a topological Ramsey space. It suffices to prove the Pigeonhole Principle **A.4** for $\mathcal{E}_B$.

Let $X \in \mathcal{E}_B$, $u = r_k(X)$, and $\mathcal{O} \subseteq \mathcal{AE}_{k+1}^B$, and let X_u denote $\{v(k) : v \in r_{k+1}[u, X]\}$. Recalling that a_u denotes $\rho_B^{-1}(w_u)$, we point out that $X_u \in \mathcal{E}_{B_{a_u}}$ and $\{\{w\} : w \in X_u\} \subseteq \mathcal{AE}_1^{B_{a_u}}$. If $w_u \neq \emptyset$, then by the induction hypothesis, $\mathcal{E}_{B_{a_u}}$ is a topological Ramsey space. Define a coloring $c : X_u \to 2$ by $c(w) = 0$ if and only if $u \cup \{w\} \in \mathcal{O}$. By the Abstract Nash-Williams Theorem for $\mathcal{AE}_1^{B_{a_u}}$, there is a $Y_u \in \mathcal{E}_{B_{a_u}} | X_u$ such that each $w \in Y_u$ has the same color. By the Fusion Lemma 24, there is a $Z \in [u, X]$ such that $Z_u \subseteq Y_u$. Thus, Z satisfies **A.4**.

Suppose now that $w_u = \emptyset$. Then X_u equals the set of all $w \in X$ for which $\min(w) > \max(u)$. Let N denote the set of $n > \max(u)$ for which $X_n \neq \emptyset$. For each $n \in N$, let X_n denote $\{w \in X : w \rhd \{n\}\}$ and let a_n denote $\rho_B^{-1}(\{n\})$. Since $\mathrm{rank}(B_{a_n}) < \alpha$, the induction hypothesis yields that $\mathcal{E}_{B_{a_n}}$ is a topological Ramsey space. Let $c_n : X_n \to 2$ be the coloring such that $c_n(w) = 0$ if and only if $u \cup w \in \mathcal{O}$, for $w \in X_n$. Since the set of singletons $\{\{w\} : w \in X_n\}$ is actually a subset of $\mathcal{AE}_1^{B_{a_n}}$, the Abstract Nash-Williams Theorem for $\mathcal{E}_{B_{a_n}}$ implies that there is a $Y_n \in \mathcal{E}_{B_{a_n}}$ such that $Y_n \subseteq X_n$ and Y_n is homogeneous for the coloring c_n. Take an infinite set $M \subseteq N$ such that $c_n \upharpoonright Y_n$ have the same color for all $n \in M$. By the Fusion Lemma 24, there is a $Z \in [u, X]$ such that for each $n > \max(u)$ for which $Z_n \neq \emptyset$, n must be in M and $Z_n \subseteq Y_n$. Thus, **A.4** is satisified by Z. $\qquad\square$

6. Uniform projections and canonical equivalence relations on 1-extensions

In [1], we proved that, for $2 \leq k < \omega$, each equivalence relation on $\mathcal{AE}_1^{[\omega]^k}|X$ is essentially one of $k+1$ canonical types: projections to $[\omega]^n$ for $n \leq k$. That is, there is some $n \leq k$ and some $Y \leq X$ such that for all pairs $w = \{i_0, \ldots, i_{k-1}\}$ and $w' = \{j_0, \ldots, j_{k-1}\}$ of elements in Y, w and w' are equivalent if and only if their projections to length n, $\pi_n(w) = \{i_l : l < n\}$ and $\pi_n(w') = \{j_l : l < n\}$, are equal.

For uniform barriers B of infinite rank, there are no longer only finitely many canonical equivalence relations on $\mathcal{AE}_1^B$; in fact, there are continuum many. However, just as the projections π_n, $n \leq k$, on $\mathcal{AE}_1^{[\omega]^k}$ gave images which were of the form of the uniform barrier $[\omega]^n$, so too the structure of a canonical equivalence relation on $\mathcal{AE}_1^B$ for an infinite rank uniform barrier B closely resembles the structure of a projection of B to some uniform barrier consisting of initial segments of members of B. We make this precise below.

The following notation will be used throughout.

Notation. We shall often use the meet symbol $w \wedge w'$ to denote $w \cap w'$ for $w, w' \in \widehat{\mathbb{W}}_B$, since the tree structure of $\widehat{\mathbb{W}}_B$ under initial segments is quite important. For $X \in \mathcal{E}_B$, $\hat{X}$ denotes $\{w' \in [\omega]^{<\omega} : \exists w \in X \, (w \trianglerighteq w')\}$. For $w_* \in \hat{X} \setminus X$, let $X_{w_*} = \{w \in \hat{X} : w \triangleright w_*\}$. Note that $X_{w_*} \in \mathcal{E}_{B_{a_*}}$, where $a_* = \rho_B^{-1}(w_*)$. For $u \in \mathcal{AE}^B$, let w_u denote the $\vartriangleleft$-maximal member of $\widehat{\mathbb{W}}_B$ such that for all $X \in [u, \mathbb{W}_B]$, $X(|u|) \triangleright w_u$. Note then that for any $X \geq_{\mathrm{fin}} u$, w_u is equal to the meet $\bigwedge \{v(|u|) : v \in r_{|u|+1}[u, X]\}$. Let a_u denote $\rho_B^{-1}(w_u)$. For $u \leq_{\mathrm{fin}} X$, recall that $X_u = \{w \in X_{w_u} : \min(w \setminus w_u) > \max(u)\}$ is a member of $\mathcal{E}_{B_{a_u}}$, and is exactly the set of those $w \in X$ such that $u \cup \{w\} \in r_{|u|+1}[u, X]$, and that $X_u \in \mathcal{E}_{B_{a_u}}$.

Given a uniform barrier B, for each $X \in \mathcal{E}_B$ and $w_* \in \hat{X} \setminus X$, we will define the set of all *uniform projections on X_{w_*}* by induction on $\mathrm{rank}(B_a)$, where $a = \rho_B^{-1}(w_*)$.

Definition 29 (Uniform Projections UP(B, X, w))**.** Let $X \in \mathcal{E}_B$ and $w \in \hat{X} \setminus X$, and let a denote $\rho_B^{-1}(w)$. If $\mathrm{rank}(B_a) = 1$, define UP$(B, X, w) = \{\{w\}, X_w\}$. Assume now that for all triples (B, X, w) with $\mathrm{rank}(B_a) < \alpha$, the collection UP$(B, X, w)$ has been defined. Let (B, X, w) be a triple with $\mathrm{rank}(B_a) = \alpha$. A subset $P \subseteq \hat{X}_w$ is in UP(B, X, w) if and only if the following (1) - (4) hold:

    (1) $\forall p \in P$, $p \trianglerighteq w$;

    (2) P is a *front on X_w*; by this we mean that for each $w' \in X_w$, there is a $p \in P$ such that $p \trianglelefteq w'$, and for all $p, p' \in P$, $p \trianglelefteq p'$ implies $p = p'$.

Enumerating those $n > \max(a)$ such that $a \cup \{n\} \in \rho_B^{-1}(\hat{X}_w)$ as $n_0 < n_1 < \ldots$, and letting $p_i = \rho_B(a \cup \{n_i\})$,

    (3) For all $i < \omega$, $P_i := \{p \in P : p \trianglerighteq p_i\}$ is in UP$(B_{a \cup \{n_i\}}, X, p_i)$; and

(4) Either
   (a) there is a $\beta < \alpha$ such that for all $i < \omega$, $\mathrm{rank}(P_i) = \beta$; or
   (b) there is a strictly increasing sequence β_i with $\sup_{i<\omega} \beta_i = \alpha$ such that for each $i < \omega$, $\mathrm{rank}(P_i) = \beta_i$.

Remark. $\rho_B^{-1}(P)$ is usually not a barrier or even a front on some infinite subset of ω, but it does have the structure of a uniform barrier. The above construction recursively defines the notion of uniformness. Further, the construction recursively defines the notion of rank of a uniform projection in the same way that the recursive definition of uniform barrier defines the notion of rank of a uniform barrier.

Given $u \leq_{\mathrm{fin}} X \in \mathcal{E}_B$ let $\mathrm{UP}(B, X, u)$ denote $\{P \in \mathrm{UP}(B, X, w_u) : P \subseteq \hat{X}_u\}$. This is the set of *uniform projections on X_u*. Given $P \in \mathrm{UP}(B, X, u)$ and $Y \leq X$ with $u \leq_{\mathrm{fin}} Y$, we shall write $P|Y$ to denote $P \cap \hat{Y}_u$. Note that $P|Y \in \mathrm{UP}(B, Y, u)$.

Given $P, Q \in \mathrm{UP}(B, X, u)$, write $P \lhd Q$ if and only if for all pairs $p \in P$ and $q \in Q$, either $p \lhd q$ or else p and q are $\unlhd$-incomparable. The next lemma shows that and two members of $\mathrm{UP}(B, X, u)$ are $\lhd$-comparable or equal on some smaller $Y \leq X$.

Lemma 30. *Let $u \leq_{\mathrm{fin}} X \in \mathcal{E}_B$, and let $P, Q \in \mathrm{UP}(B, X, u)$. Then there is a $Y \in [u, X]$ such that $P|Y \lhd Q|Y$, $P|Y = Q|Y$, or $P|Y \rhd Q|Y$.*

Proof. The proof is by induction on rank of B_{a_u}, over all pairs $u \leq_{\mathrm{fin}} X \in \mathcal{E}_B$. If $\mathrm{rank}(B_{a_u}) = 1$, then $\mathrm{UP}(B, X, u)$ is equal to the set $\{\{w_u\}, X_u\}$, these uniform projections having rank 0 and 1, respectively. The lemma immediately follows for this case.

Now suppose that for all $u \leq_{\mathrm{fin}} X$ with $\mathrm{rank}(B_{a_u}) < \alpha$, there is a $Y \in [u, X]$ such that the pair $P|Y$, $Q|Y$ satisfy the lemma. Fix $u \leq_{\mathrm{fin}} X \in \mathcal{E}_B$ with $\mathrm{rank}(B_{a_u}) = \alpha$. Let $P, Q \in \mathrm{UP}(B, X, u)$, and let N denote the set of all $n > \max(u)$ for which $w_u \cup \{n\} \in \hat{X}$. For each $n \in N$, let $\gamma_n = \mathrm{rank}(P|X_{w_u \cup \{n\}})$, $\delta_n = \mathrm{rank}(Q|X_{w_u \cup \{n\}})$, and $a_n = \rho_B^{-1}(w_u \cup \{n\})$. Then γ_n and δ_n are less than α, so apply the induction hypothesis to obtain $Y_n \in \mathcal{E}_{B_{a_n}}$ such that $Y_n \leq X_{w_u \cup \{n\}}$ and one of $P|Y_n \lhd Q|Y_n$, $P|Y_n = Q|Y_n$, $P|Y_n \rhd Q|Y_n$ holds. Let M be an infinite subset of N such that the same one of the three possibilities holds for each $n \in M$. Apply the Fusion Lemma 24 to obtain a $Z \in [u, X]$ such that for each $n \in \omega$, if $Z_{w_u \cup \{n\}} \neq \emptyset$, then $n \in M$ and $Z_{w_u \cup \{n\}} \leq Y_n$ in $\mathcal{E}_{B_{a_n}}$. Then Z satisfies the lemma. $\square$

Definition 31 (Canonical Projection Maps)**.** For $X \in \mathcal{E}_B$, $w_* \in \hat{X} \setminus X$, and $P \in \mathrm{UP}(B, X, w_*)$, define the *projection map* $\pi_P : X_{w_*} \to P$ as follows. If $P = \{w_*\}$, then define $\pi_P(w) = \emptyset$ for all $w \in X_{w_*}$. Otherwise, for $w \in X_{w_*}$, let $\pi_P(w)$ be *the* $p \in P$ such that $p \unlhd w$. The set $\{\pi_P : P \in \mathrm{UP}(B, X, w_*)\}$ is the collection of *canonical projections* on X_{w_*}.

Definition 32 (Canonical Equivalence Relations on X_{w_*})**.** Given $P \in$ $\mathrm{UP}(B, X, w_*)$, define the equivalence relation E_P on X_{w_*} by $w \ E_P \ w'$ if and only if $\pi_P(w) = \pi_P(w')$.

Since for $X \in \mathcal{E}_B$, $\mathcal{A}\mathcal{E}_1^B | X = \{\{w\} : w \in X\}$, an equivalence relation on $\mathcal{A}\mathcal{E}_1^B | X$ is essentially the same as an equivalence relation on X, and vice versa. Thus, we may consider the members of $\mathrm{UP}(B, X) := \mathrm{UP}(B, X, \emptyset)$ as canonical projections on X.

For $\{n\} \in \rho_B^{-1}(X)$, let $w_n = \rho_B(\{n\})$ and let $\mathrm{UP}(B_n, X_n)$ denote the appropriate shift of $\mathrm{UP}(C, Y)$, where $C = \delta_n(B_n \backslash \{n\})$ and $Y = \rho_C(\delta_n(\rho_B^{-1}(X_n) \backslash \{n\}))$. Precisely, $\mathrm{UP}(B_n, X_n)$ is the set of all $\rho_B(\{n\} \cup (\rho_C^{-1}(Q) + (n+1))$, for $Q \in \mathrm{UP}(C, Y)$.

Theorem 33 (Canonization for $\mathcal{A}\mathcal{E}_1^B$)**.** *Let $u \leq_{\mathrm{fin}} X \in \mathcal{E}_B$, and let E be an equivalence relation on the members of X. Then there is a $Z \leq X$ and a $P \in \mathrm{UP}(B, X)$ such that $E|Z = E_P|Z$.*

Proof. The proof proceeds by induction on the rank of B, using the fact that, if $C = \{\delta_n(b \backslash \{n\}) : b \in B_n\}$, then there is an isomorphic correspondence between $\mathcal{E}_C$ and $\mathcal{E}_{B_n}$. If $\mathrm{rank}(B) = 1$, then $\mathcal{E}_B$ is the Ellentuck space. Given any equivalence relation E on some $X \in \mathcal{E}$, the Erdős-Rado Theorem in [29] implies there is a $Y \leq X$ such that either for all $w, w' \in Y$, $w \ E \ w'$, or else for all $w, w' \in Y$, $w \ \not{E} \ w'$. In the first case, $P = \{\emptyset\}$, and in the second case, $P = Y$ provide the uniform projections in $\mathrm{UP}(B, Y)$ such that E_P canonizes E on Y.

Now suppose that for all C with $\mathrm{rank}(C) < \alpha$, the lemma holds, and suppose that $\mathrm{rank}(B) = \alpha$. Let $X \in \mathcal{E}_B$ and E be an equivalence relation on X. Let $\mathcal{H} = \{\mathfrak{a}_2^B(Y) : Y \leq X\}$, the set of all second full approximations of members below X. Note that for $x \in \mathcal{H}$, $x(0) \wedge x(|x| - 1) = \emptyset$, and moreover, for any $w, w' \in X$ with $w \wedge w' = \emptyset$, there is an $x \in \mathcal{H}$ such that $\{w, w'\} = \{x(0), x(|x| - 1)\}$.

Let $\mathcal{H}_0 = \{x \in \mathcal{H} : x(0) \ E \ x(|x| - 1)\}$, and apply the Abstract Nash-Williams Theorem to obtain $Y \leq X$ homogeneous for $\mathcal{H}_0$. Suppose that $\mathcal{H}|Y \subseteq \mathcal{H}_0$, and let $w, w' \in Y$. Take $w'' \in Y$ so that $\min(w'') > \max(w \cup w')$. Then there are $x, y \in \mathcal{H}|Y$ such that $x(0) = w$, $y(0) = w'$, and $x(|x| - 1) = y(|y| - 1) = w''$. Since $x(0) \ E \ x(|x| - 1)$ and $y(0) \ E \ y(|y| - 1)$, it follows that $w \ E \ w'$. Therefore, each pair of members of Y are equivalent, and $P = \{\emptyset\}$ is the uniform projection canonizing $E|Y$.

Now suppose that $\mathcal{H}|Y \cap \mathcal{H}_0 = \emptyset$. It follows that whenever $w, w' \in Y$ with $w \cap w' = \emptyset$, then $w \ \not{E} \ w'$. Let w_n denote $\rho_B(\{n\})$ and Y_n denote $\{w \in Y : w \trianglerighteq w_n\}$, and let $N \subseteq \omega$ be the set of all n such that $Y_n \neq \emptyset$ (equivalently, $Y_n \in \mathcal{E}_{B_n}$). By the induction hypothesis, for each $n \in N$, there is a uniform projection $P_n \in \mathrm{UP}(B_n, Y_n)$ such that for any two $w, w' \in Y_n$, $w \ E \ w'$ if and only if $\pi_{P_n}(w) = \pi_{P_n}(w')$.

There is an infinite subset $N' \subseteq N$ such that either for all $m, n \in N$, $m < n$ in N' implies $\mathrm{rank}(P_m) = \mathrm{rank}(P_n)$, or else for all $m, n \in N$, $m < n$

in N' implies $\text{rank}(P_m) < \text{rank}(P_n)$. Apply the Fusion Lemma 23 to obtain a $Z \leq X$ such that whenever $Z_n \neq \emptyset$, then $n \in N'$ and $Z_n \subseteq Y_n$. Let $P = \bigcup\{P_n : n \in N' \text{ and } Z_n \neq \emptyset\}$. Then P is in $\text{UP}(B, Z)$.

We claim that π_P canonizes $E|Z$. Let $w, w' \in Z$. Notice that for each $n \in N'$, $w_n \trianglelefteq P_n$; so if $w \cap w' = \emptyset$, then $\pi_P(w) \neq \pi_P(w')$ and also $w \not\mathrel{E} w'$. If $w \cap w' \neq \emptyset$, then let n be such that $w_n \trianglelefteq w, w'$. Then $w \mathrel{E} w'$ if and only if $\pi_{P_n}(w) = \pi_{P_n}(w')$ if and only if $\pi_P(w) = \pi_P(w')$. Therefore, P canonizes $E|Z$. $\hfill\square$

Notation. Given $u \leq_{\text{fin}} X \in \mathcal{E}_B$, let w_u denote the $\lhd$-maximum in $\hat{X}$ such that $w_u \lhd v(|u|)$ for every $v \in r_{|u|+1}[u, X]$. Let X_{w_u} denote $\{w \in X : w \rhd w_u\}$ and X_u denote $\{w \in X : u \cup \{w\} \in r_{|u|+1}[u, X]\}$. Note that $X_u = \{w \in X_{w_u} : \min(w \setminus w_u) > \max(u)\}$, which we also denote as X_{w_u}/u.

This section concludes with the next theorem, laying some groundwork for the Ramsey-classification Theorem in the next section.

Theorem 34 (Canonization for 1-Extensions). *Let $u \leq_{\text{fin}} X \in \mathcal{E}_B$, and let E be an equivalence relation on $r_{|u|+1}[u, X]$. Then there is a $Z \in [u, X]$ and a $P \in \text{UP}(B, Z, u)$ such that E is canonical on $r_{|u|+1}[u, Z]$.*

Proof. Let a denote $\rho_B^{-1}(w_u)$. By Theorem 33, there is a $Y_{w_u} \in \mathcal{E}_{B_a}$ such that $Y_{w_u} \subseteq X_u$ and a $Q \in \text{UP}(B_a, Y_{w_u})$ such that $E \upharpoonright Y_{w_u} = E_Q \upharpoonright Y_{w_u}$. Fusion Lemma 24 implies that there is a $Z \in [u, X]$ such that $Z_u \subseteq Y_{w_u}$. Then $Q \cap Z_u$ is in $\text{UP}(B, Z, u)$ and canonizes E on $r_{|u|+1}[u, Z]$. $\hfill\square$

7. Ramsey-classification Theorems

The main theorem of this section is the Ramsey-classification Theorem 45 showing that for each uniform barrier B, any given equivalence relation on a front on $\mathcal{E}_B$ is canonized by a projection map which is Nash-Williams and further satisfies a certain property $(*)$, when restricted to some member of $\mathcal{E}_B$ (see Definitions 42 and 43 below). A projection map projects each member $u \in \mathcal{A}\mathcal{E}^B$ to some subtree of $\hat{u}$. The main theorem extends Theorem 33 of [1], which in turn extends the Pudlák-Rödl Theorem in the following sense: If $C \lhd B$ and R is an equivalence relation on a front $\mathcal{F}$ consisting of full approximations on $\mathcal{E}_C$, then taking $\mathcal{F}'$ to be front of all full approximations on $\mathcal{E}_B$ extending members of $\mathcal{F}$ and taking R' to be the induced equivalence relation on $\mathcal{F}'$, the canonical map for R' on $\mathcal{F}'|X$ projects to a canonical map for R on $\mathcal{F}|\pi_{C,B}(X)$.

The following notation will be used throughout.

Notation. Let $X \in \mathcal{E}_B$, $u, v \in \mathcal{A}\mathcal{E}^B$, and $k, n < \omega$. We shall leave the superscript B off of $r_k^B(X)$ whenever no confusion will arise. $X(n)$ denotes the $\prec$-n-th member of X, and $r_k(X) = \{X(n) : n < k\}$. Let $\text{Ext}(u, v) = \{Y \in \mathcal{E}_B : u \leq_{\text{fin}} Y\}$, $\text{Ext}(u, X) = \{Y \leq X : u \leq_{\text{fin}} Y\}$, and $\text{Ext}(u, v, X) = \text{Ext}(u, X) \cap \text{Ext}(v, X)$.

Let $X/u = \{X(n) : n < \omega \text{ and } \max X(n) > \max u\}$ and $v/u = \{v(n) : n < |v| \text{ and } \max v(n) > \max u\}$. Let $[v, X/u]$ denote $\{Y \in \mathcal{E}_B : v \sqsubset Y \text{ and } Y/v \subseteq X/u\}$. Let $r_k[u, X/v]$ denote $\{z \in \mathcal{AE}_k^B : z \sqsupseteq u \text{ and } z/u \subseteq X/v\}$, and $r[u, X/v]$ denote $\bigcup\{r_k[u, X/v] : k \geq |u|\}$. Let $\operatorname{depth}_X(u, v) = \max\{\operatorname{depth}_X(u), \operatorname{depth}_X(v)\}$.

Fact 35. *Suppose $Y \leq X$ are in $\mathcal{E}_B$ and $u, v \in \mathcal{AE}^B$ satisfy $v \leq_{\mathrm{fin}} u$, $\max u = \max v$, $u \leq_{\mathrm{fin}} X$, and $v \leq_{\mathrm{fin}} Y$. Then there is a $Z \in [u, X]$ such that for each $y \leq_{\mathrm{fin}} v$ and each $z \in r[y, Z/v]$, $z/v \subseteq Y$.*

Proof. Let $d = |u|$ and $z_d = u$. Given z_n for $n \geq d$, if $w_{z_n} \in \hat{Y}$, then choose $z_{n+1} \in r_{n+1}[z_n, Y/u]$. If $w_{z_n} \notin \hat{Y}$, then choose any $z_{n+1} \in r_{n+1}[z_n, X]$. It is straightforward to check that $Z = \bigcup_{n \geq d} z_n$ satisfies the conclusion. $\square$

Recall Definition 5 of front on a topological Ramsey space from Section 3.

Definition 36. Let $\mathcal{F}$ be a front on $\mathcal{E}_B$ and let R be an equivalence relation on $\mathcal{F}$. Let $\hat{\mathcal{F}} = \{r_k(u) : u \in \mathcal{F} \text{ and } k \leq |u|\}$. Suppose $u, v \in \hat{\mathcal{F}}$ and $X \in \operatorname{Ext}(u, v)$. We say that X *separates* u and v if and only if for all $y \in \mathcal{F} \cap r[u, X/v]$ and $z \in \mathcal{F} \cap r[v, X/u]$, $y \not\mathrel{R} z$. We say that X *mixes* u and v if and only if no $Y \in \operatorname{Ext}(u, v, X)$ separates u and v. We say that X *decides* for u and v if and only if either X mixes u and v or else X separates u and v.

Note that $X \in \operatorname{Ext}(u, v)$ mixes u and v if and only if for each $Y \in \operatorname{Ext}(u, v, X)$, there are $y \in \mathcal{F} \cap r[u, Y/v]$ and $z \in \mathcal{F} \cap r[v, Y/u]$ for which $y \mathrel{R} z$. The following fact and lemma follow from similar proofs to those of Facts 24 and 25 in [1]. We do point out that the proof of the transitivity of mixing is the one place in this article where Abstract Ellentuck Theorem is used and the Abstract Nash-Williams Theorem does not seem to be sufficient.

Fact 37. *The following are equivalent for $X \in \operatorname{Ext}(u, v)$:*
  1. *X mixes u and v.*
  2. *For all $Y \in \operatorname{Ext}(u, v, X)$, there are $y \in \mathcal{F} \cap r[u, Y/v]$ and $z \in \mathcal{F} \cap r[v, Y/u]$ for which $y \mathrel{R} z$.*
  3. *For all $Y \in [\operatorname{depth}_X(u, v), X]$, there are $y \in \mathcal{F} \cap r[u, Y/v]$ and $z \in \mathcal{F} \cap r[v, Y/u]$ for which $y \mathrel{R} z$.*

Lemma 38 (Transitivity of Mixing). *Suppose that X mixes u and v and X mixes v and z. Then X mixes u and z.*

Next, we define the notion of a hereditary property, and give a general lemma about fusion to obtain a member of $\mathcal{E}_B$ on which a hereditary property holds.

Definition 39. A property $P(u, X)$ defined on $\mathcal{AE}^B \times \mathcal{E}_B$ is *hereditary* if whenever $X \in \operatorname{Ext}(u)$ and $P(u, X)$ holds, then also $P(u, Y)$ holds for all $Y \in [\operatorname{depth}_X(u), X]$. Similarly, a property $P(u, v, X)$ defined on $\mathcal{AE}^B \times \mathcal{AE}^B \times \mathcal{E}_B$

is *hereditary* if whenever $P(u, v, X)$ holds, then also $P(u, v, Y)$ holds for all $Y \in [\text{depth}_X(u, v), X]$.

Lemma 40. *Let $P(\cdot, \cdot)$ be a hereditary property on $\mathcal{AE}^B \times \mathcal{E}_B$. If whenever $u \leq_{\text{fin}} X$ there is a $Y \in [\text{depth}_X(u), X]$ such that $P(u, Y)$, then for each $Z \in \mathcal{E}_B$, there is a $Z' \leq Z$ such that for all $u \in \mathcal{AE}^B | Z'$, $P(u, Z')$ holds.*

Likewise, suppose $P(\cdot, \cdot, \cdot)$ is a hereditary property on $\mathcal{AE}^B \times \mathcal{AE}^B \times \mathcal{E}_B$. If whenever $X \in \text{Ext}(u, v)$ there is a $Y \in [\text{depth}_X(u, v), X]$ such that $P(u, v, Y)$, then for each $Z \in \mathcal{R}$, there is a $Z' \leq Z$ such that for all $u, v \in \mathcal{AE}^B | Z'$, $P(u, v, Z')$ holds.

The proof of Lemma 40 is straightforward, using the Fusion Lemma 24; being very similar to that of Lemma 4.6 in [12], we omit it.

Lemma 41. *Given any front $\mathcal{F}$ and function $f : \mathcal{F} \to \omega$, there is an $X \in \mathcal{E}_B$ such that for all $u, v \in \hat{\mathcal{F}} | X$, X decides u and v.*

Lemma 41 follows immediately from Lemma 40 and the fact that mixing and separating are hereditary properties.

Given a set T' of $\trianglelefteq$-incomparable members of $\widehat{\mathbb{W}}_B$, we call T a *projection of T'* if T is a collection of $\trianglelefteq$-incomparable members of $\hat{T}'$. In other words, T is the set of maximal nodes of some subtree of $\hat{T}'$. T is a *proper projection of T'* if T is a projection of T' which is not equal to T'.

Definition 42. We call a map φ on a front $\mathcal{F} \subseteq \mathcal{AE}^B$ on X
   (1) *inner* if for each $u \in \mathcal{F}$, $\varphi(u)$ is a projection of u.
   (2) *Nash-Williams* if whenever $u, v \in \mathcal{F}$ and there is some $n \leq |u|$ such that $\varphi(u) \cap \widehat{r_n(u)} = \varphi(v)$, then $\varphi(u) = \varphi(v)$.
   (3) *irreducible* if it is inner and the following property $(*)$ holds:
       $(*)$ There is a $Y \leq X$ such that given $u \in \mathcal{F} | Y$, there is a $v \in \mathcal{F}$ such that

$$(15) \qquad\qquad \varphi(u) = \varphi(v) = u \wedge v,$$

   where $u \wedge v$ denotes the set of maximal nodes in $\hat{u} \cap \hat{v}$.

Definition 43 (Canonical equivalence relations on a front). Let $\mathcal{F}$ be a front on $\mathcal{E}_B$. An equivalence relation R on $\mathcal{F}$ is *canonical* if and only if there is an irreducible map φ canonizing R on $\mathcal{F}$, meaning that for all $u, v \in \mathcal{F}$, $u \, R \, v \longleftrightarrow \varphi(u) = \varphi(v)$.

Irreducible maps on $\mathcal{E}_B$ are unique in the following sense.

Theorem 44. *Let R be an equivalence relation on some front $\mathcal{F}$ on $\mathcal{E}_B$. Suppose φ and φ' are irreducible maps canonizing R. Then there is an $X \in \mathcal{E}_B$ such that for each $u \in \mathcal{F} | X$, $\varphi(u) = \varphi'(u)$.*

Theorem 44 will be proved after Theorem 45.

We now prove the Ramsey-classification theorem for equivalence relations on fronts. Like the proof of the Ramsey-classification theorem for the spaces

$\mathcal{E}_k$, $k < \omega$, in [1], the proof follows the same general outline as that of Theorem 4.14 in [12], and the facts which have proofs similar to those in [12] will be stated here without proof. However, the majority of the proof of this theorem involves new arguments, required because of the infinitely splitting structure of the spaces and the infinite rank of the barriers.

Theorem 45 (Ramsey-classification Theorem). *Let B be a uniform barrier on ω. Given $V \in \mathcal{E}_B$ and an equivalence relation R on a front $\mathcal{F}$ on V, there is a $W \leq V$ such that R restricted to $\mathcal{F}|W$ is canonical.*

Proof. By Lemma 41, shrinking V if necessary, we may assume that for all $u, v \in \hat{\mathcal{F}}|V$, V decides for u and v. Recall that for any $u \leq_{\text{fin}} X \in \mathcal{E}_B$, mixing on $r_{|u|+1}[u, X]$ is an equivalence relation on $X_u := \{y(|u|) : y \in r_{|u|+1}[u, X]\}$. For a uniform projection $P_u \in \text{UP}(B, Y, u)$, let π_{P_u} denote the projection map which takes $w \in Y_u$ and sends it to *the* $p \in P_u$ such that $p \trianglelefteq w$.

Claim 1. *Given $X \in \mathcal{E}_B$, there is a $Y \leq X$ such that for each $u \in (\hat{\mathcal{F}} \setminus \mathcal{F})|Y$, there is a uniform projection $P_u \in \text{UP}(B, Y, u)$ such that for all $y, z \in r_{|u|+1}[u, Y]$, Y mixes y and z if and only if $\pi_{P_u}(y(|u|)) = \pi_{P_u}(z(|u|))$.*

Proof. Let $u \leq_{\text{fin}} Z \leq X$ be given, let $n = |u|$, and let E_u denote the equivalence relation on Z_u induced by the equivalence relation of mixing on $r_{|u|+1}[u, Z]$. By the Canonization Theorem for 1-Extensions 34, there is a $Z' \in [u, Z]$ and a $P_u \in \text{UP}(B, Z', u)$ such that $E_u|Z'_u$ is canonical, given by π_{P_u}. Since mixing is a hereditary property, Lemma 40 implies there is a $Y \leq X$ such that for all $u \leq_{\text{fin}} Y$, E_u is canonical on Y_u. $\square$

Thus, by shrinking V if necessary, we may also assume that V satisfies Claim 1. For $u \in \mathcal{A}\mathcal{E}^B|V$, let P_u denote the uniform projection $P \in \text{UP}(B, V, u)$ given by Claim 1 which determines the mixing relation on $r_{|u|+1}[u, V]$, and let π_u denote π_{P_u}.

Definition 46 (The maps φ and φ'). For $v \in \hat{\mathcal{F}}|V$, define $\varphi(v)$ to be the set of all $\trianglelefteq$-maximal nodes in $\{\pi_{r_n(v)}(v(n)) : n < |v|\}$; and define $\varphi'(v)$ to be the set $\{\pi_{r_n(v)}(v(n)) : n < |v|\}$.

Thus, φ is by definition an inner map.

Remark. We shall work mostly with φ' and show that it canonizes the equivalence relation R on some W. Toward the end of the proof of this theorem, we shall show that for all $u, v \in \hat{\mathcal{F}}|W$, $\varphi(u) = \varphi(v)$ if and only if $\varphi'(u) = \varphi'(v)$.

The next fact is straightforward, its proof closely resembling that of Claim 4.17 in [12].

Fact 47. *Suppose $u \in (\hat{\mathcal{F}} \setminus \mathcal{F})|V$ and $v \in \hat{\mathcal{F}}|V$.*
  (1) *Suppose $u \in \mathcal{A}\mathcal{E}_n^B|V$ and $y, z \in r_{n+1}[u, V]$. If V mixes y and v and V mixes z and v, then $\pi_u(y(n)) = \pi_u(z(n))$.*
  (2) *If $u \sqsubset v$ and $\varphi'(u) \setminus \{\emptyset\} = \varphi'(v) \setminus \{\emptyset\}$, then V mixes u and v.*

Recall that X_u/v denotes $\{y(m) : y \in r_{m+1}[u, X/v]\}$, which is the same as $\{w \in X : w \triangleright w_u$ and $\min(w \setminus w_u) > \max(u \cup v)\}$.

Fact 48. *Let $u, v \in \mathcal{AE}^B|V$, $m = |u|$ and $n = |v|$. If $w_u = w_v$, then $V_u/v = V_v/u$. If $w_u \neq w_v$, then V_u/v and V_v/u are disjoint.*

Proof. If $w_u = w_v$, then it is clear that $V_u/v = \{w \in V : w \triangleright w_u$ and $\min(w \setminus w_u) > \max(u \cup v)\} = V_v/u$.

If w_u and w_v are $\trianglelefteq$-incomparable, then $V_u/v \subseteq \{w \in V : w \triangleright w_u\}$ and $V_v/u \subseteq \{w \in V : w \triangleright w_v\}$, so they are disjoint. If $w_u \triangleleft w_v$, then for each $y \in r_{m+1}[u, V/v]$, $\min(y(m) \setminus w_u) > \max(v) > \max(w_v)$, and hence $y(m) \not\triangleright w_v$. Likewise, if $w_v \triangleleft w_u$, then each $z \in r_{n+1}[v, V/u]$ satisfies $z(n) \not\triangleright w_u$. Thus, $w_u \neq w_v$ implies that V_u/v and V_v/u are disjoint. $\square$

Lemma 49. *For each pair $u, v \leq_{\mathrm{fin}} V$, if $P_u \neq \{w_u\}$, $P_v \neq \{w_v\}$ and $w_u \neq w_v$, then V separates u and v.*

Proof. Let $d_0 = \mathrm{depth}_V(u, v)$ and let $d \geq d_0$ be the least integer such that $w_{r_d(V)}$ equals w_u or w_v. Without loss of generality, assume that $w_{r_d(V)} = w_u$. Then for each $X \in [d, V]$, $X(d) \triangleright w_u$ and $X(d) \not\triangleright w_v$, since $w_v \neq w_u$. Let $\mathcal{H}$ be the collection of all $x \in \mathcal{AE}^B|V$ such that

    (1) $x \sqsupset r_d(V)$;
    (2) $x \cap V_v/u$ is a second full approximation in $\mathcal{E}_{B_{a_v}}$; and
    (3) $|x|$ is minimal with respect to (2).

In other words, (2) and (3) state that $\{w \in x : w \triangleright w_v$ and $\min(w \setminus w_v) > \max(r_d(V))\} = \mathfrak{a}_2^{B_{a_v}}(Z)$ for some $Z \in \mathcal{E}_{B_{a_v}}$, (since for $w \triangleright w_v$, $\min(w \setminus w_v) > \max(r_d(V))$ if and only if $\min(w \setminus w_v) > \max(u)$). For $x \in \mathcal{H}$, let $k_x < l_x$ denote the numbers such that $x(k_x)$ is minimal in $x \cap (V_v/u)$ and $x(l_x)$ is maximal in $x \cap (V_v/u)$. Then in fact $l_x = |x| - 1$ and $\pi_v(x(k_x)) \neq \pi_v(x(l_x))$. Notice that $\mathcal{H}$ is a front on $[d, V]$.

Let $\mathcal{H}_0$ denote the set of those $x \in \mathcal{H}$ such that V mixes $u \cup x(d)$ and $v \cup x(k_x)$. Apply the Abstract Nash-Williams Theorem to obtain an $X \in [d, V]$ homogeneous for the partition of $\mathcal{H}$ induced by $\mathcal{H}_0$. If $\mathcal{H}|X \subseteq \mathcal{H}_0$, then in fact V also mixes $u \cup x(d)$ and $v \cup x(l_x)$ for each $x \in \mathcal{H}|X$, since $x(l_x)$ is the last member of some full approximation in $\mathcal{E}_{B_{a_v}}$ and hence is equal to $y(k_y)$ for some $y \in \mathcal{H}|X$. Then taking any $x \in \mathcal{H}|X$, we have that $\pi_v(k_x) \neq \pi_v(l_x)$, V mixes $u \cup x(d)$ and $v \cup x(k_x)$, and V mixes $u \cup x(d)$ and $v \cup x(l_x)$. Transitivity of mixing implies that V mixes $v \cup x(k_x)$ and $v \cup x(l_x)$, contradicting that $\pi_v(x(k_x)) \neq \pi_v(x(l_x))$. Therefore, it must be the case that $(\mathcal{H}|X) \cap \mathcal{H}_0 = \emptyset$. For each pair $w \in X_u/v$ and $w' \in X_v/u$ with $w = X(m)$ and $w' = X(n)$ for some $m < n$, there is an $x \in \mathcal{H}|X$ such that $x(d) = w$ and $x(k_x) = w'$. Hence, V separates $u \cup w$ and $v \cup w'$.

To take care of the case when $w \in X_u/v$ and $w' \in X_v/u$ with $w = X(n)$ and $w' = X(m)$ for some $n > m$, let $\mathcal{K}$ be the collection of all $x \in \mathcal{AE}^B|X$ such that

    (1) $x \sqsupset r_d(V)$;

(2) $x(|x|-1) \vartriangleright w_u$ and there are $i_x < j_x \in (d, |x|-1)$ such that $x(i_x) \vartriangleright w_v$ and $x(j_x) \vartriangleright w_v$; and

(3) $|x|$ is minimal with respect to (2).

Let $\mathcal{K}_0$ consist of those $x \in \mathcal{K}$ for which V mixes $u \cup x(|x|-1)$ and $v \cup x(i_x)$. Apply the Abstract Nash-Williams Theorem to obtain an $X' \in [d, X]$ homogeneous for the partition of $\mathcal{K}$ induced by $\mathcal{K}_0$. Since, $\pi_v(x(i_x)) \neq \pi_v(x(j_x))$ for each $x \in \mathcal{K}$, a similar argument to the one above yields that $(\mathcal{K}_0|X') \cap \mathcal{K} = \emptyset$. Hence, X' separates u and v, which implies the Lemma, since V already decided u and v. $\qquad\square$

Lemma 50. *Let $u, v \in \mathcal{AE}^B|V$. Suppose that V mixes u and v, and both $P_u \neq \{w_u\}$ and $P_v \neq \{w_v\}$. Then for each $X \in \mathrm{Ext}(u, v, V)$, there is a $Y \in [d, V]$ such that $P_u|Y_u/v = P_v|Y_v/u$, where $d \geq \mathrm{depth}_X(u, v)$ is least such that $w_{r_d(X)} = w_u$.*

Proof. The hypotheses along with Lemma 49 imply that $w_u = w_v$, which we denote by w_*. Let $X \in \mathrm{Ext}(u, v, V)$ be given, and let $d \geq \mathrm{depth}_X(u, v)$ be least such that $w_{r_d(X)} = w_*$. Let a_* denote $\rho_B^{-1}(w_*)$. Without loss of generality, suppose that $\mathrm{depth}_X(v) = d$. Then $X_u/v = X_v/u = X_v$, which we shall denote as X_*. By Lemma 30 and the Fusion Lemma 24, there is an $X' \in [d, X]$ such that $P_u|X_u'/v$ and $P_v|X_v'/u$ are related by exactly one of $\vartriangleleft$, $=$, or $\vartriangleright$.

Let X_*' denote $X' \cap X_*$. Let $\mathcal{H}$ be the collection of all $x \in \mathcal{AE}^B|X'$ such that

(1) $x \sqsupset r_d(X)$;

(2) $x \cap X_*'$ is a second full approximation in $\mathcal{E}_{B_{a_*}}$; and

(3) $|x|$ is minimal with respect to (2).

Then $\mathcal{H}$ is a front on $[d, X']$. For $x \in \mathcal{H}$, let k_x denote the index of the least member of $x \cap X_*'$. Let $\mathfrak{a}_1^*(x)$ denote $\{x(k_x)\}$ and $\mathfrak{a}_2^*(x)$ denote $x \cap X_*'$; these are the exactly first and second full approximations of $x \cap X_*'$ in $\mathcal{E}_{B_{a_*}}$, respectively.

Let $\mathcal{H}_0$ be the subset of those $x \in \mathcal{H}$ such that V mixes $u \cup x(k_x)$ and $v \cup x(k_x)$. Let $\mathcal{H}_1$ be the collection of those $x \in \mathcal{H}$ such that for some $w \in \mathfrak{a}_2^*(x) \setminus \mathfrak{a}_1^*(x)$, V mixes $u \cup x(k_x)$ and $v \cup w$. Let $\mathcal{H}_2$ the collection of those $x \in \mathcal{H}$ such that for some $w \in \mathfrak{a}_2^*(x) \setminus \mathfrak{a}_1^*(x)$, V mixes $v \cup x(k_x)$ and $u \cup w$. Applying the Abstract Nash-Williams Theorem three times, obtain a $Y \in [d, X']$ which is homogeneous for all three partitions of $\mathcal{H}$ induced by the $\mathcal{H}_i$, $i < 3$.

The rest of the proof of this lemma proceeds by showing that for each $i < 3$, it must be the case that $(\mathcal{H}|Y) \cap \mathcal{H}_i = \emptyset$, and then arguing that this implies V separates u and v, contradicting the hypothesis of the lemma. This will lead to the conclusion that $P_u|X_u'/v \not\vartriangleleft P_v|X_v'/u$. The same argument, interchanging u and v, yields that $P_u|X_u'/v \not\vartriangleright P_v|X_v'/u$. Therefore, the only possibility is $P_u|X_u'/v = P_v|X_v'/u$, which proves the lemma.

Let $x, y \in \mathcal{H}|Y$ such that $\pi_u(x(k_x)) = \pi_u(y(k_y))$ but $\pi_v(x(k_x)) \neq \pi_v(y(k_x))$. Then V separates $v \cup x(k_x)$ and $v \cup y(k_y)$. Now if $\mathcal{H}|Y \subseteq \mathcal{H}_0$, then V mixes $u \cup x(k_x)$ and $u \cup y(k_y)$, V mixes $u \cup x(k_x)$ and $v \cup x(k_x)$, and V mixes $u \cup y(k_y)$ and $v \cup y(k_y)$; hence V mixes $v \cup x(k_x)$ and $v \cup y(k_y)$, a contradiction. Thus, $(\mathcal{H}|Y) \cap \mathcal{H}_0 = \emptyset$.

Next, suppose that $\mathcal{H}|Y \subseteq \mathcal{H}_1$. Fix any $x \in \mathcal{H}|Y$. Recall that $x(k_x)$ determines the number p such that every second full approximation in $\mathcal{AE}_{B_{a_*}}$ extending $x(k_x)$ has cardinality p. For $y \in \mathcal{H}|Y$ such that $y \sqsupset r_{k_x+1}(x)$, y_* is a second full approximation in $\mathcal{AE}_{B_{a_*}}$; hence $\{y_*(0), \ldots, y_*(p)\} = y_*$.

Fix $r_{k_x+1}(x)$, and let $\mathcal{K} = \{y \in \mathcal{H}|Y : y \sqsupset r_{k_x+1}(x)\}$. Thus, $\mathcal{K}$ is a front on $[r_{k_x+1}(x), Y]$. For each $j < p$, let $\mathcal{K}_j$ be the collection of $y \in \mathcal{H}|Y$ such that $y \sqsupset r_{k_x+1}(x)$ and V' mixes $u \cup x(k_x)$ and $v \cup y_*(j+1)$. Apply the Abstract Nash-Williams Theorem p times to obtain a $Y' \in [r_{k_x+1}(x), Y]$ which is homogeneous for each of the partitions of $\mathcal{K}$ induced by $\mathcal{K}_j$, $j < p$.

Since $\mathcal{K}|Y' \subseteq \mathcal{H}|Y \subseteq \mathcal{H}_1$, for each $y \in \mathcal{K}|Y'$, there is some $j < p$ for which $u \cup x(k_x)$ and $v \cup y_*(j+1)$ are mixed by V. Fixing any such pair y, j, it follows that $\mathcal{K}|Y' \subseteq \mathcal{K}_j$, since $\mathcal{K}|Y'$ is homogeneous for the partition $\mathcal{K}_j$, $\mathcal{K} \setminus \mathcal{K}_j$. Let $w_\wedge$ denote the meet of $x(k_x)$ and $y_*(j+1)$. Then $w_\wedge$ is the meet of $x(k_x)$ and $z_*(j+1)$ for each $z \in \mathcal{K}|Y'$. There are two cases to consider.

Case 1. $w_\wedge \lhd \pi_v(x(k_x))$. Since $P_u|Y_* \lhd P_v|Y_*$ and $w_\wedge \lhd y_*(j+1)$, it follows that $\pi_v(z_*(j+1)) \neq \pi_v(x(k_x))$, for each $z \in \mathcal{K}|Y'$. Thus, there is a $z \in \mathcal{K}|Y'$ such that $\pi_v(z_*(j+1)) \neq \pi_v(y_*(j+1))$. Then $\mathcal{K} \setminus \mathcal{K}_j$ implies that V mixes $u \cup x(k_x)$ and $v \cup y_*(j+1)$, and V mixes $u \cup x(k_x)$ and $v \cup z_*(j+1)$. But this implies that V mixes $v \cup y_*(j+1)$ and $v \cup x_*(j+1)$, contradicting $\pi_v(y_*(j+1)) \neq \pi_v(z_*(j+1))$.

Case 2. $w_\wedge \unrhd \pi_v(x(k_x))$. Then $\pi_v(y_*(j+1)) = \pi_v(x(k_x))$, so V mixes $v \cup x(k_x))$ and $v \cup y_*(j+1)$. At the same time, $\mathcal{K} \setminus \mathcal{K}_j$ implies that V mixes $u \cup x(k_x)$ and $v \cup y_*(j+1)$. Hence, V mixes $u \cup x(k_x)$ and $v \cup x(k_x)$. But this implies that $\mathcal{K}|Y' \subseteq \mathcal{H}_0$, whereas we already showed that $(\mathcal{H}|Y) \cap \mathcal{H}_0 = \emptyset$, a contradiction.

Since in both cases we reach a contradiction, it must be the case that $(\mathcal{H}|Y) \cap \mathcal{H}_1 = \emptyset$. A similar argument, shows that $(\mathcal{H}|Y) \cap \mathcal{H}_2$ must be empty. Thus, for each $i < 3$, $(\mathcal{H}|Y) \cap \mathcal{H}_i$ is empty. But this implies that Y separates u and v, a contradiction. Therefore, the assumption that $P_u|X_u'/v \lhd P_v|X_v'/u$ was incorrect. As stated above, this leads to the conclusion of the lemma. $\square$

The next lemma is the crux of the proof of the theorem.

Lemma 51. *There is a $W \leq V$ such that for all $u, v \in (\hat{\mathcal{F}} \setminus \mathcal{F})|W$ which are mixed by W, the following holds: For all $y \in r_{|u|+1}[u, W/v]$ and $z \in r_{|v|+1}[v, W/u]$, W mixes y and z if and only if $\pi_u(y(|u|)) = \pi_v(z(|v|))$.*

Proof. We will show that for all pairs $u, v \in (\hat{\mathcal{F}} \setminus \mathcal{F})|V$ which are mixed by V, for each $X \in \text{Ext}(u, v, V)$, there is a $Z \in [\text{depth}_X(u, v), X]$ such that for all $y \in r_{|u|+1}[u, Z/v]$ and $z \in r_{|v|+1}[v, Z/u]$, V mixes y and z if and only of

$\pi_u(y(|u|)) = \pi_v(z(|v|))$. The lemma then follows from Fact 37 and Lemma 40.

Suppose $u, v \in (\hat{\mathcal{F}} \setminus \mathcal{F})|V$ are mixed by V. Let $m = |u|$ and $n = |v|$.

Claim 2. $P_u = \{w_u\}$ if and only if $P_v = \{w_v\}$.

Proof. Suppose toward a contradiction that $P_u = \{w_u\}$ but $P_v \neq \{w_v\}$. Fact 47 (1) implies that there is at most one E_v equivalence class of members z of $r_{n+1}[v, V/u]$ for which z is mixed with each member of $r_{m+1}[u, V/v]$. If each $z \in r_{n+1}[v, V/u]$ is not mixed with any $y \in r_{m+1}[u, V/z]$, then V separates u and v, a contradiction. So, suppose $z \in r_{n+1}[v, V/u]$ is mixed with some $y \in r_{m+1}[u, V/v]$. By Fact 47 (2), every two members y, y' of $r_{m+1}[u, V/v]$ are mixed, since $P_u = \{w_u\}$. Hence, V mixes z with every $y \in r_{m+1}[u, V/v]$. Let $d' = \text{depth}_V(u, v)$, and take $Z \in [d', V]$ such that the minimum number in $\bigcup Z \setminus \bigcup r_{d'}(V)$ is larger than $\max(z)$. Then for each $z' \in r_{n+1}[v, Z/u]$, $\pi_v(z') > \pi_v(z)$, so $z' E_v z$. Hence, z' is separated from each $y \in r_{m+1}[u, Z/v]$. But this contradicts that V mixes u and v. Therefore, $P_v = \{w_v\}$. $\square$

If $P_u = \{w_u\}$ and $P_v = \{w_v\}$, then given any $X \in \text{Ext}(u, v, V)$, for all $y \in r_{m+1}[u, X/v]$ and $z \in r_{n+1}[v, X/u]$, V mixes y and z, by Fact 47 (2) and transitivity of mixing. At the same time, $\pi_u(y(m)) = \pi_v(z(n)) = \emptyset$. In this case simply let $Z = X$.

For the rest of the proof of this lemma, assume that $P_u \neq \{w_u\}$ and $P_v \neq \{w_v\}$. Let $X \in \text{Ext}(u, v, V)$ be given and let $d = \text{depth}_X(u, v)$. Since we are assuming that V mixes u and v, Lemma 49 implies that $w_u = w_v$, which we shall simply denote by w_*. Let X_* denote X_u/v, which, we point out, is equal to X_v/u, and let a_* denote $\rho_B^{-1}(w_*)$. By Lemma 50, there is an $X' \in [d, X]$ such that $P_u|X_u'/v = P_v|X_v'/u$. Without loss of generality, assume X already satisfies this, and let P denote $P_u|X_*$.

Claim 3. There is a $Z \in [d, X]$ such that for all $y \in r_{m+1}[u, Z/v]$ and $z \in r_{n+1}[v, Z/u]$, Z mixes y and z if and only if $\pi_u(y(m)) = \pi_v(z(n))$.

Proof. Let $d_0 \geq d$ be least such that $X(d_0) \in X_*$. Then $w_{r_{d_0}(X)} = w_*$ and $\{Y(d_0) : Y \in [d_0, X]\} = X_*$. For $x \in \mathcal{AE}^B|X$, let x_* denote $x \cap X_*$. Let $\mathcal{K}$ be the set of those $x \in \mathcal{AE}^B|X$ such that $x \sqsupset r_{d_0}(X)$, x_* is a second full approximation in $\mathcal{E}_{B_{a_*}}$, and $x(|x| - 1) \in x_*$. For $x \in \mathcal{K}$, let $l(x_*)$ denote the cardinality of x_*; hence $x_* = \{x_*(l) : l < l(x_*)\}$. Let $\mathcal{K}_0$ consist of those $x \in \mathcal{K}$ such that there is an $l < l(x_*)$ such that either (a) $\pi_P(x_*(0)) = \pi_P(x_*(l))$ and V separates $u \cup x_*(0)$ and $v \cup x_*(l)$; or (b) $\pi_P(x_*(0)) \neq \pi_P(x_*(l))$ and V mixes $u \cup x_*(0)$ and $v \cup x_*(l)$. Take $X' \in [d_0, X]$ homogeneous for the partition of $\mathcal{K}$ induced by $\mathcal{K}_0$.

Construct $Z \in [d_0, X']$ inductively as follows. Noting that $w_{r_{d_0}(X')} = w_*$, let $X_0 = X'$ and let w_0 denote $X_0(d_0)$. For the inductive step, suppose we are given $w_i = X_i(d_i) \in X_*$, and let $l(w_i)$ denote the cardinality of any (hence every) second full approximation in $\mathcal{E}_{B_{a_*}}$ with w_i as its least member. Let $\mathcal{K}^{w_i}$ denote the set of all $x \in \mathcal{K}|X'$ such that $x_*(0) = w_i$. For each $l < l(w_i)$,

let $\mathcal{K}_l^{w_i}$ denote the set of all $x \in \mathcal{K}^{w_i}$ such that $\pi_P(x_*(l)) = \pi_P(w_i)$ and V separates $u \cup \{w_i\}$ and $v \cup x_*(l)$. Apply the Abstract Nash-Williams Theorem to each of the partitions of $\mathcal{K}^{w_i}$ induced by $\mathcal{K}_l^{w_i}$, $l < l(w_i)$, to obtain an $X_{i+1} \in [d_i + 1, X_i]$ homogeneous for each of these partitions. Let d_{i+1} be least above d_i such that $X_{i+1}(d_{i+1}) \in X_*$ and let w_{i+1} denote $X_{i+1}(d_{i+1})$. Let $Z = \bigcup_{i<\omega} r_{d_i+1}(X_i)$.

Subclaim 1. For each $x \in \mathcal{K}|Z$ and each $l < l(x_*)$, if V mixes $u \cup x_*(0)$ and $v \cup x_*(l)$, then $\pi_P(x_*(0)) = \pi_P(x_*(l))$.

Proof. Suppose there is an $x \in \mathcal{K}|Z$ and $l < l(x_*)$ such that V mixes $u \cup x_*(0)$ and $v \cup x_*(l)$, but $\pi_P(x_*(0)) \neq \pi_P(x_*(l))$. Then letting $w_\wedge = x_*(0) \wedge x_*(l)$, the meet of $x_*(0)$ and $x_*(l)$, we see that $w_\wedge \lhd \pi_P(x_*(l))$. Thus, there is a $y \in \mathcal{K}|Z$ such that $y_*(0) = x_*(0)$ (hence $l(y_*) = l(x_*)$) and $\pi_P(y_*(l)) \neq \pi_P(x_*(l))$. Let i be the integer such that $x_*(0) = w_i$. Since Z is homogeneous for the partition of $\mathcal{K}^{w_i}$ induced by $\mathcal{K}_l^{w_i}$, it follows that V mixes $u \cup y_*(0)$ and $v \cup y_*(l)$. Since $x_*(0) = y_*(0)$, transitivity of mixing implies that V mixes $v \cup x_*(l)$ and $v \cup y_*(l)$, contradicting that $\pi_P(y_*(l)) \neq \pi_P(x_*(l))$. $\square$

It follows from Subclaim 1 that no $x \in \mathcal{K}|Z$ can satisfy (b).

Subclaim 2. For each $x \in \mathcal{K}|Z$, if there is an $l \leq l(x_*)$ satisfying (a), then for all $j \leq l(x_*)$ with $\pi_P(x_*(j)) = \pi_P(x_*(0))$, V separates $u \cup x_*(0)$ and $v \cup x_*(j)$.

Proof. Suppose that $x \in \mathcal{K}|Z$ and $l \leq l(x_*)$ satisfy $\pi_P(x_*(0)) = \pi_P(x_*(l))$ and V separates $u \cup x_*(0)$ and $v \cup x_*(l)$. Let $j < l(x_*)$ be any index such that $\pi_P(x_*(j)) = \pi_P(x_*(l))$. Then V mixes $v \cup x_*(l)$ and $v \cup x_*(j)$. Then V must separate $u \cup x_*(0)$ and $v \cup x_*(j)$ (for the alternative would imply V mixes $u \cup x_*(0)$ and $v \cup x_*(l)$). $\square$

Now if $\mathcal{K}|Z \subseteq \mathcal{K}_0$, then it cannot be because of satisfying (b), by Subclaim 1, so it must be because (a) holds for each $x \in \mathcal{K}|Z$. Subclaim 2 then implies that Z separates u and v, a contradiction. Therefore, $\mathcal{K}|Z \cap \mathcal{K}_0$ must be empty. This proves the claim. $\square$

By Claim 3 and Lemma 40, the Lemma holds. $\square$

This last section of the proof of the Ramsey-classification Theorem involves putting the previous lemmas, claims, and facts together to show that $u \, R \, v$ if and only if $\varphi'(u) \setminus \{\emptyset\} = \varphi'(v) \setminus \{\emptyset\}$ if and only if $\varphi(u) = \varphi(v)$, for all $u, v \in \mathcal{F}|W$, and checking that φ is in fact irreducible.

It is important to point out the following facts about the projection maps.

Facts 52. *Let $u \in \hat{\mathcal{F}}|W$. For $i < |u|$, let p_i^u denote $\pi_{r_i(u)}(u(i))$.*

(1) *If $p_i^u \neq \emptyset$, then $p_i^u \rhd w_{r_i(u)}$.*
(2) *If $p_i^u \neq \emptyset$, then whenever $j < |u|$ with $j \neq i$ and $p_j^u \unrhd p_i^u$, it must be the case that $j > i$ and $p_j^u \rhd p_i^u$.*

(3) *Let I_u be the subset of $i < |u|$ for which $p_i^u \neq \emptyset$. Then the set $\{p_i^u : i \in I_u\}$ has no duplicates; that is, $i \neq j$ in I_u implies $p_i^u \neq p_j^u$. Thus, $\varphi'(u) \setminus \{\emptyset\} = \{p_i^u : i \in I_u\}$.*
(4) *For $i, j \in I_u$, $i < j$ implies $p_i^u \prec p_j^u$.*

Proof. Suppose $p_i^u \neq \emptyset$. Then $P_{r_i(u)} \rhd \{w_{r_i(u)}\}$, so $p_i^u \rhd w_{r_i(u)}$ and hence (1) holds.

Fix $i < |u|$ and suppose $j < |u|$ is given satisfying $i \neq j$ and $p_j^u \unrhd p_i^u$. Recall that the definition of $w_{r_i(u)}$ implies that $w_{r_i(u)} = u(i) \wedge r_i(u)$. Further, there are no members $w \in u$ such that both $w \unrhd p_i^u$ and $w <_{\text{lex}} u(i)$, since that would contradict the definition of $w_{r_i(u)}$ and that $w_{r_i(u)} \lhd p_i^u$. So if $j < i$ and $u(j) <_{\text{lex}} u(i)$, then p_j^u cannot equal p_i^u. For the other case, if $j < i$ and $u(i) <_{\text{lex}} u(j)$, then $u(i) \wedge u(j) \unlhd w_{r_i(u)}$, which is $\lhd p_i^u$. Hence, $u(i) \wedge u(j) \not\unrhd p_i^u$, so again, p_u^j cannot equal p_i^u.

For $j > i$, if $p_j^u = p_i^u$, then $w_{r_j(u)} \unrhd p_i^u$, since $w_{r_j(u)} = u(j) \wedge r_j(u)$. But this implies p_j^u must be $\emptyset$, since projection of $u(j)$ to $w_{r_j(u)}$ or some initial segment of it implies that $\pi_{r_j(u)}$ is the projection to the emptyset. Thus, (2) holds. (3) follows from (2).

For (4), notice that for any $j \in I_u$, $p_j^u \neq \emptyset$ implies that $p_j^u \rhd w_{r_j(u)}$, and since $\min(u(j) \setminus w_{r_j(u)})$ is strictly greater than $\max(r_j(u))$, it follows that $p_j^u \succ$ every member of $\widehat{r_j(u)}$. Thus, if $i < j$, then $p_i^u \in \widehat{r_j(u)}$ implies $p_i^u \prec p_j^u$. $\qquad\square$

Let $\varphi''(u)$ denote $\varphi'(u) \setminus \{\emptyset\}$. Then $\varphi''(u) = \{p_i^u : i \in I_u\}$.

Claim 4. *For $u, v \in \hat{\mathcal{F}}|W$, if $\varphi''(u) = \varphi''(v)$, then W mixes u and v. Hence, for $u, v \in \mathcal{F}|W$, if $\varphi''(u) = \varphi''(v)$, then $u \, R \, v$.*

Proof. Suppose that $\varphi''(u) = \varphi''(v)$. Using the notation in Fact 52, let $\langle p_i^u : i \in I_u \rangle$ and $\langle p_j^v : j \in I_v \rangle$ denote the $\prec$-increasing enumerations of $\varphi''(u)$ and $\varphi''(v)$, respectively. Since $\varphi''(u) = \varphi''(v)$, $|I_u|$ must equal $|I_v|$, so let $k = |I_u|$. Enumerate I_u as $i_0 < \cdots < i_{k-1}$ and I_v as $j_0 < \cdots < j_{k-1}$. Since the sequences $\langle p_i^u : i \in I_u \rangle$ and $\langle p_j^v : j \in I_v \rangle$ are both in $\prec$-increasing order, the only way the sets $\varphi''(u)$ and $\varphi''(v)$ can be equal is if for each $l < k$, $p_{i_l}^u = p_{j_l}^v$, which we will denote by p_l. Given $l < k$, let x_l denote $r_{i_l}(u)$ and let y_l denote $r_{j_l}(v)$. Note that $\varphi''(x_l) = \varphi''(y_l)$. Since $p_l \neq \emptyset$, it follows that $w_{x_l} \lhd p_l \unlhd u(m_l)$ and $w_{y_l} \lhd p_l \unlhd v(n_l)$. Therefore, $x_l \prec p_l \preceq u(m_l)$ and $y_l \prec p_l \preceq v(n_l)$.

Now $\varphi''(x_0) = \varphi''(y_0) = \emptyset$, so W mixes x_0 and y_0, by Fact 47 (2). Then Lemma 51 implies that W mixes $r_{m_0+1}(u)$ and $r_{n_0+1}(v)$, since $p_{i_0}^u = p_{j_0}^v$ (and hence $u(i_0)$ and $v(j_0)$ are $\prec$-above both x_0 and y_0). Then for all $m \in (i_0+1, i_1]$ and all $n \in (j_0+1, j_1]$, $\varphi''(r_m(u)) = \varphi''(x_0) = \varphi''(y_0) = \varphi''(r_n(v))$, so Fact 47 (2) implies that W mixes x_1 and y_1. Continuing in this manner, one proves by induction on $l < k$ that W mixes $r_{m_k+1}(u)$ and $r_{n_k+1}(v)$. Since $\varphi''(u) = \varphi''(r_{i_{k-1}+1}(u)) = \varphi''(r_{j_{k-1}+1}(v)) = \varphi''(v)$, apply Fact 47 (2)

as necessary to conclude that W mixes u and v. In particular, if $u, v \in \mathcal{F}|W$, then W mixing u and v implies that $u \, R \, v$. $\qquad\square$

We point out the following useful observation: For $u \in \mathcal{F}|W$ and $i < |u|$, if $\pi_{r_i(u)}(u(i)) \neq \emptyset$, then $\pi_{r_i(u)}(u(i)) \notin \widehat{r_i(u)}$. Thus, $\varphi''(r_i(u)) = \varphi''(u) \cap \widehat{r_i(u)}$.

Claim 5. φ'' *is Nash-Williams on* $\mathcal{F}|W$.

Proof. Let $u, v \in \mathcal{F}|W$ and suppose $m < |u| - 1$ is maximal such that $\varphi''(r_m(u)) = \varphi''(v)$ but $\varphi''(r_{m+1}(u)) \neq \varphi''(v)$. Then $P_{r_m(u)} \neq \{w_{r_m(u)}\}$. Letting $k = |\varphi''(u)|$ and $l = |\varphi''(v)|$, and $\langle p_i^u : i < k \rangle$ and $\langle p_i^v : i < l \rangle$ being their $\prec$-increasing enumerations, we see that $k > l$, $\langle p_i^u : i < l \rangle = \langle p_i^v : i < l \rangle$, and $p_{l-1}^v = p_{l-1}^u \prec p_l^u$.

Let n be maximal such that $r_n(v) \prec p_l^u$. Then $\varphi''(r_n(v)) = \varphi''(v)$, and if $n < |v|$, then $\pi_{r_n(v)} = \pi_\emptyset$. Since $\varphi''(r_m(u)) = \varphi''(r_n(v))$, Claim 4 implies that W mixes $r_m(u)$ and $r_n(v)$. By Fact 47 (1), W mixes $r_n(v)$ and $r_m(u) \cup \{w\}$ for at most one $\pi_{r_m(u)}$-projection class of w's in $W_{r_m(u)}$. Since $p_l^u \succ r_n(v), r_m(u)$, letting $d = \mathrm{depth}_W(r_n(v), r_m(u))$, there is a $W' \in [d, W]$ such that the least member in W' above $r_n(v)$ and $r_m(u)$ is also $\prec$-above p_l^u. By Lemma 40, we may assume that we already thinned W to satisfy that no 1-extension of $r_m(u)$ into W is mixed with $r_n(v)$. But then W separates $r_n(v)$ and $r_m(u)$, a contradiction. $\qquad\square$

Claim 6. *For all* $u, v \in \mathcal{F}|W$, *if* $u \, R \, v$, *then* $\varphi''(u) = \varphi''(v)$.

Proof. Let $u, v \in \mathcal{F}|W$ and suppose that $\varphi''(u) \neq \varphi''(v)$. Let $\langle p_i^u : i < |\varphi''(u)| \rangle$ and $\langle p_i^v : i < |\varphi''(v)| \rangle$ enumerate $\varphi''(u)$ and $\varphi''(v)$ in $\prec$-increasing order. Since by Claim 5 φ'' is Nash-Williams, neither of $\varphi''(u)$, $\varphi''(v)$ end-extends the other; so let $j < \min(|\varphi''(u)|, |\varphi''(v)|)$ be least such that $p_j^u \neq p_j^v$. Let m_j, n_j be the integers such that $\pi_{r_{m_j}}(u(m_j)) = p_j^u$ and $\pi_{r_{n_j}}(v(n_j)) = p_j^v$.

If both $u(m_j)$ and $v(n_j)$ are $\prec$-above both $r_{m_j}(u)$ and $r_{n_j}(v)$, then Lemma 51 along with $p_j^u \neq p_j^v$ imply that $r_{m_j+1}(u)$ and $r_{n_j+1}(v)$ are separated by W; and hence, $u \, \not\!R \, v$. Otherwise, without loss of generality, suppose $u(m_j) \preceq r_{n_j}(v)$. Take n least such that $v(n) \succeq u(m_j)$. Then $n < n_j$, $r_n(v) \prec u(m_j)$, and $v(n) \succ r_{m_j}(u)$. Assume that W mixes $r_n(v)$ and $r_{m_j}(u)$, for if not, then already we have $u \, \not\!R \, v$. Since $n < n_j$, $\pi_{r_n(v)}(v(n))$ cannot equal p_j^u. By Lemma 51, W separates $r_n(v)$ and $r_{m_j}(u)$; hence $u \, \not\!R \, v$. $\qquad\square$

Claims 4 and 6 yield that φ'' canonizes the equivalence relation R on $\mathcal{F}|W$; that is, for each pair $u, v \in \mathcal{F}|W$, $u \, R \, v$ if and only if $\varphi''(u) = \varphi''(v)$.

Next, we show that φ holds the same information about R that φ'' does. The following observation will be useful: For each $u \in \hat{\mathcal{F}}|W$,

$$(16) \qquad \pi_{r_0(u)}(u(0)) \preceq u(0) \prec \pi_{r_1(u)}(u(1)) \preceq u(1) \prec \cdots .$$

Recall that for any two sets D, E, $D \triangle E$ denotes their symmetric difference, $(D \setminus E) \cup (E \setminus D)$.

Lemma 53. *For $u, v \in \hat{\mathcal{F}}|W$, $\varphi''(u) = \varphi''(v)$ if and only if $\varphi(u) = \varphi(v)$. Therefore, for $u, v \in \mathcal{F}|W$, $u \, R \, v$ if and only if $\varphi(u) = \varphi(v)$.*

Proof. Since $\varphi(u)$ is the set of all maximal nodes in $\varphi''(u)$, the forward direction is trivial. So now assume that $\varphi''(u) \neq \varphi''(v)$. Let w denote the $\prec$-least member of $\varphi''(u) \triangle \varphi''(v)$, and without loss of generality, assume that $w \in \varphi''(u) \setminus \varphi''(v)$. Let $i < |u|$ and $j < |v|$ be maximal such that $r_i(u) \prec w$ and $r_j(v) \prec w$. Then $w \preceq u(i)$ and $w \preceq v(j)$.

Since w is $\prec$-minimal in $\varphi''(u) \triangle \varphi''(v)$, $w \in \varphi''(u)$, and $r_i(u)$ is $\prec$-maximal below w, it follows from the observation (16) that $w = \pi_{r_i(u)}(u(i))$. Hence, the set $\{\pi_{r_l(u)}(u(l)) \prec w : l < |u|\}$ equals $\varphi''(r_i(u))$. Let w' denote $\pi_{r_j(v)}(v(j))$. Then w' is the $\prec$-least member of $\varphi''(v)$ satisfying $w \prec w'$, so by observation (16), we see that $\{\pi_{r_l(v)}(v(l)) \prec w : l < |v|\} = \varphi''(r_j(v))$. Therefore, $\varphi''(r_i(u)) = \varphi''(r_j(v))$, since w is the $\prec$-least member of $\varphi''(u) \triangle \varphi''(v)$.

Note that $r_j(v) \prec w = \pi_{r_i(u)}(u(i)) \preceq u(i)$ and $r_i(u) \prec w \prec w' \preceq v(j)$, so $u(i) \succ r_j(v)$ and $v(j) \succ r_i(u)$. Since $\varphi''(r_i(u)) = \varphi''(r_j(v))$, Claim 4 implies that W mixes $r_i(u)$ and $r_j(v)$. Hence, $P_{r_i(u)}$ and $P_{r_j(v)}$ are equal on W above $r_i(u)$ and $r_j(v)$, by the proof of Lemma 51. Since $P_{r_i(u)}$ is a uniform projection, $w \neq w'$ implies that w and w' are $\trianglelefteq$-incomparable. Hence, $w \ntrianglelefteq v(j)$. This along with the fact that $r_j(v) \prec w$ (and hence $w \notin \widehat{r_j(v)}$) imply that $w \notin \widehat{r_{j+1}(v)}$. For $j < l < |v|$, $\min(v(l) \setminus w_{r_l(v)}) = \min(v(l) \setminus r_l(v)) \geq \max(w') > \max(w)$. Thus, $\pi_{r_j(v)}(v(j)) \ntrianglerighteq w$, since $w \notin \widehat{r_{j+1}(v)}$ and every new number in $v(j) \setminus \widehat{r_{j+1}(v)}$ is strictly greater than $\max(w)$. Hence, $w \notin \widehat{\varphi''(v)}$, which implies that $\varphi(u) \setminus \varphi(v) \neq \emptyset$. Thus, we have proved that whenever $\varphi''(u) \neq \varphi''(v)$, then also $\varphi(u) \neq \varphi(v)$, which proves the first statement of the lemma. By Claims 4 and 6, φ canonizes R on $\mathcal{F}|W$. $\square$

Fact 54. *φ is Nash-Williams on $\mathcal{F}|W$.*

Proof. If $u, v \in \mathcal{F}|W$ and there is some $j < |v|$ such that $\varphi(u) = \varphi(v) \cap \widehat{r_j(v)}$, then noting that this implies that $\varphi(u) = \varphi(r_j(v))$, Lemma 53 implies that $\varphi''(u) = \varphi''(r_j(v))$. Since φ'' is Nash-Williams, by Claim 5, $\varphi''(u)$ and $\varphi''(v)$ must be equal. Then Lemma 53 implies that $\varphi(u) = \varphi(v)$. $\square$

The next lemma will conclude the proof of the Ramsey-classification Theorem. Essentially, the property $(*)$ follows from taking W' thin enough below W and then applying Claim 4 and Lemma 51.

Lemma 55. *φ is irreducible.*

Proof. Since φ is inner, it suffices to show that φ satisfies property $(*)$. Take $W' \leq W$ thin enough so that the following holds: There are $n(0, 0) < \cdots < n(0, l_0)$, where $l_0 = \mathrm{lh}(W'(0)) - 1$, such that for each $i \leq l_0$, $W(n(0, i)) \prec W'(0)$ and $\mathrm{lh}(W(n(0, i)) \wedge W'(0)) = i$. In general, we require W' thin enough that for each $k \geq 1$, there are $n(k, m_k) < \cdots < n(k, l_k)$, where

$m_k = \mathrm{lh}(w_{r_k(W)})$ and $l_k = \mathrm{lh}(W'(k)) - 1$, such that for each $m_k \leq i \leq l_k$, $W'(k-1) \prec W(n(k,i)) \prec W'(k)$ and $\mathrm{lh}(W(n(k,i)) \wedge W'(k)) = i$.

Let $u \in \mathcal{F}|W'$ and build $v \in \mathcal{F}|W$ as follows. If $\pi_{r_0(u)}(u(0)) = u(0)$, then let $v(0) = u(0)$. Otherwise, take $v(0) \prec u(0)$ in W such that $v(0) \wedge u(0) = \pi_{r_0(u)}(u(0))$. Since $P_{r_0(u)}$ is a uniform projection and $v(0) \trianglerighteq \pi_{r_0(u)}(u(0))$, it follows that $\pi_{r_0(u)}(v(0)) = \pi_{r_0(u)}(u(0))$. Thus, in both cases, letting $r_1(v)$ denote $\{v(0)\}$, we see that $\varphi(r_1(v)) = \varphi(r_1(u))$ and $r_1(v) \wedge u = \varphi(r_1(u))$.

Suppose that $n \leq |u|$ and we have chosen $r_n(v) = \{v(0), \dots, v(n-1)\}$ so that for each $i < n$, $\pi_{r_i}(v(i)) = \pi_{r_i}(u(i))$ and $v(i) \wedge u = \pi_{r_i}(u(i))$. Thus, $\varphi(r_n(v)) = \varphi(r_n(u)) = r_n(v) \wedge u$. There are three possibilities.

If $r_n(v) \in \mathcal{F}$, then let $v = r_n(v)$. By the induction hypothesis, $\varphi(v) = \varphi(r_n(u)) = v \wedge u$. Since φ is Nash-Williams, $\varphi(u)$ must equal $\varphi(v)$.

If $n < |u|$ and $r_n(v) \notin \mathcal{F}$, recall that since $\varphi(r_n(v)) = \varphi(r_n(u))$, the proof of Lemma 51 shows that $\pi_{r_n(v)} = \pi_{r_n(u)}$. Thus, choose $v(n) \triangleright w_{r_n(v)}$ so that $u(n-1) \prec v(n) \prec u(n)$, $\pi_{r_n(v)}(v(n)) = \pi_{r_n(u)}(u(n))$, and $v(n) \wedge u = \pi_{r_n(v)}(v(n))$. (This last condition is possible by taking $v(n)$ so that $\min(v(n) \setminus \pi_{r_n(v)}(v(n)))$ is not in u.) Then $\varphi(r_{n+1}(v)) = \varphi(r_{n+1}(u)) = r_{n+1}(v) \wedge u$.

If $n = |u|$ and $r_n(v) \notin \mathcal{F}$, let w_* denote $w_{r_n(v)} \wedge u$. Note that w_* is in $\widehat{r_n(v)} \wedge \hat{u}$, since $w_{r_n(v)} \in \widehat{r_n(v)}$. Thus, by choosing any $v(n) \triangleright w_{r_n(v)}$ with $\min(v(n) \triangleright w_{r_n(v)}) > \max(u \cup r_n(v))$, we obtain an $r_{n+1}(v) = r_n(v) \cup \{v(n)\}$ satisfying $v(n) \wedge u = w_*$. Thus, $r_{n+1}(v) \wedge u = r_n(v) \wedge u$, which by the induction hypothesis is equal to $\varphi(u)$ and $\varphi(r_n(v))$. Since φ is Nash-Williams, any extension of $r_n(v)$ to some $v \in \mathcal{F}$ will have $\varphi(v) = \varphi(r_n(v)))$. Hence, also $\varphi(r_{n+1}(v)) = \varphi(r_n(v)) = \varphi(u)$. □

Thus, φ is an irreducible map canonizing R on $\mathcal{F}|W$. Thus concludes the proof of Theorem 45. □

This section concludes with proving that the irreducible map in Theorem 45 is unique. First, some useful observations.

Fact 56. $u(i) \wedge \varphi(u) \triangleright w_{r_i(u)}$ *if and only if* $\pi_{r_i(u)}(u(i)) \neq \emptyset$.

Fact 57. φ *is* $\trianglelefteq$*-maximal among projection maps canonizing R on $\mathcal{F}|W$. That is, if γ is another projection map such that for all $u, v \in \mathcal{F}|W$, $\gamma(u) = \gamma(v) \longleftrightarrow u\,R\,v$, then $\gamma(u) \trianglelefteq \varphi(u)$, for each $u \in \mathcal{F}|W$.*

Proof. Let $u \in \mathcal{F}|W$, and suppose $n < |u|$ is such that $\gamma(u) \cap \widehat{u(n)} \neq \varphi(u) \cap \widehat{u(n)}$. Let w denote $u(n)$. Since γ is a projection map, then either $\gamma(u) \cap \hat{w} \lhd \varphi(u) \cap \hat{w}$ or $\gamma(u) \cap \hat{w} \rhd \varphi(u) \cap \hat{w}$. Suppose toward a contradiction that $\gamma(u) \cap \hat{w} \rhd \varphi(u) \cap \hat{w}$. Let x denote $r_n(u)$. Take $w' \in W$ such that $x \cup \{w'\} \in r_{n+1}[x, W]$ and $\pi_x(w') = \pi_x(w)$. Then $x \cup \{w\}$ and $x \cup \{w'\}$ are mixed, so we may extend $x \cup \{w\}$ and $x \cup \{w'\}$ to some $y, y' \in \mathcal{F}|W$, respectively, such that $\varphi(y) = \varphi(y')$. Then $y\,R\,y'$. Now $\gamma(y) \cap \hat{w} \neq \gamma(y') \cap \hat{w}'$; so since γ is a projection map, it follows that $\gamma(y) \neq \gamma(y')$, which contradicts that γ canonizes R on $\mathcal{F}|W$. □

We now prove that irreducible maps are unique, up to restriction below some member of the space.

Proof of Theorem 44. Let γ be any irreducible map canonizing R on $\mathcal{F}|W$. Take $W' \leq W$ witnessing $(*)$ for γ. Suppose toward a contradiction that there is a $u \in \mathcal{F}|W'$ such that $\gamma(u) \neq \varphi(u)$. Then $\gamma(u) \lhd \varphi(u)$, by Fact 57. Take a $v \in \mathcal{F}|W$ such that $\gamma(u) = \gamma(v) = u \wedge v$. Then $u \wedge v \lhd \varphi(u)$. Since $\varphi(u)$ and $\varphi(v)$ are projections of u and v, respectively, if $\varphi(u)$ and $\varphi(v)$ are to be equal, then they must both be contained in $u \wedge v$, which is not the case. Thus, $\varphi(u) \neq \varphi(v)$, a contradiction. $\square$

8. Conclusion and Remarks

We close this paper with a few brief remarks. The canonical equivalence relations on fronts of the form $\mathcal{AE}_k^B$ are obtained as a corollary to Theorem 45. We do not present it here, since, unlike the analogous situation for all other known topological Ramsey spaces, here we do not obtain simply a sequence of k many canonical equivalence relations on the 1-extensions. This happens because, fixing $i < k$, for $X, Y \in \mathcal{E}_B$, $r_{i+1}[i, X]$ and $r_{i+1}[i, Y]$ may be isomorphic to $\mathcal{E}_C$ and $\mathcal{E}_D$ for very different uniform barriers C and D.

The topological Ramsey spaces $\mathcal{E}_B$ and the Ramsey-classification Theorem 45 form the basis for forthcoming work [30] on the Rudin-Keisler and Tukey structures of the associated ultrafilters $\mathcal{G}_B$. One of the interesting results there is that the ultrafilters associated with these spaces produce downwards-closed structures of size continuum in both the Rudin-Keisler and Tukey structures, in contrast to the countable initial structures associated with all other topological Ramsey spaces investigated so far. In another vein, the spaces $\mathcal{E}_B$ will also likely lead to some interesting constructions in Banach space theory, as the spaces $\mathcal{E}_k$ from [1] have led to some interesting new Banach spaces in forthcoming work of Arias, Dobrinen, Giron, and Mijares.

We have shown in Theorem 44 that the irreducible map φ in Theorem 45 is unique; more to the point, the property $(*)$ is sufficient for uniqueness among inner maps canonizing the equivalence relation R. We ask whether the following property is sufficient for uniqueness: We say that a projection map γ on a front $\mathcal{F}$ on $\mathcal{E}_B$ is *strongly Nash-Williams* if for each pair $u, v \in \widehat{\mathcal{F}}$ and each $i \leq \operatorname{depth}_W(u, v)$, $\gamma(u) \cap \widehat{r_i(W)} \trianglelefteq \gamma(v) \cap \widehat{r_i(W)}$ implies $\gamma(u) \cap \widehat{r_i(W)} = \gamma(v) \cap \widehat{r_i(W)}$. If a projection map γ canonizing R on $\mathcal{E}_B|W$ is strongly Nash-Williams, is it necessarily equal to φ? We leave this question for future work.

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