OPEN DETERMINACY FOR CLASS GAMES

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ABSTRACT. The principle of open determinacy for class games—two-player games of perfect information with plays of length ω, where the moves are chosen from a possibly proper class, such as games on the ordinals—is not provable in Zermelo-Fraenkel set theory ZFC or Gödel-Bernays set theory GBC, if these theories are consistent, because provably in ZFC there is a definable open proper class game with no definable winning strategy. In fact, the principle of open determinacy and even merely clopen determinacy for class games implies Con(ZFC) and iterated instances Con^α(ZFC) and more, because it implies that there is a satisfaction class for first-order truth, and indeed a transfinite tower of truth predicates Tr_α for iterated truth-about-truth, relative to any class parameter. This is perhaps explained, in light of the Tarskian recursive definition of truth, by the more general fact that the principle of clopen determinacy is exactly equivalent over GBC to the principle of transfinite recursion over well-founded class relations. Meanwhile, the principle of open determinacy for class games is provable in the stronger theory $\mathrm{GBC} + \Pi^1_1$-comprehension, a proper fragment of Kelley-Morse set theory KM.

One of the intriguing lessons of the past half-century of set theory is that there is a robust connection between infinitary game theory and fundamental set-theoretic principles, including the existence of certain large cardinals, and the existence of strategies in infinite games has often turned out to have an unexpected set-theoretic power. In this article, we should like to exhibit another such connection in the case of games of proper class size, by proving that the principle of clopen determinacy for class games is exactly equivalent to the principle of transfinite recursion along well-founded class relations. Since this principle implies Con(ZFC) and iterated instances of Con^α(ZFC) and more, the principles of open determinacy and clopen determinacy both transcend ZFC in consistency strength.

We consider two-player games of perfect information, where two players alternately play elements from an allowed space X of possible moves, which in our case may be a proper class such as the class of all ordinals $X = \mathrm{Ord}$. Together, the players build an infinite sequence $\vec{\alpha} = \langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ in X^ω, which is the play resulting from this particular instance of the game. The winner of this play is determined by consulting a fixed class of plays $A \subseteq X^\omega$, possibly a proper class: if $\vec{\alpha} \in A$, then the first player has won this particular instance of the game, and otherwise the second

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player has won. A strategy for a player is a (class) function $\sigma : X^{<\omega} \to X$, which tells a player how to move next, given a finite position in the game. Such a strategy is winning for that player, if following the instructions of the strategy leads to a winning play of the game, regardless of how the other player has moved. The game is determined, if one of the players has a winning strategy. We may formalize all talk of classes here in Gödel-Bernays GBC set theory, or in ZFC if one prefers to regard classes as definable from parameters.

The case of open games, generalizing the finite games, is an attractive special case, which for set-sized games has been useful in many arguments. Specifically, a game is *open* for a particular player, if for every winning play of the game for that player, there occurred during the course of play a finite position where the winning outcome was already ensured, in the sense that all plays extending that position are winning for that player. This is equivalent to saying that the winning condition set for that player is open in the product topology on X^ω, where we put the discrete topology on X. Similarly, a game is *clopen*, if it is open for each player; these are the games for which every play of the game has a finite stage where the outcome is already known.

It is a remarkable elementary fact, the Gale-Stewart theorem [GS53], that in the context of set-sized games, every open game is determined. An elegant proof of open determinacy can be undertaken using the theory of ordinal game values. (For an accessible discussion of ordinal game values, using infinite chess amongst other games as illustration, please see the second author's articles [EH14, EHP].) Suppose that we have a game that is open for player I, with open winning condition $A \subseteq X^\omega$, where X is the *set* of possible moves. Consider the collection of positions that arise in this game at a finite stage of the game. Such a position p is said to have value 0, if player I has essentially already won by achieving that position, in the sense that every play extending p is in A; in other words, the entire basic open neighborhood of p is contained in the open set A. A position q with player I to play has value 1, if it isn't yet winning in that sense, but player I can make a move to $q \frown x$ so as to have value 0. These game values measure the distance, in a sense, to a win for player I. Continuing recursively, define that a position p with player I to play has value $\alpha + 1$, if α is minimal such that player I can play to a position $p \frown x$ with value α. Finally, the value of a position p with player II to play is defined only when every position $p \frown y$ that player II can reach already has a value, and in this case the value of p is the supremum of those values. There are two key observations to make. First, if a position has a value, then on his turns player I can play so as to decrease this value and player II cannot play so as to increase it or make it become undefined; thus, by means of this value-reducing strategy, the first player can ensure that eventually the value will become zero, since there is no infinite descending sequence of ordinals, and so this strategy is winning for player I. Second, if a position does not have a defined value, then player I cannot play so as to give it a value and player II can play so as to maintain the fact that it is unvalued; thus, by means of this maintaining strategy, the second player can ensure that a position with value zero is never reached, and so this strategy is winning for player II. Thus, we have proved the Gale-Stewart theorem: every open game is determined.

The reader should observe that this proof of open determinacy relied on the space X of possible moves being a set, because when defining the value function of a position where it was player II's turn, we took a supremum over the ordinal values of the positions arising from the possible moves, and if X were a proper class, we couldn't necessarily be sure to stay within the class of ordinals with this supremum. With a proper class X, the recursive procedure might break down, and it would seem that we could be pushed to consider class well-orderings having an order-type taller than Ord. What we should like to do in this article, therefore, is consider more seriously the case where X is a proper class. In this case, the strategies $\sigma : X^{<\omega} \to X$ will also be proper classes, and the winning condition $A \subseteq X^\omega$ may also be a proper class.

Question 1. *Can we prove open determinacy for class games?*

For example, does every definable open class game in ZFC admit a definable winning strategy for one of the players? In GBC, must every open class game have a winning strategy?

We shall prove that the answers to both of these questions is no. The basic reason, established by theorem 2, is that open determinacy implies Con(ZFC) and much more (but see also the remarks after theorem 4). Thus, the principle of open determinacy and even of clopen determinacy goes strictly beyond the strength of ZFC and GBC, if these are consistent. This result is generalized and clarified by theorem 6, which shows that the principle of clopen determinacy is exactly equivalent over GBC to the principle of transfinite recursion over well-founded class relations. After this, we shall prove in theorem 7 that open determinacy is provable in Kelley-Morse set theory KM, and even in the theory $\mathrm{GBC} + \Pi^1_1$-comprehension, which is a proper fragment of KM.

Theorem 2. *The principle of clopen determinacy for class games implies* Con(ZFC)*, as well as iterated consistency assertions* $\mathrm{Con}^\alpha(\mathrm{ZFC})$ *and more. Specifically, there is a definable clopen game, whose determinacy is equivalent over* GBC *to the existence of a satisfaction class for first-order truth.*

The proof shows that every model of ZFC has a definable clopen game with no definable winning strategy, and so one cannot prove clopen determinacy in ZFC or GBC, if these theories are consistent.

Proof. Consider what we call the *truth-telling* game, which will be a definable open game with no definable winning strategy. The truth-telling game has two players, the *interrogator* and the *truth-teller*, who we may imagine—in the style of a similar game described by Adrian Mathias [Mat15] in the context of arithmetic and extensions of PA—play out the game in a court of law, with the truth-teller in the witness box answering tricky pointed questions posed by the opposing counsel. On each turn, the interrogator puts an inquiry to the the truth-teller concerning the truth of a particular set-theoretic formula $\varphi(\vec{a})$ with parameters. The truth-teller must reply to the inquiry by making a truth pronouncement either that it is *true* or that it is *false*, not necessarily accurately, and in the case that the formula φ is an existential assertion $\exists x\, \psi(x, \vec{a})$ declared to be true, then the truth teller must additionally identify a particular witness b and pronounce that $\psi(b, \vec{a})$ is also true. So a play of the game consists of a sequence of such inquiries and truth pronouncements.

The truth-teller need not necessarily answer truthfully to win! Rather, the truth-teller wins a play of the game, provided that she does not violate the recursive

Tarskian truth conditions during the course of play. What we mean, first, is that when faced with an atomic formula, she must pronounce it true or false in accordance with the actual truth or falsity of that atomic formula; similarly, she must pronounce that $\varphi \wedge \psi$ is true just in case she pronounces both φ and ψ separately to be true, if those inquiries had been issued by the interrogator during play; she must pronounce opposite truth values for φ and $\neg\varphi$, if both are inquired about; and she must pronounce $\exists x\, \varphi(x, \vec{a})$ to be true if and only if she ever pronounces $\varphi(b, \vec{a})$ to be true of any particular b (the forward implication of this is already ensured by the extra pronouncement in the existential case of the game). This is an open game for the interrogator, because if the truth-teller ever should violate the Tarskian conditions, then this violation will be revealed at finite stage of play.

We remind the reader that a *satisfaction class* or *truth predicate* for first-order truth is a class Tr of pairs $\langle \varphi, \vec{a} \rangle$ consisting of a formula φ and a list of parameters $\vec{a}$ assigned to the free variables of that formula, which obeys the Tarskian recursive definition of truth (for simplicity we shall write the pair simply as $\varphi(\vec{a})$, suppressing the variable assignment, but keep in mind that these are mentions of formulas rather than uses). So in the atomic case, we'll have $(a = b) \in \mathrm{Tr}$ if and only if $a = b$, and $(a \in b) \in \mathrm{Tr}$ if and only if $a \in b$; for negation, $\neg\varphi(\vec{a}) \in \mathrm{Tr}$ if and only if $\varphi(\vec{a}) \notin \mathrm{Tr}$; for conjunction, $(\varphi \wedge \psi)(\vec{a}) \in \mathrm{Tr}$ if and only if $\varphi(\vec{a}) \in \mathrm{Tr}$ and $\psi(\vec{a}) \in \mathrm{Tr}$; and for quantifiers, $\exists x\, \varphi(x, \vec{a}) \in \mathrm{Tr}$ just in case there is b for which $\varphi(b, \vec{a}) \in \mathrm{Tr}$. Tarski proved that in any sufficiently strong first-order theory no such truth predicate for first-order truth is definable in the same language. Meanwhile, in the second-order Kelley-Morse set theory and even in the weaker theory GBC plus the principle of transfinite recursion over well-founded class relations, we can define a truth predicate for first-order truth, simply because the Tarskian recursion itself is a well-founded recursion on the complexity of the formulas, where we define the truth of $\varphi(\vec{a})$ in terms of $\psi(\vec{b})$ for simpler formulas ψ.

Lemma 2.1. *The truth-teller has a winning strategy in the truth-telling game if and only if there is a satisfaction class for first-order truth.*

Proof. We may understand this lemma as formalized in Gödel-Bernays GBC set theory, which includes the global choice principle. Clearly, if there is a satisfaction class for first-order truth, then the truth-teller has a winning strategy, which is simply to answer all questions about truth in accordance with that satisfaction class, using the global choice principle to pick Skolem witnesses in the existential case. Since by definition that class obeys the Tarskian conditions, she will win the game, no matter which challenges are issued by the interrogator.

Conversely, suppose that the truth-teller has a winning strategy τ in the game. We shall use τ to build a satisfaction class for first-order truth. Specifically, let Tr be the collection of formulas $\varphi(\vec{a})$ that are pronounced true by τ in any play according to τ, including the supplemental truth pronouncements made in the existential case about the particular witnesses. We claim that Tr is a satisfaction class. Since the truth-teller was required to answer truthfully to all inquiries about atomic formulas, it follows that Tr contains all and only the truthful atomic assertions. In particular, the answers provided by the strategy τ on inquiries about atomic formulas are independent of the particular challenges issued by the interrogator and of the order in which they are issued. Next, we generalize this to all formulas, arguing by induction that the truth pronouncements made by τ on a formula is always independent of the play in which that formula arises. We have already noticed this for atomic formulas. In the case of negation, if inductively all plays in which $\varphi(\vec{a})$ is issued as a challenge or arises as a witness case come out true, then all plays in which $\neg\varphi(\vec{a})$ arises will result in false, or else we could create a play in which τ would violate the Tarskian truth conditions, simply by asking about $\varphi(\vec{a})$ after $\neg\varphi(\vec{a})$ was answered affirmatively. Similarly, if φ and ψ always come out the same way, then so must $\varphi \wedge \psi$. We don't claim that τ must always issue the same witness b for an existential $\exists x\, \psi(x, \vec{a})$, but if the strategy ever directs the truth-teller to pronounce this statement to be true, then it will provide some witness b and pronounce $\psi(b, \vec{a})$ to be true, and by induction this truth pronouncement for $\psi(b, \vec{a})$ is independent of the play on which it arises, forcing $\exists x\, \varphi(x, \vec{a})$ to always be pronounced true. Thus, by induction on formulas, the truth pronouncements made by the truth-teller strategy τ allow us to define from τ a satisfaction predicate for first-order truth. □

It follows by Tarski's theorem on the non-definability of truth that there can be no definable winning strategy for the truth-teller in this game, because there can be no definable satisfaction predicate.

Lemma 2.2. *The interrogator has no winning strategy in the truth-telling game.*

Proof. Suppose that σ is a strategy for the interrogator. So σ is a proper class function that directs the interrogator to issue certain challenges, given the finite sequence of previous challenges and truth-telling answers. By the reflection theorem, there is a closed unbounded proper class of cardinals θ, such that $\sigma\,"\, V_\theta \subseteq V_\theta$. That is, V_θ is closed under σ, in the sense that if all previous challenges and responses come from V_θ, then the next challenge will also come from V_θ. Since $\langle V_\theta, \in\rangle$ is a set, we have a satisfaction predicate on it, as well as a Skolem function selecting existential witnesses. Consider the play, where the truth-teller replies to all inquiries by consulting truth in V_θ, rather than truth in V, and using the Skolem function to provide the witnesses in the existential case. The point is that if the interrogator follows σ, then all the inquiries will involve only parameters $\vec{a}$ in V_θ, provided that the truth-teller also always gives witnesses in V_θ, which in this particular play will be the case. Since the satisfaction predicate on V_θ does satisfy the Tarskian truth conditions, it follows that the truth-teller will win this instance of the game, and so σ is not a winning strategy for the interrogator. □

Thus, if open determinacy holds for classes, then there is a satisfaction predicate Tr for first-order truth. Let us explain how this implies Con(ZFC) and more. Working in Gödel-Bernays set theory, we may apply the reflection theorem to the class Tr and thereby find a proper class club C of cardinals θ for which $\langle V_\theta, \in, \mathrm{Tr} \restriction V_\theta\rangle \prec_{\Sigma_1} \langle V, \in, \mathrm{Tr}\rangle$. In particular, this implies that $\mathrm{Tr} \cap V_\theta$ is a satisfaction class on V_θ, which therefore agrees with truth in that structure, and so these models form a continuous elementary chain, whose union is the entire universe:

$$V_{\theta_0} \prec V_{\theta_1} \prec \cdots \prec V_\lambda \prec \cdots \prec V.$$

There is a subtle point here concerning ω-nonstandard models, namely, to see that all instances of ZFC axioms are declared true by Tr, it is inadequate merely to note that we have assumed ZFC to be true in V, because this will give us only the standard-finite instances of those axioms in Tr, but perhaps we have nonstandard natural numbers in V, beyond the natural numbers of our metatheory. Nevertheless, because in GBC we have the collection axiom relative to the truth predicate itself, we may verify that all instances of the collection axiom (including nonstandard instances, if any)

$$\forall b\,\forall z\,\big(\forall x\in b\,\exists y\,\varphi(x,y,z)\to\exists c\,\forall x\in b\,\exists y\in c\,\varphi(x,y,z)\big)$$

must be declared true by Tr, because we may replace the assertion of $\varphi(x,y,z)$ with the assertion $\varphi(x,y,z)\in\mathrm{Tr}$, which reduces the instance of collection for the (possibly nonstandard) formula φ to an instance of standard-finite collection in the language of Tr, using the Gödel code of φ as a parameter, thereby collecting sufficient witnesses y into a set c. So even the nonstandard instances of the collection axiom must be declared true by Tr. It follows that each of these models V_θ for $\theta\in C$ is a transitive model of ZFC, understood in the object theory of V, and so we may deduce Con(ZFC) and $\mathrm{Con}\big(\mathrm{ZFC}+\mathrm{Con}(\mathrm{ZFC})\big)$ and numerous iterated consistency statements of the form $\mathrm{Con}^\alpha(\mathrm{ZFC})$, which must be true in all such transitive models for quite a long way. Alternatively, one can make a purely syntactic argument for Con(ZFC) from a satisfaction class, using the fact that the satisfaction class is closed under deduction and does not assert contradictions.

We have not yet quite proved the theorem, because the truth-telling game is an open game, rather than a clopen game, whereas the hypothesis of the theorem allows only clopen determinacy. The truth-teller wins the truth-telling game only by playing the game out for infinitely many steps, and this is not an open winning condition for her, since at any point the play could have continued in such a way so as to produce a loss for the truth-teller, if the players cooperated in order to achieve that. So let us describe a modified game, the *counting-down truth-telling game*, which will be clopen. Specifically, this game is just like the truth-telling game, except that we insist that the interrogator must also state on each move a specific ordinal α_n, which descend during play $\alpha_0>\alpha_1>\cdots>\alpha_n$. If the interrogator gets to 0, then the truth-teller is declared the winner. For this modified game, the winner will be known in finitely many moves, because either the truth-teller will violate the Tarskian conditions or the interrogator will hit zero; and so this is a clopen game. Since the counting-down version of the game is harder for the interrogator, it follows that the interrogator still can have no winning strategy. We modify the proof of lemma 2.1 for this game by claiming that if τ is a winning strategy for the truth-teller in the counting-down truth-telling game, then the truth pronouncements made by τ in response to all plays *with sufficiently large ordinals* all agree with one another independently of the interrogator's play. The inductive argument of lemma 2.1 still works under the assumption that the counting-down ordinal is sufficiently large, because there will be enough time to reduce a problematic case. For example, if $\varphi(\vec{a})$ always gets the same truth pronouncement for plays in which it arises with sufficiently large ordinals, then so also does $\neg\varphi(\vec{a})$, with a slightly larger ordinal, because in a play with the wrong value for $\neg\varphi(\vec{a})$ we may direct the interrogator to inquire next about $\varphi(\vec{a})$ and get a violation of the Tarskian recursion. Similar reasoning works in the other cases, and so we may define a satisfaction class from a strategy in the modified game. Since that game is clopen, we have proved that clopen determinacy for class games implies the existence of a satisfaction class for first-order truth, and this implies Con(ZFC) and more, as we have explained. □

We didn't really need the interrogator to count down in the ordinals, since it would in fact have sufficed to have him count down merely in the natural numbers;

the amount of time remaining required for the truth pronouncements to stabilize is essentially related to the syntactic complexity of φ.

We may easily modify the game by allowing a fixed class parameter B, so that clopen determinacy implies that there is a satisfaction class relative to truth in $\langle V, \in, B\rangle$. For example, we may get a truth predicate Tr_1 for the structure $\langle V, \in, \mathrm{Tr}\rangle$ itself, so that Tr_1 concerns truth-about-truth. Iterating this idea further, let us consider the *iterated-truth-telling game*, where we expand the language beyond the usual language of set theory by adding a hierarchy of predicates Tr_α, one for every ordinal α, which will serve as iterated-truth predicates. In this version of the game, we allow the interrogator to ask about formulas in this expanded language, while counting down in the ordinals, and the truth-teller is required to obey not only the usual Tarskian recursive truth conditions, but also the iterated-truth condition that $\mathrm{Tr}_\alpha\left(\varphi(\vec{a})\right)$ is pronounced true just in case $\varphi(\vec{a})$ is a formula-parameter pair using only truth predicates Tr_β for $\beta < \alpha$ and also $\varphi(\vec{a})$ is itself pronounced true, if this challenge was issued. The iterated-truth-telling game is open for the interrogator, but we may modify it to the counting-down iterated-truth-telling game, by insisting that the interrogator count down in the natural numbers during play, with the truth-teller winning when the clock expires. This results in a clopen game, and the arguments used in the proof of theorem 2 generalize easily to show that the interrogator cannot have a winning strategy in this counting-down iterated-truth-telling game, and the truth-teller has a strategy just in case there is a satisfaction predicate for truth-about-truth iterated through the ordinals:

Theorem 3. *If the principle of clopen determinacy holds, then there is a system of iterated-truth satisfaction classes Tr_α for ordinals α, obeying the Tarskian recursion and the iterated-truth conditions.*

Since this theorem can also be obtained as a direct consequence of theorem 6, as the iterated-truth predicates can be defined by a first-order recursion, we shall not give a separate detailed proof.

Let us briefly clarify the role of the global choice principle.

Theorem 4. *In Gödel-Bernays set theory* GB*, the principle of clopen determinacy implies the global axiom of choice.*

Proof. Consider the game where player I plays a nonempty set b an player II plays a set a, with player II winning if $a \in b$. This is a clopen game, since it is over after one move for each player. Clearly, player I can have no winning strategy, since if b is nonempty, then player II can win by playing any element $a \in b$. But a winning strategy for player II amounts exactly to a global choice function, selecting uniformly from each nonempty set an element. □

We find it interesting to notice that the set analogue of the proof of theorem 4 shows in ZF that clopen determinacy for set-sized games implies the axiom of choice, and so over ZF the principle of clopen determinacy for set-sized games is equivalent to the axiom of choice. As a consequence, we may prove in ZF that the *universal axiom of determinacy*, which asserts that every game on every set is determined, is simply false: either there is some clopen game that is not determined, or the axiom of choice holds and there is a game on the natural numbers that is not determined.

A second simple observation about theorem 4 is that this argument answers a special case of question 1. Namely, since there are models of ZFC where the global axiom of choice fails, there must be some models of ZFC having definable clopen games with no definable winning strategy. Our earlier theorem 2, in contrast, established the stronger result that *every* model of ZFC, including those with global choice, has a definable clopen game with no definable strategy.

We shall now generalize the argument of theorem 2 to prove a stronger result, which we believe explains the phenomenon of theorem 2. Specifically, in theorem 6 we shall prove that clopen determinacy is exactly equivalent over GBC to the principle of first-order transfinite recursion over well-founded class relations. This explains the result of theorem 2 because, as we have mentioned, truth itself is defined by such a recursion, the familiar Tarskian recursive definition of truth defined by recursion on formulas, and so the principle of transfinite recursion over well-founded class relations implies the existence of a satisfaction class for first-order truth. Note that even though the tower of formula complexity has height ω, nevertheless the relation underlying this recursion is not set-like, since the truth value of an existential assertion $\exists x\, \varphi(x, \vec{a})$ relies on the truth values of $\varphi(b, \vec{a})$, for all objects b, and this is a proper class of predecessors.

Definition 5. The principle of *first-order transfinite recursion over well-founded class relations* is the assertion that every first-order recursive definition along any well-founded binary class relation (not necessarily set-like) has a solution.

Let us explain in more detail. A binary relation $\lhd$ on a class I is well-founded, if every nonempty subclass $B \subseteq I$ has a $\lhd$-minimal element. This is equivalent in GBC to the assertion that there is no infinite $\lhd$-descending sequence, and indeed one can prove this equivalence in $\mathrm{GB}+\mathrm{DC}$, meaning the dependent choice principle for set relations: clearly, if there is an infinite $\lhd$-descending sequence, then the set of elements on that sequence is a set with no $\lhd$-minimal element; conversely, if there is a nonempty class $B \subseteq I$ with no $\lhd$-minimal element, then by the reflection principle relativized to the class B, there is some V_θ for which $B \cap V_\theta$ is nonempty and has no $\lhd$-minimal element; but using DC for $\lhd \cap V_\theta$ we may successively pick $x_{n+1} \lhd x_n$ from $B \cap V_\theta$, leading to an infinite $\lhd$-descending sequence. We find it interesting to notice that in this class context, therefore, well-foundedness for class relations becomes a first-order concept, which is a departure from the analogous situation in second-order number theory, where of course well-foundedness is Π^1_1-complete and definitely not first-order expressible in number theory. Continuing with our discussion of recursion, suppose that we have a well-founded binary relation $\lhd$ on a class I, and suppose further that $\varphi(F, b, y, Z)$ is a formula describing the recursion rule we intend to implement, where φ involves only first-order quantifiers, F is a class variable for a partial solution and Z is a fixed class parameter, henceforth suppressed. We assume that this recursion rule is functional in the sense that for any $b \in I$ and any class F, there is a unique y such that $\varphi(F, b, y)$. The idea is that $\varphi(F, b, y)$ expresses the recursive rule to be iterated: given a partial solution F defined up to b, then $\varphi(F, b, y)$ instructs us to put object y at node b. A *solution* of the recursion is a class function $F : I \to V$ such that $\varphi\left(F \restriction b, b, F(b)\right)$ holds for every $b \in I$, where $F \restriction b$ means the restriction of F to the class $\{\, c \in I \mid c \lhd b \,\}$. Thus, the value $F(b)$ is determined by the (possibly proper) class of previous values $F(c)$ for $c \lhd b$. The principle of first-order transfinite recursion over well-founded class relations is the assertion that for every such well-founded relation $\langle I, \lhd \rangle$ and any first-order recursive rule φ as above, there is a solution. One may equivalently consider only well-founded partial order relations, simply by taking the transitive closure of the relation, and then proving that the original recursion has a solution if and only if the corresponding recursion on the partial order has a solution.

Although the principle of transfinite recursion defined above may appear to be a scheme, in fact it is fully expressible in a single second-order assertion in the language of Gödel-Bernays set theory. The reason is that for any class parameters Z and F, the principle of transfinite recursion implies that there is a unique satisfaction class for first-order truth relative to Z, and we may use these satisfaction classes to refer uniformly to the truth of $\varphi(F, b, y, Z)$, allowing us to quantify over φ rather than treating it scheme-theoretically. Basically, the principle asserts, "for every class, there is a satisfaction class relative to it, and for every well-founded relation $\lhd$, every formula φ and every parameter Z, if the formula $\varphi(F, b, y, Z)$ is functional (which means for every class function F and every truth-predicate for first-order truth with respect to F and Z, the corresponding relation is functional), then there is a solution of the recursion (using the unique truth-predicate relative to Z and that solution)."

In the special case for which the relation $\lhd$ is set-like, which means that the predecessors $\{\, c \mid c \lhd b \,\}$ of any point b form a set (rather than a proper class), then GBC easily proves that there is a unique solution class, which furthermore is definable from $\lhd$. One simply follows the usual proof of transfinite recursion in ZFC, showing that every $b \in I$ is in the domain of a partial solution that obeys the recursive rule on its domain, because there can be no minimal counterexample to this; all such partial solutions agree on their common domains, and the union of them is a total solution of the recursion. Similarly, GBC can prove that there are solutions to other transfinite recursion instances for which the well-founded relation is not necessarily set-like, such as a recursion of length $\mathrm{Ord} + \mathrm{Ord}$ or even much longer.

Meanwhile, if GBC is consistent, then it cannot in general prove that transfinite recursions along non-set-like well-founded relations always succeed, since as we mentioned this principle implies that there is a truth predicate for first-order truth, which implies Con(ZFC) and therefore also Con(GBC). Thus, GBC plus transfinite recursion is strictly stronger than GBC in consistency strength, although it is provable in Kelley-Morse set theory KM, in essentially the same way that GBC proves the set-like special case.

Theorem 6. *In Gödel-Bernays set theory* GBC*, the following are equivalent.*

(1) *Clopen determinacy for class games. That is, in any two-player game of perfect information whose winning condition class is both open and closed, there is a winning strategy for one of the players.*
(2) *The principle of first-order transfinite recursion over well-founded class relations: every such recursion has a solution.*

Proof. $(2 \to 1)$ Assume the principle of first-order transfinite recursion over well-founded class relations, and suppose we are faced with a clopen game. Consider the game tree T, consisting of positions arising during play, up to the moment that a winner is known, orienting the tree so that the root is at the top and play proceeds downward. This tree is well-founded precisely because the game is clopen. Let us label the terminal nodes of the tree with I or II according to who has won the game in that position, and more generally, let us label all the nodes of the tree with I or II according to the following transfinite recursion: if a node has I to play, then it will have label I if there is a move to a node already labeled I, and otherwise II;

similarly, when it is player II's turn to play, then if she can play to a node labeled II, we label the original node with II, and otherwise I. By the principle of transfinite recursion, there is a labeling of the entire tree that accords with this recursive rule. It is now easy to see that if the initial node is labeled with I, then player I has a winning strategy, which is simply to stay on the nodes labeled I. (We use the global choice principle to choose a particular such node with the right label; this use can be avoided if the space X of possible moves is already well-ordered, such as in the case of games on the ordinals $X = \mathrm{Ord}$.) Note that player II cannot play in one move from a node labeled I to one labeled II. Similarly, if the initial node is labeled II, then player II has a winning strategy, which is simply to stay on the nodes labeled II. And so the game is determined, and we have established clopen determinacy.

($1 \to 2$) Conversely, let us assume the principle of clopen determinacy for class games. Suppose we are faced with a recursion along a class relation $\lhd$ on a class I, using a first-order recursion rule $\varphi(F, b, y)$, possibly with a fixed class parameter, which we suppress. We shall define a certain clopen game, and prove that any winning strategy for this game will produce a solution for the recursion. It will be convenient for us to assume that $\varphi(F, b, y)$ is absolutely functional, meaning that not only does it define a function as we have mentioned in V, but also that $\varphi(F, b, y)$ defines a function $(F, b) \mapsto y$ when used over any model of the form $\langle V_\theta, \in, F\rangle$, regardless of the theory of this model, for any $F \subseteq V_\theta$ (we are viewing the class function as a predicate, a class of pairs). The strongly functional property can be achieved simply by implementing a default value, replacing the formula with the assertion that $\varphi(F, b, y)$, if y is unique such that this holds, and otherwise $y = \emptyset$.

At first, we consider a simpler open game, the *recursion game*, which will be much like the truth-telling game used in theorem 2, except that in this game, the truth-teller will also provide information about the putative solution of the recursion in question; later, we shall revise this game to a clopen game. In the recursion game, we have the same two players again, the interrogator and the truth-teller, but now the interrogator will make inquiries about truth in a structure of the form $\langle V, \in, \lhd, F\rangle$, where $\lhd$ is the well-founded class relation and F is a class function (considered as a predicate for a class of ordered pairs), not yet specified, but which we hope will become a solution of the recursion. Specifically, the interrogator is allowed to ask about the truth of any first-order formula $\varphi(\vec{a})$ in the language of this structure, and also to inquire as to the value of $F(b)$ for any particular b. The truth-teller, as before, will answer the inquiries by pronouncing either that $\varphi(\vec{a})$ is true or that it is false, and in the case $\varphi(\vec{a}) = \exists x\, \psi(x, \vec{a})$ and the formula was pronounced true, then the truth-teller shall also provide as before a witness b for which she also pronounces $\psi(b, \vec{a})$ to be true. The truth-teller loses immediately, if she should ever violate Tarski's recursive definition of truth, and she also is required to pronounce any instance of the recursion rule $\varphi\left(F \restriction b, b, F(b)\right)$ to be true and also to assert that F is a class function on I. Since violations of any of these requirements, if they occur at all, do so at a finite stage of play, it follows that the game is open for the interrogator.

Lemma 6.1. *The interrogator has no winning strategy in the recursion game.*

Proof. To prove this lemma, we use the idea of lemma 2.2, modified with a sneaky trick. Suppose that σ is a strategy for the interrogator. So σ is a class function that instructs the interrogator how to play next, given a position of partial play.

By the reflection theorem, there is an ordinal θ such that V_θ is closed under σ, and using the satisfaction class that comes from clopen determinacy, we may actually also arrange that $\langle V_\theta, \in, \lhd \cap V_\theta, \sigma \cap V_\theta\rangle \prec \langle V, \in, \lhd, \sigma\rangle$. Consider the relation $\lhd \cap V_\theta$, which is a well-founded relation on $I \cap V_\theta$. The important point is that this relation is now a set, and in GBC we may certainly undertake transfinite recursions along well-founded set relations. Thus, there is a (unique) function $f : I \cap V_\theta \to V_\theta$ such that the structure $\langle V_\theta, \in, \lhd \cap V_\theta, f\rangle$ satisfies $\varphi\left(f \upharpoonright b, b, f(b)\right)$ for all $b \in I \cap V_\theta$, where $f \upharpoonright a$ means restricting f to the predecessors of b that happen to be in V_θ. There is a unique such function f, precisely because by our assumption that φ was strongly functional, if we have defined f on the hereditary predecessors of some point $b \in I \cap V_\theta$, then there is unique value y to place at b itself that satisfies $\varphi(f \upharpoonright b, b, y)$ in the structure $\langle V_\theta, \in, f \upharpoonright b\rangle$, and this unique y therefore will become the value $y = f(b)$. We use our assumption that φ was strongly functional here, since we want to ensure that it can still be used to define a valid recursion over $\lhd \cap V_\theta$. (We are not claiming that $\langle V_\theta, \in, \lhd \cap V_\theta, f\rangle$ models ZFC$(\lhd, f)$.) Consider now the play of the recursion game in V, where the interrogator uses the strategy σ and the truth-teller plays in accordance with truth in the structure $\langle V_\theta, \in, \lhd \cap V_\theta, f\rangle$, which is a sneaky trick because the function f is a solution of the recursion rule φ only on the relation $\lhd \cap V_\theta$, rather than the full relation $\lhd$. But since V_θ was closed under σ, the interrogator will never issue challenges outside of V_θ in this play; and since the function f fulfills the recursion $\varphi\left(f \upharpoonright b, b, f(b)\right)$ in this structure, the truth-teller will not be trapped in any violation of the Tarski conditions or the recursion condition. Thus, the truth-teller will win this instance of the game, and so σ was not a winning strategy for the interrogator, as desired. □

Lemma 6.2. *The truth-teller has a winning strategy in the recursion game if and only if there is a solution of the recursion.*

Proof. If there is a solution F of the recursion, then by clopen determinacy we know there is also a satisfaction class Tr for first order truth in the structure $\langle V, \in, \lhd, F\rangle$, and the truth-teller can answer all queries of the interrogator in the recursion game by referring to what Tr asserts is true in this structure. This will be winning for the truth-teller, since Tr obeys the Tarskian conditions and makes all instances of the recursive rule true.

Conversely, suppose that τ is a winning strategy for the truth-teller in the recursion game. We may see as before that the truth pronouncements made by τ about truth in the structure $\langle V, \in, \lhd\rangle$ are independent of the play in which they occur, and they provide a satisfaction predicate for this structure. This is proved just as for the truth-telling game by induction on the complexity of the formulas: the strategy must correctly answer all atomic formulas, and the answers to more complex formulas must be independent of the play since violations of this would lead to violations of the Tarski conditions by reducing to simpler formulas, as before, and this would contradict our assumption that τ is a winning strategy for the truth-teller. Now let us consider truth pronouncements made by τ in the language involving the class predicate symbol F. We shall actually need this property only in the restricted languages, where for each $b \in I$, we consider formulas in the corresponding language that make reference only to the predicate $F \upharpoonright b$, rather than the full predicate F; we consider $F \upharpoonright b$ as it is naturally defined in the language with F. We claim by induction on b, with an embedded induction on formulas, that for every $b \in I$, the truth pronouncements provided by the strategy τ in this language are independent of the play in which they are made and furthermore provide a truth predicate for a structure of the form $\langle V, \in, \lhd, F \restriction b\rangle$. The case where b is $\lhd$-minimal is essentially similar to the case we already handled, where no reference to F is made, since τ must assert that F is a class function on I and so τ must also assert that $F \restriction b = \emptyset$ whenever b is minimal. Suppose inductively that our claim is true for assertions in the language with $F \restriction c$, whenever c is $\lhd$-hereditarily below b, and consider the language with $F \restriction b$. (Note that the claim we are proving by induction is first-order expressible in the class parameter τ, and so this induction can be legitimately undertaken in GBC; we haven't allowed an instance of Π^1_1-comprehension to sneak in here.) It is not difficult to see that τ must pronounce that each $F \restriction c$ is a function on the class of $\lhd$-predecessors of c, which furthermore agrees with $F \restriction b$ on their common domain, since any violation of this will amount to a contradiction to the assertion that F is a function on I, which τ must assert by the rules of the game. So our induction assumption ensures that τ has determined a well-defined class function $F \restriction b$. Furthermore, since τ is required to affirm that the symbol F obeys the recursive rule, it follows that τ asserts that $F \restriction b$ obeys the recursive rule up to b. We now argue by induction on formulas that the truth pronouncements made by τ about the structure $\langle V, \in, \lhd, F \restriction b\rangle$ forms a satisfaction class for this structure. In the atomic case, the truth pronouncements about this structure are independent of the play of the game in which they occur, since this is true for atomic formulas in the language of set theory and for atomic assertions about $\lhd$, by the rules of the game, and it true for atomic assertions about $F \restriction b$ by our induction hypothesis on b. Continuing the induction, it follows that the truth pronouncements made about compound formulas in this structure are similarly independent of the play and obey the Tarskian conditions, since any violation of this can be easily exposed by having the interrogator inquire about the constituent formulas, just as in the truth-telling game. So the claim is also true for $F \restriction b$. Thus, for every $b \in I$, the strategy τ is providing a satisfaction class for $\langle V, \in, \lhd, F \restriction b\rangle$, which furthermore verifies that the resulting class function $F \restriction b$ determined by this satisfaction class fulfills the desired recursion relation up to b. Since these restrictions of F also all agree with one another, the union of these class functions is a class function $F : I \to V$ that for every b obeys the desired recursive rule $\varphi\left(F \restriction b, b, F(b)\right)$. So the recursion has a solution, and this instance of the principle of first-order transfinite recursion along well-founded class relations is true. □

So far, we have established that the principle of open determinacy implies the principle of transfinite recursion along well-founded class relations. In order to improve this implication to use only clopen determinacy rather than open determinacy, we modify the game as in lemma 2.1 by requiring the interrogator to count-down during play. Specifically, the *count-down recursion game* proceeds just like the recursion game, except that now we also insist that the interrogator announce on the first move a natural number n, such that the interrogator loses if the truth-teller survives for at least n moves (we could have had him count down in the ordinals instead, which would have made things more flexible for him, but the analysis is essentially the same). This is now a clopen game, since the winner will be known by the time this clock expires, either because the truth-teller will violate the Tarski conditions or the recursion condition before that time, in which case the interrogator wins, or else because she did not and the clock expired, in which case the truth-teller wins. So this is a clopen game.

Since the modified version of the game is even harder for the interrogator, there can still be no winning strategy for the interrogator. So by the principle of clopen determinacy, there is a winning strategy τ for the truth-teller. This strategy is allowed to make decisions based on the number n announced by the interrogator on the first move, and it will no longer necessarily be the case that the theory declared true by τ will be independent of the interrogator's play, since the truth-teller can relax as the time is about to expire, knowing that there isn't time to be caught in a violation. Nevertheless, it will be the case, we claim, that the theory pronounced true by τ for all plays with sufficiently many remaining moves will be independent of the interrogator's play. One can see this by observing that if an assertion $\psi(\vec{a})$ is independent in this sense, then also $\neg\psi(\vec{a})$ will be independent in this sense, for otherwise there would be plays with a large number of plays remaining giving different answers for $\neg\psi(\vec{a})$ and we could then challenge directly afterward with $\psi(\vec{a})$, which would have to give different answers or else τ would not win. Similarly, since τ is winning for the truth-teller, one can see that allowing the interrogator to specify a bound on the total length of play does not prevent the arguments above showing that τ describes a coherent solution function $F : I \to V$ satisfying the recursion $\varphi\left(F \upharpoonright b, b, F(b)\right)$, provided that one looks only at plays in which there are sufficiently many moves remaining. There cannot be a $\lhd$-least b where the value of $F(b)$ is not determined in this sense, and so on just as before. So the strategy must give us a function F and a truth predicate for $\langle V, \in, \lhd, F\rangle$ witnessing that it solves the desired recursion, as desired.

In conclusion, the principle of clopen determinacy for class games is equivalent to the principle of first-order transfinite recursion along well-founded class relations. □

We find a certain symmetry between theorem 6, which shows that clopen determinacy is exactly equivalent over GBC to the principle of first-order transfinite recursion over well-founded class relations, with the theorem of Steel and Simpson [Sim09, Thm V.8.7] in second-order arithmetic, showing that clopen determinacy for games on the natural numbers is exactly equivalent in reverse mathematics to the theory of arithmetical transfinite recursion ATR_0. In both cases, we have equivalence of clopen determinacy with a principle of first-order transfinite recursion. In the case of games on the natural numbers, however, Steel and Simpson proved that open determinacy is also equivalent with ATR_0, whereas the corresponding situation of open determinacy for class games is not yet completely settled.

It follows from theorems 2 and 6 that the principle of open determinacy for class games cannot be proved in set theories such as ZFC or GBC, if these theories are consistent, since there are models of those theories that have no satisfaction class for first-order truth. We should now like to prove, in contrast, that the principle of open determinacy for class games *can* be proved in stronger set theories, such as Kelley-Morse set theory KM, as well as in $\mathrm{GBC} + \Pi^1_1$-comprehension, which is a proper fragment of KM.

In order to undertake this argument, however, it will be convenient to consider the theory KM^+, a natural strengthening of Kelley-Morse set theory KM that we consider in [GHJa]. The theory KM^+ extends KM by adding the *class choice scheme*, which asserts of any second-order formula φ, that for every class parameter Z, if for every set x there is a class X with property $\varphi(x, X, Z)$, then there is a class $Y \subseteq V \times V$, such that for every x we have $\varphi(x, Y_x, Z)$, where Y_x denotes the x^{th}

slice of Y. Thus, the axiom asserts that if every set x has a class X with a certain property, then we can choose particular such classes and put them together into a single class Y in the plane, such that the x^{th} slice Y_x is a witness for x. In [GHJa], we prove that this axiom is not provable in KM itself, thereby revealing what may be considered an unfortunate weakness of KM. The class choice scheme can also naturally be viewed as a class collection axiom, for the class Y gathers together a sufficient collection of classes Y_x witnessing the properties $\varphi(x, Y_x, Z)$. In this light, the weakness of KM in comparison with KM^+ is precisely analogous to the weakness of the theory ZFC- in comparison with the theory ZFC^- that we identified in [GHJb]—these are the theories of ZFC without power set, using replacement or collection + separation, respectively—since in each case the flawed weaker theory has replacement but not collection, which leads to various unexpected failures for the respective former theories.

The natural weakening of the class choice scheme to the case where φ is a first-order assertion, having only set quantifiers, is called the *first-order class choice principle*, and it is expressible as a single assertion, rather than only as a scheme, in KM and indeed in GBC+ the principle of first-order transfinite recursion over well-founded class relations, since in these theories we have first-order truth-predicates available relative to any class. A still weaker axiom makes the assertion only for choices over a fixed set, such as the first-order class ω-choice principle:

$$\forall Z\, \big(\forall n \in \omega\, \exists X\, \varphi(n, X, Z) \to \exists Y \subseteq \omega \times V\, \forall n \in \omega\, \varphi(n, Y_n, Z)\big)\,,$$

where φ has only first-order quantifiers, and this is also finitely expressible in GBC + the principle of first-order transfinite recursion over well-founded class relations. In our paper [GHJa], we separate these axioms from one another and prove that none of them is provable in KM, assuming the consistency of an inaccessible cardinal.

The Π^1_1-*comprehension axiom* is the assertion that for any Π^1_1 formula $\varphi(x, Z)$, with class parameter Z, we may form the class $\{\, a \mid \varphi(a, Z)\,\}$. By taking complements, this is equivalent to Σ^1_1-comprehension.

Theorem 7. *Kelley-Morse set theory* KM *proves the principle of open determinacy for class games. Indeed, this conclusion is provable in the subtheory consisting of* GBC *plus* Π^1_1*-comprehension.*

Proof. Assume GBC plus Π^1_1-comprehension. In order to make our main argument more transparent, we shall at first undertake it with the additional assumption that the first-order class choice principle holds (thus, we work initially in a fragment of KM^+). Afterwards, we shall explain how to eliminate our need for the class choice principle, and thereby arrive at a proof using just GBC plus Π^1_1-comprehension.

Consider any open class game $A \subseteq X^\omega$, where A is the open winning condition and X is the class of allowed moves. We shall show the game is determined. To do so, notice that for any position $p \in X^{<\omega}$, the assertion that a particular class function $\sigma : X^{<\omega} \to X$ is a winning strategy for player I in the game proceeding from position p is an assertion about σ involving only first-order quantifiers; one must say simply that every play of the game that proceeds from p and follows σ on player I's moves after that, is in A. Thus, the assertion that player I has a winning strategy for the game starting from position p is a Σ^1_1 assertion about p. Using Π^1_1-comprehension, therefore, we may form the class

$$W = \left\{\; p \in X^{<\omega} \;\middle|\; \begin{array}{l}\text{Player I has a winning strategy in the}\\ \text{game proceeding from position } p\end{array} \right\}.$$

With this class, we may now carry out a class analogue of one of the usual soft proofs of the Gale-Stewart theorem. Namely, if the initial (empty) position of the game is in W, then player I has a winning strategy, and we are done. Otherwise, the initial node is not in W, and we simply direct player II to avoid the nodes of W during play. If this is possible, then it is clearly winning for player II, since he will never land on a node all of whose extensions are in the open class, since such a node is definitely in W, and so he will win. To see that player II can avoid the nodes of W, observe simply that at any position p, if it is player II's turn to play, and player I does not have a strategy in the game proceeding from p, then we claim that there must be at least one move that player II can make, to position $p \frown x$ for some $x \in X$, such that $p \frown x \notin W$. If not, then $p \frown x \in W$ for all moves x, and so for each such x there is a strategy τ_x that is winning for player I in the game proceeding from $p \frown x$. By the first-order class choice principle (and this is precisely where we use our extra assumption), we may gather such strategies τ_x together into a single class and thereby construct a strategy for player I that proceeds from position p in such a way that if player II plays x, then player I follows τ_x, which is winning for player I. Thus, there is a winning strategy for player I from position p, contradicting our assumption that $p \notin W$, and thereby establishing our claim. So if $p \notin W$ and it is player II's turn to play, then there is a play $p \frown x$ that remains outside of W. Similarly, if $p \notin W$ and it is player I's turn to play, then clearly there can be no next move $p \frown y$ placing it inside W, for then player I would have also had a strategy from position p. Thus, if the initial position is not in W, then player II can play so as to retain that property (using global choice to pick a particular move realizing that situation), and player I cannot play so as to get inside W, and this is therefore a winning strategy for player II. So the game is determined.

The argument above took place in the theory $\mathrm{GBC} + \Pi^1_1$-comprehension + the first-order class choice principle. And although it may appear at first to have made a fundamental use of the class choice principle, we shall nevertheless explain how to eliminate this use. The first observation to make is that Π^1_1-comprehension implies the principle of first-order transfinite recursion along any well-founded relation. To see this, suppose that $\lhd$ is any well-founded relation on a class I and $\varphi(F, b, y, Z)$ is a functional recursive rule, asserting that if F is a partial solution up to b, then we should put value y at node b. The class of $b \in I$ that are in the domain of some partial solution to the recursion is Σ^1_1-definable. And furthermore, all such partial solutions must agree on their common domain, by an easy inductive argument along $\lhd$. It follows that the union of all the partial solutions is Σ^1_1-definable and therefore exists as a class, and it is easily seen to obey the recursion rule on its domain. So it is a maximal partial solution. If it is not total, then there must be a $\lhd$-minimal element b not in the domain; but this is impossible, since we could apply the recursive rule once more to get a value y to place at b, thereby producing a partial solution that includes b. So the maximal partial solution is actually a total solution, verifying this instance of the transfinite recursion principle.

Next, we shall explain how to continue the constructibility hierarchy beyond Ord. This construction has evidently been discovered and rediscovered several times in set theory, but rarely published; the earliest reference appears to be the dissertation of Leslie Tharp [Tha65], although Bob Solovay reportedly also undertook the construction as an undergraduate student, without publishing it; see also current work [AF]. In the countable realm, of course, the analogous construction is routine, where one uses reals to code arbitrary countable structures including models of set theory of the form $\langle L_\alpha, \in\rangle$. For classes, suppose that $\Gamma = \langle \mathrm{Ord}, \leq_\Gamma\rangle$ is a *meta-ordinal*, which is to say, a well-ordered class relation $\leq_\Gamma$ on Ord; this relation need not necessarily be set-like, and the order type can reach beyond Ord. By the principle of first-order transfinite recursion, we may iterate the constructible hierarchy up to Γ, and thereby produce a class model $\langle L_\Gamma, \in_\Gamma\rangle$ of $V = L$, whose ordinals have order-type Γ. Specifically, we reserve a class of nodes to be used for representing the new "(meta-)sets" at each level of the L_Γ hierarchy, and define $\in_\Gamma$ recursively, so that at each level, we add all and only the sets that are definable (from parameters) over the previous structure. To be clear, the domain of the structure $\langle L_\Gamma, \in_\Gamma\rangle$ is a class $L_\Gamma \subseteq V$, and the relation $\in_\Gamma$ is not the actual $\in$ relation, but nevertheless $\in_\Gamma$ is a well-founded extensional relation in our original model, and the structure $\langle L_\Gamma, \in_\Gamma\rangle$ looks internally like the constructible universe. Thus, we have what might be termed merely a code for or presentation of the fragment L_Γ of the constructibility hierarchy up to Γ, which someone outside the universe might prefer to think of as an actual transitive set. In order to speak of L_Γ in our GBC context, then, we must be aware that different choices of Γ will lead to different presentations, with sets being represented differently and by different sets. Nevertheless, we may assume without loss that the actual sets in $L = L_{\mathrm{Ord}}$ are represented in this presentation in some highly canonical way, so that the ordinals are represented by themselves, for example, and the other sets are represented by their own singletons (say), and so in particular, all the various L_Γ will agree on their $\in_\Gamma$ relations for sets constructed before Ord. Also, using the principle of first-order transfinite recursion, it is easy to see that any two meta-ordinals Γ and Γ' are comparable, in the sense that one of them is (uniquely) isomorphic to an initial segment of the other, and similarly the structures L_Γ and $L_{\Gamma'}$ admit such coherence as well; in particular, if Γ and Γ' are isomorphic, then so also are the structures L_Γ and $L_{\Gamma'}$. Consider the meta-ordinals Γ for which there is a larger meta-ordinal Θ, such that L_Θ has as an element a well-ordered structure $\Gamma' = \langle \mathrm{Ord}, \leq_{\Gamma'}\rangle$ with order-type isomorphic to Γ. In a sense, these are the meta-ordinals below $(\mathrm{Ord}^+)^L$. In this case, there will be an L-least such code Γ' in L_Θ. And furthermore, any other meta-ordinal Θ' which constructs such a code will agree on this L-least code. Since we assumed that the ordinals were represented as themselves in L_Θ, we may view Γ' as a meta-ordinal in our original model. Thus, the meta-ordinals Γ realized by a relation in some L_Θ have *canonical* codes, the meta-ordinals that are L-least with respect to some (and hence all sufficiently large) L_Θ. There is exactly one such code for each meta-ordinal order type that is realized inside any L_Θ. Now, let $\mathcal{L}$ be the collection of classes $B \subseteq L$ that are realized as an element in some L_Γ—these are the classes of the meta-L that are contained in the actual L—and consider the model $\mathscr{L} = \langle L, \in, \mathcal{L}\rangle$. It is not difficult to see that this is a model of GBC, precisely because the L-hierarchy closes under definability at each step of the recursion. Furthermore, the existence of canonical codes will allow us to show that this model satisfies the first-order class choice principle. Suppose that $\mathscr{L} \models \forall b\, \exists X\, \varphi(b, X)$, where φ has only first-order quantifiers. For each set b, there is a class X with property $\varphi(b, X)$, and such a class X exists as a set in some L_Γ for some meta-ordinal Γ. We may consider Γ to be a canonical code for a meta-ordinal which is minimal with respect to the property of having such an X, and in this case, the class $\Gamma = \Gamma_b$ is Σ^1_1-definable (and actually Δ^1_1-definable) from b. So the map $b \mapsto \Gamma_b$ exists as a class in the ground model, and we may therefore form a meta-ordinal Θ that is larger than all the resulting meta-ordinals Γ_b. Inside L_Θ, we may select the L-least X_b witnessing $\varphi(b, X_b, Z)$, and thereby form the class $\{\,(b,c) \mid c \in X_b\,\}$, which fulfills this instance of the first-order class choice principle (and the argument easily accommodates class parameters). A similar idea shows that $\mathscr{L}$ satisfies Π^1_1-comprehension, provided that this was true in the original model (and indeed $\mathscr{L}$ satisfies KM, if this was true in the original model, and in this case one can also verify the class choice scheme in $\mathscr{L}$, without requiring this in V, which shows that $\mathrm{Con(KM)} \to \mathrm{Con(KM^+)}$; see [GHJa].) It will be more convenient to establish Σ^1_1-comprehension, which is equivalent. Consider a formula of the form $\exists X\, \varphi(b, X)$, where φ has only first-order quantifiers. The class $B = \{\, b \in L \mid \exists X \in \mathcal{L}\ \varphi(b, X)\,\}$ is Σ^1_1-definable and therefore exists as a class in our original universe. We need to show it is in $\mathcal{L}$. For each $b \in B$, there is a class X such that $X \in \mathcal{L}$ and $\varphi(b, X)$, and such a class X is constructed in some L_Γ at some minimal meta-ordinal stage Γ, which we may assume is a canonical code. Thus, the map $b \mapsto \Gamma_b$ is Σ^1_1-definable, and so it exists as a class. Thus, we may form a single meta-ordinal Θ larger than all the Γ_b, and in L_Θ, we may define the set B. So $B \in \mathcal{L}$, verifying this instance of Σ^1_1-comprehension in $\mathscr{L}$, as desired.

One may now check that the construction of the previous paragraph relativizes to any class $Z \subseteq \mathrm{Ord}$, leading to a model $\langle L[Z], \in, \mathcal{S}\rangle$ that satisfies $\mathrm{GBC} + \Pi^1_1$-comprehension + the first-order class choice principle, in which Z is a class. One simply carries the class parameter Z through all of the previous arguments. If Z codes all of V, then the result is a model $\langle V, \in, \mathcal{S}\rangle$, whose first-order part has the same sets as the original model V.

Using this, we may now prove the theorem. Consider any open class game $A \subseteq X^\omega$, where X is the class of allowed moves. Let $Z \subseteq \mathrm{Ord}$ be a class that codes in some canonical way every set in V and also the classes X and A. The resulting structure $\langle V, \in, \mathcal{S}\rangle$ described in the previous paragraph therefore satisfies $\mathrm{GBC}+\Pi^1_1$-comprehension + the first-order class choice principle. The game $A \subseteq X^\omega$ exists in this structure, and since it is open there, it follows by the first part of the proof of this theorem that this game is determined in that structure. So there is a strategy $\sigma \in \mathcal{S}$ for the game A that is winning for one of the players. But this is absolute to our original universe, because the two universes have exactly the same sets and therefore exactly the same plays of the game. So the game is also determined in our original universe, and we have thus verified this instance of the principle of open determinacy for class games. $\square$

One might view the previous argument as a proper class analogue of Blass's result [Bla72] that computable games have their winning strategies appearing in the L-hierarchy before the next admissible set, since we found the winning strategies for the open class game in the meta-L hierarchy on top of the universe. Nevertheless, we are unsure sure exactly what it takes in the background theory to ensure that the meta-L structure is actually admissible.

The work of this article suggests numerous questions for further investigation. Can we weaken the assumption of Π^1_1-comprehension in theorem 7 to use only the principle of first-order transfinite recursion over well-founded class relations? If so, it would follow that open determinacy and clopen determinacy for class games are both equivalent over GBC to the principle of transfinite recursion, which would resonate with the corresponding situation in reverse mathematics for games on the natural numbers, where both open determinacy and clopen determinacy are equivalent to ATR_0. Which strengthening of GBC suffices to prove the meta-L structure we construct in theorem 7 is admissible? If this was possible in GBC plus transfinite recursion, then the proper class analogue of the Blass result mentioned in the previous paragraph might show that open determinacy and clopen determinacy for classes are equivalent over GBC. What is the relation between transfinite recursion over well-founded class relations and Δ^1_1-comprehension? Is there a class game analogue of Martin's proof [Mar75] of Borel determinacy? What does it take to prove the class analogue of Borel determinacy for class games? There is a natural concept of class Borel codes, which in KM^+ gives rise to a collection of classes that is the smallest collection of classes containing the open classes and closed under countable unions and complements. Are all such class games determined? If κ is an inaccessible cardinal, then the full second-order structure $\langle V_\kappa, \in, V_{\kappa+1}\rangle$ is a model of KM^+ that satisfies Borel determinacy for class games. Is there an proper class analogue of Harvey Friedman's famous proof [Fri71] that Borel determinacy requires strength? We have taken up all these questions in current work.

References

[AF] Carlin Antos and Sy-David Friedman. Hyperclass forcing and minimal models of MK. in preparation.

[Bla72] Andreas Blass. Complexity of winning strategies. *Discrete Mathematics*, 3(4):295 – 300, 1972.

[EH14] C. D. A. Evans and Joel David Hamkins. Transfinite game values in infinite chess. *Integers*, 14:Paper No. G2, 36, 2014.

[EHP] C. D. A. Evans, Joel David Hamkins, and Norman Lewis Perlmutter. A position in infinite chess with game value ω^4. manuscript in preparation.

[Fri71] Harvey M. Friedman. Higher set theory and mathematical practice. *Annals of Mathematical Logic*, 2(3):325 – 357, 1971.

[GHJa] Victoria Gitman, Joel David Hamkins, and Thomas A. Johnstone. Kelley-morse set theory and choice principles for classes. manuscript in preparation.

[GHJb] Victoria Gitman, Joel David Hamkins, and Thomas A. Johnstone. What is the theory ZFC without Powerset? *Mathematical Logic Quarterly*. in press, to appear.

[GS53] D. Gale and F. M. Stewart. Infinite games with perfect information. *Ann. Math. Studies*, 28:245–266, 1953.

[Mar75] Donald A. Martin. Borel determinacy. *Ann. of Math. (2)*, 102(2):363–371, 1975.

[Mat15] A. R. D. Mathias. In lode della logica. *La Matematica nella Societ e nella Cultura-Rivista dell'U.M.I.*, 8:43–74, April 2015. in Italian.

[Sim09] Stephen G. Simpson. *Subsystems of second order arithmetic*. Perspectives in Logic. Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, second edition, 2009.

[Tha65] Leslie Howard Tharp. *Constructibility in impredicative set theory*. Ph.d. dissertation, Massachusetts Institute of Technology, June 1965.

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