Game characterizations and lower cones in the Weihrauch degrees*

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Abstract

We introduce generalized Wadge games and show that each lower cone in the Weihrauch degrees is characterized by such a game. These generalized Wadge games subsume the original Wadge games, the eraser and backtrack games as well as variants of Semmes' tree games. As a new example we introduce the tree derivative games which characterize all even finite levels of the Baire hierarchy, and a variant characterizing the odd finite levels.

1 Introduction

The use of infinite games in set theory has a well-established tradition, going back to work by Banach, Borel, Zermelo, Kőnig, and others (see [12, §27] for a thorough historical account of the subject), and taking a prominent role in the field with the work of Gale and Stewart on the determinacy of certain types of such games [11].

In this paper, we will focus on infinite games which have been used to characterize classes of functions in descriptive set theory. Interest in this particular area began with the seminal work of Wadge [27], who introduced what is now known as the *Wadge game*, an infinite game in which two players, **I** and **II**, are

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given a partial function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and play with perfect information. In each run of this game, at each round player **I** first picks a natural number and player **II** responds by either picking a natural number or passing, although she must pick natural numbers at infinitely many rounds. Thus, in the long run **I** and **II** build elements $x \in \mathbb{N}^\mathbb{N}$ and $y \in \mathbb{N}^\mathbb{N}$, respectively, and **II** wins the run if and only if $x \notin \mathrm{dom}(f)$ or $f(x) = y$. Wadge proved that this game *characterizes* the continuous functions, in the following sense.

Theorem 1 (Wadge)**.** *A partial function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is relatively continuous iff player* **II** *has a winning strategy in the Wadge game for f.*

By adding new possibilities for player **II** at each round, one can obtain games characterizing larger classes of functions. For example, in the *eraser game* characterizing the Baire class 1 functions, player **II** is allowed to erase past moves, the rule being that she is only allowed to erase each position of her output sequence finitely often. In the *backtrack game* characterizing the functions which preserve the class of $\mathbf{\Sigma}^0_2$ sets under preimages, player **II** is allowed to erase *all* of her past moves at any given round, the rule in this case being that she only do this finitely many times. See [15] for a survey of these and other related results.

In his PhD thesis [26], Semmes introduced the *tree game* characterizing the (total) Borel functions in Baire space. Player **I** plays as in the Wadge game, and therefore builds some $x \in \mathbb{N}^\mathbb{N}$ in the long run, but at each round n player now **II** plays a finite *labeled tree*, i.e., a pair (T_n, ϕ_n) of a finite tree $T_n \subseteq \mathbb{N}^{<\mathbb{N}}$ and a function $\phi_n : T_n \setminus \{\langle\rangle\} \to \mathbb{N}$, where $\langle\rangle$ denotes the empty sequence. The rules are that $T_n \subseteq T_{n+1}$ and $\phi_n \subseteq \phi_{n+1}$ must hold for each n, and that the *final* labeled tree $(T, \phi) = (\bigcup_{n\in\mathbb{N}} T_n, \bigcup_{n\in\mathbb{N}} \phi_n)$ must be an infinite tree with a unique infinite branch. Player **II** then wins if the sequence of labels along this infinite branch is exactly $f(x)$. By providing suitable extra requirements on the structure of the final tree, Semmes was able to obtain the *multitape game* characterizing the classes of functions which preserve $\mathbf{\Sigma}^0_3$ under preimages, the *multitape eraser game* characterizing the class of functions for which the preimage of any $\mathbf{\Sigma}^0_2$ set is a $\mathbf{\Sigma}^0_3$ set, and a game characterizing the Baire class 2 functions.

As examples of applications of these games, Semmes found a new proof of a theorem of Jayne and Rogers characterizing the class of functions which preserve $\mathbf{\Sigma}^0_2$ under preimages, and extended this theorem to the classes characterized by the multitape and multitape eraser games, by performing a detailed analysis of the corresponding game in each case.

In this work, we exhibit a very general view on how such games can characterize classes of functions. Just as *nice* classes of sets can be understood as lower cones in the Wadge degrees, *nice* classes of functions are found in the lower cones in the Weihrauch degrees. Weihrauch reducibility (in its modern form) was introduced by Brattka and Gherardi [4, 3] based on earlier work by Weihrauch on a reducibility between sets of functions analogous to Wadge reducibility [28, 29]. As an application, we use this general framework to obtain new games characterizing each Baire class n for finite n.

While the traditional scope of descriptive set theory is restricted to Polish spaces, their subsets and functions between them, these restrictions are immaterial for the approach presented here. Our results naturally hold for multi-valued functions between represented spaces. As such, this work is part of a larger development to extend descriptive set theory to a more general setting, cf. e.g. [8, 22, 24, 14, 18].

We shall freely use standard concepts and notation from descriptive set theory, and refer to [13] for an introduction.

2 Preliminaries on represented spaces and Weihrauch reducibility

Represented spaces and continuous/computable maps between them form the setting for computable analysis [30]. For a comprehensive modern introduction we refer to [20].

A *represented* space $\mathbf{X} = (X, \delta_{\mathbf{X}})$ is given by a set X and a partial surjection $\delta_{\mathbf{X}} :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. A (multivalued) function between represented spaces is just a (multivalued) function on the underlying sets. We say that a partial function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a *realizer* for a multivalued function $f :\subseteq \mathbf{X} \to \mathbf{Y}$ (in symbols: $F \vdash f$) if $\delta_{\mathbf{Y}}(F(p)) \in f(\delta_{\mathbf{X}}(p))$ for all $p \in \mathrm{dom}(f\delta_{\mathbf{X}})$. We call f computable (continuous), if it admits some computable (continuous) realizer.

Represented spaces and continuous functions do indeed generalize Polish spaces and continuous functions. Let (X, τ) be some Polish space. Fix a countable dense sequence $(a_i)_{i\in\mathbb{N}}$ and a compatible metric d. Now define $\delta_{\mathbf{X}}$ by $\delta_{\mathbf{X}}(p) = x$ iff $d(a_{p(i)}, x) < 2^{-i}$ holds for all $i \in \mathbb{N}$. In words: We represent a point by a sequence of basic points converging to it with prescribed speed. It is a foundational result in computable analysis that the notion of continuity for the represented space $(X, \delta_{\mathbf{X}})$ coincides with that for the Polish space (X, τ).

Definition 2. Let f and g be partial, multivalued functions between represented spaces. Say that f is *Weihrauch reducible* to g, in symbols $f \leq_{\mathrm{W}} g$, if there are computable functions $K :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and $H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that whenever $G \vdash g$, the function $F := (p \mapsto K(p, G(H(p))))$ is a realizer for f.

If there are computable functions $K, H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that whenever $G \vdash g$ then $KGH \vdash f$, then we that that f is *strongly Weihrauch reducible* to g ($f \leq_{\mathrm{sW}} f$). We write $f \leq_{\mathrm{W}}^{\mathrm{c}} g$ and $f \leq_{\mathrm{sW}}^{\mathrm{c}} g$ for the variations where computable is replaced with continuous.

A multivalued function f *tightens* g, denoted by $f \preceq g$, if $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and whenever $x \in \mathrm{dom}(g)$, then $f(x) \subseteq g(x)$, cf. [23, 19].

Proposition 3 (e.g. [17, Chapter 4])**.** *Let $f :\subseteq \mathbf{A} \rightrightarrows \mathbf{B}$ and $g :\subseteq \mathbf{C} \rightrightarrows \mathbf{D}$. We have*

1. *$f \leq_{\mathrm{sW}} g$ ($f \leq_{\mathrm{sW}}^{\mathrm{c}} g$) iff there exist computable (continuous) $k :\subseteq \mathbf{A} \rightrightarrows \mathbf{C}$ and $h :\subseteq \mathbf{D} \rightrightarrows \mathbf{B}$ such that $hgk \preceq f$; and*
2. *$f \leq_{\mathrm{W}} g$ ($f \leq_{\mathrm{W}}^{\mathrm{c}} g$) iff there exist computable (continuous) $k :\subseteq \mathbf{A} \rightrightarrows \mathbb{N}^{\mathbb{N}} \times \mathbf{C}$ and $h :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbf{D} \rightrightarrows \mathbf{B}$ such that $h(\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times g)k \preceq f$.*

There are plenty of interesting operations defined on Weihrauch degrees (see e.g. the introduction of [5] for a recent overview), here we only require the sequential composition operator $\star$ from [6, 7]. Rather than defining it explicitly as in [7], we will make use of the following characterization:

Theorem 4. $f \star g \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}}\{f' \circ g' \mid f \leq_{\mathrm{W}} f \ \wedge\ g' \leq_{\mathrm{W}} g\}$

We conclude this section by introducing a particularly important family of Weihrauch degrees. Let $\langle\ ,\ \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be some standard computable pairing function. Now, given $p \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $(p)_n \in \mathbb{N}^{\mathbb{N}}$ be defined by $(p)_n(k) = p(\langle n, k\rangle)$. Then let $\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be defined via $\lim(p) = \lim_{n\to\infty}(p)_n$, with the limit on the right-hand side being taken pointwise. Then let $\lim^{(0)} := \mathrm{id}_{\mathbb{N}^{\mathbb{N}}}$ and $\lim^{(n+1)} := \lim \star \lim^{(n)}$.

Fact 5. $f \leq^{\mathrm{c}}_{\mathrm{W}} \lim^{(n)}$ iff f is of Baire class n.

The preceding fact is a theorem for functions between Polish spaces, see e.g. [2]. For more general spaces, this is the appropriate definition of Baire class n that makes things work as expected. For some more discussion on this, see [21, 22], in particular the synthetic Banach-Lebesgue-Hausdorff theorem.

3 Transparent cylinders

We call $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ a *cylinder* if $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times f \leq_{\mathrm{sW}} f$. Note that f is a cylinder iff $g \leq_{\mathrm{W}} f$ and $g \leq_{\mathrm{sW}} f$ are equivalent for all g. This notion is from [4].

Definition 6. Call $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ *transparent* iff for any computable (continuous) $g :\subseteq \mathbf{Y} \rightrightarrows \mathbf{Y}$ there is a computable (continuous) $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{X}$ with $T \circ f \preceq g \circ T$.

A represented space $\mathbf{Z} = (Z, \delta_{\mathbf{Z}})$ is a *subspace* of $\mathbf{Y} = (Y, \delta_{\mathbf{Y}})$ if $Z \subseteq Y$ and $\delta_{\mathbf{Z}} = \delta_{\mathbf{Y}} \upharpoonright \{p \in \mathrm{dom}(\delta_{\mathbf{Y}})\,;\, \delta_{\mathbf{Y}}(p) \in Z\}$.

Lemma 7. *Let $T :\subseteq (X, \delta_{\mathbf{X}}) \rightrightarrows (Y, \delta_{\mathbf{Y}})$ be transparent, and let $(Z, \delta_{\mathbf{Z}})$ be a subspace of $(Y, \delta_{\mathbf{Y}})$. Then $S :\subseteq (X, \delta_{\mathbf{X}}) \rightrightarrows (Z, \delta_{\mathbf{Z}})$ given by*

$$S = T \upharpoonright \{x \in \mathrm{dom}(T)\,;\, T(x) \subseteq Z\}$$

is also transparent.

Proof. Let $f :\subseteq (Z, \delta_{\mathbf{Z}}) \rightrightarrows (Z, \delta_{\mathbf{Z}})$ be computable. Then $f :\subseteq (Y, \delta_{\mathbf{Y}}) \rightrightarrows (Y, \delta_{\mathbf{Y}})$, and therefore there exists a computable $g :\subseteq (X, \delta_{\mathbf{X}}) \rightrightarrows (X, \delta_{\mathbf{X}})$ such that

1. $\mathrm{dom}(f \circ T) \subseteq \mathrm{dom}(T \circ g)$, and
2. for all $x \in \mathrm{dom}(f \circ T)$ we have $T \circ g(x) \subseteq f \circ T(x)$.

Claim 1. $\mathrm{dom}(f \circ S) = \mathrm{dom}(f \circ T)$.

Indeed, the left-to-right inclusion is immediate from the definition of S. Conversely, suppose $x \in \mathrm{dom}(f \circ T)$. Therefore $x \in \mathrm{dom}(T)$ and $T(x) \subseteq \mathrm{dom}(f)$. Thus, since $\mathrm{dom}(f) \subseteq Z$, it follows that $x \in \mathrm{dom}(S)$ and $S(x) \subseteq \mathrm{dom}(f)$, as desired.

Claim 2. $\mathrm{dom}(f \circ S) \subseteq \mathrm{dom}(S \circ g)$.

Indeed, let $x \in \mathrm{dom}(f \circ S) = \mathrm{dom}(f \circ T)$. Then $x \in \mathrm{dom}(T \circ g)$, i.e., $x \in \mathrm{dom}(g)$ and $g(x) \subseteq \mathrm{dom}(T)$, and $T \circ g(x) \subseteq f \circ T(x) \subseteq Z$. Thus $g(x) \subseteq \mathrm{dom}(S)$, i.e. $x \in \mathrm{dom}(S \circ g)$.

Claim 3. For all $x \in \mathrm{dom}(f \circ S)$ we have $S \circ g(x) \subseteq f \circ S(x)$.

Indeed, let $x \in \text{dom}(f \circ S) \subseteq \text{dom}(S \circ g)$. We have

$$\begin{array}{rcl} S \circ g(x) & = & T \circ g(x) \\ & \subseteq & f \circ T(x) \\ & = & f \circ S(x) \end{array}$$

■

The transparent (singlevalued) functions on Baire space where studied by de Brecht under the name *jump operator* in [9]. These are relevant because they induce endofunctors on the category of represented spaces, which in turn can characterize function classes in DST ([21]). The term *transparent* was coined in [6]. Our extension of the concept to multivalued functions between represented spaces is rather straight-forward, but requires the use of the notion of tightening.

Note that if $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ is transparent, then for every $y \in \mathbf{Y}$ there is some $x \in \text{dom}(T)$ with $T(x) = \{y\}$, i.e. T is *slim* in the terminology of [6, Definition 2.7].

Theorem 8 (Brattka & P. [7]). *For every multivalued function g there is a multivalued function $g^t \equiv_W g$ which is a transparent cylinder.*

Proposition 9. *Let $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ and $S :\subseteq \mathbf{Y} \rightrightarrows \mathbf{Z}$ be cylinders. If T is transparent then $S \circ T$ is a cylinder and $S \circ T \equiv_W S \star T$. Furthermore, if S is also transparent then so is $S \circ T$.*

Proof. Suppose that T is transparent.

($S \circ T$ is a cylinder) As S is a cylinder, there are computable $h :\subseteq \mathbf{Z} \rightrightarrows \mathbb{N}^\mathbb{N} \times \mathbf{Z}$ and $k :\subseteq \mathbb{N}^\mathbb{N} \times \mathbf{Y} \rightrightarrows \mathbf{Y}$ such that $\text{id}_{\mathbb{N}^\mathbb{N}} \times S \succeq h \circ S \circ k$. Likewise, there are computable $h' :\subseteq \mathbf{Y} \rightrightarrows \mathbb{N}^\mathbb{N} \times \mathbf{Y}$ and $k' :\subseteq \mathbb{N}^\mathbb{N} \times \mathbf{X} \rightrightarrows \mathbf{X}$ such that $\text{id}_{\mathbb{N}^\mathbb{N}} \times T \succeq h' \circ S \circ k'$. As composition respects tightening ([23, Lemma 2.4.1.b]), we conclude that $(\text{id}_{\mathbb{N}^\mathbb{N}} \times S) \circ (\text{id}_{\mathbb{N}^\mathbb{N}} \times T) = \text{id}_{\mathbb{N}^\mathbb{N}} \times (S \circ T) \succeq h \circ S \circ k \circ h' \circ T \circ k'$. Note that $(k \circ h') :\subseteq \mathbf{Y} \rightrightarrows \mathbf{Y}$ is computable, and as T is transparent, there is some computable $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{X}$ with $(k \circ h') \circ T \succeq T \circ f$. But then $\text{id}_{\mathbb{N}^\mathbb{N}} \times (S \circ T) \succeq h \circ S \circ k \circ h' \circ T \circ k' \succeq h \circ S \circ T \circ f \circ k'$, thus h and $f \circ k'$ witness that $\text{id}_{\mathbb{N}^\mathbb{N}} \times (S \circ T) \leq_{sW} (S \circ T)$, i.e. $S \circ T$ is a cylinder.

($S \circ T \equiv_W S \star T$) The direction $S \circ T \leq_W S \star T$ is immediate. Assume $S' \leq_W S$ and $T' \leq_W T$. We need to show that $S' \circ T' \leq_W S \circ T$ (if the composition exists). As S and T are cylinders, we find that already $S' \leq_{sW} S$ and $T' \leq_{sW} T$. Let h, k witness the former and h', k' the latter. We conclude $h \circ S \circ k \circ h' \circ T \circ k' \preceq S' \circ T'$. As above, there then is some computable f with $k \circ h' \circ T \succeq T \circ f$. Then h and $f \circ k'$ witness that $S' \circ T' \leq_{sW} S \circ T$.

Now suppose that S is also transparent.

($S \circ T$ is transparent) Let $h :\subseteq \mathbf{Z} \rightrightarrows \mathbf{Z}$ be computable. By assumption that S is transparent, there is some computable $g :\subseteq \mathbf{Y} \rightrightarrows \mathbf{Y}$ such that $S \circ g \preceq h \circ S$. Then there is some computable $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{X}$ with $T \circ f \preceq g \circ T$. As composition respects tightening ([23, Lemma 2.4.1.b]), we find that $h \circ S \circ T \preceq S \circ g \circ T \preceq S \circ T \circ f$, which is what we need. ■

It is easy to see that lim is a transparent cylinder, and therefore we have

$$\lim \circ \lim \equiv_W \lim \star \lim .$$

4 Generalized Wadge games

Definition 10. A *probe* for $\mathbf{Y}$ is a computable partial function $\zeta :\subseteq \mathbf{Y} \to \mathbb{N}^{\mathbb{N}}$ such that for every computable (continuous) $f :\subseteq \mathbf{Y} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ there is a computable (continuous) $e :\subseteq \mathbf{Y} \rightrightarrows \mathbf{Y}$ such that $\zeta e \preceq f$.

Note that being a probe is just the dual notion to be being an admissible representation as in the approach taken by Schröder in [25]. As each constant function is continuous, a probe has to be surjective. Moreover, a probe is always transparent.

Definition 11. Let $\zeta :\subseteq \mathbf{Y} \to \mathbb{N}^{\mathbb{N}}$ be a probe, $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ and $f :\subseteq \mathbf{A} \rightrightarrows \mathbf{B}$. The (ζ, T)*-Wadge game for* f is played by two players, **I** and **II**, who take turns in infinitely many rounds. At each round of a run of the game, player **I** first plays a natural number and player **II** then either plays a natural number or passes, as long as she plays natural numbers infinitely often. Therefore, in the long run player **I** builds $x \in \mathbb{N}^{\mathbb{N}}$ and **II** builds $y \in \mathbb{N}^{\mathbb{N}}$, and player **II** *wins* the run of the game if $x \notin \mathrm{dom}(f\delta_{\mathbf{A}})$, or $y \in \mathrm{dom}(\delta_{\mathbf{B}}\zeta T\delta_{\mathbf{X}})$ and $\delta_{\mathbf{B}}\zeta T\delta_{\mathbf{X}}(y) \subseteq f\delta_{\mathbf{A}}(x)$.

For example, it is easy to see that the Wadge game is the $(\mathrm{id}, \mathrm{id})$-Wadge game, the eraser game is the $(\mathrm{id}, \lim)$-Wadge game, and the backtrack game is the $(\mathrm{id}, \lim_\Delta)$-Wadge game, where $\lim_\Delta(p) = \lim(p)$ with $\mathrm{dom}(\lim_\Delta) = \{p \in \mathbb{N}^{\mathbb{N}} ; \exists n \forall m, k \geq n.\ (p)_m = (p)_k\}$. In the next section we will show how one can view Semmes's tree game and some of its variations as (ζ, T)-Wadge games for appropriate ζ and T.

Theorem 12. *Let T be a transparent cylinder. Then player* **II** *has a (computable) winning strategy in the (ζ, T)-Wadge game for f iff $f \leq^{\mathrm{c}}_{\mathrm{W}} T$ ($f \leq_{\mathrm{W}} T$).*

Proof. ($\Rightarrow$) Any (computable) strategy for player **II** gives rise to a continuous (computable) function $k :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. If the strategy is winning, then $\delta_{\mathbf{B}}\zeta T\delta_{\mathbf{X}}k \preceq f\delta_{\mathbf{A}}$, which implies $\delta_{\mathbf{B}}\zeta T\delta_{\mathbf{X}}k\delta_{\mathbf{A}}^{-1} \preceq f\delta_{\mathbf{A}}\delta_{\mathbf{A}}^{-1} = f$. Thus the continuous (computable) maps $\delta_{\mathbf{B}} \circ \zeta$ and $\delta_{\mathbf{X}}k\delta_{\mathbf{A}}^{-1}$ witness that $f \leq^{\mathrm{c}}_{\mathrm{sW}} T$ ($f \leq_{\mathrm{sW}} T$).

($\Leftarrow$) As T is a cylinder, if $f \leq^{\mathrm{c}}_{\mathrm{W}} T$ ($f \leq_{\mathrm{W}} T$), then already $f \leq^{\mathrm{c}}_{\mathrm{sW}} T$ ($f \leq_{\mathrm{sW}} T$). Thus, there are continuous (computable) h, k with $h \circ T \circ k \preceq f$. As $\delta_{\mathbf{B}} \circ \delta_{\mathbf{B}}^{-1} = \mathrm{id}_{\mathbf{B}}$, we find that $\delta_{\mathbf{B}} \circ \delta_{\mathbf{B}}^{-1} \circ h \circ T \circ k \preceq f$. Now $\delta_{\mathbf{B}}^{-1} \circ h :\subseteq \mathbf{Y} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is continuous (computable), so by definition of a probe, there is some continuous (computable) $e :\subseteq \mathbf{Y} \rightrightarrows \mathbf{Y}$ with $\delta_{\mathbf{B}} \circ \zeta \circ e \circ T \circ k \preceq f$. As T is a cylinder, there is some continuous (computable) g with $e \circ T \succeq T \circ g$, thus $\delta_{\mathbf{B}} \circ \zeta \circ T \circ g \circ k \preceq f$.

As $(g \circ k) :\subseteq \mathbf{A} \rightrightarrows \mathbf{X}$ is continuous (computable), it has some (continuous) computable realizer $K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. By Theorem 1, player **II** has a winning strategy in the Wadge game for K. This strategy also wins the (ζ, T)-Wadge game for f for her. ■

Corollary 13. *Let T and S be transparent cylinders. If the (ζ, T)-Wadge games characterize the class $\underset{\sim}{\boldsymbol{\Gamma}}$ and the (ζ', S)-Wadge games characterize the class $\underset{\sim}{\boldsymbol{\Gamma}}'$, then the $(\zeta', S \circ T)$-Wadge games characterize $\underset{\sim}{\boldsymbol{\Gamma}}' \circ \underset{\sim}{\boldsymbol{\Gamma}}$.*

Proof. By combining Proposition 9 with Theorem 12. ■

The converse of Theorem 12 is almost true, as well:

Proposition 14. *If the (ζ, T)-Wadge games characterize a lower cone in the Weihrauch degrees, then it is the lower cone of $\zeta \circ T$, and $\zeta \circ T$ is a transparent cylinder.*

Proof. Similar to the corresponding observation in Theorem 12, note that whenever player **II** has a (computable) winning strategy in the (ζ, T)-Wadge game for f, this induces a (strong) Weihrauch reduction $f \leq^{c}_{sW} \zeta \circ T$ ($f \leq_{sW} \zeta \circ T$). Conversely, by simply copying player **I**'s moves, player **II** wins the (ζ, T)-Wadge game for $\zeta \circ T$. This establishes the first claim.

Now, as $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times (\zeta \circ T) \leq_W \zeta \circ T$, the assumption that the (ζ, T)-Wadge game characterize a lower cone in the Weihrauch degrees implies that player **II** wins the (ζ, T)-Wadge game for $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times (\zeta \circ T)$. Thus, $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times (\zeta \circ T) \leq_{sW} \zeta \circ T$ follows, and we find $\zeta \circ T$ to be a cylinder.

For the remaining claim that $\zeta \circ T$ is transparent, let $G :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ be continuous (computable). Then $G \circ \zeta \circ T \leq^{c}_{W} \zeta \circ T$ ($G \circ \zeta \circ T \leq_W \zeta \circ T$), hence player **II** has a (computable) winning strategy in the (ζ, T)-Wadge game for $G \circ \zeta \circ T$. This strategy induces some continuous (computable) $H :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ with $G \circ \zeta \circ T \circ \delta_\mathbf{X} \succeq \zeta \circ T \circ \delta_\mathbf{X} \circ H$. Thus, $\delta_\mathbf{X} \circ H \circ \delta_\mathbf{X}^{-1}$ is the desired witness. ■

5 Games for the Baire hierarchy

In order to view Semmes's tree games in the general framework of the preceding section, first note that in those games there is a certain notion of *equivalence* of labeled trees, informally due to the fact that the only relevant information about the labeled tree that **II** builds are its *structure* or *shape*, along with the labels of its nodes. For example, suppose in a certain run of the tree game where player **I** builds $x \in \mathbb{N}^\mathbb{N}$, player **II** follows a strategy in which the final labeled tree is composed of all of the finite sequences with constant value 0, each labeled 0. Furthermore consider another strategy for **II**, which when played against x results in a final tree composed of all of the finite sequences with constant value 1, each one again labeled 0. Clearly, player **II** wins the run where she follows the first strategy if, and only if, the same holds for the second strategy.

Let us call a labeled tree as defined above a *concrete* labeled tree, or simply a *concrete tree*, in contrast with the *abstract* trees which we will define later. For technical reasons, we exclude the case of the tree composed solely of a root; thus a concrete tree is either empty or has at least two elements. A concrete tree (T, ϕ) is called

- *linear* if each $\sigma \in T$ has at most one child;
- *finitely branching* if each $\sigma \in T$ has only finitely many children;
- *pruned* if each $\sigma \in T$ has at least one child; and
- *proper* if $[T] \neq \emptyset$, and for every $x, y \in [T]$ we have $\phi(x \upharpoonright (n+1)) = \phi(y \upharpoonright (n+1))$ for all $n \in \mathbb{N}$.

A concrete tree (T, ϕ) is a *subtree* of a concrete tree (T', ϕ') if $T \subseteq T'$ and $\phi = \phi' \upharpoonright T$.

A sequence $s \in \mathbb{N}^{\leq\mathbb{N}}$ is an *induced label* of (T, ϕ) if

1. $s \in \mathbb{N}^{<\mathbb{N}}$ and there exists $\sigma \in T$ with $|\sigma| = |s| + 1$ such that $s(n) = \phi(\sigma \upharpoonright (n+1))$ holds for all $n < |s|$; or

2. $s \in \mathbb{N}^{\mathbb{N}}$ and there exists $x \in [T]$ such that $s(n) = \phi(x \upharpoonright (n+1))$ holds for all $n \in \mathbb{N}$.

In order to see Semmes's tree games as (ζ, T)-Wadge games, the first task is to define a representation $\delta_{\mathbb{CT}}$ of concrete labeled trees. Given $s \in \mathbb{N}^{\leq \mathbb{N}}$ with $|s| > 0$ let the *left shift* of s, denoted by $\mathrm{shift}(s)$, be the unique $t \in \mathbb{N}^{\leq \mathbb{N}}$ such that $s = \langle n \rangle ^\frown t$ for some $n \in \mathbb{N}$. We now define a sequence $\langle X_n \rangle_{n \in \mathbb{N}}$ of subsets of $\mathbb{N}^{\mathbb{N}}$ by letting $X_0 := \{p \in \mathbb{N}^{\mathbb{N}} \,;\, \forall k \in \mathbb{N}.\ (p)_k(0) = 0 \Rightarrow (p)_k = 0^\omega\}$ and $X_{n+1} := \{p \in X_n \,;\, \forall k \in \mathbb{N}.\ (p)_k(0) \neq 0 \Rightarrow \mathrm{shift}((p)_k) \in X_n\}$. The representation $\delta_{\mathbb{CT}}$ of concrete trees has

$$\mathrm{dom}(\delta_{\mathbb{CT}}) = \bigcap_{n \in \mathbb{N}} X_n.$$

Given $p \in \mathrm{dom}(\delta_{\mathbb{CT}})$ and $\sigma \in \mathbb{N}^{<\mathbb{N}} \setminus \{\langle\rangle\}$, let us say that σ is a *path through* p if

1. $(p)_{\sigma(0)}(0) \neq 0$ and

2. if $|\sigma| > 1$ then $\mathrm{shift}(\sigma)$ is a path through $\mathrm{shift}((p)_{\sigma(0)})$.

If σ is a path through p, then its *encoded label* in p is defined recursively as $(p)_{\sigma(0)}(0) - 1$, if $|\sigma| = 1$, or as the encoded label of $\mathrm{shift}(\sigma)$ in $\mathrm{shift}((p)_{\sigma(0)})$ if $|\sigma| > 1$. Finally, we define $\delta_{\mathbb{CT}}(p) = (T, \phi)$ iff

$$\begin{aligned} T &= \begin{cases} \{\sigma \in \mathbb{N}^{<\mathbb{N}} \,;\, \sigma \text{ is a path through } p\} \cup \{\langle\rangle\}, & \text{if } p \neq 0^\omega \\ \emptyset, & \text{otherwise} \end{cases} \\ \phi(\sigma) &= \text{the encoded label of } \sigma \text{ in } p. \end{aligned}$$

The corresponding represented space is denoted by $\mathbb{CT}$. Note that each concrete labeled tree has a unique code.

As indicated above, we want to work in a quotient space of concrete trees by an appropriate notion of equivalence. We say that concrete trees $\mathcal{T}_0 = (T_0, \phi_0)$ and $\mathcal{T}_1 = (T_1, \phi_1)$ are *bisimilar*, denoted by $\mathcal{T}_0 \simeq \mathcal{T}_1$, if $T_0 = T_1 = \emptyset$ or there exists a relation $Z \subseteq T_0 \times T_1$ such that $\langle\rangle Z \langle\rangle$ and such that whenever $\sigma Z \tau$:

1. $|\sigma| = |\tau|$, and $\phi_0(\sigma) = \phi_1(\tau)$ in case $\sigma \neq \langle\rangle$;

2. for every child σ' of σ in T_0 there exists a child τ' of τ in T_1 such that $\sigma' Z \tau'$; and

3. for every child τ' of τ in T_1 there exists a child σ' of σ in T_0 such that $\sigma' Z \tau'$.

An *abstract* labeled tree, or simply *abstract tree*, is an equivalence class of concrete trees under bisimilarity. These are therefore naturally represented by the function $\delta_{\mathbb{AT}}$ with $\mathrm{dom}(\delta_{\mathbb{AT}}) = \mathrm{dom}(\delta_{\mathbb{CT}})$ defined by $\delta_{\mathbb{AT}}(p) = \delta_{\mathbb{CT}}(p)/\simeq$. We denote by $\mathbb{AT}$ the space of all abstract trees, represented by $\delta_{\mathbb{AT}}$.

Note that, by this definition, formally speaking an abstract tree is not itself a tree but only a certain type of set of concrete trees. However, it can be helpful to think of an abstract tree as an unordered tree without any concrete underlying set of vertices, as follows. An *informal tree* is a (possibly empty) countable set I of objects of the form (n, J), where n is a natural number and J is again an informal tree – the intuition being that each such object (n, J) represents a child of the root of a tree with label $n$and whose subtree is exactly J. See Figure 1 for a depiction of an informal tree.

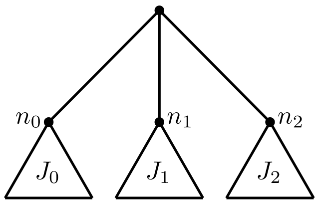

Figure 1: Depiction of the informal tree $\{(n_0, J_0), (n_1, J_1), (n_2, J_2)\}$.

To see how these intuitively correspond to abstract trees, let $\delta_{\mathbb{IT}}$ be the function informally defined by corecursion with

$$\begin{aligned} \mathrm{dom}(\delta_{\mathbb{IT}}) &= \mathrm{dom}(\delta_{\mathbb{AT}}) \\ \delta_{\mathbb{IT}}(p) &= \{(n, \delta_{\mathbb{IT}}(q))\,;\, \exists k.\ (p)_k = \langle n+1\rangle^\frown q\}. \end{aligned}$$

Then an informal tree I corresponds to an abstract tree A if $\delta_{\mathbb{IT}}(p) = I$ and $\delta_{\mathbb{AT}}(p) = A$ for some $p \in \mathrm{dom}(\delta_{\mathbb{IT}}) = \mathrm{dom}(\delta_{\mathbb{AT}})$.

The reason why this correspondence is informal is of course that, due to the Axiom of Foundation, in ZFC the definition of $\delta_{\mathbb{IT}}(p)$ will fail whenever $\delta_{\mathbb{CT}}(p)$ is an ill-founded tree, since in this case there would have to exist an infinite $\in$-descending chain of sets starting at $\delta_{\mathbb{IT}}(p)$. However, this definition would work in a completely satisfactory fashion in a system of non-wellfounded set theory such as $\mathrm{ZFC}^- + \mathrm{AFA}$, where AFA is the Axiom of Anti-Foundation first formulated by Forti and Honsell [10] and later popularised by Aczel [1] — in the style of Aczel [1, Chapter 6], in $\mathrm{ZFC}^- + \mathrm{AFA}$ the set of informal trees is defined as the greatest fixed point of the class operator Φ defined by letting $\Phi(X)$ be the class of all countable sets of elements of the form (n, T), with $n \in \mathbb{N}$ and T a countable subset of X. Thus in this context the set of informal trees is exactly

$$\bigcup \{x\,;\, x \text{ is a set and } x \subseteq \Phi(x)\}.$$

For this reason we will often refer to informal trees for intuition in the rest of this section.

Any property of concrete trees can be extended to abstract trees by stipulating that an abstract tree has the property in question if one of its concrete representatives does. Note that for some properties this extension behaves better than for some others. For example, the property of *having height* α for $\alpha \leq \omega$ behaves well, since any two bisimilar concrete trees have the same height. On the other hand, the property of *being finitely branching* does not behave as well, since every finitely branching concrete tree is bisimilar to an infinitely branching one. Note also that, in terms of informal trees, we can express this last property by saying that an informal tree I is finitely branching if it is a finite set and for each element $(n, J) \in I$ we have that J is finitely branching. An abstract tree A is a subtree of an abstract tree A' if some concrete representative of A is a subtree of some concrete representative of A'. We denote by $\mathbb{AT}_{\mathrm{pp}}$ the space of proper pruned abstract trees, and by $\mathbb{AT}_{\text{fb}}$ the space of finitely branching abstract trees, each represented by the appropriate restriction of $\delta_{\mathbb{AT}}$.

Note that if $s \in \mathbb{N}^{\leq\mathbb{N}}$ is an induced label of $\mathcal{T}$ and $\mathcal{T} \simeq \mathcal{T}'$, then s is also an induced label of $\mathcal{T}'$. For this reason, in this case we also say that s is an induced label of the abstract tree $\mathcal{T}/\simeq$.

Proposition 15. *The operation* $\text{IndLabel} :\subseteq \mathbb{AT} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ *which outputs any infinite induced label of a pruned abstract tree with non-empty body is a probe for* $\mathbb{AT}$*.*

Proof. Indeed, that IndLabel is computable follows easily from the fact that dom(IndLabel) is composed only of pruned trees with non-empty bodies. Furthermore, given a computable realizer F of some $f :\subseteq \mathbb{AT} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ and $p \in \text{dom}(F)$, it is easy to computably define $E(p)$ so that it is the code of a linear tree whose infinite induced label is exactly $F(p)$, and therefore E is the realizer of $e :\subseteq \mathbb{AT} \rightrightarrows \mathbb{AT}$ such that

$$\text{IndLabel} \circ e \preceq f. \qquad \blacksquare$$

In what follows, IndLabel will be used as the probe when characterizing variations of tree game as a (ζ, T)-Wadge game, so the task now is to find the appropriate T in each case.

Given a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, let its *pruning derivative* PD(T) be the subtree of T composed of those nodes whose subtrees have infinite height. Since PD(T) is a subtree of T, it is easy to extend PD to concrete labeled trees. Furthermore, it is again not difficult to see that bisimilar concrete trees have bisimilar pruning derivatives, so that PD can be seen as a map

$$\text{PD} : \mathbb{AT} \to \mathbb{AT}.$$

Proposition 16. $\text{PD} \equiv_{\text{W}} \widehat{\text{PD}}$.

Proof. Given a concrete tree $\mathcal{T} = (T, \phi)$, let $\mathcal{T}^* = (T^*, \phi^*)$ be defined by

$$\begin{aligned} T^* &= \{\langle n+1\rangle^\frown\sigma \,;\, n \in \mathbb{N} \text{ and } \langle n\rangle^\frown\sigma \in T\} \cup \{0^n \,;\, n \in \mathbb{N}\} \\ \phi^*(\langle n+1\rangle^\frown\sigma) &= \phi(\langle n\rangle^\frown\sigma) + 1 \\ \phi^*(0^{n+1}) &= 0. \end{aligned}$$

Note that given the code of $\mathcal{T}$ we can compute the code of $\mathcal{T}^*$, and conversely given the code of a concrete tree of the form $\mathcal{T}^*$, we can compute the code of $\mathcal{T}$. Note also that the derivative of a tree of the form $\mathcal{T}^*$ is never the empty tree.

To see that $\widehat{\text{PD}} \leq_{\text{W}} \text{PD}$, let p be a code for a sequence of abstract trees. Define $K(p)$ so that $(K(p))_n = \langle n+1\rangle^\frown((p)_n)^*$, and define $H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ so that $(H(q'))_n = q$ for the unique q such that $(q')_k = \langle n+1\rangle^\frown q^*$ for some $k \in \mathbb{N}$. It is now easy to see that $H \circ \text{PD} \circ K(p)$ is a code for the sequence of derivatives of the trees coded by p. $\blacksquare$

In order to establish where PD sits on the Weihrauch hierarchy, let isFinite : $\{0,1\}^{\mathbb{N}} \to \{0,1\}$ be given by isFinite(p) = 1 iff $\{n \in \mathbb{N}\,;\, p(n) = 1\}$ is a finite set. Since isFinite is the characteristic function of a $\mathbf{\Sigma}^0_2$-Wadge-complete set, it follows that $\widehat{\text{isFinite}}$ is Weihrauch-complete for the Baire class 2 functions [16], which in view of Fact 5 can be restated as:

Lemma 17. $\widehat{\text{isFinite}} \equiv_{\mathrm{W}} \lim \circ \lim$.

We now have

Proposition 18. $\mathrm{PD} \equiv_{\mathrm{W}} \widehat{\text{isFinite}}$

Proof. ($\mathrm{PD} \leq_{\mathrm{W}} \widehat{\text{isFinite}}$) Fix some computable bijection b between $\mathbb{N}$ and $\mathbb{N}^{<\mathbb{N}}$. Define a computable $K :\subseteq \mathbb{N}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ so that for any $p \in \mathrm{dom}(\delta_{\mathbb{CT}})$ and any $n \in \mathbb{N}$, we have that the number of 1s in the sequence $(K(p))_n$ is the same as the height of the subtree of $\delta_{\mathbb{CT}}(p)$ rooted at $b(n)$. Thus $b(n)$ is in the pruning derivative of $\delta_{\mathbb{CT}}(p)$ iff $\text{isFinite}((K(p))_n) = 0$ iff $\widehat{\text{isFinite}} \circ K(p)(n) = 0$. It is now an easy but tedious task to computably define $H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ so that $H(p, \widehat{\text{isFinite}} \circ K(p))$ is the code of the pruning derivative of $\delta_{\mathbb{CT}}(p)$.

($\widehat{\text{isFinite}} \leq_{\mathrm{W}} \mathrm{PD}$) By Proposition 16, it is enough to show $\text{isFinite} \leq_{\mathrm{W}} \mathrm{PD}$. To this end, given $p \in \{0,1\}^{\mathbb{N}}$ let $K(p)$ be a code for a tree which has a unique infinite branch, and whose unique infinite infinite label is the constant sequence 0^ω, if $\text{isFinite}(p) = 1$, or $n + 1^\omega$ if $n = \{k \in \mathbb{N}\,;\, p(k) = 1\}$. It is easy to see that K is computable — when going through p, extend the branch labeled 0 whenever you see a new k such that $p(k) = 1$, and extend the branch labeled $n+1$ whenever you see k such that $p(k) = 0$, where n is the cardinality of $\{m < k\,;\, p(m) = 1\}$. Thus, the derivative of this tree has a unique infinite branch, and we can computably decide whether $\text{isFinite}(p) = 1$ or not by checking its infinite label. ∎

Proposition 19. PD *is a cylinder.*

Proof. Given a code p of an abstract tree A let $K(p)$ be a code for the abstract tree A' obtained from A by changing the label ℓ of each of its descendants to $\ulcorner 1, \ell \urcorner$ and adding an infinite branch A'' with infinite label $\langle \ulcorner 0, p(n) \urcorner \rangle_{n\in\mathbb{N}}$. Note that K is computable, and that the tree derivative of A' is the abstract tree obtained from $\mathrm{PD}(L)$ by modifying labels as above and adding A'' as an infinite branch. We can also computably define an *inverse* operation $H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by $H(q) = \ulcorner q', q'' \urcorner$, where q' is the unique element of $\mathbb{N}^{\mathbb{N}}$ such that $\langle \ulcorner 0, q'(0) \urcorner, \ulcorner 0, q'(1) \urcorner, \ldots \rangle$ is the infinite label of an infinite branch B of the tree A_q coded by q, and q'' is a code of the tree obtained from A_q by removing B and changing each remaining label of the form $\ulcorner 1, \ell \urcorner$ to ℓ.

It is now easy to see that $H \circ G \circ K(p) = \ulcorner p, G(p) \urcorner$ whenever $G \vdash \mathrm{PD}$, and thus

$$\mathrm{id} \times \mathrm{PD} \leq_{\mathrm{sW}} \mathrm{PD}.$$

∎

Proposition 20. PD *is transparent.*

Proof. Let F be a computable realizer of $f :\subseteq \mathbb{AT} \rightrightarrows \mathbb{AT}$. We will define a computable $g :\subseteq \mathbb{AT} \rightrightarrows \mathbb{AT}$ such that

$$\mathrm{dom}(f \circ \mathrm{PD}) \subseteq \mathrm{dom}(\mathrm{PD} \circ g) = \mathrm{dom}(g)$$

and

$$\mathrm{PD} \circ g(A) \subseteq f \circ \mathrm{PD}(A)$$

for all $A \in \mathrm{dom}(f \circ \mathrm{PD})$ by explicitly defining a computable realizer G of g.

The construction is based on the observation that, since F is computable, the presence of any particular node with any particular label in the tree coded by $F(q)$ is triggered by the presence of one of possibly countably many finite configurations in the tree coded by $q \in \mathrm{dom}(F)$. Since we are interested in the case $q = H(p)$ where $H \vdash \mathrm{PD}$, we will build $G(p)$ in such a way that the nodes of the tree $\mathcal{T}_G$ coded by $G(p)$ are associated to all possible configurations which, after $\mathcal{T}_G$ is derived, result in a configuration that triggers F, in the sense just mentioned. Our task will then be to keep track of which of these configurations are present in the tree $\mathcal{T}_p$ coded by p, and what happens to them after $\mathcal{T}_p$ is derived, i.e., whether or not they will indeed trigger F.

For convenience, given $s, t \in \mathbb{N}^{\leq \mathbb{N}}$ let us write $s \sqsubseteq t$ in case $|s| = |t|$ and for all $n < |s|$ we have that $s(n) \neq 0$ implies $s(n) = t(n)$. Note that if $\mathcal{T} \subseteq \mathcal{T}'$ then $p \sqsubseteq p'$ where p and p' are the respective codes of $\mathcal{T}$ and $\mathcal{T}'$. Therefore, let us fix a realizer H of PD with the property that $H(p) \sqsubseteq p$ for all $p \in \mathrm{dom}(\mathrm{PD} \circ \delta_{\mathbb{AT}})$. Given $\sigma \in \mathbb{N}^{<\mathbb{N}}$ such that $\sigma^\frown 0^\omega \in \mathrm{dom}(\delta_{\mathbb{CT}})$, let us abuse notation slightly and write $\delta_{\mathbb{CT}}(\sigma)$ for $\delta_{\mathbb{CT}}(\sigma^\frown 0^\omega)$.

Now, suppose $\sigma \in \mathbb{N}^{<\mathbb{N}}$ is such that $\delta_{\mathbb{CT}}(\sigma)$ is a linear tree. For each $\tau \sqsupseteq \sigma$ let $X_\tau \subseteq \mathbb{N}^{<\mathbb{N}}$ be a computable prefix-free set such that

$$F^{-1}[\tau] = \bigcup_{\tau' \in X_\tau} [\tau'].$$

In other words, we know that if a concrete tree $\mathcal{T}$ with code p is such that the code $H(p)$ of $\mathrm{PD}(\mathcal{T})$ has some $\tau' \in X_\tau$ for $\tau \sqsupseteq \sigma$ as a prefix, then $\delta_{\mathbb{CT}}(\sigma)$ is a subtree of the concrete tree coded by $F \circ H(p)$. Note that this happens exactly when for some such $\tau' \in X_\tau$, letting $\tau'' := p \restriction |\tau'|$ we have

1. $\tau'' \sqsupseteq \tau'$, and
2. any node of the tree $\mathcal{T}''$ coded by $\tau''^\frown 0^\omega$ that is not in the tree coded by $\tau'^\frown 0^\omega$ is also not in $\mathrm{PD}(\mathcal{T}'')$, i.e., any such node has a subtree of bounded height in $\mathcal{T}''$. Note that there are only finitely many such nodes, so there is a common upper bound $B_{\tau''} \in \mathbb{N}$ on the heights of all of their subtrees.

Now, for each $n \geq 1$ let W_n be the set of all tuples $(\sigma, \tau, \tau', \tau'', N)$ where

1. $\delta_{\mathbb{CT}}(\sigma)$ is a linear tree of height n;
2. $\tau \sqsupseteq \sigma$;
3. $\tau' \in X_\tau$;
4. $\tau'' \sqsupseteq \tau'$; and
5. $N \in \mathbb{N}$,

and note that each W_n is a countable set.

We are now ready to define G. Given $p \in \mathrm{dom}(F \circ H)$, let $\mathcal{T}_p$ be the concrete tree coded by p. We can now computably define $G(p)$ in such a way that the concrete tree $\mathcal{T}_G = (T_G, \phi_G)$ coded by $G(p)$ has the following properties.

1. the nodes at level 1 of $\mathcal{T}_G$ are bijectively associated to the elements $(\sigma, \tau, \tau', \tau'', N) \in W_1$ such that $\tau'' \subset p$;
2. if $\sigma \neq \langle\rangle$ is in $\mathcal{T}_G$ and is associated to an element $(\sigma_0, \tau_0, \tau_0', \tau_0'', N_0)$ of $W_{|\sigma|}$ then

(a) $\phi_G(\sigma)$ is the label of the element of $\delta_{\mathbb{CT}}(\sigma_0)$ at height $|\sigma|$,

(b) $\tau_0'' \subseteq p$,

(c) if some node of $\delta_{\mathbb{CT}}(\tau'')$ that is not a node of $\delta_{\mathbb{CT}}(\tau')$ is the root of a subtree of $\mathcal{T}_p$ of height greater than N_0, then the height h_σ of the subtree of $\mathcal{T}_G$ rooted at σ is finite,

(d) if the antecedent of (c) above does not happen, then h_σ is equal to the minimum of the heights of the subtrees of $\mathcal{T}_p$ rooted at the elements of $\delta_{\mathbb{CT}}(\tau')$, and

(e) if $h_\sigma > 1$ and for each antecessor τ of σ in T_G we have $h_\tau > 1+|\sigma|-|\tau|$, then the children of σ in $\mathcal{T}_G$ are bijectively associated to the elements of $W_{|\sigma|+1}$ of the form $(\sigma_1, \tau_1, \tau_1', \tau_1'', N_1)$ with $\sigma_1 \supset \sigma_0$ and $\tau_1'' \subset p$.

All that remains to be proved now is that $\mathcal{T}_{HG} = \delta_{\mathbb{CT}} \circ H \circ G(p)$ and $\mathcal{T}_{FH} = \delta_{\mathbb{CT}} \circ F \circ H(p)$ are bisimilar trees. Define a relation Z between $\mathcal{T}_{HG}$ and $\mathcal{T}_{FH}$ by $\sigma Z \tau$ iff $\sigma = \tau = \langle\rangle$, or σ is associated to $(\sigma_0, \tau_0, \tau_0', \tau_0'', N_0) \in W_{|\sigma|}$ and τ is the element of $\delta_{\mathbb{CT}}(\sigma_0)$ at level $|\sigma|$. We will show that Z is a bisimulation.

Suppose $\sigma Z \tau$. This implies both $|\sigma| = |\tau|$ and that σ and τ have the same labels in $\mathcal{T}_{HG}$ and $\mathcal{T}_{FH}$. Note that since σ is a node of $\mathcal{T}_{HG}$, it follows that σ is the root of a subtree of $\mathcal{T}_G$ of infinite height.

Now let σ' be a child of σ in $\mathcal{T}_{HG}$. By construction, σ' is associated to some $(\sigma_1, \tau_1, \tau_1', \tau_1'', N_1) \in W_{|\sigma|+1}$ with $\sigma_1 \supset \sigma_0$ such that $\delta_{\mathbb{CT}}(\sigma_1)$ is a linear tree of height $|\sigma| + 1$. Let τ' be the element of $\delta_{\mathbb{CT}}(\sigma_1)$ at level $|\sigma| + 1$. What must be shown is that τ' is in $\mathcal{T}_{FH}$, so that $\sigma' Z \tau'$ will follow. The fact that σ' is in $\mathcal{T}_{HG}$ implies that σ' is the root of a subtree of $\mathcal{T}_G$ of infinite height. By condition 2(c) of the construction, this implies that every node of $\delta_{\mathbb{CT}}(\tau_1'')$ that is not a node of $\delta_{\mathbb{CT}}(\tau_1')$ is the root of a subtree of $\mathcal{T}_p$ of height at most N_0, and by condition 2(d) every node of $\delta_{\mathbb{CT}}(\tau_1')$ is the root of a subtree of $\mathcal{T}_p$ of infinite height. Together with $\tau_1'' \sqsupseteq \tau_1'$, these facts imply exactly that $\tau_1' \subset H(p)$, and therefore $\delta_{\mathbb{CT}}(\sigma')$ is a subtree of $F \circ H(p)$ as desired.

Finally, let τ' be a child of τ in $\mathcal{T}_{FH}$. Let $\sigma_1 \in \mathbb{N}^{<\mathbb{N}}$ be such that $\delta_{\mathbb{CT}}(\sigma_1)$ is the linear subtree of $\mathcal{T}_{FH}$ which has τ' as its longest element, and let $\tau_1 = H(p) \upharpoonright (|\sigma_1|)$, so that $\tau_1 \sqsupseteq \sigma_1$. Since

$$F^{-1}[\tau_1] = \bigcup_{\tau_1' \in X_{\tau_1}} [\tau_1'],$$

there is some $\tau_1' \in X_{\tau_1}$ such that $\tau_1' \subset H(p)$. Now let $\tau_1'' = p \upharpoonright (|\tau_1'|)$, so that again we have $\tau_1'' \sqsupseteq \tau_1'$, and finally let $N_1 \in \mathbb{N}$ be some common upper bound on the heights of the subtrees of $\mathcal{T}_p$ rooted at nodes of $\delta_{\mathbb{CT}}(\tau_1'')$ which are not in $\delta_{\mathbb{CT}}(\tau_1')$. We therefore have $(\sigma_1, \tau_1, \tau_1', \tau_1'', N_1) \in W_{|\tau'|} = W_{|\sigma|+1}$. If $\sigma = \langle\rangle$ then we are done, since some σ' at level 1 is associated to $(\sigma_1, \tau_1, \tau_1', \tau_1'', N_1)$ and thus $\sigma' Z \tau'$. Otherwise, by construction σ is associated to some $(\sigma_0, \tau_0, \tau_0', \tau_0'', N_0) \in W_{|\sigma|}$, and the fact that we have $\sigma Z \tau$ implies that τ is the longest element of $\delta_{\mathbb{CT}}(\sigma_0)$ and therefore that $\sigma_1 \supset \sigma_0$. Therefore some child σ' of σ is associated to $(\sigma_1, \tau_1, \tau_1', \tau_1'', N_1)$ and $\sigma' Z \tau'$ follows. ∎

From Theorem 12, Lemma 17, and Proposition 18, it now follows that the $(\mathrm{IndLabel}, \mathrm{PD}^n)$-Wadge game characterizes Baire class $2n$, i.e., that **II** has a winning strategy in the $(\mathrm{IndLabel}, \mathrm{PD}^n)$-Wadge game for f iff f is of Baire class $2n$. In other words, for the case where $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, the modification of Semmes's tree game in which in the long run player **I** builds $x \in \mathbb{N}^{\mathbb{N}}$ and player **II** builds a labeled tree (T, ϕ) such that

1. T has at least one infinite branch;
2. all of the infinite branches of T have the same induced label y by ϕ;
3. the n^{th} pruning derivative of T is a pruned tree,

and where **II** wins if $y = f(x)$, characterizes the Baire class $2n$ functions.

Proposition 21. *The operation* Linearize $:\subseteq \mathbb{AT} \to \mathbb{AT}_{\text{pp}}$ *which maps a proper finitely branching abstract tree to its unique proper pruned subtree is Weihrauch-equivalent to* lim.

Proof. ($\lim \leq_{\mathrm{W}}$ Linearize) Given $p \in \mathrm{dom}(\lim)$, we can computably build the code $K(p)$ of a concrete tree $\mathcal{T} = (T, \phi)$ with the property that all and only the sequences of the form $(p)_n \restriction n$ are induced labels of $\mathcal{T}$, and such that no node of T has two different children with the same label. Since $p \in \mathrm{dom}(\lim)$, it follows that $\mathcal{T}$ is a proper finitely branching concrete tree, and $\lim(p)$ is exactly the infinite induced label of $\mathcal{T}$, so

$$\mathrm{IndLabel} \circ \mathrm{Linearize} \circ K = \lim .$$

(Linearize $\leq_{\mathrm{W}} \lim$) Let p be the code of a proper finitely branching abstract tree A, let $\mathcal{T} = (T, \phi)$ be the concrete tree coded by p, and for each $n \in \mathbb{N}$ let $\mathcal{T}_n = (T_n, \phi_n)$ be the concrete tree coded by $(p \restriction n)^\frown 0^\omega$.

Given $n, k \in \mathbb{N}$, let $\mathrm{guess}(n, k) \in \mathbb{N}^{<\mathbb{N}}$ be

- $\langle\rangle$, if $n = 0$;
- $\mathrm{guess}(n-1, k)$, if $n > 0$ and there are no induced labels of T_n of length at least k which are not induced labels of T_{n-1}; and
- the longest $\sigma \in \mathbb{N}^{<\mathbb{N}}$ such that all induced labels of T_n of length at least k which are not induced labels of T_{n-1} have σ as prefix, otherwise.

Now let $K(p) \in \mathbb{N}^{\mathbb{N}}$ be such that $(K(p))_n = \mathrm{guess}(n, n)^\frown 0^\omega$. Note that K is computable.

Claim 1. $x := \lim(K(p))$ is equal to the unique infinite induced label y of $\mathcal{T}$ (and thus of A).

Indeed, let $\mathcal{T}' = (T', \phi')$ be a concrete finitely branching representative of A. For any $n \in \mathbb{N}$ and $\tau \in T'$ with $|\tau| = n$, if the induced label of τ is not a prefix of y then since T' is finitely branching we have that the subtree of T' rooted at τ is finite, and thus has finite height. Again since T' is finitely branching, there are only finitely many such τ, and therefore there exists $N \in \mathbb{N}$ which is an upper bound on the heights of all such subtrees of T'. Since the height of a tree is invariant under bisimilarity, the same holds for T, i.e., N is an upper bound on the heights of the subtrees of T rooted at the (possibly infinitely many) elements $\tau \in T$ with $|\tau| = n$ whose induced labels are not prefixes of y. Since $[T] \neq \emptyset$ and T_N is finite, there exists $M > N$ such that T_M has an induced label of length at least N which is not an induced label of T_{M-1}. Hence $y \restriction n$ is a prefix of $(K(p))_m$ for all $m \geq M$, which concludes the proof of the claim.

Now letting $H(q)$ be the code of any proper pruned tree whose unique infinite induced label is q, it follows that H can be taken to be computable and

$$H \circ \lim \circ K \vdash \text{Linearize} \qquad \blacksquare$$

Note that Linearize is just the restriction of PD to the space of proper finitely branching trees, and therefore a simple analysis of the proof of Proposition 19 shows that it also gives us the following.

Proposition 22. Linearize *is a cylinder.*

The following can also be proved with a simplified version of the proof of Proposition 20.

Proposition 23. Linearize *is transparent.*

Proof. Let $f :\subseteq \mathbb{AT}_{\mathrm{pp}} \rightrightarrows \mathbb{AT}_{\mathrm{pp}}$ be computable. Note that in this case f has a computable realizer F with the property that for any $\sigma \in \mathbb{N}^{<\mathbb{N}}$ there exists a computable, prefix-free $X_\sigma \subseteq \mathbb{N}^{<\mathbb{N}}$ such that σ occurs as a sequence of labels in the (abstract or concrete) tree coded by $F(p)$ iff some $\tau \in X_\sigma$ occurs as a sequence of labels in the tree coded by p. Let us call any $\tau \in X_\sigma$ a *trigger* for σ, and let $W_n := \{(\sigma, \tau)\,;\, |\sigma| = n \text{ and } \tau \text{ is a trigger for } \sigma\}$.

Let H be a realizer of PD as in the proof of Theorem 20, so that $H(p) \sqsubseteq p$ for all $p \in \mathrm{dom}(H)$, and let $p \in \mathrm{dom}(F \circ H) \cap \mathrm{dom}(\delta_{\mathbb{AT}})$ be such that the concrete tree $\mathcal{T}_p$ coded by p is bisimilar to some finitely branching tree.

We can computably define $G(p)$ in such a way that the concrete labeled tree $\mathcal{T}_G = \delta_{\mathbb{CT}} \circ G(p)$ has the following properties.

1. the nodes at level 1 of $\mathcal{T}_G$ are bijectively associated to the elements $(\sigma, \tau) \in W_1$ such that τ occurs as a sequence of labels of $\mathcal{T}_p$;
2. if $\sigma \neq \langle\rangle$ is in $\mathcal{T}_G$ and is associated to an element (σ_0, τ_0) of $W_{|\sigma|}$ then
   (a) the induced label of σ in $\mathcal{T}_G$ is σ_0,
   (b) τ_0 occurs as a sequence of labels of $\mathcal{T}_p$,
   (c) the height h_σ of the subtree of $\mathcal{T}_G$ rooted at σ is equal to the maximum height of a subtree of $\mathcal{T}_p$ rooted at some node whose induced label is τ_0 (note that this maximum is attained, since $\mathcal{T}_p$ is bisimilar to a finitely branching tree),
   and
   (d) if $h_\sigma > 1$ and for each antecessor σ' of σ in $\mathcal{T}_G$ we have $h_{\sigma'} > 1 + |\sigma| - |\sigma'|$, then the children of σ in $\mathcal{T}_G$ are bijectively associated to the elements of $W_{|\sigma|+1}$ of the form (σ_1, τ_1) with $\sigma_0 \subset \sigma_1$.

To see that $\delta_{\mathbb{CT}} \circ H \circ G(p) \simeq \delta_{\mathbb{CT}} \circ F \circ H(p)$, let $\sigma Z \tau$ iff $\sigma = \tau = \langle\rangle$ or σ and τ have the same induced labels in $\delta_{\mathbb{CT}} \circ H \circ G(p)$ and $\delta_{\mathbb{CT}} \circ F \circ H(p)$, respectively. Now suppose $\sigma Z \tau$, and let $(\sigma_0, \tau_0) \in W_{|\sigma|}$ be associated to σ.

Let σ' be a child of σ in $\delta_{\mathbb{CT}} \circ H \circ G(p)$. It follows that σ' is associated to some pair $(\sigma_1, \tau_1) \in W_{|\sigma|+1}$, and that the induced label of σ' in $\mathcal{T}_G$ is σ_1. Since σ' is in the derivative of $\delta_{\mathbb{CT}} \circ G(p)$, by condition 2(c) of the construction it follows that some node ν of $\mathcal{T}_p$ with induced label τ_1 is the root of a subtree of $\mathcal{T}_p$ of infinite height. Thus ν is in $\delta_{\mathbb{CT}} \circ H(p)$, and by definition of $(\sigma_1, \tau_1) \in W_{|\sigma|+1}$

it follows that some node with induced label σ_1 is in $\delta_{\mathbb{CT}} \circ F \circ H(p)$. Finally, since $\delta_{\mathbb{CT}} \circ F \circ H(p)$ is a proper pruned tree, it follows that τ' has a child τ' with induced label σ_1, and thus $\sigma' Z \tau'$.

Conversely, let τ' be a child of τ in $\delta_{\mathbb{CT}} \circ F \circ H(p)$, and let σ_1 be its induced label. Therefore, some $\tau_1 \in X_{\sigma_1}$ occurs as an induced label in $\delta_{\mathbb{CT}} \circ H(p)$, and thus some node ν of $\mathcal{T}_p$ has σ_1 as its induced label and is the root of a subtree of $\mathcal{T}_p$ of infinite height. This implies that some child σ' of σ in $\mathcal{T}_G$ is associated to $(\sigma_1, \tau_1) \in W_{|\sigma|+1}$, and that such σ' is also in $\delta_{\mathbb{CT}} \circ H \circ G(p)$. Therefore $\sigma' Z \tau'$.

All that remains to be proved is that $\mathcal{T}_G$ is bisimilar to a finitely branching tree. Note that, by construction, if some node σ of $\mathcal{T}_G$ has infinitely many children σ_n which are roots of non-bisimilar subtrees of $\mathcal{T}_G$, then the labels of the σ_n are pairwise distinct, and therefore these must be associated to elements $(\sigma'_n, \tau_n) \in W_{|\sigma|+1}$ such that the τ_n are pairwise $\subseteq$-incomparable. Note that $\mathcal{T}_p$ is bisimilar to a finitely branching tree, thus in particular only finitely many different labels occur on each of its levels. This implies that $\lim_{n\in\mathbb{N}} |\tau_n| = \infty$, and therefore arbitrarily long prefixes of the infinite induced label of $\mathcal{T}_p$ occur among the prefixes of the τ_n. But then we cannot have that all σ'_n have the same length $|\sigma| + 1$, a contradiction. ■

As before, it now follows that the $(\mathrm{IndLabel}, \mathrm{Linearize} \circ \mathrm{PD}^n)$-Wadge game characterizes Baire class $2n + 1$. Thus, in the case where $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, the modification of Semmes's tree game in which in the long run player **I** builds $x \in \mathbb{N}^\mathbb{N}$ and player **II** builds a labeled tree (T, ϕ) such that

1. T has at least one infinite branch;
2. all of the infinite branches of T have the same induced label y by ϕ;
3. each node of the n^{th} pruning derivative of T has only finitely many children which are not on some infinite branch of T,

and where **II** wins if $y = f(x)$, characterizes the Baire class $2n + 1$ functions.

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