Does ultra-slow diffusion survive in a three dimensional cylindrical comb?

A. Iomin[a,*], V. Méndez[b]

[a]*Department of Physics, Technion, Haifa, 32000, Israel*
[b]*Grup de Física Estadística, Departament de Física. Universitat Autònoma de Barcelona.*
Edifici Cc. 08193 Cerdanyola (Bellaterra) Spain

Abstract

We present an exact analytical result on ultra-slow diffusion by solving a Fokker-Planck equation, which describes anomalous transport in a three dimensional (3D) comb. This 3D cylindrical comb consists of a cylinder of discs of ether infinite or finite radius, threaded on a backbone. It is shown that the ultra-slow particle spreading along the backbone is described by the mean squared displacement (MSD) of the order of $\ln(t)$. This phenomenon takes place only for normal two dimensional diffusion inside the infinite secondary branches (discs). When the secondary branches have finite boundaries, the ultra-slow motion is a transient process and the asymptotic behavior is normal diffusion. In another example, when anomalous diffusion takes place in the secondary branches, a destruction of ultra-slow (logarithmic) diffusion takes place as well. As the result, one observes "enhanced" subdiffusion with the MSD $\sim t^{1-\alpha} \ln t$, where $0 < \alpha < 1$.

Keywords: Comb model, Cylindrical comb, Subdiffusion, Ultra-slow diffusion

1. Introduction

The transport of particles in inhomogeneous media exhibits anomalous diffusion. This phenomenon is well established [1, 2, 3, 4] and reviewed (see *e.g.*,

---

*Corresponding author
Email addresses:* `iomin@physics.technion.ac.il` (A. Iomin), `vicenc.mendez@uab.cat` (V. Méndez)

[5, 6, 7, 8, 9]). A comb model is a simple description of anomalous diffusion, which however reflects many important transport properties of inhomogeneous media. The comb model was introduced as a toy model for understanding anomalous transport in low dimensional percolation clusters [10, 11, 12]. It is a particular example of a non-Markovian phenomenon, which is also explained in the framework of continuous time random walks [11, 13, 14].

Anomalous diffusion on the two dimensional (2D) comb is described by the 2D probability distribution function (pdf) $P = P(x, y, t)$ of finding a particle at time t at position y along the secondary branch that crosses the backbone at point x. The transport is inhomogeneous, namely, for $y \neq 0$ diffusion in the x direction is absent, and diffusion on a 2D comb-like structure is described by the Fokker-Planck equation [12]

$$\partial_t P(x, y, t) = \mathcal{D}_x \delta(y) \partial_x^2 P(x, y, t) + \mathcal{D}_y \partial_y^2 P(x, y, t) . \tag{1}$$

Here $\mathcal{D}_x \delta(y)$ is the diffusion coefficient in the x direction, and $\mathcal{D}_y$ is the diffusion coefficient in the y direction. The δ-function in the diffusion coefficient in the x direction implies that diffusion occurs along the x direction at $y = 0$ only. Thus, this equation describes diffusion along the backbone (at $y = 0$) where the secondary branches (fingers) play a role of traps. The comb model with infinite secondary branches (fingers) describes subdiffusion in the x direction[1] with the mean squared displacement (MSD) $\langle x^2(t) \rangle$, which grows by power law $\sim t^{1/2}$ [10, 11, 12]. For the finite secondary branches, subdiffusion is a transient process until time t_0, and after a transient time scale $t > t_0$ the transport along the backbone corresponds to normal diffusion with the MSD $\langle x^2(t) \rangle \sim t$ [5].

It is convenient to work with dimensionless variables and parameters. This can be obtained by the re-scaling with relevant combinations of the comb parameters $[D_x] = \frac{cm^3}{sec}$ and $[D_y] = \frac{cm^2}{sec}$, such that the dimensionless time and coordinates are

$$D_x^2 t / D_y^3 \rightarrow t , \quad D_x x / D_y \rightarrow x , \quad D_x y / D_y \rightarrow y / \sqrt{D} , \tag{2}$$

---

[1]Note that there is normal diffusion in the y direction in Eq. (1).

where D can be considered as a dimensionless diffusion coefficient for the secondary branch dynamics.
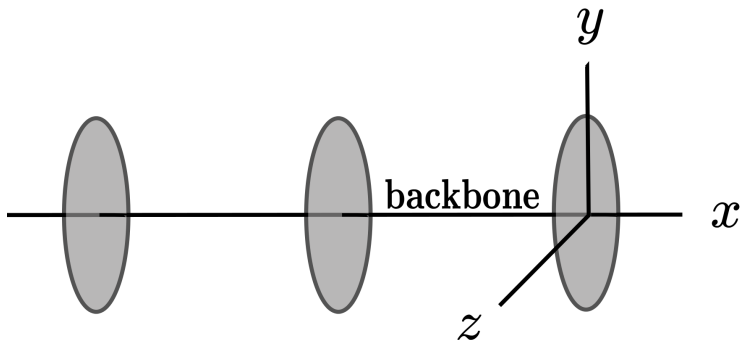

Figure 1: Schematic representation of the 3D cylindrical comb. The backbone coincides with axis x and the discs, belonging to the y-z plane, are threaded on the backbone and form a continuous cylinder.

In this paper we consider anomalous ultra-slow diffusion in a three dimensional cylindrical comb, which consists of a continuous cylinder of discs threaded on the x axis, as it is shown in Fig. 1. Recently, the ultra-slow phenomena were attracted much attention in biological search problems with long-range memories [15, 16]. In the continuous time random work, ultra-slow diffusion is known as a result of super-heavy-tailed distributions of waiting times, see details of a discussion in Refs. [17, 18, 20] and numerical results in Refs. [20, 19]. These heavy-tailed distributions result in the ratio $\langle x^2(t) \rangle / t^\alpha \to 0$ at $t \to \infty$, which tends to zero, in contrast with subdiffusion, where this ratio limits to a constant value. We show that the MSD exhibits a logarithmic behavior in time $\langle x^2(t) \rangle \sim \ln(t)$, which is a result of the transversal branch dynamics in the 2D space. This behavior has been discussed in the framework of scaling arguments for the return probability [5, 21], which are based on the fractal dimension d of the transversal branch structure and the spectral dimension d_s, which is defined by decay of the return probability $\sim t^{-d_s/2}$ [8, 22] and $\langle x^2(t) \rangle \sim t^{1-\frac{d_s}{2}}$ for $d_s < 2$ [21]. As it is shown in Ref. [21], $\langle x^2(t) \rangle \sim \ln t$ for $d_s = 2$. Therefore, it is instructive to present a rigorous result by solving analytically the Fokker-Planck

equation in the three dimensional space within the cylindrical comb geometry constraint, when $d = 2$.

2. Dynamics in a cylindrical comb: an infinite comb model

We consider a 3D cylindrical comb [7, 21], shown in Fig. 1, in the framework of a standard formulation of the comb model (1) for the 3D case. Therefore, the random dynamics on this structure is describe by the 3D distribution function $P = P(x, y, z, t)$, where the x-axis corresponds to the backbone, while the dynamics on the two dimensional secondary branches is described by the y and z coordinates. The diffusion equation in the dimensionless variables and parameters reads

$$\partial_t P = \delta(y)\delta(z)\partial_x^2 P + D(\partial_y^2 + \partial_z^2)P. \tag{3}$$

The natural boundary conditions are taken at infinity, where the distribution function and its first space derivatives vanish. The initial condition is

$$P_0 \equiv P(x, y, z, t = 0) = \delta(x)\delta(y)\delta(z). \tag{4}$$

2.1. Analysis in the time domain

The formal solution of Eq. (3) can be presented in a convolution form

$$P(x, y, z, t) = \int_0^t G(y, z, t - t')F(x, t')dt', \tag{5}$$

where $G(y, z, t)$ is the propagator for two dimensional diffusion in the secondary branches, while $F(x, t)$ relates to the solution along the backbone[2]. Taking into

---

[2]This relation in the time and Laplace domains is established in Eqs. (7) and (8). By analogy with the 2D comb, this formal presentation of the solution in the convolution form follows from the Laplace transform of (3). It is tempting to understand this case in complete analogy with the 2D comb of Eq. (1), as well, where one assumes that the solution is the multiplication of the independent probability distribution functions in the Laplace domain $\tilde{G}(y, z, s) \cdot \tilde{F}(x, s)$. However, for the 3D comb the situation is more sophisticated. Namely, as we show in Sec. 2.2, the solution $\tilde{G}(y, z, s)$ is singular at $(y, z) = (0, 0)$. Therefore, $\tilde{F}(x, s)$

account the cylindrical symmetry, one obtains for the $y - z$ surface

$$G(y, z, t) \equiv G(r, t) = \frac{1}{4\pi Dt} \exp\left(-\frac{r^2}{4Dt}\right), \tag{6}$$

where $r^2 = y^2 + z^2$. To define the MSD in the x direction, one needs to find a reduced distribution $P_1(x, t)$ by integrating (5) over y and z taking into account that the element of differential area is $dydz = d\theta r dr$. From Eq. (6), one obtains

$$
\begin{aligned}
P_1(x, t) &= \int_{-\infty}^{\infty} P(x, y, z, t) dy dz \\
&= \int_0^t \left[\int_0^{\infty} r dr \int_0^{2\pi} d\theta G(r, t - t')\right] F(x, t') dt' \\
&= \int_0^t F(x, t') dt'. \tag{7}
\end{aligned}
$$

In the Laplace space, this expression establishes a relation between $\tilde{P}_1(x, s) = \hat{\mathcal{L}}[P_1(x, t)]$ and $\tilde{F}(x, s) = \hat{\mathcal{L}}[F(x, t)]$. This relation reads

$$\tilde{F}(x, s) = s\tilde{P}_1(x, s). \tag{8}$$

The initial condition for the reduced distribution is $P_1(x, t = 0) = \delta(x)$. Using relation (8), one obtains an equation for $\tilde{P}_1(x, t)$. Integrating Eq. (3) over y and z, and taking into account Eqs. (5), (6) and (8), one obtains

$$s\tilde{P}_1(x, s) = \frac{1}{4\pi D}\partial_x^2 \hat{\mathcal{L}}[t^{-1}]s\tilde{P}_1(x, s) + \delta(x). \tag{9}$$

Note that the Laplace transform of t^{-1} exists as a principal value integral [23]. Fourier transforming Eq. (9), one obtains

$$\bar{\tilde{P}}_1(k, s) = \frac{4\pi D}{s(4\pi D + k^2\hat{\mathcal{L}}[t^{-1}])}. \tag{10}$$

This yields the MSD in the form

$$\langle x^2(t)\rangle = \hat{\mathcal{L}}^{-1}\left[-\frac{d^2}{dk^2}\bar{\tilde{P}}_1(k, s)\right]_{k=0} = \frac{1}{2\pi D}\int_{-i\infty}^{+i\infty} \hat{\mathcal{L}}[t^{-1}]\frac{e^{st}ds}{s}. \tag{11}$$

---

is not simply related to the distribution function $\tilde{P}(x, y = 0, z = 0, s)$. However, in the time domain this relation can be easily established, using formal solution in the convolution form (5).

Taking into account that $\hat{\mathcal{L}}^{-1}\hat{\mathcal{L}}\left[t^{-1}\right] \equiv t^{-1}$ and $e^{st}/s = \int^t e^{st}dt + C$, where C is an integration constant of the indefinite integration, one obtains

$$
\begin{aligned}
\langle x^2(t)\rangle &= \frac{1}{2\pi D}\ln(t) + \frac{C}{2\pi D\,t} \\
&= \frac{1}{2\pi D}\ln(t)\,, \quad \text{as} \quad t \to \infty\,.
\end{aligned}
\tag{12}
$$

Therefore, for the large time dynamics, ultra-slow diffusion takes place with the MSD growing as $\ln(t)$.

2.2. Consideration in the Laplace domain

Let us consider a relation between the temporal dynamics and the dynamics in the Laplace space. Performing the Laplace transform of Eq. (3), one obtains

$$
s\tilde{P} = \delta(y)\delta(z)\partial_x^2\tilde{P} + D(\partial_y^2 + \partial_z^2)\tilde{P} + P_0\,.
\tag{13}
$$

Correspondingly, Eq. (5) reads in the Laplace domain

$$
\tilde{P}(x, y, z, s) = \tilde{G}(y, z, s)\tilde{F}(x, s) \equiv \tilde{G}(r, s)\tilde{F}(x, s)\,,
\tag{14}
$$

where the cylindrical symmetry is taken into account in the last term. Taking into account Eqs. (13) and (14), and introducing a new variable in the form of a scaled radius $u = r\sqrt{s/D}$, one finds the solution for $\tilde{G}(r, s) \equiv \tilde{G}(u)$ from the equation

$$
u^2\tilde{G}'' + u\tilde{G}' - u^2\tilde{G} = 0\,,
\tag{15}
$$

where prime means the derivative over u. This is an equation for the modified Bessel functions $I_0(u)$ and $K_0(u)$ (see e.g., [24]). The solution, which satisfied the boundary condition at infinity $r = \infty$, is the modified Bessel function of the second kind

$$
\tilde{G}(r, s) = A \cdot K_0\left(r\sqrt{s/D}\right)\,.
\tag{16}
$$

It should be stressed that the Laplace inversion of $K_0\left(r\sqrt{s/D}\right)$ is exactly the solution $G(r, t)$ in Eq. (6):

$$
\int_0^\infty \frac{1}{4\pi Dt}\exp\left(-\frac{r^2}{4Dt} - st\right)dt = \frac{1}{2\pi D}K_0\left(r\sqrt{s/D}\right)\,.
\tag{17}
$$

Therefore, $A = 1/2\pi D$ that satisfies the normalization condition and the initial condition for $G(r,t)$. Taking into account solution (16), we establish the relation (8) by integrating Eq. (14) over y and z. Using a property of integration of the modified Bessel function:

$$\int_0^\infty u K_0(au) du = 1/a^2 \,, \tag{18}$$

one obtains

$$\tilde{P}_1(x,s) = \tilde{F}(x,s) \int dy dz \tilde{G}(x,y,s) = \frac{\tilde{F}}{D} \int_0^\infty K_0\Big(r\sqrt{s/D}\Big) r dr = \frac{\tilde{F}(x,s)}{s} \,, \tag{19}$$

which coincides exactly with the result in Eq. (8), and where we also use the Laplace transform of $\tilde{P}_1(x,s) = 2\pi\tilde{F} \int_0^\infty \tilde{G}(r,s) r dr$ in Eq. (7).

Now we admit an important point of the analysis: *namely Eq. (13) cannot be integrated over the y and z, because $\tilde{G}(r,s)$ is singular at $r = 0$. In this case, $\tilde{F}(x,s)$ cannot be defined from this procedure in the laplace domain[3]. Therefore, to continue the analysis, one has to return to the time domain and repeat the analysis for the temporal dynamics, performed in section 2.1.* This situation differs cardinally from the analysis for the 2D comb model in Eq. (1), where the finite expression for MSD can be obtained in the Fourier-Laplace domain.

3. Comb dynamics with finite discs: Transition to normal diffusion

To consider anomalous diffusion on finite combs, we consider reflecting boundary conditions at $r = R$, such that $\partial_r \tilde{G}(r = R, s) = 0$, which determines the absence of the probability flux in the direction normal to the boundary surface. In this case, solution of Eq. (15) is found in the form of modified

---

[3]It is worth noting that this does not mean that the solution does not exists, and that the singularity of $\tilde{G}(r,s)$ at $r = 0$ does not mean the absence of the solution as well. Moreover, the existence of the Laplace transform (17) ensures the existence of the Laplace solution $\tilde{G}(r,s)$ in the entire disc, including $r = 0$ as well as Eq. (17) ensures the conditions when Plancherel's theorem holds.

Bessel function of the first kind $I_0(u)$. Namely, making a shift of the argument $u \to u = (R - r)\sqrt{s/D}$ in Eq. (15), one obtains[4]

$$\tilde{G}(r, s) = I_0\left[(R - r)\sqrt{s/D}\right] / I_0\left(R\sqrt{s/D}\right), \tag{20}$$

which satisfies the boundary condition

$$\frac{d}{dr} I_0\left[(R - r)\sqrt{s/D}\right]\Big|_{r=R} = I_1\left[(R - r)\sqrt{s/D}\right]|_{r=R} = 0, \tag{21}$$

while for $r = 0$, one obtains $\tilde{G}(0, s) = 1$. This corresponds to a standard construction of the solution [12, 25].

Integration of Eq. (13) over the y and z yields

$$s\tilde{P}_1(x, s) = \partial_x^2 \tilde{F}(x, s) + \delta(x). \tag{22}$$

Again, the relation between $\tilde{P}_1$ and $\tilde{F}$ can be established by integration of $I_0(u)$ over the $y - z$ surface of discs, which yields

$$2\pi \int_0^R r I_0\left[(R - r)\sqrt{s/D}\right] dr = 2\pi R^2 \int_0^1 u I_0[a(1 - u)] du.$$

Here we used the following variable changes $u = r/R$ and $a = R\sqrt{s/D}$. Then using another variable change $w = 1 - u$, one obtains two integrals [26]

$$\frac{R^2}{a} \int_0^a I_0(w) dw - R^2 \int_0^1 w I_0(aw) dw = \frac{R^2}{a}\left[2\sum_{n=0}^{\infty} I_{2n+1}(a) - I_1(a)\right].$$

Here, we are interested in the long time dynamics, when $s \to 0$ and $a \ll 1$, correspondingly. In this case $I_n(a) \approx \left(\frac{a}{2}\right)^n / \Gamma(n + 1)$, and we take into account the first term with $n = 0$ in the sum that yields $I_1(a)$ in the squared brackets. Finally, one obtains for the long time asymptotics[5]

$$2\pi \int_0^R \tilde{G}(r, s) r dr \approx \pi R^2. \tag{23}$$

---

[4]Obviously, this corresponds to shifted arguments y and z in the input Eq. (3) without touching the comb geometry $\partial_t P = \delta(y)\delta(z)\partial_x^2 P + D[\partial_{(y_0-y)}^2 + \partial_{(z_0-z)}^2]P$, where $y_0^2 + z_0^2 = R^2$. Since y_0 and z_0 are constant values, this shift does not change the dynamics, and this procedure can be done.

[5]This result can be obtained from Eq. (20) taking into account that $\tilde{G}(r, s) \approx 1$ for the small argument in the limit $s \to 0$.

This yields the following relation

$$\tilde{F}(x, s) = \frac{\pi}{R^2} \tilde{P}_1(x, s) \,. \tag{24}$$

Substituting relation (24) in Eq. (22) and performing the Laplace inversion, one obtains the Fokker-Planck equation for normal diffusion with the diffusion coefficient π/R^2

$$\partial_t P_1 = \frac{\pi}{R^2} \partial_x^2 P_1 \,. \tag{25}$$

It is worth stressing that this long time diffusion takes place only for times larger then a transient time $t > t_0$, where $t_0 = R^2/D$. This situation is different from the long-time asymptotics observed in [17].

This result is generic for combs with finite secondary branches (either fingers in the 2D comb, or discs in the 3D comb). However, the finite boundary conditions for the $y - z$ discs result in the destruction of ultra-slow diffusion in the x direction, as well. Mathematically, this fact follows immediately from the Laplace image $\tilde{G}(r, s)$, which depends on the boundary conditions. The ultra-slow motion takes place only for normal diffusion in the infinite discs.

3.1. Scenarios of transitions from ultra-slow to normal diffusion

It is worth noting that solution (20) is the most easy way to obtain the result on normal diffusion. However, it does not explain the transition from ultra-slow to normal diffusion. Moreover, from here we cannot see that this ultra-slow motion is a transient process. To observe this, let us return to the initial Eq. (15)

$$u^2 \tilde{G}'' + u\tilde{G}' - u^2 \tilde{G} = 0 \,,$$

For the reflecting boundary conditions, the solution of Eq. (15) is found in the form of a superposition of the modified Bessel functions $I_0(u)$ and $K_0(u)$. This reads

$$G(u) = AK_0(u) + BI_0(u) \,, \tag{26}$$

where the boundary condition yields

$$G'(u)\Big|_{r=R} = -AK_1(a) + BI_1(a) = 0 \,, \tag{27}$$

9

where $a = R\sqrt{s/D}$. This boundary condition establishes a relation between coefficients $A = A(s)$ and $B = B(s)$ as well. There are two possibilities for the coefficients; (i) $B = AK_1(a)/I_1(a)$ and A is a constant, and (ii) $A = BI_1(a)/K_1(a)$, and B is a constant. These yield two solutions with different combinations of the coefficients,

$$G_1(u) = \frac{1}{2\pi D}K_0(u) + \frac{K_1(a)}{I_1(a)}I_0(u),\tag{28}$$

$$G_2(u) = \frac{I_1(a)}{K_1(a)}K_0(u) + I_0(u).\tag{29}$$

However, this ambiguity between $G_1(u)$ and $G_2(u)$ is easily resolved by the physical meaning of the solutions at different time scales. The first solution $G_1(u)$ is valid for the initial times with the large argument when $s \gg 1$. Since $R > r$, the second term in this solution can be neglected, and $G_1(u) = \frac{1}{2\pi D}K_0(u)$ corresponds to ultra-slow diffusion, studied in Sec. 2. This process is transient and takes place for $t \ll R^2/D$.

It is reasonable to consider the second solution for the small argument, when $s \to 0$. In this case the first term of the order $s\ln(s)$ is much smaller than $I_0(u)$, which is of the order of 1. Now taking integration over y and z, we have

$$2\pi \int_0^R rI_0(ar/R)dr = 2\pi R^2 I_1(a)/a.$$

In the limit $a \to 0$, the Bessel function is $I_1(a) \sim a/2$ [24]. Finally, one obtains exactly the result in Eq. (23) $\int_0^R G_2(r,s)d^2r = \pi R^2$ (here $d^2r \equiv dydz$). This solution corresponds to normal diffusion at times $t \gg R^2/D$, considered in the previous section.

These two solutions $G_1(u)$ and $G_2(u)$ correspond to the initial time scale and the asymptotically large time scale correspondingly. The corresponding solutions for the reduced distribution $P_1(x,t)$ in the form of ultra-slow diffusion (8) and normal diffusion (24) as the limiting cases, are anticipated as well and both are obtained above, in the previous sections in Eqs. (9) and (25) correspondingly.

Note that diffusion in the side branch discs can be anomalous as well. Does

this ultra-slow diffusion survives in this case? The answer is it does not. We prove this statement in the next section.

4. Anomalous diffusion in discs

What happens with this ultra-slow diffusion if diffusion in the transversal disks is anomalous and described by a memory kernel $\mathcal{K}(t)$? The comb model (3) now reads as an extension of the Generalized Master Equation

$$\partial_t P = \delta(y)\delta(z)\partial_x^2 P + D(\partial_y^2 + \partial_z^2)\int_0^t \mathcal{K}(t-t')P(t')dt'. \tag{30}$$

The temporal kernel $\mathcal{K}(t)$ is defined in the Laplace domain through a waiting time pdf $\psi(t)$ [14, 27]

$$\tilde{\mathcal{K}} = s\tilde{\psi}(s)/[1 - \tilde{\psi}(s)]. \tag{31}$$

Repeating the analysis in the Laplace domain of Sec. 2.2, one obtains solution (16) in the form

$$\tilde{G}(r,s) = A \cdot K_0(Br), \tag{32}$$

where $B = \sqrt{s/D\mathcal{K}(s)}$ and A is a normalization constant. Therefore integration (18) yields

$$\tilde{F}(x,s) = \frac{B^2}{A}\tilde{P}_1(x,s). \tag{33}$$

As already admitted above (in Sec. 2.2), $\tilde{G}(r,s)$ is singular function at $r = 0$. Therefore, as in Eq. (13), straightforward integration of Eq. (30) over the y and z coordinates can be performed only in the real time domain. However, the function $G(r = 0, t)$ does exists and correspondingly $\hat{\mathcal{L}}[G(r = 0, t)](s)$ exists as well, at least as a principal value integral like in Eq. (9). Therefore, to obtain equation for P_1, one has to return to the time domain consideration for $P_1(r,t)$ by integrating Eq. (30) over y and z. To be specific, let us consider subdiffusion in the $y - z$ discs, described by $\tilde{\psi}(s) = \frac{1}{1+(\tau s)^\alpha}$ and correspondingly with the memory kernel

$$\tilde{\mathcal{K}}(s) = s^{1-\alpha}/\tau^\alpha,$$

11

where τ is a dimensionless characteristic time scale and $0 < \alpha < 1$. In this case $B^2 = s^\alpha \tau^\alpha / D$. Therefore, in the limit $r \to 0$ the argument $Br \ll 1$ and Eq. (32) reads for this small argument [24]

$$K_0[(bs)^{\alpha/2}] \approx \ln\frac{2}{\gamma} - \frac{\alpha}{2}\ln(bs),\qquad(34)$$

where γ is an Euler constant and $b = \tau(r^2/D)^{1/\alpha}$. Now, we perform the Laplace inversion at $r \to 0$

$$G(r,t) = A\int_{-i\infty}^{i\infty} K_0[(bs)^{\alpha/2}]e^{st}ds \approx A\delta(t)\ln\frac{2}{\gamma} - Ab^{-1}\frac{\alpha}{2}\int_{-i\infty}^{i\infty}\ln(p)e^{pt_b}dp,\qquad(35)$$

where $t_b = t/b$. The last term can be presented in a form of an integral [23]

$$\int_{-i\infty}^{i\infty}\ln(p)e^{pt_b}dp = \frac{d}{dt_b}\int_{-i\infty}^{i\infty} p^{-1}\ln(p)e^{pt_b}dp = -\frac{1}{t_b}.$$

Finally, one obtains for $t > 0$ and $r \to 0$

$$G(r \to 0, t) \approx \frac{A\alpha}{2}\cdot\frac{1}{t}.\qquad(36)$$

This result is independent of r and therefore, the limit $r = 0$ is correct. For this approximate solution, the constant $A = \frac{1}{2\pi D}$ is taken to satisfy the limit $\alpha = 1$, which corresponds to solution (6) at $r = 0$.

Repeating procedures of Sec. 2, namely performing first integration over y and z in Eq. (30) and then the Laplace transform over time, and Fourier transform over x, and taking into account the result of Eq. (33), one obtains a modification of Eq. (10). This reads for $\bar{\tilde{P}}_1(k,s) = \hat{\mathcal{L}}\hat{\mathcal{F}}[P_1(x,t)]$

$$\bar{\tilde{P}}_1(k,s) = \frac{1}{s + \bar{\mathcal{D}}_\alpha s^\alpha k^2 \hat{\mathcal{L}}[t^{-1}])},\qquad(37)$$

where $\bar{\mathcal{D}}_\alpha = \alpha\tau^\alpha/D$. Repeating the argument for the inferring Eq. (12), one obtains for the MSD

$$\langle x^2(t)\rangle = \frac{2\bar{\mathcal{D}}_\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{\ln(t')dt'}{(t-t')^\alpha} \simeq \mathcal{D}_\alpha t^{1-\alpha}\ln(t).\qquad(38)$$

Note that we omit here a term $\sim \gamma t^{1-\alpha}$, which is a slower contribution to anomalous diffusion than the term accounted in Eq. (38). Here $\mathcal{D}_\alpha = 2\bar{\mathcal{D}}_\alpha/\Gamma(2-\alpha)$

is a generalized transport coefficient. Subdiffusion with the transport exponent $1 - \alpha$ is dominant. Therefore, we conclude that ultra-slow diffusion $\sim \ln(t)$ takes place only for $\alpha = 1$ that can be realized as the result of normal diffusion in the infinite secondary branched discs.

5. Conclusion

We presented an exact analytical result on ultra-slow diffusion by solving the Fokker-Planck equation in the 3D cylindrical comb geometry. It is shown that the ultra-slow motion with the MSD on the x backbone is of the order of $\ln(t)$, and it results from normal diffusion in the secondary branched discs of the infinite radius. The technically specific point of the analysis is the singularity of $\tilde{G}(r, s)$ at $r = 0$. The integration of the comb equations (3) and (30) over the y and z coordinates is performed in the time domain, while the relation between the reduced pdf $P_1(x, t)$ and the backbone pdf $P(x, r = 0, t)$ is established in the Laplace domain.

An important modification of the model is a choice of the boundary conditions at finite radius of the discs, or finite boundary conditions, which is a realistic situation. In this case, the physical realization of the ultra-slow transport is restricted by the transient time scale $t < t_0 = R^2/D$. In the continuous time random walk (CTRW) theory [14] this transient time is determined from the exponential decay of the tempered waiting time distributions[6]. In particular, the introduction of the finite boundary conditions for the 2D comb model (1) is a geometrical realization of such tempered waiting time pdf [31], when before time t_0, both the waiting times and subdiffusion correspond to the boundary condition at infinities, while for the time $t > t_0$, the waiting times have finite scale and normal diffusion takes place. The transition over time in this tempered waiting time distribution from the short time scale $t < t_0$ to the long time

---

[6]This situation can take place in Lévy walks in random [28] and chaotic [29] systems (see also recent discussion in Ref. [30]).

asymptotics $t > t_0$ is continuous [31]. For the 3D comb model, we found solution in the composition forms (28) and (29), which contain both ultra-slow and normal diffusions: $G(u) = A(s)K_0(u) + B(s)I_0(u)$. The reflecting boundary condition chooses the coefficients in such a way that for the short time scale $s \gg 1$ ($t < t_0$), the second term $B(s)I_0(u)$ vanishes. The first term $A(s)K_0(u)$ corresponds to the "super-heavy-tailed" waiting time pdf that eventually leads to ultra-slow diffusion along the backbone. The latter is a transient process, since it takes place only for $t < t_0$. In the opposite case of the large time asymptotics, the boundary condition leads to vanishing of the first term $A(s)K_0(u)$ for $s \to 0$. This realizes in the exponential time decay of the tempered waiting time pdf, which can be estimated by taking into account the first two terms[7] in the expansion of $I_0(u)$ in Eq. (29) for $s \to 0$. In this case $I_0\left(r\sqrt{s/D}\right) \simeq 1 + sr^2/4D$ [24]. Taking integration over d^2r, one obtains relation (24) in the form

$$\tilde{F}(x, s) = \frac{6D}{\pi R^3} \cdot \frac{\tilde{P}_1(x, s)}{s + 12D/R^2} .$$

After the Laplace inversion, one obtains the exponential decay of the tempered waiting time pdf

$$\psi(t) = \frac{6D}{\pi R^3} \exp\left(-\frac{12Dt}{R^2}\right),$$

which is specified by the comb parameters[8] R and D, and valid for the large times $t \gg R^2/D$.

If the transport in the secondary branches is anomalously diffusive (subdiffusive), the anomalous transport becomes dominant in the backbone, as well. As the result, ultra-slow diffusion is replaced by the anomalous transport with the MSD $\sim t^{1-\alpha}\ln(t)$, which is more sophisticated than usual power law subdiffusion, and we call it *enhanced subdiffusion*. This continuous transition from the ultra-slow motion for $\alpha = 1$ to enhanced subdiffusion with $0 < \alpha \leq 1$ is due to anomalous diffusion in the secondary branched dynamics, which is con-

---

[7]We thank the anonymous referee for this useful comment.

[8]Note that R and D are dimensionless parameters, which are scaled by the transport properties D_x, D_y, D_z of media.

trolled by the transport exponent α. Since the side-branched $y - z$ dynamics acts as traps for the projected motion along the x axis, the analytical form of the solution $G(r,t)$ is crucial for the realization of ultra-slow diffusion. The latter takes place only for the Gaussian solution (6), when $\alpha = 1$. When $\alpha < 1$, the solution of $G(r,t)$ corresponds to subdiffusion in the form of Fox function [14]. This subdiffusion leads to less extensive invasion of the side branches in comparison with normal diffusion that increases the probability to return to the backbone at $r = 0$. Eventually, this enhances the transport along the backbone, and therefore, the continuous transition, by α, from ultra-slow diffusion to enhanced subdiffusion takes place. In the opposite case, when transport in the discs is enhanced, for example by additional radial advection, one anticipates that the transport along the backbone can be saturated. This result was obtained numerically for the 2D comb model in the framework of the CTRW consideration [32]. From the CTRW point of view, the analytical expression of the solution $G(r,t)$ relates to the form of the waiting time distribution for the transport along the backbone. In particular, only the Gaussian solution of $G(r,t)$ in Eq. (6) corresponds to the "super-heavy-tailed" waiting time pdf, which leads to ultra-slow diffusion along the backbone. This mechanism can be helpful to model ecological processes like in generalization of an elephant random walk model [15].

References

[1] E.W. Montroll and H. Scher, Random walks on lattices. J. Stat. Phys. 9: 101 (1973).

[2] Y. Gefen, A. Aharony, and S. Alexander, Anomalous diffusion on percolating clusters. Phys. Rev. Lett. 50: 77 (1983).

[3] J. Klafter, A. Blumen, and M. F. Shlesinger, Stochastic pathway to anomalous diffusion. Phys. Rev. A 35: 3081 (1987).

[4] S. Fedotov and V. Méndez, Non-Markovian model for transport and reactions of particles in spiny dendrites. Phys. Rev. Lett. 101: 218102, (2008).

[5] J. Bouchaud and A. Georges, Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications. Phys. Rep. 195: 127 (1990).

[6] P. S. Isichenko, Percolation, statistical topography, and transport in random media. Rev. Mod. Phys. 64: 961 (1992).

[7] O. A. Dvoretskaya, P. S. Kondratenko, and L. V. Matveev, Anomalous diffusion in generalized Dykhne model. JETP 110: 58 (2010) [Zh. Eksp. Teor. Fiz. 137: 67 (2010), Rus.].

[8] D. ben Avraam and S. Havlin, *Diffusion and Reactions in Fractals and Disordered Systems*. Cambridge University Press, Cambridge, 2000.

[9] I.M. Sokolov, Models of anomalous diffusion in crowded environments. Soft Matter 8: 9043 (2012).

[10] S.R. White and M. Barma, Field-induced drift and trapping in percolation networks. J. Phys. A 17: 2995 (1984).

[11] G.H. Weiss and S. Havlin, Some properties of a random walk on a comb structure. Physica A 134: 474 (1986).

[12] V.E. Arkhincheev and E.M. Baskin, Anomalous diffusion and drift in a combmodel of percolation clasters. Sov. Phys. JETP 73: 161 (1991).

[13] E.W. Montroll and M.F. Shlesinger, The wonderful world of random walks. in *Studies in Statistical Mechanics*, v. 11, eds J. Lebowitz and E.W. Montroll. Noth–Holland, Amsterdam, 1984.

[14] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach. Phys. Rep. 339: 1 (2000).

[15] M.A.A. da Silva, G.M. Viswanathan, and J.C. Cressoni, Ultraslow diffusion in an exactly solvable non-Markovian random. Phys. Rev. E 89: 052110 (2014).

[16] D. Boyer and C. Solis-Salas, Random walks with preferential relocations to places visited in the past and their application to biology. Phys. Rev. Lett. 112: 240601 (2014).

[17] A.V. Chechkin, J. Klafter, and I.M. Sokolov, Fractional Fokker-Planck equation for ultraslow kinetics. Europhys. Lett., 63: 326 (2003).

[18] S.I Denisov and H. Kantz, Continuous-time random walk theory of super-slow diffusion. Europhys. Lett. 92: 30001 (2010).

[19] A.S Bodrova, A.V. Chechkin, A.G. Cherstvy, and R. Metzler, Ultraslow scaled Brownian motion. New J. Phys. 17: 063038 (2015).

[20] A. Godec, A.V. Chechkin, E. Barkai, H. Kantz, and R. Metzler, Localisation and universal fluctuations in ultraslow diffusion processes, J. Phys. A: Math. Theor. 47: 492002 (2014).

[21] G. Forte, R. Burioni, F. Cecconi, and A. Vulpiani, Anomalous diffusion and response in branched systems: a simple analysis. J. Phys.: Condens. Matter 25: 465106 (2013).

[22] S. Alexander and R. Orbach, Density of states on fractals : "fractons". J. Physique Lett. 43: 625 (1982).

[23] H. Bateman and A. Erdèlyi, *Tables of Integral Transforms*. McGraw-Hill, New York, 1954, V 1.

[24] E. Jahnke, F. Emde, and F. Lösch, *Tables of Higher Functions*. McGraw-Hill, New York, 1960.

[25] E. Baskin and A. Iomin, Superdiffusion on a comb structure. Phys. Rev. Lett. 93: 120603 (2004).

[26] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products.* Elsevier Academic Press, Amsterdam, 2007.

[27] V. Méndez, S. Fedotov, and W. Horsthemke, *Reaction-Transport Systems, Mesoscopic Foundations, Fronts and Spatial Instabilities.* Springer, Berlin, 2010.

[28] J. Klafter, A. Blumen, and M. F. Shlesinger, Stochastic pathway to anomalous diffusion. Phys. Rev. A 35: 3081 (1987).

[29] G.M. Zaslavsky, Fractional kinetic equation for Hamiltonian chaos. Physica D 76: 110 (1994).

[30] A. Godec and R. Metzler, Linear response, fluctuation-dissipation, and finite-system-size effects in superdiffusion. Phys. Rev. E 88: 012116 (2013).

[31] A. Iomin, V. Zaburdaev, T. Pfohl, Actin polymerization front propagation in a comb-reaction system. arXiv:1502.06710 [cond-mat.soft], submitted in Phys. Rev. E.

[32] V. Méndez, A. Iomin, D. Campos, W. Horsthemke, Mesoscopic description of random walks on combs. arXiv:1509.01670 [cond-mat.stat-mech], submitted in Phys. Rev. E.