

A multiplication operation for the hierarchy of norms[☆]

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Abstract

Assuming $\text{AD} + \text{DC}$, the hierarchy of norms is a wellordered structure of equivalence classes of ordinal-valued maps. We define operations on the hierarchy of norms, in particular an operation that acts as multiplication of the ranks of norms, and use these operations to establish a considerably improved lower bound for the length of the hierarchy of norms.

1. Introduction

As usual in set theory, we refer to the elements of Baire space ω^ω as *real numbers* and use the notation \mathbb{R} for ω^ω . A surjective function φ from \mathbb{R} onto some ordinal α is called a *norm*. In analogy to the usual Wadge ordering of sets of reals, we can order the norms by setting $\varphi \leq_N \psi$ if and only if there is a continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have $\varphi(x) \leq \psi(f(x))$. The First Periodicity Theorem [1] shows that in the theory $\text{ZF} + \text{AD} + \text{DC}$, this ordering is a prewellordering (cf. Theorem 3).

As a consequence, in $\text{ZF} + \text{AD} + \text{DC}$, we can define an ordinal Σ , the *length of the hierarchy of norms*, to be the order-type of \leq_N . Defining as usual $\Theta := \sup\{\alpha; \text{there is a surjection from } \mathbb{R} \text{ to } \alpha\}$, the second author proved in [6, Corollary 3 and Theorem 5] that $\Theta^2 \leq \Sigma < \Theta^+$.

In this paper, we shall improve the lower bound to $\Theta^{(\Theta^\Theta)}$. In §2, we give the basic definitions needed for this paper. The structure theory of the Wadge hierarchy will serve as a template for the later sections; in §3, we give a brief

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overview of this structure theory. One particularly important notion in this context is the notion of self-duality; in §4, we introduce the analogous notion for the hierarchy of norms and some basic operations that are specific for the hierarchy of norms. In §5, we discuss operations on the hierarchy of norms that can be directly transferred from the Wadge hierarchy (such as addition).

The heart of the paper are §§6 and 7, where we define a multiplication operation for the hierarchy of norms (which is the analogue of the operation for the Wadge hierarchy defined by Steel in [7, §III.D]) and prove the main theorem about it (Theorem 22). Finally, in §8, we apply the main theorem to get the improved lower bound for Σ (Theorem 27) and finish with some open questions.

The content of this paper is based on the first author's Master's thesis [3].

2. Basic definitions & properties

Our axiomatic framework will be ZF; if we assume additional axioms, we shall state them explicitly.

Our main object of study will be the set of functions from the reals to Θ , i.e., $\Theta^{\mathbb{R}}$. We write

$$\text{lh}(\varphi) := \sup\{\alpha + 1; \alpha \in \varphi[\mathbb{R}]\}$$

for the *length* of a function $\varphi : \mathbb{R} \rightarrow \Theta$. A function φ is called a *weak norm* if $\text{lh}(\varphi) < \Theta$. We have the following fact:

Lemma 1. *The ordinal Θ is singular if and only if there is some $\varphi : \mathbb{R} \rightarrow \Theta$ with $\text{lh}(\varphi) = \Theta$; thus, Θ is regular if and only if the set of weak norms coincides with $\Theta^{\mathbb{R}}$.*

Proof. If $\text{cf}(\Theta) = \alpha < \Theta$, then there are a cofinal $f : \alpha \rightarrow \Theta$ and a surjection $g : \mathbb{R} \rightarrow \alpha$, thus $\text{lh}(f \circ g) = \Theta$. Conversely, if $\varphi : \mathbb{R} \rightarrow \Theta$, then $\alpha := \text{otyp}(\text{ran}(\varphi)) < \Theta$ by definition of Θ . For $\beta < \alpha$, we define $f(\beta)$ to be the β th element of $\text{ran}(\varphi)$. If $\text{lh}(\varphi) = \Theta$, then $f : \alpha \rightarrow \Theta$ is cofinal, and hence $\text{cf}(\Theta) \leq \alpha < \Theta$. \square

A weak norm is called a *norm* if its range is an ordinal. If φ is a norm, then $\text{lh}(\varphi) = \text{ran}(\varphi)$. We denote the set of weak norms by $\text{w}\mathcal{N}$ and the set of norms by \mathcal{N} .

A relation is called a *preorder* if it is transitive and reflexive. If \leq is a preorder on a set X , then we can define the *corresponding equivalence relation* \equiv by $a \equiv b :\Leftrightarrow a \leq b \wedge b \leq a$ for all $a, b \in X$, and the *corresponding strict preorder relation* $<$ by $a < b :\Leftrightarrow a \leq b \wedge \neg a \equiv b$ for all $a, b \in X$. A preorder \leq induces a partial order on the \equiv -equivalence classes; we denote this partial order with the same symbol \leq .

As mentioned before, for $\varphi, \psi \in \Theta^{\mathbb{R}}$, we write $\varphi \leq_{\text{N}} \psi$ if and only if there is a continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have $\varphi(x) \leq \psi(f(x))$. We write $\varphi \leq_{\text{NL}} \psi$ if there is a Lipschitz function with the same property. These relations are preorders and we denote the corresponding equivalence relations by \equiv_{N} and \equiv_{NL} and their corresponding strict preorder relations by $<_{\text{N}}$ and $<_{\text{NL}}$.

It is easy to see that if $\text{lh}(\varphi) < \text{lh}(\psi)$, then $\varphi <_{\text{NL}} \psi$; furthermore, any two norms of length $\alpha+1$ for some α are Lipschitz-equivalent. As usual, for $x, y \in \mathbb{R}$, we define their *Turing sum* $x * y$ as follows:

$$(x * y)(n) := \begin{cases} x(k), & \text{if } n = 2k, \\ y(k), & \text{if } n = 2k + 1. \end{cases}$$

Proposition 2. *For any weak norm φ there is a norm ψ such that $\text{lh}(\varphi) = \text{lh}(\psi)$ and $\varphi \equiv_{\text{N}} \psi$.*

Proof. Let $\alpha := \text{lh}(\varphi)$ and let $\pi : \mathbb{R} \rightarrow \alpha + 1$ be a surjection. Then the following $\psi : \mathbb{R} \rightarrow \Theta$ is a norm as claimed:

$$\psi(x * y) := \begin{cases} \pi(x), & \text{if } \pi(x) \leq \varphi(y), \\ \varphi(y), & \text{otherwise.} \end{cases}$$

□

The relations \leq_{N} and \leq_{NL} can be defined in terms of games as follows: let R be any binary relation on ordinals and let $\varphi, \psi \in \mathcal{N}$. Then we define $G_{\text{W}}^R(\varphi, \psi)$ as follows: players **I** and **II** play by turns; player **I** always plays a natural number, and player **II** plays either a natural number or a special token \mathfrak{p} , which shall symbolize passing a turn. Player **II** loses if she plays only finitely many natural numbers. Otherwise, let $a \in \mathbb{R}$ be the sequence of moves of player **I and let $b \in \mathbb{R}$ be the sequence of natural number moves of player **II**. Then player **II** wins if and only if $\langle \varphi(a), \psi(b) \rangle \in R$. The game $G_{\text{L}}^R(\varphi, \psi)$ is the special case where player **II** is not allowed to play \mathfrak{p} -moves at all.**

It is immediate that $\varphi \leq_{\text{N}} \psi$ if and only if Player **II** has a winning strategy in $G_{\text{W}}^{\leq}(\varphi, \psi)$ and $\varphi \leq_{\text{NL}} \psi$ if and only if Player **II** has a winning strategy in $G_{\text{L}}^{\leq}(\varphi, \psi)$. Via this game representation, the proof of the First Periodicity Theorem yields:

Theorem 3. *Assume AD and DC. Then both $(\Theta^{\mathbb{R}}, \leq_{\text{N}})$ and $(\Theta^{\mathbb{R}}, \leq_{\text{NL}})$ are prewellorders, i.e., \leq_{N} and \leq_{NL} are linear and well-founded.*

We let $\langle \lambda_{\alpha}; \alpha < \Theta \rangle$ be the strictly increasing sequence of all limit ordinals below Θ . If AD + DC holds, we write $|\varphi|_{\text{N}}$ and $|\varphi|_{\text{NL}}$ for the ranks of φ in the wellfounded relations given by Theorem 3. For $\alpha < \Theta$, we define

$$\Sigma_{\alpha} := \sup\{|\varphi|_{\text{N}}; \text{lh}(\varphi) \leq \lambda_{\alpha}\}$$

and $\Sigma := \bigcup\{\Sigma_{\alpha}; \alpha < \Theta\}$. By Proposition 2, it does not matter whether we take the supremum over all norms or over all weak norms as the structures

$$(\{\varphi \in \mathcal{N}; \text{lh}(\varphi) \leq \lambda_{\alpha}\} / \equiv_{\text{N}}, \leq_{\text{N}}) \text{ and } (\{\varphi \in \text{w}\mathcal{N}; \text{lh}(\varphi) \leq \lambda_{\alpha}\} / \equiv_{\text{N}}, \leq_{\text{N}})$$

are isomorphic via the inclusion function. We call $(\mathcal{N} / \equiv_{\text{N}}, \leq_{\text{N}})$ the *hierarchy of norms* and $(\mathcal{N} / \equiv_{\text{NL}}, \leq_{\text{NL}})$ the *Lipschitz hierarchy of norms*; the ordinal Σ is

the length of the hierarchy of norms. As stated in Lemma 1, if (and only if) Θ is regular, then $\text{w}\mathcal{N} = \Theta^{\mathbb{R}}$, and hence under the assumption of $\text{AD} + \text{DC} + \text{“}\Theta \text{ is regular”}$, $\text{otyp}(\Theta^{\mathbb{R}}/\equiv_{\mathbb{N}}, \leq_{\mathbb{N}}) = \Sigma$.

The stratification of Σ in terms of the Σ_{α} is an important feature of the hierarchy of norms (that does not have an analogue in the Wadge hierarchy) and was used in [6, Theorem 4] to give the lower bound of Θ^2 . We shall illustrate this in the following argument which is implicit in the proof of the main result [6, Corollary 3]:

In [6, Lemma 4], it was proved that there is a function $(\cdot)^+ : \mathcal{N} \rightarrow \mathcal{N}$ such that for any norm φ we have that $\text{lh}(\varphi) = \text{lh}(\varphi^+)$ and $\varphi <_{\mathbb{N}} \varphi^+$. We say that a norm φ is *embedded in a norm* ψ iff there is some $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $\psi(x * y) = \varphi(y)$. It is easy to see that if φ is embedded in ψ , then $\varphi \leq_{\text{NL}} \psi$.

Lemma 4. *Let $\alpha, \beta < \Theta$ be arbitrary. Then there is a strictly $\leq_{\mathbb{N}}$ -increasing sequence $\langle \varphi_{\nu} ; \nu < \alpha \rangle$ of norms such that for all $\nu < \alpha$ we have $\text{lh}(\varphi_{\nu}) = \lambda_{\beta}$.*

Proof. We fix a surjection $s : \mathbb{R} \rightarrow \alpha$ and a norm φ with $\text{lh}(\varphi) = \lambda_{\beta}$. Then we define φ_{ν} for $\nu < \alpha$ recursively as follows. We set $\varphi_0 := \varphi$. For $\nu \neq 0$ we first define a norm φ_{ν}^* by setting for all $x, y \in \mathbb{R}$:

$$\varphi_{\nu}^*(x * y) := \begin{cases} \varphi_{s(x)}(y), & \text{if } s(x) < \nu, \\ \varphi(y), & \text{otherwise.} \end{cases}$$

Based on this we let $\varphi_{\nu} := (\varphi_{\nu}^*)^+$. It is clear that $\text{lh}(\varphi_{\nu}) = \text{lh}(\varphi) = \lambda_{\beta}$ for all $\nu < \alpha$. Furthermore for any $\nu < \alpha$ and any $\xi < \nu$ we have that φ_{ξ} is embedded in φ_{ν}^* and so we get that $\varphi_{\xi} \leq_{\text{NL}} \varphi_{\nu}^* <_{\mathbb{N}} (\varphi_{\nu}^*)^+ = \varphi_{\nu}$. \square

It is crucial here that the operations used in the definition of the sequence do not change the length of the norms, and therefore produce a sequence below Σ_{β} , thus inductively giving $\Sigma_{\beta} \geq \Theta \cdot \beta$, and consequently $\Sigma \geq \Theta^2$ (cf. [6, Theorem 4]).

To conclude this section we introduce a few pieces of general notation that we shall use in the following. Let s be a finite or countable sequence of natural numbers (i.e., $s \in \omega^{<\omega}$ or $s \in \mathbb{R}$) and $n \in \omega$. Then we denote by $s + n$ the sequence obtained by increasing each member of s by n . Let s be a finite sequence and x a finite or countable sequence. Then $s \hat{\ } x$ refers to the sequence obtained by concatenating s with x . For $K \leq \omega$ we use $0^{(K)}$ to denote the sequence of length K that is constantly 0.

Let $s \in \omega^{<\omega}$ and $p \in \Theta^{\mathbb{R}}$. Then we define $\varphi_{[s]} : \mathbb{R} \rightarrow \Theta$ by setting for all $x \in \mathbb{R}$

$$\varphi_{[s]}(x) := \varphi(s \hat{\ } x).$$

Finally, if τ and σ are strategies for Players **I** and **II**, respectively, in an infinite game, and if x and y are infinite plays for Players **I** and **II**, respectively, we write $(x * \sigma)_{\text{II}}$ for the play of Player **II** resulting from playing according to σ against x and $(\tau * y)_{\text{I}}$ for the play of Player **I** resulting from reacting to y according to τ .

3. Operations for the Wadge hierarchy

Since our main result is motivated by an analogous result for the Wadge hierarchy, we shall give an overview of the basic structure theory of the Wadge hierarchy in this section. The theorems in this section are due to Wadge, Steel and Van Wesep. The organization of this section follows very closely an unpublished monograph by Alessandro Andretta.

As usual, we denote Lipschitz and Wadge reducibility by \leq_L and \leq_W , respectively. These are preorders on $\wp(\mathbb{R})$, and we call their quotient structures modulo their corresponding equivalence relations the *Lipschitz hierarchy* and the *Wadge hierarchy*, respectively. Assuming $\text{AD} + \text{DC}$, these hierarchies are well-founded (by the *Martin-Monk Theorem*, cf. [2, Corollary 9]). Although these hierarchies are not linear, they satisfy the so-called *Semi-Linear Ordering Principle* that is the statement of the following theorem (which in particular implies that antichains in the Wadge hierarchy cannot have more than two elements):

Theorem 5. *Assume AD. Let $A, B \subseteq \mathbb{R}$. Then either $A \leq_W B$ or $\mathbb{R} \setminus B \leq_W A$. An analogous result holds for \leq_W replaced by \leq_L . [10, Theorem IV.1]*

We call A L-self-dual iff $A \equiv_L \mathbb{R} \setminus A$. We call A W-self-dual iff $A \equiv_W \mathbb{R} \setminus A$. The nontrivial antichains are all of the form $([A]_L, [\mathbb{R} \setminus A]_L)$ in the Lipschitz hierarchy or $([A]_W, [\mathbb{R} \setminus A]_W)$ in the Wadge hierarchy for some set $A \subseteq \mathbb{R}$, i.e., they consist of the equivalence classes for a non-self-dual set and for its complement. These two notions of self-duality coincide, and we can use the word “self-dual” to refer to both L-self-duality and W-self-duality:

Theorem 6 (Steel-Van Wesep). *Assume AD. Let $A \subseteq \mathbb{R}$. Then A is L-self-dual iff A is W-self-dual. [9, Theorem 3.1]*

Self-duality is crucial for the structure theory of the Wadge hierarchy since many of the properties of operations on the Wadge hierarchy depend on whether you apply them to self-dual or non-self-dual sets:

Definition 7. Let $A, B \in \wp(\mathbb{R})$, $\langle A_i; i \in \omega \rangle \in (\wp(\mathbb{R}))^\omega$.

1. $A \oplus B := \{(2n) \frown a; n \in \omega, a \in A\} \cup \{(2n+1) \frown b; n \in \omega, b \in B\}$.
2. $\bigoplus_{n \in \omega} A_n := \{ \langle n \rangle \frown a; n \in \omega, a \in A_n \}$.
3. $A^* := \{0^{(n)} \frown \langle m+1 \rangle \frown a; n, m \in \omega, a \in A\}$.
4. $A^\circ := A^* \cup \{0^{(\omega)}\}$.
5. $A \boxplus B := \{(s+1) \frown \langle 0 \rangle \frown a; s \in \omega^{<\omega}, a \in A\} \cup \{b+1; b \in B\}$.
6. $A^\sharp := \{s \frown \langle 0 \rangle \frown (a+1); s \in \omega^{<\omega}, a \in A\} \cup \{a+1; a \in A\}$.
7. $A^\flat := A^\sharp \cup \{x \in \mathbb{R}; \exists^\infty n(x(n) = 0)\}$.

Proposition 8. *Assume AD and DC. Let $A, B \in \wp(\mathbb{R})$, $\langle A_i; i \in \omega \rangle \in (\wp(\mathbb{R}))^\omega$.*

1. Then $|\bigoplus_{n \in \omega} A_n|_{\mathbb{L}} = \sup_{n \in \omega} |A_n|_{\mathbb{L}}$. If, furthermore, for all $n \in \omega$ there is $m \in \omega$ such that $A_n <_{\mathbb{L}} A_m$, then $\bigoplus_{n \in \omega} A_n$ is self-dual. [10, Theorems V.1 & V.2]
2. If A is non-self-dual, then $|A \oplus (\mathbb{R} \setminus A)|_{\mathbb{W}} = |A|_{\mathbb{W}} + 1$ and $A \oplus (\mathbb{R} \setminus A)$ is self-dual. [9, Lemma 2.3]
3. If A is self-dual, then $|A^*|_{\mathbb{W}} = |A^\circ|_{\mathbb{W}} = |A|_{\mathbb{W}} + 1$ and $([A^*]_{\mathbb{W}}, [A^\circ]_{\mathbb{W}})$ is a non-self-dual pair. [10, Theorem V.6]
4. If A is self-dual, then $|A \boxplus B|_{\mathbb{W}} = |A|_{\mathbb{W}} + 1 + |B|_{\mathbb{W}}$. [10, Proposition V.7 & Theorems V.8 & V.9]
5. If A is self-dual, then $|A^\sharp|_{\mathbb{W}} = |A^\flat|_{\mathbb{W}} = |A|_{\mathbb{W}} \cdot \omega_1$ and $([A^\sharp]_{\mathbb{W}}, [A^\flat]_{\mathbb{W}})$ is a non-self-dual pair. [10, Theorems V.16 & V.17]

In his doctoral dissertation, Steel introduced an operation that acts as multiplication on Wadge ranks [7, Definition III.D.6]:

Definition 9. We define three binary relations M_1, M_2, M_3 on \mathbb{R} as follows:

$$\begin{aligned}
M_1(x, y) &:\Leftrightarrow \exists n \in \omega \exists \langle s_0, \dots, s_{2n} \rangle \in (\omega^{<\omega} \setminus \{\emptyset\})^{2n+1} \\
&\quad [x = (s_0 + 1) \wedge \langle 0 \rangle \wedge (s_1 + 1) \wedge \langle 0 \rangle \wedge \dots \wedge \langle 0 \rangle \wedge (s_{2n} + 1) \wedge \langle 0 \rangle \wedge (y + 1)] \\
M_2(x, y) &:\Leftrightarrow x = y + 1 \vee \\
&\quad \exists n \in \omega \exists \langle s_0, \dots, s_{2n+1} \rangle \in (\omega^{<\omega} \setminus \{\emptyset\})^{2n+2} \exists z \in \mathbb{R} \\
&\quad [x = (s_0 + 1) \wedge \langle 0 \rangle \wedge (s_1 + 1) \wedge \langle 0 \rangle \wedge \dots \wedge \langle 0 \rangle \wedge (s_{2n+1} + 1) \wedge \langle 0 \rangle \wedge (z + 1) \\
&\quad \wedge y = s_0 \wedge s_2 \wedge \dots \wedge s_{2n} \wedge z] \\
M_3(x, y) &:\Leftrightarrow \exists \langle s_i; i \in \omega \rangle \in (\omega^{<\omega} \setminus \{\emptyset\})^\omega \\
&\quad [x = (s_0 + 1) \wedge \langle 0 \rangle \wedge (s_1 + 1) \wedge \langle 0 \rangle \wedge \dots \wedge \langle 0 \rangle \wedge (s_n + 1) \wedge \dots \\
&\quad \wedge y = s_0 \wedge s_2 \wedge s_4 \wedge \dots]
\end{aligned}$$

Now let $A, B \in \wp(\mathbb{R})$. Then

$$A \boxtimes B := \{x \in \mathbb{R}; \exists a \in A(M_1(x, a))\} \cup \{x \in \mathbb{R}; \exists b \in B(M_2(x, b) \vee M_3(x, b))\}.$$

Using the concept of backtrack reducibility introduced in Van Wesep's doctoral dissertation [8, § II.3], Steel proved:

Proposition 10. Assume AD and DC. Let $A, B \in \wp(\mathbb{R})$, A self-dual and B such that $|B|_{\mathbb{W}}$ is a limit ordinal of uncountable cofinality. Then

$$|A \boxtimes B|_{\mathbb{W}} = |A|_{\mathbb{W}} \cdot \omega_1 \cdot |B|_{\mathbb{W}}.$$

[7, Theorem III.D.4]

4. Self-duality in the hierarchy of norms

The aim of the remainder of this paper will be to translate the operations introduced in § 3 to the hierarchy of norms in order to show that the length of this hierarchy is closed under the corresponding ordinal operations. By Proposition 2, we can work with weak norms instead of norms (i.e., we do not need to make sure that the operations preserve surjectivity).

We stressed in § 3 that some of the operations in the Wadge case use that Wadge degrees come in two types: self-dual degrees and non-self-dual degrees. In the Wadge hierarchy, non-self-dual pairs were the two-element antichains. But the hierarchy of norms is linear and hence does not have any two-element antichains. Consequently, we shall need a different notion of self-duality.

Our notion is a special case of the general notion of self-duality for Wadge-like hierarchies on sets of functions from \mathbb{R} to a fixed better quasi-order (BQO) from [5, Definition 5]. In the context of the hierarchy of norms, it was studied by Duparc in [4], where, however, the term “self-dual” was not used.

Definition 11. Let φ be a weak norm. Then φ is *L-self-dual* if and only if Player **II** wins the game $G_L^<(\varphi, \varphi)$. We call φ *W-self-dual* if and only if Player **II** wins the game $G_W^<(\varphi, \varphi)$.

We have an analogue of the Steel-Van Wesep Theorem in this context:

Theorem 12. *Assume AD. Let φ be a weak norm. Then φ is L-self-dual if and only if φ is W-self-dual.*

Proof. Assume towards a contradiction that φ is W-self-dual, but not L-self-dual. We take winning strategies for Player **II** in the games $G_W^<(\varphi, \varphi)$, $G_W^<(\varphi, \varphi)$ and, using AD, a winning strategy for Player **I** in the game $G_L^<(\varphi, \varphi)$. Using these winning strategies we play a global game exactly as in the proof of the Steel-Van Wesep Theorem for the Wadge hierarchy in [9, Theorem 3.1]. From this global game we then obtain a set of reals $\{x_n; n \in \omega\}$ such that $\varphi(x_{n+1}) < \varphi(x_n)$, which gives us an infinitely descending chain of ordinals. \square

As in the Wadge case, we drop the designations “W-” and “L-” in light of Theorem 12 and just speak of “self-dual” and “non-self-dual” norms. In [4], Duparc proves a number of useful facts about these notions restricted to Borel weak norms; however, the proofs immediately translate to the hierarchy of weak norms under the assumption of AD:

Proposition 13 (Duparc). *Assume AD. Let φ, ψ be weak norms such that ψ is non-self-dual. Then $\varphi <_N \psi$ if and only if Player **II** wins $G_W^<(\varphi, \psi)$. Also $\varphi <_{NL} \psi$ if and only if Player **II** wins $G_L^<(\varphi, \psi)$. [4, Remark 5.b]*

Corollary 14. *Assume AD. Let φ, ψ be weak norms such that φ is non-self-dual. Then $\varphi \equiv_{NL} \psi$ if and only if $\varphi \equiv_N \psi$.*

Proof. Assume that $\varphi \equiv_N \psi$, but $\varphi <_{NL} \psi$. Then ψ is non-self-dual and so Player **II** has a winning strategy in $G_L^<(\varphi, \psi)$. But with the same winning strategy, Player **II** can also win $G_W^<(\varphi, \psi)$ and so $\varphi <_N \psi$, contradicting $\varphi \equiv_N \psi$. \square

For any weak norm φ we define the tree $\mathbf{T}(\varphi) \subseteq \omega^{<\omega}$ by

$$\mathbf{T}(\varphi) := \{s \in \omega^{<\omega}; \varphi|_s \equiv_{\mathbb{N}} \varphi\}.$$

For weak norms φ_i for $i \in \omega$, we define a weak norm $\bigoplus_{n \in \omega} \varphi_n$ by setting for any $n \in \omega$ and $x \in \mathbb{R}$:

$$\left[\bigoplus_{n \in \omega} \varphi_n \right] (\langle n \rangle \frown x) = \varphi_n(x).$$

Proposition 15 (Duparc). *Assume AD. Let φ be a weak norm.*

1. *The function φ is non-self-dual if and only if $\mathbf{T}(\varphi)$ is ill-founded [4, Proposition 10].*
2. *If φ is self-dual, then there is a sequence $\langle \varphi_i; i \in \omega \rangle$ of non-self-dual weak norms such that $\varphi \equiv_{\mathbb{W}} \bigoplus_{n \in \omega} \varphi_n$ [4, Proposition 15].*
3. *Assume DC. Let $\langle \varphi_i; i \in \omega \rangle$ be a sequence of non-self-dual weak norms. Then $|\bigoplus_{n \in \omega} \varphi_n|_{\mathbb{N}} = \sup_{n \in \omega} |\varphi_n|_{\mathbb{N}}$ [4, Claim 17].*
4. *Assume DC. Then φ is self-dual if and only if $|\varphi|_{\mathbb{N}}$ is a limit ordinal of countable cofinality [4, Proposition 16].*

Lemma 16. *Assume AD and DC. Let $\langle \varphi_i; i \in \omega \rangle$ be a sequence of weak norms such that for all $m \in \omega$ there is $n \in \omega$ such that $\varphi_m <_{\mathbb{L}} \varphi_n$. Then*

$$\left| \bigoplus_{i \in \omega} \varphi_i \right|_{\mathbb{NL}} = \sup_{i \in \omega} |\varphi_i|_{\mathbb{NL}}.$$

Proof. Similar to the proof of [4, Proposition 15]. □

5. Basic operations on the hierarchy of norms

In the following, we shall give a list of operations on the hierarchy of norms used in this paper; for a real x such that there are infinitely many $n \in \omega$ with $x(n) \neq 0$, we write $\text{unstretch}(x)$ to denote the real y obtained by erasing all occurrences of 0 from x and decreasing all remaining members of the sequence by 1.

Definition 17. Let φ, ψ be weak norms, $n \in \omega \setminus \{0\}$, $\alpha < \omega_1$, and $x \in \mathbb{R}$. We define

$$\varphi^{\text{succ}}(x) := \begin{cases} \varphi(\text{unstretch}(x)) + 1, & \text{if } \exists^\infty n \in \omega (x(n) \neq 0), \\ 0, & \text{otherwise;} \end{cases}$$

$$\varphi^{+n}(x) := \begin{cases} \varphi(\text{unstretch}(x)) + n, & \text{if } \exists^\infty n \in \omega (x(n) \neq 0), \\ n - 1, & \text{otherwise;} \end{cases}$$

$$\begin{aligned}
\varphi^{+\omega}(\langle n \rangle \wedge x) &:= \begin{cases} \varphi(\text{unstretch}(x)) + n, & \text{if } \exists^\infty n \in \omega(x(n) \neq 0), \\ n, & \text{otherwise;} \end{cases} \\
\varphi^\vee(\langle n \rangle \wedge x) &:= \varphi(x) + n; \\
(\varphi + 1)(x) &:= \varphi(x) + 1; \\
\varphi^{\text{stretch}}(x) &:= \begin{cases} \varphi(\text{unstretch}(x)), & \text{if } \exists^\infty n \in \omega(x(n) \neq 0), \\ 0, & \text{otherwise;} \end{cases} \\
(\varphi \boxplus \psi)(x) &:= \begin{cases} \varphi(y), & \text{if there is } s \in \omega^{<\omega} \text{ such that } x = (s+1) \wedge \langle 0 \rangle \wedge y, \\ \psi(y), & \text{if } x = y + 1; \end{cases} \\
\varphi^\sharp(x) &:= \begin{cases} \varphi(y), & \text{if } \exists s \in \omega^{<\omega} (x = s \wedge \langle 0 \rangle \wedge (y+1)), \\ \varphi(y), & \text{if } x = (y+1), \\ 0, & \text{if } \exists^\infty n \in \omega(y(n) = 0). \end{cases}
\end{aligned}$$

Lemma 18. *We assume AD and DC; let φ and ψ be weak norms.*

1. *We have that $|\varphi^{\text{succ}}|_{\mathbb{N}} = |\varphi|_{\mathbb{N}} + 1$ and φ^{succ} is non-self-dual.*
2. *For any $n \geq 1$, we have that $|\varphi^{+n}|_{\mathbb{N}} = |\varphi|_{\mathbb{N}} + n$.*
3. *We have that $\varphi^{+\omega} \equiv_{\text{NL}} \bigoplus_{n \in \omega} \varphi^{+(n+1)}$. As a consequence, $|\varphi^{+\omega}|_{\mathbb{N}} = |\varphi|_{\mathbb{N}} + \omega$ and $\varphi^{+\omega}$ is self-dual.*
4. *If φ is non-self-dual, then $\varphi^{\text{stretch}} \equiv_{\text{NL}} \varphi$.*
5. *If φ is non-self-dual, then $\varphi^\vee \equiv_{\text{NL}} \varphi^{+\omega}$.*
6. *The operations \boxplus and $(\cdot)^\sharp$ are monotone: for any four weak norms $\varphi, \varphi', \psi, \psi'$ with $\varphi \leq_{\mathbb{N}} \varphi'$ and $\psi \leq_{\mathbb{N}} \psi'$ we have that $\varphi \boxplus \psi \leq_{\mathbb{N}} \varphi' \boxplus \psi'$ and $\varphi^\sharp \leq (\varphi')^\sharp$.*
7. *If φ is self-dual, then $|\varphi \boxplus \psi|_{\mathbb{N}} = |\varphi|_{\mathbb{N}} + 1 + |\psi|_{\mathbb{N}}$.*
8. *If φ is self-dual, then φ^\sharp is non-self-dual and $|\varphi^\sharp|_{\mathbb{N}} = |\varphi|_{\mathbb{N}} \cdot \omega_1$.*
9. *We have that $(\varphi^\sharp)^\sharp \equiv_{\text{NL}} \varphi^\sharp$.*

Proof. The proofs of claims 1. to 6. are easy exercises for the reader (using Proposition 15 and Lemma 16; furthermore, 4. is an auxiliary result to prove 5.). The proofs of claims 7. to 9. are analogous to the proofs for addition and the sharp operation on the Wadge degrees which can be found in Wadge's doctoral dissertation [10, Theorems V.9 and V.17]. \square

All operations in Definition 17 make only minor changes to the length of a weak norm: if $\text{lh}(\varphi), \text{lh}(\psi) \leq \lambda_\alpha$ and χ is the result of applying one of the operations from Definition 17 to φ and ψ , then $\text{lh}(\chi) \leq \lambda_\alpha$. Thus, Lemma 18 implies that the ordinals Σ_α are closed under ordinal addition and ordinal multiplication with ω_1 .

We stress that the first four operations are quite different from the case of the Wadge hierarchy: in the Wadge hierarchy, self-dual and non-self-dual degrees alternate, and therefore, the successor operation consists of the operations $A \mapsto A^*$, $A \mapsto A^\circ$, and $A \mapsto A \oplus (\mathbb{R} \setminus A)$. In the hierarchy of norms, the operations $\varphi \mapsto \varphi^{\text{succ}}$ and $\varphi \mapsto \varphi^{+\omega}$ give a non-self-dual norm and a self-dual norm strictly above a given arbitrary norm, respectively. This will be of central importance in §6.

The operations $(\varphi, \psi) \mapsto \varphi \boxplus \psi$ and $\varphi \mapsto \varphi^\sharp$ are straightforward adaptations from the Wadge context (cf. §3); the operations $(\cdot)^{\text{stretch}}$ and $(\cdot) + 1$ are purely auxiliary.

If $\vec{n} : \mathbb{R} \rightarrow \Theta$ is the constant function with range $\{n\}$, then $|\vec{n}|_{\mathbb{N}} = n$, and hence $|\bigoplus_{n \in \omega} \vec{n}|_{\mathbb{N}} = \omega$. Thus, Proposition 18 gives that $|\widehat{\omega}_1|_{\mathbb{N}} = \omega_1$ for

$$\widehat{\omega}_1 := \left(\bigoplus_{n \in \omega} \vec{n} \right)^\sharp.$$

We shall make use of $\widehat{\omega}_1$ in the following two sections.

6. Multiplication of norms

We now define the multiplication operation for the hierarchy of weak norms.

Definition 19. Let φ and ψ be weak norms. Then we define a weak norm $\varphi \boxtimes \psi$ by setting for any $x \in \mathbb{R}$:

$$(\varphi \boxtimes \psi)(x) := \begin{cases} \varphi(y), & \text{if } M_1(x, y), \\ \psi(y), & \text{if } M_2(x, y) \text{ or } M_3(x, y), \\ 0, & \text{otherwise,} \end{cases}$$

where M_1, M_2 and M_3 are defined as in Definition 9.

As the operations defined in §5 this definition is a straightforward adaptation of Steel's multiplication operation in the Wadge hierarchy. However, the proof of our main result, Theorem 22, will not be a straightforward adaptation due to the structural differences between the two hierarchies.

In this section, we give the motivation behind Steel's (and our) definition of the multiplication operation, together with a discussion of the basic notion of a *product conforming real*, used in §7. Our motivation follows the exposition in Steel's doctoral dissertation [7, Section III.D]; since it is not readily available, we believe that repeating it here serves the reader.

We call an $x \in \mathbb{R}$ such that there is $y \in \mathbb{R}$ such that $M_1(x, y)$ or $M_2(x, y)$ or $M_3(x, y)$ holds a *product conforming real*. It is easy to see that an $x \in \mathbb{R}$ is product conforming if and only if $x(0) \neq 0$ and x contains no subsequent 0's. We can consider a product conforming real $x = (s_0 + 1) \wedge \langle 0 \rangle \wedge (s_1 + 1) \wedge \langle 0 \rangle \wedge (s_2 + 1) \wedge \langle 0 \rangle \wedge \dots$ as a scheme of sequences of natural numbers entered into two rows as in

$$\begin{array}{c|ccccccccc}
 \text{Row 1} & s_0 & & s_2 & & & & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & & \dots\dots \\
 \text{Row 2} & & s_1 & & & & & & & \\
 & & & & & & & & & \dots\dots
 \end{array},$$

where the s_{2n} 's are entered into row 1, the s_{2n+1} 's are entered into row 2 and we use \downarrow to denote changes of the row (corresponding to 0's in x).

Then we can understand, e.g., the game $G_L^{\leq}(\chi, \varphi \boxtimes \psi)$ as a game in which Player **I** plays natural numbers and Player **II** builds up a two-row scheme of sequences of natural numbers as above. Schematically a game of this form looks as follows:

$$\begin{array}{c|cccccccccc}
 \text{I} & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & \\
 \hline
 \text{II} & \text{Row 1} & b_0 & b_1 & & & & & b_2 & & \dots\dots \\
 & \text{Row 2} & & & \downarrow & c_0 & c_1 & \downarrow & & \downarrow & c'_0
 \end{array}$$

Then the winning conditions for $G_L^{\leq}(\chi, \varphi \boxtimes \psi)$ translate as follows, when we denote Player **I**'s play in a given match by x :

Case 1 is that Player **II** settles for row 2 eventually, i.e., Player **II** plays as in

$$\begin{array}{c|cccccccc}
 \text{Row 1} & a_0 & \dots & a_n & & & & & \\
 \text{Row 2} & & & \downarrow & c_0 & c_1 & \dots & c_m & c_{m+1} & \dots\dots
 \end{array}.$$

Then let $c \in \mathbb{R}$ be the real entered into row 2 after Player **II**'s last change to row 2. In this case Player **II** wins if and only if

$$\chi(x) \leq \varphi(c).$$

This case corresponds to M_1 .

Case 2 is that Player **II** settles for row 1 eventually, i.e., Player **II** plays as in

$$\begin{array}{c|ccccccc}
 \text{Row 1} & a_0 & \dots & a_n & a_{n+1} & \dots\dots \\
 \text{Row 2} & & & & & & &
 \end{array}$$

or Player **II** changes rows indefinitely as indicated in the following schematic

$$\begin{array}{c|cccccccc}
 \text{Row 1} & a_0 & \dots & a_n & & & a_{n+1} & \dots & a_{n'} & & \\
 \text{Row 2} & & & \downarrow & c_0 & \dots & c_m & \downarrow & & \downarrow & c'_0 & \dots\dots
 \end{array}.$$

Then let $a \in \mathbb{R}$ be the real entered into row 1 throughout the whole match. In this case Player **II** wins if and only if

$$\chi(x) \leq \psi(a).$$

This case corresponds to M_2 and M_3 .

In all these considerations we neglected the possibility that Player **II** plays a real that is not product conforming. But this is without loss of generality by the following result:

Lemma 20. *Let φ, ψ be weak norms. If Player **II** wins the game $G_{\mathbb{W}}^{\leq}(\chi, \varphi \boxtimes \psi)$, then there is a winning strategy τ for Player **II** such that for any $x \in \mathbb{R}$ the real $(x * \tau)_{\mathbf{II}}$ played by Player **II** as a response is product conforming.*

Analogous results hold for $G_{\mathbb{W}}^{\leq}(\chi, \varphi \boxtimes \psi)$ replaced by $G_{\mathbb{W}}^{\leq}(\chi, \varphi \boxtimes \psi)$, $G_{\mathbb{L}}^{\leq}(\chi, \varphi \boxtimes \psi)$ or $G_{\mathbb{L}}^{\leq}(\chi, \varphi \boxtimes \psi)$.

Proof. We note that $y \in \mathbb{R}$ is *not* product conforming if and only if either $y(0) = 0$ or y contains two consecutive 0's. This can be easily avoided by Player **II** in all the above games by just playing 1 instead of any move that will lead to her playing a real that is not product conforming. \square

Lemma 21. *Assume AD and DC. Let $\varphi, \varphi', \psi, \psi', \chi, \psi_n$ be weak norms (for $n \in \omega$).*

1. *If $\varphi \leq_{\text{NL}} \varphi'$ and $\psi \leq_{\text{N}} \psi'$, then $\varphi \boxtimes \psi \leq_{\text{N}} \varphi' \boxtimes \psi'$.*
2. *If φ is self-dual and $\varphi \boxtimes \psi \leq_{\text{N}} \varphi \boxtimes \psi'$, then $\psi \leq_{\text{N}} \psi'$.*
3. *If φ is self-dual and $\varphi \boxtimes \psi \leq_{\text{NL}} \varphi \boxtimes \psi'$, then $\psi \leq_{\text{NL}} \psi'$.*
4. *If φ is self-dual, then $\varphi \boxtimes \psi$ is self-dual if and only if ψ is self-dual.*
5. $\varphi \boxtimes (\psi \boxplus \chi) \equiv_{\text{NL}} (\varphi \boxtimes \psi) \boxplus (\varphi \boxtimes \chi)$.
6. $\varphi \boxtimes (\bigoplus_{n \in \omega} \psi_n) \equiv_{\text{N}} \bigoplus_{n \in \omega} ((\varphi \boxtimes \psi_n) \boxplus \varphi)$.
7. $\varphi \boxtimes (\psi^{\sharp}) \equiv_{\text{N}} (\varphi \boxtimes \psi)^{\sharp}$.

Proof. Easy unravelling of the definitions. \square

7. The Main Theorem

The overall aim of this section will be to prove the following theorem.

Theorem 22. *Assume AD and DC. If φ is a self-dual weak norm and ψ is a weak norm such that $|\psi|_{\mathbb{N}}$ is a limit ordinal of uncountable cofinality, then*

$$|\varphi \boxtimes \psi|_{\mathbb{N}} = |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\psi|_{\mathbb{N}}.$$

Theorem 22 allows us to show that the ordinals Σ_{α} are closed under multiplication: Let φ, ψ be weak norms with $\text{lh}(\varphi), \text{lh}(\psi) \leq \lambda_{\alpha}$. Then also $\text{lh}(\varphi^{+\omega}) \leq \lambda_{\alpha}$ and $\text{lh}(\psi^{+\omega}) \leq \lambda_{\alpha}$. Using the explicitly given weak norm $\widehat{\omega}_1$ defined in § 5, we get that $\text{lh}(\varphi^{+\omega} \boxplus (\psi^{+\omega} \boxtimes \widehat{\omega}_1)) \leq \lambda_{\alpha}$ and also $|\varphi^{+\omega} \boxplus (\psi^{+\omega} \boxtimes \widehat{\omega}_1)|_{\mathbb{N}} \geq |\varphi|_{\mathbb{N}} \cdot |\psi|_{\mathbb{N}}$. So we get for any $\beta, \gamma < \Sigma_{\alpha}$ that $\beta \cdot \gamma < \Sigma_{\alpha}$, once we have established Theorem 22.

The proof of Theorem 22 follows Steel's proof of Proposition 10 in [7, Theorem III.4]; it is a proof by induction on $|\psi|_{\mathbb{N}}$. For the benefit of the reader we shall give the complete argument, even in those parts which are only minor modifications of Steel's argument. However, some of the cases of the inductive

proof cannot be directly transferred from Steel's proof, since he uses the operations $A \mapsto A \oplus \mathbb{R} \setminus A$ and $A \mapsto A^\flat$ on Wadge degrees which do not have analogues in the hierarchy of norms. Moreover, if you replace $A \mapsto A \oplus \mathbb{R} \setminus A$ by $\varphi \mapsto \varphi^{+\omega}$ (since both assign the next self-dual degree to a given non-self-dual degree), then the ordinal behavior on the level of ranks is different, since $|A \oplus \mathbb{R} \setminus A|_N = |A|_N + 1$, whereas $|\varphi^{+\omega}|_N = |\varphi|_N + \omega$. Also, the operation $A \mapsto A^\flat$ is used by Steel in the base case of the induction to define $A \mapsto A^{\flat\sharp}$ and $A \mapsto A^{\sharp\flat}$, corresponding on the level of ranks roughly to multiplication with ω_1^2 . We need an appropriate operation replacing these in the context of the hierarchy of norms. This operation will be $\varphi \mapsto \varphi^\spadesuit := ((\varphi^\sharp)^\vee)^\sharp$, which has the property that $|\varphi^\spadesuit|_N = |\varphi|_N \cdot \omega_1^2$ for any self-dual norm φ . Using this operation and the weak norm $\widehat{\omega}_1$, we can establish the base case of the proof as follows.

Proposition 23. *Assume AD. Let φ be self-dual weak norm. Then $\varphi \boxtimes \widehat{\omega}_1 \equiv_N \varphi^\spadesuit$. Therefore, additionally assuming DC, if φ is a self-dual weak norm and ψ is a weak norm such that $|\psi|_N = \omega_1$, then $|\varphi \boxtimes \psi|_N = |\varphi|_N \cdot \omega_1^2$.*

Proof. We only show that $\varphi^\spadesuit \leq_N \varphi \boxtimes \widehat{\omega}_1$. The other direction is similar, but a bit easier.

We note that by construction of φ^\spadesuit and $\widehat{\omega}_1$ the game $G_L^{\leq}(\varphi^\spadesuit, \varphi \boxtimes \widehat{\omega}_1)$ is equivalent to the two-player game $\tilde{G}(\varphi)$, which we define as follows. Player **I** and Player **II** take turns at their moves as usual, but in addition to playing natural numbers both players have special moves. Player **I** can announce to erase his target, a move which we denote by $\epsilon\mathfrak{t}$, or to start recording a real, a move which we denote by $\mathfrak{s}\mathfrak{r}$. Player **II** can change rows—in the sense of a norm of the form $\varphi \boxtimes \psi$ —or announce to erase her target, which we also denote by $\epsilon\mathfrak{t}$. However, we impose a few side conditions on possible plays as follows:

1. Player **I** cannot start recording in his first move.
2. Player **II** must play product conforming, i.e., not changing the row in her first move and not changing rows in immediate succession.
3. If Player **I** erases his target, his next move must be a natural number.
4. Player **II** can only erase her target while playing in row 1.

Now we fix the winning conditions for the game $\tilde{G}(\varphi)$ by assigning ordinal values to plays by Players **I** and **II** and comparing these values.

A play by Player **I** is evaluated as follows:

Case 1. If Player **I** starts recording a real at least once, but only finitely often and after his last move of this kind he only plays natural numbers, then let $y \in \mathbb{R}$ be the real composed of these natural numbers. Also we fix a natural number n as follows: If Player **I** makes the move $\epsilon\mathfrak{t}$ at least once, then n is the natural number occurring immediately after Player **I**'s last such move. Otherwise n is just Player **I**'s first move, which by the side conditions is a natural number. We then assign to **I**'s play the value $\varphi(y) + n$. We call y the *real recorded by Player I* and n the *target number of Player I*.

Case 2. If the first case is not the case, but Player **I** makes the move \mathfrak{ct} at most finitely often, then the value of his play is n , where n is the natural number occurring immediately after the last \mathfrak{ct} -move, or just his first move at all, if none such exists. Again we call n the *target number of Player I*.

Case 3. Otherwise **I**'s play gets the value 0.

A play by Player **II** is evaluated as follows:

Case 1. If Player **II** settles for row 2, we let y be the real recorded in row 2 after the last change to row 2 and assign to **II**'s play the value $\varphi(y)$. We call y the *real recorded by Player II*.

Case 2. If Player **II** does not settle for row 2 and Player **II** erases her target only finitely often, then we assign to her play the value n , where n is either the first natural number occurring in row 2 after her last \mathfrak{ct} -move, or simply her first move, if none such last occurrence exists. We call n the *target number of Player II*.

Case 3. Otherwise **II**'s play gets the value 0.

Now given a match of $\tilde{G}(\varphi)$, Player **II** wins if and only if the value of her play is greater or equal the value of Player **I**'s play. Otherwise Player **I** wins.

We give a winning strategy for Player **II** in $\tilde{G}(\varphi)$, thereby showing that $\varphi \spadesuit \leq_N \varphi \boxtimes \widehat{\omega}_1$. For this we first note that Player **II** wins all games of the form $G_L(\varphi + n, \varphi)$ for $n \in \omega$, since φ is self-dual. So (using $\text{AC}_\omega(\mathbb{R})$, a consequence of AD) we choose for $n \in \omega$ winning strategies σ_n for Player **II** in $G_L(\varphi + n, \varphi)$. Now we let Player **II** move in $\tilde{G}(\varphi)$ as follows. She starts by copying Player **I**'s moves into her row 1, as long as Player **I** only plays natural numbers. If Player **I** erases his target, Player **II** stays in row 1 or changes to row 1 (if necessarily, entering a dummy 0 into row 2 first to avoid having consecutive row changes) and announces to change her target there.

Afterwards Player **II** resumes copying Player **I**'s moves as long as the latter plays natural numbers. If **I** starts recording, Player **II** resets row 2 by either changing to row 1 and afterwards to row 2 or directly changing from row 1 to row 2, while playing dummy 0-moves to avoid successive row changes. Then Player **II** begins reacting to Player **I**'s moves directly after his move of \mathfrak{st} according to the strategy σ_n , where n is the target of **I**'s play up to this point as long as Player **I** continues playing natural numbers.

This strategy is indeed winning for Player **II** in $\tilde{G}(\varphi)$, since on the one hand it ensures that the target of **II**'s play is identical to the target of **I**'s play, whenever it exists. On the other hand it ensures that if **I** records a real y , then **II** records a real y' such that $\varphi(y) + n \leq \varphi(y')$, where n is the target of **I**'s play.

For the “therefore”, we just note that for any two norms χ, χ' of rank ω_1 we have $\chi \equiv_{\text{NL}} \chi'$ by Propositions 13 and 15. \square

In the inductive proof on the rank of the right hand side of the \boxtimes -term, Steel considers the continuity in this right hand side, his [7, Lemma III.6], as the core of the argument. The analogue for the hierarchy of norms is the next proposition. We note that the proof in the case of norms is far simpler than the one for the Wadge hierarchy; Steel's proof distinguishes four cases, the fourth

of which uses the concept of backtrack-reducibility, while the argument for the hierarchy of norms collapses to Steel's case 4 while at the same time getting rid of the need to consider any other reducibility concept than norm reducibility.

Proposition 24. *Assume AD and DC. Let φ be a self-dual weak norm and ψ a non-self-dual weak norm. Then we have that*

$$|\varphi \boxtimes \psi|_{\mathbb{N}} = \sup_{\psi' <_{\mathbb{N}} \psi} |\varphi \boxtimes \psi'|_{\mathbb{N}}$$

Proof. It already follows from Lemma 21 that $|\varphi \boxtimes \psi|_{\mathbb{N}} \leq \sup_{\psi^* <_{\mathbb{N}} \psi} |\varphi \boxtimes \psi^*|_{\mathbb{N}}$. Thus we take a regular norm χ with $\chi <_{\mathbb{N}} \varphi \boxtimes \psi$ and show that there is a regular norm ψ^* with $\psi^* <_{\mathbb{N}} \psi$ such that $\chi \leq_{\mathbb{N}} \varphi \boxtimes \psi^*$. Since ψ is non-self-dual we get that also $\varphi \boxtimes \psi$ is non-self-dual. Thus by Proposition 13 Player **II** has a winning strategy τ in the game $G_{\mathbb{W}}^{\leq}(\chi, \varphi \boxtimes \psi)$. Without loss of generality we assume that the strategy τ only produces product conforming reals. Then we define a set $Z \subseteq \omega^{\omega}$ as follows:

$$Z := \{x \in \omega^{\omega} ; \text{Player II settles for row 1, when playing with } \tau \text{ against } x\}.$$

Using this we define a regular norm ψ^* by setting for any $x \in \omega^{\omega}$

$$\psi^*(x) := \begin{cases} \chi(x), & \text{if } x \in Z, \\ 0, & \text{otherwise.} \end{cases}$$

We note that Player **II** can win the game $G_{\mathbb{W}}^{\leq}(\psi^*, \psi^{\text{stretch}})$ according to the following strategy: Player **II** follows her winning strategy τ for the game $G_{\mathbb{W}}^{\leq}(\chi, \varphi \boxtimes \psi)$, whenever τ tells her to play into row 1. Otherwise she just plays 0.

Since ψ is non-self-dual, we have that $\psi \equiv_{\mathbb{N}} \psi^{\text{stretch}}$ and so it follows (again by non-self-duality of ψ) that $\psi^* <_{\mathbb{N}} \psi$.

Finally we have to show that $\chi \leq_{\mathbb{N}} \varphi \boxtimes \psi^*$. To this end we give a winning strategy for Player **II** in the game $G_{\mathbb{W}}^{\leq}(\chi, \varphi \boxtimes \psi^*)$ as follows: To determine Player **II**'s reaction in the game $G_{\mathbb{W}}^{\leq}(\chi, \varphi \boxtimes \psi^*)$ to moves made by Player **I**, we play a shadow match of the game $G_{\mathbb{W}}^{\leq}(\chi, \varphi \boxtimes \psi)$, in which Player **I** makes the same moves and Player **II** reacts with her winning strategy τ . As long as in the shadow match Player **II** passes or stays in row 1, in the actual match player **II** just copies Player **I**'s moves into row 1. If x is a real such that $(x * \tau)_{\mathbb{II}}$ never leaves row 1, then clearly $x \in Z$, Player **II** produces the real $(x + 1)$ in the actual game and so Player **II** wins, since $(\varphi \boxtimes \psi^*)(y) = \psi^*(x) = \chi(x) \geq \chi(x)$.

If, however, at some point in the shadow match Player **II** changes to row 2, then in the actual match we let Player **II** also change to row 2 and from then on follow the strategy τ , i.e., copy the moves she makes in row 2 in the shadow match. If x is a real such that $(x * \tau)_{\mathbb{II}}$ only changes to row 2 once and then stays there and y is the real played by Player **II** in the game $G_{\mathbb{W}}^{\leq}(\chi, \varphi \boxtimes \psi^*)$ following the strategy we just specify, then $(\varphi \boxtimes \psi)((x * \tau)_{\mathbb{II}}) = (\varphi \boxtimes \psi^*)(y)$ and so Player **II** wins again, since τ is a winning strategy in $G_{\mathbb{W}}^{\leq}(\chi, \varphi \boxtimes \psi)$. We refer to the sequence of moves specified in this paragraph by $(*)$.

If in the shadow match Player **II** changes back from row 2 to row 1, then in the actual match Player **II** resumes copying the moves of Player **I** in the shadow match, but starting from the first move that she hasn't already copied, so possibly lagging a finite amount of moves behind. Then, again, if x is a real such that $(x * \tau)_{\mathbf{II}}$ settles for row 1 from this point on, then $x \in Z$ and as we have already seen, Player **II** wins according to the strategy we just specify in the game $G_{\mathbb{W}}^{\leq}(\chi, \varphi \boxtimes \psi^*)$. If, however, Player **II** changes back to row 2, then our strategy proceeds exactly according to $(*)$.

Inductively we get that we have just specified a winning strategy for Player **II** in $G_{\mathbb{W}}^{\leq}(\chi, \varphi \boxtimes \psi^*)$ and so $\chi \leq_{\mathbb{N}} \varphi \boxtimes \psi^*$. \square

Now we get to the last technical lemma before we can prove the main theorem.

Lemma 25. *Assume AD and DC. Let φ, ψ be weak norms such that φ is self-dual and ψ is non-self-dual. Then we have that*

$$|\varphi \boxtimes \psi^{+\omega}|_{\mathbb{N}} \leq |\varphi \boxtimes \psi|_{\mathbb{N}} + \omega + |\varphi|_{\mathbb{N}}.$$

Proof. Unraveling the relevant definitions one can easily check that for any two norms χ, χ' we have that $\chi \boxtimes (\chi' + 1) \equiv_{\mathbb{N}} (\chi \boxtimes \chi') + 1$. Also for any non-self-dual χ we have that $\chi^{\text{stretch}} \equiv_{\mathbb{N}} \chi$. Now in total we can calculate that for self-dual φ and non-self-dual ψ we get

$$\begin{aligned} \varphi \boxtimes \psi^{\text{succ}} &= \varphi \boxtimes (\psi^{\text{stretch}} + 1) \equiv_{\mathbb{N}} \varphi \boxtimes (\psi + 1) \equiv_{\mathbb{N}} (\varphi \boxtimes \psi) + 1 \\ &\equiv_{\mathbb{N}} (\varphi \boxtimes \psi)^{\text{stretch}} + 1 = (\varphi \boxtimes \psi)^{\text{succ}}, \end{aligned}$$

where for the penultimate equivalence we used Lemma 21.4. Inductively we get from this that $(\varphi \boxtimes \psi)^{+n} \equiv_{\mathbb{N}} \varphi \boxtimes (\psi^{+n})$ for all $n \in \omega \setminus \{0\}$. Noting that $\psi^{+\omega} \equiv_{\mathbb{N}} \bigoplus_{n \in \omega \setminus \{0\}} \psi^{+n}$ we obtain that

$$\varphi \boxtimes (\psi^{+\omega}) \equiv_{\mathbb{N}} \varphi \boxtimes \left(\bigoplus_{n \in \omega \setminus \{0\}} \psi^{+n} \right) \equiv_{\mathbb{N}} \bigoplus_{n \in \omega \setminus \{0\}} ((\varphi \boxtimes \psi^{+n}) \boxplus \varphi),$$

which implies that

$$\begin{aligned} |\varphi \boxtimes (\psi^{+\omega})|_{\mathbb{N}} &= \sup_{n \in \omega \setminus \{0\}} |(\varphi \boxtimes \psi^{+n}) \boxplus \varphi|_{\mathbb{N}} \\ &\leq |((\varphi \boxtimes \psi)^{+\omega}) \boxplus \varphi|_{\mathbb{N}} \\ &= |\varphi \boxtimes \psi|_{\mathbb{N}} + \omega + |\varphi|_{\mathbb{N}}, \end{aligned}$$

as claimed. \square

Now we can prove the main theorem.

Proof of Theorem 22. The proof proceeds by induction on $\alpha := |\psi|_{\mathbb{N}}$. The base case of $|\psi|_{\mathbb{N}} = \omega_1$ is already dealt with in Proposition 23. Thus let $\alpha > \omega_1$ and take the unique sequences $\langle \xi_m; m \leq k \rangle$ and $\langle \eta_m; m \leq k \rangle$ of ordinals with $\xi_m > \xi_{m+1}$ for all $m < k$ and $0 < \eta_m < \omega_1$ for all $m \leq k$ such that

$$\alpha = \sum_{m \leq k} \omega_1^{\xi_m} \cdot \eta_m.$$

We distinguish two cases with several subcases.

Case 1 is that $k > 0$. Then we fix a weak norm χ of minimal \leq_{NL} -rank such that $|\chi|_{\mathbb{N}} = \omega_1^{\xi_0} \cdot \eta_0$.

Subcase 1.1 is that $\text{cf}(\omega_1^{\xi_0} \cdot \eta_0) = \omega$, i.e., that χ is self-dual. Then by Proposition 18 we take a weak norm ψ^* with $\chi \boxplus \psi^* \equiv \psi$ and get that $|\psi^*|_{\mathbb{N}} = \sum_{0 < m \leq k} \omega_1^{\xi_m} \cdot \eta_m$ and also that $\text{cf}(|\psi^*|_{\mathbb{N}}) > \omega$. It follows by induction hypothesis that $|\varphi \boxtimes \psi^*|_{\mathbb{N}} = |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\psi^*|_{\mathbb{N}}$.

Next we take a strictly increasing sequence $\langle \nu_n; n \in \omega \rangle$ of limit ordinals cofinal in $\omega_1^{\xi_0} \cdot \eta_0$ such that for all $n \in \omega$, $\text{cf}(\nu_n) > \omega$. Then we choose (using AC_ω) for any $n \in \omega$ a weak norm χ_n such that $|\chi_n|_{\mathbb{N}} = \nu_n$. Then by induction hypothesis, $|\varphi \boxtimes \chi_n|_{\mathbb{N}} = |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\psi^*|_{\mathbb{N}}$. Also by construction we have that $\chi \equiv_{\text{NL}} \bigoplus_{n \in \omega} \chi_n$ and so we get that

$$\begin{aligned} |\varphi \boxtimes \chi|_{\mathbb{N}} &= \left| \varphi \boxtimes \left(\bigoplus_{n \in \omega} \chi_n \right) \right|_{\mathbb{N}} \\ &= \left| \bigoplus_{n \in \omega} ((\varphi \boxtimes \chi_n) \boxplus \varphi) \right|_{\mathbb{N}} \\ &= \sup_{n \in \omega} |(\varphi \boxtimes \chi_n) \boxplus \varphi|_{\mathbb{N}}. \end{aligned}$$

But now for any $n \in \omega$ we calculate that

$$\begin{aligned} |(\varphi \boxtimes \chi_n) \boxplus \varphi|_{\mathbb{N}} &\leq |(\varphi \boxtimes \chi_n)^{+\omega} \boxplus \varphi|_{\mathbb{N}} \\ &\leq |\varphi \boxtimes \chi_n|_{\mathbb{N}} + \omega + |\varphi|_{\mathbb{N}} \\ &= |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\chi_n|_{\mathbb{N}} + \omega + |\varphi|_{\mathbb{N}} \\ &< |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\chi_{n+1}|_{\mathbb{N}}. \end{aligned}$$

Thus we get that

$$\sup_{n \in \omega} |(\varphi \boxtimes \chi_n) \boxplus \varphi|_{\mathbb{N}} = \sup_{n \in \omega} (|\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\chi_n|_{\mathbb{N}}) = |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\chi|_{\mathbb{N}},$$

which lets us conclude that

$$\begin{aligned} |\varphi \boxtimes \psi|_{\mathbb{N}} &= |\varphi \boxtimes (\chi \boxplus \psi^*)|_{\mathbb{N}} \\ &= |(\varphi \boxtimes \chi) \boxplus (\varphi \boxtimes \psi^*)|_{\mathbb{N}} = |\varphi \boxtimes \chi|_{\mathbb{N}} + |\varphi \boxtimes \psi^*|_{\mathbb{N}} \\ &= |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\chi|_{\mathbb{N}} + |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\psi^*|_{\mathbb{N}} \\ &= |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\psi|_{\mathbb{N}}. \end{aligned}$$

Subcase 1.2 is that $\text{cf}(\omega_1^{\xi_0} \cdot \eta_0) > \omega$, i.e., that χ is non-self-dual. Then we distinguish two subsubcases.

Subsubcase 1.2.1 is that there is $n \in \omega$ such that $|\psi|_{\mathbb{N}} = \omega_1^{\xi_0} + \omega_1 \cdot (n+1)$. Then we define weak norms χ_ℓ for $\ell < \omega$ recursively by setting $\chi_0 := \chi$, $\chi_{\ell+1} := \chi_\ell^{+\omega} \boxplus \widehat{\omega_1}$ and get for all $\ell \in \omega$ that $|\chi_\ell|_{\mathbb{N}} = |\chi|_{\mathbb{N}} + \omega_1 \cdot \ell$. In particular for any $\ell < \omega$, $|\chi_\ell|_{\mathbb{N}}$ is a limit ordinal of uncountable cofinality and hence χ_ℓ is non-self-dual. So by comparing norm ranks we get that $\psi \equiv_{\text{NL}} \chi_{n+1}$. Thus we get the following:

$$\varphi \boxtimes \psi \equiv_{\mathbb{N}} \varphi \boxtimes \chi_{n+1} = \varphi \boxtimes (\chi_n^{+\omega} \boxplus \widehat{\omega_1}) \equiv_{\mathbb{N}} (\varphi \boxtimes \chi_n^{+\omega}) \boxplus (\varphi \boxtimes \widehat{\omega_1}).$$

But by the induction hypothesis we get that

$$\begin{aligned} |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\psi|_{\mathbb{N}} &= |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\chi_n|_{\mathbb{N}} + |\varphi|_{\mathbb{N}} \cdot \omega_1^2 \\ &= |\varphi \boxtimes \chi_n|_{\mathbb{N}} + |\varphi \boxtimes \widehat{\omega_1}|_{\mathbb{N}} \\ &\leq |\varphi \boxtimes \chi_n^{+\omega}|_{\mathbb{N}} + |\varphi \boxtimes \widehat{\omega_1}|_{\mathbb{N}} \\ &= |\varphi \boxtimes (\chi_n^{+\omega} \boxplus \widehat{\omega_1})|_{\mathbb{N}} \\ &= |\varphi \boxtimes \psi|_{\mathbb{N}}. \end{aligned}$$

Since $|\varphi \boxtimes \chi_n^{+\omega}|_{\mathbb{N}} \leq |\varphi \boxtimes \chi_n|_{\mathbb{N}} + \omega + |\varphi|_{\mathbb{N}}$, we furthermore get that

$$\begin{aligned} |\varphi \boxtimes \psi|_{\mathbb{N}} &= |\varphi \boxtimes \chi_n^{+\omega}|_{\mathbb{N}} + |\varphi \boxtimes \widehat{\omega_1}|_{\mathbb{N}} \\ &\leq |\varphi \boxtimes \chi_n|_{\mathbb{N}} + \omega + |\varphi|_{\mathbb{N}} + |\varphi|_{\mathbb{N}} \cdot \omega_1^2 \\ &= |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot (|\chi_n|_{\mathbb{N}} + \omega_1) \\ &= |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\psi|_{\mathbb{N}}. \end{aligned}$$

In total this shows that $|\varphi \boxtimes \psi|_{\mathbb{N}} = |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\psi|_{\mathbb{N}}$, as claimed.

Subsubcase 1.2.2 is that $|\psi|_{\mathbb{N}} \geq \omega_1^{\xi_0} \cdot \eta_0 + \omega_1 \cdot \omega$. Then we consider $\chi^{+\omega}$ and fix by Proposition 18 a weak norm ψ^* such that $\psi \equiv_{\mathbb{N}} \chi^{+\omega} \boxplus \psi^*$ and so by non-self-duality of ψ furthermore $\psi \equiv_{\text{NL}} \chi^{+\omega} \boxplus \psi^*$. Also clearly $|\psi|_{\mathbb{N}} = |\chi|_{\mathbb{N}} + |\psi^*|_{\mathbb{N}}$ and by induction hypothesis we have that $|\varphi \boxtimes \psi^*|_{\mathbb{N}} = |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\psi^*|_{\mathbb{N}}$. Since $\varphi \boxtimes \chi^{+\omega}$ is self-dual we now get that $|\varphi \boxtimes \psi|_{\mathbb{N}} = |\varphi \boxtimes \chi^{+\omega}|_{\mathbb{N}} + |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\psi^*|_{\mathbb{N}}$. Also, by induction hypothesis we have that

$$|\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\chi|_{\mathbb{N}} \leq |\varphi \boxtimes \chi|_{\mathbb{N}} \leq |\varphi \boxtimes (\chi^{+\omega} + \widehat{\omega_1})|_{\mathbb{N}} = |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot (|\chi|_{\mathbb{N}} + \omega_1).$$

So finally, using that $|\psi^*|_{\mathbb{N}} \geq \omega_1 \cdot \omega$, we get that

$$\begin{aligned} |\varphi \boxtimes \psi|_{\mathbb{N}} &= |\varphi \boxtimes \chi^{+\omega}|_{\mathbb{N}} + |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\psi^*|_{\mathbb{N}} \\ &\leq |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot (|\chi|_{\mathbb{N}} + \omega_1) + |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\psi^*|_{\mathbb{N}} \\ &= |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot (|\chi|_{\mathbb{N}} + \omega_1 + |\psi^*|_{\mathbb{N}}) \\ &= |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot (|\chi|_{\mathbb{N}} + \omega_1 + |\psi^*|_{\mathbb{N}}) \\ &= |\varphi|_{\mathbb{N}} \cdot \omega_1 \cdot |\psi|_{\mathbb{N}} \end{aligned}$$

and

$$\begin{aligned}
|\varphi|_N \cdot \omega_1 \cdot |\psi|_N &= |\varphi|_N \cdot \omega_1 \cdot |\chi|_N + |\varphi|_N \cdot \omega_1 \cdot |\psi^*|_N \\
&= |\varphi \boxtimes \chi|_N + |\varphi \boxtimes \psi^*|_N \\
&\leq |\varphi \boxtimes \chi^{+\omega}|_N + |\varphi \boxtimes \psi^*|_N \\
&= |\varphi \boxtimes \psi|_N.
\end{aligned}$$

Case 2 is that $k = 0$ and that $|\psi|_N = \omega_1^{\xi_0} \cdot \eta_0$. Since $\text{cf}(|\psi|_N) > \omega$, then η_0 is a successor and ξ_0 is either a successor or a limit of uncountable cofinality.

Subcase 2.1 is that $\eta_0 > 1$, i.e., there is $\gamma > 0$ such that $\eta_0 = \gamma + 1$. But then $|\psi|_N = \omega_1^{\xi_0} \cdot \gamma + \omega_1^{\xi_0}$ and we take regular norms χ, ψ^* of minimal \leq_{NL} -rank such that $|\chi|_N = \omega_1^{\xi_0} \cdot \gamma$ and $|\psi^*|_N = \omega_1^{\xi_0}$. Arguing from here exactly as in Subcase 1.2 we can show that $|\varphi \boxtimes \psi|_N = |\varphi|_N \cdot \omega_1 \cdot |\psi|_N$.

Subcase 2.2 is that $\eta_0 = 1$ and ξ_0 is a successor ordinal, i.e., there is an ordinal ϑ such that $\xi_0 = \vartheta + 1$. Then $|\psi|_N = \omega_1^{\vartheta} \cdot \omega_1$ and we fix a weak norm χ of minimal \leq_{NL} -rank such that $|\chi|_N = \omega_1^{\vartheta}$.

If χ is self-dual, then we have that $\psi \equiv_{\text{NL}} \chi^\#$ and so

$$|\varphi \boxtimes \psi|_N = |\varphi \boxtimes (\chi^\#)|_N = |(\varphi \boxtimes \chi)^\#|_N = |\varphi \boxtimes \chi|_N \cdot \omega_1.$$

Now since χ is self-dual, we have that $\text{cf}(|\chi|_N) = \omega$ and so arguing as in Subcase 1.1 we get that $|\varphi \boxtimes \chi|_N = |\varphi|_N \cdot \omega_1 \cdot |\chi|_N$ and so in total that $|\varphi \boxtimes \psi|_N = |\varphi|_N \cdot \omega_1 \cdot |\psi|_N$.

If, however, χ is non-self-dual, then we have that $\psi \equiv_{\text{NL}} (\chi^{+\omega})^\#$. Since $\chi^{+\omega}$ is self-dual, we get as in the case that χ is self-dual that

$$|\varphi \boxtimes \psi|_N = |\varphi \boxtimes (\chi^{+\omega})|_N \cdot \omega_1.$$

By induction hypothesis and Lemma 25 we get that

$$|\varphi|_N \cdot \omega_1 \cdot |\psi|_N \leq |\varphi \boxtimes \chi|_N \leq |\varphi \boxtimes \chi^{+\omega}|_N \leq |\varphi|_N \cdot \omega_1 \cdot |\chi|_N + \omega + |\varphi|_N.$$

Using this we calculate that

$$|\varphi|_N \cdot \omega_1 \cdot |\psi|_N = |\varphi|_N \cdot \omega_1 \cdot |\chi|_N \cdot \omega_1 \leq |\varphi \boxtimes \chi^{+\omega}|_N \cdot \omega_1 = |\varphi \boxtimes \psi|_N$$

and

$$\begin{aligned}
|\varphi \boxtimes \psi|_N &= |\varphi \boxtimes \chi^{+\omega}|_N \cdot \omega_1 \\
&\leq (|\varphi|_N \cdot \omega_1 \cdot |\chi|_N + \omega + |\varphi|_N) \cdot \omega_1 \\
&\leq |\varphi|_N \cdot \omega_1 \cdot (|\chi|_N + 1) \cdot \omega_1 \\
&= |\varphi|_N \cdot \omega_1 \cdot |\chi|_N \cdot \omega_1 \\
&= |\varphi|_N \cdot \omega_1 \cdot |\psi|_N.
\end{aligned}$$

This shows that $|\varphi \boxtimes \psi|_N = |\varphi|_N \cdot \omega_1 \cdot |\psi|_N$, as claimed.

Subcase 2.3 is that $\eta_0 = 1$ and ξ_0 is a limit ordinal of uncountable cofinality. Then by Proposition 24 and the induction hypothesis we get that

$$\begin{aligned} |\varphi \boxtimes \psi|_N &= \sup_{\psi' <_N \psi} |\varphi \boxtimes \psi'|_N = \sup_{\substack{\psi' <_N \psi \\ \text{cf}(|\psi'|_N) > \omega}} |\varphi \boxtimes \psi'|_N \\ &= \sup_{\substack{\psi' <_N \psi \\ \text{cf}(|\psi'|_N) > \omega}} (|\varphi|_N \cdot \omega_1 \cdot |\psi'|_N) = |\varphi|_N \cdot \omega_1 \cdot |\psi|_N \end{aligned}$$

□

8. Application

We have already seen that by Theorem 22 all ordinals Σ_α with $\alpha < \Theta$ are closed under multiplication. However, we can even strengthen this a bit by showing that in fact the ordinals Σ_α are closed under exponentiation with any ordinal $\beta < \Theta$ as exponent.

Proposition 26. *Assume AD and DC. For any $\alpha < \Theta$ and any $\beta < \Theta$ we have that $\gamma < \Sigma_\alpha$ implies $\gamma^\beta < \Sigma_\alpha$. As a consequence for every $\gamma < \Sigma_\alpha$ we have that $\gamma^\Theta \leq \Sigma_\alpha$.*

Proof. Take a weak norm with $\text{lh}(\varphi) \leq \lambda_\alpha$, i.e., such that $|\varphi|_N < \Sigma_\alpha$. Fix $\beta < \Theta$ and a surjection $\pi : \mathbb{R} \twoheadrightarrow \beta + 1$. Now for any $\delta \leq \beta$ we construct weak norms φ_δ with $\text{lh}(\varphi_\delta) \leq \lambda_\alpha$ and $|\varphi_\delta|_N \geq |\varphi|_N^\delta$ recursively on δ as follows. First we set $\varphi_0 := \varphi$. Then for a successor ordinal δ , say $\delta = \vartheta + 1$, we set $\varphi_\delta := \varphi_\vartheta^{+\omega} \boxtimes (\varphi^{+\omega} \boxplus \widehat{\omega}_1)$ and get by induction hypothesis that

$$|\varphi_\delta|_N = (|\varphi_\vartheta|_N + \omega) \cdot \omega_1 \cdot (|\varphi|_N + \omega_1) \geq |\varphi|_N^\vartheta \cdot |\varphi|_N = |\varphi|_N^\delta.$$

If δ is a limit ordinal, then we define φ_δ by setting for any $x, y \in \mathbb{R}$:

$$\varphi_\delta(x * y) := \begin{cases} \varphi_{\pi(x)}(y), & \text{if } \pi(x) < \delta, \\ \varphi(y), & \text{otherwise.} \end{cases}$$

Then for any $\delta' < \delta$ we have by construction that φ_δ embeds into $\varphi_{\delta'}$ and so we get by induction hypothesis that

$$|\varphi_\delta|_N \geq \sup_{\delta' < \delta} |\varphi_{\delta'}|_N \geq \sup_{\delta' < \delta} |\varphi|_N^{\delta'} = |\varphi|_N^\delta.$$

By construction it is clear that for any $\delta < \beta$, $\text{lh}(\varphi_\delta) \leq \lambda_\alpha$. So in particular we have that $|\varphi|_N^\beta \leq |\varphi_\beta|_N < \Sigma_\alpha$. □

This gives us a new lower bound for the hierarchy of norms as follows.

Theorem 27. *Assume AD and DC. Then we have that $\Sigma \geq \Theta^{(\Theta^\Theta)}$.*

Proof. We show by induction on $\alpha < \Theta$ that $\Sigma_\alpha \geq \Theta^{(\Theta^\alpha)}$. For the base case we just note that $\Theta^{(\Theta^0)} = \Theta \leq \Sigma_0$. For a successor ordinal $\gamma + 1$ we note that by induction hypothesis $\Theta^{(\Theta^\gamma)} \leq \Sigma_\gamma < \Sigma_{\gamma+1}$ and so by Proposition 26 we get that $(\Theta^{(\Theta^\gamma)})^\Theta = \Theta^{(\Theta^{\gamma+1})} \leq \Sigma_{\gamma+1}$. The limit case is immediate by continuity of the ordinal function $\alpha \mapsto \Theta^{(\Theta^\alpha)}$. \square

Putting this together with the known upper bound for Σ we thus get that $\Theta^{(\Theta^\Theta)} \leq \Sigma < \Theta^+$. It would be desirable to improve the lower and upper bounds to calculate the exact value of Σ . We close the paper with a list of concrete open questions:

Question 28. Is there an operation on weak norms acting like ordinal exponentiation?

To the best of our knowledge, the answer to this question is unknown even for the Wadge hierarchy. If this is the case (in fact, it would be enough to have an operation that dominates ordinal exponentiation for a $<_{\mathbb{N}}$ -unbounded set of norms), then Σ is closed under ordinal exponentiation. Using the ideas of the proofs of Proposition 26 and Theorem 27, we would get $\Sigma \geq \varepsilon_{\Theta+\Theta}$ (the $(\Theta + \Theta)$ th epsilon-number).

Question 29. If $\Sigma' := \text{otyp}(\Theta^{\mathbb{R}}/\equiv_{\mathbb{N}}, \leq_{\mathbb{N}})$, then we had observed that $\Sigma' = \Sigma$ if and only if Θ is regular (cf. Lemma 1). Can we say more about the relationship between Σ and Σ' under the assumption $\text{AD} + \text{DC} + \text{“}\Theta \text{ is singular”}$?

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