Normal Measures and Strongly Compact Cardinals *†

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Abstract

We prove four theorems concerning the number of normal measures a non-$(\kappa + 2)$-strong strongly compact cardinal κ can carry.

1 Introduction and Preliminaries

We consider in this paper the number of normal measures a non-$(\kappa + 2)$-strong strongly compact cardinal κ can carry. It follows from a theorem of Solovay [6, Corollary 20.20(i)] that if κ is $(\kappa+2)$-strong, then κ is a measurable limit of measurable cardinals with 2^{2^κ} many normal measures over

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κ, the maximal number of normal measures a measurable cardinal can have. It is known, however, that there can be strongly compact cardinals κ which are not $(\kappa+2)$-strong. A result of Menas [11, Theorem 2.21] shows that if κ is a measurable limit of strongly compact cardinals (which might or might not also be supercompact), then κ itself must be strongly compact. By the arguments of [11, Theorem 2.22], the smallest such κ cannot be $(\kappa+2)$-strong. In addition, Magidor's famous theorem of [9] establishes that it is consistent, relative to the existence of a strongly compact cardinal, for the least strongly compact cardinal κ to be the least measurable cardinal. Under these circumstances, by the previously mentioned theorem of Solovay, κ also cannot be $(\kappa + 2)$-strong. The work of Menas and Magidor therefore raises the following

Question: Suppose κ is a strongly compact cardinal which is not $(\kappa+2)$-strong. How many normal measures is it consistent for κ to carry?

In trying to provide answers to this question, we will begin by examining what occurs when the strongly compact cardinals being considered are either the least measurable limit of supercompact cardinals or the least measurable cardinal. Specifically, we will prove the following two theorems.

Theorem 1 *Suppose $V \models$ "ZFC + GCH + κ is the least measurable limit of supercompact cardinals + $\lambda \geq \kappa^{++}$ is a regular cardinal". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + κ is the least measurable limit of supercompact cardinals + $2^\kappa = \kappa^+$ + $2^{\kappa^+} = 2^{2^\kappa} = \lambda$ + κ carries 2^{2^κ} many normal measures".*

Theorem 2 *Suppose $V \models$ "ZFC + GCH + κ is supercompact + $\lambda \geq \kappa^{++}$ is a regular cardinal". There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + κ is both the least measurable and least strongly compact cardinal + $2^\kappa = \kappa^+$ + $2^{\kappa^+} = 2^{2^\kappa} = \lambda$ + κ carries 2^{2^κ} many normal measures".*

Theorems 1 and 2 handle the case where the non-$(\kappa+2)$-strong strongly compact cardinal κ in question carries 2^{2^κ} many normal measures and $2^\kappa = \kappa^+$. We may also ask if it is possible to have $2^\kappa > \kappa^+$. The next two theorems take care of this situation for certain non-$(\kappa+2)$-strong strongly compact cardinals κ. Specifically, we will also prove the following two theorems.

Theorem 3 *Con(ZFC + There is a supercompact limit of supercompact cardinals)* $\Longrightarrow$ *Con(ZFC + There is a strongly compact cardinal* κ *which is not the least measurable limit of supercompact cardinals but is both a measurable limit of supercompact cardinals and is not* $(\kappa+2)$*-strong +* $2^\kappa = \kappa^{+17}$ *+* $2^{\kappa^{+17}} = 2^{2^\kappa} = \kappa^{+95}$ *+* κ *carries* 2^{2^κ} *many normal measures).*

Theorem 4 *Con(ZFC + There is a supercompact limit of supercompact cardinals)* $\Longrightarrow$ *Con(ZFC + The least strongly compact cardinal* κ *is a limit of measurable cardinals but is not* $(\kappa+2)$*-strong +* $2^\kappa = \kappa^{+17}$ *+* $2^{\kappa^{+17}} = 2^{2^\kappa} = \kappa^{+95}$ *+* κ *carries* 2^{2^κ} *many normal measures).*

In Theorems 3 and 4, there is nothing special about "κ^{+17}" and "κ^{+95}". They stand for any "reasonably definable cardinals appropriate for reverse Easton iterations" (i.e., values of an Easton function defined as in [10, Theorem, Section 18, pages 83–88]). Also, Theorems 3 and 4 are in some ways "weaker" than Theorems 1 and 2. This is in the sense that in Theorem 3, unlike in Theorem 1, κ is not the least measurable limit of supercompact cardinals. In Theorem 4, unlike in Theorem 2, κ is not the least measurable cardinal. We will discuss this further towards the end of the paper.

Before beginning the proofs of our theorems, we briefly discuss some preliminary information. Essentially, our notation and terminology are standard. When exceptions occur, these will be clearly noted. In particular, when forcing, $q \geq p$ means that q *is stronger than* p. If $\mathbb{P}$ is a notion of forcing for the ground model V and G is V-generic over $\mathbb{P}$, then we will abuse notation somewhat by using both $V[G]$ and $V^{\mathbb{P}}$ to denote the generic extension when forcing with $\mathbb{P}$. We will also occasionally abuse notation by writing x when we actually mean $\dot{x}$ or $\check{x}$.

Suppose κ is a regular cardinal. As in [7], we will say that $\mathbb{P}$ is κ*-directed closed* if every directed subset of $\mathbb{P}$ of size less than κ has an upper bound. We note that for λ any ordinal, $\mathrm{Add}(\kappa, \lambda)$, the standard partial ordering for adding λ many Cohen subsets of κ, is κ-directed closed.

We presume a basic knowledge and understanding of large cardinals and forcing, as found in, e.g., [6]. We do mention that the cardinal κ is $(\kappa+2)$*-strong* if there is an elementary embedding $j : V \to M$ having critical point κ such that $V_{\kappa+2} \subseteq M$. It is the case that if κ is supercompact, then κ is $(\kappa+2)$-strong (and much more).

A corollary of Hamkins' work on gap forcing found in [4, 5] will be employed in the proof of

Theorem 1. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [4, 5] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ^+-directed closed". In Hamkins' terminology of [4, 5], $\mathbb{P}$ *admits a gap at* δ. Also, as in the terminology of [4, 5] and elsewhere, an embedding $j : \overline{V} \to \overline{M}$ is *amenable to* $\overline{V}$ when $j \restriction A \in \overline{V}$ for any $A \in \overline{V}$. The specific corollary of Hamkins' work from [4, 5] we will be using is then the following.

Theorem 5 (Hamkins) *Suppose that $V[G]$ is a generic extension obtained by forcing with $\mathbb{P}$ that admits a gap at some regular $\delta < \kappa$. Suppose further that $j : V[G] \to M[j(G)]$ is an elementary embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^\delta \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V.*

A consequence of Theorem 5 is that if $\mathbb{P}$ admits a gap at some regular $\delta < \kappa$ and κ is either supercompact or measurable in $V^{\mathbb{P}}$, then κ is supercompact or measurable in V as well.

2 The Proofs of Theorems 1 – 4

We turn now to the proofs of our theorems.

Proof: To prove Theorem 1, let $V \vDash$ "ZFC + GCH + κ is the least measurable limit of supercompact cardinals". Without loss of generality, by doing a preliminary forcing as in [1] if necessary, we assume in addition that $V \vDash$ "Every supercompact cardinal $\delta < \kappa$ has its supercompactness indestructible under δ-directed closed forcing + GCH holds at and above κ".

Let $V_1 = V^{\mathrm{Add}(\kappa^+, \lambda)}$. Because $\mathrm{Add}(\kappa^+, \lambda)$ is κ^+-directed closed, V_1 and V contain the same subsets of κ. Thus, $V_1 \vDash$ "κ is measurable". In addition, standard arguments show that $V_1 \vDash$ "$2^\kappa = \kappa^+ + 2^{\kappa^+} = 2^{2^\kappa} = \lambda$". Further, if $V \vDash$ "$\delta < \kappa$ is supercompact", then because $V \vDash$ "δ has its supercompactness indestructible under δ-directed closed forcing", $V_1 \vDash$ "δ is supercompact". We may consequently infer that $V_1 \vDash$ "κ is a measurable limit of supercompact cardinals". To show κ is in fact the least measurable limit of supercompact cardinals in V_1, observe that the closure properties of $\mathrm{Add}(\kappa^+, \lambda)$ tell us forcing with $\mathrm{Add}(\kappa^+, \lambda)$ creates no new measurable cardinals below κ. To see that forcing with $\mathrm{Add}(\kappa^+, \lambda)$ creates no new supercompact cardinals below κ, let $\delta < \gamma < \kappa$ be such that $V \models$ "δ is not supercompact but γ is supercompact". If $V_1 \models$ "δ is supercompact", then again by the closure properties of $\mathrm{Add}(\kappa^+, \lambda)$, it must be the case that $V \models$ "δ is η supercompact for every $\eta < \gamma$". Since $V \models$ "γ is supercompact", the argument found in [3, page 31, paragraph 4] tells us that $V \models$ "δ is supercompact", a contradiction.

We now know that $V_1 \models$ "ZFC + κ is the least measurable limit of supercompact cardinals + $2^\kappa = \kappa^+$ + $2^{\kappa^+} = 2^{2^\kappa} = \lambda$". Working in V_1, let $\mathbb{P}^*$ be the (possibly proper class) reverse Easton iteration which begins by forcing with $\mathrm{Add}(\omega, 1)$ and then does nontrivial forcing only at inaccessible cardinals δ which are not limits of inaccessible cardinals, where the forcing done is $\mathrm{Add}(\delta, 1)$.[1] Standard arguments (see, e.g., the proof of [1, Theorem]) show that every V_1-supercompact cardinal is preserved to $V_2 = V_1^{\mathbb{P}^*}$. Since forcing with $\mathbb{P}^*$ does not change either cofinalities or the size of power sets, $V_2 \models$ "$2^\kappa = \kappa^+$ + $2^{\kappa^+} = 2^{2^\kappa} = \lambda$". In addition, if we write $\mathbb{P}^* = \mathbb{P}_\kappa * \dot{\mathbb{Q}}$, it is the case that $\Vdash_{\mathbb{P}_\kappa}$ "$\dot{\mathbb{Q}}$ is (at least) $(2^\kappa)^+$-directed closed". This means that the number of normal measures κ carries is the same in both $V_2 = V_1^{\mathbb{P}^*} = V_1^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}}$ and $V_1^{\mathbb{P}_\kappa}$. Therefore, since $\mathbb{P}_\kappa \subseteq V_\kappa$, $|\mathbb{P}_\kappa| = \kappa$, and every $p \in \mathbb{P}_\kappa$ has at least two incompatible extensions, by [2, Lemma 1.1], κ is a measurable cardinal carrying 2^{2^κ} many normal measures not concentrating on measurable cardinals in both $V_1^{\mathbb{P}_\kappa}$ and V_2.

The proof of Theorem 1 will consequently be finished if we can show that in V_2, κ is the least measurable limit of supercompact cardinals. To do this, note that it is possible to write $\mathbb{P}^* = \mathrm{Add}(\omega, 1) * \dot{\mathbb{R}}$, where $\Vdash_{\mathrm{Add}(\omega,1)}$ "$\dot{\mathbb{R}}$ is (at least) $\aleph_2$-directed closed". By Theorem 5, this means that forcing with $\mathbb{P}^*$ creates no new measurable or supercompact cardinals. Since the work of the preceding paragraph yields that $V_2 \models$ "κ is a measurable limit of supercompact cardinals" and $V_1 \models$ "κ is the least measurable limit of supercompact cardinals", it consequently follows that $V_2 \models$ "κ is the least measurable limit of supercompact cardinals" as well. By taking $\mathbb{P} = \mathrm{Add}(\kappa^+, \lambda) * \dot{\mathbb{P}}^*$, the proof of Theorem 1 has been completed.

[1] $\mathbb{P}^*$ is a proper class if there are class many inaccessible cardinals, but is a set otherwise.

$\square$

Theorem 2 is proven similarly. Let $V \models$ "ZFC + GCH + κ is supercompact". Without loss of generality, by first doing the forcing of [7], we assume in addition that $V \models$ "κ's supercompactness is indestructible under κ-directed closed forcing + GCH holds at and above κ". If we as before let $V_1 = V^{\mathrm{Add}(\kappa^+, \lambda)}$, as in the proof of Theorem 1, it is then the case that $V_1 \models$ "ZFC + κ is supercompact + $2^\kappa = \kappa^+$ + $2^{\kappa^+} = 2^{2^\kappa} = \lambda$". Working in V_1, again as in the proof of Theorem 1, let $\mathbb{P}^*$ be the (possibly proper class) reverse Easton iteration which begins by forcing with $\mathrm{Add}(\omega, 1)$ and then does nontrivial forcing only at inaccessible cardinals δ which are not limits of inaccessible cardinals, where the forcing done is $\mathrm{Add}(\delta, 1)$.[2] By the same arguments as in the proof of Theorem 1, in $V_2 = V_1^{\mathbb{P}^*}$, it is the case that $V_2 \models$ "ZFC + κ is supercompact + $2^\kappa = \kappa^+$ + $2^{\kappa^+} = 2^{2^\kappa} = \lambda$ + κ carries 2^{2^κ} many normal measures not concentrating on measurable cardinals".

Now, working in V_2, let $\mathbb{Q}$ be the Magidor iteration of Prikry forcing [9] which adds a cofinal ω sequence to each measurable cardinal below κ, with $V_3 = V_2^{\mathbb{Q}}$. As in [9], $V_3 \models$ "κ is both the least strongly compact and least measurable cardinal". Since by the proof of [9, Lemma 4.4], forcing with $\mathbb{Q}$ neither collapses any cardinals nor changes the size of any power sets, $V_3 \models$ "$2^\kappa = \kappa^+$ + $2^{\kappa^+} = 2^{2^\kappa} = \lambda$". Because $V_2 \models$ "κ carries 2^{2^κ} many normal measures not concentrating on measurable cardinals", by the proof of [9, Theorem 2.5] and the fact forcing with $\mathbb{Q}$ collapses no cardinals, $V_3 \models$ "κ carries 2^{2^κ} many normal measures" as well. V_3 is thus as desired. By taking $\mathbb{P} = \mathrm{Add}(\kappa^+, \lambda) * \dot{\mathbb{P}}^* * \dot{\mathbb{Q}}$, the proof of Theorem 2 has been completed.

$\square$

To prove Theorem 3, let $V \models$ "κ is the least supercompact limit of supercompact cardinals". Assume without loss of generality that a reverse Easton iteration has been done as in [10, Theorem, Section 18, pages 83–88] so that in addition, $V \models$ "For every inaccessible cardinal δ, $2^\delta = \delta^{+17}$ and $2^{\delta^{+17}} = 2^{2^\delta} = \delta^{+95}$". By assuming $j(\kappa)$ has been chosen to be minimal, in analogy to the proof of [11, Proposition 2.7], let $j : V \to M$ be an elementary embedding witnessing the $(\kappa+2)$-strongness

[2]Strictly speaking, unlike the proof of Theorem 1, the proof of Theorem 2 does not require that $\mathbb{P}^*$ be defined by starting by forcing with $\mathrm{Add}(\omega, 1)$. This is since no use of Theorem 5 is made, so there is no need to introduce a gap at $\aleph_1$. However, for uniformity in presentation, the same definition of $\mathbb{P}^*$ is used.

of κ such that $M \models$ "κ is not $(\kappa+2)$-strong". We will show that in M, κ is our desired strongly compact cardinal.

To do this, since κ is the critical point of j, for any $\delta < \kappa$ such that $V \models$ "δ is supercompact", $M \models$ "$j(\delta) = \delta$ is supercompact" as well. In addition, because $V_{\kappa+2} \subseteq M$ and a measure over κ is a member of $V_{\kappa+2}$, M contains every (normal or non-normal) measure over κ. Therefore, since $V \models$ "κ carries 2^{2^κ} many normal measures as κ is $(\kappa+2)$-strong", $M \models$ "κ is measurable and carries 2^{2^κ} many normal measures". Because V and M are elementarily equivalent, $M \models$ "$2^\kappa = \kappa^{+17}$ and $2^{\kappa^{+17}} = 2^{2^\kappa} = \kappa^{+95}$". Since $M \models$ "κ is a measurable limit of supercompact cardinals", by Menas' theorem of [11], $M \models$ "κ is strongly compact". Putting the above together, we now have that $M \models$ "κ is a strongly compact cardinal which is a measurable limit of supercompact cardinals and is not $(\kappa+2)$-strong + $2^\kappa = \kappa^{+17}$ + $2^{\kappa^{+17}} = 2^{2^\kappa} = \kappa^{+95}$ + κ carries 2^{2^κ} many normal measures". By reflection, $A = \{\delta < \kappa \mid \delta$ is a measurable limit of supercompact cardinals which is not $(\delta+2)$-strong, $2^\delta = \delta^{+17}$, $2^{\delta^{+17}} = 2^{2^\delta} = \delta^{+95}$, and δ carries 2^{2^δ} many normal measures$\}$ is unbounded in κ in V. Since for any $\delta \in A$, $M \models$ "$j(\delta) = \delta$ is a measurable limit of supercompact cardinals", κ is not the least measurable limit of supercompact cardinals in either V or M. This completes the proof of Theorem 3.

$\square$

Turning now to the proof of Theorem 4, we will use the cardinal κ witnessing the conclusions of Theorem 3 in our proof. First, let us observe that in M, since $\kappa < j(\kappa)$ and $M \models$ "$j(\kappa)$ is the least supercompact limit of supercompact cardinals", κ is below the least supercompact limit of supercompact cardinals. Keeping this in mind, we take M as our ground model. Let $\mathbb{P}$ be the Magidor iteration of Prikry forcing [9] which adds a cofinal ω sequence to each supercompact cardinal below κ. By [9, Theorem 3.4], $M \models$ "κ is strongly compact". Because V and M are elementarily equivalent, $M \models$ "For every inaccessible cardinal δ, $2^\delta = \delta^{+17}$". Consequently, since the supercompact cardinals are unbounded in κ in M, as in the proof of [9, Theorem 4.5], $M^{\mathbb{P}} \models$ "There are unboundedly in κ many singular strong limit cardinals violating GCH". By Solovay's theorem of [12], this means we may now infer that $M^{\mathbb{P}} \models$ "No cardinal $\delta < \kappa$ is strongly compact", i.e.,

$M^{\mathbb{P}} \models$ "κ is the least strongly compact cardinal".

For any M-measurable cardinal δ which is not supercompact, write $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{R}}$. By the definition of $\mathbb{P}$, we have that $\Vdash_{\mathbb{P}_\delta}$ "Forcing with $\dot{\mathbb{R}}$ adds no new subsets of 2^δ". If $|\mathbb{P}_\delta| < \delta$, then by the Lévy-Solovay results [8], $M^{\mathbb{P}_\delta} \models$ "δ is measurable". If $|\mathbb{P}_\delta| = \delta$, then by [9, Theorem 2.5], it again follows that $M^{\mathbb{P}_\delta} \models$ "δ is measurable". It is thus the case that $M^{\mathbb{P}_\delta * \dot{\mathbb{R}}} = M^{\mathbb{P}} \models$ "δ is measurable" as well. In addition, because κ is in M a limit of supercompact cardinals, there are in M unboundedly many in κ measurable cardinals which are not supercompact. Consequently, we may now infer that $M^{\mathbb{P}} \models$ "κ is a limit of measurable cardinals".

Because κ is below the least supercompact limit of supercompact cardinals, no normal measure over κ concentrates on supercompact cardinals.[3] Consequently, as in the last paragraph of the proof of Theorem 2, we may now infer that $M^{\mathbb{P}} \models$ "κ carries 2^{2^κ} many normal measures", as well as $M^{\mathbb{P}} \models$ "$2^\kappa = \kappa^{+17} + 2^{\kappa^{+17}} = 2^{2^\kappa} = \kappa^{+95}$". We have therefore completed the proof of Theorem 4, unless it also happens to be true that $M^{\mathbb{P}} \models$ "κ is $(\kappa+2)$-strong". If this is the case, then let $j : M^{\mathbb{P}} \to N^{j(\mathbb{P})}$ be an elementary embedding witnessing the $(\kappa+2)$-strongness of κ such that $N^{j(\mathbb{P})} \models$ "κ is not $(\kappa+2)$-strong". Note that $j(\mathbb{P})$ factors as $\mathbb{P} * \dot{\mathbb{Q}}$. We will show that in $N^{\mathbb{P}}$, κ is our desired strongly compact cardinal.

We first show that $N \models$ "κ is a measurable limit of supercompact cardinals" (and hence is strongly compact in N, by Menas' theorem from [11]). To do this, consider $j \restriction M : M \to N$, which is still an elementary embedding having critical point κ. As before, for any $\delta < \kappa$ such that $M \models$ "δ is supercompact", $N \models$ "$j(\delta) = \delta$ is supercompact". Thus, since $M \models$ "κ is a limit of supercompact cardinals", $N \models$ "κ is a limit of supercompact cardinals". Further, exactly as in the proof of Theorem 3, because $M^{\mathbb{P}} \models$ "κ is a measurable cardinal carrying 2^{2^κ} many normal measures" and j is an elementary embedding witnessing that κ is $(\kappa+2)$-strong in $M^{\mathbb{P}}$, $N^{j(\mathbb{P})} \models$ "κ is a measurable cardinal carrying 2^{2^κ} many normal measures" as well. In N, as $\Vdash_{\mathbb{P}}$ "Forcing with $\dot{\mathbb{Q}}$ adds no new

[3] This is since otherwise, if μ were a normal measure over κ concentrating on supercompact cardinals, with $j_\mu : M \to N$ the associated elementary embedding, then $N \models$ "κ is supercompact". Further, as j has critical point κ, for any $\delta < \kappa$ such that $M \models$ "δ is supercompact", $N \models$ "$j_\mu(\delta) = \delta$ is supercompact". Since the supercompact cardinals are unbounded in κ in M, this means that $N \models$ "κ is a supercompact limit of supercompact cardinals". By reflection, the set of supercompact limits of supercompact cardinals is unbounded below κ in M. This contradicts that in M, κ is below the least supercompact limit of supercompact cardinals.

subsets of 2^κ, preserves all cardinals, and does not change the size of power sets", $N^{\mathbb{P}} \vDash$ "κ is a measurable cardinal carrying 2^{2^κ} many normal measures". Therefore, since by [9, Theorem 3.1], forcing with $\mathbb{P}$ creates no new measurable cardinals, $N \vDash$ "κ is a measurable cardinal".

We now know that $N \vDash$ "κ is strongly compact and is a limit of supercompact cardinals". In addition, by elementarity, it is again the case that $N \vDash$ "For every inaccessible cardinal δ, $2^\delta = \delta^{+17}$". This means that as in the first two paragraphs of the proof of this theorem, we may infer that $N^{\mathbb{P}} \vDash$ "κ is the least strongly compact cardinal and is a limit of measurable cardinals". Because M and N are elementarily equivalent and forcing with $\mathbb{P}$ preserves both cardinals and the size of power sets, $N^{\mathbb{P}} \vDash$ "$2^\kappa = \kappa^{+17} + 2^{\kappa^{+17}} = 2^{2^\kappa} = \kappa^{+95}$". In N, since $\Vdash_{\mathbb{P}}$ "Forcing with $\dot{\mathbb{Q}}$ adds no new subsets of the least inaccessible cardinal above κ" and $N^{j(\mathbb{P})} \vDash$ "κ is not $(\kappa + 2)$-strong", $N^{\mathbb{P}} \vDash$ "κ is not $(\kappa + 2)$-strong" as well. Since we have already seen that $N^{\mathbb{P}} \vDash$ "κ carries 2^{2^κ} many normal measures", this completes the proof of Theorem 4.

□

3 Concluding Remarks

We conclude with a few observations. As we remarked in Section 1, the non-$(\kappa+2)$-strong strongly compact cardinals κ witnessing the conclusions of Theorems 3 and 4 are neither the least measurable limit of supercompact cardinals nor the least measurable cardinal. This is since the methods used in the proofs of Theorems 1 and 2 do not seem to be adaptable to the situation where $2^\kappa > \kappa^+$. The reason is that in the proofs of Theorems 1 and 2, we need to know the partial ordering $\mathbb{P}_\kappa$ of Theorem 1 increases the number of normal measures over the strongly compact cardinal κ in question not concentrating on measurable cardinals to 2^{2^κ}. In order to show that this is indeed the case, as the proof of [2, Lemma 1.1] indicates, we have to be able to construct a generic object for a certain κ^+-directed closed partial ordering $\mathbb{Q}$ by meeting all of the dense open subsets (or maximal antichains) of $\mathbb{Q}$ present in a generic extension $M[G]$ of a κ-closed inner model M of the ground model V. Here, G is V-generic over $\mathbb{P}_\kappa$, $\mathbb{Q} \in M[G] \subseteq V[G]$, the construction takes place in $V[G]$, and $j : V \to M$ is an ultrapower embedding generated by a normal measure over κ not concentrating on measurable cardinals. If $2^\kappa = \kappa^+$, then this is not a problem, since $M[G] \models$ "$|\mathbb{Q}| = j(\kappa)$", $M[G]$ remains κ-closed with respect to $V[G]$, and we must only meet $|j(\kappa^+)| = |2^{j(\kappa)}| = |\{f \mid f : \kappa \to \kappa^+\}| = |[\kappa^+]^\kappa| = 2^\kappa = \kappa^+$ many dense open subsets. We can do this by letting $\langle D_\alpha \mid \alpha < \kappa^+ \rangle$ enumerate in $V[G]$ all of the dense open subsets of $\mathbb{Q}$ present in $M[G]$ and defining via an induction of length κ^+ an increasing sequence $\langle p_\alpha \mid \alpha < \kappa^+ \rangle$ of members of $\mathbb{Q}$ such that $p_\alpha \in D_\alpha$. Because $\mathbb{Q}$ is κ^+-directed closed, there is no problem whatsoever in achieving this goal. However, if $2^\kappa > \kappa^+$, then the preceding calculation of $|2^{j(\kappa)}|$ and hence the number of dense open subsets of $\mathbb{Q}$ present in $M[G]$ yields some $\lambda \geq \kappa^{++}$. Building the generic object for $\mathbb{Q}$ via the preceding induction does not work, as there are λ many dense open subsets which must be met. The construction will break down at stage κ^+, because $\mathbb{Q}$ is only κ^+-directed closed. It is not at all clear at the moment how to overcome this obstacle.

Theorems 1 – 4 only barely scratch the surface of what we feel is possible for non-$(\kappa+2)$-strong strongly compact cardinals κ. We finish by making this precise via the following

Conjecture: For any non-$(\kappa + 2)$-strong strongly compact cardinal κ (such as the ones considered earlier), it is relatively consistent for κ to carry exactly δ many normal measures. Here, $1 \leq \delta \leq 2^{2^\kappa}$ is any cardinal, and the values of both 2^κ and 2^{2^κ} can be freely manipulated in a way compatible with the value of δ. In particular, it is relatively consistent to have a non-$(\kappa + 2)$-strong strongly compact cardinal κ which carries exactly 1, 2, 3, 98, $\aleph_{64}$, δ for δ the least inaccessible cardinal, κ^{+99}, etc. many normal measures, with arbitrary values for either 2^κ or 2^{2^κ} which are compatible with δ many normal measures over κ.

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