

FORCING THE TRUTH OF A WEAK FORM OF SCHANUEL'S CONJECTURE

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ABSTRACT. Schanuel's conjecture states that the transcendence degree over \mathbb{Q} of the $2n$ -tuple $(\lambda_1, \dots, \lambda_n, e^{\lambda_1}, \dots, e^{\lambda_n})$ is at least n for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ which are linearly independent over \mathbb{Q} ; if true it would settle a great number of elementary open problems in number theory, among which the transcendence of e over π .

Wilkie [8], and Kirby [3, Theorem 1.2] have proved that there exists a smallest countable algebraically and exponentially closed subfield K of \mathbb{C} such that Schanuel's conjecture holds relative to K (i.e. \mathbb{Q} is replaced by K in the statement of Schanuel's conjecture). We prove a slightly weaker result (i.e. that there exists such a countable field K without specifying that there is a smallest such) using the forcing method and Shoenfield's absoluteness theorem.

This result suggests that forcing can be a useful tool to prove theorems (rather than independence results) and to tackle problems in domains which are apparently quite far apart from set theory.

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A BRIEF INTRODUCTION

We want to give an example of how we might use forcing to study a variety of expansions of the complex (or real) numbers enriched by *arbitrary* Borel predicates, still maintaining certain “tameness” properties of the theory of these expansions. We clarify what we intend by “tameness” as follows: in contrast with what happens for example with \mathcal{o} -minimality in the case of real closed fields, we do not have to bother much with the complexity of the predicate P we wish to add to the real numbers (we can allow P to be an arbitrary Borel predicate), but we pay a price reducing significantly the variety of elementary superstructures (M, P_M) for which we are able to lift P to P_M so that $(\mathbb{R}, P) \prec (M, P_M)$ and for which we are able to use the forcing method to say something significant on the first order theory of (M, P_M) . Nonetheless the family of superstructures M for which this is possible is still a large class, as we can combine (Woodin and) Shoenfield's absoluteness for the theory of projective sets of reals with a duality theorem relating certain spaces of functions to forcing constructions, to obtain the following:

Theorem 1 (Folklore?— V. and Vaccaro [7]). *Let X be an extremally (extremely) disconnected compact space.*

Let $C^+(X)$ be the space of continuous functions $f : X \rightarrow \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ such that the preimage of ∞ is nowhere dense (\mathbb{S}^2 is the one point compactification of \mathbb{C}).

Given any Borel predicate R on \mathbb{C}^n , there is a predicate $R_X \subseteq C^+(X)^n \times X$ (equivalently a boolean predicate $R_X : C^+(X)^n \rightarrow \text{CL}(X)$ where $\text{CL}(X)$ is the boolean algebra given by clopen subsets of X) such that for all $p \in X$

$$(\mathbb{C}, R) \prec_{\Sigma_2} (C^+(X)/p, R_X/p),$$

where $C^+(X)/p$ is the ring of germs in p of functions in $C^+(X)$, and $R_X/p([f_1], \dots, [f_n])$ holds if there is a neighborhood U of p such that $R(f_1(x), \dots, f_n(x))$ holds on a dense set of $x \in U$.

Moreover if we assume the existence of class many Woodin cardinals we get that

$$(\mathbb{C}, R) \prec (C^+(X)/p, R_X/p).$$

It turns out that the above spaces of functions are intrinsically intertwined with the forcing method: they provide an equivalent description of the forcing names for complex numbers for the notion of forcing given by the non-empty clopen subsets of X . Moreover these spaces are universal among the spaces of the form $C^+(Y)$ with Y compact Hausdorff, in the sense that for any such Y there is an isometric $*$ -homomorphism of the unital C^* -algebra $C(Y)$ into a C^* -algebra of the form $C(X)$ with X compact and extremely disconnected; this homomorphism extends to a $*$ -monomorphism of the ring $C^+(Y)$ into the ring $C^+(X)$ (we refer the reader to [7, Chapter 4] for more details).

Playing with the choice of the compact space X and of the Borel predicate R we can cook up spaces in which it is possible to compute the solution of certain projective statements. Using the elementarity of these structures with respect to the standard complex numbers, we can conclude that the solution we computed in these expansions is the correct solution. This is exactly what we plan to do in the following for a weakening of the well known Schanuel's conjecture.

1. MAIN RESULT

For a vector $\vec{v} = (v_1, \dots, v_n)$ and a function E we let $\vec{v}(c) = (v_1(c), \dots, v_n(c))$ if each v_i is a function and c is in the domain. $E(\vec{v}) = (E(v_1), \dots, E(v_n))$ if each v_i is in the domain of E .

Definition 1.1. Given rings $K \subset R$ and $\vec{\lambda} \in R^n$,

- $\text{Ldim}_K(\vec{\lambda})$ is the linear dimension of the K -module spanned by $\vec{\lambda}$.
- $\text{Ldim}_K(\vec{\lambda}/Y)$ is the linear dimension over K of the K -module which is the quotient of the K -module spanned by $\vec{\lambda} \cup Y$ and the K -module spanned by Y .
- $\text{Trdg}_K(\vec{\lambda})$ is the transcendence degree over K of the ring $K[\vec{\lambda}] \subset R$, i.e. the largest size of a subset A of $\vec{\lambda}$ such that no polynomial with coefficients in K and $|A|$ -many variables vanishes on the elements of the subset.
- Let $(F, +, \cdot, 0, 1)$ be a field and $E : F \rightarrow F^*$ be an homomorphism of the additive group $(F, +)$ on the multiplicative group (F^*, \cdot) . Let

$$Z(F) = \{a \in F : \forall x E(x) = 1 \rightarrow E(ax) = 1\}.$$

Then $Z(F)$ is a ring.

Given a field K with $Z(F) \subseteq K \subseteq F$:

- The *Ax character* of the pair (E, K) is the function:

$$\text{AC}_{E,K}(\vec{\lambda}) = \text{Trdg}_K(\vec{\lambda}, E(\vec{\lambda})) - \text{Ldim}_{Z(F)}(\vec{\lambda}/K).$$

- The *Schanuel character* of the pair (E, K) is the function:

$$\text{SC}_{E,K}(\vec{\lambda}) = \text{Trdg}_K(\vec{\lambda}, E(\vec{\lambda})) - \text{Ldim}_K(\vec{\lambda}).$$

1.1. Exponential fields. We introduce axioms suitable to formulate our results on Zilber conjectures on the exponential function relative to some algebraically closed field K .

Definition 1.2. Consider a language for algebraically closed fields augmented by predicate symbols for an exponential map E , and for a special sub-field K .

$$(F, K, E, \cdot, +, 0, 1)$$

is a model of $T_{\text{WSP}(K)}$ if it satisfies¹:

- (1) **AC FIELD:** F is an algebraically closed field of characteristic 0.
- (2) **EXP FIELD:** The exponential map $E : F \rightarrow F^*$ is a surjective homomorphism of the additive group $(F, +)$ into the multiplicative group (F^*, \cdot) with

$$\ker(E) = \omega \cdot Z(F) = \{\omega \cdot \lambda : \lambda \in Z(F)\}$$

for some $\omega \in F$ transcendental over $Z(F)$ ([4, Axioms 2'a, 2'b, Section 1.2]).

- (3) **K -SP (Schanuel property for K):** $K \subseteq F$ is a field containing $Z(F)$ and $\text{SC}_{E,K} : F^{<\omega} \rightarrow \mathbb{N}$ cannot get negative values.

An exponential field is a pair (F, E) satisfying the field axioms and axiom (2).

Zilber [9] showed that there is a natural axiom system T_{Zilber} expanding $T_{\text{WSP}(\mathbb{Q})}$ and axiomatizable in the logic $L_{\omega_1, \omega}(Q)$ (where Q stands for the quantifier for uncountably many elements) such that for each uncountable cardinal κ there is exactly one field \mathbb{B} and one exponential function $E : \mathbb{B} \rightarrow \mathbb{B}^*$ with $\ker(E) = \omega \cdot \mathbb{Z}$ for some $\omega \in \mathbb{B}$ transcendental over \mathbb{Q} and such that $(\mathbb{B}, \mathbb{Q}, E, +, 0, 1)$ is a standard model of T_{Zilber} . Roughly T_{Zilber} extends $T_{\text{WSP}(\mathbb{Q})}$ requiring that certain kinds of irreducible algebraic varieties V (the so called *rotund (or normal) and free* irreducible varieties on $F^n \times (F^*)^n$) admit F_0 -generic points² of the form $(x_1, \dots, x_n, E(x_1), \dots, E(x_n))$, where $F_0 \subseteq \mathbb{B}$ is a finitely generated field containing a set of generators for the prime ideal defining the irreducible variety V . Moreover the axioms require that the number of such generic points is at most countable for the irreducible rotund varieties of dimension n (see [MR2102856 \(2006a:03051\)](#) for a short account of the axiom system). Zilber's conjecture is that $(\mathbb{C}, \mathbb{Q}, e^x, +, \cdot)$ is a model of T_{Zilber} .

We shall give a proof based on forcing and generic absoluteness of the following:

Theorem 1.3 (Kirby [3], Wilkie [8]). *There exists a countable (algebraically closed) field $K_0 \subseteq \mathbb{C}$ such that $(\mathbb{C}, K_0, e^x, +, \cdot)$ is a model of $T_{\text{WSP}(K_0)}$.*

Actually what we will prove is the following:

¹The axioms we introduce are mostly taken from [4, Section 1.2], specifically axiom (2) corresponds to axioms 2'a and 2'b of [4, Section 1.2], we do not insist on the axiom 2'c, while axiom (3) is a variation of the axiom 3' of [4, Section 1.2]. In order to be fully consistent with their axiomatization the Schanuel character in axiom (3) should be replaced by the ‘‘predimension’’ function $\Delta_K(\vec{\lambda}) = \text{Trdg}_{Z(F)}(\vec{\lambda}, E(\vec{\lambda})/K) - \text{Ldim}_{Z(F)}(\vec{\lambda}/K)$. Nonetheless the fields $K \subseteq F$ we will look at are such that $Z(F) \cup \ker(E) \subseteq K$ and it can be checked that for these fields $\text{SC}_K(\vec{\lambda}) \geq \text{AC}_K(\vec{\lambda}) - 1$ while $\Delta_K(\vec{\lambda}) \geq \text{AC}_K(\vec{\lambda})$. In our analysis we will focus on the properties of the function AC_K , however we chose to formulate the Schanuel property at K in terms of the function SC_K to make transparent the correspondence with Schanuel's conjecture.

²Generic according to [6, Def. 1.3].

Theorem 1.4. *There exists a countable (algebraically and exponentially closed) field $K_0 \subseteq \mathbb{C}$ such that*

$$\text{AC}_{K_0, \text{exp}}(\vec{\lambda}) > 0$$

for all $\vec{\lambda} \in \mathbb{C}^{<\mathbb{N}}$ (where $\text{exp}(\lambda) = e^\lambda$).

The first theorem will be an immediate corollary of the second one by the following elementary argument: Assume $\lambda_1, \dots, \lambda_n = \vec{\lambda}$ are K_0 -linearly independent. Then either $\lambda_1, \dots, \lambda_{n-1}$ are \mathbb{Z} -linearly independent modulo K_0 or $\lambda_2, \dots, \lambda_n$ are \mathbb{Z} -linearly independent modulo K_0 . Assume $\lambda_1, \dots, \lambda_{n-1}$ are \mathbb{Z} -linearly independent modulo K_0 . By the second theorem we get that

$$\begin{aligned} \text{Trdg}_{K_0}(\vec{\lambda}, e^{\vec{\lambda}}) &\geq \\ &\geq \text{Trdg}_{K_0}(\lambda_1, \dots, \lambda_{n-1}, e^{\lambda_1}, \dots, e^{\lambda_{n-1}}) = \\ &= \text{AC}_{K_0, \text{exp}}(\lambda_1, \dots, \lambda_{n-1}) + \text{Ldim}_{\mathbb{Q}}(\lambda_1, \dots, \lambda_{n-1}/K_0) > \\ &> \text{Ldim}_{\mathbb{Q}}(\lambda_1, \dots, \lambda_{n-1}/K_0) = n - 1 \end{aligned}$$

and we are done.

The proof of the second theorem is articulated in three steps and runs as follows:

- (1) The above theorem is expressible by the lightface Σ_2^1 -formula

$$\text{WSP} \equiv \exists f \in \mathbb{C}^{\mathbb{N}} (\text{ran}(f) = K_0 \text{ is a field} \wedge \forall \vec{\lambda} \in \mathbb{C}^{<\mathbb{N}} \text{AC}_{\text{exp}, K_0}(\vec{\lambda}) > 0),$$

since it is a rather straightforward calculation to check that the formulae

$$\phi(f) \equiv (f \in \mathbb{C}^{\mathbb{N}} \wedge \text{ran}(f) = K_0 \text{ is a field})$$

and

$$\text{WSP}(\vec{\lambda}, f) \equiv \phi(f) \wedge (\vec{\lambda} \in \mathbb{C}^{<\mathbb{N}} \rightarrow \text{AC}_{\text{exp}, K_0}(\vec{\lambda}) > 0)$$

are Borel statements definable over the parameters $f, \vec{\lambda}$ which require only to quantify over the countable sets $f, \mathbb{N}, \mathbb{Q}$. It is a classical result of set theory (known as Shoenfield's absoluteness) that any Σ_2^1 -property known to hold in some forcing extension is actually true. So in order to establish the theorem it is enough to prove the above formula consistent by means of forcing i.e. to prove that $\llbracket \text{WSP} \rrbracket_{\mathbb{B}} = 1_{\mathbb{B}}$ in the boolean valued model for set theory $V^{\mathbb{B}}$ for some complete boolean algebra \mathbb{B} .

- (2) The second step relies on the following observation: whenever \mathbb{B} is any complete boolean algebra and V is the universe of sets (i.e. the standard model of ZFC), the family of \mathbb{B} -names for complex numbers in the boolean valued model $V^{\mathbb{B}}$ (which we denote by $\dot{\mathbb{C}}$) “corresponds” to the space of continuous functions

$$C^+(\text{St}(\mathbb{B})) = \{f : \text{St}(\mathbb{B}) \rightarrow \mathbb{S}^2 : f \text{ is continuous and } f^{-1}[\{\infty\}] \text{ is nowhere dense}\},$$

where $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ is the one point compactification of \mathbb{C} with the euclidean topology, and $\text{St}(\mathbb{B})$ is the space of ultrafilters on $\text{St}(\mathbb{B})$ (equivalently of ring homomorphisms of \mathbb{B} onto the boolean algebra $\{0, 1\}$). More precisely there is a natural embedding of the structure $C^+(\text{St}(\mathbb{B}))$ into the boolean valued model $V^{\mathbb{B}}$ which identifies $C^+(\text{St}(\mathbb{B}))$ with $\dot{\mathbb{C}}$.

Various facets of this identification are common knowledge for the set theory scholars, but until now —at least to my knowledge— nobody has

ever written down it in due form, presently a complete account appears in Vaccaro's master thesis [7].

The reader is averted that these spaces of functions may not be exotic: for example if \mathbf{MALG} is the complete boolean algebra given by Lebesgue-measurable sets modulo Lebesgue null sets, $C(\text{St}(\mathbf{MALG}))$ is isometric to $L^\infty(\mathbb{R})$ via the Gelfand-transform of the C^* -algebra $L^\infty(\mathbb{R})$ and consequently $\text{St}(\mathbf{MALG})$ is homeomorphic to the space of characters of $L^\infty(\mathbb{R})$ endowed with the *weak-** topology inherited from the dual of $L^\infty(\mathbb{R})$.

What is more important to us is that for all complete boolean algebras \mathbf{B} and for all $G \in \text{St}(\mathbf{B})$ the space of germs given by $C^+(\text{St}(\mathbf{B}))/G$ is an algebraically closed field to which any "natural" (i.e. for example Borel) relation defined on \mathbb{C}^n can be extended: for example the exponential function can be extended to $C^+(\text{St}(\mathbf{B}))/G$ by the map $[f]_G \mapsto [e^f]_G$. Moreover we can identify \mathbb{C} inside $C^+(\text{St}(\mathbf{B}))/G$ as the subfield given by germs of constant functions. We invite the reader to skim through [7, Chapters 2,3,4] to get a thorough presentation of the properties of the spaces $C^+(\text{St}(\mathbf{B}))$ seen as \mathbf{B} -valued extensions of the complex numbers.

In this paper we are also interested in canonical subfields of $C^+(\text{St}(\mathbf{B}))/G$ which give the correct lift to $C^+(\text{St}(\mathbf{B}))/G$ of $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$, these are respectively:

- The field $\check{\mathbb{C}}/G$ given by germs of locally constant functions, i.e. functions f in $C^+(\text{St}(\mathbf{B}))$ such that

$$\bigcup \{f^{-1}[\{\lambda\}] : \lambda \in \mathbb{C}, f^{-1}[\{\lambda\}] \text{ is clopen}\}$$

is an open dense subset of $\text{St}(\mathbf{B})$.

- The subfield $\check{\mathbb{Q}}/G$ (respectively the subring $\check{\mathbb{Z}}/G$) of $\check{\mathbb{C}}/G$ given by germs of locally constant functions with range contained in \mathbb{Q} (respectively in \mathbb{Z}).

These rings corresponds in the forcing terminology of set theory respectively: to the \mathbf{B} -names for complex numbers of the ground model, to the \mathbf{B} -names for rational numbers of the ground model, to the \mathbf{B} -names for integer numbers of the ground model. This characterization will play an important role in our proof.

The second step of our proof will show that if $G \in \text{St}(\mathbf{B})$ and \mathbf{B} is a complete boolean algebra, the structure

$$(C^+(\text{St}(\mathbf{B}))/G, \check{\mathbb{C}}/G, [f]_G/[g]_G \mapsto [e^{f/g}]_G, \dots, [0]_G, [1]_G)$$

is a model of $T_{\text{WSP}(\check{\mathbb{C}}/G)}$ for any $G \in \text{St}(\mathbf{B})$.

The key arguments in this second step do not require any specific training in set theory and needs just a certain amount of familiarity with first order logic, the basic properties of algebraic varieties, and with the combinatorics of forcing as expressed in terms of complete atomless boolean algebras. In particular there is no need to be acquainted with forcing or set theory to follow the proof of the above results (such a familiarity will nonetheless be of great help to follow the arguments).

The basic ideas for the proof are the following:

- (A) For any $[\vec{f}]_G = ([f_1]_G, \dots, [f_n]_G) \in (C^+(\text{St}(\mathbf{B}))/G)^n$, the variety $V(\bar{I}_G(\vec{f}, e^{\vec{f}}))$ on $(C^+(\text{St}(\mathbf{B}))/G)^{2n}$ given by the 0-set of polynomials in $\check{\mathbb{C}}/G[\vec{x}, \vec{y}]$ vanishing at $[\vec{f}, e^{\vec{f}}]_G$ has dimension equal to the transcendence degree of

the tuple $[\vec{f}, e^{\vec{f}}]_G$ over $\check{\mathbb{C}}/G$. To establish the Schanuel property for $[\vec{f}]_G$ it is enough to study the algebraic dimension of this variety on $(C^+(\text{St}(\mathbf{B}))/G)^{2n}$.

- (B) For a dense open set of G , the ideal $\bar{I}_G(\vec{f}, e^{\vec{f}})$ is generated by polynomials p_1, \dots, p_k with *complex coefficients*, consequently the algebraic dimension of $V(\bar{I}_G(\vec{f}, e^{\vec{f}}))$ as a variety on $(C^+(\text{St}(\mathbf{B}))/G)^{2n}$ is equal to the algebraic dimension of the complex variety $V(p_1, \dots, p_k)$ given by points in \mathbb{C}^{2n} on which all the p_j vanish.
 - (C) Let $[\vec{f}]_G = ([f_1]_G, \dots, [f_n]_G)$ be given by nowhere locally constant functions which are \mathbb{Q}/G -linearly independent modulo $\check{\mathbb{C}}/G$, by (B) above the transcendence degree of the $2n$ -tuple $[\vec{f}, e^{\vec{f}}]_G$ over $\check{\mathbb{C}}/G$ is equal to the transcendence degree of the same $2n$ -tuple over \mathbb{C} (seen as a subfield of $C^+(\text{St}(\mathbf{B}))/G$).
 - (D) For an n -tuple $[\vec{f}]_G$ as above we can show that the transcendence degree over \mathbb{C} of the $2n$ -tuple $[\vec{f}, e^{\vec{f}}]_G$ is at least $n + 1$ as follows: we can find ϕ_1, \dots, ϕ_n analytic functions from $[0, 1]$ to \mathbb{C} linearly independent over \mathbb{Q} modulo \mathbb{C} with the following property: Let $[\phi]$ denote the germ of ϕ at 0. Then the map $[\phi_i] \mapsto [f_i]_G, [e^{\phi_i}] \mapsto [e^{f_i}]_G$ extends to an isomorphism of the corresponding finitely generated subfields. The desired conclusion follows, since the field of germs at 0 of analytic functions from $[0, 1]$ to \mathbb{C} is a field to which Ax's theorem on Schanuel's property for functions fields apply.
- (3) The third step of our paper combines steps (1) and (2) as follows: We choose a boolean algebra \mathbf{B} such that in the boolean valued model $V^{\mathbf{B}}$, $\llbracket \check{\mathbb{C}} \text{ is countable} \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}}$ (for example we can choose \mathbf{B} to be the boolean algebra of regular open subsets of $\mathbb{C}^{\mathbb{N}}$ where \mathbb{C} is endowed with the discrete topology). In particular in $V^{\mathbf{B}}$ we will have that

$$\llbracket \check{\mathbb{C}} \text{ is countable as witnessed by } \dot{f} \wedge \text{WSP}(\dot{f}, \dot{\mathbb{C}}) \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}},$$

i.e. $\llbracket \text{WSP} \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}}$ holds in $V^{\mathbf{B}}$. By the results of step (1), we thus get that WSP holds in V concluding the proof of Theorem 1.3.

We will not expand any further on step (1), the core of the paper concerns the proof of the results in step (2), we add some more comments in the last part regarding step (3). We try (as much as possible) to make the arguments in step (2) accessible to persons which are not acquainted with the forcing techniques and more generally with logic. For this reason we shall limit the use of techniques which are specific of set theory just to the last step.

2. STEP (2)

2.1. Results from complex analysis and algebraic geometry. We need just classical results in the field and we use as a general reference text [6], though some of the results we shall need may not be covered in that textbook. We will use the following theorems:

- (1) The following corollary of Ax's theorem [1, Theorem 3]:

Theorem 2.1. *Assume (F, E) is an exponential field which is algebraically closed. Let $D : F \rightarrow F$ be a differential map (i.e. $D(f + g) = D(f) + D(g)$ and $D(fg) = D(f)g + fD(g)$ for all $f, g \in F$) such that $\ker(D)$ is a field and $D(E(f)) = D(f) \cdot E(f)$ for all $f \in F$.*

Then for all $\vec{f} = (f_1, \dots, f_n) \in F^n$ which are \mathbb{Q} -linearly independent over $\ker(D)$ we have that

$$\text{Trdg}_{\ker(D)}(f_1, \dots, f_n, E(f_1), \dots, E(f_n)) \geq n + 1.$$

- (2) The field of fractions \mathcal{O}^Ω given by germs at 0 of analytic functions $f : [0, 1] \rightarrow \mathbb{C}$ with differential $D([f]/[g]) = \frac{[f]g - g[f]}{[g]^2}$ satisfies the assumptions of Ax's theorem with $\ker(D) = \mathbb{C}$.
- (3) Any irreducible affine algebraic variety on K^n with K algebraically closed field is of the form $V(I)$ with I a finitely generated prime ideal on $K[x_1, \dots, x_n]$. Moreover the set of generic points for $V(I)$ over a finitely generated subfield of K containing generators for I is of second category in the Zariski topology on $V(I)$.
- (4) The linear dimension of the ambient affine space K^n minus the minimal cardinality of a set of generators for a prime ideal I is the geometric dimension of the irreducible variety $V(I) \subseteq K^n$.
- (5) Any quasi-projective and smooth irreducible variety contained in \mathbb{C}^n (i.e. a Zariski open set of an irreducible algebraic variety in \mathbb{C}^n contained in the non-singular points of the variety) is also an analytic manifold.
- (6) The regular (or smooth) points of an irreducible quasi-projective variety are an open non-empty Zariski subset of the variety and any generic point of the variety is smooth.
- (7) Any complex analytic manifold of dimension 0 contained in \mathbb{C}^n is a discrete set of points.
- (8) Any family of m distinct points $\{p_0, \dots, p_m\}$ in an $n + 1$ -dimensional complex analytic manifold can be connected by an analytic path i.e. an analytic map $\vec{\phi} : [0, 1] \rightarrow V$ which is injective in $[0, 1)$ and is such that $\{p_0, \dots, p_m\} \subseteq \text{ran}(\vec{\phi})$ and $\vec{\phi}(0) = p_0$.

2.2. Forcing on $C^+(\text{St}(\mathbf{B}))$. We refer the reader to [7, Chapters 2, 3, 4] for a detailed account on the material presented here.

- A topological space (X, τ) is 0-dimensional, if its clopen sets form a base for τ .
- A compact topological space (X, τ) is extremally (extremely) disconnected if its algebra of clopen sets $\text{CL}(X)$ overlaps with its algebra of regular open sets $\text{RO}(X)$.

For a boolean algebra \mathbf{B} we let $\text{St}(\mathbf{B})$ be the Stone space of its ultrafilters with topology generated by the clopen sets

$$N_b = \{G \in \text{St}(\mathbf{B}) : b \in G\}.$$

We remark the following:

- $\text{St}(\mathbf{B})$ is a compact 0-dimensional Hausdorff space and any 0-dimensional compact space (X, τ) is isomorphic to $\text{St}(\text{Cl}(X))$,

- A compact Hausdorff space (X, τ) is extremely disconnected if and only if its algebra of clopen sets is a complete boolean algebra. In particular $\text{St}(\mathbf{B})$ is extremely disconnected if and only if $\mathbf{B} = \text{CL}(\text{St}(\mathbf{B}))$ is complete.

Recall also that the algebra of regular open sets of a topological space (X, τ) is always a complete boolean algebra with operations

- $\bigvee \{A_i : i \in I\} = \overline{\bigcup \{A_i : i \in I\}}$,
- $\neg A = X \setminus A_i$,
- $A \wedge B = A \cap B$.

An antichain on a boolean algebra \mathbf{B} is a subset A such that $a \wedge b = 0_{\mathbf{B}}$ for all $a, b \in A$, $\mathbf{B}^+ = \mathbf{B} \setminus \{0_{\mathbf{B}}\}$ is the family of positive elements of \mathbf{B} and a dense subset of \mathbf{B}^+ is a subset D such that for all $b \in \mathbf{B}^+$ there is $a \in D$ such that $a \leq_{\mathbf{B}} b$. In a complete boolean algebra \mathbf{B} any dense subset D of \mathbf{B}^+ contains an antichain A such that $\bigvee A = \bigvee D = 1_{\mathbf{B}}$.

Another key observation on Stone spaces of complete boolean algebras we will often need is the following:

Fact 2.2. *Assume \mathbf{B} is a complete atomless boolean algebra, then on its Stone space $\text{St}(\mathbf{B})$:*

- $N_{\bigvee_{\mathbf{B}} A} = \overline{\bigcup_{a \in A} N_a}$ for all $A \subseteq \mathbf{B}$.
- $N_{\bigvee_{\mathbf{B}} A} = \bigcup_{a \in A} N_a$ for all finite sets $A \subseteq \mathbf{B}$.
- For any infinite antichain $A \subseteq \mathbf{B}^+$, $\bigcup_{a \in A} N_a$ is properly contained in $N_{\bigvee_{\mathbf{B}} A}$ as a dense open subset.

Given a compact Hausdorff topological space X we let $C^+(X)$ be the space of continuous functions

$$f : X \rightarrow \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$$

(where \mathbb{S}^2 is seen as the one point compactification of \mathbb{C}) with the property that $f^{-1}[\{\infty\}]$ is a closed nowhere dense subset of X . In this manner we can endow $C^+(X)$ of the structure of a commutative ring of functions with involution letting the operations be defined pointwise on all points whose image is in \mathbb{C} and be undefined on the preimage of ∞ . More precisely $f + g$ is the unique continuous function

$$h : X \rightarrow \mathbb{S}^2$$

such that $h(x) = f(x) + g(x)$ whenever this makes sense (it makes sense on an open dense subset of X , since the preimage of the point at infinity under f, g is closed nowhere dense) and is extended by continuity on the points on which $f(x) + g(x)$ is undefined. Thus $f + g \in C^+(X)$ if $f, g \in C^+(X)$. Similarly we define the other operations. We take the convention that constant functions are always denoted by their constant value, and that $0 = 1/\infty$.

Definition 2.3. Let G be an ultrafilter on \mathbf{B} . For $f, g \in C^+(\text{St}(\mathbf{B}))$ let $[f]_G = [h]_G$ iff for some $a \in G$, $f \upharpoonright N_a = g \upharpoonright N_a$.

$C^+(\text{St}(\mathbf{B}))/G$ is the quotient ring of $C^+(\text{St}(\mathbf{B}))$ by G given by the equivalence classes $[f]_G$ for $f \in C^+(\text{St}(\mathbf{B}))$.

In the sequel given a vector $\vec{f} = (f_1, \dots, f_n) \in C^+(\text{St}(\mathbf{B}))^n$, $b \in \mathbf{B}$, $G \in \text{St}(\mathbf{B})$:

- $[\vec{f}]_G$ is a shorthand for $([f_1]_G, \dots, [f_n]_G)$,
- $\vec{f}(G)$ is a shorthand for $(f_1(G), \dots, f_n(G))$,

- $\vec{f} \upharpoonright N_b$ is a shorthand for $(f_1 \upharpoonright N_b, \dots, f_n \upharpoonright N_b)$,
- For $g : \mathbb{C} \rightarrow \mathbb{C}$, $g(\vec{f})$ is a shorthand for $(g \circ f_1, \dots, g \circ f_n)$.

We also define the following family of rings indexed by positive elements of a complete boolean algebra:

Definition 2.4. Let \mathbf{B} be a complete boolean algebra and $b \in \mathbf{B}^+$.

- $\check{C}_c \subseteq C^+(N_c)$ is the ring of functions $f \in C^+(N_c)$ which are locally constant i.e. such that

$$\bigcup \{f^{-1}[\{\lambda\}] : f^{-1}[\{\lambda\}] \text{ is clopen}\}$$

is open dense in N_c . \check{C} stands for $\check{C}_{1_{\mathbf{B}}}$.

- Let K be a structure among $\mathbb{Q}, \mathbb{Z}, \mathbb{N}$, we define \check{K}_c to be the family of functions $f \in \check{C}_c$ such that $\text{ran } f \subseteq K$. \check{K} stands for $\check{K}_{1_{\mathbf{B}}}$.

As a warm-up for the sequel we can already prove the following:

Fact 2.5. Assume \mathbf{B} is a complete boolean algebra. Then:

- (1) $(C^+(\text{St}(\mathbf{B}))/G, [f]_G \mapsto [e^f]_G)$ and $(\check{C}/G, [f]_G \mapsto [e^f]_G)$ are exponential fields with kernel $2\pi \cdot (\mathbb{Z}/G)$ for all $G \in \text{St}(\mathbf{B})$.
- (2) $\check{\mathbb{Q}}/G$ is a field for all $G \in \text{St}(\mathbf{B})$.

Proof. Left to the reader. For what concerns the field structure of $C^+(\text{St}(\mathbf{B}))/G$, it is not hard to check that for a non-zero $[f]_G \in C^+(\text{St}(\mathbf{B}))/G$, we can find some N_b with $b \in G$ so that $g \in C^+(N_b)$ and $g \cdot (f \upharpoonright N_b) = 1$ in $C^+(N_b)$. We can then extend g arbitrarily to a continuous function in $C^+(\text{St}(\mathbf{B}))$ out of N_b . The rest is similar or easier. \square

Germs of continuous functions on Stone spaces and forcing. We need to consider $C^+(\text{St}(\mathbf{B}))$ as a \mathbf{B} -boolean valued model. This is done as follows:

Definition 2.6. We identify a cba \mathbf{B} with the complete boolean algebra of clopen (regular open) sets of $\text{St}(\mathbf{B})$. The equality relation on $C^+(\text{St}(\mathbf{B}))$ is the map

$$\begin{aligned} = : C^+(\text{St}(\mathbf{B}))^2 &\rightarrow \mathbf{B} \\ (f, g) &\mapsto \overline{\{H : f(H) = g(H)\}} \end{aligned}$$

We denote $=(f, g)$ by $\llbracket f = g \rrbracket$.

This equality boolean relation satisfies:

$$\llbracket f = g \rrbracket \wedge \llbracket h = g \rrbracket \leq \llbracket f = h \rrbracket$$

and

$$\llbracket f = g \rrbracket = \llbracket g = f \rrbracket$$

for all f, g, h .

A forcing relation on $C^+(\text{St}(\mathbf{B}))$ is a map

$$R : C^+(\text{St}(\mathbf{B}))^n \rightarrow \mathbf{B}$$

such that

$$R(f_1, \dots, f_n) \wedge \llbracket f_i = h \rrbracket \leq R(f_1, \dots, f_{i-1}, h, f_{i+1}, \dots, f_n)$$

for all f_1, \dots, f_n, h .

Let R_1, \dots, R_n be forcing relations on $C^+(\text{St}(\mathbf{B}))^n$ and ϕ be a formula in the language $\{R_1, \dots, R_n\}$. We define:

- $\llbracket R_i(\vec{f}) \rrbracket = R_i(\vec{f})$ for all $i \leq n$,
- $\llbracket \phi \wedge \psi \rrbracket = \llbracket \phi \rrbracket \wedge \llbracket \psi \rrbracket$,
- $\llbracket \phi \vee \psi \rrbracket = \llbracket \phi \rrbracket \vee \llbracket \psi \rrbracket$,
- $\llbracket \neg \phi \rrbracket = \neg \llbracket \phi \rrbracket$,
-

$$\llbracket \exists x \phi(x, \vec{f}) \rrbracket = \bigvee \{ \llbracket \phi(g, \vec{f}) \rrbracket : g \in C^+(\text{St}(\mathbf{B})) \}.$$

Given G ultrafilter on \mathbf{B} we make $C^+(\text{St}(\mathbf{B}))/G$ a structure for the language $\{R_1, \dots, R_n\}$ letting

$$C^+(\text{St}(\mathbf{B}))/G \models R_i/G(\llbracket \vec{f} \rrbracket_G)$$

if and only if $R_i(\vec{f}) \in G$.

We have the following Theorems:

Lemma 2.7 (Mixing Lemma). *Assume \mathbf{B} is a complete boolean algebra and $A \subseteq \mathbf{B}$ is an antichain. Then for all family $\{f_a : a \in A\} \subseteq C^+(\text{St}(\mathbf{B}))$, there exists $f \in C^+(\text{St}(\mathbf{B}))$ such that*

$$a \leq \llbracket f = f_a \rrbracket$$

for all $a \in A$.

Proof. Sketch: Let $f \in C^+(\text{St}(\mathbf{B}))$ be the unique function such that $f \upharpoonright N_{(\neg \bigvee A)} = 0$ and $f \upharpoonright N_a = f_a \upharpoonright N_a$ for all $a \in A$. Check that f is well defined and works. \square

Lemma 2.8 (Fullness Lemma). *Let R_1, \dots, R_n be forcing relations on $C^+(\text{St}(\mathbf{B}))^{<\omega}$. Then for all formulae $\phi(x, \vec{y})$ in the language $\{R_1, \dots, R_n\}$ and all $\vec{f} \in C^+(\text{St}(\mathbf{B}))^n$ there exists $g \in C^+(\text{St}(\mathbf{B}))$ such that*

$$\llbracket \exists x \phi(x, \vec{f}) \rrbracket = \llbracket \phi(g, \vec{f}) \rrbracket.$$

Proof. Sketch: Find A maximal antichain among the b such that $\llbracket \phi(g_b, \vec{f}) \rrbracket > 0_{\mathbf{B}}$ for some g_b . Now apply the Mixing Lemma to patch together all the g_a for $a \in A$ in a g . Check that

$$\llbracket \exists x \phi(x, \vec{f}) \rrbracket = \llbracket \phi(g, \vec{f}) \rrbracket.$$

\square

Theorem 2.9 (Cohen's forcing Theorem). *Let R_1, \dots, R_n be forcing relations on $C^+(\text{St}(\mathbf{B}))$. Then for all $\vec{f} \in C^+(\text{St}(\mathbf{B}))^n$ and all formulae $\phi(\vec{x})$ in the language $\{R_1, \dots, R_n\}$:*

- (1) $C^+(\text{St}(\mathbf{B}))/G \models \phi(\llbracket \vec{f} \rrbracket_G)$ if and only if $\llbracket \phi(\vec{f}) \rrbracket \in G$,
- (2) for all $a \in \mathbf{B}$ the following are equivalent:
 - (a) $\llbracket \phi(f_1, \dots, f_n) \rrbracket \geq a$,
 - (b) $C^+(\text{St}(\mathbf{B}))/G \models \phi(\llbracket \vec{f} \rrbracket_G)$ for all $G \in N_a$,
 - (c) $C^+(\text{St}(\mathbf{B}))/G \models \phi(\llbracket \vec{f} \rrbracket_G)$ for densely many $G \in N_a$.

Proof. Sketch: Proceed by induction on the complexity of ϕ using the Mixing Lemma and the Fullness Lemma to handle the quantifier's cases. \square

2.3. $T_{\text{WSP}(\check{C}/G)}$ holds in $C^+(\text{St}(\mathbf{B}))/G$.

Theorem 2.10. *Assume \mathbf{B} is a cba and $G \in \text{St}(\mathbf{B})$. Then*

$$\text{AC}_{\check{C}/G, \text{exp}/G}([\vec{f}]_G) > 0$$

for all $[\vec{f}]_G \in (C^+(\text{St}(\mathbf{B}))/G)^n$ (where $\text{exp}/G([\vec{f}]_G) = [e^{\vec{f}}]_G$).

Before embarking in the proof of the above Theorem, let us show how the forcing theorem simplifies our task and outlines some caveat.

For any $b \in \mathbf{B}$ we can consider $C^+(N_b)$ both as a ring of functions in the usual sense, or as a boolean valued model on the boolean algebra $\mathbf{B} \upharpoonright b$ in which we consider the sum and product operations as forcing relations, imposing for example for the sum:

$$[[f + g = h]] = \overline{\{H \in N_b : f(H) + g(H) = h(H)\}}$$

and similarly for the other field operations. By the forcing theorem, we will get that $[[\phi]] = 1_{\mathbf{B}}$ for all field axioms ϕ expressed in the language with ternary relation symbols to code the operations, since each $C^+(\text{St}(\mathbf{B}))/G$ is a field for all $G \in \text{St}(\mathbf{B})$. Notice in sharp contrast that $C^+(\text{St}(\mathbf{B}))$ is *not* a field when we consider it as an algebraic ring. This starts to outline a serious distinction between the theory of $C^+(\text{St}(\mathbf{B}))$ seen as a boolean valued model and its theory seen as an algebraic ring.

Moreover in the sequel we do not work simply with the boolean valued model $C^+(\text{St}(\mathbf{B}))$ in the language for fields. We will consider it as a boolean valued model in the language with predicate symbols for the relations and operations \check{C} , exp , $+$, \cdot , we will also add a predicate symbol for the ring $\check{\mathbb{Q}}(\check{\mathbb{Z}})$ given by the locally constant \mathbb{Q} -valued (\mathbb{Z} -valued) functions and for the forcing relations expressing $\check{\mathbb{Z}}$ -linear independence over \check{C} and the \check{C} -transcendence degree forcing relation.

Definition 2.11. Let \mathbf{B} be a complete boolean algebra. For all $c \in \mathbf{B}$:

- $\check{C}_c \subseteq C^+(N_c)$ is the ring of functions which are locally constant and \check{C} stand for $\check{C}_{1_{\mathbf{B}}}$.
- Let K be a structure among $\mathbb{Q}, \mathbb{Z}, \mathbb{N}$, we define \check{K}_c to be the family of functions given by $f \in \check{C}_c$ such that $\text{ran } f \subseteq K$ and \check{K} stand for $\check{K}_{1_{\mathbf{B}}}$.

Given $\vec{f} = (f_1, \dots, f_n) \in C^+(\text{St}(\mathbf{B}))^n$ and $c \in \mathbf{B}$, let:

- $[[\text{Trdg}_{\check{C}}(\vec{f}) = m]] =$

$$= \bigvee_{\mathbf{B}} \{b \in \mathbf{B} : \forall c \leq b (\text{Trdg}_{\check{C}}(\vec{f} \upharpoonright N_c) = m)\},$$
- $[[\text{Ldim}_{\check{\mathbb{Q}}}(\vec{f}/\check{C}) = m]] =$

$$= \bigvee_{\mathbf{B}} \{b \in \mathbf{B} : \forall c \leq b (\text{Ldim}_{\check{\mathbb{Q}}}(\vec{f} \upharpoonright N_c/\check{\mathbb{C}}) = m)\}.$$

Fact 2.12. *The above relations are forcing relation for $C^+(\text{St}(\mathbf{B}))$.*

Proof. Left to the reader. □

On the face of the definitions we get that

$$[[\text{Ldim}_{\check{\mathbb{Q}}}(\vec{f}/\check{C}) = m]] = \bigvee_{\mathbf{B}} \{b \in \mathbf{B} : \forall c \leq b (\text{Ldim}_{\check{\mathbb{Q}}}(\vec{f} \upharpoonright N_c/\check{\mathbb{C}}) = m)\}$$

entails that

$$\text{Ldim}_{\check{\mathbb{Q}}/H}([\vec{f}]_H/\check{\mathbb{C}}/H) = m$$

only on an open dense subset of

$$H \in N_{\llbracket \text{Ldim}_{\check{\mathbb{Q}}}(\vec{f}/\check{\mathbb{C}}) = m \rrbracket}.$$

Similarly for the boolean predicate $\llbracket \text{Trdg}_{\check{\mathbb{C}}}(\vec{f}) = m \rrbracket$. First of all we observe that for these two boolean predicates this open dense subset is the whole of $N_{\llbracket \text{Ldim}_{\check{\mathbb{C}}}(\vec{f}) = m \rrbracket}$ ($N_{\llbracket \text{Trdg}_{\check{\mathbb{C}}}(\vec{f}) = m \rrbracket}$):

Fact 2.13. *Let \mathbf{B} be a complete boolean algebra and $\vec{f} = (f_1, \dots, f_n) \in C^+(\text{St}(\mathbf{B}))^n$. Then for all $G \in \text{St}(\mathbf{B})$:*

(1) $\llbracket \text{Trdg}_{\check{\mathbb{C}}}(\vec{f}) = m \rrbracket \in G$ if and only if

$$\text{Trdg}_{\check{\mathbb{C}}/G}([\vec{f}]_G) = m.$$

(2) $\llbracket \text{Ldim}_{\check{\mathbb{Q}}}(\vec{f}/\check{\mathbb{C}}) = m \rrbracket \in G$ if and only if

$$\text{Ldim}_{\check{\mathbb{Q}}/G}([\vec{f}]_G/(\check{\mathbb{C}}/G)) = m.$$

Proof. The proof is a standard application of the forcing method. To get the reader acquainted with what we shall be doing in the remainder we give some of its parts. Let $\vec{f} = (f_1, \dots, f_n)$ be a tuple of $C^+(\text{St}(\mathbf{B}))$ -functions. Assume towards a contradiction that

$$\text{Trdg}_{\check{\mathbb{C}}/G}([\vec{f}]_G) < m$$

but $\llbracket \text{Trdg}_{\check{\mathbb{C}}}(\vec{f}) = m \rrbracket \in G$. Then there is a polynomial $p(\vec{x})$ in $\check{\mathbb{C}}/G[\vec{x}]$ such that

$$p([\vec{f}]_G) = [0]_G.$$

By the forcing theorem we get that $\llbracket p(\vec{f}) = 0 \rrbracket \in G$. Assume

$$p(\vec{x}) = \sum_{\alpha} f_{\alpha} \vec{x}^{\alpha},$$

where α ranges over the appropriate multiindexes and each $f_{\alpha} \in C^+(\text{St}(\mathbf{B}))$. Then we also get that $(f_{\alpha} \upharpoonright N_b) \in \check{\mathbb{C}}_b$ for all α for some $b \in G$ refining $\llbracket p(\vec{f}) = 0 \rrbracket$.

This gives that $\text{Trdg}_{\check{\mathbb{C}}_c}(\vec{f} \upharpoonright N_c) < m$ as witnessed by

$$\sum_{\alpha} (f_{\alpha} \upharpoonright N_c) \vec{x}^{\alpha}$$

for all $c \leq_{\mathbf{B}} b$.

On the other hand, $d = \llbracket \text{Trdg}_{\check{\mathbb{C}}}(\vec{f}) = m \rrbracket \in G$ means that for an open dense subset A of N_d we have that for all non-empty $N_e \subseteq A$ $\text{Trdg}_{\check{\mathbb{C}}}(\vec{f} \upharpoonright N_e) = m$. Notice that $0_{\mathbf{B}} < b \wedge d \in G$. Thus we can find $c \leq b \wedge d$ such that $N_c \subseteq A$ is non-empty and $f_{\alpha} \upharpoonright N_c$ is constant for all multiindexes α ; such a c can be found as follows: list all relevant multiindexes α in the expression of $p(\vec{x})$ with a non-null coefficient f_{α} as $\alpha_1, \dots, \alpha_m$. First refine $b \wedge d$ to a $c_{\alpha_1} > 0_{\mathbf{B}}$ such that $f_{\alpha_1} \upharpoonright N_{c_1}$ is constant, given c_i for $i < m$, let $0_{\mathbf{B}} < c_{i+1} \leq c_i$ be such that $f_{\alpha_{i+1}} \upharpoonright N_{c_{i+1}}$ is constant, set $c = c_m$.

Then on N_c we have at the same time that $\text{Trdg}_{\check{\mathbb{C}}}(\vec{f} \upharpoonright N_c) < m$ as witnessed by the polynomial

$$p_c(\vec{x}) = \sum_{j=1, \dots, m} (f_{\alpha_j} \upharpoonright N_c) \vec{x}^{\alpha_j}$$

vanishing on $\vec{f} \upharpoonright N_c$ and also that $\text{Trdg}_{\mathbb{C}}(\vec{f} \upharpoonright N_c) = m$ as witnessed by the fact that $N_c \subseteq A$. We reached a contradiction.

The converse direction for Trdg and the proof for the other predicate are left to the reader. \square

2.3.1. *Key Lemmas.* Let $b \in \mathbf{B}$, and $\vec{f} = (f_1, \dots, f_n)$ be a tuple of $C^+(\text{St}(\mathbf{B}))$ -functions.

- $I_b(\vec{f})$ is the ideal on $\mathbb{C}[\vec{x}]$ given by polynomials $p(\vec{x})$ with coefficients in \mathbb{C} such that

$$p(\vec{f}(H)) = 0 \text{ for all } H \in N_b.$$

- $I_G(\vec{f})$ is the ideal on $\mathbb{C}[\vec{x}]$ of polynomials $p(\vec{x})$ with coefficients in \mathbb{C} such that

$$p([\vec{f}]_G) = 0.$$

- $\bar{I}_b(\vec{f})$ is the ideal on $\check{\mathbb{C}}_b[\vec{x}]$ given by polynomials $p(\vec{x})$ with coefficients in $\check{\mathbb{C}}_b$ such that

$$p(\vec{f} \upharpoonright N_b) = 0.$$

- $\bar{I}_G(\vec{f})$ is the ideal on $\check{\mathbb{C}}/G[\vec{x}]$ of polynomials $p(\vec{x})$ with coefficients in $\check{\mathbb{C}}_b$ for some $b \in G$ such that

$$[p]_G([\vec{f}]_G) = 0.$$

If no confusion can arise we let I_b denote $I_b(\vec{f})$ and similarly for all the other ideals defined above.

Notice the following:

- $I_b \subseteq I_G$ for all $G \in N_b$,
- $I_b \subseteq \bar{I}_b$,
- $I_G \subseteq \bar{I}_G$ for all $G \in N_b$,
- $[p]_G \in \bar{I}_G$ for all $p \in \bar{I}_b$ and for all $G \in N_b$, where

$$[p]_G = \sum_{\alpha} [f_{\alpha}]_G x^{\alpha} \text{ if } p = \sum_{\alpha} f_{\alpha} x^{\alpha}.$$

Fact 2.14. $V(I_G)$ and $V(\bar{I}_G)$ are irreducible algebraic varieties.

Proof. Assume $p(\vec{x})q(\vec{x}) \in I_G(\vec{f})$. Then $[p \circ \vec{f}]_G [q \circ \vec{f}]_G = 0$ in $C^+(\text{St}(\mathbf{B}))/G$. Since the latter is a field we get that $[p \circ \vec{f}]_G$ or $[q \circ \vec{f}]_G$ must be 0, which yields the desired conclusion. The proof for $V(\bar{I}_G)$ is identical. \square

Lemma 2.15. Assume \mathbf{B} is a complete boolean algebra. For each $b \in \mathbf{B}^+$ and $\vec{f} = (f_1, \dots, f_n)$ tuple of $C^+(\text{St}(\mathbf{B}))$ -functions, there exists $c \leq_{\mathbf{B}} b$ in \mathbf{B}^+ such that for all $G \in N_c$:

- $I_c(\vec{f}) = I_G(\vec{f})$,
- $[\vec{f}]_G$ is a generic point for $V(I_G(\vec{f}))^{C^+(\text{St}(\mathbf{B}))/G}$, where for any ideal I on $C^+(\text{St}(\mathbf{B}))/G[\vec{x}]$, $V(I)^{C^+(\text{St}(\mathbf{B}))/G}$ is the variety given by points in $(C^+(\text{St}(\mathbf{B}))/G)^n$ which annihilate all polynomials in I .

Proof. Assume the first conclusion of the Lemma fails for b and \vec{f} . Let $b_0 = b$ and $I_0 = I_b(\vec{f})$ and build by induction a strictly increasing chain of ideals I_n on C and a decreasing chain of elements $b_n >_{\mathbf{B}} 0_{\mathbf{B}}$ as follows:

Given $I_n = I_{b_n}(\vec{f})$, find —if possible— some $p(\vec{x}) \in \mathbb{C}[\vec{x}]$ which is not in I_n and vanishes on $[\vec{f}]_G$ for some $G \in N_{b_n}$. Then

$$p([\vec{f}]_G) = [0]_G$$

if and only if

$$\llbracket p(\vec{f}) = 0 \rrbracket \in G.$$

If we can proceed for all n ,

$$\{I_n : n \in \mathbb{N}\}$$

is a strictly increasing chain of ideals on the Noetherian ring $\mathbb{C}[x_1, \dots, x_n]$. This is impossible, so we can find $b_n = c$ such that $I_G(\vec{f}) = I_c(\vec{f})$ for any $G \in N_c$.

We are left to prove that $[\vec{f}]_G \in (C^+(\text{St}(\mathbb{B}))/G)^n$ is a generic point for $V(I_c(\vec{f}))^{C^+(\text{St}(\mathbb{B}))/G}$ for any $G \in N_c$. This is immediate for all $G \in N_c$, since:

$$p([\vec{f}]_G) = 0 \text{ iff } p(\vec{x}) \in I_G(\vec{f}) = I_c(\vec{f}).$$

The proof of the Lemma is completed. \square

Lemma 2.16. *Assume \mathbb{B} is a complete boolean algebra. Let $\vec{f} = (f_1, \dots, f_n)$ be a tuple of $C^+(\text{St}(\mathbb{B}))$ -functions, and $c \in \mathbb{B}$ be such that $I_c(\vec{f}) = I_G(\vec{f})$ for all $G \in N_c$. Then $I_d(\vec{f})$ is a set of generators for $\bar{I}_d(\vec{f} \upharpoonright N_d)$ in $\check{C}_d[\vec{x}]$ for all $d \leq_{\mathbb{B}} c$ and $I_G(\vec{f})$ is a set of generators for $\bar{I}_G(\vec{f})$ for all $G \in N_c$. In particular*

$$V(I_G(\vec{f}))^{C^+(\text{St}(\mathbb{B}))/G} = V(\bar{I}_G(\vec{f}))^{C^+(\text{St}(\mathbb{B}))/G}.$$

Proof. Let $p_1, \dots, p_k \in \mathbb{C}[\vec{x}]$ be a family of generators for $I_c(\vec{f})$. We claim that p_1, \dots, p_k is also a family of generators for $\bar{I}_c(\vec{f})$ in $\check{C}_c[\vec{x}]$: Pick some $p \in \check{C}_c[\vec{x}]$ such that $p \in \bar{I}_c(\vec{f})$. Since the coefficients of p are locally constant functions defined on N_c , we can find a maximal antichain $\{d_j : j \in J\}$ such that each d_j refines c and is such that

$$p \upharpoonright N_{d_j} = \sum_{\alpha} f_{\alpha} \upharpoonright N_{d_j} x^{\alpha} \in \mathbb{C}[\vec{x}].$$

This gives that

$$p \upharpoonright N_{d_j}(\vec{f}) \in I_{d_j}(\vec{f}) = I_c(\vec{f})$$

for all $j \in J$. Find thus $q_j^1, \dots, q_j^k \in \mathbb{C}[\vec{x}]$ such that

$$p \upharpoonright N_{d_j} = \sum_{l=1, \dots, k} q_j^l p_l.$$

Define for each $l = 1, \dots, k$ $q_l \in C^+(N_c)$ by the requirement that

$$q_l \upharpoonright N_{d_j} = q_j^l$$

for all $j \in J$.

Then $q_l \in \check{C}_c[\vec{x}]$ for all $l = 1, \dots, k$ and

$$p = \sum_{l=1, \dots, k} q_l \cdot p_l \in \bar{I}_c(\vec{f}).$$

Since $p \in I_c(\vec{f})$ was chosen arbitrarily, we conclude that p_1, \dots, p_k are a set of generators for $\bar{I}_c(\vec{f})$ in $\check{C}_c[\vec{x}]$. This proves the first part of the Lemma.

For the second part observe that p_1, \dots, p_k are a family of generators for $I_G(\vec{f})$ for all $G \in N_c$.

Now pick $[p]_G \in \bar{I}_G(\vec{f})$ for $G \in N_c$. Then for some $d \leq_{\mathbf{B}} c$ in $G \upharpoonright N_d \in \bar{I}_d(\vec{f})$. But since $c \geq_{\mathbf{B}} d$ it is immediate to check that p_1, \dots, p_k are generators also for $\bar{I}_d(\vec{f})$. We conclude that $p \upharpoonright N_d$ can be obtained as a linear combination of p_1, \dots, p_k with coefficients in $\check{\mathbb{C}}_d[\vec{x}]$. Thus this occurs as well for $[p]_G$ taking the germs of these coefficients in $C^+(\text{St}(\mathbf{B}))/G$. The proof of the Lemma is completed. \square

Lemma 2.17. *Let $b \in \mathbf{B}$ and $\vec{f} = (f_1, \dots, f_n)$ be a tuple of $C^+(\text{St}(\mathbf{B}))$ -functions. Assume that*

$$\llbracket \text{Ldim}_{\check{\mathbb{Q}}} (f_1, \dots, f_n / \check{\mathbb{C}}) = n \rrbracket \geq_{\mathbf{B}} b$$

(i.e. $[f_1]_H, \dots, [f_n]_H$ are $\check{\mathbb{Q}}/H$ -linearly independent modulo $\check{\mathbb{C}}/H$ for all $H \in N_b$).

Then there exists an ultrafilter $G \in N_b$ such that

$$\text{Trdg}_{\check{\mathbb{C}}/G}([f]_G, [e^{\vec{f}}]_G) \geq n + 1.$$

Clearly the proof of this Lemma concludes the proof of Theorem 2.10 since it shows that the statement

$$\text{AC}_{\check{\mathbb{Q}}/H, \text{exp}/H}([f]_H) > 0$$

holds for a dense set of H for any fixed $\vec{f} \in (C^+(\text{St}(\mathbf{B}))^{<\mathbb{N}})$. In particular we get that the statement

$$\llbracket \text{AC}_{\check{\mathbb{Q}}, \text{exp}}(\vec{f}) > 0 \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}}$$

for all $\vec{f} \in (C^+(\text{St}(\mathbf{B}))^{<\mathbb{N}})$. Using the observations regarding the properties of the forcing predicates $\llbracket \text{Ldim}_{\check{\mathbb{Q}}}(\vec{f}/\check{\mathbb{C}}) \rrbracket_{\mathbf{B}}$ and $\llbracket \text{Trdg}_{\check{\mathbb{C}}}(\vec{f}) \rrbracket_{\mathbf{B}}$ and once again the forcing theorem, we get that

$$\text{AC}_{\check{\mathbb{Q}}/H, \text{exp}/H}([f]_H) > 0$$

holds for all H and for any fixed $\vec{f} \in (C^+(\text{St}(\mathbf{B}))^{<\mathbb{N}})$, which is the desired conclusion.

We now prove the Lemma:

Proof. First of all we choose $c \leq b$ such that

$$I_c(\vec{f}, e^{\vec{f}}) = I_G(\vec{f}, e^{\vec{f}})$$

for all $G \in N_c$, which is possible by Lemma 2.15. We let $I = I_c = I_G$ in what follows and $p_1, \dots, p_m \in \mathbb{C}[\vec{x}, \vec{y}]$ be a set of generators of minimal size for I in the appropriate ring.

We immediately notice —by standard arguments on the dimension of algebraic varieties— that the dimension of $V(I)$ as a variety over \mathbb{C}^{2n} and of $V(\bar{I}_G)$ as a variety over $(C^+(\text{St}(\mathbf{B}))/G)^{2n}$ is always equal to $2n - m$ for all $G \in N_c$.

Moreover $2n - m$ is also equal to the transcendence degree of $([f], e^{\vec{f}}]_G)$ over \mathbb{C} as well as over $\check{\mathbb{C}}$, since —by Lemma 2.16— the latter is a generic point of the variety

$$V(I)^{C^+(\text{St}(\mathbf{B}))/G} = V(\bar{I}_G)^{C^+(\text{St}(\mathbf{B}))/G}$$

for any $G \in N_c$ for the field $\check{\mathbb{C}}/G$.

So in order to prove the Lemma it is enough to study the geometric dimension of $V(I)^{C^+(\text{St}(\mathbf{B}))/G}$ as a subvariety of $(C^+(\text{St}(\mathbf{B}))/G)^{2n}$ and to prove that it is at least n for some $G \in N_c$.

We start this task remarking the following:

Fact 2.18. $([f], e^{\vec{f}}]_G)$ is a smooth point of $V(I)^{C^+(\text{St}(\mathbf{B}))/G}$ for all $G \in N_c$.

Proof. This follows from the fact that $([\vec{f}, e^{\vec{f}}]_G)$ is a generic point of $V(I)^{C^+(\text{St}(\mathbf{B}))/G}$. \square

Now observe that letting $U(I)$ be the set of smooth points of $V(I)$, $U(I)$ is a quasi-projective algebraic variety. Let

$$\mathbf{Exp}(n) = \{(\vec{\lambda}, e^{\vec{\lambda}}) : \lambda \in \mathbb{C}^n\}.$$

Remark that $U(I) \cap \mathbf{Exp}(n)$ is the zero-set of a finite set of analytic functions. Thus it can be split in disjoint closed (in $U(I)$) connected components. Let V' be the connected component of $U(I) \cap \mathbf{Exp}(n)$ to which $(\vec{f}(G), e^{\vec{f}(G)})$ belongs for some $G \in N_c$.

Claim 1. $\dim(V') > 0$.

Proof. Assume $\dim V' = 0$. Then by standard argument regarding the properties of analytic manifolds, we get that V' is an isolated point of $U(I) \cap \mathbf{Exp}(n)$, since any analytic variety contained in some open neighborhood of \mathbb{C}^{2n} having dimension 0 is a discrete set of points for the Euclidean topology on \mathbb{C}^{2n} . In particular V' , being a connected component of the dimension 0-part of the analytic variety $U(I) \cap \mathbf{Exp}(n)$, must be an isolated point of this variety. Hence we can find an open neighborhood $B \subseteq \mathbb{C}^{2n}$ of $\{(\vec{f}(G), e^{\vec{f}(G)})\}$ such that $(\vec{f}(H), e^{\vec{f}(H)}) \in V'$ for all H such that $(\vec{f}(H), e^{\vec{f}(H)}) \in B \cap V(I)$. However

$$I = I_H(\vec{f}, e^{\vec{f}}) = I_G(\vec{f}, e^{\vec{f}}) = I_c(\vec{f}, e^{\vec{f}})$$

for all $H \in N_c$. In particular $p(\vec{f}(H), e^{\vec{f}(H)}) = 0$ for all $p \in I$ and all $H \in N_c$, i.e. $(\vec{f}(H), e^{\vec{f}(H)}) \in V(I)$ for all $H \in N_c$.

Since V' consist of just one point we get that $(\vec{f}(H), e^{\vec{f}(H)}) = (\vec{f}(G), e^{\vec{f}(G)})$ for all $H \in N_c$ with $(\vec{f}(H), e^{\vec{f}(H)}) \in B$. We conclude that \vec{f} is constant with value $\vec{f}(G)$ on an open subset of N_c , contradicting our assumptions that \vec{f} is nowhere locally constant on $N_b \supseteq N_c$. \square

We now come to the heart of the proof of this Lemma:

Claim 2. For some $G \in N_c$

$$\text{Trdg}_{\mathbb{C}}([f_1]_G, \dots, [f_n]_G, [e^{f_1}]_G, \dots, [e^{f_n}]_G) \geq n + 1.$$

Proof. Let $c_1 \leq c$ be such that $(\vec{f}(H), e^{\vec{f}(H)}) \in V'$ for all $H \in N_{c_1}$. Our assumptions give that

$$(f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)}) \in V'$$

for all $H \in N_{c_1}$ and that V' is connected and of positive dimension.

Let $C^\Omega([0, 1], V')$ (in the sequel -for the sake of brevity- $C^\Omega(V')$) denote the vector valued paths which are analytic with range in $V' \subseteq \mathbb{C}^{2n}$.

We will use the following standard fact:

Fact 2.19. For any distinct H_1, \dots, H_k with $\vec{f}(H_i) \neq \vec{f}(H_j)$ for all $0 < i \neq j \leq k$ in N_{c_1} there is a path in $C^\Omega(V')$ passing through

$$(f_1(H_j), \dots, f_n(H_j), e^{f_1(H_j)}, \dots, e^{f_n(H_j)})$$

for all $0 < j \leq k$.

For each $H \in N_{c_1}$ consider the family Path_H of $C^\Omega(V')$ -paths

$$\vec{\phi} : [0, 1] \rightarrow V' \subseteq \mathbb{C}^{2n}$$

with

$$\vec{\phi}(0) = (f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)})$$

Let \mathcal{H} be the family of hypersurfaces given by points (\vec{x}, \vec{y}) satisfying

$$\sum_{i=1, \dots, n} m_i x_i = a; \quad \prod_{i=1, \dots, n} y_i^{m_i} = e^a$$

for some $a \in \mathbb{C}$ and some vector $(m_1, \dots, m_n) \in \mathbb{N}^n$.

Subclaim 1. *For all $G \in N_{c_1}$ the set D_G of $H \in N_{c_1}$ such that any $C^\Omega(V')$ -path in Path_G passing through H is contained in some hypersurface in \mathcal{H} is nowhere dense.*

Proof. Assume not for some G . Let $d \in \mathbb{B}$ be such that $D_G \cap N_d$ is dense in N_d .

By our assumptions, any $C^\Omega(V')$ -path contained in V' starting in the point

$$(f_1(G), \dots, f_n(G), e^{f_1(G)}, \dots, e^{f_n(G)})$$

and passing through some element of D is contained in an hypersurface in \mathcal{H} . Since V' is connected, for any $G_1, \dots, G_k \in D_G$ there is a $C^\Omega(V')$ -path in Path_G passing through

$$(f_1(G_j), \dots, f_n(G_j), e^{f_1(G_j)}, \dots, e^{f_n(G_j)}).$$

By our assumptions this path is contained in some hypersurface of the form

$$\sum_{i=1, \dots, n} m_i x_i = a; \quad \prod_{i=1, \dots, n} y_i^{m_i} = e^a$$

belonging to \mathcal{H} . Now select for as long as it is possible for each $0 \leq j < n$ some $G_j \in D_G$ so that $G_0 = G$ and

$$(f_1(G_{j+1}), \dots, f_n(G_{j+1}), e^{f_1(G_{j+1})}, \dots, e^{f_n(G_{j+1})}).$$

does not belong to the unique $(j-1)$ -dimensional hypersurface E_j determined as follows: Let A_j be the unique $(j-1)$ -dimensional hyperplane in \mathbb{C}^n passing for the points

$$(f_1(G_k), \dots, f_n(G_k))$$

with $k \leq j$. Let E_j consists of the points of the form $(\vec{\lambda}, e^{\vec{\lambda}})$ with $\vec{\lambda} \in A_j$. E_j is an hypersurface contained in some element of \mathcal{H} for each $0 \leq j \leq n-1$. To proceed in the construction notice that E_j is a closed subset of \mathbb{C}^{2n} for all $j < n$, thus

$$U_j = \{H \in N_d : (f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)}) \in E_j\}$$

is a closed subset of N_d . So either the latter set overlaps with N_d , or its complement has open and non-empty intersection with N_d , in which case we can find $G_{j+1} \in D_G \setminus U_j$ since D_G is dense in N_d . Continue this way for all $0 \leq j < n$ for which this is possible until $j = n-1$, if possible.

We show that this j cannot exist, reaching a contradiction.

- If we stop at stage $j < n-1$, this occurs only if for all $H \in D_G \setminus \{G_0, \dots, G_j\}$

$$(f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)}) \in E_j.$$

However $E_j \subseteq M$ for some hypersurface $M \in \mathcal{H}$. This M is therefore the 0-set of equations of the form

$$\sum_{i=1,\dots,n} m_i x_i = a, \quad \prod_{i=1,\dots,n} y_i^{m_i} = e^a.$$

In particular we get that for a dense set of $H \in N_d$

$$(f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)}) \in E_j.$$

Since belonging to E_j is a closed property of \mathbb{C}^{2n} , and the map $H \mapsto (f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)})$ is continuous on N_d , we get that for all $H \in N_d$

$$(f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)}) \in E_j.$$

Then in $C^+(N_d)$

$$\sum_{i=1,\dots,n} m_i f_i \upharpoonright N_d = a,$$

This contradicts the \mathbb{C} -linear independence of the vector $1, f_1 \upharpoonright N_d, \dots, f_n \upharpoonright N_d$ on N_d for a $d \leq b$, which was an assumption of the Lemma.

- Otherwise we can continue up to stage $j = n - 1$. This gives that

$$\{(f_1(G_k), \dots, f_n(G_k)) : 0 \leq k < n\}$$

are points in \mathbb{C}^n in general position, i.e. they are not contained in any proper affine subspace of \mathbb{C}^n . Since $(f_1(G_k), \dots, f_n(G_k), e^{f_1(G_k)}, \dots, e^{f_n(G_k)})$ are all in V' for all $k < n$ and V' is an analytic and connected variety, there is a $C^\Omega(V')$ -path $(\phi_1, \dots, \phi_{2n})$ connecting all of them and starting in $(\vec{f}(G_0), e^{\vec{f}(G_0)})$. Now observe that $\vec{\phi} = (\phi_1, \dots, \phi_n)$ is an analytic path passing through n -points in \mathbb{C}^n in general position. Thus it cannot be contained in any hyperplane of \mathbb{C}^n . In particular $(\vec{\phi}, e^{\vec{\phi}}) \in \text{Path}_G$ cannot be contained in any hypersurface belonging to \mathcal{H} , which is a contradiction.

The subclaim is proved. \square

By the above subclaim we can fix $G \in N_{c_1}$ and find $H \in N_{c_1} \setminus D_G$ (since this latter set contains a dense open subset of N_{c_1}). Then we can pick an analytic path $(\vec{\phi}, e^{\vec{\phi}})$ in Path_G passing through $(\vec{f}(H), e^{\vec{f}(H)})$ and not contained in any hyperplane in \mathcal{H} .

Consider finally the field of fractions of germs $[f]$ of analytic functions $f : [0, 1] \rightarrow \mathbb{C}$ around the point 0, where $[f] = [g]$ are equivalent germs if f and g overlap on $[0, t)$ for some $t \leq 1$. This is a differential field \mathcal{O}^Ω with differential

$$D : \mathcal{O}^\Omega \rightarrow \mathcal{O}^\Omega$$

mapping

$$[f]/[g] \rightarrow [f'g - g'f]/[g^2]$$

and $\ker(D) = \mathbb{C}$ given by the germs of constant functions.

Since we chose $\vec{\phi}$ not contained in E for any hypersurface $E \in \mathcal{H}$, we get that $[\vec{\phi}]$ is a vector of elements of the differential field \mathcal{O}^Ω which are \mathbb{Q} -linearly independent modulo \mathbb{C} , so that the hypothesis of Ax's theorem apply to these elements. By Ax's result 2.1, we get that

$$\text{Trdg}_{\mathbb{C}}([\vec{\phi}, e^{\vec{\phi}}]) \geq n + 1.$$

Now let

$$J = \{p \in \mathbb{C}[\vec{x}, \vec{y}] : p([\vec{\phi}, e^{\vec{\phi}}]) = 0\},$$

we get that $I = I_{N_{c_2}} \subseteq J$ since $(\vec{\phi}, e^{\vec{\phi}})$ has range contained in $V(I)$. In particular

$$\text{Trdg}_{\mathbb{C}/G}([\vec{f}]_G, [e^{\vec{f}}]_G) = \dim(V(I)) \geq \dim(V(J)) = \text{Trdg}_{\mathbb{C}}([\vec{\phi}, e^{\vec{\phi}}]) \geq n + 1.$$

This concludes the proof of the claim and of the Lemma³. \square

The proof of the Lemma is completed. \square

3. STEP 3

From now on we shall assume the reader has some familiarity with the boolean valued model approach to forcing in set theory. Standard references for the material of this section can be [2] or [5], and a detailed account of the results we sketch here can be found in [7]. We briefly sketch the general picture of the forcing theory in the next subsection.

3.1. A brief outline of forcing over the standard model of set theory. Recall that for (V, \in) the standard model of ZFC for the first order language $\{\in, =\}$ and \mathbf{B} a complete boolean algebra in V we can define (by transfinite recursion) the class of \mathbf{B} -names $V^{\mathbf{B}}$ given by $\tau \in V$ if τ is a function with domain contained in $V^{\mathbf{B}}$ and range contained in \mathbf{B} . We can also define forcing relations

$$\begin{aligned} \in_{\mathbf{B}} : (V^{\mathbf{B}})^2 &\rightarrow \mathbf{B} \\ (\tau, \sigma) &\mapsto \llbracket \tau \in \sigma \rrbracket \end{aligned}$$

$$\begin{aligned} =_{\mathbf{B}} : (V^{\mathbf{B}})^2 &\rightarrow \mathbf{B} \\ (\tau, \sigma) &\mapsto \llbracket \tau = \sigma \rrbracket \end{aligned}$$

such that $(V^{\mathbf{B}}, \in_{\mathbf{B}}, =_{\mathbf{B}})$ is a full \mathbf{B} -valued model for the language of set theory and $\llbracket \phi \rrbracket = 1_{\mathbf{B}}$ for all axioms ϕ of ZFC.

Letting

$$[\tau]_G = \{\sigma : \llbracket \tau = \sigma \rrbracket \in G\}$$

and

$$[\tau]_G \in [\sigma]_G \text{ if and only if } \llbracket \tau \in \sigma \rrbracket \in G$$

We also have that

$$\llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket \in G \text{ if and only if } V^{\mathbf{B}}/G \models \phi([\tau_1]_G, \dots, [\tau_n]_G)$$

for all formulae $\phi(x_1, \dots, x_n)$ in this language and all $G \in \text{St}(\mathbf{B})$.

Finally we recall that G is V -generic for a cba \mathbf{B} if $G \cap D$ is nonempty for all D dense subset of \mathbf{B}^+ and that for such a G and all $\tau \in V^{\mathbf{B}}$ we can define:

$$\tau_G = \{\sigma_G : \tau(\sigma) \in G\}$$

and let

$$V[G] = \{\tau_G : \tau \in V^{\mathbf{B}}\}.$$

With this choice of G we have that the map $[\tau]_G \mapsto \tau_G$ define an isomorphism of $(V^{\mathbf{B}}/G, \in_G)$ with $(V[G], \in)$.

³With some extra work one can check that $J = I$ for an open dense set of $H \in N_{c_1}$.

Moreover any element $u \in V$ has a canonical name $\check{u} \in V^{\mathbf{B}}$ such that $\check{u}_G = u$ whenever G is V -generic for \mathbf{B} .

It is well known that V -generic filter cannot exist for atomless complete boolean algebra, nonetheless there is a wide spectra of solutions to overcome this issue and work under the assumption that for any such \mathbf{B} V -generic filters can be found.

3.2. The relation between $C^+(\text{St}(\mathbf{B}))$ and $V^{\mathbf{B}}$. We have the following theorem linking the boolean valued model $C^+(\text{St}(\mathbf{B}))$ to the set theoretic boolean valued model $V^{\mathbf{B}}$ (see [7, Theorem 4.3.5]):

Theorem 3.1. *Let \mathbf{B} be a cba, $b \in \mathbf{B}$, and $\{U_n : n \in \omega\}$ be a countable base for the euclidean topology on \mathbb{C} . Given $f \in C^+(N_b)$ for some $b \in \mathbf{B}$, let $\tau_f \in V^{\mathbf{B}}$ be a \mathbf{B} -name for the unique object in $V^{\mathbf{B}}$ satisfying in $V^{\mathbf{B}}$:*

$$\llbracket \tau_f \in U_n \rrbracket = \overline{f^{-1}[U_n]}.$$

Given R a forcing relation on $C^+(N_b)^n$ let $\bar{R} \in V^{\mathbf{B}}$ be a \mathbf{B} -name for a n -ary relation on the n -tuples of complex numbers \mathbb{C}^n as computed in $V^{\mathbf{B}}$ such that

$$\llbracket \bar{R}(\tau_{f_1}, \dots, \tau_{f_n}) \rrbracket^{V^{\mathbf{B}}} = R(f_1, \dots, f_n).$$

Then the assignment $f \mapsto \tau_f$, $R \mapsto \bar{R}$ is an embedding of the boolean valued models $C^+(\text{St}(\mathbf{B}))$ and $C^+(N_b)$ for $b \in \mathbf{B}$ in the boolean valued model $V^{\mathbf{B}}$ such that:

- the equality forcing relation on $C^+(\text{St}(\mathbf{B}))$ is mapped to the equality relation on $V^{\mathbf{B}}$;
- for all $\tau \in V^{\mathbf{B}}$ such that

$$\llbracket \tau \text{ is a complex number} \rrbracket^{V^{\mathbf{B}}} = b,$$

there exists $f \in C^+(N_b)$ such that

$$\llbracket \tau = \tau_f \rrbracket^{V^{\mathbf{B}}} = b;$$

- for all forcing relations R on $C^+(\text{St}(\mathbf{B}))^n$ and all $f_1, \dots, f_n \in C^+(\text{St}(\mathbf{B}))$

$$\llbracket \bar{R}(\tau_{f_1}, \dots, \tau_{f_n}) \rrbracket^{V^{\mathbf{B}}} = R(f_1, \dots, f_n).$$

3.3. Shoenfield's absoluteness. We say that $A \subseteq \mathbb{C}^m$ is a Σ_2^1 -property if there is a Borel predicate $R \subseteq \mathbb{C}^{<\omega}$ and $\vec{a} \in \mathbb{C}^{<\omega}$ such that $A(\vec{a})$ holds if and only if

$$\exists x \forall y R(x, y, \vec{a}).$$

Given a Borel predicate $R \subseteq \mathbb{C}^n$ and a complete boolean algebra \mathbf{B} , we let

$$R_{\mathbf{B}} : C^+(\text{St}(\mathbf{B}))^n \rightarrow \mathbf{B}$$

$$(f_1, \dots, f_n) \mapsto \overline{\{H : R(f_1(H), \dots, f_n(H))\}}$$

and

$$\bar{R}_{\mathbf{B}} : (V^{\mathbf{B}})^n \rightarrow \mathbf{B}$$

$$(\tau_1, \dots, \tau_n) \mapsto \bigwedge_{j=1, \dots, n} \llbracket \tau_j \text{ is a complex number} \rrbracket \wedge R_{\mathbf{B}}(f_{\tau_1}, \dots, f_{\tau_n})$$

Theorem 3.2 (Shoenfield's absoluteness). *Assume A is a Σ_2^1 -property defined by the Borel predicate R as $\exists y \forall x R(x, y, \vec{a})$. Then $A(a_1, \dots, a_n)$ holds in V for complex numbers a_1, \dots, a_n if and only if*

$$\llbracket \exists x \forall y \bar{R}_{\mathbf{B}}(x, y, \check{a}_1, \dots, \check{a}_n) \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}}$$

for some complete boolean algebra \mathbf{B} .

3.4. WSP holds for \mathbb{C} relative to a countable subfield. We can now prove Theorem 1.3: Shoenfield's absoluteness gives a simple proof of the following:

Corollary 3.3. *$C^+(St(\mathbf{B}))/G$ is an algebraically closed field for any $G \in St(\mathbf{B})$ and for any complete boolean algebra \mathbf{B} .*

Proof. The graph of the multiplication and of the addition are Borel relations on \mathbb{C}^3 , and the field axioms and the algebraic closure axioms are expressible as Σ_2 -properties of these operations. \square

Now let \mathbf{B} be the complete boolean algebra of regular sets in $\mathbb{C}^{\mathbb{N}}$ where \mathbb{C} is endowed with the discrete topology. In $V[G]$ there is a new bijection f of $\mathbb{C}^V = \mathbb{C}$ with \mathbb{N} given $f(n) = a$ if and only if

$$\{g \in \mathbb{C}^{\mathbb{N}} : g(n) = a\}$$

is in G . Moreover

$$V[G] \models \phi((\tau_1)_G, \dots, (\tau_n)_G) \text{ if and only if } \llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket \in G.$$

Now we observe that the following holds if G is V -generic in $V[G]$:

- $C^+(St(\mathbf{B}), \mathbb{C})/G$ is isomorphic to the complex numbers of $V[G]$ via the map

$$[f]_G = (\tau_f)_G$$

- $e^{V[G]}$ is the unique analytic function on the field

$$\mathbb{C}^{V[G]} = \{\tau_G : \llbracket \tau \text{ is a complex number} \rrbracket \in G\}$$

whose power series expansion is

$$\sum_{n=0, \infty} x^n / n!.$$

Moreover $e^{V[G]}$ is the graph of $[f]_G \mapsto [e^f]_G$ modulo the isomorphism of $C^+(St(\mathbf{B}), \mathbb{C})/G$ with $\mathbb{C}^{V[G]}$,

- $\mathbb{C} \cap V = \mathbb{C}^V = \check{\mathbb{C}}_G$ is identified with $\check{\mathbb{C}}/G$ modulo the above isomorphism and $\mathbb{N}^{V[G]} \cap V = \mathbb{N}^V = \check{\mathbb{N}}^{V[G]} = \check{\mathbb{N}}_G$ are the natural numbers both in V and in $V[G]$.
- The Key Lemmas for \vec{f} give that

$$\text{Trdg}_{\mathbb{C}^V}([\vec{f}]_G, e^{[\vec{f}]_G}) \geq n$$

whenever $[\vec{f}]_G$ is a family of \mathbb{C}^V -linearly independent vectors, since the boolean value of this statement is $1_{\mathbf{B}}$ (notice that such vectors are identified to complex numbers of $V[G] \setminus V$, since the complex numbers of V are represented by the locally constant functions).

- $V[G]$ models that \mathbb{C}^V is a countable exponentially and algebraically closed subfield of $\mathbb{C}^{V[G]}$ and the latter is the field of complex numbers in $V[G]$.

In particular $V[G]$ models that:

There exists \mathbb{C}^V , countable algebraically and exponentially closed subfield of $\mathbb{C}^{V[G]}$, such that for all $\vec{f} \in (\mathbb{C}^{V[G]})^n$

$$\text{Trdg}_{\mathbb{C}^V}([\vec{f}]_G, e^{[\vec{f}]_G}) \geq \text{Ldim}_{\mathbb{C}^V}(\vec{f}).$$

This is a Σ_2^1 -statement in no parameters and a few (lightface definable) Borel predicates which holds in

$$(\mathbb{C}^{V[G]}, \mathbb{C}^V, \mathbb{N}^V, e^{V[G]}, \text{Trdg}_{\mathbb{C}^V}, \text{Ldim}_{\mathbb{C}^V}).$$

By Shoenfield's absoluteness it holds in V , since all of the above predicates are Borel.

More precisely the forcing theorem gives that $V^{\mathbb{B}}$ models the above statement with boolean value $1_{\mathbb{B}}$ and Shoenfield's absoluteness shows that it also holds in V .

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