C^*-algebras and B-names for Complex Numbers

Andrea Vaccaro, Matteo Viale

Abstract

We outline a duality existing between commutative C^*-algebras and the family of B-names for complex numbers in a boolean valued model for set theory V^{B} given by a complete boolean algebra B. In particular we can describe these families of boolean names as spaces of functions obtained by a natural limit process over the commutative and unital C^*-algebras whose spectrum is extremely disconnected. We also outline how this duality could be combined with generic absoluteness results in the forcing theory to yield new methods to analyze the theory of the complex numbers as well as that of commutative C^*-algebras.

1. Introduction

This paper outlines a duality between the theory of commutative unital C^*-algebras, a specific domain of functional analysis, and the theory of Boolean valued models, which pertains to logic and set theory. More specifically, the main purpose will be to show that a commutative unital C^*-algebra $\mathcal{A}$, whose spectrum is extremely disconnected, can be identified with the B-names for complex numbers in the boolean valued model for set theory V^{B}, where B is the complete boolean algebra given by clopen sets on the spectrum of $\mathcal{A}$.

We will also show how to transform generic absoluteness results, such as Shoenfield's absoluteness and Woodin's proof of the invariance of the theory of $L(\mathbb{R})$ under set forcing in the presence of class many Woodin cardinals, in tools to describe the degree of elementarity between the complex numbers and the ring of germs at points of the spectrum of these C^*-algebras. In this respect the major outcome of the results we will present can be summarized in the following:

Theorem 1. *Let X be an extremally (extremely) disconnected compact Hausdorff space.*

Let $C^+(X)$ be the space of continuous functions $f : X \to \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ such that the preimage of ∞ is nowhere dense ($\mathbb{S}^2$ is the one point compactification of $\mathbb{C}$).

Given any *Borel predicate R on $\mathbb{C}^n$, there is a predicate $R^X \subseteq C^+(X)^n \times X$ (equivalently a boolean predicate $R^X : C^+(X)^n \to \mathsf{CL}(X)$ where $\mathsf{CL}(X)$ is the boolean algebra given by clopen subsets of X) such that for all $p \in X$*

$$\langle \mathbb{C}, R\rangle \prec_{\Sigma_2} \langle C^+(X)/p, R^X/p\rangle,$$

where $C^+(X)/p$ is the ring of germs in p of functions in $C^+(X)$, and $R^X/p([f_1], \dots [f_n])$ holds if there is a neighborhood U of p such that $R(f_1(x), \dots, f_n(x))$ holds on a dense set of $x \in U$.

Moreover if we assume the existence of class many Woodin cardinals we get that

$$\langle \mathbb{C}, R\rangle \prec \langle C^+(X)/p, R^X/p\rangle.$$

Our main purpose is to provide a language suitable to translate deep results in the theory of forcing in useful tools to analyze certain spaces of functions which can be of interest in various domains of mathematics. We try to make the statements of the theorems comprehensible to most readers with a fair acquaintance with first order logic. On the other hand the proofs will require a great familiarity with the forcing method.

We organize the paper as follows: In section 2 we introduce the spaces of functions $C^+(X)$ with X compact, Hausdorff and extremally (extremely) disconnected and we outline their simplest properties. In section 3 we introduce the notion of B-valued model for a first order signature and we show how to endow $C^+(X)$ of the structure of a B-valued model for B the boolean algebra given by regular open sets of X. In section 4 we show the natural isomorphism existing between these B-valued models and the family of boolean names for complex numbers in the boolean valued model for set theory V^{B}. In section 5 we show how to translate generic absoluteness results in a proof of the above theorem. This paper outlines the original parts of the master thesis of the first author [7]. A thorough presentation of all the results (and the missing details) presented here can be found there. We encounter a problem in the exposition: those familiar with forcing arguments will find most of the proofs redundant or trivial, those unfamiliar with forcing will find the paper far too sketchy. We aim to address readers of both kinds, so the current presentation tries to cope with this tension at the best of our possibilities.

2. The space of functions $C^+(St(\mathsf{B}))$

We refer the reader to [7, Chapter 2] for a detailed account on the material presented in this section.

- A topological space (X, τ) is 0-dimensional, if its clopen sets form a base for τ.
- A compact topological space (X, τ) is extremally (extremely) disconnected if its algebra of clopen sets $\mathsf{CL}(X)$ overlaps with its algebra of regular open sets $\mathsf{RO}(X)$.

For a boolean algebra B we let $St(\mathsf{B})$ be the Stone space of its ultrafilters with topology generated by the clopen sets

$$\mathcal{O}_b = \{G \in St(\mathsf{B}) : b \in G\}.$$

The following holds:

- $St(\mathsf{B})$ is a compact 0-dimensional Hausdorff space and any 0-dimensional compact space (X, τ) is isomorphic to $St(\mathsf{CL}(X))$,
- A compact Hausdorff space (X, τ) is extremely disconnected if and only if its algebra of clopen sets is a complete boolean algebra. In particular $St(\mathsf{B})$ is extremely disconnected if and only if $\mathsf{B} = \mathsf{CL}(St(\mathsf{B}))$ is complete.

Recall also that the algebra of regular open sets of a topological space (X, τ) is always a complete boolean algebra with operations

- $\bigvee\{A_i : i \in I\} = \overset{\circ}{\overline{\bigcup\{A_i : i \in I\}}}$,
- $\neg A = X \overset{\circ}{\setminus} A$,
- $A \wedge B = A \cap B$.

An antichain on a boolean algebra B is a subset A such that $a \wedge b = 0_{\mathsf{B}}$ for all $a, b \in A$, $\mathsf{B}^+ = \mathsf{B} \setminus \{0_{\mathsf{B}}\}$ is the family of positive elements of B and a dense subset of B^+ is a subset D such that for all $b \in \mathsf{B}^+$ there is $a \in D$ such that $a \leq_{\mathsf{B}} b$. In a complete boolean algebra B any dense subset D of B^+ contains an antichain A such that $\bigvee A = \bigvee D = 1_{\mathsf{B}}$.

Another key observation on Stone spaces of complete boolean algebras we will often need is the following:

Fact 2.1. *Assume* B *is a complete atomless boolean algebra, then on its Stone space* $St(\mathsf{B})$*:*

- $\mathcal{O}_{\bigvee_{\mathsf{B}} A} = \overset{\circ}{\overline{\bigcup_{a \in A} \mathcal{O}_a}}$ *for all* $A \subseteq \mathsf{B}$.
- $\mathcal{O}_{\bigvee_{\mathsf{B}} A} = \bigcup_{a \in A} \mathcal{O}_a$ *for all finite sets* $A \subseteq \mathsf{B}$.
- *For any infinite antichain* $A \subseteq \mathsf{B}^+$, $\bigcup_{a \in A} \mathcal{O}_a$ *is properly contained in* $\mathcal{O}_{\bigvee_{\mathsf{B}} A}$ *as a dense open subset (*$\{1_{\mathsf{B}} - a : a \in A\}$ *has the finite intersection property and can be extended to an ultrafilter disjoint form* A*).*

Given a compact Hausdorff topological space X, we let $C^+(X)$ be the space of continuous functions

$$f : X \to \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$$

(where $\mathbb{S}^2$ is seen as the one point compactification of $\mathbb{C}$) with the property that $f^{-1}[\{\infty\}]$ is a closed nowhere dense (i.e. with a dense open complement) subset of X. In this manner we can endow $C^+(X)$ of the structure of a commutative ring of functions with involution letting the operations be defined pointwise on all points whose image is in $\mathbb{C}$ and be undefined on the preimage of ∞. More precisely $f + g$ is the unique continuous function

$$h : X \to \mathbb{S}^2$$

such that $h(x) = f(x) + g(x)$ whenever this makes sense (it makes sense on an open dense subset of X, since the preimage of the point at infinity under f, g is closed nowhere dense) and is extended by continuity on the points on which $f(x) + g(x)$ is undefined. Thus $f + g \in C^+(X)$ if $f, g \in C^+(X)$. Similarly we define the other operations. We take the convention that constant functions are always denoted by their constant value, and that $0 = 1/\infty$. We leave to the reader as an instructive exercise the following:

Lemma 2.2. *Let* X *be compact Hausdorff extremally disconnected. Then for any* $p \in X$ *the ring of germs* $C^+(X)/p$ *is an algebraically closed field.*

Its proof will be an immediate corollary of the main theorem we stated in the introduction, since the theory of algebraically closed fields is axiomatizable by means of Π_2-formulae using simple Borel predicates on $\mathbb{C}^n$ for all n. However, as a warm up for the sequel, the reader can try to prove that it is a field.

Remark 2.3. The reader is averted that the spaces of functions $C^+(X)$ we are considering may not be exotic: for example if MALG is the complete boolean algebra given by Lebesgue-measurable sets modulo Lebesgue null sets, $C(St(\mathsf{MALG}))$ is isometric to $L^\infty(\mathbb{R})$ via the Gelfand-transform of the C^*-algebra $L^\infty(\mathbb{R})$ and consequently $St(\mathsf{MALG})$ is homeomorphic to the space of characters of $L^\infty(\mathbb{R})$ endowed with the *weak-$*$ topology* inherited from the dual of $L^\infty(\mathbb{R})$. $C^+(St(\mathsf{MALG})) = L^{\infty+}(\mathbb{R})$ is obtained adding to $L^\infty(\mathbb{R})$ the measurable functions which are essentially bounded on all sets of finite Lebesgue measure.

Moreover by means of Gelfand transform the spaces $C^+(X)$ we are considering are always obtained canonically from a commutative unital C^*-algebras with extremely disconnected spectrum by a completion procedure as the one described above for $L^{\infty+}(\mathbb{R})$.

3. Boolean Valued Models

In a first order model a formula can be interpreted as true or false. Given a complete boolean algebra B, B-boolean valued models generalize Tarski semantics associating to each formula a value in B, so that there are no more only true and false propositions (those associated to 1_{B} and 0_{B} respectively), but also other "intermediate values" of truth. The classic definition of boolean valued models for set theory and of their semantic for the language $\mathcal{L} = \{\in\}$ may be found in [3, Chapter 14]. As mentioned earlier, we need to generalize the definition to any first order language and to any theory of the language. A complete account of the theory of these boolean valued models can be found in [6]. Since this book is a bit out of date, we recall below the basic facts we will need and we invite the reader to consult [7, Chapter 3] for a detailed account on the material of this section.

Definition 3.1. Given a complete boolean algebra B and a first order language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\}$$

a B-*boolean valued model* (or B-valued model) $\mathcal{M}$ in the language $\mathcal{L}$ is a tuple

$$\langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, f_j^{\mathcal{M}} : j \in J\rangle$$

where:

1. M is a non-empty set, called *domain* of the B-boolean valued model, whose elements are called B-*names*;
2. $=^{\mathcal{M}}$ is the *boolean value* of the equality:
$$\begin{aligned} =^{\mathcal{M}} \colon M^2 &\to \mathsf{B} \\ (\tau, \sigma) &\mapsto [\![\tau = \sigma]\!]^{\mathcal{M}}_{\mathsf{B}} \end{aligned}$$
3. The forcing relation $R_i^{\mathcal{M}}$ is the *boolean interpretation* of the n-ary relation symbol R_i:
$$\begin{aligned} R_i^{\mathcal{M}} : M^n &\to \mathsf{B} \\ (\tau_1, \ldots, \tau_n) &\mapsto [\![R_i(\tau_1, \ldots, \tau_n)]\!]^{\mathcal{M}}_{\mathsf{B}} \end{aligned}$$
4. $f_j^{\mathcal{M}}$ is the *boolean interpretation* of the n-ary function symbol f_j:
$$\begin{aligned} f_j^{\mathcal{M}} : M^{n+1} &\to \mathsf{B} \\ (\tau_1, \ldots, \tau_n, \sigma) &\mapsto [\![f_j(\tau_1, \ldots, \tau_n) = \sigma]\!]^{\mathcal{M}}_{\mathsf{B}} \end{aligned}$$

We require that the following conditions hold:

for $\tau, \sigma, \chi \in M$,

(i) $[\![\tau = \tau]\!]^{\mathcal{M}}_{\mathsf{B}} = 1_{\mathsf{B}}$;
(ii) $[\![\tau = \sigma]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![\sigma = \tau]\!]^{\mathcal{M}}_{\mathsf{B}}$;
(iii) $[\![\tau = \sigma]\!]^{\mathcal{M}}_{\mathsf{B}} \wedge [\![\sigma = \chi]\!]^{\mathcal{M}}_{\mathsf{B}} \leq [\![\tau = \chi]\!]^{\mathcal{M}}_{\mathsf{B}}$;

for $R \in \mathcal{L}$ with arity n, and $(\tau_1, \ldots, \tau_n), (\sigma_1, \ldots, \sigma_n) \in M^n$,

(iv) $(\bigwedge_{h\in\{1,\ldots,n\}} [\![\tau_h=\sigma_h]\!]^{\mathcal{M}}_{\mathsf{B}}) \wedge [\![R(\tau_1,\ldots,\tau_n)]\!]^{\mathcal{M}}_{\mathsf{B}} \leq [\![R(\sigma_1,\ldots,\sigma_n)]\!]^{\mathcal{M}}_{\mathsf{B}}$;

for $f_j \in \mathcal{L}$ with arity n, and $(\tau_1,\ldots,\tau_n),(\sigma_1,\ldots,\sigma_n) \in M^n$ and $\mu,\nu \in M$,

(v) $(\bigwedge_{h\in\{1,\ldots,n\}} [\![\tau_h=\sigma_h]\!]^{\mathcal{M}}_{\mathsf{B}}) \wedge [\![f_j(\tau_1,\ldots,\tau_n)=\mu]\!]^{\mathcal{M}}_{\mathsf{B}} \leq [\![f_j(\sigma_1,\ldots,\sigma_n)=\mu]\!]^{\mathcal{M}}_{\mathsf{B}}$;

(vi) $\bigvee_{\mu\in M} [\![f_j(\tau_1,\ldots,\tau_n)=\mu]\!]^{\mathcal{M}}_{\mathsf{B}} = 1_{\mathsf{B}}$;

(vii) $[\![f_j(\tau_1,\ldots,\tau_n)=\mu]\!]^{\mathcal{M}}_{\mathsf{B}} \wedge [\![f_j(\tau_1,\ldots,\tau_n)=\nu]\!]^{\mathcal{M}}_{\mathsf{B}} \leq [\![\mu=\nu]\!]^{\mathcal{M}}_{\mathsf{B}}$.

If no confusion can arise, we will omit the pedix B and we will confuse a function or predicate symbol with its interpretation.

Given a B-model $(M,=^M)$ for equality a forcing relation R on M is a map $R: M^n \to \mathsf{B}$ satisfying condition (iv) above for boolean predicates.

We now define the relevant maps between those objects.

Definition 3.2. Let $\mathcal{M}$ be a B-valued model and $\mathcal{N}$ a C-valued model in the same language $\mathcal{L}$. Let

$$i : \mathsf{B} \to \mathsf{C}$$

be a morphism of boolean algebras and $\Phi \subseteq M \times N$ a relation. The couple $\langle i, \Phi\rangle$ is a *morphism* of boolean valued models if:

1. $\mathrm{dom}\Phi = M$;
2. given $(\tau_1,\sigma_1),(\tau_2,\sigma_2) \in \Phi$:
$$i([\![\tau_1=\tau_2]\!]^{\mathcal{M}}_{\mathsf{B}}) \leq [\![\sigma_1=\sigma_2]\!]^{\mathcal{N}}_{\mathsf{C}},$$
3. given R an n-ary relation symbol and $(\tau_1,\sigma_1),\ldots,(\tau_n,\sigma_n) \in \Phi$:
$$i([\![R(\tau_1,\ldots,\tau_n)]\!]^{\mathcal{M}}_{\mathsf{B}}) \leq [\![R(\sigma_1,\ldots,\sigma_n)]\!]^{\mathcal{N}}_{\mathsf{C}},$$
4. given f an n-ary function symbol and $(\tau_1,\sigma_1),\ldots,(\tau_n,\sigma_n),(\mu,\nu) \in \Phi$:
$$i([\![f(\tau_1,\ldots,\tau_n)=\mu]\!]^{\mathcal{M}}_{\mathsf{B}}) \leq [\![f(\sigma_1,\ldots,\sigma_n)=\nu]\!]^{\mathcal{N}}_{\mathsf{C}},$$

An *injective morphism* is a morphism such that in 2 equality holds.

An *embedding* of boolean valued models is an injective morphism such that in 3 and 4 equality holds.

An embedding $\langle i, \Phi\rangle$ from $\mathcal{M}$ to $\mathcal{N}$ is called *isomorphism* of boolean valued models if i is an isomorphism of boolean algebras, and for every $b \in N$ there is a $a \in M$ such that $(a,b) \in \Phi$.

Suppose $\mathcal{M}$ is a B-valued model and $\mathcal{N}$ a C-valued model (both in the same language $\mathcal{L}$) such that B is a complete subalgebra of C and $M \subseteq N$. Let J be the immersion of $\mathcal{M}$ in $\mathcal{N}$. $\mathcal{N}$ is said to be a *boolean extension* of $\mathcal{M}$ if $\langle id_{\mathsf{B}}, J\rangle$ is an embedding of boolean valued models.

Remark 3.3. When $\mathsf{B} = \mathsf{C}$ we will consider $i = id_{\mathsf{B}}$ unless otherwise stated.

Since we are allowing function symbols in $\mathcal{L}$, the definition of the semantic is a bit more intricate than in the case of a purely relational language.

Definition 3.4. Given a B-valued model $\mathcal{M}$ in a language $\mathcal{L}$, let φ be a $\mathcal{L}$-formula whose free variables are in $\{x_1, \ldots, x_n\}$, and let ν be a valuation of the free variables in $\mathcal{M}$ whose domain contains $\{x_1, \ldots, x_n\}$. We denote with $[\![\varphi(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}}$ the *boolean value* of $\varphi(\nu)$.

First, let t be an $\mathcal{L}$-term and $\tau \in M$; we define recursively $[\![(t = \tau)(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}} \in \mathsf{B}$ as follows:

- if t is a variable x, then

$$[\![(x = \tau)(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![\nu(x) = \tau]\!]^{\mathcal{M}}_{\mathsf{B}}$$

- if $t = f(t_1, \ldots, t_n)$ where t_i are terms and f is an n-ary function symbol, then

$$[\![(f(t_1, \ldots, t_n) = \tau)(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}} = \bigvee_{\sigma_1, \ldots, \sigma_n \in M} \left(\bigwedge_{1 \le i \le n} [\![(t_i = \sigma_i)(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}} \right) \wedge [\![f(\sigma_1, \ldots, \sigma_n) = \tau]\!]^{\mathcal{M}}_{\mathsf{B}}$$

Given a formula φ, we define recursively $[\![\varphi(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}}$ as follows:

- if $\varphi \equiv t_1 = t_2$, then

$$[\![(t_1 = t_2)(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}} = \bigvee_{\tau \in M} [\![(t_1 = \tau)(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}} \wedge [\![(t_2 = \tau)(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}}$$

- if $\varphi \equiv R(t_1, \ldots, t_n)$, then

$$[\![(R(t_1, \ldots, t_n))(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}} = \bigvee_{\tau_1, \ldots, \tau_n \in M} \left(\bigwedge_{1 \le i \le n} [\![(t_i = \tau_i)(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}} \right) \wedge [\![R(\tau_1, \ldots, \tau_n)]\!]^{\mathcal{M}}_{\mathsf{B}}$$

- if $\varphi \equiv \neg\psi$, then

$$[\![\varphi(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}} = \neg [\![\psi(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}}$$

- if $\varphi \equiv \psi \wedge \theta$, then

$$[\![\varphi(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![\psi(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}} \wedge [\![\theta(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}}$$

- if $\varphi \equiv \exists y \psi(y)$, then

$$[\![\varphi(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}} = \bigvee_{\tau \in M} [\![\psi(y/\tau, \nu)]\!]^{\mathcal{M}}_{\mathsf{B}}$$

If no confusion can arise, we omit the index $\mathcal{M}$ and the pedix B, and we simply denote the boolean value of a formula φ with parameters in $\mathcal{M}$ by $[\![\varphi]\!]$.

By definition, an isomorphism of boolean valued models preserves the boolean value of the atomic formulas. Proceeding by induction on the complexity, one can get the result for any formula.

Proposition 3.5. *Let $\mathcal{M}$ be a B-valued model and $\mathcal{N}$ a C-valued model in the same language $\mathcal{L}$. Assume $\langle i, \Phi \rangle$ is an isomorphism of boolean valued models. Then for any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, and for every $(\tau_1, \sigma_1), \ldots, (\tau_n, \sigma_n) \in \Phi$ we have that:*

$$i([\![\varphi(\tau_1, \ldots, \tau_n)]\!]^{\mathcal{M}}_{\mathsf{B}}) = [\![\varphi(\sigma_1, \ldots, \sigma_n)]\!]^{\mathcal{N}}_{\mathsf{C}}$$

With elementary arguments it is possible prove the Soundness Theorem also for boolean valued models.

Theorem 3.6 (Soundness Theorem)**.** *Assume that φ is a $\mathcal{L}$-formula which is syntactically provable by a $\mathcal{L}$-theory T, and that each formula in T has boolean value at least $b \in \mathsf{B}$ in a B-valued model $\mathcal{M}$. Then $[\![\varphi(\nu)]\!]_{\mathcal{M}} \geq b$ for all valuations ν in $\mathcal{M}$.*

We get a first order model from a B-valued model passing to a quotient by a ultrafilter $G \subseteq \mathsf{B}$. This corresponds for spaces of type $C^+(St(\mathsf{B}))$ to a specialization of the space to the ring of germs in G. In the general context we are considering it is defined as follows.

Definition 3.7. Let B a complete boolean algebra, $\mathcal{M}$ a B-valued model in the language $\mathcal{L}$, and G a ultrafilter over B. Consider the following equivalence relation on M:

$$\tau \equiv_G \sigma \Leftrightarrow [\![\tau = \sigma]\!] \in G$$

The first order model $\mathcal{M}/G = \langle M/G, =^{\mathcal{M}/G}, R_i^{\mathcal{M}/G} : i \in I, f_j^{\mathcal{M}/G} : j \in J \rangle$ is defined letting:

- For any n-ary relation symbol R in $\mathcal{L}$

$$R^{\mathcal{M}/G} = \{([\tau_1]_G, \ldots, [\tau_n]_G) \in (M/G)^n : [\![R(\tau_1, \ldots, \tau_n)]\!] \in G\}.$$

- For any n-ary function symbol f in $\mathcal{L}$

$$\begin{aligned} f^{\mathcal{M}/G} &: (M/F)^n \to M/G \\ ([\tau_1]_G, \ldots, [\tau_n]_G) &\mapsto [\sigma]_G. \end{aligned}$$

  where σ is such that $[\![f(\tau_1, \ldots, \tau_n) = \sigma]\!] \in G$. Def. 3.1(vii) guarantees that this function is well defined.

If we require $\mathcal{M}$ to satisfy a key additional condition, we get an easy way to control the truth value of formulas in $\mathcal{M}/G$.

Definition 3.8. A B-valued model $\mathcal{M}$ for the language $\mathcal{L}$ is *full* if for every $\mathcal{L}$-formula $\varphi(x, \overline{y})$ and every $\overline{\tau} \in M^{|\overline{y}|}$ there is a $\sigma \in M$ such that

$$[\![\exists x \varphi(x, \overline{\tau})]\!] = [\![\varphi(\sigma, \overline{\tau})]\!]$$

Theorem 3.9 (Boolean Valued Models Łoś's Theorem)**.** *Assume $\mathcal{M}$ is a full B-valued model for the language $\mathcal{L}$ Then for every formula $\varphi(x_1, \ldots, x_n)$ in $\mathcal{L}$ and $(\tau_1, \ldots, \tau_n) \in M^n$:*

(i) $\mathcal{M}/G \models \varphi([\tau_1]_G, \ldots, [\tau_n]_G)$ *if and only if* $[\![\varphi(\tau_1, \ldots, \tau_n)]\!] \in G$ *for all G a ultrafilter over* B.

(ii) For all $a \in \mathsf{B}$ the following are equivalent:

  (a) $[\![\varphi(f_1, \ldots, f_n)]\!] \geq a$,
  (b) $\mathcal{M}/G \models \varphi([\tau_1]_G, \ldots, [\tau_n]_G)$ *for* all $G \in \mathcal{O}_a$,
  (c) $\mathcal{M}/G \models \varphi([\tau_1]_G, \ldots, [\tau_n]_G)$ *for* densely many $G \in \mathcal{O}_a$.

3.1. $C^+(St(\mathsf{B}))$ as a boolean valued extension of $\mathbb{C}$

[2, Chapter 10] is a reference for the proofs of all the facts mentioned above, else we refer the reader to [7, Chapters 2, 3].

The following example shows how to obtain a boolean extension of a topological space X for a language composed of symbols which are interpreted as Borel subsets of X^n.

Example 3.10. Fix a complete boolean algebra B, a topological space X such that

$$\Delta_X = \{(x,x) \in X \times X : x \in X\}$$

is Borel[1] on $X \times X$. Consider $M = C(St(\mathsf{B}), X)$, the set of continuous functions from $St(\mathsf{B})$ to X.

We define a structure of B-valued extension of X on M for the language with equality as follows: Given $f, g \in M$, the set

$$W = \{G \in St(\mathsf{B}) : f(G) = g(G)\}$$

is a Borel subset of $St(\mathsf{B})$ since both f and g are continuous. Recall that $A \subseteq X$ is meager if it is contained in the countable union of closed nowhere dense sets and A has the Baire property if $U \Delta A$ is meager for some (unique) regular open set U. Since every Borel set B has the Baire property [3, Lemma 11.15], and $St(\mathsf{B})$ is compact Hausdorff, by [2, Chapter 29, Lemma 5], we get that

$$\overline{\{G \in St(\mathsf{B}) : f(G) R g(G)\}}^{\circ}$$

is the unique regular open with a meager symmetric difference with W. Identifying B with $\mathsf{RO}(St(\mathsf{B}))$ (B is complete), we have that

$$=^{St(\mathsf{B})} (f, g) = [\![f = g]\!]^{St(\mathsf{B})} = \overline{\{G \in St(\mathsf{B}) : f(G) R g(G)\}}^{\circ}$$

is a well defined element of B and satisfies the clauses of Def. 3.1 for the equality relation. For any Borel $R \subseteq X^n$, the predicate $R^{St(\mathsf{B})} : C(St(\mathsf{B}), X)^n \to \mathsf{B}$ defined by

$$R^{St(\mathsf{B})}(f_1, \dots, f_n) = [\![R(f_1, \dots, f_n)]\!]^{St(\mathsf{B})} = \overline{\{G \in St(\mathsf{B}) : R(f_1(G), \dots, f_n(G))\}}^{\circ}$$

is a forcing relation R satisfying the clauses of Def. 3.1 for an n-ary relation on $C(St(\mathsf{B}), X)$. Similarly we can lift Borel functions $F : X^n \to X$.

With these definitions it can be checked that

$$\mathcal{M} = \langle C(St(\mathsf{B}), X), =^{St(\mathsf{B})}, R_i^{St(\mathsf{B})} : i \in I, F_j^{St(\mathsf{B})} : j \in J \rangle$$

is a B-valued model for the signature given by the Borel relations $R_i : i \in I$ and Borel functions $F_j : j \in J$ chosen on X. Moreover the set $\{c_x \in M : x \in X\}$, where c_x is the constant function with value x, is a copy of X in M, i.e: the complete homomorphism given by the inclusion of 2 in B induces an embedding of the 2-valued model $\langle X, =, R_i : i \in I, F_j : j \in J \rangle$ into the B-valued model $\mathcal{M}$ mapping $x \mapsto c_x$ (however we do not *as yet* assert that this embedding preserves the truth of formulae with quantifiers). Thus we can infer that $\mathcal{M}$ is a B-valued extension of an isomorphic copy of X seen as a 2-valued model.

Finally if G is a ultrafilter on $St(\mathsf{B})$, i.e. a point of $St(\mathsf{B})$, we can define the ring $C(X, Y)/G$ of germs in $C(X, Y)$ letting

$$[f]_G = \{h : [\![f = g]\!]^{St(\mathsf{B})} \in G\}$$

[1] I.e. belonging to the smallest σ-algebra on X^2 containing the open sets.

and $R^{St(\mathsf{B})}([f_1]_G, \dots, [f_n]_G])$ iff $R^{St(\mathsf{B})}(f_1, \dots, f_n) \in G$. We can easily check that the map $x \mapsto [c_x]_G$ defines an embedding of 2-valued models of $\langle X, =, R_i : i \in I, F_j : j \in J\rangle$ into $\mathcal{M}/G$.

If X is Polish (i.e. second countable and completely metrizable), Δ_X is closed (X is Hausdorff), therefore, for any fixed language $\mathcal{L}$ whose elements are Borel relations and functions on X, we can define a structure of B-valued extension of X for the language $\mathcal{L}$. If $X = \mathbb{C}$, the domain of such extension is the C^*-algebra $C(St(\mathsf{B}))$ with extremely disconnected spectrum.

It can be checked that if X is compact $C(St(\mathsf{B}), X)$ endowed with suitable lifting of Borel predicates is a full B-valued model, while if X is not compact and contains an infinite set with discrete relative topology (i.e. $\mathbb{N}$ as a subset of $\mathbb{C}$) $C(St(\mathsf{B}), X)$ is not a full B-valued model (see Remark 4.4 below).

The latter observation is one of the compelling reasons which leads us to associate to $\mathbb{C}$ (which is Polish non-compact, locally compact) the space of functions $C^+(St(\mathsf{B}))$ (which we will show to be a *full* B-valued model). Similar tricks will be needed to properly describe the full boolean extensions of arbitrary (non-compact) Polish spaces by means of spaces of functions.

We resume the above observations in the following definition:

Definition 3.11. Let X be a compact Hausdorff extremely disconnected topological space.

(i) Let Y be a topological space such that Δ_Y is Borel in Y^2. For any Borel relation R on Y^n, $R^X : C(X, Y)^n \to \mathsf{CL}(X)$ maps $(f_1, \dots, f_n)$ to the clopen set

$$\overline{\mathring{\{G \in X : R(f_1(G), \dots, f_n(G))\}}}.$$

The lifting of Borel functions on Y to $C(X, Y)$ is obtained by lifting their graph to a forcing relation on $C(X, Y)$.

(ii) We let $C^+(X)$ be the space of continuous functions

$$f : X \to \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$$

(where $\mathbb{S}^2$ is seen as the one point compactification of $\mathbb{C}$) with the property that $f^{-1}[\{\infty\}]$ is a closed nowhere dense subset of X. We lift Borel relations $R \subseteq \mathbb{C}^n$ to R^X again letting

$$R^X(f_1, \dots, f_n) = \overline{\mathring{\{G \in X : R(f_1(G), \dots, f_n(G))\}}}.$$

We let $\langle C(X)/G, R^X/G\rangle$ and $\langle C^+(X)/G, R^X/G\rangle$ be the associated ring of germs with R^X/G defined for both rings by the requirement: $R^X([f_1]_G, \dots, [f_n]_G])$ iff $R^X(f_1, \dots, f_n) \in G$.

We have the following Theorems:

Lemma 3.12 (Mixing Lemma)**.** *Assume B is a complete boolean algebra and $A \subseteq \mathsf{B}$ is an antichain. Then for all family $\{f_a : a \in A\} \subseteq C^+(St(\mathsf{B}))$, there exists $f \in C^+(St(\mathsf{B}))$ such that*

$$a \leq [\![f = f_a]\!]$$

for all $a \in A$.

Proof. Sketch: Let $f \in C^+(St(\mathsf{B}))$ be the unique function such that $f \restriction \mathcal{O}_{(\neg \bigvee A)} = 0$ and $f \restriction \mathcal{O}_a = f_a \restriction \mathcal{O}_a$ for all $a \in A$. Check that f is well defined and works. □

Lemma 3.13 (Fullness Lemma)**.** *Let $R_1, \ldots, R_n$ be forcing relations on $C^+(St(\mathsf{B}))^{<\mathbb{N}}$. Then for all formulae $\varphi(x, \vec{y})$ in the language $\{R_1, \ldots, R_n\}$ and all $\vec{f} \in C^+(St(\mathsf{B}))^n$ there exists $g \in C^+(St(\mathsf{B}))$ such that*

$$\left[\!\left[\exists x \varphi(x, \vec{f})\right]\!\right] = \left[\!\left[\varphi(g, \vec{f})\right]\!\right].$$

Proof. Sketch: Find A maximal antichain among the b such that $\left[\!\left[\varphi(g_b, \vec{f})\right]\!\right] \geq b$ for some g_b. Now apply the Mixing Lemma to patch together all the g_a for $a \in A$ in a g. Check that

$$\left[\!\left[\exists x \varphi(x, \vec{f})\right]\!\right] = \left[\!\left[\varphi(g, \vec{f})\right]\!\right].$$

□

4. B-names for complex numbers

We refer the reader to [3] for a comprehensive treatment of the forcing method and to [7, Chapter 3] for a sketchy presentation covering in more details the results of this section. All over this section we assume the reader has some familiarity with the standard presentations of forcing and we follow notation standard in the set theoretic community (for example $\mathbb{N}$ is often denoted as the ordinal ω). Through this section we will assume V (the universe of sets) to be a transitive model of ZFC, and $\mathsf{B} \in V$ a boolean algebra which V models to be complete. V^{B} will denote the boolean valued model of set theory as defined in [3, Chapter 14] and $\check{a} \in V^{\mathsf{B}}$ will denote the canonical B-names for sets $a \in V$. If G is a V-generic ultrafilter in B, $V[G]$ will denote the generic extension of V and σ_G the interpretations of B-names in V^{B} by G. In this situation there is a natural isomorphism between $(V^{\mathsf{B}}/G, \in^{\mathsf{B}}/G)$ and $(V[G], \in)$ defined by $[\sigma]_G \mapsto \sigma_G$. Cohen's forcing theorem in this setting states the following for any formula $\varphi(x_1, \ldots, x_n)$ in the language of set theory:

- $V[G] \models \varphi((\sigma_1)_G, \ldots, (\sigma_n)_G)$ if and only if $[\![\varphi(\sigma_1, \ldots, \sigma_n)]\!] \in G$,
- $[\![\varphi(\sigma_1, \ldots, \sigma_n)]\!] \geq b$ if and only if $V[G] \models \varphi((\sigma_1)_G, \ldots, (\sigma_n)_G)$ for all V-generic filters G to which b belongs.

It is well known that V-generic filter cannot exists for atomless complete boolean algebra, nonetheless there is a wide spectra of solutions to overcome this issue and work under the assumption that for any such B V-generic filters can be found[2]. We will also use in several points the following forms of absoluteness for Δ_1-properties: for all provably Δ_1-definable property $\varphi(x, r)$ with $r \subseteq \omega$ over the theory $\mathsf{ZFC} + (r \subseteq \omega)$ the following holds:

- $\varphi(a, r)$ holds in a transitive N which is a model of (a large enough fragment of) ZFC with $a, r \in N$ if and only if $[\![\varphi(\check{a}, \check{r})]\!] = 1_{\mathsf{B}}$ holds in N for all boolean algebra $\mathsf{B} \in N$ which N models to be complete.
- $\varphi(a, r)$ holds in V if and only if it holds in any (some) transitive set N which is a model of (a large enough fragmnt of) ZFC with $a, r \in N$.

[2]See also how we handle this issue recurring to generic filters for countable elementary substructures of some V_θ in the proofs to follow.

Let X be a Polish space. Then X can be identified to a G_δ-subset of the Hilbert cube $\mathcal{H} = [0,1]^{\mathbb{N}}$ [4, Theorem 4.14].

Let

$$\hat{\mathcal{B}} = \{B_r(q) : r \in \mathbb{Q}, q \in D\}$$

where $B_r(q)$ is the open ball of radius r and center q and D is the set of points in $\mathcal{H}$ with rational coordinates which are non-zero just on a finite set. Then $\hat{\mathcal{B}}$ is a countable basis for the topology on $\mathcal{H} = [0,1]^{\mathbb{N}}$ described by a provably Δ_1-definable property defined by a lightface Borel predicate.

Definition 4.1. Let X be a Polish space in V, w.l.o.g.

$$X = \bigcap_{n\in\mathbb{N}} \bigcup \{B_{r_{mn}}(q_{mn}) : m \in \mathbb{N}\}$$

for a suitably chosen family of elements $B_{r_{mn}}(q_{mn})$ of $\hat{\mathcal{B}}$. $\sigma \in V^{\mathsf{B}}$ is a B-*name for an element of X* if

$$\left[\!\!\left[\sigma \in \bigcap_{n\in\mathbb{N}} \bigcup \{\dot{B}_{r_{mn}}(q_{mn}) : m \in \mathbb{N}\}\right]\!\!\right]^{V^{\mathsf{B}}} = 1_{\mathsf{B}},$$

where $(\dot{B}_r(q))_G$ is in $V[G]$ the ball of radius r and center q of the space $\mathcal{H}$ as defined in $V[G]$ for all V-generic filter G.

We denote by X^{B} the set of all B-names (of minimal rank) for elements in X modulo the equivalence relation:

$$\sigma \equiv \tau \Leftrightarrow [\![\sigma = \tau]\!] = 1_{\mathsf{B}}$$

We will write B-name for a complex numbers to denote an element of the family $\mathbb{C}^{\mathsf{B}}$.

We can similarly lift Borel relations on X^n to boolean relations on $(X^{\mathsf{B}})^n$:

Remark 4.2. Let X be a Polish space. Without loss of generality, X is a G_δ-subset of $\mathcal{H} = [0,1]^{\mathbb{N}}$. $\hat{\mathcal{B}}$ induces a countable open basis on X:

$$\hat{\mathcal{B}}_X = \{B_r(q) \cap X : r \in \mathbb{Q}, q \in D\}.$$

Every Borel subset of X is obtained, in fewer than $\aleph_1$ steps, from the elements of $\hat{\mathcal{B}}_X$ by taking countable unions and complements. It is possible to code these operations with r a subset of ω (see [3, Chapter 25]). For our purposes it is enough to say that if R is a Borel subset of X^n, there is some $r \subseteq \omega$ and a (ZFC provably) Δ_1-property $P(\vec{x}, y)$ such that

$$\vec{x} \in R \Leftrightarrow P(\vec{x}, r)$$

Suppose $r \in V$. We denote with R^V the set $\{\vec{x} \in V : P(\vec{x}, r)\}$. Guided by these considerations, we define in V the following B-name:

$$R^{\mathsf{B}} = \{(\vec{\tau}, [\![P(\vec{\tau}, \check{r})]\!]) : \vec{\tau} \text{ is a } \mathsf{B}\text{-name for an element of } X^n\}$$

$R^{\mathsf{B}} \in V^{\mathsf{B}}$ is a canonical name to interpret the Borel relation R in any generic extension of V by a generic filter G.

Definition 4.3. Given R a Borel n-ary relation on X we define, for $\sigma_1, \dots, \sigma_n \in X^{\mathsf{B}}$, we let $\vec{\sigma} \in V^{\mathsf{B}}$ denote the canonical name for the tuple $(\sigma_1, \dots, \sigma_n)$ and we set:

$$R^{\mathsf{B}}(\sigma_1, \dots, \sigma_n) = \left[\!\!\left[\vec{\sigma} \in R^{\mathsf{B}}\right]\!\!\right]^{V^{\mathsf{B}}},$$

similarly we define Borel functions $f : X^n \to X$.

With these definitions

$$\langle X^{\mathsf{B}}, R_1^{\mathsf{B}}, \ldots, R_k^{\mathsf{B}}, F_1^{\mathsf{B}}, \ldots, F_l^{\mathsf{B}} \rangle$$

is a B-valued extension of X, where each R_i (F_j) is an arbitrary Borel relation (function) on X^{n_i} (from X^{m_j} to X).

Remark 4.4. So far we have defined a structure of B-valued models for Borel relations and functions on both X^{B} and $C(St(\mathsf{B}), X)$. However, whenever X is not compact, we cannot exhibit a natural isomorphism between these two models, unless we enlarge $C(St(\mathsf{B}), X)$. The problem (that can be appreciated by the reader familiar with forcing) is the following. Assume we split a complete atomless boolean algebra B in a countable maximal antichain $A = \{a_n : n \in \omega\}$. Then $\bigvee_{n\in\omega} a_n = 1_{\mathsf{B}}$ but $\bigcup_{n\in\omega} \mathcal{O}_{a_n}$ is just an open dense subset of $St(\mathsf{B})$, as the family $\{1 - a_n : n \in \omega\}$ has the finite intersection property and can be extended to an ultrafilter H missing the antichain A. Now consider the function $f : G \mapsto n$ iff $a_n \in G$. This should naturally correspond to the B-name for a natural number

$$\sigma_f = \{\langle \check{n}, a_n \rangle : n \in \omega\}.$$

Notice also that the function is continuous on its domain since the target is a discrete subspace of $\mathbb{C}$ and the preimage of each point is clopen. Moreover this function naturally extends to a continuous function in $C^+(St(\mathsf{B})) \setminus C(St(\mathsf{B})$ mapping the G out of its domain to ∞. This shows that $C(St(\mathsf{B}))$ is a space of functions too small to capture all possible B-names for complex numbers. The reader who has grasped the content of this remark will find the proofs of the following Lemmas almost self-evident, however we decided to include them in full details, since at some points there are delicate issues regarding the way to formulate certain simple properties of Polish spaces in an absolute (i.e Δ_1-definable) manner which can be tricky for those who are not fully familiar with forcing.

Definition 4.5. Let X be a Polish space presented as a G_δ-subset of the Hilbert cube $\mathcal{H} = [0,1]^{\mathbb{N}}$. Let B be a complete boolean algebra.

$C^+(St(\mathsf{B}), X)$ is the family of continuous functions $f : St(\mathsf{B}) \to \mathcal{H}$ such that $f^{-1}[\mathcal{H} \setminus X]$ is nowhere dense in $St(\mathsf{B})$.

We can define a structure of B-valued extension on X over $C^+(St(\mathsf{B}), X)$ repeating verbatim what we have done in Section 3.1 for $C(St(\mathsf{B}), X)$. Everything will work smoothly since for all Borel $R \subseteq X^n$ and $f_1, \ldots, f_n \in C^+(St(\mathsf{B}), X)$, the set of $H \in St(\mathsf{B})$ such that $R(f_1(H), \ldots, f_n(H))$ is not defined is always a meager subset of $St(\mathsf{B})$. We are ready to prove the following theorem.

Theorem 4.6. *Let X be a Polish space. Then $\langle C^+(St(\mathsf{B}), X), =^{St(\mathsf{B})} \rangle$ and $\langle X^{\mathsf{B}}, =^{\mathsf{B}} \rangle$ are isomorphic B-valued models.*

We are mainly interested in what happens for $X = \mathbb{C}$, and we will give the full proof of the theorem above for this special case. However, with minimal modifications, the reader will be able to generalize by himself the proof to any Polish space (for spaces admitting a one point compactification it suffices to replace all occurrences of $\mathbb{C}$ with X in the proof to follow, for other spaces this is slightly more delicate).

Remark 4.7. In the following given a complete boolean algebra B, we will often confuse it with $\mathsf{RO}(St(\mathsf{B}))$. If U is a regular open set of $St(\mathsf{B})$ and $G \in St(\mathsf{B})$, we may write equivalently

$$G \in U, U \in G$$

depending on whether we are considering U as an element of $\mathsf{RO}(St(\mathsf{B}))$ or as the correspondent element in B.

Remark 4.8. The definitions given in Remark 4.2 and Definition 4.5 can be simplified when working in $\mathbb{C}$. Instead of $\hat{\mathcal{B}}_{\mathbb{C}}$ from Remark 4.2, we will work directly with $\mathcal{B} = \{U_n : n \in \omega\}$, the countable basis of $\mathbb{C}$ whose elements are the open balls with rational radius and whose centre has rational coordinates. Moreover, instead of Definition 4.5, we shall work with $C^+(St(\mathsf{B}))$ as defined in Def. 3.11(ii).

Proof of Theorem 4.6

The proof splits in several Lemmas.

The first Lemma gives a characterization of the B-name to associate to an $f \in C^+(St(\mathsf{B}))$ which we will need to define the boolean isomorphism we look for.

Lemma 4.9. *Assume $f \in V$ is an element of $C^+(St(\mathsf{B}))$. For $H \in St(\mathsf{B})$ we define*

$$\Sigma_f^H = \{\overline{U}_n : \overset{\circ}{\overline{f^{-1}[U_n]}} \in H\}$$

Then, for $H \in St(\mathsf{B})$, we have:

$$f(H) = \sigma_f^H$$

where σ_f^H it is the only element in $\bigcap \Sigma_f^H$ if Σ_f^H is non-empty, and $\sigma_f^H = \infty$ otherwise.

Remark 4.10. The Lemma shows that in ZFC, given $f \in C^+(St(\mathsf{B}))$, it holds that

$$f(H) = x \Leftrightarrow x = \sigma_f^H.$$

The latter is a (ZFC provably) Δ_1-property with ω, B, and $\{a_n = \overset{\circ}{\overline{f^{-1}[U_n]}} : n \in \mathbb{N}\}$ as parameters. Thus, given V a transitive model of ZFC, B a complete boolean algebra in V, G a V-generic filter in B, any $f \in V$ element of $C^+(St(\mathsf{B}))^V$ can be extended in an absolute manner to $V[G]$ by the rule:

$$\begin{aligned} f^{V[G]} : St(\mathsf{B})^{V[G]} &\to \mathbb{C}^{V[G]} \\ H &\mapsto \sigma_f^H \end{aligned}$$

where σ_f^H is defined as in the previous lemma through the set $\Sigma_f^H = \{\overline{U}_n : a_n \in H\}$.

This observation is used in the following proposition defining the boolean isomorphism between $\mathbb{C}^{\mathsf{B}}$ and $C^+(St(\mathsf{B}))$.

Proposition 4.11. *Fix V a transitive model of ZFC and $\mathsf{B} \in V$ a boolean algebra which V models to be complete. Let $f \in C^+(St(\mathsf{B}))$ and consider*

$$\mathcal{B} = \{U_n : n \in \omega\}$$

the countable basis of $\mathbb{C}$ defined in Remark 4.8. For each $n \in \omega$ let

$$a_n = \overset{\circ}{\overline{f^{-1}[U_n]}}$$

There exists a unique $\tau_f \in \mathbb{C}^{\mathsf{B}}$ such that[3]

$$[\![\tau_f \in \dot{U}_n]\!] = a_n.$$

[3] $\dot{U}_n$ denotes the B-name for the complex numbers in the open ball of the generic extension determined by the rational coordinates and rational radius of the ball U_n.

Corollary 4.12. *With the hypotheses of Proposition 4.11, if G is a V-generic filter in B then:*

$$f^{V[G]}(G) = (\tau_f)_G.$$

By Proposition 4.11 we conclude that the map $f \mapsto \tau_f$ defines a function between $C^+(St(\mathsf{B}))$ and $\mathbb{C}^{\mathsf{B}}$. We still need to show that the function is a surjective boolean map i.e. it maps boolean equality on $C^+(St(\mathsf{B}))$ to boolean equality on $\mathbb{C}^{\mathsf{B}}$ and is surjective (in the sense of boolean embeddings). The latter is performed by the following Lemma:

Lemma 4.13. *Assume $\tau \in \mathbb{C}^{\mathsf{B}}$. Consider*

$$\begin{aligned} f_\tau : St(\mathsf{B}) &\to \mathbb{C} \cup \{\infty\} \\ H &\mapsto \sigma_\tau^H \end{aligned}$$

where, given

$$\Sigma_\tau^H = \{\overline{U}_n : \left[\!\left[\tau \in \dot{U}_n \right]\!\right] \in H\}$$

σ_τ^H is the only element in $\bigcap \Sigma_\tau^H$ if Σ_f^H is non-empty, $\sigma_\tau^H = \infty$ otherwise. The function f belongs to $C^+(St(\mathsf{B}))$ and $\tau_{f_\tau} = \tau$.

Finally we need to show that $f \mapsto \sigma_f$ respects boolean equality, i.e. that:

$$[\![f = g]\!]^{C^+(St(\mathsf{B}))} = [\![\tau_f = \tau_g]\!]^{\mathbb{C}}. \tag{1}$$

Since it makes no difference to prove the equality for this relation or for an arbitrary Borel relation (or function), we will prove the following stronger result:

Lemma 4.14. *Assume $R \subseteq \mathbb{C}^n$ is a Borel relation. Then $R^{St(\mathsf{B})}(f_1, \ldots, f_n) = R^{\mathsf{B}}(\sigma_{f_1}, \ldots, \sigma_{f_n})$.*

It is clear that these Lemmas entails the conclusion of the theorem. We prove all of them in the next subsection.

Proof of the key Lemmas

Proof of Lemma 4.9. Assume Σ_f^H is empty. If $f(H) \in U_n$ for some $n \in \omega$ it follows that:

$$H \in f^{-1}[U_n] \subseteq \overset{\circ}{\overline{f^{-1}[U_n]}}$$

hence $\overset{\circ}{\overline{f^{-1}[U_n]}} \in \Sigma_f^H$, which is absurd. Suppose now that Σ_f^H is non-empty.

Claim 4.14.1. *Assume Σ_f^H is non-empty. Then $\bigcap \Sigma_f^H$ is a singleton.*

Proof. Let $m \in \omega$ be such that $\overline{U}_m \in \Sigma_f^H$.

Existence: The family

$$\hat{\Sigma}_f^H = \{\overline{U}_m \cap \overline{U}_n : \overset{\circ}{\overline{f^{-1}[U_n]}} \in H\}$$

is a family of closed subsets of $\overline{U}_m$. Σ_f^H inherits the finite intersection property from H, hence so does $\hat{\Sigma}_f^H$. We can conclude that

$$\emptyset \neq \bigcap \hat{\Sigma}_f^H \subseteq \bigcap \Sigma_f^H$$

Uniqueness: Suppose there are two different points $x, y \in \bigcap \Sigma_f^H$. There exists $p \in \omega$ such that $x \in U_p, y \notin \overline{U}_p$. The last relation guarantees that $\overline{U}_p \notin \Sigma_f^H$. Now we show that for $w \in \bigcap \Sigma_f^H$, $w \in U_n$ implies $\overline{\mathring{f^{-1}[U_n]}} \in H$. Therefore $x \in U_p$ implies $\overline{U}_p \in \Sigma_f^H$, which is absurd. Suppose $\overline{\mathring{f^{-1}[U_p]}} \notin H$, we have that:

$$H \in \neg\overline{\mathring{f^{-1}[U_p]}} \cap \overline{\mathring{f^{-1}[U_m]}} \subseteq f^{-1}[\overline{U}_m \setminus U_p]$$

For each $z \in \overline{U}_m \setminus U_p$ there exists U_{n_z} such that

$$z \in U_{n_z} \wedge x \notin \overline{U}_{n_z}$$

This family of open balls covers the compact space $\overline{U}_m \setminus U_p$, so that there are $z_1, \cdots, z_k \in \overline{U}_m \setminus U_p$ which verify the following chain of inclusions:

$$f^{-1}[\overline{U}_m \setminus U_p] \subseteq \bigcup_{1 \leq i \leq k} f^{-1}[U_{n_{z_i}}] \subseteq \bigcup_{1 \leq i \leq k} \overline{\mathring{f^{-1}[U_{n_{z_i}}]}}$$

There is therefore a z_j such that $\overline{\mathring{f^{-1}[U_{n_{z_j}}]}} \in H$, hence $\overline{U}_{z_j} \in \Sigma_f^H$. This is absurd since $x \notin \overline{U}_{z_j}$.

□

Suppose $f(H) \neq \sigma_f^H$ and consider two open balls U_1, U_2 in $\mathcal{B}$ such that

$$\overline{U}_1 \cap \overline{U}_2 = \emptyset$$

$$f(H) \in U_1$$

$$\sigma_f^H \in U_2$$

It easily follows that both $\overline{\mathring{f^{-1}[U_1]}}$ and $\overline{\mathring{f^{-1}[U_2]}}$ are in H (the second assertion, can be shown along the same lines of the uniqueness proof in Claim 4.14.1). These two sets are disjoint, a contradiction follows.

The Lemma is proved. □

In order to prove Proposition 4.11, we need to generalize what we have exposed in Remark 4.2 about Borel codes. Something similar can be performed in $St(\mathsf{B})$, here the parameters for the "code" have to be taken among real numbers (to code the complexity of the Borel relation) and elements of B (to code the basic open sets), since a basis for $St(\mathsf{B})$ is $\{\mathcal{O}_a : a \in \mathsf{B}\}$. The following can be shown starting from the elements clopen basis and then extending the proof to cover the case of arbitrary open or closed sets.

Fact 4.15. *Let G a V-generic filter over B. Assume R^V, S^V are two open or closed sets in $St(\mathsf{B})^V$. Then*

$$R^V \subseteq S^V \Leftrightarrow R^{V[G]} \subseteq S^{V[G]}$$

Proof. See [7, Lemma 4.3.3]. □

Proof of Proposition 4.11. Consider the B-name

$$\Sigma_f = \{(\dot{U}_n, a_n) : n \in \omega\}$$

Standard argument in forcing give that

$$\left[\!\left[\exists! x (x \in \bigcap \Sigma_f) \right]\!\right] = 1_{\mathsf{B}}, \tag{2}$$

we give a proof of this equality for the sake of completeness:

Proof of equation (2).

Claim 4.15.1. *Assume*[4] $M \prec V$ *in* V *is a countable model of* ZFC *such that* $\omega \cup \{a_n : n \in \omega\} \cup \{\mathsf{B}, f\} \subseteq M$*, and* $\pi : M \to N$ *is the Mostowski's Collapse. Let* G *be an* N*-generic filter for* $\pi(\mathsf{B})$*. Then:*

$$N[G] \models \exists! x \left(x \in \bigcap \Sigma^G_{\pi(f)} \right)$$

where $\Sigma^G_{\pi(f)} = \{\overline{U}^{N[G]}_n : \pi(a_n) \in G\}$*.*

Proof of the claim. Notice that since $\omega \subseteq M$ is transitive, rational and complex numbers (the power-set of a transitive set is transitive) are preserved by π, and $\mathbb{C}^N = \mathbb{C} \cap N$. First, we prove that $\Sigma^G_{\pi(f)}$ is non-empty (notice that $\pi(f)$ preserves all the properties of f since π is an isomorphism). The preimage of $\mathbb{C}^N$ through $\pi(f)$ contains an open dense subset of $St(\pi(\mathsf{B}))^N$ in N, hence (observe that $\pi(f)^{-1}[U^N_n] = \pi(f^{-1}[U_n])$) it follows that

$$\{\pi(a_n) : n \in \omega\}$$

is an open dense subset of $\pi(\mathsf{B})^+$ in N as well. Since G is N-generic, $G \cap D \neq \emptyset$. Thus $\pi(a_m) \in G$ and $\overline{U}^{N[G]}_m \in \Sigma^G_{\pi(f)}$ for some $m \in \omega$. The proof that $\bigcap \Sigma^G_{\pi(f)}$ is a singleton can be carried in N as in Claim 4.14.1. □

Since the Claim holds for all N-generic filters G for $\pi(\mathsf{B})$, by the forcing theorem applied to N and $\pi(\mathsf{B})$, we get that N models

$$\left[\!\left[\exists! x (x \in \bigcap \Sigma_{\pi(f)}) \right]\!\right] = 1_{\pi(\mathsf{B})}.$$

Thus M models

$$\left[\!\left[\exists! x (x \in \bigcap \Sigma_f) \right]\!\right] = 1_{\mathsf{B}}.$$

Since $M \prec V$, we get that the latter holds in V, completing the proof of equation (2). □

Now V^{B} is full, there is therefore a B-name τ_f such that

$$\left[\!\left[\tau_f \in \bigcap \Sigma_f \right]\!\right] = 1_{\mathsf{B}}.$$

This is a B-name for a complex number. Moreover, if τ is a B-name for a complex number and

$$\left[\!\left[\tau \in \bigcap \Sigma_f \right]\!\right] = 1_{\mathsf{B}},$$

then, from

$$(\tau_f \in \bigcap \Sigma_f) \wedge (\tau \in \bigcap \Sigma_f) \wedge (\exists! x (x \in \bigcap \Sigma_f)) \to \tau = \tau_f$$

follows that:

$$[\![\tau = \tau_f]\!] = 1_{\mathsf{B}}.$$

This shows that the map $f \mapsto \tau_f$ is well defined.

To conclude the proof of Proposition 4.11 we still must show that

$$\left[\!\left[\tau_f \in \dot{U}_n \right]\!\right] = \overset{\circ}{\overline{f^{-1}[U_n]}} = a_n \tag{3}$$

[4] More precisely $M \prec V_\theta$ for some V_θ large enough to reflect all the relevant properties of V and to contain all relevant objects of V we are interested in. $M \in V$ is a countable subset of V_θ which is elementary in $(V_\theta, \in)$ and contains all the relevant objects. In particular V_θ and M might not be models of ZFC but of enough of its axioms to carry on the arguments to follow.

Proof of equation (3). Let again $M \prec V$ be a countable structure as in Claim 4.15.1, $\pi : M \to N$ its Mostowski's Collapse, and G an N-generic filter for $\pi(\mathsf{B})$. On the one hand we have (using the same proof of the uniqueness part in Claim 4.14.1) that if $(\tau_{\pi(f)})_G \in U_n^{N[G]}$ then $\pi(a_n) \in G$, which gives

$$\left[\!\left[\tau_f \in \dot{U}_n\right]\!\right] \leq a_n.$$

On the other hand

$$G \in \pi(f)^{N[G]^{-1}}[U_n^{N[G]}] \Rightarrow \pi(f)^{N[G]}(G) = \tau^G_{\pi(f)} \in U_n^{N[G]} \Rightarrow \left[\!\left[\tau_{\pi(f)} \in \dot{U}_n\right]\!\right]^{N^{\pi(\mathsf{B})}} \in G$$

which means, interpreting $\left[\!\left[\tau_{\pi(f)} \in \dot{U}_n\right]\!\right]^{N^{\pi(\mathsf{B})}}$ as a clopen subset of $St(\pi(\mathsf{B}))^{N[G]}$, that

$$\pi(f)^{N[G]^{-1}}[U_n^{N[G]}] \subseteq \left(\left[\!\left[\tau_{\pi(f)} \in \dot{U}_n\right]\!\right]^{N^{\pi(\mathsf{B})}}\right)^{N[G]}$$

Lemmas 4.15 and 4.9 guarantee that this is equivalent to

$$\pi(f)^{-1}[U_n^N] \subseteq \left[\!\left[\tau_{\pi(f)} \in \dot{U}_n\right]\!\right]^{N^{\pi(\mathsf{B})}}$$

Since $\left[\!\left[\tau_{\pi(f)} \in \dot{U}_n\right]\!\right]^{N^{\pi(\mathsf{B})}}$ is clopen this implies that

$$\pi(a_n) \leq \left[\!\left[\tau_{\pi(f)} \in \dot{U}_n\right]\!\right]^{N^{\pi(\mathsf{B})}}$$

and therefore:

$$\left(a_n \leq \left[\!\left[\tau_f \in \dot{U}_n\right]\!\right]\right)^M$$

The thesis in V follows from $M \prec V$. □

Proposition 4.11 is proved. □

Proof of Lemma 4.13. The proof that Σ^H_τ is non-empty when its intersection has one single point can be carried as in Claim 4.14.1 substituting all over the proof $\overline{\mathring{f^{-1}[U_n]}}$ with $\left[\!\left[\tau \in \dot{U}_n\right]\!\right]$.

Preimage of $\{\infty\}$ is nowhere dense: We show that the preimage of $\mathbb{C}$ through f_τ contains an open dense set. Set

$$a_n = \left[\!\left[\tau \in \dot{U}_n\right]\!\right]$$

and consider the set $A = \{a_n : n \in \omega\}$. We show that:

$$\bigvee_{n\in\omega} a_n = 1_{\mathsf{B}}.$$

Let $M \prec V$ be a countable structure such that $\mathsf{B}, \tau \in M$, $\omega, A \subseteq M$, and as usual let $\pi : M \to N$ be the Mostowski's Collapse.

Notice that (since $\omega, \mathbb{C}, \mathbb{Q}, U_n$ are all defined by lightface Δ_1-properties) $\omega^N = \omega = \omega^{N[G]}$, $\mathbb{Q}^N = \mathbb{Q} = \mathbb{Q}^{N[G]}$, $\mathbb{C} \cap N[G] = \mathbb{C}^{N[G]}$, and $U_n \cap N[G] = \pi(\dot{U}_n)_G$ for all G N-generic for $\pi(\mathsf{B})$.

Since τ is a B-name for a complex number in M, $\pi(\tau)$ is a $\pi(\mathsf{B})$-name for a complex number in N. Let G be an N-generic filter over $\pi(\mathsf{B})$, we have therefore:

$$N[G] \models \pi(\tau)^G \in \mathbb{C}.$$

We can thus infer

$$N[G] \models \exists n \in \omega(\pi(\tau)^G \in U_n = \pi(\dot{U}_n)_G)$$

for all N-generic filter G, since $\mathbb{C} \cap N[G] = \bigcup_{n\in\omega} U_n \cap N[G]$. Thus:

$$\bigvee_{n\in\omega} \pi(a_n) = \left[\!\!\left[\exists n \in \check{\omega}(\pi(\tau) \in \pi(\dot{U}_n))\right]\!\!\right] \geq 1_{\pi(\mathsf{B})}.$$

Pulling back the above to $M \prec V$ we get that

$$\bigvee_{n\in\omega} a_n = \left[\!\!\left[\exists n \in \check{\omega}(\tau \in \dot{U}_n)\right]\!\!\right] \geq 1_{\mathsf{B}}$$

holds in M and thus in V. This implies that A is predense and therefore that $\bigcup_{n\in\omega} \mathcal{O}_{a_n}$ is dense in $St(\mathsf{B})$.

Continuous: Let $H \in St(\mathsf{B})$ be in the preimage of $\mathbb{C}$ and let U be an open subset of $\mathbb{C}$ containing $f_\tau(H)$. Consider $U_k \in \mathcal{B}$ such that

$$f_\tau(H) \in U_k$$

$$\overline{U}_k \subseteq U$$

Since

$$f_\tau(H) \in U_k \Rightarrow a_k \in H, \tag{1}$$

(this can be proved as in the uniqueness part in Claim 4.14.1 substituting $\overline{\overset{\circ}{f^{-1}[U_n]}}$ with $\left[\!\!\left[\tau \in \dot{U}_n\right]\!\!\right]$), and since the following inclusion holds

$$\mathcal{O}_{a_k} \subseteq f_\tau^{-1}(U),$$

the continuity of f_τ for points in the preimage of $\mathbb{C}$ is proved.

Consider now $H \in f_\tau^{-1}(\{\infty\})$. Let A be an open neighborhood of ∞, and let $U_k \in \mathcal{B}$ be such that:

$$\overline{U}_k^c \subseteq A$$

We also consider U_l such that

$$\overline{U}_k \subseteq U_l$$

By definition of f_τ we have that $H \in \mathcal{O}_{a_l}^c$, and by equation (1) the image of any element in the open set $\mathcal{O}_{a_l}^c$ can not belong to U_l. Thus

$$\mathcal{O}_{a_l}^c \subseteq f_\tau^{-1}[U_l^c] \subseteq f_\tau^{-1}[\overline{U}_k^c] \subseteq f_\tau^{-1}[A]$$

$\tau_{f_\tau} = \tau$: We already know that (see equation (1)):

$$f_\tau^{-1}[U_n] \subseteq \mathcal{O}_{a_n}$$

The second set is clopen, therefore:

$$\left[\!\left[\tau_{f_\tau} \in \dot{U}_n\right]\!\right] = \overline{\overset{\circ}{f_\tau^{-1}[U_n]}} \subseteq \mathcal{O}_{a_n} \tag{2}$$

Toward a contradiction, assume $[\![\tau = \tau_{f_\tau}]\!] \neq 1_{\mathsf{B}}$. Let $M \prec V$ a countable structure with $\mathsf{B}, \tau, f \in M$, $\omega \subseteq M$, let $\pi : M \to N$ is the Mostowski's Collapse, then there is an N-generic filter G which verifies

$$N[G] \models \pi(\tau)_G \neq \pi(\tau_{f_\tau})_G$$

Thus there exists $n \in \omega$ such that:

$$\pi(\tau_{f_\tau})_G \in U_n^{N[G]}$$

$$\pi(\tau)_G \notin U_n^{N[G]}$$

The inclusion relation (2) implies

$$\left[\!\left[\pi(\tau_{f_\tau}) \in \pi(\dot{U}_n)\right]\!\right] \leq \pi(a_n) = \left[\!\left[\pi(\tau) \in \pi(\dot{U}_n)\right]\!\right],$$

but by Cohen's Forcing Theorem $\left[\!\left[\pi(\tau_{f_\tau}) \in \pi(\dot{U}_n)\right]\!\right] \in G$. This is a contradiction.

The Lemma is proved. □

Proof of Lemma 4.14. We will consider in detail the case of $R \subseteq \mathbb{C}$ a unary Borel relation in $\mathbb{C}$, the general case for n-ary R is immediate. Given $f \in C^+(St(\mathsf{B}))$, consider $[\![R(f)]\!]$ and $\left[\!\left[\tau_f \in \dot{R}\right]\!\right]$ as regular open subsets of $St(\mathsf{B})$. In order to show that they overlap, it is sufficient to prove that their symmetric difference is meager. By definition, we already know that $[\![R(f)]\!]$ has meager difference with the set

$$\{H \in St(\mathsf{B}) : f(H) \in R\} = f^{-1}[R].$$

Therefore it suffices to prove that $\left[\!\left[\tau_f \in \dot{R}\right]\!\right]$ and $f^{-1}[R]$ have meager difference. The proof proceeds step by step on the hierarchy of Borel sets Σ^0_α, Π^0_α, for α a countable ordinal.

<u>Σ^0_1</u>: Let R be an element of the basis

$$\mathcal{B} = \{U_n : n \in \omega\}$$

defined in Remark 4.2. The thesis follows from Proposition 4.11, in fact

$$\left[\!\left[\tau_f \in \dot{U}_n\right]\!\right] = \overline{\overset{\circ}{f^{-1}[U_n]}}$$

which has meager difference with $f^{-1}[U_n]$. Consider now

$$R = \bigcup_{i \in \mathcal{I}} U_i$$

where $\mathcal{I}$ is a countable set of indexes. In this case we have that

$$f^{-1}[R] = \bigcup_{i \in \mathcal{I}} f^{-1}[U_i]$$

and

$$\left\llbracket \tau_f \in \dot{R} \right\rrbracket = \bigvee_{i \in \mathcal{I}} \left\llbracket \tau_f \in \dot{U}_i \right\rrbracket = \overline{\mathring{A}}$$

where $A = \bigcup_{i \in \mathcal{I}} \left\llbracket \tau_f \in \dot{U}_i \right\rrbracket$. For each $i \in \mathcal{I}$, the sets $f^{-1}[U_i]$ and $\left\llbracket \tau_f \in \dot{U}_i \right\rrbracket$ have meager difference, thus $f^{-1}[R] \Delta A$ is meager. The proof is therefore concluded because $A \Delta \overline{\mathring{A}}$ is meager.

$\underline{\Sigma^0_\alpha \Rightarrow \Pi^0_\alpha}$: Suppose $R \in \Pi^0_\alpha$, and that the thesis holds for Borel sets in Σ^0_α. By definition $R^c \in \Sigma^0_\alpha$, therefore:

$$f^{-1}[R^c] \Delta \left\llbracket \tau_f \in \dot{R^c} \right\rrbracket \text{ is meager}$$

hence

$$f^{-1}[R] \Delta \left\llbracket \tau_f \in \dot{R} \right\rrbracket \text{ is meager}$$

$\underline{\Pi^0_\alpha \Rightarrow \Sigma^0_{\alpha+1}}$: This item can be proved as the second part of the case $\alpha = 1$, substituting the U_n with Borel sets in Π^0_α.

$\underline{\Sigma^0_\beta \text{ for } \beta \text{ limit ordinal}}$: If the thesis holds for $\alpha < \beta$, then the proof can be carried similarly to the case $\Pi^0_\alpha \Rightarrow \Sigma^0_{\alpha+1}$.

The Lemma is proved. □

4.1. $C(St(\mathsf{B}))/G$ and $C^+(St(\mathsf{B}))/G$ in generic extensions

The following proposition shows that if we restrict our attention to V-generic filters for B then $C(St(\mathsf{B}))$ is a family of names large enough to describe all complex numbers of $V[G]$.

Proposition 4.16. *Assume V is a model of ZFC, B a complete boolean algebra in V and G a V-generic filter in B. Then*

$$C^+(St(\mathsf{B}))/G \cong C(St(\mathsf{B}))/G$$

Proof. We need to show that for each $f \in C^+(St(\mathsf{B}))$ we can find a $\tilde{f} \in C(St(\mathsf{B}))$ such that

$$\left\llbracket f = \tilde{f} \right\rrbracket \in G$$

which, by Corollary 4.12, is equivalent to

$$f^{V[G]}(G) = \tilde{f}^{V[G]}(G)$$

We denote again:

$$a_n = \overline{\mathring{f^{-1}[U_n]}}$$

Proceeding as in Claim 4.15.1, we can find $m \in \omega$ such that $a_m \in G$. For each $H \in \mathcal{O}_{a_m}$ we have that:

$$f(H) \in \overline{U}_m$$

by Lemma 4.9. We can therefore consider the restriction of f to $\mathcal{O}_{a_m}$ (which is clopen) and extend it to a $\tilde{f} \in C(St(\mathsf{B}))$ setting it to be constantly 0 on $\mathcal{O}_{\neg a_m}$. The implication

$$f \restriction_{\mathcal{O}^V_{a_m}} = \tilde{f} \restriction_{\mathcal{O}^V_{a_m}} \Rightarrow f^{V[G]} \restriction_{\mathcal{O}^{V[G]}_{a_m}} = \tilde{f}^{V[G]} \restriction_{\mathcal{O}^{V[G]}_{a_m}}$$

guarantees the thesis since $G \in \mathcal{O}^{V[G]}_{a_m}$. □

4.2. Extensions of the boolean isomorphism

In general any boolean predicate or function on the B-valued model $C^+(St(\mathsf{B}), X)$ can be transferred to a corresponding boolean predicate on X^{B} using the above isomorphism $f \mapsto \sigma_f$.

Definition 4.17. Let X be a Polish space and B a complete boolean algebra. For any boolean relation $R^{St(\mathsf{B})} : C^+(St(\mathsf{B}), X)^n \to \mathsf{B}$ (and boolean function $F^{St(\mathsf{B})}$)

$$R^{\mathsf{B}}(\sigma_1, \dots, \sigma_n) = R^{St(\mathsf{B})}(f_{\sigma_1}, \dots, f_{\sigma_n}),$$

(accordingly we can define the boolean function F^{B}).

By Theorem 4.6, we immediately have the following.

Theorem 4.18. *Fix a signature*

$$\mathcal{L} = \{R_i : i \in I\} \cup \{F_j : j \in J\}.$$

and assume $R_i^{St(\mathsf{B})} : i \in I$, $F_j^{St(\mathsf{B})} : j \in J$ are boolean interpretations of the signature making $C^+(St(\mathsf{B}))$ a B-valued model. The map

$$\begin{aligned} \Gamma : C^+(St(\mathsf{B}), X) &\to X^{\mathsf{B}} \\ f &\mapsto \tau_f \end{aligned}$$

is an isomorphism of the B-valued model

$$\langle C^+(St(\mathsf{B}), X), R_i^{St(\mathsf{B})} : i \in I, F_j^{St(\mathsf{B})} : j \in J \rangle$$

with the B-valued model

$$\langle X^{\mathsf{B}}, R_i^{\mathsf{B}} : i \in I, F_j^{\mathsf{B}} : j \in J \rangle.$$

5. Generic absoluteness

We can now show that for any polish space X the B-valued models $(C^+(St(\mathsf{B}), R^{St(\mathsf{B})})$, with R a Borel (universally Baire) relation on X^n, is an elementary superstructure of (X, R). By Lemma 4.14, whenever R is a Borel relation on X^n with X Polish, $R^{\mathsf{B}}(\sigma_1, \dots, \sigma_n) = R^{St(\mathsf{B})}(f_{\sigma_1}, \dots, f_{\sigma_n})$ (where R^{B} is defined as in Def. 4.2). This equality is a special case of the much more general result which can be proved for Universally Baire relations.

Definition 5.1 (Feng, Magidor, Woodin [1])**.** Let X be a Polish space. $A \subseteq X^n$ is Universally Baire if $f^{-1}[Y]$ has the Baire property in Y for all continuous $f : Y \to X^n$ and all compact Hausdorff spaces Y.

UB denote the class of universally Baire subsets of $\mathcal{H}$ (or any other Polish space).

Fact 5.2. *Let X be a Polish space. $A \subseteq X^n$ is Universally Baire if and only if $f^{-}[Y]$ has the Baire property in Y for all continuous $f : Y \to X^n$ with Y compact and extremely disconnected.*

Proof. We need to prove just one direction, and we prove it as follows: Assume $f : Y \to X^n$ is continuous for some Y compact Hausdorff but not extremally disconnected. Set $Y^* = St(\mathsf{RO}(Y))$ and define $\pi : Y^* \to Y$ by $\pi(G) = x$ if x is the unique point in Y belonging to

$$\Sigma_G = \{\overline{U} : U \in G\}.$$

The same arguments we encountered in the proof of the isomorphism of $C^+(St(\mathsf{B}))$ with $\mathbb{C}^{\mathsf{B}}$ show that π is continuous, open and surjective. In particular $f^{-1}[X^n]$ has the Baire property in Y iff $g^{-1}[X^n]$ has the Baire property in Y^*, where $g = f \circ \pi$. □

By [2, Chapter 29, Lemma 5] Borel sets are universally Baire as already observed in Example 3.10. Woodin [5, Theorem 3.4.5, Remark 3.4.7] showed that the theory of $L(\mathbb{R}, \mathsf{UB})$ is generically invariant in the presence of class many Woodin cardinals which are a limit of Woodin, and moreover that these assumptions entail that any Σ^1_n-property defines a universally Baire relation. Shoenfield [3, Lemma 25.20], (or [7, Theorem 3.5.3, Remark 3.5.4] or [8, Lemma 1.2]) showed that the Σ^1_2-theory of any Polish space X is generically invariant under set forcing. This translates by the results of this paper in the following:

Theorem 5.3. *Assume $R_i : i \in I$ and $F_j : j \in J$ are Borel predicates and functions on some Polish space Y. Let X be a compact Hausdorff extremely disconnected space and $p \in X$. Then*

$$\langle Y, R_i : i \in I, F_j : j \in J\rangle \prec_{\Sigma_2} \langle C^+(X,Y)/p, R_i^X/p : i \in I, F_j^X/p : j \in J\rangle.$$

Moreover if we assume the existence of class many Woodin cardinals which are a limit of Woodin cardinal then we can let each R_i and F_j be arbitrary universally Baire relations and functions and we have the stronger conclusion that

$$\langle Y, R_i : i \in I, F_j : j \in J\rangle \prec \langle C^+(X,Y)/p, R_i^X/p : i \in I, F_j^X/p : j \in J\rangle.$$

Proof. By Shoenfield (or Woodin's) theorem we have that for all Σ^1_2 (Σ^1_n for any n) property $\varphi(\vec{x})$ in the parameters $R_i : i \in I$, $F_j : j \in J$ with each R_i, F_j Borel (Universally Baire) the following are equivalent:

- $\varphi(\vec{r})$ holds in $\langle Y, R_i : i \in I, F_j : j \in J\rangle$,
- $[\![\varphi(\vec{r})]\!]^{V^{\mathsf{B}}} = 1_{\mathsf{B}}$ in $\mathbb{C}^{\mathsf{B}}$ for some complete boolean algebra B,
- $[\![\varphi(\vec{r})]\!]^{V^{\mathsf{B}}} = 1_{\mathsf{B}}$ in $\mathbb{C}^{B}$ for all complete boolean algebras B.

Since X is compact Hausdorff and extremely disconnected, $\mathsf{CL}(X)$ is a complete boolean algebra and X is homeomorphic to $St(\mathsf{CL}(X))$. By Theorem 4.18 $C^+(X,Y)$ and Y^{B} are isomorphic B-valued models. In particular $C^+(X,Y)$ is full. By the first two equivalent items we get that $[\![\varphi(\vec{r})]\!]^{C^+(X,Y)} = 1_{\mathsf{B}}$ in $C^+(X,Y)$ if and only if $\varphi(\vec{r})$ holds in Y. Since the above holds for all relevant property φ we can apply Los theorem to the full B-valued models $C^+(X,Y)$ and in the point (ultrafilter) p to conclude that

$$\langle Y, R_i : i \in I, F_j : j \in J\rangle \prec_{(\Sigma_2)} \langle C^+(X,Y)/p, R_i^X/p : i \in I, F_j^X/p : j \in J\rangle.$$

□

We remark that these results suggest the following "original" proof strategy. Prove that a certain problem regarding for example complex numbers and analytic functions has a solution in some forcing extension. Then argue that its solution can be formalized as a first order property of the structure $C^+(X)/p$. Conclude using elementarity that the solution of the problem for the complex numbers is really the one computed in $C^+(X)/p$. We have already successfully applied the above to prove a result related to Schanuel's conjecture in number theory (unfortunately for us already proved by other means): the interested reader is referred to [9].

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References

[1] Qi Feng, Menachem Magidor, and Hugh Woodin, *Universally Baire sets of reals*, Set theory of the continuum (Berkeley, CA, 1989) (H. Judah, W. Just, and H. Woodin, eds.), Math. Sci. Res. Inst. Publ., vol. 26, Springer, New York, 1992, pp. 203–242. MR MR1233821 (94g:03095)

[2] Steven Givant and Paul Halmos, *Introduction to Boolean algebras*, Undergraduate Texts in Mathematics, Springer, New York, 2009. MR 2466574 (2009j:06001)

[3] Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513 (2004g:03071)

[4] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597 (96e:03057)

[5] Paul B. Larson, *The stationary tower*, University Lecture Series, vol. 32, American Mathematical Society, Providence, RI, 2004, Notes on a course by W. Hugh Woodin. MR 2069032

[6] Helena Rasiowa and Roman Sikorski, *The mathematics of metamathematics*, Monografie Matematyczne, no. 41, Polish Scientific Publishers, 1963.

[7] Andrea Vaccaro, *C*-algebras and B-names for complex numbers*, (2015), Unpublished Master Thesis, (https://etd.adm.unipi.it/t/etd-08302015-193406/).

[8] Matteo Viale, *Martin's maximum revisited*, (2012), (http://arxiv.org/abs/1110.1181).

[9] ———, *Forcing the truth of a weak form of Schanuel's conjecture*, (2015), submitted.