Power Kripke-Platek set theory and the axiom of choice

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Abstract

Whilst Power Kripke-Platek set theory, $\mathbf{KP}(\mathcal{P})$, shares many properties with ordinary Kripke-Platek set theory, $\mathbf{KP}$, in several ways it behaves quite differently from $\mathbf{KP}$. This is perhaps most strikingly demonstrated by a result, due to Mathias, to the effect that adding the axiom of constructibility to $\mathbf{KP}(\mathcal{P})$ gives rise to a much stronger theory, whereas in the case of $\mathbf{KP}$ the constructible hierarchy provides an inner model, so that $\mathbf{KP}$ and $\mathbf{KP} + V = L$ have the same strength.

This paper will be concerned with the relationship between $\mathbf{KP}(\mathcal{P})$ and $\mathbf{KP}(\mathcal{P})$ plus the axiom of choice or even the global axiom of choice, $\mathbf{AC}_{global}$. Since L is the standard vehicle to furnish a model in which this axiom holds, the usual argument for demonstrating that the addition of $\mathbf{AC}$ or $\mathbf{AC}_{global}$ to $\mathbf{KP}(\mathcal{P})$ does not increase proof-theoretic strength does not apply in any obvious way. Among other tools, the paper uses techniques from ordinal analysis to show that $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$ has the same strength as $\mathbf{KP}(\mathcal{P})$, thereby answering a question of Mathias. Moreover, it is shown that $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$ is conservative over $\mathbf{KP}(\mathcal{P})$ for Π_4^1 statements of analysis.

The method of ordinal analysis for theories with power set was developed in an earlier paper. The technique allows one to compute witnessing information from infinitary proofs, providing bounds for the transfinite iterations of the power set operation that are provable in a theory.

As the theory $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$ provides a very useful tool for defining models and realizability models of other theories that are hard to construct without access to a uniform selection mechanism, it is desirable to determine its exact proof-theoretic strength. This knowledge can for instance be used to determine the strength of Feferman's operational set theory with power set operation as well as constructive Zermelo-Fraenkel set theory with the axiom of choice.

1   Introduction

A previous paper [37] gave a characterization of the smallest segment of the von Neumann hierarchy which is closed under the provable power-recursive functions of $\mathbf{KP}(\mathcal{P})$. It also furnished a proof-theoretic reduction of $\mathbf{KP}(\mathcal{P})$ to Zermelo set theory plus iterations of the powerset operation to any ordinal below the Bachmann-Howard ordinal.[1] The same bound also holds for the theory $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}$, where $\mathbf{AC}$ stands for the *axiom of choice*. These theorems considerably sharpen results of H. Friedman to the extent that $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}$ does not prove the existence of the first non-recursive ordinal ω_1^{CK} (cf. [8, Theorem 2.5] and [15, Theorem 10]). However, [37] did not explicitly address the question of the proof-theoretic strength of $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}$. In the present paper it will be shown that even adding global choice, $\mathbf{AC}_{global}$, to $\mathbf{KP}(\mathcal{P})$ does not increase its proof-theoretic strength. This is in stark contrast to the axiom of constructibility $V = L$ which increases it as was shown by Mathias [14]. That this increase is enormous will also been borne out by the results of this paper.

Since the global axiom of choice, $\mathbf{AC}_{global}$, is less familiar, let us spell out the details. By $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$ we mean an extension of $\mathbf{KP}(\mathcal{P})$ where the language contains a new binary relation symbol R and the axiom schemes of $\mathbf{KP}(\mathcal{P})$ are extended to this richer language and the following axioms pertaining to R are added:

$$(i) \qquad \forall x \forall y \forall z [R(x,y) \wedge R(x,z) \rightarrow y = z] \tag{1}$$

$$(ii) \qquad \forall x [x \neq \emptyset \rightarrow \exists y \in x \, R(x,y)]. \tag{2}$$

2   Power Kripke-Platek set theory

Before stating the axioms of $\mathbf{KP}(\mathcal{P})$, let us recall the axioms of $\mathbf{KP}$. Roughly speaking, $\mathbf{KP}$ arises from $\mathbf{ZF}$ by completely omitting the power set axiom and restricting separation and collection to set bounded formulae but adding set induction (or class foundation). Quantifiers of the forms $\forall x \in a$, $\exists x \in a$

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[1]The theories share the same $\Sigma_1^{\mathcal{P}}$ theorems, but are still distinct since Zermelo set theory does not prove $\Delta_0^{\mathcal{P}}$-Collection whereas $\mathbf{KP}(\mathcal{P})$ does not prove full Separation.

are called *set bounded. Set bounded* or Δ_0-*formulae* are formulae wherein all quantifiers are set bounded. The axioms of **KP** consist of *Extensionality, Pair, Union, Infinity,* Δ_0-*Separation*

$$\exists x\,\forall u\,[u \in x \leftrightarrow (u \in a\,\wedge\,A(u))]$$

for all Δ_0-formulae $A(u)$, Δ_0-*Collection*

$$\forall x \in a\,\exists y\,G(x,y)\,\rightarrow\,\exists z\,\forall x \in a\,\exists y \in z\,G(x,y)$$

for all Δ_0-formulae $G(x,y)$, and *Set Induction*

$$\forall x\,[(\forall y \in x\,C(y)) \rightarrow C(x)]\,\rightarrow\,\forall x\,C(x)$$

for all formulae $C(x)$.

A transitive set A such that $(A, \in)$ is a model of **KP** is called an *admissible set*. Of particular interest are the models of **KP** formed by segments of Gödel's *constructible hierarchy* **L**. An ordinal α is *admissible* if the structure $(\mathbf{L}_\alpha, \in)$ is a model of **KP**.

KP is an important set theory as a great deal of set theory requires only the axioms of **KP** and its standard models, the admissible sets, have been a major source of interaction between model theory, recursion theory and set theory (cf. [2]). *Power Kripke-Platek set theory* is obtained from **KP** by also viewing the creation of the powerset of any set as a basic operation performed on sets. In the classical context, subsystems of **ZF** with Bounded Separation and Power Set have been studied by Thiele [42], Friedman [8] and more recently in great depth by Mathias [15]. They also occur naturally in power recursion theory, investigated by Moschovakis [17] and Moss [18], where one studies a notion of computability on the universe of sets which regards the power set operation as an initial function. Semi-intuitionistic set theories with Bounded Separation but containing the Power Set axiom were proposed by Pozsgay [19, 20] and then studied more systematically by Tharp [41], Friedman [7] and Wolf [44]. Such theories are naturally related to systems derived from topos-theoretic notions and to type theories (e.g., see [36]). Mac Lane has singled out and championed a particular fragment of **ZF**, especially in his book *Form and Function* [14]. *Mac Lane Set Theory*, christened **MAC** in [15], comprises the axioms of Extensionality, Null Set, Pairing, Union, Infinity, Power Set, Bounded Separation, Foundation, and Choice.

To state the axioms of **KP**$(\mathcal{P})$ it is convenient to introduce another type of bounded quantifiers.

Definition 2.1. We use subset bounded quantifiers $\exists x \subseteq y \ldots$ and $\forall x \subseteq y \ldots$ as abbreviations for $\exists x (x \subseteq y \wedge \ldots)$ and $\forall x (x \subseteq y \rightarrow \ldots)$, respectively.

The $\Delta_0^{\mathcal{P}}$-formulae are the smallest class of formulae containing the atomic formulae closed under $\wedge, \vee, \rightarrow, \neg$ and the quantifiers

$$\forall x \in a, \ \exists x \in a, \ \forall x \subseteq a, \ \exists x \subseteq a.$$

A formula is in $\Sigma^{\mathcal{P}}$ if belongs to the smallest collection of formulae which contains the $\Delta_0^{\mathcal{P}}$-formulae and is closed under $\wedge, \vee$ and the quantifiers $\forall x \in a, \ \exists x \in a, \ \forall x \subseteq a$ and $\exists x$. A formula is $\Pi^{\mathcal{P}}$ if belongs to the smallest collection of formulae which contains the $\Delta_0^{\mathcal{P}}$-formulae and is closed under $\wedge, \vee$, the quantifiers $\forall x \in a, \ \exists x \in a, \ \forall x \subseteq a$ and $\forall x$.

Definition 2.2. $\mathbf{KP}(\mathcal{P})$ has the same language as $\mathbf{ZF}$. Its axioms are the following: Extensionality, Pairing, Union, Infinity, Powerset, $\Delta_0^{\mathcal{P}}$-Separation, $\Delta_0^{\mathcal{P}}$-Collection and Set Induction (or Class Foundation).

The transitive models of $\mathbf{KP}(\mathcal{P})$ have been termed **power admissible** sets in [8].

Remark 2.3. Alternatively, $\mathbf{KP}(\mathcal{P})$ can be obtained from $\mathbf{KP}$ by adding a function symbol $\mathcal{P}$ for the powerset function as a primitive symbol to the language and the axiom

$$\forall y \, [y \in \mathcal{P}(x) \leftrightarrow y \subseteq x]$$

and extending the schemes of Δ_0 Separation and Collection to the Δ_0-formulae of this new language.

Lemma 2.4. $\mathbf{KP}(\mathcal{P})$ *is* **not** *the same theory as* $\mathbf{KP} + \mathbf{Pow}$. *Indeed,* $\mathbf{KP} + \mathbf{Pow}$ *is a much weaker theory than* $\mathbf{KP}(\mathcal{P})$ *in which one cannot prove the existence of* $V_{\omega+\omega}$.

Proof: See [37, Lemma 2.4]. □

Remark 2.5. The system $\mathbf{KP}(\mathcal{P})$ in the present paper is not quite the same as the theory $\mathbf{KP}^{\mathcal{P}}$ in Mathias' paper [15, 6.10]. The difference between $\mathbf{KP}(\mathcal{P})$ and $\mathbf{KP}^{\mathcal{P}}$ is that in the latter system set induction only holds for $\Sigma_1^{\mathcal{P}}$-formulae, or what amounts to the same, $\Pi_1^{\mathcal{P}}$ foundation ($A \neq \emptyset \rightarrow \exists x \in A \ x \cap A = \emptyset$ for $\Pi_1^{\mathcal{P}}$ classes A).

Friedman [8] includes only Set Foundation in his formulation of a formal system $\mathbf{PAdm}^s$ appropriate to the concept of recursion in the power set operation $\mathcal{P}$.

3 Extracting additional explicit results from ordinal analysis

[37] featured an ordinal analysis of $\mathbf{KP}(\mathcal{P})$. As it turns out the technique can be augmented to also yield an ordinal analysis of $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$. The changes mainly concern the infinitary system $RS_\Omega^\mathcal{P}$ of [37, §5]. The modified infinitary system $RS_\Omega^\mathcal{P}(\mathsf{R})$ results from $RS_\Omega^\mathcal{P}$ by the following changes:

(i) $RS_\Omega^\mathcal{P}(\mathsf{R})$-terms are defined as in Definition 5.1 of [37], except that in clause 3, $F(\vec{x}, y)$ is allowed to be any $\Delta_0^\mathcal{P}$-formula of $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$, i.e., it may contain the relation symbol R.

(ii) The axioms and rules of $RS_\Omega^\mathcal{P}$ in Definition 5.3 of [37], have to be formulated with respect to the richer language, i.e., $\Delta_0^\mathcal{P}$ and $\Sigma^\mathcal{P}$ refer to the language with R as a basic symbol. This affects (A1), (A3), (A6), (A7), and the rule $\Sigma^\mathcal{P}\text{-}Ref$. Moreover, to these axioms one adds two new ones:

$$(A8) \qquad \Gamma, \neg(\exists y \in t)(y \in t), (\exists y \in t)\,\mathsf{R}(t, y) \qquad\qquad (3)$$

$$(A9) \qquad \Gamma, \neg\mathsf{R}(t, s), \neg\mathsf{R}(t, r), s = r\,. \qquad\qquad (4)$$

With these changes, the embedding of $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$ into the infinitary proof system (cf. [37, Theorem 6.9]) and cut elimination in $RS_\Omega^\mathcal{P}(\mathsf{R})$ proceed in exactly the same way as for $RS_\Omega^\mathcal{P}$ in [37, §7], yielding the following result:

Corollary 3.1. *Let A be a $\Sigma^\mathcal{P}$-sentence of $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$. Suppose that $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global} \vdash A$. Then there exists an operator $\mathcal{H}$ and an ordinal $\rho < \psi_\Omega(\varepsilon_{\Omega+1})$ such that*
$$\mathcal{H} \,\vdash_\rho^\rho A\,.$$

    Proof: The same proof as in [37, Corollary 7.7] works here, too. $\mathcal{H}$ and ρ can be explicitly constructed from the proof of A in $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$. $\square$

    A refinement of [37, Theorem 8.1] then yields partial conservativity of $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$ over $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}$.

Theorem 3.2. *Let A be a $\Sigma^\mathcal{P}$ sentence of the language of set theory without R. If $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global} \vdash A$ then $\mathbf{KP}(\mathcal{P}) + \mathbf{AC} \vdash A$.*

    Proof: Suppose $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global} \vdash \theta$, where θ is a $\Sigma^\mathcal{P}$-sentence. It follows from Corollary 3.1 that one can explicitly find $\mathcal{H}$ and $\tau < \psi_\Omega(\varepsilon_{\Omega+1})$ such that $\mathcal{H} \,\vdash_\rho^\rho A$. The refinement of [37, Theorem 8.1] consists in noticing

that if $\mathcal{H}(\emptyset) \subseteq C^\Omega(\omega_m(\omega^{\Omega+m}), 0)$ and $\Gamma(s_1, \ldots, s_n)$ is a sequent consisting only of $\Sigma^\mathcal{P}$-formulae and

$$\mathcal{H} \vdash^\alpha_\rho \Gamma(s_1, \ldots, s_n)$$

with $\alpha, \rho < \Omega$, then, for all variable assignments $\ell : VAR \to V_{\psi_\Omega(\omega_m(\omega^{\Omega+m}))}$,

$$V_{\psi_\Omega(\omega_m(\omega^{\Omega+m}))} \models \Gamma(\ell(s_1), \ldots, \ell(s_n)),$$

if the predicate R is interpreted as the graph of a choice function on $V_{\psi_\Omega(\omega_m(\omega^{\Omega+m}))}$. The entire ordinal analysis can thus be carried out in $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}$ since this theory proves the existence of $V_{\psi_\Omega(\omega_m(\omega^{\Omega+m}))}$ as well as a choice function on this set (i.e. a function f defined on this set satisfying $f(x) \in x$ whenever $x \neq \emptyset$). As A is $\Sigma^\mathcal{P}$ and R does not occur in A, A must be true. This shows that $\mathbf{KP}(\mathcal{P}) + \mathbf{AC} \vdash A$. $\qquad\square$

The next result extracts further information from infinitary proofs in $RS^\mathcal{P}_\Omega$.

Theorem 3.3. *If* $\mathbf{KP}(\mathcal{P}) + \mathbf{AC} \vdash \theta$, *where* θ *is a* $\Sigma^\mathcal{P}$*-sentence, then one can explicitly find an ordinal (notation)* $\tau < \psi_\Omega(\varepsilon_{\Omega+1})$ *such that*

$$\mathbf{KP} + \mathbf{AC} + \texttt{the von Neumann hierarchy } (V_\alpha)_{\alpha \leq \tau} \texttt{ exists} \vdash \theta.$$

Proof: First note that $\mathbf{AC}$ can be formulated as a $\Pi^\mathcal{P}_1$-sentence. From $\mathbf{KP}(\mathcal{P}) + \mathbf{AC} \vdash \theta$ one obtains $\mathbf{KP}(\mathcal{P}) \vdash \mathbf{AC} \to \theta$. As the latter statement is equivalent to a $\Sigma^\mathcal{P}$-sentence, it follows from [37, Corollary 7.7] and a refinement of [37, Theorem 8.1] (as above) that one can explicitly find a $\tau < \psi_\Omega(\varepsilon_{\Omega+1})$ such that

$$V_\tau \models \mathbf{AC} \to \theta. \tag{5}$$

As before, the refinement of [37, Theorem 8.1] consists in noticing that if $\mathcal{H}(\emptyset) \subseteq C^\Omega(\omega_m(\omega^{\Omega+m}), 0)$ and $\Gamma(s_1, \ldots, s_n)$ is a sequent consisting only of $\Sigma^\mathcal{P}$-formulae and

$$\mathcal{H} \vdash^\alpha_\rho \Gamma(s_1, \ldots, s_n)$$

with $\alpha, \rho < \Omega$, then, for all variable assignments $\ell : VAR \to V_{\psi_\Omega(\omega_m(\omega^{\Omega+m}))}$,

$$V_{\psi_\Omega(\omega_m(\omega^{\Omega+m}))} \models \Gamma(\ell(s_1), \ldots, \ell(s_n)).$$

The entire ordinal analysis can be carried out in

$$\mathbf{KP} + \texttt{the hierarchy } (V_\delta)_{\delta < \tau} \texttt{ exists}$$

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for a suitable τ, e.g. $\tau = \psi_\Omega(\omega_m(\omega^{\Omega+m})) + \omega$. Thus

$$\mathbf{KP} + \mathbf{AC} + \texttt{the hierarchy } (V_\delta)_{\delta<\tau} \texttt{ exists} \vdash \theta.$$

$\square$

Below we shall talk about well-orderings $\prec$. The field of $\prec$ is the set $\{u \mid \exists v\, (u \prec v \vee v \prec u)\}$. If $u \in \mathrm{Field}(\prec)$ we denote by $\prec\!\!\restriction u$ the ordering $\prec$ restricted to the set $\{v \mid v \prec u\}$. $\prec\!\!\restriction u$ is said to be an **initial segment** of $\prec$. We say that $\prec$ is a **cardinal** if there is no bijection between $\mathrm{Field}(\prec)$ and $\{v \mid v \prec u\}$ for any $u \in \mathrm{Field}(\prec)$.

Theorem 3.4. *Let τ be a limit ordinal. If*

$$\mathbf{KP} + \mathbf{AC} + \texttt{the von Neumann hierarchy } (V_\alpha)_{\alpha<\tau} \texttt{ exists}$$

proves a Π^1_4 statements Φ of second order arithmetic, then

$$\mathbf{Z} + \texttt{the von Neumann hierarchy } (V_\alpha)_{\alpha<\tau\cdot4+4} \texttt{ exists}$$

proves Φ.

Proof: We briefly recall the proof that $\mathbf{ZF}$ and $\mathbf{ZFC}$ prove the same Π^1_4 statements of second order arithmetic. Assume $\mathbf{ZFC} \vdash \Phi$, where

$$\Phi \;=\; \forall x \subseteq \omega \exists y \subseteq \omega \forall u \subseteq \omega \exists v \subseteq \omega\, \theta(x,y,u,v)$$

with $\theta(x,y,u,v)$ an arithmetic formula. Now fix an arbitrary $x \subseteq \omega$ and build the relativized constructible hierarchy $L(x)$ which is a model of $\mathbf{ZFC}$, assuming $\mathbf{ZF}$ in the background. Then $L(x) \models \forall u \subseteq \omega \exists v \subseteq \omega\theta(x,y,u,v)$ for some $y \subseteq \omega$ with $y \in L(x)$. It then follows from a version of Shoenfield's Absoluteness Lemma with subsets of ω as parameters that $\forall u \subseteq \omega \exists v \subseteq \omega\theta(x,y,u,v)$ holds in V and hence Φ holds in V.

We would like to simulate the foregoing proof in the background theory

$$\mathbf{Z} + \texttt{the von Neumann hierarchy } (V_\alpha)_{\alpha\leq\tau\cdot4+4} \texttt{ exists}.$$

The idea is basically that for a given $x \subseteq \omega$ and a well-ordering $\prec$ one can simulate an initial segment of the constructible hierarchy $L(x)$ along $\prec$. Let's denote this by $L_\prec(x)$. $L_\prec(x)$ is basically a set of formal terms built from the elements of the field of $\prec$. However, we also need to equip $L_\prec(x)$ with an equivalence relation $\approx$ such that $s \approx s'$ signifies that s and s' denote the 'same' set, and an elementhood relation ε such that $s\varepsilon t \wedge s \approx s' \to s'\varepsilon t$.

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This is a well known procedure in proof theory, so we shall not dwell on the details. For another approach based on the Gödel functions see [15].

We need a sufficiently long well-ordering to guarantee that $(L_\prec(x), \approx, \varepsilon)$ is a model of T. We resort to Hartog's construction (see e.g. [16, 7.34]). Given a set A let $<_{h(A)}$ be the well-ordering that arises from the set of all well-orderings on subsets of A by singling out equivalence classes

$$[\lessdot]_A = \{\lhd \mid \lhd \text{ is a well-ordering of a subset of } A \text{ order-isomorphic to } \lessdot\}$$

and setting

$$[\lessdot_1]_A <_{h(A)} [\lessdot_2]_A \text{ iff } \lessdot_1 \text{ is isomorphic to a proper intial segment of } \lessdot_2.$$

Since $\mathbf{Z}$ has the powerset axiom and proves comparability of well-orderings, one can show that $<_{h(A)}$ is a well-ordering; moreover, $<_{h(A)}$ is the smallest well-ordering such that there is no injection of its field into A (see [16, 7.34]). As a result, there is no bijection between the field of $<_{h(A)}$ and the field of a proper initial segment of $<_{h(A)}$, that is, $<_{h(A)}$ behaves like a cardinal.

Now, if $\lessdot \subseteq V_\alpha \times V_\alpha$ then $\lessdot \in V_{\alpha+3}$ and hence $[\lessdot]_{V_\alpha} \in V_{\alpha+4}$. Therefore $<_{h(V_\alpha)}$ is a well-ordering on a subset of $V_{\alpha+4}$ and hence $<_{h(V_\alpha)}$ has a smaller order-type than $<_{h(V_{\alpha+4})}$, that is $<_{h(V_\alpha)}$ is order-isomorphic to a proper initial segment of $<_{h(V_{\alpha+4})}$. Finally set

$$\begin{aligned} \prec \;&:=\; <_{h(V_{\tau \cdot 4 + \omega})} \\ L^*(x) \;&:=\; (L_\prec(x), \approx, \varepsilon). \end{aligned}$$

Our goal is to show that $L^*(x)$ is a model of

$$T := \mathbf{KP} + \mathbf{AC} + \texttt{the von Neumann hierarchy } (V_\alpha)_{\alpha<\tau} \texttt{ exists.}$$

It is a straightforward matter to show that $L^*(x)$ satisfies Pairing, Union, Infinity, and Δ_0-separation, similarly as one shows that any L_λ with λ a limit $> \omega$ satisfies these axioms. Next we address Δ_0-Collection. Note that $L^*(x)$ is a set with a definable well-ordering. So we have definable Skolem functions on L^*. Moreover, for any $v \in \text{Field}(\prec)$ which is a limit one has a bijection between $L_{\prec \upharpoonright v}(x)$ and $\prec \upharpoonright v$. Now suppose $L^*(x) \models \forall u \in a \, \exists y \, \Psi(u, y, \vec{b})$ with all parameters exhibited. Then there exists $v \in \text{Field}(\prec)$ of limit type such that $a, \vec{b} \in L_{\prec \upharpoonright v}(x)$. Let Y be the Σ_1 Skolem hull of $L_{\prec \upharpoonright v}(x)$ in L^*. Note that there is a bijection between Y and and $\text{Field}(\prec \upharpoonright v)$. Then, similarly as in the Condensation Lemma (e.g. [5, II.5.2]) one proves that Y is isomorphic to some $L_{\prec \upharpoonright y}(x)$ with $v \preceq y$ and $y \in \text{Field}(\prec)$. The latter follows since $\prec$

is a cardinal. As a result, $L^*(x) \models \forall u \in a \, \exists y \in L_{\prec\restriction y}(x) \, \Psi(u, y, \vec{b})$, showing Δ_0-Collection.

Finally we have to show that in L^* the powerset operation can be iterated at least τ times. By the above we know that all orderings $<_{h(V_{\alpha \cdot 4})}$ for $\alpha < \tau$ are isomorphic to initial segments $\prec\restriction u_\alpha$ of $\prec$ and that they form an increasing sequence of cardinals. Using again a condensation argument (e.g. [5, II.5.5]) one shows that

$$L^*(x) \models \mathcal{P}(L_{\prec\restriction u_\alpha}(x)) \subseteq L_{\prec\restriction u_{\alpha+1}}(x)$$

from which it follows that

$$L^* \models \text{the von Neuman hierarchy } (V_\beta)_{\beta < \tau} \text{ exists.}$$

Since $L^*(x)$ also has a Δ_1 definable well-ordering $\mathbf{AC}$ holds in $L^*(x)$. □

Corollary 3.5. *If Φ is Π_4^1 sentence such that $\mathbf{KP}(\mathcal{P}) + \mathbf{AC} \vdash \Phi$ then $\mathbf{KP}(\mathcal{P}) \vdash \Phi$.*

Proof: Suppose $\mathbf{KP}(\mathcal{P}) + \mathbf{AC} \vdash \Phi$ with Φ being Π_4^1. Then Φ is also a $\Sigma^{\mathcal{P}}$-sentence. By Theorem 3.3 one can explicitly find an ordinal (notation) $\tau < \psi_\Omega(\varepsilon_{\Omega+1})$ such that

$\mathbf{KP} + \mathbf{AC} + \texttt{the von Neumann hierarchy } (V_\alpha)_{\alpha \leq \tau} \texttt{ exists} \vdash \theta.$

With Theorem 3.4 we have

$\mathbf{Z} + \texttt{the von Neumann hierarchy } (V_\alpha)_{\alpha < \tau \cdot 4 + 4} \texttt{ exists} \vdash \Phi.$

Now, $\mathbf{KP}(\mathcal{P})$ proves the existence of the ordinal $\tau \cdot 4 + \omega$ and also proves that $V_{\tau \cdot 4 + \omega}$ is a model of $\mathbf{Z}$, it follows that $V_{\tau \cdot 4 + \omega} \models \Phi$, whence Φ holds. □

4 On Mathias' theory $\mathbf{KP}^{\mathcal{P}}$ and a miscellany of related work and questions

As explained in the introduction, the difference between $\mathbf{KP}(\mathcal{P})$ and $\mathbf{KP}^{\mathcal{P}}$ (see [15, 6.10]) is that in the latter system set induction only holds for $\Sigma_1^{\mathcal{P}}$-formulae.

Let φ be the usual two place Veblen function (see [43]) used by proof theorists. It is by now well-known how one deals with $\Sigma_1^{\mathcal{P}}$ set induction proof-theoretically, employing partial cut elimination and an asymmetric

interpretation of quantifiers. Without the powerset function this is dealt with in [13, §3]. With the powerset function it is carried out in [4] and also follows from [23, 6.1] if one substitutes for G the powerset function. The upshot of this work is the following Theorem.

Theorem 4.1. *If* $\mathbf{KP}^{\mathcal{P}} + \mathbf{AC} \vdash \theta$, *where* θ *is a* $\Sigma^{\mathcal{P}}$-*sentence, then one can explicitly find an ordinal (notation)* $\tau < \varphi_{\omega}(0)$ *such that*

$\quad$ $\mathbf{KP} + \mathbf{AC} + $ `the von Neumann hierarchy` $(V_{\alpha})_{\alpha \leq \tau}$ `exists` $\vdash \theta$.

As a consequence, one arrives at the following:

Corollary 4.2. *If* Φ *is* Π^1_4 *sentence such that* $\mathbf{KP}^{\mathcal{P}} + \mathbf{AC} \vdash \Phi$ *then* $\mathbf{KP}^{\mathcal{P}} \vdash \Phi$.

$\quad$ **Proof**: Suppose $\mathbf{KP}^{\mathcal{P}} + \mathbf{AC} \vdash \Phi$ with Φ being Π^1_4. Then Φ is also a $\Sigma^{\mathcal{P}}$-sentence. By Theorem 4.1 one can explicitly find an ordinal (notation) $\tau < \varphi_{\omega}(0)$ such that

$\quad$ $\mathbf{KP}^{\mathcal{P}} + \mathbf{AC} + $ `the von Neumann hierarchy` $(V_{\alpha})_{\alpha \leq \tau}$ `exists` $\vdash \theta$.

With Theorem 3.4 we have

$\quad$ $\mathbf{Z} + $ `the von Neumann hierarchy` $(V_{\alpha})_{\alpha < \tau \cdot 4 + 4}$ `exists` $\vdash \Phi$.

Now, since $\mathbf{KP}^{\mathcal{P}}$ proves the existence of the ordinal $\tau \cdot 4 + \omega$ and also proves that $V_{\tau \cdot 4 + \omega}$ is a model of $\mathbf{Z}$, it follows that $V_{\tau \cdot 4 + \omega} \models \Phi$, whence Φ holds. $\quad\square$

One can slightly strengthen the previous result by also adding $\Pi^{\mathcal{P}}_1$ set induction which is the same as $\Sigma^{\mathcal{P}}_1$ foundation.

Let $\Delta^{\mathcal{P}}_0$-DC be the schema saying that whenever $\forall x \exists y R(x, y)$ holds for a $\Delta^{\mathcal{P}}$ predicate R then for every set z there exists a function f with domain ω such that $f(0) = z$ and

$$\forall n \in \omega\, R(f(n), f(n+1)).$$

Lemma 4.3. $\mathbf{KP}^{\mathcal{P}} + \mathbf{AC} \vdash \Delta^{\mathcal{P}}_0$-DC.

$\quad$ **Proof**: We argue in $\mathbf{KP}^{\mathcal{P}} + \mathbf{AC}$. Assume $\forall x \exists y R(x, y)$. In $\mathbf{KP}^{\mathcal{P}}$ we have the function $\alpha \mapsto V_{\alpha}$. Fix a set z. Using $\Delta^{\mathcal{P}}_0$ Collection we get

$$\forall \alpha \exists \beta\, [z \in V_{\beta} \ \wedge\ \forall x \in V_{\alpha}\, \exists y \in V_{\beta}\, R(x, y)].$$

By taking the smallest β we get a $\Sigma^{\mathcal{P}}_1$ class function F such that

$$\forall \alpha [z \in V_{F(\alpha)} \ \wedge\ \forall x \in V_{\alpha}\, \exists y \in V_{F(\alpha)}\, R(x, y)].$$

Now let $\alpha_0 := 0$, $\alpha_{n+1} = F(\alpha_n)$, and $\tau := \sup_{n \in \omega} \alpha_n$. Then $z \in V_\tau \wedge \forall x \in V_\tau \exists y \in V_\tau R(x, y)$. Using **AC** there is a choice function $h : V_\tau \to V_\tau$ such that $R(x, h(x))$ holds for all $x \in V_\tau$. Now define $f : \omega \to V_\tau$ by $f(0) = z$ and $f(n+1) = f(f(n))$. Then $R(f(n), f(n+1))$ holds for all $n \in \omega$. $\qquad \square$

Lemma 4.4. $\mathbf{KP}^\mathcal{P} + \mathbf{AC} \vdash \Pi_1^\mathcal{P}$ *set induction.*

Proof: We argue in $\mathbf{KP}^\mathcal{P} + \mathbf{AC}$. Suppose we have a counterexample to $\Pi_1^\mathcal{P}$ set induction. Then there is a $\Pi_1^\mathcal{P}$ predicate $Q(x) = \forall u\, S(u, x)$ with S $\Delta_0^\mathcal{P}$ and a set a_0 such that $\neg Q(a_0)$ and $\forall x\, [\forall y \in x\, Q(y) \to Q(x)]$. The latter is equivalent to $\forall x\, [\neg Q(x) \to \exists y \in x\, \neg Q(y)]$ which yields

$$\forall x\, \forall u\, \exists y\, \exists v\, [\neg S(u, x) \to y \in x \wedge \neg S(y, v)].$$

The bracketed part of the latter statement is $\Delta_0^\mathcal{P}$. $\neg Q(a_0)$ entails that there exists b_0 such that $\neg S(a_0, b_0)$. Thus by Lemma 4.3 there exists a function f with domain ω such that $f(0) = \langle a_0, b_0 \rangle$ and, for all $n \in \omega$, $f(n)$ is a pair $\langle a_n, b_n \rangle$ such that

$$\neg S(a_n, b_n) \to a_{n+1} \in a_n \wedge \neg S(a_{n+1}, b_{n+1}).$$

By induction on n we get $S(a_n, b_n) \wedge a_{n+1} \in a_n$ for all n, contradicting $\Delta_0^\mathcal{P}$ foundation. $\qquad \square$

Corollary 4.5. *If Φ is Π_4^1 sentence such that $\mathbf{KP}^\mathcal{P} + \mathbf{AC} + \Pi_1^\mathcal{P}$ set induction $\vdash \Phi$ then $\mathbf{KP}^\mathcal{P} \vdash \Phi$.*

Adrian Mathias [15, Theorem 9] proved that adding $V = L$ to $\mathbf{KP}^\mathcal{P}$ results in a stronger theory that proves the consistency of $\mathbf{KP}^\mathcal{P}$. This can be strengthened further.

Theorem 4.6. *[15, Theorem 9]* $\mathbf{KP}(\mathcal{P}) + V = L$ *is much stronger than* $\mathbf{KP}(\mathcal{P})$. *Even* $\mathbf{KP}^\mathcal{P} + V = L$ *is much stronger than* $\mathbf{KP}(\mathcal{P})$.

Proof: We work in $\mathbf{KP}^\mathcal{P} + V = L$. Pick a limit ordinal κ such that $\mathcal{P}(\omega) \in L_\kappa$. There exists a $\Sigma_1(L_\kappa)$ map of κ onto L_κ (see [5, II.6.8]). In particular there exists a map f of κ onto $\mathcal{P}(\omega)$. We claim that κ is uncountable, meaning that there is no surjective function that maps ω onto κ. This follows since otherwise there would exist a surjection of ω onto the powerset of ω, contradicting Cantor.

Next we aim at showing that $\mathbf{KP}^\mathcal{P} + V = L$ proves that the Howard-Bachmann ordinal exists. To this end we invoke [31] section 4. Let B be the

set of ordinal notations and $<_{hb}$ be its ordering as defined in [31, §4]. The class **Acc** of [31, Definition 4.2] consists of those ordinal notations $a <_{hb} \Omega$ such that there exists an ordinal α and an order isomorphism f between α and the initial segment of B determined by a. However, since B is a countable set such an α will always be $< \kappa$. As a result, **Acc** is actually a set in our background theory. One can then show that all the results in [31, §4] hold in our background theory. However, we can prove more. The class $\mathfrak{M}$ of [31, Definition 4.7] will also be a set in our background theory and therefore the metainduction of [31, Theorem 4.13] can be carried out as a formal induction in our background theory, and thus it can be show that the Bachmann-Howard ordinal exists as a set-theoretic ordinal. Consequently we can carry out the ordinal analysis of $\mathbf{KP}(\mathcal{P})$ from [37] inside our background theory. As a result, we get that $\mathbf{KP}^{\mathcal{P}} + V = L$ proves the $\Sigma_1^{\mathcal{P}}$ reflection principle for $\mathbf{KP}(\mathcal{P})$. $\square$

Corollary 4.7. $\mathbf{KP}^{\mathcal{P}} + V = L$ *proves the* $\Sigma_1^{\mathcal{P}}$ *reflection principle for* $\mathbf{KP}(\mathcal{P})$.

5   Some applications

The theory $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$ provides a very useful tool for defining models and realizability models of other theories that are hard to construct without access to a uniform selection mechanism. Since its exact proof-theoretic strength has been determined, this knowledge can be used to determine the strength of other theories, too. We give two examples, the first being Feferman's operational set theory with power set operation.

   Jäger, in [11], raised the following question at the very end of §4 in connection with the strength of OST($\mathbb{P}$).

> Unfortunately, the combination of Theorem 18 and Theorem 22 does not completely settle the question about the consistency strength of OST($\mathbb{P}$) yet. So far we have an interesting lower and an interesting upper bound, but it still has to be determined what the relationship between $\mathbf{KP}(\mathcal{P})$ and $\mathbf{KP}(\mathcal{P}) + (V = L)$ is.

   The theory $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}$ has a translation into OST($\mathbb{P}$) by [11, Theorem 18] and by [11, Theorem 22] OST($\mathbb{P}$) can be interpreted in $\mathbf{KP}(\mathcal{P}) + V = L$. However, by Theorem 4.6 this leaves a huge gap. The main difficulty is posed by the choice operator $\mathbb{C}$ of OST. The inductive definition of the relation ternary $R(a, b, c)$ that serves to interpret the application of OST($\mathbb{P}$) in $\mathbf{KP}(\mathcal{P})$ is given in stages R^{α} in [11, Definition 7]. The assumption $V = L$

plays a central role in clause 22 when picking a $<_L$-least c that fulfills the pertaining requirements. Here, instead of L_α one can use V_α, and instead of picking the $<_L$-least that works one can take the set C of all possible candidates in V_α and then apply the global choice function to C to select a particular one. Thus together with [11, Theorem 18] we have

Corollary 5.1. *The theories* $\text{OST}(\mathbb{P})$ *and* $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}_{global}$ *are mutually interpretable in each other and have the same strength as* $\mathbf{KP}(\mathcal{P})$.

Similar results hold for versions of these three theories where the amount of $\in$-induction is restricted. The result that $\text{OST}(\mathbb{P})$ and $\mathbf{KP}(\mathcal{P})$ have the same strength was also obtained by Sato and Zumbrunnen [38, Theorem 114], though using rather different methods.

A question left open in [30] was that of the strength of constructive Zermelo-Fraenkel set theory with the axiom of choice. There $\mathbf{CZF} + \mathbf{AC}$ was interpreted in $\mathbf{KP}(\mathcal{P}) + V = L$ ([30, Theorem 3.5]). However, the (realizability) interpretation works with $\mathbf{AC}_{global}$ as well. Thus, as $\mathbf{CZF} + \mathbf{AC}$ proves the power set axiom, it follows from [36, Theorem 15.1] that we have the following:

Corollary 5.2. $\mathbf{CZF} + \mathbf{AC}$ *and* $\mathbf{KP}(\mathcal{P})$ *have the same strength.*

Acknowledgement

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