THE REARRANGEMENT NUMBER

ANDREAS BLASS, JÖRG BRENDLE, WILL BRIAN,
JOEL DAVID HAMKINS, MICHAEL HARDY, AND PAUL B. LARSON

Abstract. How many permutations of the natural numbers are needed so that every conditionally convergent series of real numbers can be rearranged to no longer converge to the same sum? We show that the minimum number of permutations needed for this purpose, which we call the rearrangement number, is uncountable, but whether it equals the cardinal of the continuum is independent of the usual axioms of set theory. We compare the rearrangement number with several natural variants, for example one obtained by requiring the rearranged series to still converge but to a new, finite limit. We also compare the rearrangement number with several well-studied cardinal characteristics of the continuum. We present some new forcing constructions designed to add permutations that rearrange series from the ground model in particular ways, thereby obtaining consistency results going beyond those that follow from comparisons with familiar cardinal characteristics. Finally we deal briefly with some variants concerning rearrangements by a special sort of permutations and with rearranging some divergent series to become (conditionally) convergent.

1. Introduction

Let $\sum_n a_n$ be a conditionally convergent series of real numbers. Permutations p of the summands, producing rearrangements $\sum_n a_{p(n)}$ of

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Commentary concerning this article can be made on the fourth author's blog at `http://jdh.hamkins.org/the-rearrangement-number`.

the original series, can disrupt the convergence in several ways. Riemann [15] showed that one can obtain rearrangements converging to any prescribed real number. (At the end of this introduction, we record some information about the history of Riemann's Rearrangement Theorem.) Minor modifications of Riemann's argument produce rearrangements that diverge to $+\infty$ or to $-\infty$, as well as rearrangements that diverge by oscillation.

Instead of arbitrary permutations of $\mathbb{N}$, can some limited class C of permutations suffice to disrupt, in one way or another, the convergence of all conditionally convergent series? In particular, how small can the cardinality $|C|$ of such a class C be? This question was raised on MathOverflow by the fifth author of this paper [9], and there were partial answers and comments from several of the other authors [4, 8, 12].

It turns out that the question has substantial set-theoretic ramifications, and the present paper reports on these, as well as some related matters. We begin, in Section 2, by defining the *rearrangement numbers,* the minimal cardinalities of families of permutations needed to achieve various sorts of disruption of the convergence of conditionally convergent series. We point out easy connections between some of these cardinal numbers. This section also contains a few notational conventions that we use throughout the paper.

In Section 3, we introduce a technique of padding series by inserting many zero terms, and we use this method to demonstrate that the rearrangement numbers are uncountable. That is, countably many permutations cannot suffice to disrupt all conditionally convergent series. The same method is applied in a more sophisticated way in Section 6 to establish stronger lower bounds for the rearrangement numbers.

In the two intervening sections, we introduce and exploit another technique, to "mix" several given permutations into a single one that accomplishes some of the disruption of the given ones. In Section 4, we use this technique to show that several of the rearrangement numbers coincide. In Section 5, we extend the technique to relate rearrangement to properties of Baire category.

In Section 7, we relate rearrangement to properties of Lebesgue measure, using information about series in which the signs of the terms are chosen at random.

Sections 8 and 9 are devoted to showing that even the largest of the rearrangement numbers can consistently be strictly smaller than the cardinality of the continuum. (For the smallest of the rearrangement numbers, this consistency already follows from the result in Section 5.)

The last two sections treat related questions in modified contexts. Section 10 looks at divergent series that can be rearranged to (conditionally) converge. We show that, to achieve this sort of rearrangement for all such series, one must use as many permutations as the cardinality of the continuum. Section 11 is about the situation where, as in the usual proof of Riemann's Rearrangement Theorem, the permutations leave the relative order of the positive terms and the relative order of the negative terms unchanged, i.e., they affect only the interleaving between the positive and negative terms.

Except for the last two sections, we have tried to arrange the material in order of increasing set-theoretic prerequisites. Sections 2, 3, and 4 use only elementary cardinal arithmetic. Sections 5, 6, and 7 involve cardinal characteristics of the continuum. We give the relevant definitions, but we refer to [2] for some facts about these characteristics. The results in Sections 8 and 9 presuppose general knowledge of forcing, including finite-support iterations. In the last two sections, 10 and 11, where we work in modified contexts, we return to more elementary methods. Most of the section headings indicate not only the results obtained in the section but also the methods used to obtain them.

History of Riemann's theorem. On page 97 of [15], Riemann gives the nowadays familiar proof that a conditionally convergent series can be rearranged to converge to any prescribed finite sum. Riemann died in 1866, and the publication of this paper in 1867 was arranged by Dedekind. A footnote by Dedekind on the first page says that the paper was submitted by Riemann for his habilitation at Göttingen in 1854. Dedekind further explains that Riemann apparently didn't intend to publish it, but that its publication is justified by its intrinsic interest and its method of treating the basic principles of infinitesimal analysis. The footnote is dated 1867. (The next paper in the journal, also dating from 1854 and published posthumously through the efforts of Dedekind, is the one where Riemann introduces what is now called Riemannian geometry.)

Riemann attributes to Dirichlet the observation that there is a crucial difference between (what are nowadays called) absolutely and conditionally convergent series. Riemann immediately continues with the construction of a rearrangement achieving any prescribed sum for a conditionally convergent series. The possibility of such rearrangements does not seem to be in the cited paper of Dirichlet [7]. What Dirichlet does point out there is that a conditionally convergent series can become divergent when the terms are (not rearranged but) multiplied by factors that approach 1. This is on page 158, and the mention of conditional convergence is only in a parenthetical comment. Specifically, Dirichlet refers to an argument in which Cauchy inferred the convergence of a series from the convergence of another series whose corresponding terms differ by factors approaching 1; Dirichlet gives an example showing that this inference is not valid "(du moins lorsque, comme il arrive ici, les termes n'ont pas tous le même signe)." Incidentally, this paper introduced what is now called the Dirichlet kernel in the theory of Fourier series.

Finally, we briefly mention some more recent history for readers interested in constructive mathematics. Diener and Lubarsky [6] have studied a version Riemann's Rearrangement Theorem in the context of constructive mathematics. Since classically equivalent formulations need not be constructively equivalent, it it appropriate to specify that they use the formulation "If every permutation of a series of real numbers converges, then the series converges absolutely." (In constructive logic, it seems that this formulation neither implies nor is implied by "If a series converges conditionally (i.e., not absolutely), then some rearrangement diverges.") Diener and Lubarsky show that this form of Riemann's Rearrangement Theorem is not constructively provable; they exhibit a topological model where it fails.

2. Definitions, Conventions, and Basic Facts

We define the *rearrangement numbers*, denoted by $\mathfrak{rr}$, sometimes with subscripts, as the minimum number of permutations of $\mathbb{N}$ needed to disrupt, in various ways, the convergence of all conditionally convergent series.

Definition 1. $\mathfrak{rr}$ is the smallest cardinality of any family C of permutations of $\mathbb{N}$ such that, for every conditionally convergent series $\sum_n a_n$ of real numbers, there is some permutation $p \in C$ for which the rearrangement $\sum_n a_{p(n)}$ no longer converges to the same limit.

In this definition, the rearranged series might converge to a (finite) sum different from $\sum_n a_n$, might diverge to $+\infty$ or to $-\infty$, or might diverge by oscillation. If we specify one of these options, we get more specific rearrangement numbers, as follows.

Definition 2.

- $\mathfrak{rr}_f$ is defined like $\mathfrak{rr}$ except that $\sum_n a_{p(n)}$ is required to converge to a finite sum (different from $\sum_n a_n$).
- $\mathfrak{rr}_i$ is defined like $\mathfrak{rr}$ except that $\sum_n a_{p(n)}$ is required to diverge to $+\infty$ or to $-\infty$.

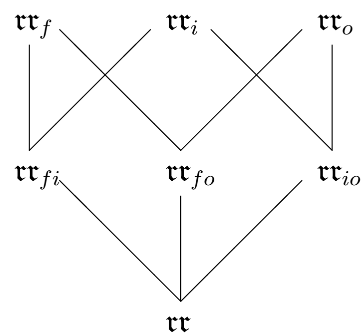

FIGURE 1. The seven *a priori* rearrangement numbers

- $\mathfrak{rr}_o$ is defined like $\mathfrak{rr}$ except that $\sum_n a_{p(n)}$ is required to diverge by oscillation.

The subscripts f, i, and o are intended as mnemonics for “finite,” “infinite,” and “oscillate.”

We shall not discuss variants in which the disruption of convergence is even more specific, for example by distinguishing oscillation between finite bounds from oscillation over the whole real line. We shall, however, have use for variants in which two of the three sorts of disruption are allowed; we denote these by $\mathfrak{rr}$ with two subscripts. Thus, for example, $\mathfrak{rr}_{fi}$ is the minimum size of a set C of permutations of $\mathbb{N}$ such that, for every conditionally convergent $\sum_n a_n$, there is $p \in C$ for which $\sum_n a_{p(n)}$ either converges to a different finite sum or diverges to $+\infty$ or to $-\infty$. The definitions of $\mathfrak{rr}_{fo}$ and $\mathfrak{rr}_{io}$ are analogous. Of course, if we allow all three sorts of disruption, we could write $\mathfrak{rr}_{fio}$, but we have chosen (in Definition 1) to denote this cardinal simply by $\mathfrak{rr}$, because it seems to be the most natural of all these rearrangement numbers and because it was the subject of the original question in [9].

We have, a priori, seven rearrangement numbers: The original $\mathfrak{rr}$, three variants with one subscript, and three with two subscripts. Clearly, if one variant allows more modes of disruption than another, then every family of permutations adequate for the latter variant is also adequate for the former, and therefore the former cardinal is less than or equal to the latter. Figure 1 is a Hasse diagram showing the seven variants and the order relationships resulting from this elementary observation. Later, we shall see that some of these variants coincide, so the diagram will become simpler.

Before proceeding, it will be convenient to adopt the following conventions, which are standard in set theory.

Convention 3. Each natural number n is identified with the set of strictly smaller natural numbers, $n = \{0, 1, \ldots, n-1\}$. If f is a function and S is a subset of its domain, then $f[S] = \text{Range}(f \restriction S) = \{f(x) : x \in S\}$. Note that we use round parentheses for the value of a function at a point in its domain and square brackets for the image of a subset of the domain. In particular, if f has domain $\mathbb{N}$, then, for example, $f[3] = \{f(0), f(1), f(2)\}$ and this is usually different from $f(3)$.

Notation 4. The cardinality $2^{\aleph_0}$ of the continuum is denoted by $\mathfrak{c}$.

Riemann's Rearrangement Theorem and the minor modifications that achieve divergence to $\pm\infty$ and divergence by oscillation immediately imply that $\mathfrak{c}$ is an upper bound for all seven of our rearrangement numbers.

3. Padding with Zeros, Uncountability

In this section, we obtain our first lower bound for rearrangement numbers. Stronger lower bounds will be obtained later. Recall that $\mathfrak{rr}$ is the smallest of the rearrangement numbers, so the following result, stated for $\mathfrak{rr}$, implies the same for all the other rearrangement numbers.

Theorem 5. *$\mathfrak{rr}$ is uncountable.*

Proof. We must show that, given any countable set $C = \{p_n : n \in \mathbb{N}\}$ of permutations of $\mathbb{N}$, there is a conditionally convergent series $\sum_n a_n$ such that, for each permutation $p \in C$, the rearranged series $\sum_n a_{p(n)}$ converges to the same sum as the original $\sum_n a_n$. For this purpose, we start with any conditionally convergent series $\sum_n b_n$, for example the alternating harmonic series $\sum_n (-1)^n/n$, and we modify it by inserting a large number of zeros between consecutive terms. The purpose of the zeros is to put the non-zero terms so far apart that the permutations in C leave their ordering essentially unchanged.

To make this strategy precise, we begin by defining a rapidly increasing function $l : \mathbb{N} \to \mathbb{N}$ by the following induction; the intention is that $l(k)$ tells the location where b_k should go in the padded series. Begin by setting $l(0) = 0$. After $l(k)$ has been defined, choose $l(k+1)$ larger than $l(k)$ and different from the finitely many numbers of the form $p_m(j)$ for $m \le k$ and $j \le p_m^{-1}(l(k))$. This definition ensures that

$$(\forall m \le k)\ p_m^{-1}(l(k)) < p_m^{-1}(l(k+1)).$$

That is, the relative order of $l(k)$ and $l(k+1)$ is not changed if we apply to them the inverses of any of the first $k+1$ elements $p_0, \ldots, p_k$

of C. Equivalently, the inverse of any particular $p_m \in C$ preserves the relative order of all but the first m elements of the range of l. (Our preoccupation here with the inverses of the permutations from C, rather than with the permutations themselves, is explained by the fact that the summand that appears in position k in a series $\sum_n a_n$ gets moved to position $p^{-1}(k)$ in the rearranged series $\sum_n a_{p(n)}$.)

As indicated above, we let $l(k)$ tell us the location where the k^{th} term b_k of our original series $\sum_n b_n$ should go in our padded series $\sum_n a_n$. That is, we define

$$a_n = \begin{cases} b_k & \text{if } n = l(k), \\ 0 & \text{if } n \notin \text{Range}(l). \end{cases}$$

Thus, the series $\sum_n a_n$ has the same nonzero terms in the same order as $\sum_n b_n$; the only difference is that many zeros have been inserted. So $\sum_n a_n$ is conditionally convergent (to the same sum as $\sum_n b_n$). Furthermore, if we apply any permutation $p_m \in C$ to $\sum_n a_n$, all but the first m nonzero terms will remain in their original order. Thus, the rearranged series $\sum_n a_{p_m(n)}$ still converges to the same sum.

That is, the countable set C is not as required in the definition of $\mathfrak{rr}$. □

Theorem 5 and the observations in the preceding section tell us that all our rearrangement numbers lie between $\aleph_1$ and $\mathfrak{c}$, inclusive. So they qualify as cardinal characteristics of the continuum [2]. Like all such characteristics, they are uninteresting if the continuum hypothesis (CH) holds, i.e., if $\aleph_1 = \mathfrak{c}$. But in the absence of CH, it is reasonable to ask how they compare with more familiar cardinal characteristics like those described in [2]. We shall obtain numerous such comparisons in the following sections.

4. Oscillation Is Easy, Mixing Permutations

In this section, we show that our seven rearrangement numbers are in fact only four, because several of them provably coincide.

Theorem 6. $\mathfrak{rr} = \mathfrak{rr}_{fo} = \mathfrak{rr}_{io} = \mathfrak{rr}_o$.

Proof. In view of the observations in Section 2 (see Figure 1), it suffices to prove $\mathfrak{rr}_o \leq \mathfrak{rr}$. The key ingredient in the proof is the following lemma, which shows how to "mix" two permutations. Recall from Convention 3 that $g[n]$ means $\{g(0), g(1), \ldots, g(n-1)\}$.

Lemma 7. *For any permutation p of $\mathbb{N}$, there exists a permutation g of $\mathbb{N}$ such that*

- *$g[n] = p[n]$ for infinitely many n, and*
- *$g[n] = n$ for infinitely many n*

Proof of Lemma. Let an arbitrary permutation p of $\mathbb{N}$ be given. Notice that, for any n and any injective function $h : n \to \mathbb{N}$, there is some $M > n$ such that h can be extended to an injective function $h' : M \to \mathbb{N}$ with $\mathrm{Range}(h') = p[M]$. Indeed, since p maps onto $\mathbb{N}$, we can choose M so large that $h[n] \subseteq p[M]$, and then we can extend h to the required h' by sending the elements of $M \setminus n$ bijectively to the $M - n$ elements of $p[M] \setminus h[n]$.

Similarly, using the identity function on $\mathbb{N}$ instead of p, we can extend any injective $h : n \to \mathbb{N}$ to an injective $h' : M \to \mathbb{N}$ such that $\mathrm{Range}(h') = M$.

Applying these two constructions, one for p and one for the identity map, alternately, we obtain the desired g. That is, we define, by induction on k, injective functions $g_k : n_k \to \mathbb{N}$ to serve as initial segments of g. We start with the empty function as g_0. If g_k is already defined for an odd number k, then we properly extend it to some $g_{k+1} : n_{k+1} \to \mathbb{N}$ such that $\mathrm{Range}(g_{k+1}) = p[n_{k+1}]$. If g_k is already defined for an even number k, then we properly extend it to some $g_{k+1} : n_{k+1} \to \mathbb{N}$ such that $\mathrm{Range}(g_{k+1}) = n_{k+1}$. Since each g_{k+1} is injective and extends g_k, and since every natural number eventually appears in the range of some g_k, there is a permutation g of $\mathbb{N}$ that extends all of the g_k's. Our construction for odd k ensures that g satisfies the first conclusion of the lemma, and our construction for even k ensures that it satisfies the second. This completes the proof of the lemma. □

Returning to the proof of the theorem, suppose C is a set of permutations of $\mathbb{N}$ as in the definition of $\mathfrak{rr}$. For each $p \in C$, let g_p be a permutation of $\mathbb{N}$ that mixes p and the identity, as in the lemma. Define $C' = C \cup \{g_p : p \in C\}$. Because C is infinite (in fact uncountable, by Theorem 5), C and C' have the same cardinality, so the theorem will be proved if we show that C' is as in the definition of $\mathfrak{rr}_o$.

Let $\sum_n a_n$ be an arbitrary conditionally convergent series of real numbers; we must find a permutation in C' that makes the series diverge by oscillation. There is a permutation $p \in C$ that disrupts the convergence of $\sum_n a_n$. If $\sum_n a_{p(n)}$ diverges by oscillation, then we are done, since $p \in C'$.

Suppose $\sum_n a_{p(n)}$ converges to a finite sum t different from the sum s of $\sum_n a_n$. Then, by our choice of g_p, infinitely many of the partial sums of $\sum_n a_{g_p(n)}$ agree with the corresponding partial sums of $\sum_n a_{p(n)}$ and thus approach t, while infinitely many other partial sums of $\sum_n a_{g_p(n)}$

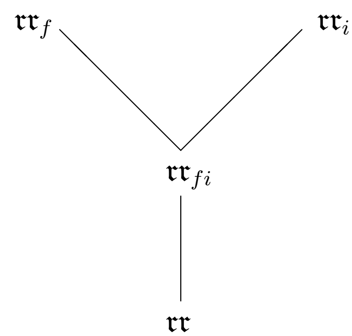

FIGURE 2. The four rearrangement numbers

agree with those of $\sum_n a_n$ and thus approach s. Since $s \neq t$, we conclude that $\sum_n a_{g_p(n)}$ diverges by oscillation.

Finally, if $\sum_n a_{p(n)}$ diverges to $+\infty$ or to $-\infty$, the same argument as in the preceding paragraph again shows that $\sum_n a_{g_p(n)}$ diverges by oscillation. □

In view of Theorem 6, the Hasse diagram of rearrangement numbers in Figure 1 simplifies to Figure 2.

5. Baire Category, More Mixing

This is the first of several sections in which certain well-known cardinal characteristics of the continuum play a role. We define these characteristics here (even though some will be needed only in later sections), and we list the known inequalities relating them. See [2] for more information about these (and other) cardinal characteristics.

We begin with two characteristics related to the ordering of sequences of natural numbers by eventual domination.

Definition 8. For functions $f, g : \mathbb{N} \to \mathbb{N}$, define $f \leq^* g$ to mean that $f(n) \leq g(n)$ for all but finitely many $n \in \mathbb{N}$. The *bounding number* $\mathfrak{b}$ is the minimum cardinality of a family $\mathcal{B}$ of functions $f : \mathbb{N} \to \mathbb{N}$ such that no single g is $\geq^*$ all $f \in \mathcal{B}$. The *dominating number* is the minimum cardinality of a family $\mathcal{D}$ of functions $f : \mathbb{N} \to \mathbb{N}$ such that every $g : \mathbb{N} \to \mathbb{N}$ is $\leq^*$ at least one member of $\mathcal{D}$.

Next are two characteristics arising from the Baire Category Theorem.

Definition 9. A subset M of a complete metric space X is *meager* (also called *first category*) if it can be covered by countably many closed sets with empty interiors in X. A *comeager* set is the complement of a meager set; equivalently, it is a set that includes the intersection of countably many dense open subsets of X. When X is the space $\mathbb{N}^{\mathbb{N}}$ of functions $\mathbb{N} \to \mathbb{N}$, equipped with the product topology, we denote the family of meager subsets of $\mathbb{N}^{\mathbb{N}}$ by $\mathcal{M}$. The *covering number for Baire category*, $\mathbf{cov}(\mathcal{M})$, is the minimum number of meager sets needed to cover $\mathbb{N}^{\mathbb{N}}$. Equivalently, it is the minimum number of dense open subsets of $\mathbb{N}^{\mathbb{N}}$ with empty intersection. The *uniformity of Baire category*, $\mathbf{non}(\mathcal{M})$, is the minimum cardinality of a non-meager subset of $\mathbb{N}^{\mathbb{N}}$.

Both $\mathbf{cov}(\mathcal{M})$ and $\mathbf{non}(\mathcal{M})$ would be unchanged if we used the real line in place of $\mathbb{N}^{\mathbb{N}}$ in the definition.

These characteristics for Baire category have analogs for Lebesgue measure.

Definition 10. By *Lebesgue measure* we mean the product measure on the set $2^{\mathbb{N}}$ of functions $\mathbb{N} \to 2$ induced by the uniform measure on 2. This is equivalent to the usual Lebesgue measure on the real interval [0,1] via binary expansions of reals. We write $\mathcal{N}$ for the family of subsets of measure zero in $2^{\mathbb{N}}$. The *covering number for Lebesgue measure*, $\mathbf{cov}(\mathcal{N})$, is the minimum number of measure-zero sets needed to cover $2^{\mathbb{N}}$. The *uniformity of Lebesgue measure*, $\mathbf{non}(\mathcal{N})$, is the smallest cardinality of a subset of $2^{\mathbb{N}}$ that does not have measure zero (i.e., that has positive outer measure).

All six of the cardinal characteristics defined here, $\mathfrak{b}$, $\mathfrak{d}$, $\mathbf{cov}(\mathcal{M})$, $\mathbf{non}(\mathcal{M})$, $\mathbf{cov}(\mathcal{N})$, and $\mathbf{non}(\mathcal{N})$, lie between $\aleph_1$ and $\mathfrak{c}$, inclusive. There are several known inequalities between them: Obviously $\mathfrak{b} \leq \mathfrak{d}$. An analysis of the nature of meager sets in $\mathbb{N}^{\mathbb{N}}$ shows that $\mathfrak{b} \leq \mathbf{non}(\mathcal{M})$ and $\mathbf{cov}(\mathcal{M}) \leq \mathfrak{d}$. Finally, a result of Rothberger [17] ([2, Theorem 5.11] is a more accessible reference) asserts that $\mathbf{cov}(\mathcal{N}) \leq \mathbf{non}(\mathcal{M})$ and $\mathbf{cov}(\mathcal{M}) \leq \mathbf{non}(\mathcal{N})$. No further inequalities between these six cardinals are provable in ZFC. In fact, given any assignment of values $\aleph_1$ or $\aleph_2$ to these cardinals and to $\mathfrak{c}$, if the assignment is consistent with the inequalities stated here then it is realized in some models of ZFC; the relevant models can be found in [1, Chapter 7].

The main result of this section is our only nontrivial (i.e., other than $\mathfrak{c}$) upper bound on a rearrangement number. Unfortunately, it applies only to the smallest of the four rearrangement numbers, $\mathfrak{rr}$.

Theorem 11. $\mathfrak{rr} \leq \mathbf{non}(\mathcal{M})$.

Proof. Although $\mathbf{non}(\mathcal{M})$ was defined using the space $\mathbb{N}^{\mathbb{N}}$ of all functions $\mathbb{N} \to \mathbb{N}$, we shall use the fact that $\mathbf{non}(\mathcal{M})$ would be unchanged if we used instead the subspace consisting of only the permutations of $\mathbb{N}$. In fact, the subspace is homeomorphic to the whole space. This follows, via [11, Theorem 7.7], from the easily verified facts that the subspace is a G_δ set in $\mathbb{N}^{\mathbb{N}}$ and that it has no nonempty compact open subset. A direct construction of a homeomorphism can be given by coding permutations as follows. Think of a permutation p as a set of ordered pairs $(n, p(n))$, and list these ordered pairs in a sequence $\langle (a_k, b_k) : k \in \mathbb{N} \rangle$ in such a way that, for each even k, a_k is the smallest number that has not occurred as a_j for any $j < k$, while for each odd k, b_k is the smallest number that has not occurred as b_j for any $j < k$. For each even k, consider where b_k occurs in the increasing enumeration of all the natural numbers different from b_j for all $j < k$; let c_k be the number of this position. Similarly, for each odd k, consider where a_k occurs in the increasing enumeration of all the natural numbers different from a_j for all $j < k$, and let c_k be the number of this position. Then the correspondence between permutations p and the sequences $\bar{c} = \langle c_k : k \in \mathbb{N} \rangle \in \mathbb{N}^{\mathbb{N}}$ defined in this manner is easily seen to be a bijection. Since any finite part of $\bar{c}$ is determined by a finite part of p and vice versa, this bijection is continuous in both directions, i.e., it is a homeomorphism.

Thus, to prove the theorem, it suffices to show that any nonmeager set C of permutations suffices to disrupt the convergence of all conditionally convergent series.

Consider any conditionally convergent series $\sum_n a_n$. We claim that the permutations p such that $\sum_n a_{p(n)}$ fails to converge to the same sum as $\sum_n a_n$ form a comeager set. Since C isn't meager, it must meet this comeager set, and that suffices to complete the proof of the theorem.

In fact, we prove a stronger claim, namely that the set of permutations p such that a subsequence of the partial sums of $\sum_n a_{p(n)}$ diverges to $+\infty$ is the intersection of countably many dense open subsets of the space of all permutations. Indeed, this set is

$$\bigcap_{k \in \mathbb{N}} \bigcup_{m \geq k} \{p : \sum_{n=0}^{m} a_{p(n)} \geq k\},$$

so we need only prove that, for each k, the set

$$U_k = \bigcup_{m \geq k} \{p : \sum_{n=0}^{m} a_{p(n)} \geq k\}$$

is dense and open. Taking into account the definition of the topology on the set of permutations, as a subspace of the product space $\mathbb{N}^{\mathbb{N}}$, we see that openness is immediate: If $p \in U_k$, then there is $m \geq k$ with $\sum_{n=0}^{m} a_{p(n)} \geq k$, and any p' that agrees with p up to m is also in U_k.

It remains to show that U_k is dense and, in view of the definition of the topology, what we must show is that every injective function $h : d \to \mathbb{N}$, for any $d \in \mathbb{N}$, can be extended to an injective function $h' : d' \to \mathbb{N}$ with $\sum_{n=0}^{m} a_{h'(n)} \geq k$ for some $m \geq k$. But this is easy. First extend h, if necessary, so that its domain is $\geq k$. Then extend it further, using successive values $h'(n)$ for which $a_{h'(n)}$ is positive, until the sum exceeds k. The sum will eventually exceed k because, in a conditionally convergent series like $\sum_n a_n$, the sum of the positive terms diverges to $+\infty$. $\square$

A slight modification of this proof establishes the comeagerness of the set of permutations p for which $\sum_n a_{p(n)}$ has not only arbitrarily high positive partial sums but also arbitrarily low negative partial sums. Thus, any nonmeager set of permutations suffices to convert any conditionally convergent series to a series that diverges by oscillation over the whole real line, i.e., has partial sums ranging from $-\infty$ to $+\infty$ and thus (since the terms of the series approach zero) dense in the real line.

The argument in the proof cannot, however, be modified to obtain convergence of $\sum_n a_{p(n)}$ to a different finite sum or divergence to $+\infty$ or $-\infty$. That is, the argument does not make $\mathbf{non}(\mathcal{M})$ an upper bound for $\mathfrak{rr}_f$ or $\mathfrak{rr}_i$ or even $\mathfrak{rr}_{fi}$. We shall see later that this is not a defect of the argument; these larger rearrangement numbers can consistently be larger than $\mathbf{non}(\mathcal{M})$.

To close this section, we establish a Baire category lower bound for $\mathfrak{rr}_{fi}$. Even though this bound will be superseded by a stronger one in the next section, the argument seems to be of sufficient interest to justify mentioning it here.

Theorem 12. $\mathfrak{rr}_{fi} \geq \mathbf{cov}(\mathcal{M})$.

Proof. We begin by looking more closely at the proof of Lemma 7. We showed there that, given a permutation p of $\mathbb{N}$, we can extend any injective $h : n \to \mathbb{N}$ to an injective function $h' : M \to \mathbb{N}$ such that $\mathrm{Range}(h') = p[M]$. From this it follows that the set A_p defined as

$$\{g : g \text{ a permutation of } \mathbb{N} \text{ and } g[M] = p[M] \text{ for infinitely many } M\}$$

is a comeager subset of the space of all permutations of $\mathbb{N}$.

Now suppose, toward a contradiction, that $\mathfrak{rr}_{fi} < \mathbf{cov}(\mathcal{M})$, and let C witness this. That is, $|C| < \mathbf{cov}(\mathcal{M})$ and every conditionally convergent series can be rearranged by a permutation in C so as to either converge to a different sum or diverge to $+\infty$ or $-\infty$. Without loss of generality, let the identity permutation id be a member of C. The set $\bigcap_{p\in C} A_p$, being the intersection of fewer than $\mathbf{cov}(\mathcal{M})$ comeager sets, is nonempty, so let g be a member of it.

Consider any conditionally convergent series $\sum_n a_n$ and, by our choice of C, let $p \in C$ be such that $\sum_n a_{p(n)}$ has either a different finite sum from $\sum_n a_n$ or an infinite sum. Since $g \in A_p$, we have $g[M] = p[M]$ for infinitely many M and also $g[M] = M$ for another infinitely many M. As a result, the same argument as in the proof of Theorem 6 shows that $\sum_n a_{g(n)}$ diverges by oscillation.

Thus, $\{g\}$ witnesses that $\mathfrak{rr}_o = 1$, which is absurd, by Theorem 5. This contradiction completes the proof that $\mathfrak{rr}_{fi} \geq \mathbf{cov}(\mathcal{M})$. □

It follows from the preceding results that the rearrangment numbers are not all provably equal.

Corollary 13. *It is consistent with ZFC that* $\mathfrak{rr} < \mathfrak{rr}_{fi}$.

Proof. Cohen's original model for the negation of the continuum hypothesis has $\mathbf{non}(\mathcal{M}) = \aleph_1$ and $\mathbf{cov}(\mathcal{M}) = \mathfrak{c}$. It follows, by Theorems 11 and 12 that this model satisfies

$$\mathfrak{rr} \leq \mathbf{non}(\mathcal{M}) = \aleph_1 < \mathfrak{c} = \mathbf{cov}(\mathcal{M}) \leq \mathfrak{rr}_{fi}.$$ □

6. Bounding and Dominating, More Padding

In this section, we extend the method of padding with zeros, used in the proof of Theorem 5, to obtain stronger lower bounds for rearrangement numbers. Recall that the key idea in the padding method was to spread out the nonzero terms in a series so far that the permutations under consideration do not change their relative order (up to finitely many exceptions). The following definition introduces a cardinal characteristic intended to capture this idea.

Definition 14. A set $A \subseteq \mathbb{N}$ is *preserved* by a permutation p of $\mathbb{N}$ if p does not change the relative order of members of A except for finitely many elements. That is, for all but finitely many elements of A, we have $x < y \iff p(x) < p(y)$. If A is not preserved by p we say that it is *jumbled* by p. A *jumbling family* is a family of permutations such that every infinite $A \subseteq \mathbb{N}$ is jumbled by at least one member of the family. The *jumbling number*, $\mathfrak{j}$, is the smallest cardinality of a jumbling family.

The proof of Theorem 5 shows that $\mathfrak{j}$ is uncountable. We shall see later that $\mathfrak{j} = \mathfrak{b}$, but first we check that the concept of jumbling provides a lower bound for the rearrangement numbers.

Theorem 15. $\mathfrak{j} \leq \mathfrak{rr}$.

Proof. Let C be any family of fewer than $\mathfrak{j}$ permutations; we shall find a conditionally convergent series whose sum is unchanged under all the permutations in C. Let $\sum_n b_n$ be any conditionally convergent series, and, since $|C| < \mathfrak{j}$, let $A \subseteq \mathbb{N}$ be an infinite set preserved by the inverses of all the permutations in C. Let $\sum_n a_n$ be the series obtained by putting the b_k's at positions in A, in order, and filling the remaining positions with zeros. That is,

$$a_n = \begin{cases} b_k & \text{if } n \text{ is the } k^{\text{th}} \text{ element of } A, \\ 0 & \text{if } n \notin A. \end{cases}$$

If $p \in C$ then, since A is preserved by p^{-1}, the orders of the nonzero terms of $\sum_n a_n$ and of its rearrangement $\sum_n a_{p(n)}$ are the same, with at most finitely many exceptions. Therefore, the sums agree. We have a conditionally convergent series, $\sum_n a_n$, whose sum is unchanged when it is rearranged by any of the permutations in C. $\square$

To connect this theorem with familiar cardinal characteristics, we show next that the jumbling number is the same as the unbounding number.

Theorem 16. $\mathfrak{j} = \mathfrak{b}$. *Consequently,* $\mathfrak{b} \leq \mathfrak{rr}$.

Proof. We need only prove $\mathfrak{j} = \mathfrak{b}$, because the "consequently" part of the theorem then follows immediately via Theorem 15.

We begin by proving $\mathfrak{b} \leq \mathfrak{j}$, i.e., a family C of fewer than $\mathfrak{b}$ permutations cannot be a jumbling family; it must preserve some infinite set. Consider any such family C and associate to each permutation $p \in C$ a function $f_p : \mathbb{N} \to \mathbb{N}$ with the property that, for every $n \in \mathbb{N}$, we have $n < f_p(n)$ and

$$(\forall x \leq n)(\forall y \geq f_p(n))\, p(x) < p(y).$$

Notice that we are only requiring $f_p(n)$ to be larger than finitely many numbers, namely n and all the numbers $p^{-1}(z)$ for $z \leq \max(p[n+1])$. So there is no difficulty obtaining such a function f_p.

Because $|C| < \mathfrak{b}$, there is a function $g : \mathbb{N} \to \mathbb{N}$ such that $f_p \leq^* g$ for all $p \in C$. Increasing the values of g, we can arrange that g is strictly increasing. Now define an infinite set $A = \{a_0 < a_1 < \dots\}$ of natural numbers inductively, starting with an arbitrary a_0 (say 0) and ensuring at every step that $a_{n+1} \geq g(a_n)$. We claim that all the permutations $p \in C$ preserve A.

To see this, consider any $p \in C$ and, since $f_p \leq^* g$, fix k such that $f_p(n) \leq g(n)$ for all $n \geq k$. For elements $a_r < a_s$ of A that are larger than k, we have, since $r+1 \leq s$,

$$f_p(a_r) \leq g(a_r) \leq a_{r+1} \leq a_s,$$

and therefore, by our choice of f_p, $p(a_r) < p(a_s)$. This shows that every $p \in C$ preserves A, and so it completes the proof that $\mathfrak{b} \leq \mathfrak{j}$. (Note that this inequality suffices to give the "consequently" part of the theorem.)

It remains to show that $\mathfrak{j} \leq \mathfrak{b}$. For this purpose, it is convenient to invoke an alternative characterization of $\mathfrak{b}$, essentially due to Solomon [19]; for the version used here, see [2, Theorem 2.10]. An *interval partition* is a partition of $\mathbb{N}$ into (infinitely many) finite intervals $I_n = [i_n, i_{n+1})$, where $0 = i_0 < i_1 < \dots$. A second interval partition $\{J_n : n \in \mathbb{N}\}$ is said to *dominate* the interval partition $\{I_n : n \in \mathbb{N}\}$ if, for all but finitely many k, the interval J_k includes some I_n as a subset. Then $\mathfrak{b}$ is the smallest cardinality of a family of interval partitions such that no single interval partition dominates them all.

Fix an undominated family $\mathcal{F}$ of $\mathfrak{b}$ interval partitions. To each of the partitions $I = \{I_n : n \in \mathbb{N}\} \in \mathcal{F}$, associate the permutation p_I that flips each of the intervals I_n upside down. That is, if $x \in I_n = [i_n, i_{n+1})$ then $p_I(x) = i_n + i_{n+1} - x - 1$. Let $C = \{p_I : I \in \mathcal{F}\}$. So C has cardinality $\mathfrak{b}$, and we shall complete the proof by showing that C is a jumbling family.

Consider any infinite $A = \{a_0 < a_1 < \dots\} \subseteq \mathbb{N}$, and assume without loss of generality that $a_0 = 0$. Form an interval partition $J = \{J_k : k \in \mathbb{N}\}$ by setting $J_k = [a_{3k}, a_{3k+3})$. By our choice of $\mathcal{F}$, it contains an interval partition $I = \{I_n : n \in \mathbb{N}\}$ that is not dominated by J. That is, for infinitely many values of k, there is no interval I_n included in J_k.

Temporarily fix one such k, and let I_n be the interval of I that contains a_{3k+1}. If I_n contained neither a_{3k} nor a_{3k+2}, then I_n would be included in J_k, contrary to our choice of k. Therefore, I_n must contain at least one element of A in addition to a_{3k+1}. We thus have two elements of A whose order is reversed by p_I, since p_I flips I_n upside down.

Now un-fix k. The preceding paragraph applies to infinitely many values of k, so we have infinitely many pairs of elements of A whose order is reversed by p_I. So p_I jumbles A. Since A was arbitrary, C is a jumbling family. $\square$

Remark 17. Readers familiar with Cichoń's diagram of cardinal charactersitics (see for example [2, Section 5]) will notice that the bounds for

$\mathfrak{rr}$, namely $\mathfrak{b} \leq \mathfrak{rr} \leq \mathbf{non}(\mathcal{M})$, given by Theorems 11 and 16 sandwich $\mathfrak{rr}$ between two adjacent characteristics in that diagram. So, unless $\mathfrak{rr}$ turns out to equal one of those bounds, we cannot expect stronger inequalities comparing $\mathfrak{rr}$ to the ten characteristics in Cichoń's diagram.

Since $\mathfrak{rr}$ is the smallest of the rearrangement numbers, any lower bound for it, such as $\mathfrak{b}$ from Theorem 16, automatically applies to the variants $\mathfrak{rr}_{fi}$, $\mathfrak{rr}_f$, and $\mathfrak{rr}_i$ as well. These three variants, however, admit the following stronger lower bound.

Theorem 18. $\mathfrak{d} \leq \mathfrak{rr}_{fi}$. *Consequently,* $\mathfrak{d} \leq \mathfrak{rr}_f$ *and* $\mathfrak{d} \leq \mathfrak{rr}_i$.

Proof. We need only prove that $\mathfrak{d} \leq \mathfrak{rr}_{fi}$, since the "consequently" part of the theorem then follows by virtue of the trivial inequalities pointed out in Section 2.

Consider any family C of fewer than $\mathfrak{d}$ permutations of $\mathbb{N}$. We must find a conditionally convergent series $\sum_n a_n$ such that none of the permutations in C make this series converge to a different finite sum or diverge to $+\infty$ or $-\infty$.

Begin by associating to each $p \in C$ a function $f_p : \mathbb{N} \to \mathbb{N}$ as in the proof of Theorem 16 except that we use p^{-1} instead of p. That is, for every $n \in \mathbb{N}$, we have $n < f_p(n)$ and

$$(\forall x \leq n)(\forall y \geq f_p(n))\, p^{-1}(x) < p^{-1}(y).$$

Because $|C| < \mathfrak{d}$, there exists $g : \mathbb{N} \to \mathbb{N}$ that is not eventually dominated by any of these functions f_p for $p \in C$. We can arrange, by increasing its values if necessary, that g is strictly increasing and that $g(n) > n$ for all n. By iterating g, we obtain a strictly increasing sequence

$$0 < g(0) < g(g(0)) < \cdots < g^k(0) < g^{k+1}(0) < \dots.$$

This sequence will be used to apply the padding-with-zeros technique as follows.

Fix a conditionally convergent series $\sum_n b_n$, and define the padded version $\sum_n a_n$ by

$$a_n = \begin{cases} b_k & \text{if } n = g^k(0) \\ 0 & \text{if } n \text{ is not of the form } g^l(0) \text{ for any } l. \end{cases}$$

We shall show that no $p \in C$ can have $\sum_n a_{p(n)}$ converging to a finite sum other than $\sum_n a_n$, nor can that rearranged sum diverge to $+\infty$ or to $-\infty$.

Consider any $p \in C$. By our choice of g, there are infinitely many numbers $x \in \mathbb{N}$ such that $f_p(x) < g(x)$. Temporarily concentrate on one such x. Let k be the smallest integer such that $x < g^k(0)$. So we have both $g^{k-1}(0) \le x$ and $g(x) < g^{k+1}(0)$ (the former because k is smallest and the latter because g is increasing). Therefore,

$$g^{k-1}(0) \le x < f_p(x) < g(x) < g^{k+1}(0),$$

where the second inequality comes from our choice of f_p and the third from our choice of x. In view of the definition of f_p, we see that all the numbers $p^{-1}(0), p^{-1}(g(0)), \dots, p^{-1}(g^{k-1}(0))$ are smaller than all the numbers $p^{-1}(g^{k+1}(0)), p^{-1}(g^{k+2}(0)), \dots$. Notice that, for any r, $p^{-1}(g^r(0))$ is the location where b_r appears in the series $\sum_n a_{p(n)}$. so in this series, all of $b_0, b_1, \dots, b_{k-1}$ occur before all of $b_{k+1}, b_{k+2}, \dots$. (Nothing is said here about b_k; it could occur out of order anywhere.) Therefore, this rearranged series has a partial sum that differs from $\sum_{n=0}^k b_n$ by at most $|b_k|$.

This discussion was based on a particular x where $f_p(x) < g(x)$. But there are infinitely many such x's and infinitely many k's associated to them as above. For each of these k's, we have seen that $\sum_n a_{p(n)}$ has a partial sum differing from $\sum_{n=0}^k b_n$ by at most $|b_k|$. But, as k tends to infinity, $\sum_{n=0}^k b_n$ tends to the infinite sum $\sum_n b_n = \sum_n a_n$, and $|b_k|$ tends to zero. Thus, in the sequence of partial sums of $\sum_n a_{p(n)}$, there is an infinite subsequence tending to $\sum_n a_n$, which means that the whole sequence of partial sums cannot tend to a different value or to $\pm\infty$. □

Note that the argument at the end of this proof does not contradict the possibility that $\sum_n a_{p(n)}$ might diverge by oscillation, with its partial sums $\sum_{n=0}^k a_{p(n)}$ coming close to $\sum_n a_n$ at the infinitely many k's under consideration, but wandering around for other values of k. This is why the theorem gives a lower bound for $\mathfrak{rr}_{fi}$ but not for $\mathfrak{rr}$. In fact, it is not provable in ZFC that $\mathfrak{d} \le \mathfrak{rr}$. This is a consequence of Theorem 11 and the fact that $\mathbf{non}(\mathcal{M}) < \mathfrak{d}$ is known to be consistent with ZFC. (The basic Cohen model satisfies $\mathbf{non}(\mathcal{M}) = \aleph_1 < \mathfrak{d} = \mathfrak{c}$.)

Note also that Theorem 18 supersedes Theorem 12 because it is provable that $\mathbf{cov}(\mathcal{M}) \le \mathfrak{d}$, and it is consistent that this inequality is strict. (For example, strict inequality holds in the Laver and Miller models.)

7. Measure, Random Signs

In this section, we relate the rearrangement numbers to the covering number for measure. We shall need a result of Rademacher [14], stated as a lemma below, about infinite series with randomly chosen signs.

Lemma 19 (Rademacher). *Let $(c_n : n \in \mathbb{N})$ be any sequence of real numbers. Let $A \subseteq 2^{\mathbb{N}}$ be the set of those sequences s of zeros and ones for which $\sum_n (-1)^{s(n)} c_n$ converges. Then the Lebesgue measure of A is 1 if $\sum_n {c_n}^2$ converges and 0 otherwise.*

In other words, if we attach signs randomly to the terms of the series $\sum_n c_n$, the result will converge almost surely if $\sum_n {c_n}^2$ converges, and it will diverge almost surely otherwise.

Theorem 20. $\mathbf{cov}(\mathcal{N}) \leq \mathfrak{rr}$.

Proof. Consider first a single permutation p of $\mathbb{N}$. Since $\sum_n 1/p(n)^2$ is a rearrangement of the convergent series $\sum_n 1/n^2$, it converges, and therefore, by the lemma, the set

$$A_p = \{s \in 2^{\mathbb{N}} : \sum_n (-1)^{s(n)}/p(n) \text{ diverges}\}$$

has measure zero. Therefore, so does

$$B_p = \{s \in 2^{\mathbb{N}} : \sum_n (-1)^{s(p(n))}/p(n) \text{ diverges}\},$$

since it is just the pre-image of A_p under the measure-preserving bijection $s \mapsto s \circ p : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.

Now consider any family C of fewer than $\mathbf{cov}(\mathcal{N})$ permutations of $\mathbb{N}$. By definition of $\mathbf{cov}(\mathcal{N})$, the associated measure-zero sets B_p for $p \in C$ cannot cover $2^{\mathbb{N}}$, so there is some $s \in 2^{\mathbb{N}}$ such that $\sum_n (-1)^{s(p(n))}/p(n)$ converges for all $p \in C$. That is, the rearrangements of $\sum_n (-1)^{s(n)}/n$ by permutations from C will not diverge to $\pm\infty$ or by oscillation.

This proves that $\mathbf{cov}(\mathcal{N}) \leq \mathfrak{rr}_{io}$, and it remains only to recall from Theorem 6 that $\mathfrak{rr}_{io} = \mathfrak{rr}$. □

There is no provable inequality in either direction between $\mathbf{cov}(\mathcal{N})$ and $\mathfrak{b}$. Specifically, $\mathbf{cov}(\mathcal{N}) < \mathfrak{b}$ in the Laver model and $\mathfrak{b} < \mathbf{cov}(\mathcal{N})$ in the random real model. Thus, the lower bounds for $\mathfrak{rr}$ in Theorems 16 and 20 are independent, and each of them can consistently be strict.

Furthermore, since the random model has $\mathfrak{d} < \mathbf{cov}(\mathcal{N})$, we find that the lower bounds $\mathfrak{d}$ and $\mathbf{cov}(\mathcal{N})$ for $\mathfrak{rr}_{fi}$ are independent and can each consistently be strict. We summarize these consistency observations for future reference.

Corollary 21. *None of the inequalities*

- $\mathbf{cov}(\mathcal{N}) \leq \mathfrak{rr}$,
- $\mathfrak{b} \leq \mathfrak{rr}$,
- $\mathbf{cov}(\mathcal{N}) \leq \mathfrak{rr}_{fi}$,
- $\mathfrak{d} \leq \mathfrak{rr}_{fi}$

is provably reversible. That is, in each case, strict inequality is consistent with ZFC.

8. Forcing New Finite Limits

In this section and the next, we show that it is consistent with ZFC that all the rearrangement numbers are strictly smaller than the cardinality of the continuum. In view of the order relationships between the various rearrangement numbers (see Figure 2), it suffices to prove the consistency of the two inequalities $\mathfrak{rr}_f < \mathfrak{c}$ and $\mathfrak{rr}_i < \mathfrak{c}$. In this section, we construct a model for $\mathfrak{rr}_f < \mathfrak{c}$. In Section 9, we shall construct a model for $\mathfrak{rr}_i < \mathfrak{c}$, and we shall point out that the two constructions can be combined to produce a model that satisfies both inequalities simultaneously. In these two sections, we assume a familiarity with the forcing method of building models of ZFC, including the technique of finite-support iterated forcing.

Both models will be obtained by starting with a model in which $\mathfrak{c} > \aleph_1$ (we can take $\mathfrak{c}$ as large as we wish), and then performing an ω_1-step, finite-support iteration of forcings that satisfy the countable chain condition. Because of the chain condition, cardinals in the final model will be the same as in the ground model, so the final model will have $\mathfrak{c} > \aleph_1$. At each step of the iteration, we shall adjoin a permutation of $\mathbb{N}$ that disrupts the convergence of all conditionally convergent series in the model produced by the previous steps; in the present section, the disruption consists of producing a new, finite sum for the rearranged series, and in Section 9 it consists of making the rearranged series diverge to $+\infty$ or to $-\infty$. Thanks to the countable chain condition, every conditionally convergent series in the final model is already in the intermediate model after some countably many steps, and so its convergence will be disrupted by the permutation added at the next step. Thus, the $\aleph_1$ permutations that we added, one per step of the iteration, suffice to disrupt the convergence of all conditionally convergent series in the final model. That is, the final model will satisfy $\mathfrak{rr}_f < \mathfrak{c}$ in the construction from the present section, $\mathfrak{rr}_i < \mathfrak{c}$ in the construction from Section 9, and both inequalities in a construction that interleaves the two iterations.

We turn now to the construction of a forcing notion that satisfies the countable chain condition and adds a permutation of $\mathbb{N}$ that rearranges all conditionally convergent series in the ground model to have new, finite sums. In fact, the new sums will be very new, in that they are outside the ground model. For this proof, we shall need a classical result of Lévy [13] and Steinitz [18] and the following associated definitions.

Definition 22. Let d be a natural number, and let $\bar{a} = \langle a^i : i < d\rangle$ be a d-tuple of infinite series of real numbers. Define $K(\bar{a})$ to be the set of d-tuples of real numbers $\langle s_i : i < d\rangle$ for which the series $\sum_{i<d} s_i a^i$ converges absolutely. Define $R(\bar{a})$ to be the orthogonal complement of $K(\bar{a})$ in $\mathbb{R}^d$. The d-tuple of series $\langle a^i : i < d\rangle$ is said to be *independent* if $K(\bar{a}) = \{0\}$. An arbitrary set I of infinite series of real numbers is said to be *independent* if every finite tuple of distinct elements of I is independent.

Notice that independence as defined here is the ordinary notion of linear independence applied to the quotient of the vector space of all infinite series of real numbers modulo the subspace of absolutely convergent series.

The Lévy-Steinitz Theorem extends the Riemann Rearrangement Theorem to the context of infinite series of vectors in a finite-dimensional space $\mathbb{R}^d$.

Theorem 23 (Lévy [13], Steinitz [18])**.** *If $\bar{a} = \langle a^i : i < d\rangle$ is a finite tuple of convergent series of real numbers, then the set of d-tuples $\langle \sum_n a^i_{p(n)} : i < d\rangle$ obtainable by rearrangements p coincides with the set of vector sums $\langle \sum_n a^i_n : i < d\rangle + \bar{x}$ with $\bar{x} \in R(\bar{a})$.*

In other words, the alterations of the sum $\langle \sum_n a^i_n : i < d\rangle$ obtainable by permuting the summands are the same as the alterations that simply add an arbitrary vector from $R(\bar{a})$.

More modern sources than [13] and [18] for information about the Lévy-Steinitz Theorem include [3, 10, 16].

We shall make use of this theorem via the following corollary.

Corollary 24. *Let $\bar{a} = \langle a^i : i < d\rangle$ be an independent d-tuple of convergent series of real numbers. Let $\bar{v}$ be an arbitrary vector in $\mathbb{R}^d$. Let $f : n \to \mathbb{N}$ be an injective function from some natural number n into $\mathbb{N}$. Then there is a permutation p of $\mathbb{N}$, extending f, such that $\langle \sum_n a^i_{p(n)} : i < d\rangle = \bar{v}$.*

Proof. Since $\bar{a}$ is independent, we have $R(\bar{a}) = \mathbb{R}^d$, so $\bar{v}$ is, by the Lévy-Steinitz theorem, obtainable as the rearranged sum $\langle \sum_n a^i_{p(n)} : i < d\rangle$ for some permutation p of $\mathbb{N}$. To make p extend f, it suffices to alter p at only finitely many places, and this will not affect the sum of the rearrangement. $\square$

We shall also need the following result of Steinitz, which is used in one of the proofs of the Lévy-Steinitz Theorem.

Theorem 25 (Polygonal Confinement, [18])**.** *For each natural number d, there exists a constant C_d such that, if $\bar{v}_0, \bar{v}_1, \dots, \bar{v}_{n-1}$ are any n vectors in $\mathbb{R}^d$, each of length $\|\bar{v}_m\| \leq 1$, with sum zero, $\sum_{m<n} \bar{v}_m = 0$, then there is a permutation p of $n \setminus \{0\}$ such that*

$$\Big\|\bar{v}_0 + \sum_{m \in k \setminus \{0\}} \bar{v}_{p(m)}\Big\| \leq C_d$$

for all $k \leq n$.

In other words, given a closed polygonal path, starting and ending at the origin in $\mathbb{R}^d$, whose sides have lengths ≤ 1, one can reorder the sides (except for the first) so that the entire polygon stays within C_d of the origin. The essential point is that C_d depends only on the dimension, not on the number of steps $\bar{v}_m$ in the path.

Fix, once and for all, a nondecreasing sequence of constants C_d satisfying the conclusion of the theorem.

The following corollary is a slight variant of the theorem and will be more convenient for our application.

Corollary 26. *Let $\bar{v}_0, \bar{v}_1, \dots, \bar{v}_{n-1}$ be n vectors in $\mathbb{R}^d$, let $\bar{b}$ be their sum, and let ρ be a positive real number with all $\|\bar{v}_i\| \leq \rho$ and $\|\bar{b}\| \leq \rho$. Then there is a permutation p of $n \setminus \{0\}$ such that*

$$\Big\|\bar{v}_0 + \sum_{i \in k \setminus \{0\}} \bar{v}_{p(i)}\Big\| \leq \rho C_d + \|\bar{b}\|$$

for all $k \leq n$.

Proof. Dividing all the vectors $\bar{v}_i$ and their sum $\bar{b}$ by ρ, we may assume without loss of generality that $\rho = 1$. Then, since the sequence $\bar{v}_0, \bar{v}_1, \dots, \bar{v}_{n-1}, -\bar{b}$ has sum zero and consists of vectors of length at most 1, we can apply the Polygonal Confinement Theorem to obtain a permutation of this sequence in which $\bar{v}_0$ is still first, and all partial sums (starting at the beginning of the sequence) have length at most C_d. If $-\bar{b}$ were still at the end of the sequence, after this permutation, then these partial sums would be the partial sums in the statement of the corollary, and we would be done (even without the $\|\bar{b}\|$ term on the right side of the inequality). If $-\bar{b}$ is not at the end after the permutation, then some of the partial sums that we know to be shorter than C_d would differ from the partial sums in the statement of the corollary; the former would include $-\bar{b}$ while the latter would not. But that difference affects the lengths of these partial sums by at most $\|\bar{b}\|$, by the triangle inequality. So we get the inequality claimed in the corollary. □

We are now ready to define the partial order $\mathbb{P}$ that will add a permutation making all conditionally convergent series from the ground model converge to new (finite) sums not in the ground model.

Convention 27. Throughout this section, I is an independent set of convergent series of real numbers.

Independence is not needed for the definition of $\mathbb{P}$, but it is involved in the subsequent lemmas showing that $\mathbb{P}$ has the desired properties. Note that independence implies that all the series in I are conditionally convergent.

Definition 28. $\mathbb{P}_I$ is the partially ordered set whose elements are triples (f, A, ε) such that:

- f is an injective function from some $n \in \mathbb{N}$ into $\mathbb{N}$.
- A is a finite nonempty subset of I.
- ε is a positive rational number.
- If $\langle a^i : i < d \rangle$ is an enumeration of A, then
$$\big\|\langle a^i_m : i < d \rangle\big\| < \varepsilon / C_d$$
for all $m \in \mathbb{N} \setminus \mathrm{Range}(f)$.

The order on $\mathbb{P}_I$ is defined by setting $(g, B, \delta) \leq (f, A, \varepsilon)$ when:

- g extends f.
- B is a superset of A.
- If $\langle a^i : i < d \rangle$ is an enumeration of A then
$$\Big\| \sum_{k \in m \setminus \mathrm{Dom}(f)} \langle a^i_{g(k)} : i < d \rangle \Big\| < \varepsilon$$
for all $m \in \mathrm{Dom}(g) + 1$.
- $$\delta + \Big\| \sum_{k \in \mathrm{Dom}(g) \setminus \mathrm{Dom}(f)} \langle a^i_{g(k)} : i < d \rangle \Big\| \leq \varepsilon.$$

Remark 29. This remark is an attempt to aid the reader's intuition about this notion of forcing; it can be skipped by those readers who are willing to simply work with the formal definition of $\mathbb{P}_I$.

In any condition (f, A, ε), the first component $f : n \to \mathbb{N}$ is intended to be an initial segment of the generic permutation π added by the forcing. Thus, the first n terms of a rearranged series $\sum_n t_{\pi(n)}$ will be the terms of the original series in the locations specified by f, namely $t_{f(0)}, \dots, t_{f(n-1)}$.

The second component, A, specifies finitely many series $\sum_m a^i_m$ in I over which our condition wants to exercise some control. The last clause in the definition of conditions says that, except for those terms whose position in the π-rearrangement has already been specified by f, the remaining terms in the series in A are small compared to ε. In fact they are very small in two senses. First, the inequality applies to these terms not just individually but "jointly" across all elements of A. That is, it does not just bound the individual terms a^i_m but the d-component vectors $\langle a^i_m : i < d\rangle$. Second, the bound is not merely ε but ε/C_d, where C_d is the constant from the Lévy-Steinitz Theorem. The point of this is that it provides, via Corollary 26, a bound for sums of these vectors if we are willing to suitably rearrange them.

The use of an enumeration $\langle a^i : i < d\rangle$ in the last requirement for conditions, and also later in the definition of the ordering and elsewhere, is unimportant in the sense that, if the statements are true for one enumeration of A, then they are also true for all other enumerations. The only reason enumerations are involved at all is to have an ordering of the components in vectors like $\langle a^i_m : i < d\rangle$. If we stretched the meaning of "vector" to allow the components to be indexed by finite sets other than natural numbers, then no such enumeration would be needed; A itself could serve as the index set.

In the definition of the ordering, the first two clauses are standard; a stronger condition tells us more about the generic permutation π (i.e., it specifies a longer initial segment of π), and it tries to control more of the series in I. The last two clauses are more subtle, but it helps to notice first that they refer only to the series in A, the ones that the weaker condition (f, A, ε) wants to control. B is not mentioned in these clauses. Furthermore, these clauses are about the vectors $\langle a^i_q : i < d\rangle$ associated to locations q in $\mathrm{Dom}(g) \setminus \mathrm{Dom}(f)$, i.e., locations for which f did not say where they will go in the π-rearrangement but g did. If we think of these vectors as listed in a sequence, in the order assigned to them by g, then all initial segments of this sequence are required to have small sums, i.e., shorter than ε; so, intuitively, g arranged these vectors in an intelligent order, as suggested by polygonal confinement. And furthermore, the amount by which the final sum of all these vectors is shorter than ε is an upper bound for the third component δ in the stronger condition. The point of that is that further extensions will be subject to bounds given by this δ and that will prevent them from combining with the extension given by g to achieve sums greater than ε.

It is not difficult to check that the definition of the ordering of $\mathbb{P}_I$ is legitimate; it is, in particular, transitive. (For some intuition behind transitivity, see the last sentence of the preceding remark.) It is also not difficult to see that $\mathbb{P}_I$ is nonempty. In fact, for any nonempty finite subset A of I, there is an ε such that $(\varnothing, A, \varepsilon)$ is a condition. To prove it, use the fact that all the series in A converge, so their terms are bounded, and then just choose ε large enough.

We next prove several lemmas establishing density properties of $\mathbb{P}_I$. All of them depend on our convention that I is an independent family of convergent series. The first lemma lets us extend the initial segment f of the generic permutation and tighten the constraint ε.

Lemma 30. *For any condition (f, A, ε) and any positive integer n, there is an extension $(g, A, \delta) \leq (f, A, \varepsilon)$ (with the same second component A) with the following properties:*

- *$n \subseteq Dom(g) \cap Range(g)$.*
- *$\delta < 1/n$.*
- *If $\langle a^i : i < d\rangle$ is an enumeration of A then*
$$\big\|\langle a^i_m : i < d\rangle\big\| < \frac{\delta}{C_{2d}}$$
for all $m \in \mathbb{N} \setminus Range(g)$.

Proof. Let (f, A, ε) and n be given, let $r = \mathrm{Dom}(f)$, and fix an enumeration $\langle a^i : i < d\rangle$ of A. By Corollary 24, there is a permutation p of $\mathbb{N}$, extending f, such that
$$\sum_{n=0}^{\infty}\langle a^i_{p(n)} : i < d\rangle = \sum_{n\in\mathrm{Dom}(f)}\langle a^i_{f(n)} : i < d\rangle,$$
or in other words,
$$\sum_{m\geq r}\langle a^i_{p(m)} : i < d\rangle = 0.$$
Recall that, by definition of conditions, we have strict inequalities $\|\langle a^i_m : i < d\rangle\| < \varepsilon/C_d$ for all $m \notin \mathrm{Range}(f)$. Furthermore, the norms on the left side of these inequalities tend to zero as m increases, because the series in A are convergent. So we can fix a positive number $\eta < \varepsilon$ such that $\|\langle a^i_m : i < d\rangle\| < \eta/C_d$ for all $m \notin \mathrm{Range}(f)$. Fix a positive rational number δ smaller than both $1/n$ and$(\varepsilon - \eta)/2$.

For any sufficiently large natural number n_*, we have all of the following:

- $n_* \geq n$.
- $n \subseteq p[n_*]$.
- $\big\|\sum_{m\in n_*\setminus r}\langle a^i_{p(m)} : i < d\rangle\big\| < \delta$.
- $\|\langle a^i_m : i < d\rangle\| < \delta/C_{2d}$ for each $m \in \mathbb{N} \setminus n_*$.

The first and second of these assertions are clear, and the fourth follows from the fact that all the series in A converge. To see the third, note that the sum there is a partial sum of the series $\sum_{m\geq r}\langle a^i_{p(m)} : i < d\rangle$ whose sum is zero by our choice of p.

Choose n_* large enough so that all these statements are true, and then use Corollary 26 to produce an injection from $n_* \setminus r$ to $\mathbb{N}$ which, when combined with $f : r \to \mathbb{N}$, produces an injection[1] $g : n_* \to \mathbb{N}$, extending f, with the same range as $p \restriction n_*$, such that, for all $m \leq n_*$,

$$\Big\|\langle a^i_r : i < d\rangle + \sum_{k\in m\setminus(r+1)} \langle a^i_{g(k)} : i < d\rangle\Big\| \leq (\eta/C_d)C_d + \delta < \varepsilon - \delta.$$

Then (g, A, δ) is as required in the lemma. □

The next lemma allows us to enlarge the set A of controlled series.

Lemma 31. *For each condition $(f, A, \varepsilon) \in \mathbb{P}_I$ and each $b \in I$, there is an extension $(g, B, \delta) \leq (f, A, \varepsilon)$ with $b \in B$.*

Proof. Assume $b \notin A$, as otherwise there is nothing to prove. Enumerate $A\cup\{b\}$ as $\langle a^i : i \leq d\rangle$ with b as the last element in the enumeration, $b = a^d$. As in the proof of the preceding lemma, use Corollary 24 to extend f to a permutation p of $\mathbb{N}$ such that $\sum_{m\geq r}\langle a^i_{p(m)} : i < d\rangle = 0$, where, as before, r is the domain of f. Continuing as in the previous proof, choose η, δ, and n_* as there except that the fourth condition on n_* is strengthened to include b with the other a^i's and weakened by using C_{d+1} in place of C_{2d}, i.e.,

- $\|\langle a^i_m : i \leq d\rangle\| < \delta/C_{d+1}$ for each $m \in \mathbb{N} \setminus n_*$.

The strengthening is easy to obtain because b as well as the other a^i's are convergent series.

Finally, still proceeding as in the previous proof but with b included, use Corollary 26 to extend f to an injection $g : n_* \to \mathbb{N}$ such that, for all $m \leq n_*$,

$$\Big\|\langle a^i_r : i < d\rangle + \sum_{k\in m\setminus(r+1)} \langle a^i_{g(k)} : i < d\rangle\Big\| \leq (\eta/C_{d+1})C_{d+1} + \delta < \varepsilon - \delta.$$

Then $(g, A \cup \{b\}, \delta)$ is as required in the lemma. □

The preceding two lemmas provide the following important information about the generic object added by forcing with $\mathbb{P}_I$.

Corollary 32. *If $G \subseteq \mathbb{P}_I$ is a V-generic filter and we define*

$$\pi = \bigcup\{f : (f, A, \varepsilon) \in G\},$$

[1] Corollary 26 gives us $g(r) = r$ here. This information, though used in the formulas that follow, is not essential for the proof here or in Lemma 31.

then π is a permutation of $\mathbb{N}$ and, for every series $a \in I$, the rearrangement $\sum_n a_{\pi(n)}$ converges.

Proof. Since all the first components f of conditions in G are single-valued, injective, and pairwise compatible, π is a partial function from $\mathbb{N}$ to $\mathbb{N}$. That it is total and surjective, and thus a permutation of $\mathbb{N}$, follows from genericity and the clause $n \subseteq \mathrm{Dom}(g) \cap \mathrm{Range}(g)$ in Lemma 30.

For any series $a \in A$, genericity and Lemma 31 provide a condition $(f, A, \varepsilon) \in G$ with $a \in A$; by Lemma 30, we can further arrange that ε here is as small as we want. Then, by definition of the ordering of $\mathbb{P}_I$, extensions of (f, A, ε) cannot produce large variations in the partial sums of $\sum_{n=r}^{\infty} a_{\pi(n)}$, where $r = \mathrm{Dom}(f)$. Since any partial sum of this generic rearrangement is obtainable from a condition in G, it is also obtainable from an extension of (f, A, ε). So these partial sums cannot oscillate by more than ε. Since ε can be taken to be as small as we want, it follows that $\sum_n a_{\pi(n)}$ converges. □

The next (and last) of the density lemmas serves to ensure that the sum of the series rearranged by π is not in the ground model.

Remark 33. Any effort to impose a particular behavior (in the present situation, the behavior of convergence to new values) on arbitrary conditionally convergent series must confront the fact that two or more series might be related in such a way that their behavior under rearrangements is correlated, possibly in undesirable ways. Until now, the present argument has avoided this issue by dealing with an independent set I of series. Utimately, though, it will have to deal with arbitrary series in the ground model. The following lemma is a key step in this direction, dealing with linear combinations of series from I. Later, by taking I to be a *maximal* independent set, we shall use this lemma to deal with all series in the ground model.

Lemma 34. *Let (f, A, ε) be a condition in $\mathbb{P}_I$, let $\langle a^i : i < d\rangle$ be an enumeration of A, let $\langle s_i : i < d\rangle$ be a d-tuple of nonzero real numbers, and let r be any real number. Then there exists an extension $(g, A, \delta) \leq (f, A, \varepsilon)$ such that*

$$\Big|r - \sum_{i<d} \sum_{n \in Dom(g)} s_i a^i_{g(n)}\Big| > \delta \sum_{i<d} |s_i|.$$

Proof. As a preliminary step, we extend the given condition, if necessary, to obtain

$$r \neq \sum_{i<d} \sum_{n \in \mathrm{Dom}(f)} s_i a^i_{f(n)}.$$

If the desired inequality does not already hold, then we proceed as follows. Since I is independent and the s_i are nonzero, the series $\sum_n \left(\sum_{i<d} s_i a^i_n\right)$ is conditionally convergent. So there are arbitrarily large m with $\sum_{i<d} s_i a^i_m \neq 0$. As in previous lemmas, choose $\eta < \varepsilon$ such that $\|\langle a^i_m : i < d\rangle\| < \eta/C_d$ for all $m \notin \mathrm{Range}(f)$, and let δ be a positive rational number smaller than $(\varepsilon - \eta)/2$. Then find an $m \notin \mathrm{Range}(f)$ such that both $\sum_{i<d} s_i a^i_m \neq 0$ and $\|\langle a^i_m : i < d\rangle\| < \delta$. Such an m exists because the first of these two requirements is satisfied by infinitely many m and the second by all sufficiently large m. Then, adjoining one more point to the domain of f and extending f to take the value m there, we get a condition $(f \cup (\mathrm{Dom}(f), m), A, (\varepsilon + \eta)/2)$ that extends (f, A, ε) and has the desired inequality. This completes our preliminary step, and we assume from now on that $r \neq \sum_{i<d} \sum_{n\in\mathrm{Dom}(f)} s_i a^i_{f(n)}$. We introduce the notation

$$\zeta = \Big|r - \sum_{i<d} \sum_{n\in\mathrm{Dom}(f)} s_i a^i_{f(n)}\Big|,$$

so that we have arranged $\zeta > 0$.

As in previous proofs, Corollary 24 provides a permutation p of $\mathbb{N}$, extending f and satisfying

$$\sum_n \langle a^i_{p(n)} : i < d\rangle = \sum_{n\in\mathrm{Dom}(f)} \langle a^i_{f(n)} : i < d\rangle$$

and so

$$\sum_{n\geq\mathrm{Dom}(f)} \langle a^i_{p(n)} : i < d\rangle = 0.$$

As before, let $\eta < \varepsilon$ be such that $\|\langle a^i_m : i < d\rangle\| < \eta/C_d$ for all $m \notin \mathrm{Range}(f)$, and let δ be a positive rational number smaller than both $(\varepsilon - \eta)/2$ and $\zeta/(2\sum_{i<d} |s_i|)$. Continuing as in earlier proofs, fix n_* so large that

$$\Big\| \sum_{m\in n_*\setminus\mathrm{Dom}(f)} \langle a^i_{p(m)} : i < d\rangle \Big\| < \delta$$

and $\|\langle a^i_m : i < d\rangle\| < \delta/C_d$ for each $m \geq n_*$. By Corollary 26, there is an injection $g : n_* \to \mathbb{N}$ extending f, having range $p[n_*]$, and satisfying

$$\Big\|\langle a^i_{\mathrm{Dom}(f)} : i < d\rangle + \sum_{\mathrm{Dom}(f)<k<m} \langle a^i_{g(k)} : i < d\rangle\Big\| \leq (\eta/C_d)C_d + \delta < \varepsilon - \delta$$

for all $m \leq n_*$. As before, this ensures that (g, A, δ) is an extension of (f, A, ε) in $\mathbb{P}_I$.

Finally, comparing sums over $\mathrm{Dom}(g)$ to sums over $\mathrm{Dom}(f)$, we have that

$$\Big\| \sum_{m\in\mathrm{Dom}(g)} \langle a^i_m : i < d\rangle - \sum_{m\in\mathrm{Dom}(f)} \langle a^i_m : i < d\rangle \Big\| < \delta$$

and so

$$\Big| \sum_{m\in\mathrm{Dom}(g)} \sum_{i<d} s_i a^i_m - \sum_{m\in\mathrm{Dom}(f)} \sum_{i<d} s_i a^i_m \Big| < \delta \sum_i |s_i| < \frac{\zeta}{2}.$$

Combining this with the definition of ζ, we find that

$$\Big| r - \sum_{m\in\mathrm{Dom}(g)} \sum_{i<d} s_i a^i_m \Big| > \frac{\zeta}{2} > \delta \sum_{i<d} |s_i|,$$

as required. □

Putting the lemmas together, we obtain the following theorem describing what forcing by $\mathbb{P}_I$ accomplishes.

Theorem 35. *Let I be a maximal independent family of conditionally convergent real series, and let $G \subseteq \mathbb{P}_I$ be a V-generic filter. Let $\pi = \bigcup\{f : (f, A, \varepsilon) \in G\}$ and let b be any conditionally convergent series in V. Then $\sum_n b_{\pi(n)}$ converges to a sum not in V.*

Proof. We have already seen in Corollary 32 that π is a permutation of $\mathbb{N}$ and that the rearranged series $\sum_n a_{\pi(n)}$ converges for each $a \in I$.

Because I is a maximal independent set, any conditionally convergent series b is the sum of an absolutely convergent series c and a linear combination $\sum_{i<d} s_i a^i$ of some elements a^i of I with nonzero coefficients. It follows immediately that $\sum_n b_{\pi(n)}$ converges. Furthermore, the absolutely convergent c has the same sum after rearrangement as before; in particular, the rearranged sum is in V. So to complete the proof of the theorem, it suffices to show that $\sum_{i<d} \sum_n s_i a^i_{\pi(n)}$ is not in V. To prove this, we fix an arbitrary real $r \in V$ and show that $\sum_{i<d} \sum_n s_i a^i_{\pi(n)} \neq r$.

By Lemma 34 and genericity, G contains a condition (g, A, δ) satisfying the conclusion of that lemma. So

$$\eta = \Big| r - \sum_{i<d} \sum_{n\in\mathrm{Dom}(g)} s_i a^i_{g(n)} \Big| > \delta \sum_{i<d} |s_i|.$$

Now consider any extension $(h, B, \gamma) \leq (g, A, \delta)$ in G. We have

$$\Big\| \sum_{n\in\mathrm{Dom}(h)} \langle a^i_{h(n)} : i < d\rangle - \sum_{n\in\mathrm{Dom}(g)} \langle a^i_{g(n)} : i < d\rangle \Big\| < \delta,$$

so

$$\Big|\sum_{i<d}\sum_{n\in\mathrm{Dom}(h)} s_i a^i_{h(n)} - \sum_{i<d}\sum_{n\in\mathrm{Dom}(g)} s_i a^i_{g(n)}\Big| < \delta\sum_{i<d}|s_i|,$$

and therefore

$$\Big|r - \sum_{i<d}\sum_{n\in\mathrm{Dom}(h)} s_i a^i_{h(n)}\Big| > \eta - \delta\sum_{i<d}|s_i| > 0.$$

Because the generic filter G is directed, we know that, among the partial sums of the rearranged series $\sum_n\sum_{i<d} s_i a^i_{\pi(n)}$, cofinally many are of the form $\sum_{i<d}\sum_{n\in\mathrm{Dom}(h)} s_i a^i_{h(n)}$ for some (h, B, γ) as above. These partial sums therefore differ from r by more than the positive constant $\eta - \delta\sum_{i<d}|s_i|$. Note that this constant is independent of (h, B, γ). We therefore conclude that the infinite sum $\sum_n\sum_{i<d} s_i a^i_{h(n)}$ differs from r by at least $\eta - \delta\sum_{i<d}|s_i|$ and is therefore certainly not equal to r. $\square$

In order to iterate forcings of the form $\mathbb{P}_I$, we use the chain condition provided by the following lemma.

Lemma 36. *$\mathbb{P}_I$ satisfies the countable chain condition. In fact, it is σ-linked.*

Proof. It is straightforward to verify that, for two conditions (f, A, ε) and (g, B, δ) to be compatible, it is sufficient to have

- $f = g$,
- $\varepsilon = \delta$,
- $|A| = |B|$, and
- if A and B are enumerated as $\langle a^i : i < d\rangle$ and $\langle b^i : i < d\rangle$, then $\|\langle a^i_m : i < d\rangle\|$ and $\|\langle b^i_m : i < d\rangle\|$ are each $< \varepsilon/2C_{2d}$ for all $m \notin \mathrm{Range}(f)$.

Since there are only countably many possibilities for f, for ε (recall that ε has to be rational), and for $|A|$, it suffices to verify that every condition (f, A, ε) can be extended to a condition (h, A, γ) such that $\|\langle a^i_m : i < d\rangle\| < \gamma/2C_{2d}$ for all $m \notin \mathrm{Range}(h)$. This follows from Lemma 30. $\square$

Combining this lemma with Theorem 35, we obtain the following result for a finite-support iteration.

Theorem 37. *Suppose $\mathfrak{c} > \aleph_1$. Let $\mathbb{P}$ be a finite-support iteration of length ω_1 where each stage of the forcing is $\mathbb{P}_I$ for some maximal independent set of conditionally convergent series in the extension produced by the previous stages of the iteration. Then in the extension produced by forcing with $\mathbb{P}$, we have $\mathfrak{rr}_f = \aleph_1 < \mathfrak{c}$.*

9. Forcing Infinite Limits

In this section, we describe a notion of forcing $\mathbb{P}$ producing a permutation π that rearranges all conditionally convergent series in the ground model so that they diverge to $+\infty$ or to $-\infty$. Afterward, we iterate this forcing and the one from the previous section to show that all our rearrangement numbers can consistently be strictly smaller than $\mathfrak{c}$.

In fact, the argument here applies not only to conditionally convergent series but to a broader class of series defined as follows.

Definition 38. A series of real numbers is *potentially conditionally convergent,* abbreviated *pcc,* if some rearrangement of it is conditionally convergent.

It is easy to see that a series is pcc if and only if its terms converge to zero and the two sub-series consisting of its positive terms and its negative terms both diverge.

Remark 39. Readers who are interested only in conditionally convergent series, not in pcc ones, can safely interpret "pcc" in the rest of this section as meaning conditionally convergent. Another safe simplification of most of the the following material (all but Corollary 49) is that readers uncomfortable with the version MA(σ-centered) of Martin's Axiom used below can pretend that we refer to the ordinary, stronger version MA.

Definition 40. Let $\bar{a} = \langle a_n : n \in \mathbb{N} \rangle$ be a sequence of real numbers. Define

- $P(\bar{a}) = \{n \in \mathbb{N} : a_n > 0\}$,
- $N(\bar{a}) = \{n \in \mathbb{N} : a_n < 0\}$,
- $\mathcal{I}(\bar{a}) = \{A \subseteq \mathbb{N} : \sum_{n \in A} |a_n| \text{ converges}\}$,
- $\mathcal{I}^+(\bar{a}) = \{A \subseteq \mathbb{N} : \sum_{n \in A} |a_n| \text{ diverges}\}$, and
- $\mathcal{I}^*(\bar{a}) = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \in \mathcal{I}(\bar{a})\}$.

$\mathcal{I}(\bar{a})$ is known in the literature as the *summability ideal* for $\bar{a}$ (or, more precisely, for the sequence of absolute values $\langle |a_n| : n \in \mathbb{N} \rangle$). The terminology "ideal" is justified because $\mathcal{I}(\bar{a})$ is clearly closed under subsets and under finite unions. Its complement $\mathcal{I}^+(\bar{a})$ is the associated co-ideal and $\mathcal{I}^*(\bar{a})$ is the associated filter.

Notation 41. In preparation for defining the desired forcing $\mathbb{P}$, we fix an enumeration, of length $\mathfrak{c}$, of all the pcc series in the ground model. We regard each series $\sum_n a_n$ as the sequence $\langle a_n : n \in \mathbb{N} \rangle$ of its terms, so we are dealing with a $\mathfrak{c}$-enumeration $\langle \bar{a}^\beta : \beta < \mathfrak{c} \rangle$ of infinite sequences.

We shall also use the standard notation $\subseteq^*$ for almost-inclusion; that is, $X \subseteq^* Y$ means that $X \setminus Y$ is finite.

Next, we need a technical lemma.

Remark 42. This remark is intended to clarify the intentions behind Lemma 43 below. Of course, in principle, the lemma can stand on its own; only the lemma itself, not the intentions, will be strictly needed in what follows.

The lemma is intended to address the same issue already mentioned in Remark 33, namely that correlations between various series may constrain our options for dealing with them. In the present situation, it turns out that all the pcc series (in the ground model) can be organized into equivalence classes such that decisions about one series (for example, whether its π-rearrangement should diverge to $+\infty$ rather than $-\infty$) affect the other series in its equivalence class, but do not affect series in other equivalence classes.

The construction of the equivalence classes is complicated by the following considerations. If a series $\sum_n a_n$ is to be rearranged by π to diverge to, say, $+\infty$, then π must move some set, say X, of numbers from $P(\bar{a})$ to relatively earlier positions. We shall want to do this without disturbing series $\sum_n b_n$ from other equivalence classes. So it is desirable that the moved numbers from $P(\bar{a})$ constitute a set X in $\mathcal{I}^+(\bar{a})$ (so that we can get large partial sums this way) but in $\mathcal{I}(\bar{b})$ (so that $\sum_n b_n$ is not seriously disturbed). So we need that such an X exists. But more is needed, because we may have already chosen some set X' of numbers to be moved for the sake of some other series $\sum_n c_n$, and so we shall need an appropriate X disjoint (or at least almost disjoint) from X'. If no such X is available, then we cannot handle $\bar{b}$ independently from $\bar{a}$, so they will have to go into the same equivalence class.

Thus, the choice of the appropriate sets X depends on the equivalence relation (because elements in the same equivalence class should use the same X's) but also influences the equivalence relation. As a result, the construction of the equivalence classes and the choice of the X's need to be done in a mutual recursion. That is what Lemma 43 and its proof are about.

In terms of our fixed enumeration $\langle \bar{a}^\beta : \beta < \mathfrak{c} \rangle$ of all the pcc series, the equivalence relation described above can be viewed as an equivalence relation on the set $\mathfrak{c}$ of indices. For each equivalence class, we use its first element (smallest ordinal number) as a standard representative. In the notation of the lemma, A will be the set of these representatives, and ζ will be the function sending each ordinal $\beta < \mathfrak{c}$ to the representative of its equivalence class. The X's in the preceding discussion will be X's in the lemma also, but there is an additional complication as each equivalence class gets not a single X but an almost decreasing (modulo finite sets) $\mathfrak{c}$-sequence of X's.

Lemma 43. *Assume MA(σ-centered). There exist a set $A \subseteq \mathfrak{c}$, a function $\zeta : \mathfrak{c} \to A$, and a matrix of sets $\langle X_\alpha^\beta : \alpha \in A$ and $\alpha \leq \beta < \mathfrak{c}\rangle$ with the following properties for all $\beta < \mathfrak{c}$:*

(1) *$\zeta(\beta) \leq \beta$ with equality if and only if $\beta \in A$.*
(2) *If $\alpha \in A$ and $\alpha \leq \beta \leq \beta'$, then $X_\alpha^{\beta'} \subseteq^* X_\alpha^\beta$.*
(3) *The sets X_α^β for $\alpha \in A \cap (\beta + 1)$ are almost disjoint, i.e., the intersection of any two distinct ones is finite.*
(4) *$X_{\zeta(\beta)}^\beta$ is a subset of $P(\bar{a}^\beta)$ or of $N(\bar{a}^\beta)$.*
(5) *If $\beta \leq \beta'$ then $X_{\zeta(\beta)}^{\beta'} \in \mathcal{I}^+(\bar{a}^\beta)$.*
(6) *If $\alpha \in A$ and $\alpha < \zeta(\beta)$ then $X_\alpha^\beta \in \mathcal{I}(\bar{a}^\beta)$.*
(7) *All subsets of $X_{\zeta(\beta)}^\beta$ that belong to $\mathcal{I}^+(\bar{a}^{\zeta(\beta)})$ also belong to $\mathcal{I}^+(\bar{a}^\beta)$.*

Remark 44. Continuing from Remark 42, we comment on the ideas behind the clauses in this lemma. We regard two ordinals $\beta, \beta' < \mathfrak{c}$ as equivalent if $\zeta(\beta) = \zeta(\beta')$. By clause (1), $\zeta(\beta)$ is the first element of the equivalence class of β, and A is the set of all these first elements, for all the equivalence classes. Clause (7) describes the effect of equivalence of ordinals on the associated series. Roughly speaking, it correlates divergence of subseries of $\bar{a}^\beta$ with divergence of the corresponding subseries of $\bar{a}^{\zeta(\beta)}$. Clauses (5) and (6) act in the reverse direction for some (not all) inequivalent ordinals. Specifically, if $\alpha < \zeta(\beta)$ are the first elements of two equivalence classes then X_α^β is in $\mathcal{I}^+(\bar{a}^\alpha)$ (by (5) with α and β in place of β and β') but in $\mathcal{I}(\bar{a}^\beta)$ (by (6)).

Clause (4) implies that the matrix of X's decides a direction, positive or negative, for each pcc series $\bar{a}^\beta$. This decision will later determine whether the generic rearrangement of $\bar{a}^\beta$ will diverge to $+\infty$ or to $-\infty$.

Clauses (2) and (3) describe the general structure of the X matrix. If we regard the subscripts as the horizontal coordinate and the superscripts as vertical, then (2) says that the columns are almost decreasing, and (3) says that the rows are almost disjoint.

We now turn to the proof of the lemma.

Proof. We proceed by recursion on ordinals $\beta < \mathfrak{c}$. At stage β, we shall define $\zeta(\beta)$, $A \cap (\beta + 1)$, and the β^{th} row $\langle X_\alpha^\beta : \alpha \in A \cap (\beta + 1)\rangle$ of the X matrix.

For $\beta = 0$, we set $\zeta(0) = 0$ (as required by clause (1) of the lemma) and we put 0 into A (as required by $\zeta : \mathfrak{c} \to A$). For X_0^0, we must take a set that is in $\mathcal{I}^+(\bar{a}^0)$ (as required by (5)) and that is a subset of $P(\bar{a}^0)$ or of $N(\bar{a}^0)$ as required by (4). Such sets exist, i.e., $\bar{a}^0$ has divergent subseries consisting of only positive terms or only negative terms, because $\bar{a}^0$ is pcc. (In fact, we can choose P or N here as we wish; both sorts of sets exist.)

Next, we consider the case of successor ordinals. Suppose stage β has been completed, so, in particular, we have the almost disjoint family $\{X_\alpha^\beta : \alpha \in A \cap (\beta + 1)\}$. To produce the required items for $\beta + 1$, we proceed by a subsidiary recursion on $\alpha \in A \cap (\beta + 1)$ as follows.

At step α of this recursion, we consider two cases, according to whether or not there exists a subset Y of X_α^β that is in $\mathcal{I}^+(\bar{a}^\alpha) \cap \mathcal{I}(\bar{a}^{\beta+1})$.

If such a Y exists, then we choose one and declare it to be $X_\alpha^{\beta+1}$. Then we proceed to the next value of α.

If no such Y exists, then we stop the subsidiary recursion on α, we define $\zeta(\beta + 1) = \alpha$, and we declare $\beta + 1 \notin A$ (as required by clause (1)), so $A \cap (\beta + 2) = A \cap (\beta + 1)$. We define $X_\alpha^{\beta+1}$ to be some subset of X_α^β that is included in either $P(\bar{a}^{\beta+1})$ or $N(\bar{a}^{\beta+1})$. To see that such a set exists, notice first that X_α^β is in $\mathcal{I}^+(\bar{a}^\alpha)$ because the earlier stage β of our main recursion satisfied clause (5) (and $\alpha \in A$). Next, use the case hypothesis to infer that $X_\alpha^\beta \in \mathcal{I}^+(a^{\beta+1})$, which means that the series $\sum_{n \in X_\alpha^\beta} |a_n^{\beta+1}|$ diverges. Finally infer that, in this divergent series, either the positive terms or the negative terms form a divergent series, and the index set of such a series can serve as the desired $X_\alpha^{\beta+1}$. Finally, we set $X_\gamma^{\beta+1} = X_\gamma^\beta$ for all $\gamma \in A$ in the range $\alpha < \gamma \leq \beta$. These choices satisfy all the clauses of the lemma.

If the subsidiary recursion is not stopped at a stage where no Y is available, i.e., if this recursion continues through all ordinals in $A \cap (\beta + 1)$, then we have defined $X_\alpha^{\beta+1}$ for all $\alpha \in A \cap (\beta + 1)$; we have not defined $\zeta(\beta + 1)$ yet, nor have we added any element to A. We now put $\beta + 1$ into A and, as required by clause (1), we set $\zeta(\beta + 1) = \beta + 1$. We must still choose a set to serve as $X_{\beta+1}^{\beta+1}$. This set must be

- in $\mathcal{I}^+(\bar{a}^{\beta+1})$ (by clause (5)),
- almost disjoint from all the sets $X_\alpha^{\beta+1}$ for $\alpha \in A \cap (\beta + 1)$ (by clause (3)), and
- a subset of $P(\bar{a}^{\beta+1})$ or of $N(\bar{a}^{\beta+1})$ (by clause (4)).

If we can find such a set then, by using it as $X_{\beta+1}^{\beta+1}$, we shall satisfy all the clauses for this stage $\beta + 1$. Furthermore, any set satisfying the first two of these three requirements can be pruned to satisfy the third. This is because, as noted before, if a series diverges then either the subseries of positive terms or the subseries of negative terms (or both) will also diverge.

So to complete the successor stage of our induction on β, we must prove the existence of a set in the co-ideal $\mathcal{I}^+(\bar{a}^{\beta+1})$ that is almost disjoint from all the sets $X_\alpha^{\beta+1}$ for $\alpha \in A \cap (\beta+1)$. It is here that we must invoke MA(σ-centered).

Specifically, we apply MA(σ-centered) to Mathias forcing guided by the filter $\mathcal{I}^*(\bar{a}^{\beta+1})$. Forcing conditions are pairs (s, C) where s is a finite subset of $\mathbb{N}$ and $C \in \mathcal{I}^*(\bar{a}^{\beta+1})$ with $\min(C) > \max(s)$. Another condition (s', C') is an extension of (s, C) if $s \subseteq s'$, $C \supseteq C'$, and $s' \setminus s \subseteq C$. This forcing is σ-centered (and thus satisfies the countable chain condition) because any finitely many conditions with the same first component are compatible; just intersect their second components. So we can apply MA(σ-centered) with the following fewer than $\mathfrak{c}$ dense sets.

First, for each of the sets $X_\alpha^{\beta+1}$ that we want our $X_{\beta+1}^{\beta+1}$ to be almost disjoint from, we have the dense set

$$D_\alpha = \{(s, C) : X_\alpha^{\beta+1} \cap C = \varnothing\}.$$

There are fewer than $\mathfrak{c}$ of these sets, as they are indexed by ordinals $\alpha \in A \cap (\beta+1)$, and each of them is dense because the sets $X_\alpha^{\beta+1}$ were chosen, in our subsidiary recursion, to be in the ideal $\mathcal{I}(\bar{a}^{\beta+1})$.

Second, for each natural number k, we have the dense set

$$D'_k = \{(s, C) : |\sum_{n \in s} a_n^{\beta+1}| > k\}.$$

This is dense because the second components of our conditions are sets in $\mathcal{I}^*(\bar{a}^{\beta+1})$ and the summation of $\bar{a}^{\beta+1}$ over any such set is pcc.

By MA(σ-centered), there is a filter G of conditions meeting all these dense sets. Let $X_{\beta+1}^{\beta+1} = \bigcup\{s : (s, C) \in G\}$. The sum of $\bar{a}^{\beta+1}$ over $X_{\beta+1}^{\beta+1}$ diverges because G meets every D'_k. And the fact that $X_{\beta+1}^{\beta+1}$ is almost disjoint from each previous $X_\alpha^{\beta+1}$ follows by a routine compatibility argument from the fact that G meets every D_α.

This completes the recursion for successor steps $\beta + 1$. We turn to the limit case.

Let β be a limit ordinal, and suppose the construction has been carried out, in accordance with the requirements of the lemma, for all $\gamma < \beta$. For each $\alpha \in A \cap \beta$, we shall first produce a set $Y_\alpha \in \mathcal{I}^+(\bar{a}^\alpha)$ such that $Y_\alpha \subseteq^* X_\alpha^\gamma$ for all $\gamma < \beta$. Once this is done, we can proceed exactly as in the successor case, using Y_α in place of X_α^β and defining sets called X_α^β rather than $X_\alpha^{\beta+1}$.

To produce the desired Y_α, we consider any fixed $\alpha \in A \cap \beta$ and apply MA(σ-centered) to Mathias forcing guided by the filter generated by $\mathcal{I}^*(\bar{a}^\alpha)$ and the sets X^γ_α for $\alpha \leq \gamma < \beta$. This is a proper filter because the sets X^γ_α that we are adjoining to $\mathcal{I}^*(\bar{a}^\alpha)$ form an almost decreasing sequence (by clause (2) of the lemma for stages $\gamma < \beta$) of sets in $\mathcal{I}^+(\bar{a}^\alpha)$ (by clause (5)). The relevant dense sets are

$$D_\gamma = \{(s, C) : C \subseteq X^\gamma_\alpha\}$$

(dense because X^γ_α is in the guiding filter) and

$$D'_k = \{(s, C) : |\sum_{n \in s} a^\beta_n| > k\}$$

as in the earlier use of MA(σ-centered). A generic filter G meeting all these dense sets produces the desired $Y_\alpha = \bigcup\{s : (s, C) \in G\}$. As before $\sum_{n \in Y_\alpha} a^\beta_n$ diverges because G meets the dense sets D'_k, and $Y_\alpha \subseteq^* X^\gamma_\alpha$ because G meets D_γ.

This completes the proof of the existence of the required Y_α's, and thus completes the recursion on β that produces the sets and function required by the lemma. □

Fix A, ζ, and $\langle X^\beta_\alpha \rangle$ as in the lemma. Call an ordinal $\beta < \mathfrak{c}$ a P-ordinal or an N-ordinal according to whether the P or N alternative holds in clause (4) of the lemma. (Intuitively, the P-ordinals are those for which the generic rearrangement of $\sum_n a^\beta_n$ will diverge to $+\infty$, and the N-ordinals are those for which this rearrangement will diverge to $-\infty$.) We write $R(\beta)$ for $P(\bar{a}^\beta)$ when β is a P-ordinal and for $N(\bar{a}^\beta)$ when β is an N-ordinal.

Let $\mathbb{P}$ be the following forcing. A condition is a triple (s, F, k) such that

- s is an injective function from some $n \in \mathbb{N}$ into $\mathbb{N}$,
- F is a finite subset of $\mathfrak{c}$,
- $k \in \mathbb{N}$, and
- For all P-ordinals (resp. N-ordinals) $\beta \in F$, the sum $\sum_{i \in \mathrm{Dom}(s)} a^\beta_{s(i)}$ is positive (resp. negative) and its absolute value is $> k$.

A condition (s', F', k') extends (s, F, k) if

- $s' \supseteq s$,
- $F' \supseteq F$,
- $k' \geq k$, and
- for all $j \in \mathrm{Dom}(s') \setminus \mathrm{Dom}(s)$ and all P-ordinals (resp. N-ordinals) $\beta \in F$, $\sum_{i<j} a^\beta_{s'(i)}$ is positive (resp. negative) and its absolute value is $> k$.

Intuitively, the intended "meaning" of a condition (s, F, k) is that the generic permutation π will be an extension of s and that the finitely many series $\sum_n a^\beta_{\pi(n)}$ for $\beta \in F$ are well on their way to diverging in the intended direction. Here "well" is measured by k, and "well on their way" means that the partial sum provided by s, namely $\sum_n a^\beta_{s(n)}$, exceeds k in the intended (positive or negative) direction and will continue to do so as more terms are included in the partial sum. That $\sum_n a^\beta_{s(n)}$ is large enough in the appropriate direction is the content of the last clause in the definition of conditions; that longer partial sums will also behave in this way is the content of the last clause in the definition of extensions.

Lemma 45. *Assume MA(σ-centered).*

1. *$\mathbb{P}$ satisfies the countable chain condition; in fact, it is σ-centered.*
2. *For every $l \in \mathbb{N}$, every condition (s, F, k) has an extension (s', F', k') with $k' \geq l$.*
3. *For every $m \in \mathbb{N}$, every condition (s, F, k) has an extension (s', F', k') with $m \in Range(s')$.*
4. *For every $\gamma \in \mathfrak{c}$, every condition (s, F, k) has an extension (s', F', k') with $\gamma \in F'$.*

Proof. Part (1) is easy. Any finitely many conditions (s, F, k) with the same s and k are compatible; just take the union of the F's.

For part (2), it suffices to treat the case $l = k + 1$, since repeated extensions of this sort yield arbitrarily large l's. Let (s, F, k) be given; the desired extension will be of the form $(s', F, k + 1)$ (with the same F). Our task is to produce an $s' \supseteq s$ such that this $(s', F, k + 1)$ is an extension of (s, F, k), and that comes down to satisfying the last clause in the definition of condition and the last clause in the definition of extension.

Before proceeding with the detailed proof, we describe the idea behind it; this paragraph can be omitted by readers who just want the detailed proof. We shall extend s' in several steps, where each step serves to make the partial sum $\sum_{n \in \mathrm{Dom}(s')} a^\beta_{s'(n)}$ for some $\beta \in F$ appropriately large; these sums, taken only up to $\mathrm{Dom}(s)$, were already bigger than k in absolute value; the extension s' must make them bigger than $k+1$. The difficulty is that, when we extend s' to make one of these sums, say the one for β, large, there is a danger of making other sums, for other $\beta' \in F$, too small, and we cannot afford to do this. Not only must the final sums, over all of $\mathrm{Dom}(s')$ be at least $k+1$, but they cannot drop below k at any point between $\mathrm{Dom}(s)$ and $\mathrm{Dom}(s')$ (because of the last clause in the definition of extension). This difficulty will be overcome by means of two observations. First, if $\zeta(\beta) < \zeta(\beta')$, then clauses (5) and (6) of Lemma 43 provide a set $X^{\max\{\beta,\beta'\}}_{\zeta(\beta)}$ on which the series $\sum a^\beta_n$ diverges while $\sum a^{\beta'}_n$ converges. This means that we can append finitely many elements of that set to the range of s in such a way as to get a big (in absolute value) partial sum for the β series while making very little change in the β' series. The partial sum for β' may get smaller, but not too small. In other words, when we want to make the partial sum of a^β large, we need not worry about causing trouble for β' with larger ζ values. What about β' with smaller ζ values? They might get seriously damaged by what we do for β, but one could recover from that damage by making the partial sum of $\bar{a}^{\beta'}$ very large, not just bigger than $k+1$ (our original goal) but so much bigger that the damage from β is cancelled. We thus overcompensate at β' for the damage done by β. This does not quite solve the problem, because we need to control not only the final partial sums over all of $\mathrm{Dom}(s')$ but also the intermediate partial sums for j as in the last clause of the definition of extension. This, fortunately, can be easily handled: We do the overcompensation before the damage. By putting terms into s' in the right order, we can first make the partial sums for β' very large (overcompensation), and have the damage come later so that the partial sums stay large all the time.

Here are the formal details implementing the ideas in the preceding paragraph. Let δ be the largest element of F. (If F is empty, the construction is trivial.) Enumerate $\zeta[F]$ in increasing order as $\zeta_0 < \zeta_1 < \cdots < \zeta_{m-1}$. Since (s, F, k) is a condition, and since the last requirement in the definition of conditions demanded a strict inequality, fix an $\varepsilon > 0$ such that, for all $\beta \in F$, $\sum_{n\in\mathrm{Dom}(s)} |a^\beta_n| > k + \varepsilon$.

By backward recursion on $j < m$, find finite sets $Z_j \subseteq \mathbb{N}$ such that

- the sets Z_j are disjoint from each other and from $\mathrm{Range}(s)$,
- $Z_j \subseteq X^\delta_{\zeta_j} \cap \bigcap\{R(\beta) : \beta \in F \text{ and } \zeta(\beta) = \zeta_j\}$,
- for all $\beta \in F$ with $\zeta(\beta) = \zeta_j$,
$$\Big|\sum_{i\in Z_j} a^\beta_i\Big| > 1 + \sum_{j'>j}\sum_{i\in Z_{j'}} |a^\beta_i|,$$
- for all $\beta \in F$ with $\zeta(\beta) > \zeta_j$,
$$\sum_{i\in Z_j} |a^\beta_i| < \frac{\varepsilon}{m}.$$

To see that such sets Z_j can be chosen, consider a particular j and suppose appropriate $Z_{j'}$ have already been chosen for all j' in the range $j < j' < m$. According to the second requirement, we seek Z_j as a subset of $X^\delta_{\zeta_j}$. The rest of the second requirement prohibits only finitely many elements of this set from being in Z_j, thanks to clauses (2) and (4) of Lemma 43 (remember that $\delta \geq \beta$ for all $\beta \in F$). For the first requirement, we exclude finitely many more elements from potentially entering Z_j, namely the elements of Range(s) and the elements of the previously chosen $Z_{j'}$ for $j' > j$. The fourth requirement also excludes only finitely many elements of $X^\delta_{\zeta_j}$, because the relevant series converge absolutely when restricted to $X^\delta_{\zeta_j}$, thanks to clause (6) of Lemma 43. So we have a cofinite subset of $X^\delta_{\zeta_j}$ in which to find Z_j satisfying the third requirement. And this task is easy because the series $\sum_i a^\beta_i$ restricted to this set diverges by clause (5) of Lemma 43 (and has all its terms of the same sign by the second requirement).

Now that we have the Z_j's, we use them to define s' as follows. It is the extension of s obtained by appending the elements of all the Z_j's in order of increasing j. More formally, the domain of s' is $\mathrm{Dom}(s) + \sum_{j<m} |Z_j|$ and

$$s'(i) = \begin{cases} s(i) & \text{if } i \in \mathrm{Dom}(s), \\ t^{\text{th}} \text{ element of } Z_j & \text{if } i = \mathrm{Dom}(s) + \sum_{j'<j} |Z_{j'}| + t \text{ and } t < |Z_j|. \end{cases}$$

To check that $(s', F, k+1)$ is a condition extending (s, F, k), as required for part (2) of the lemma, we need only check the last clause in the definition of condition and the last clause in the definition of extension. So we need to consider sums of the form $\sum_{n \in \mathrm{Dom}(t)} a^\beta_{t(n)}$ where $\beta \in F$ and t is an initial segment of s' that strictly includes s. We need these sums to be $> k$ for all such t and $> k+1$ when t is all of s'.

For brevity, we shall say "block j of s'" for the segment of s' that corresponds to the j part of the definition above, that is, s' restricted to the interval

$$[\mathrm{Dom}(s) + \sum_{j'<j} |Z_j|, \mathrm{Dom}(s) + \sum_{j' \leq j} |Z_j|);$$

this is the part of s' whose range is Z_j.

Fix some $\beta \in F$ and let q be the index such that $\zeta(\beta) = \zeta_q$. For convenience, assume β is a P-ordinal; the proof for N-ordinals is the same up to signs.

Consider the sums $S(x) = \sum_{n<x} a^\beta_{s'(n)}$ beginning with $x = \mathrm{Dom}(s)$ and gradually increasing x up to $\mathrm{Dom}(s')$. We begin with $S(\mathrm{Dom}(s)) > k + \varepsilon$ by our choice of ε (ultimately coming from the fact that (s, F, k) is a condition). As x increases through blocks of s' that strictly precede block q, $S(x)$ changes by less than ε/m in any single block, by the last clause in our choice of the Z_j's. There are only m blocks altogether, so the total variation in $S(x)$ before block q is less than ε. Since $S(\mathrm{Dom}(s))$ was $> k+\varepsilon$, this variation cannot bring $S(x)$ down to k or lower.

Continuing to increase x, into block q, we find only positive summands, by the second clause in our choice of Z_j's, so $S(x)$ increases and, in particular, remains $> k$, while x is in block q. By the end of block q, $S(x)$ has become quite large, thanks to clause 3 in our choice of Z_j's. Specifically, using clause 3 and the fact that $S(x)$ was $> k$ at the beginning of block q, we see that, at the end of block q, $S(x)$ has grown to more than

$$k+1+\sum_{j'>q}\sum_{i\in Z_{j'}}|a_i^\beta|.$$

In this formula, the sum over j' and i majorizes the absolute value of any further change in $S(x)$ beyond block q. So, from the end of block q on, $S(x)$ will always be more than $k+1$. This ensures that $(s', F, k+1)$ is a condition and also that it is an extension of (s, F, k), as required. This completes the proof of part (2) of the lemma.

Part (3) is a consequence of part (2) as follows. Let (s, F, k) be any condition, and let $m \in \mathbb{N}$. The desired result is trivial if $m \in \mathrm{Range}(s)$, so assume $m \notin \mathrm{Range}(s)$. (Intuition: We cannot simply append m to the range of s, as that might ruin the requirements, in the definition of condition and extension, that certain partial sums must remain big in absolute value. So we first use part (2) to make the relevant partial sums big enough that appending m won't hurt.) Apply part (2) of the lemma with some $l > k + |a_m^\beta|$ for all $\beta \in F$. We get a condition (s', F, k') with $k' \geq l$. (Recall that the proof of part (2) did not require changing F.) Now obtain s'' by appending m to the range of s'. With F and k unchanged (note: k, not k'), we get that (s'', F, k) is an extension of (s, F, k) as required.

Finally, to prove part (4) of the lemma, let any condition (s, F, k) and any ordinal $\gamma < \mathfrak{c}$ be given; we assume $\gamma \notin F$ because otherwise the conclusion is trivial. We also assume that γ is a P-ordinal; the proof for an N-ordinal is the same except for some minus signs. It suffices to produce an extension of (s, F, k) of the form $(s', F \cup \{\gamma\}, k)$. (Intuition: We cannot in general take $s' = s$, because $\sum_{i\in\mathrm{Dom}(s)} a_{s(i)}^\gamma$ might be smaller than k, and then $(s, F\cup\{\gamma\}, k)$ would fail to satsify the last requirement in the definition of condition. We must extend s to an s' that makes $\sum_{i\in\mathrm{Dom}(s)} a_{s'(i)}^\gamma$ large enough. But we must ensure that we do not, in this extension, ruin the largeness of the sums for ordinals $\beta \in F$. The strategy for doing this is essentially the same as in the proof of part (2) above.)

As in the proof of part (2), let δ be the largest element of $F \cup \{\gamma\}$. Enumerate the ordinals in $\zeta[F] \cap \zeta(\gamma)$ (not all of $\zeta[F]$) in increasing order as $\zeta_0 < \zeta_1 < \cdots < \zeta_{m-1}$. Abbreviate $\zeta(\gamma)$ as ζ_m, so we have $\zeta_0 < \zeta_1 < \cdots < \zeta_{m-1} < \zeta_m$. By backward recursion on $j \leq m$, find finite sets $Z_j \subseteq \mathbb{N}$ such that

- the sets Z_j are disjoint from each other and from Range(s).
- $Z_j \subseteq X^{\delta}_{\zeta_j} \cap \bigcap\{R(\beta) : \beta \in F \cup \{\gamma\} \text{ and } \zeta(\beta) = \zeta_j\}$,
- For all $\beta \in F$ with $\zeta(\beta) = \zeta_j$,
$$\Big|\sum_{i \in Z_j} a^{\beta}_i\Big| > \sum_{j' > j} \sum_{i \in Z_{j'}} |a^{\beta}_i|,$$
- $$\Big|\sum_{i \in Z_m} a^{\gamma}_i\Big| > k + \Big|\sum_{i < \mathrm{Dom}(s)} a^{\gamma}_{s(i)}\Big|,$$
- for all $\beta \in F \cup \{\gamma\}$ with $\zeta(\beta) > \zeta_j$,
$$\sum_{i \in Z_j} |a^{\beta}_i| < \frac{\varepsilon}{m}.$$

These requirements resemble those in the proof of part (2). The differences are that j ranges up to and including m, that γ is included along with the elements of F in the second and fifth clauses, that the right side of the inequality in the third clause doesn't need an added 1, and that there is a fourth clause specifically about γ. That fourth clause is similar in spirit to the third, but it takes into account that we have no information about $\sum_{i \in \mathrm{Dom}(s)} a^{\gamma}_{s(i)}$; in particular this sum might be a large negative number. The right side of the inequality in the fourth clause is designed to compensate for any such negativity and to add k beyond that.

The same argument as in the proof of part (2) yields the existence of sets Z_j satisfying these requirements. Once we have these sets, we can define s' just as we did for part (2), except of course that now j ranges up to and including m where we previously had $j < m$. The proof that $(s', F \cup \{\gamma\}, k)$ is a condition extending (s, F, k) is also just as it was in part (2). This completes the proof of Lemma 45. $\square$

Assume MA(σ-centered), and suppose $G \subseteq \mathbb{P}$ is a V-generic filter. Let $\pi = \bigcup\{s : (s, F, k) \in G\}$. Since the first components s of conditions (s, F, k) are injective functions $n \to \mathbb{N}$ with $n \in \mathbb{N}$, and since compatible conditions, such as those in G, have compatible first components, π is an injective function from an initial segment of $\mathbb{N}$ into $\mathbb{N}$. By part (3) of Lemma 45 and genericity, π is surjective, and so its domain must be all of $\mathbb{N}$. Thus, π is a permutation of $\mathbb{N}$. Parts (2) and (4) of Lemma 45, genericity, and the last clause in the definition of extensions in $\mathbb{P}$ ensure that $\sum_n a^{\beta}_{\pi(n)}$ diverges to $+\infty$ for all P-ordinals β and diverges to $-\infty$ for all N-ordinals β. So we have a σ-centered forcing producing a permutation that rearranges all pcc series in the ground model to diverge to $+\infty$ or to $-\infty$.

We can achieve the same result without assuming MA(σ-centered) in the ground model. Simply use, instead of $\mathbb{P}$, a two-step iteration in which the first step forces MA(σ-centered) and the second step is $\mathbb{P}$.

Iterating such a forcing for ω_1 steps with finite supports over a model of $\mathfrak{c} > \aleph_1$ produces a model in which $\mathfrak{rr}_i = \aleph_1 < \mathfrak{c}$. Indeed, the $\aleph_1$ permutations π adjoined by the steps of the iteration witness that $\mathfrak{rr}_i = \aleph_1$ because, thanks to the countable chain condition, every conditionally convergent series appears at some intermediate stage of the iteration and is rearranged to diverge to $+\infty$ or $-\infty$ by the π added at the next step (or at any later step).

We can do even better by combining the forcings from the present section and Section 8. Start with a model where $\mathfrak{c} > \aleph_1$ and perform a finite-support iteration of length ω_1 in which the steps are alternately the forcing from Section 8 and the forcing from the present section (including, each time, the forcing of MA(σ-centered) that makes $\mathbb{P}$ forcing possible). Then each conditionally convergent series in the resulting model appears at some intermediate stage. The next stage that forces with Section 8's forcing (resp. the present section's forcing) produces a permutation making that series converge to a new finite sum (resp. diverge to $+\infty$ or to $-\infty$). Thus, we have established the following consistency result:

Theorem 46. *It is consistent with ZFC that $\mathfrak{rr}_f = \mathfrak{rr}_i = \aleph_1 < \mathfrak{c}$.*

In view of the inequalities between the various rearangement numbers (see Figure 2), it follows that all rearrangement numbers can be $\aleph_1$ while $\mathfrak{c}$ is larger. The proof shows that, in addition, we can make $\mathfrak{c}$ as large as we wish.

Zapletal [20] introduced the notion of *tame* cardinal characteristics of the continuum. These are characteristics with definitions of the form "the smallest cardinality of a set A of reals such that $\varphi(A) \wedge \psi(A)$," where all quantifiers in $\varphi(A)$ range over A or over $\mathbb{N}$ and where $\psi(A)$ has the form $(\forall x \in \mathbb{R})(\exists y \in A)\, \theta(x, y)$, where quantifiers in $\theta(x, y)$ range over $\mathbb{R}$ or $\mathbb{N}$ and where A is not mentioned in $\theta(x, y)$. By suitable coding (e.g., representing a conditionally convergent series by a single real number), one can show that our rearrangement numbers are tame in Zapletal's sense; in fact, one doesn't need the φ component in the definition of tameness.

Zapletal showed, in [20, Theorem 0.2], that if $\mathfrak{x}$ is any tame cardinal characteristic such that $\mathfrak{x} < \mathfrak{c}$ holds in some set-forcing extension, and if there is a proper class of measurable Woodin cardinals, then $\mathfrak{x} < \mathfrak{c}$ holds in the iterated Sacks model, i.e., the result of a $\mathfrak{c}^+$-step, countable support iteration of Sacks forcing. Thus, we obtain the following corollary by combining Zapletal's Theorem 0.2 with the tameness of the rearrangement numbers and the fact that Theorem 46 was proved by set-forcing.

Corollary 47. *Assume that there is a proper class of measurable Woodin cardinals. Then the rearrangement numbers are $\aleph_1$ in the iterated Sacks model.*

We note that, if one iterates Sacks forcing beyond ω_2 steps, it collapses cardinals and, as a result, the iterated Sacks model will have $\mathfrak{c} = \aleph_2$. In contrast, the models we produced to prove Theorem 46 allowed $\mathfrak{c}$ to be arbitrarily large. Also, unlike the proof of Corollary 47, our proof of Theorem 46 used no large-cardinal hypotheses.

Remark 48. The proof of Theorem 46 can be easily modified to obtain other values for the rearrangement numbers. For example, it is consistent with ZFC to have

$$\mathfrak{rr} = \mathfrak{rr}_{fi} = \mathfrak{rr}_f = \mathfrak{rr}_i = \aleph_5 \quad \text{and} \quad \mathfrak{c} = \aleph_{17}.$$

To prove this, begin with a model where $\mathfrak{c} = \aleph_{17}$ and iterate the forcings from Sections 8 and 9 for $\aleph_5$ steps (with finite support as before, and including a preliminary forcing of MA(σ-centered) before each use of the Section 9 forcing). The same arguments as before show that the cardinal of the continuum remains $\aleph_{17}$ and that now $\mathfrak{rr}_f$ and $\mathfrak{rr}_i$ are at most $\aleph_5$. That not even $\mathfrak{rr}$ can be smaller than $\aleph_5$ follows from Theorem 16, because the repeated forcing of MA(σ-centered) makes $\mathfrak{b} \geq \aleph_5$.

In the following corollary, we extract a consistency result from the fact that the forcing used in the present Section 9 is σ-centered.

Corollary 49. *It is consistent with ZFC that $\mathfrak{rr}_i < \mathbf{non}(\mathcal{N})$.*

Proof. Consider the forcing with which we obtained $\mathfrak{rr}_i < \mathfrak{c}$, namely to begin with $\mathfrak{c} > \aleph_1$ in the ground model and to iterate for ω_1 steps, with finite supports, the two-step forcing that first forces MA(σ-centered) and then forces with $\mathbb{P}$ as defined above. Let us also suppose that the ground model with which we begin the iteration satisfies MA. We saw above that the final model resulting from this iteration has $\mathfrak{rr}_i = \aleph_1 < \mathfrak{c}$. To complete the proof of the corollary, we shall show that this model has $\mathbf{non}(\mathcal{N}) = \mathfrak{c}$.

Consider, in the final model, any infinite set A of reals with cardinality $< \mathfrak{c}$; our goal is to prove that A has measure zero. For this purpose, we need to invoke several well-known facts.

First, a finite-support iteration, of length $< \mathfrak{c}^+$, of σ-centered forcing notions is σ-centered. This implies that the standard partial order for forcing MA(σ-centered) is σ-centered, and therefore that our whole iteration is σ-centered.

Second, the regular-open Boolean completion of any σ-centered partial order is σ-centered, and so are all its Boolean subalgebras.

Third, random real forcing is not σ-centered.

Combining these three facts, we find that no reals in our final model are random over the ground model. This is, in particular, the case for the reals in A. So, for each $a \in A$, there is a measure-zero Borel set N_a in the ground model such that the canonical extension $\tilde{N}_a$ with the same Borel code contains a.

Thanks to the countable chain condition, there is, in the ground model, a collection $\mathcal{C}$ of at most $|A|$ Borel sets, each of measure zero, such that all of the N_a's are elements of $\mathcal{C}$. Because the ground model satisfies MA and because $|\mathcal{C}| < \mathfrak{c}$, the ground model has a measure-zero Borel set N that includes all the sets from $\mathcal{C}$ and, in particular, all the N_a's.

For each $a \in A$, the fact that $N_a \subseteq N$ is preserved when we pass to the canonical extensions with the same Borel codes in the final model; that is, $\tilde{N}_a \subseteq \tilde{N}$. In particular, $A \subseteq \tilde{N}$. But $\tilde{N}$ has, like N, measure zero. This completes the proof that A has measure zero. $\square$

10. Almost Disjoint Signs

A natural variant of the rearrangement numbers asks for the minimum cardinality of a set C of permutations of $\mathbb{N}$ such that, for any real series $\sum_n a_n$, if it has a convergent rearrangement (i.e., if it is pcc as defined in Definition 38), then $\sum_n a_{p(n)}$ converges for some $p \in C$. We show in this section that this cardinal, unlike the rearrangement numbers, is provably equal to the cardinality of the continuum. That is an immediate consequence of the following theorem.

Theorem 50. *There is a family $\mathcal{S}$ of series such that $|\mathcal{S}| = \mathfrak{c}$, that each series in $\mathcal{S}$ has a convergent rearrangement, but that no permutation of $\mathbb{N}$ makes more than one series from $\mathcal{S}$ converge.*

Proof. We use the well-known set-theoretic result that there is a family $\mathcal{A}$ of $\mathfrak{c}$ infinite subsets of $\mathbb{N}$ such that any two distinct sets from $\mathcal{A}$ have finite intersection. One of several easy constructions of such a family proceeds as follows. Instead of looking for subsets of $\mathbb{N}$, we'll get subsets of $\mathbb{Q}$; they can be transferred to $\mathbb{N}$ by any bijection between $\mathbb{N}$ and $\mathbb{Q}$. For each real number r, pick a sequence of distinct rationals converging to r, and let A_r be the range of that sequence. Then $\mathcal{A} = \{A_r : r \in \mathbb{R}\}$ is as desired.

Fix a family $\mathcal{A}$ as above. In addition to the fact that distinct sets $X, Y \in \mathcal{A}$ have $X \cap Y$ finite, we shall need that they have $\mathbb{N} \setminus (X \cup Y)$ infinite. This is easily seen by considering a third element Z of $\mathcal{A}$ and noting that all but finitely many of its elements must be outside $X \cup Y$.

With these preliminaries out of the way, we proceed to the construction of the series required in the theorem.

For each positive integer i, let I_i be the interval $[2^i + 1, 2^{i+1}]$, and recall that $\sum_{n \in I_i} 1/n \geq 1/2$. For each subset X of $\mathbb{N}$, define a series $\sum_n a_n^X$ by

$$a_n^X = \begin{cases} -1/n & \text{if } n \in I_i \text{ for some } i \in X \\ +1/n & \text{otherwise.} \end{cases}$$

Consider the series $\sum_n a_n^X$ for $X \in \mathcal{A}$. As both X and its complement $\mathbb{N} \setminus X$ are infinite, the series $\sum_n a_n^X$ includes infinitely many blocks of negative terms of the form $\sum_{n \in I_i}(-1/n)$, each with sum $\leq -1/2$, and infinitely many blocks of positive terms of the form $\sum_{n \in I_i} 1/n$, each with sum $\geq 1/2$. So the positive and negative parts both diverge, while the individual terms approach zero, and therefore the series has a conditionally convergent rearrangement.

On the other hand, when $X \neq Y$ are distinct elements of $\mathcal{A}$, then, since $X \cap Y$ is finite, the series $\sum_n (a_n^X + a_n^Y)$ has only finitely many negative terms and infinitely many blocks of positive terms (because $\mathbb{N} \setminus (X \cup Y)$ is infinite) with sum $\geq 1/2$ in each block. So it diverges to $+\infty$ under all permutations. Therefore, the set $\mathcal{S}$ of series $\sum_n a_n^X$ for $X \in \mathcal{A}$ is as required in the theorem. $\square$

Note that the series $\sum_n a_n^X$ obtained in the proof of the theorem all diverge by oscillation, since they include blocks of terms with sums $\leq -1/2$ and blocks with sums $\geq 1/2$. Intuitively, this argument and the proof of Theorem 6 suggest that it is easy to go from conditional convergence to oscillatory divergence but difficult to go in the other direction.

11. Shuffles

The proof of Riemann's Rearrangement Theorem uses only rather special permutations of $\mathbb{N}$. Given a conditionally convergent series, one uses permutations that keep the relative order of the positive terms unchanged and also keep the relative order of the negative terms unchanged. The only time the relative order of two terms in the series is changed by the permutation is when one is positive and one is negative. The following definition formalizes this idea.

Definition 51. Let A and B be two infinite, coinfinite subsets of $\mathbb{N}$. The *shuffle* determined by A and B is the permutation $s_{A,B}$ of $\mathbb{N}$ that maps A onto B preserving order and maps $\mathbb{N} \setminus A$ onto $\mathbb{N} \setminus B$ preserving order. That is,

$$s_{A,B}(n) = \begin{cases} k^{\text{th}} \text{ element of } B & \text{if } n \text{ is the } k^{\text{th}} \text{ element of } A \\ k^{\text{th}} \text{ element of } \mathbb{N} \setminus B & \text{if } n \text{ is the } k^{\text{th}} \text{ element of } \mathbb{N} \setminus A \end{cases}$$

We shall be concerned only with the special case where we are considering a conditionally convergent series $\sum_n a_n$ and the shuffles $s_{A,B}$ under consideration have B equal to $\{n : a_n > 0\}$. Then A will be the set of locations of positive terms in the series $\sum_n a_{s_{A,B}(n)}$, because

$$n \in A \iff s_{A,B}(n) \in B \iff a_{s_{A,B}(n)} > 0.$$

In fact, we specialize even further, to alternating series where B is the set of even numbers. In this situation, we abbreviate $s_{A,B}$ to s_A. It is the permutation that puts the positive terms of $\sum_n a_n$, in order, into the positions in A and puts the remaining terms, in order, into $\mathbb{N} \setminus A$.

We could define analogs of all our rearrangement numbers using shuffles rather than arbitrary permutations, but we actually consider only the analog of $\mathfrak{rr}_f$. That is, we ask how many shuffles s_A are needed to give every conditionally convergent, alternating series a different, finite sum. The answer is $\mathfrak{c}$, as the following theorem immediately implies.

Theorem 52. *Consider the conditionally convergent, alternating series*

$$S_\alpha = \sum_n (-1)^n \frac{1}{(n+1)^\alpha}$$

for exponents $0 < \alpha < 1$. *No shuffle* s_A *makes two of these series converge to new finite sums. More precisely, if* $0 < \alpha < \beta < 1$ *and if*

$$\sum_n (-1)^{s_A(n)} \frac{1}{(s_A(n)+1)^\beta}$$

converges to a finite sum larger (resp. smaller) than S_β, then

$$\sum_n (-1)^{s_A(n)} \frac{1}{(s_A(n)+1)^\alpha}$$

diverges to $+\infty$ (resp. to $-\infty$).

Proof. We prove the part of the theorem with "larger" and $+\infty$; the proof of the other part is entirely analogous. Let Δ be a positive number such that

$$\sum_n (-1)^{s_A(n)} \frac{1}{(s_A(n)+1)^\beta} > \sum_n (-1)^n \frac{1}{(n+1)^\beta} + \Delta.$$

Let m be a large natural number; just how large m should be will be determined gradually in the following argument. Let x be the sum of the first m positive terms and the first m negative terms in S_β, i.e.,

$$x = \sum_{n=0}^{2m-1} \frac{(-1)^n}{(n+1)^\beta}.$$

We want to compare x with a certain partial sum y of the series rearranged by s_A, namely the partial sum that ends with the same negative term $-1/(2m)^\beta$. This partial sum will have the same m negative terms as the sum defining x (because s_A is a shuffle), but it may have more or fewer positive terms; say it has $m+E(m)$ positive terms, where $E(m)$, the number of excess terms, might be positive or negative. These positive terms are, again because s_A is a shuffle, the first $m+E(m)$ positive terms of the series S_β.

For sufficiently large m, x will be very close to S_β and y will be close to the sum of the series rearranged by s_A, which is more than $S_\beta + \Delta$. Therefore, taking m large enough, we have $y > x + \Delta$. In particular, $E(m)$ must be positive, i.e., there must be more positive terms in y than in x. Our next step is to estimate from below the asymptotic size of $E(m)$ for large m.

The excess terms counted by $E(m)$ begin after the m^{th} positive term in the original series S_β, so they are no larger than $1/(2m)^\beta$. The sum of these $E(m)$ terms must be more than Δ, so we have

$$\frac{E(m)}{(2m)^\beta} > \Delta \quad \text{and so} \quad E(m) > c \cdot m^\beta$$

for a suitable positive constant c.

With this estimate available, we turn to the other series S_α and its rearrangement by the same s_A. We consider large m and the partial sums x and y defined as before but with α in place of β. (We can safely use the same symbols x and y in this new context, as we shall have no further use for their old meanings.) It is important to observe that, because we are using the same shuffle s_A as before, the excess $E(m)$ is also the same as before, and in particular it obeys the asymptotic lower bound obtained above, a constant times m^β.

We use this lower bound to estimate the new $y - x$. There are two cases to consider, depending on whether $E(m) \leq m$ or not. (Actually, the "not" case is impossible, but it can be handled directly just as easily as it can be proved impossible.)

Consider first the case that $E(m) \leq m$. Then the excess terms counted by $E(m)$ begin with the m^{th} positive term of S_α and end before the $(2m)^{\text{th}}$ one. In particular, each of these terms is larger than $1/(4m)^\alpha$. The sum of the excess terms is therefore asymptotically $\geq c \cdot m^\beta/(4m)^\alpha$, which tends to $+\infty$ with m because $\alpha < \beta$. So y grows without bound as m increases, which means that the series $\sum_n (-1)^{s_A(n)}/(s_A(n)+1)^\alpha$ diverges to $+\infty$ as required.

There remains the case that $E(m) > m$. In this case, instead of adding all $E(m)$ of the excess terms in y, we obtain a lower bound by adding only the first m of them. These are, as above, greater than $1/(4m)^\alpha$, so their sum is at least $m/(4m)^\alpha$, which tends to $+\infty$ with m because $\alpha < 1$. As in the previous case, this allows us to conclude the required divergence to $+\infty$. □

12. Questions

To conclude the paper, we list some questions that remain open.

Question 53. Is it consistent with ZFC that $\mathfrak{rr}_{fi}$, $\mathfrak{rr}_f$, and $\mathfrak{rr}_i$ are different?

Recall that $\mathfrak{rr}$ can consistently be strictly smaller than $\mathfrak{rr}_{fi}$ and *a fortiori* smaller than $\mathfrak{rr}_f$ and $\mathfrak{rr}_i$, by Corollary 13. But the latter three cardinals are not separated by any of our results; as far as we know, all three might be provably equal. Note that, in this case, the forcing constructions in Sections 8 and 9 would each achieve the other's goal (as well as its own).

Question 54. Does ZFC prove that $\mathfrak{rr} = \mathbf{non}(\mathcal{M})$?

Question 55. More generally, are any of the rearrangement numbers provably equal to any previously studied cardinal characteristics of the continuum?

Question 56. Yet more generally, are there provable inequalities between the rearrangement numbers and previously studied characteristics, beyond those that follow from our results and previously known cardinal characteristic inequalities?

Question 57. In particular, are any previously studied characteristics provably $\geq \mathfrak{rr}_{fi}$?

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(A. R. Blass) Mathematics Department, University of Michigan, Ann Arbor, MI 48109–1043, U.S.A.

E-mail address: `ablass@umich.edu`

URL: `http://www.math.lsa.umich.edu/~ablass/`

(J. Brendle) Graduate School of System Informatics, Kobe University, 1–1 Rokkodai, Nada-ku, 657-8501 Kobe, Japan

E-mail address: `brendle@kobe-u.ac.jp`

(W. Brian) Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798-7328, U.S.A.

E-mail address: `wbrian.math@gmail.com`

URL: `http://wrbrian.wordpress.com`

(J. D. Hamkins) Mathematics, The Graduate Center of the City Univeristy of New York, 365 Fifth Avenue, New York, NY 10016, U.S.A. and Mathematics, College of Staten Island of CUNY, Staten Island, NY 10314, U.S.A.

E-mail address: `jhamkins@gc.cuny.edu`

URL: `http://jdh.hamkins.org`

(M. Hardy) Department of Mathematics, Hamline University, Saint Paul, MN 55104, U.S.A.

E-mail address: `drmichaelhardy@gmail.com`

(P. B. Larson) Department of Mathematics, Miami University, Oxford, OH 45056, U.S.A.

E-mail address: `larsonpb@miamioh.edu`

URL: `http://www.users.miamioh.edu/larsonpb/`