THE UNIVERSAL HOMOGENEOUS TRIANGLE-FREE GRAPH HAS FINITE BIG RAMSEY DEGREES

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Abstract. This paper proves that the universal homogeneous triangle-free graph $\mathcal{H}_3$ has finite *big Ramsey degrees*: For each finite triangle-free graph G, there is a finite number $T(\mathrm{G})$ such that for any coloring of all copies of G in $\mathcal{H}_3$ into finitely many colors, there is a subgraph of $\mathcal{H}_3$ which is again universal homogeneous triangle-free in which the coloring takes no more than $T(\mathrm{G})$ colors. This is the first result on big Ramsey degrees for a homogeneous structure omitting copies of some non-trivial finite structure.

The following structures and techniques were developed for the proof: a new flexible method for constructing trees which code the universal triangle-free graph, called strong coding trees; a new notion of strict similarity type for finite subtrees of a strong coding tree; Ramsey theorems for strong coding trees, yielding analogues of the Halpern-Läuchli and Milliken Theorems; and new notions of envelope. The proofs of the Ramsey theorems for strong coding trees involve the set-theoretic technique of forcing and new styles of forcing posets, but the theorem and its proof hold in ZFC, building on ideas from Harrington's forcing proof of the Halpern-Läuchli Theorem.

Overview

It is a central question in the theory of homogeneous relational structures as to which structures have finite big Ramsey degrees. This question, of interest for several decades, has gained recent momentum as it was brought into focus by Kechris, Pestov, and Todorcevic in [16]. An infinite structure $\mathbf{S}$ is *ultrahomogeneous*, or simply *homogeneous*, if any isomorphism between two finitely generated substructures of $\mathbf{S}$ can be extended to an automorphism of $\mathbf{S}$. A homogeneous structure $\mathbf{S}$ is said to have *finite big Ramsey degrees* if for each finite substructure A of $\mathbf{S}$, there is a number n, depending on A, such that any coloring of the copies of A in $\mathbf{S}$ into finitely many colors can be reduced down to no more than n colors on some substructure $\mathbf{S}'$ isomorphic to $\mathbf{S}$. This is interesting not only as a Ramsey property for infinite structures, but also because of its implications for topological dynamics.

Prior to work in this paper, finite big Ramsey degrees had been proved for a handful of homogeneous structures: the rationals ([2]), the Rado graph ([31]), ultrametric spaces ([25]), and enriched versions of the rationals and related circular directed graphs ([17]). According to [27], "so far, the lack of tools to represent ultrahomogeneous structures is the major obstacle towards a better understanding of their infinite partition properties." This paper addresses this obstacle by providing new tools to represent the ultrahomogeneous triangle-free graph and developing the necessary Ramsey theory to deduce finite big Ramsey degrees. The methods developed seem robust enough that correct modifications should likely apply to a large class of ultrahomogeneous structures omitting some finite substructure.

1. Introduction

Ramsey theory is the study of finding well-organized structures within a seemingly disorganized structure. By beginning with a large enough structure, it is often possible to find substructures in which order emerges and persists among all smaller structures within it. Although Ramsey-theoretic statements are often simple, they can be powerful tools for solving problems. In recent decades, the heart of many problems in mathematics have turned out to have at their core some Ramsey-theoretic content. This has been seen clearly in Banach spaces and topological dynamics.

2010 *Mathematics Subject Classification.* 05C15, 03C15, 03E02, 03E05, 03E75, 05C05, 03E45.

This research was commenced whilst the author was a visiting fellow at the Isaac Newton Institute for Mathematical Sciences in the programme 'Mathematical, Foundational and Computational Aspects of the Higher Infinite' (HIF). It was continued while the author was a visitor at the Centre de Recerca Matemàtica in the 'Intensive Research Program on Large Cardinals and Strong Logics.' The author gratefully acknowledges support from the Isaac Newton Institute, the Centre de Recerca Matemàtica, and National Science Foundation Grants DMS-1301665 and DMS-1600781.

The field of Ramsey theory opened with the following celebrated result.

Theorem 1.1 (Ramsey, [29]). *Let k and l be positive integers, and suppose P_i, $0 \leq i < k$, is a partition of all l-element subsets of $\mathbb{N}$. Then there is an infinite subset M of natural numbers and an integer $i < k$ such that all l-element subsets of M lie in P_i.*

The finite version of Ramsey's Theorem states that given positive integers k, l, m, there is a number n large enough so that given any partition of the l-element subsets of $\{0, \ldots, n-1\}$ into k pieces, there is a subset X of $\{0, \ldots, n-1\}$ of size m such that all l-element subsets of X lie in one piece of the partition. This follows from the infinite version using a compactness argument. The set X is called *homogeneous* for the given partition.

The idea of partitioning certain subsets of a given finite set and looking for a large homogeneous subset can be extended to structures. A Fraïssé class $\mathcal{K}$ of finite structures is said to have the *Ramsey property* if for any $\mathrm{A}, \mathrm{B} \in \mathcal{K}$ with A embedding into B, (written $A \leq B$), and for any finite number k, there is a finite ordered graph C such that for any coloring of the copies of A in C into k colors, there is a copy $\mathrm{B}' \leq C$ of B such that all copies of A in B' have the same color. We use the standard notation

$$\mathrm{C} \rightarrow (\mathrm{B})^{\mathrm{A}}_k \tag{1}$$

to denote that for any coloring of the copies of A in C, there is a copy B' of B inside C such that all copies of A in C have the same color. Examples of Fraïssé classes of finite structures with the Ramsey property, having no extra relations, include finite Boolean algebras (Graham and Rothschild, [12]) and finite vector spaces over a finite field (Graham, Leeb, and Rothschild, [10] and [11]). Examples of Fraïssé classes with extra structure satisfying the Ramsey property include finite ordered relational structures (independently, Abramson and Harrington, [1] and Nešetřil and Rödl, [23], [24]). In particular, this includes the class of finite ordered graphs, denoted $\mathcal{G}^<$. The papers [23] and [24] further the quite general result that all set-systems of finite ordered relational structures omitting some irreducible substructure have the Ramsey property. This includes the Fraïssé class of finite ordered graphs omitting n-cliques, denoted $\mathcal{K}_n^<$.

In contrast, the Fraïssé class of unordered finite graphs does not have the Ramsey property. However, it does posses a non-trivial remnant of the Ramsey property, called finite Ramsey degrees. Given any Fraïssé class $\mathcal{K}$ of finite structures, for each $A \in \mathcal{K}$, let $t(A, \mathcal{K})$ be the smallest number t, if it exists, such that for each $B \in \mathcal{K}$ with $A \leq B$ and for each $k \geq 2$, there is some $C \in \mathcal{K}$, into which B embeds, such that

$$C \rightarrow (B)^A_{k,t}, \tag{2}$$

where this means that for each coloring of the copies of A in C into k colors, there is a copy B' of B in C such that all copies of A in B' take no more than t colors. Then $\mathcal{K}$ has *finite (small) Ramsey degrees* if for each $\mathrm{A} \in \mathcal{K}$ the number $t(\mathrm{A}, \mathcal{K})$ exists. The number $t(\mathrm{A}, \mathcal{K})$ is called the *Ramsey degree of* A in $\mathcal{K}$ ([9]). Note that $\mathcal{K}$ has the Ramsey property if and only if $t(A, \mathcal{K}) = 1$ for each $A \in \mathcal{K}$. A strong connection between Fraïssé classes with finite Ramsey degrees and ordered expansions is made explicit in Section 10 of [16], where it is shown that if an ordered expansion $\mathcal{K}^<$ of a Fraïssé class $\mathcal{K}$ has the Ramsey property, then $\mathcal{K}$ has finite small Ramsey degrees, and the degree of $\mathrm{A} \in \mathcal{K}$ can be computed from the number of non-isomorphic order expansions it has in $\mathcal{K}^<$. A similar result holds for pre-compact expansions (see [27]). It follows from the results stated above that the classes of finite graphs and finite graphs omitting n-cliques have finite small Ramsey degrees.

At this point, it is pertinent to mention recent advances connecting Ramsey theory with topological dynamics. A new connection was established in [16] which accounts for previously known phenomena regarding universal minimal flows. In that paper, Kechris, Pestov, and Todorcevic proved several strong correspondences between Ramsey theory and topological dynamics. A Fraïssé order class is a Fraïssé class which has at least one relation which is a linear order. One of their main theorems (Theorem 4.7) shows that the extremely amenable (fixed point property on compacta) closed subgroups of the infinite symmetric group S_∞ are exactly those of the form $\mathrm{Aut}(\mathbf{F}^*)$, where $\mathbf{F}^*$ is the Fraïssé limit of some Fraïssé order class satisfying the Ramsey property. Another main theorem (Theorem 10.8) provides a way to compute the universal minimal flow of topological groups which arise as the automorphism groups of Fraïssé limits of Fraïssé classes with the Ramsey property and the ordering property. That the ordering property can be relaxed to the expansion property was proved by Nguyen Van Thé in [26].

We now turn to Ramsey theory on infinite structures. One may ask whether analogues of Theorem 1.1 can hold on more complex infinite relational structures, in particular, for Fraïssé limits of Fraïssé classes. The Fraïssé limit $\mathbf{F}$ of a Fraïssé class $\mathcal{K}$ of finite relational structures is said to have *finite big Ramsey degrees* if for each member A in $\mathcal{K}$, there is a finite number $T(\mathrm{A},\mathcal{K})$ such that for any coloring c of all the substructures of $\mathbf{F}$ which are isomorphic to A into finitely many colors, there is a substructure $\mathbf{F}'$ of $\mathbf{F}$ which is isomorphic to $\mathbf{F}$ and in which c takes no more than $T(\mathrm{A},\mathcal{K})$ colors. When this is the case, we write

$$\mathbf{F} \rightarrow (\mathbf{F})^{\mathrm{A}}_{k,T(\mathrm{A},\mathcal{K})}. \tag{3}$$

This notion has been around for several decades, but the terminology was initiated in [16].

The first homogeneous structure shown to have finite big Ramsey degrees is the rationals, which are the Fraïssé limit of the class of finite linear orders $\mathcal{LO}$. That the upper bounds exist was known by Laver, following from applications of Milliken's Theorem (see Theorem 2.5). The lower bounds were proved by Devlin in 1979 in his PhD thesis [2], where he showed that the numbers $T(k,\mathcal{LO})$ are actually tangent numbers, coefficients of the Talyor series expansion of the tangent function. In particular, $T(1,\mathbb{Q}) = 1$, as any coloring of the rationals into finitely many colors contains a copy of the rationals in one color; thus, the rationals are *indivisible*. On the other hand, $T(2,\mathbb{Q}) = 2$, so immediately for colorings of pairsets of rationals, one sees that there is no Ramsey property for the rationals when one requires that the substructure $\mathbf{Q}'$ of $\mathbb{Q}$ be "big", meaning isomorphic to the original infinite one.

The next homogeneous structure for which big Ramsey degrees have been proved is the the Rado graph, denoted $\mathcal{R}$. Also known as the random graph, $\mathcal{R}$ is the countable graph which is universal for all countable graphs, meaning each countable graph embeds into $\mathcal{R}$ as an induced substructure. Equivalently, the Rado graph is the Fraïssé limit of the class of finite graphs, denoted $\mathcal{G}$. It is an easy exercise from the defining property of the Rado graph to show that the Rado graph is indivisible, meaning that the big Ramsey degree for vertices in the Rado graph is 1. The first non-trivial lower bound result for big Ramsey degrees was proved by Erdős, Hajnal and Pósa in [6] in 1975, where they showed there is a coloring of the edges in $\mathcal{R}$ into two colors such that for any subgraph $\mathcal{R}'$ of the Rado graph such that $\mathcal{R}'$ is also universal for countable graphs, the edges in $\mathcal{R}'$ take on both colors. That this upper bound is sharp was proved over two decades later in 1996 by Pouzet and Sauer in [28], and thus, the big Ramsey degree for edges in the Rado graph is 2. The problem of whether every finite graph has a finite big Ramsey degree in the Rado graph took another decade to solve. In [31], Sauer proved that the Rado graph, and in fact a general class of binary relational homogeneous structures, have finite big Ramsey degrees. As in Laver's result, Milliken's Theorem plays a central role in obtaining the upper bounds. The sharp lower bounds were proved the same year by Laflamme, Sauer, and Vuksanovic in [18].

Sauer's result on the Rado graph in conjunction with the attention called to big Ramsey degrees in [16] sparked new interest in the field. In 2008, Nguyen Van Thé investigated big Ramsey degrees for homogeneous ultrametric spaces. Given S a set of positive real numbers, $\mathcal{U}_S$ denotes the class of all finite ultrametric spaces with strictly positive distances in S. Its Fraïssé limit, denoted $\mathbf{Q}_S$, is called the *Urysohn space associated with* $\mathcal{U}_S$ and is a homogeneous ultrametric space. In [25], Nguyen Van Thé proved that $\mathbf{Q}_S$ has finite big Ramsey degrees whenever S is finite. Moreover, if S is infinite, then any member of $\mathcal{U}_S$ of size greater than or equal to 2 does not have a big Ramsey degree. Soon after, Laflamme, Nguyen Van Thé, and Sauer proved in [17] that enriched structures of the rationals, and two related directed graphs, have finite big Ramsey degrees. For each $n \geq 1$, $\mathbb{Q}_n$ denotes the structure $(\mathbb{Q}, Q_1, \dots, Q_n, <)$, where $Q_1, \dots, Q_n$ are disjoint dense subsets of $\mathbb{Q}$ whose union is $\mathbb{Q}$. This is the Fraïssé limit of the class $\mathcal{P}_n$ of all finite linear orders equipped with an equivalence relation with n many equivalence classes. Laflamme, Nguyen Van Thé, and Sauer proved that each member of $\mathcal{P}_n$ has a finite big Ramsey degree in $\mathbb{Q}_n$. Further, using the bi-definability between $\mathbb{Q}_n$ and the circular directed graphs $\mathbf{S}(n)$, for $n = 2, 3$, they proved that $\mathbf{S}(2)$ and $\mathbf{S}(3)$ have finite big Ramsey degrees. Central to these results is a colored verision of Milliken's theorem which they proved in order to deduce the big Ramsey degrees. For a more detailed overview of these results, the reader is referred to [27].

A common theme emerges when one looks at the proofs in [2], [31], and [17]. The first two rely in an essential way on Milliken's Theorem, Theorem 2.5 in Section 2. The third proves a new colored version of Milliken's Theorem and uses it to deduce the results. The results in [25] use Ramsey's theorem. This would lead one to conclude or at least conjecture that, aside from Ramsey's Theorem itself, Milliken's Theorem contains the core combinatorial content of big Ramsey degree results. The lack of such a result applicable to homogeneous structures omitting non-trivial substructures posed the main obstacle to the investigation of their big Ramsey degrees. This is addressed in the present paper.

This article is concerned with the question of big Ramsey degrees for the universal homogeneous countable triangle-free graph, denoted $\mathcal{H}_3$. A graph G is *triangle-free* if for any three vertices in G, there is at least one pair with no edge between them; in other words, no triangle embeds into G as an induced subgraph. A graph $\mathcal{H}$ on countably many vertices is a *universal triangle-free graph* if each triangle-free graph on countably many vertices embeds into $\mathcal{H}$. Universal triangle-free graphs were first constructed by Henson in [15], and are seen to be the Fraïssé limit of $\mathcal{K}_3$, the Fraïssé class of all countable triangle-free graphs. A graph H on countably many vertices is *homogeneous* if whenever G is a finite subgraph of H, then every embedding of G into H can be extended to an automorphism of H. Henson proved that every universal triangle-free graph is homogeneous, and vice versa, and further, that any two universal homogeneous triangle-free graphs are isomorphic.

As mentioned above, Nešetřil and Rödl proved that the Fraïssé class of finite ordered triangle-free graphs, denoted $\mathcal{K}_3^<$, has the Ramsey property. It follows that the Fraïssé class of unordered finite triangle-free graphs, denoted $\mathcal{K}_3$, has finite small Ramsey degrees. In contrast, whether or not every finite triangle-free graph has a finite big Ramsey degree in $\mathcal{H}_3$ had been open until now. The first result on colorings of vertices of $\mathcal{H}_3$ was obtained by Henson in [15] in 1971. In that paper, he proved that $\mathcal{H}_3$ is weakly indivisible: Given any coloring of the vertices of $\mathcal{H}_3$ into two colors, either there is a copy of $\mathcal{H}_3$ in which all vertices have the first color, or else a copy of each member of $\mathcal{K}_3$ can be found with all vertices having the second color. From this follows a prior result of Folkman in [8], that for any finite triangle-free graph G and any number $k \geq 2$, there is a finite triangle-free graph H such that for any partition of the vertices of H into k pieces, there is a copy of G in having all its vertices in one of the pieces of the partition. In 1986, Komjáth and Rödl proved that $\mathcal{H}_3$ is indivisible; thus, the big Ramsey degree for vertex colorings is 1. It then became of interest whether this result would extend to colorings of copies of a fixed finite triangle-free graph, rather than just colorings of vertices.

In 1998, Sauer proved in [30] that edges have finite big Ramsey degree of 2 in $\mathcal{H}_3$, leaving open the following question:

Question 1.2. Does every finite triangle-free graph have finite big Ramsey degree in $\mathcal{H}_3$?

This paper answers this question in the affirmative.

Ideas from Sauer's proof in [31] that the Rado graph has finite big Ramsey degrees provided a strategy for our proof in this paper. A rough outline of Sauer's proof is as follows: Graphs can be coded by nodes on trees. Given such codings, the graph coded by the nodes in the tree consisting of all finite length sequences of 0's and 1's, denoted as $2^{<\omega}$, is bi-embeddable with the Rado graph. Only certain subsets, called strongly diagonal, need to be considered when handling tree codings of a given finite graph G. Any finite strongly diagonal set can be enveloped into a strong tree, which is a tree isomorphic to $2^{\leq k}$ for some k. The coloring on the copies of G can be extended to color the strong tree envelopes. Applying Milliken's Theorem for strong trees finitely many times, one obtains an infinite strong subtree S of $2^{<\omega}$ in which for all diagonal sets coding G with the same strong similarity type have the same color. To finish, take a strongly diagonal D subset of S which codes the Rado graph, so that all codings of G in D must be strongly diagonal. Since there are only finitely many similarity types of strongly diagonal sets coding G, this yields the finite big Ramsey degrees for the Rado graph. See Section 2 for more details.

This outline seemed to the author the most likely to succeed if indeed the universal triangle-free graph were to have finite big Ramsey degrees. However, there were difficulties involved in each step of trying to adapt Sauer's proof to the setting of $\mathcal{H}_3$, largely because $\mathcal{H}_3$ omits a substructure, namely triangles. First, unlike the bi-embeddability between the Rado graph and the graph coded by the nodes in $2^{<\omega}$, there is no bi-embeddability relationship between $\mathcal{H}_3$ and some triangle-free graph coded by some tree with a very regular structure. To handle this, rather than letting certain nodes in a tree code vertices at the very end of the whole proof scheme as Sauer does in [31], we introduce a new notion of *strong triangle-free tree* in which we distinguish certain nodes *in* the tree (called *coding nodes*) to code the vertices of a given graph, and in which the branching is maximal subject to the constraint of these distinguished nodes not coding any triangles. We further develop a flexible construction method for creating strong triangle-free trees in which the distinguished nodes code $\mathcal{H}_3$. These are found in Section 3.

Next, we wanted an analogue of Milliken's Theorem for strong triangle-free trees. While we were able to prove such a theorem for any configuration extending some fixed stem, the result simply does not hold for colorings of stems, as can be seen by an example of a bad coloring defined using interference between splitting nodes and coding nodes on the same level (Example 3.18). The means around this this was to introduce the new notion of *strong coding tree*, which is a skew tree that stretches a strong triangle-free tree while preserving all important aspects of its coding structure. Strong coding trees are defined and constructed in Section 4. There, the fundamentals of the collection of strong coding trees are charted, including sufficient conditions guaranteeing when a finite subtree A of a strong coding tree T may be end-extended into T to form another strong coding tree.

Having formulated the correct kind of trees to code $\mathcal{H}_3$, the next task is to prove an analogue of Milliken's Theorem for strong coding trees. This is accomplished in Sections 5 and 6. First, we prove analogues of the Halpern-Läuchli Theorem (Theorem 2.2) for strong coding trees. There are two cases, depending on whether the level sets being colored contain a splitting node or a coding node. In Case (a) of Theorem 5.2, we obtain the direct analogue of the Halpern-Läuchli Theorem when the level set being colored has a splitting node. A similar result is proved in Case (b) of Theorem 5.2 for level sets containing a coding node, but some restrictions apply, and these are taken care of in Section 6. The proof of Theorem 5.2 Section 5 uses the set-theoretic method of forcing, using some forcing posets created specifically for strong coding trees. However, one never moves into a generic extension; rather the forcing mechanism is used to do an unbounded search for a finite object. Once found, it is used to build the next finite level of the tree homogeneous for a given coloring. Thus, the result is a ZFC proof. This builds on ideas from Harrington's forcing proof of the Halpern-Läuchli Theorem.

In Section 6, after an initial lemma obtaining end-homogeneity, we achieve the analogue of the Halpern-Läuchli Theorem for Case (b) in Lemma 6.8. The proof introduces a third forcing which homogenizes over the possibly different end-homogeneous colorings, but again achieves a ZFC result. Then, using much induction and fusion, we obtain the first of our two Milliken-style theorems.

Theorem 6.3. *Let T be a strong coding tree and let A be a finite subtree of T satisfying the Strong Parallel 1's Criterion. Then for any coloring of all strictly similar copies of A in T into finitely many colors, there is a strong coding tree $S \leq T$ such that all strictly similar copies of A in S have the same color.*

The Strong Parallel 1's Criterion is made clear in Definition 6.1. Initial segments of strong coding trees automatically satisfy the Strong Parallel 1's Criterion. Essentially, it is a strong condition which guarantees that the finite subtree can be extended to a tree coding $\mathcal{H}_3$.

Developing the correct notion of strong subtree envelope for the setting of triangle-free graphs presented a further obstacle. The idea of extending a subset X of a strong coding tree T to an envelope which is a finite strong triangle-free tree and applying Theorem 6.3 (which would be the direct analogue of Sauer's method) simply does not work, as it can lead to an infinite regression of adding coding nodes in order to make an envelope of that form. That is, there is no upper bound on the number of similarity types of finite strong triangle-free subtrees of T which are minimal containing copies of X in T. To overcome this, in Sections 7 and 8 we develop the notions of incremental new parallel 1's and strict similarity type for finite diagonal sets of coding nodes as well as a new notion of envelope. Given any finite triangle-free graph G, there are only finitely many strict similarity types of diagonal trees coding G. Letting c be any coloring of all copies of G in $\mathcal{H}_3$ into finitely many colors, we transfer the coloring to the envelopes and apply the results in previous sections to obtain a strong coding tree $T' \leq T$ in which all envelopes encompassing the same strict similarity type have the same color. The next new idea is to thin T' to an incremental strong subtree $S \leq T'$ while simultaneously choosing a set $W \subseteq T'$ of *witnessing coding nodes*. These have the property that each finite subset X of S is incremental, and furthermore, one can add to X coding nodes from W to form an envelop satisfying the Strong Parallel 1's Criterion. Then we arrive at our second Milliken-style theorem for strong coding trees, extending the first one.

Theorem 8.9 (Ramsey Theorem for Strict Similarity Types)**.** *Let Z be a finite antichain of coding nodes in a strong coding tree T, and let h be a coloring of all subsets of T which are strictly similar to Z into finitely many colors. Then there is an incremental strong coding tree $S \leq T$ such that all subsets of S strictly similar to Z have the same h color.*

After thinning to a strongly diagonal subset $D \subseteq S$ still coding $\mathcal{H}_3$, the only sets of coding nodes in D coding a given finite triangle-free graph G are automatically antichains which are incremental and strongly diagonal. Applying Theorem 8.9 to the finitely many strict similarity types of incremental strongly diagonal sets coding G, we arrive at the main theorem.

Theorem 9.2. *The universal triangle-free homogeneous graph has finite big Ramsey degrees.*

For each $\mathrm{G} \in \mathcal{K}_3$, the number $T(\mathrm{G}, \mathcal{K}_3)$ is bounded by the number of strict similarity types of diagonal sets of coding nodes coding G, which we denote as $\mathrm{StrSim}(\mathrm{G}, \mathbb{T})$, $\mathbb{T}$ referring to any strong coding tree (see Section 4). It is presently open to see if $\mathrm{StrSim}(\mathrm{G}, \mathbb{T})$ is in fact the lower bound. If it is, then recent work of Zucker would provide an interesting connection with topological dynamics. In [37], Zucker proved that if a Fraïssé structure $\mathbf{F}$ has finite big Ramsey degrees and moreover, $\mathbf{F}$ admits a big Ramsey structure, then any big Ramsey flow of $\mathrm{Aut}(\mathbf{F})$ is a universal completion flow, and further, any two universal completion flows are isomorphic. His proof of existence of a big Ramsey structure a Fraïssé structure presently relies on the existence of colorings for an increasing sequence of finite objects whose union is $\mathbf{F}$ exhibiting all color classes which cannot be removed and which cohere in a natural way. In particular, the lower bounds for the big Ramsey numbers are necessary to Zucker's analysis. His work already applies to the rationals, the Rado graph, lower bounds being obtained by Laflamme, Sauer, and Vuksanovic in [18] and calculated for each class of graphs of fixed finite size by Larson in [19], finite ultrametric spaces with distances from a fixed finite set, $\mathbb{Q}_n$ for each $n \geq 2$, $\mathbf{S}(2)$, and $\mathbf{S}(3)$. As the strict similarity types found in this paper satisfy Zucker's coherence condition, the precise lower bounds for the big Ramsey degrees of $\mathcal{H}_3$ would provide another such example of a universal completion flow.

Acknowledgments. Much gratitude goes to Dana Bartošová for listening to and making helpful comments on early and later stages of these results and for her continued encouragement of my work on this problem; Jean Larson for listening to early stages of this work and her encouragement; Norbert Sauer for discussing key aspects of the homogeneous triangle-free graph with me during a research visit in Calgary in 2014; Stevo Todorcevic for pointing out to me in 2012 that any proof of finite Ramsey degrees for $\mathcal{H}_3$ would likely involve a new type of Halpern-Läuchli Theorem; and to the organizers and participants of the Ramsey Theory Doccourse at Charles University, Prague, 2016, for their encouragement. Most of all, I am grateful for and much indebted to Richard Laver for providing for me in 2011 the main points of Harrington's forcing proof of the Halpern-Läuchli Theorem, from which I could reconstruct the proof, setting the stage for the possibility of accomplishing this work. His spirit lives on.

2. Background: Trees coding graphs and the Halpern-Läuchli and Milliken Theorems

This section provides background and context for the developments in this paper. It contains the well-known method of using trees to code graphs as well as the Halpern-Läuchli and Milliken Theorems, and discusses their applications to previously known results on big Ramsey degrees for homogeneous structures.

2.1. **Trees coding graphs.** In [6], Erdős, Hajnal and Pósa gave the vertices in a graph a natural lexicographic order and used it to solve problems regarding strong embeddings of graphs. The set of vertices of a graph ordered by this lexicographic order can be viewed as nodes in the binary tree of finite sequences of 0's and 1's with the usual tree ordering. This view was made explicit in [30] and is described below.

The following notation is standard in mathematical logic and shall be used throughout. The set of all natural numbers $\{0, 1, 2, \dots\}$ is denoted by ω. Each natural number $k \in \omega$ is equated with the set of all natural numbers strictly less than k. Thus, 0 denotes the emptyset, $1 = \{0\}$, $2 = \{0, 1\}$, etc. For each natural number k, 2^k denotes the set of all functions from $\{0, \dots, k-1\}$ into $\{0, 1\}$. A finite *binary sequence* is a function $s : k \to 2$ for some $k \in \omega$. We may write s as $\langle s(0), \dots, s(k-1)\rangle$; for each $i < k$, $s(i)$ denotes the i-th *value* or *entry* of the sequence s. We shall use $2^{<\omega}$ to denote the collection $\bigcup_{k\in\omega} 2^k$ of all finite binary sequences. For $s \in 2^{<\omega}$, we let $|s|$ denote the *length* of s; this is exactly the domain of s. For nodes $s, t \in 2^{<\omega}$, we write $s \subseteq t$ if and only if $|s| \leq |t|$ and for each $i < |s|$, $s(i) = t(i)$. In this case, we say that s is an *initial segment* of t, or that t *extends* s. If t extends s and $|t| > |s|$, then we write $s \subset t$, and we say that s is a proper initial segment of t. For $i < \omega$, we let $s \restriction i$ denote the function s restricted to domain i. Thus, if $i < |s|$, then $s \restriction i$ is the proper initial segment of s of length i, $s \restriction i = \langle s(0), \dots, s(i-1)\rangle$; if $i \geq |s|$, then $s \restriction i$ equals s. The set $2^{<\omega}$ forms a tree when partially ordered by inclusion.

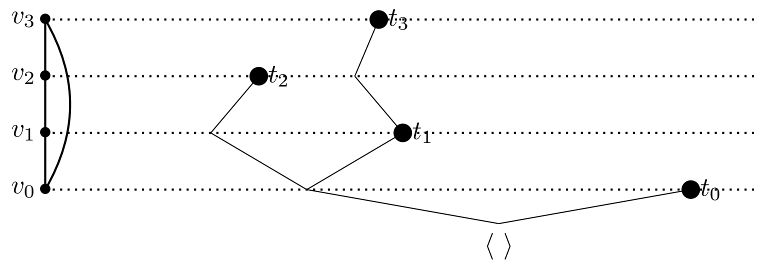

FIGURE 1. A tree with nodes $\{t_0, t_1, t_2, t_3\}$ coding the 4-cycle $\{v_0, v_1, v_2, v_3\}$

Let v, w be vertices in some graph. Two nodes $s, t \in 2^{<\omega}$ are said to *represent* v and w, respectively, if and only if, without loss of generality assuming that $|s| < |t|$, then v and w have an edge between them if and only if $t(|s|) = 1$. The number $t(|s|)$ is called the *passing number* of t at s. Thus, if t has passing number 1 at s, then s and t code an edge between v and w; and if t has passing number 0 at s, then s and t code a non-edge between v and w.

Using this idea, any graph can be coded by nodes in a binary tree as follows. Let G be a graph with N vertices, where $N \le \omega$, and let $\langle v_n : n < N\rangle$ be any enumeration of the vertices of G. Choose any node $t_0 \in 2^{<\omega}$ to represent the vertex v_0. For $n > 0$, given nodes $t_0, \dots, t_{n-1}$ in $2^{<\omega}$ coding the vertices $v_0, \dots, v_{n-1}$, take t_n to be any node in $2^{<\omega}$ such that $|t_n| > |t_{n-1}|$ and for all $i < n$, v_n and v_i have an edge between them if and only if $t_n(|t_i|) = 1$. Then the set of nodes $\{t_n : n < N\}$ codes the graph G. Note that any finite graph of size k can be coded by a collection of nodes in $\bigcup_{i<k} {}^i2$. Throughout this paper we shall hold to the convention that the nodes in a tree used to code a graph will have different lengths. Figure 1. shows a set of nodes $\{t_0, t_1, t_2, t_3\}$ from $2^{<\omega}$ coding the four-cycle $\{v_0, v_1, v_2, v_3\}$.

2.2. **The Halpern-Läuchli and Milliken Theorems.** The theorem of Halpern and Läuchli below was established as a technical lemma containing core combinatorial content of the proof that the Boolean Prime Ideal Theorem (the statement that any filter can be extended to an ultrafilter) is strictly weaker than the Axiom of Choice, assuming the Zermelo-Fraenkel axioms of set theory. (See [14].) The Halpern-Läuchli Theorem forms the basis for a Ramsey theorem on strong trees due to Milliken, which in turn forms the backbone of all previously found finite big Ramsey degrees, except where Ramsey's Theorem itself suffices. An in-depth presentation of the various versions of the Halpern-Läuchli Theorem as well as Milliken's Theorem can be found in [33]. An account focused solely on the theorems relevant to the present work can be found in [3]. Here, we merely give an overview sufficient for this article, and shall restrict to subtrees of $2^{<\omega}$, though the results hold more generally for finitely branching trees.

In this paper, we use the definition of tree which is standard for Ramsey theory on trees. The meet of two nodes s and t in $2^{<\omega}$, denoted $s \wedge t$, is the longest member $u \in 2^{<\omega}$ which is an initial segment of both s and t. Thus, $u = s \wedge t$ if and only if $u = s \restriction |u| = t \restriction |u|$ and $s \restriction (|u|+1) \ne t \restriction (|u|+1)$. In particular, if $s \subseteq t$ then $s \wedge t = s$. A set of nodes $A \subseteq 2^{<\omega}$ is *closed under meets* if $s \wedge t$ is in A, for each pair $s, t \in A$.

Definition 2.1. A subset $T \subseteq 2^{<\omega}$ is a *tree* if T is closed under meets and for each pair $s, t \in T$ with $|s| \le |t|$, $t \restriction |s|$ is also in T.

Given $n < \omega$ and a set of nodes $A \subseteq 2^{<\omega}$, define

$$A(n) = \{t \in A : |t| = n\}. \tag{4}$$

A set $X \subseteq A$ is a *level set* if $X \subseteq A(n)$ for some $n < \omega$. Note that a tree T does not have to contain all initial segments of its members, but for each $s \in T$, the level set $T(|s|)$ must equal $\{t \restriction |s| : t \in T \text{ and } |t| \ge |s|\}$.

Let $T \subseteq 2^{<\omega}$ be a tree and let $L = \{|s| : s \in T\}$. If L is infinite, then T is a *strong tree* if every node in T *splits* in T; that is, for each $t \in T$, there are $u, v \in T$ such that u and v properly extend t, and $u(|t|) = 0$ and $v(|t|) = 1$. If L is finite, then T is a *strong tree* if for each node $t \in T$ with $|t| < \max(L)$, t splits in T. A finite strong tree subtree of $2^{<\omega}$ with k many levels is called a *strong tree of height k*. Note that each finite strong subtree of $2^{<\omega}$ is isomorphic as a tree to some binary tree of height k. In particular, a strong tree of height 1 is simply a node in $2^{<\omega}$. See Figure 2. for an example of a strong tree of height 3.

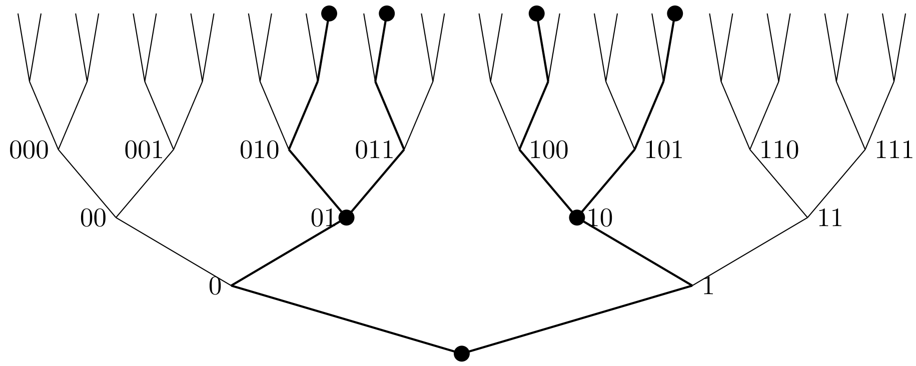

FIGURE 2. A strong subtree of $2^{<\omega}$ of height 3

The following is the strong tree version of the Halpern-Läuchli Theorem. It is a Ramsey theorem for colorings of products of level sets of finitely many trees. Here, we restrict to the case of binary trees, since that is sufficient for the exposition in this paper.

Theorem 2.2 (Halpern-Läuchli, [13]). *Let $T_i = 2^{<\omega}$ for each $i < d$, where d is any positive integer, and let*

$$c : \bigcup_{n<\omega} \prod_{i<d} T_i(n) \to k \tag{5}$$

be a given coloring, where k is any positive integer. Then there is an infinite set of levels $L \subseteq \omega$ and infinite strong subtrees $S_i \subseteq T_i$, each with nodes exactly at the levels in L, such that c is monochromatic on

$$\bigcup_{n\in L} \prod_{i<d} S_i(n). \tag{6}$$

This theorem of Halpern and Läuchli was applied by Laver in [20] to prove that given $k \geq 2$ and given any coloring of the product of k many copies of the rationals $\mathbb{Q}^k$ into finitely many colors, there are subsets X_i of the rationals which again are dense linear orders without endpoints such that $X_0 \times \cdots \times X_{k-1}$ has at most $k!$ colors. Laver further proved that $k!$ is the lower bound. Thus, the big Ramsey degree for the simplest object (single k-length sequences) in the Fraïssé class of products of finite linear orders has been found. The full result for all big Ramsey degrees for Age($\mathbb{Q}^k$) would involve applications of the extension of Milliken's theorem to products of finitely many copies of $2^{<\omega}$; such an extension has been proved by Vlitas in [35].

Harrington produced an interesting method of proof of the Halpern-Läuchli Theorem which uses the set-theoretic technique of forcing, but which takes place entirely in the standard axioms of set theory, and most of mathematics, ZFC. No new external model is actually built, but rather, finite bits of information, guaranteed by the existence of a generic filter for the forcing, are used to build the subtrees satisfying the Halpern-Läuchli Theorem. This proof is said to provide the clearest intuition into the theorem (see [33]). Harrington's proof was never published, though the ideas were well-known in certain circles. A version close to his original proof appears in [3], where a proof was reconstructed based on an outline provided to the author by Laver in 2011. This proof formed the starting point for our proofs in Sections 5 and 6 of Halpern-Läuchli style theorems for strong coding trees.

Harrington's proof for d many trees uses the forcing which adds κ many Cohen subsets of the product of level sets of d many copies of $2^{<\omega}$, where κ satisfies a certain partition relation, depending on d. For any set X and cardinal μ, $[X]^\mu$ denotes the collection of all subsets of X of cardinality μ.

Definition 2.3. Given cardinals $r, \sigma, \kappa, \lambda$,

$$\lambda \to (\kappa)^r_\sigma \tag{7}$$

means that for each coloring of $[\lambda]^r$ into σ many colors, there is a subset X of λ such that $|X| = \kappa$ and all members of $[X]^r$ have the same color.

The following ZFC result guarantees cardinals large enough to have the Ramsey property for colorings into infinitely many colors.

Theorem 2.4 (Erdős-Rado, [7])**.** *For $r < \omega$ and μ an infinite cardinal,*

$$\beth_r(\mu)^+ \to (\mu^+)^{r+1}_\mu.$$

For d many trees, letting $\kappa = \beth_{2d-1}(\aleph_0)^+$ suffices for Harrington's proof. A modified version of Harrington's proof appears in [34], where the assumption on κ is weaker, only $\beth_{d-1}(\aleph_0)^+$, but the construction is more complex. This proof informed the approach in [4] to reduce the large cardinal assumption for obtaining the consistency of the Halpern-Läuchli Theorem at a measurable cardinal. Building on this and ideas from [32] and [5], Zhang proved the consistency of Laver's result for the κ-rationals, for κ measurable, in [36].

The Halpern-Läuchli Theorem forms the essence of the next Theorem; the proof follows by induction on k, applying Theorem 2.2 to k many infinite strong trees trees.

Theorem 2.5 (Milliken, [21])**.** *Let $k \geq 1$ be given and let all strong subtrees of $2^{<\omega}$ of height k be colored by finitely many colors. Then there is an infinite strong subtree T of $2^{<\omega}$ such that all strong subtrees of T of height k have the same color.*

In the Introduction, an outline of Sauer's proof that the Rado graph has finite big Ramsey degrees was presented. Knowledge of his proof is not a pre-requisite for reading this paper, but the reader with knowledge of that paper will have better context for and understanding of the present article. A more detailed outline of the work in [31] appears in Section 3 of [3], which surveys some recent work regarding Halpern-Läuchli and Milliken Theorems and variants. Chapter 6 of [33] provides a solid foundation for understanding how Milliken's theorem is used to attain big Ramsey degrees for both Devlin's result on the rationals and Sauer's result on the Rado graph. Of course, we recommend foremost Sauer's original article [31].

We point out that Milliken's Theorem has been shown to consistently hold at a measurable cardinal by Shelah in [32], using ideas from Harrington's proof. An enriched version was proved by Džamonja, Larson, and Mitchell in [5] and applied to obtain the consistency of finite big Ramsey degrees for colorings of finite subsets of the κ-rationals, where κ is a measurable cardinal. They obtained the consistency of finite big Ramsey degrees for colorings of finite subgraphs of the κ-Rado graph for κ measurable in [5]. The uncountable height of the tree $2^{<\kappa}$ coding the κ-rationals and the κ-Rado graph renders the notion of strong similarity type more complex than for the countable cases.

There is another theorem stronger than Theorem 2.5, also due to Milliken in [22], which shows that the collection of all infinite strong subtrees of $2^{<\omega}$ forms a topological Ramsey space, meaning that it satisfies an infinite-dimensional Ramsey theorem for Baire sets when equipped with its version of the Ellentuck topology (see [33]). Though not outrightly used, this fact informed some of our intuition when approaching the present work.

3. Strong triangle-free trees coding $\mathcal{H}_3$

In the previous section, it was shown how nodes in binary trees may be used to code graphs, and strong trees and Milliken's Theorem were presented. In this section, we introduce *strong triangle-free trees*, which seem to be the correct analogue of Milliken's strong trees suitable for coding triangle-free graphs.

Sauer's proof in [31] that the Rado graph has finite big Ramsey degrees uses the fact that the Rado graph is bi-embeddable with the graph coded by the collection of all nodes in $2^{<\omega}$, where nodes with the same length code vertices with no edges between them. Colorings on the Rado graph are transfered to the graph represented by the nodes in $2^{<\omega}$, Milliken's Theorem for strong trees is applied, and then the homogeneity is transfered back to the Rado graph. In the case of the universal triangle-free graph, there is no known bi-embeddability between $\mathcal{H}_3$ and some triangle-free graph coded by nodes in a tree with some kind of uniform structure. Indeed, this may be fundamentally impossible precisely because the absence of triangles disrupts any uniformity of a coding structure. Thus, instead of looking for a uniform sort of structure which codes some triangle-free graph bi-embeddable with $\mathcal{H}_3$ and trying to prove a Milliken-style theorem for them, we define a new kind of tree in which certain nodes are distinguished to code the vertices of a given triangle-free graph. Moreover, nodes in the tree branch as much as possible, subject to the constraint that at each level of the tree, no node is extendible to another distinguished node which would code a triangle with previous distinguished nodes. The precise formulation of strong triangle-free tree appears in Definition 3.9.

Some conventions and notation are now set up. Given a triangle-free graph G, finite or infinite, let $\langle v_n : n < N \rangle$ be any enumeration of the vertices of G, where $N \leq \omega$ is the number of vertices in G. We may construct a tree T with certain nodes $\langle c_n : n < N \rangle$ in T coding the graph G as follows. Let c_0 be any node in $2^{<\omega}$ and declare c_0 to code the vertex v_0. For $n > 0$, given nodes $c_0, \dots, c_{n-1}$ in $2^{<\omega}$ coding the vertices $v_0, \dots, v_{n-1}$, let c_n be any node in $2^{<\omega}$ such that the length of c_n, denoted $|c_n|$, is strictly greater than the length of c_{n-1} and for all $i < n$, $c_n(|c_i|) = 1$ if and only if v_n and v_i have an edge between them. The set of nodes $\{c_n : n < N\}$ codes the graph G.

Definition 3.1 (Tree with coding nodes)**.** A *tree with coding nodes* is a structure $(T, N; \subseteq, <, c)$ in the language of $\mathcal{L} = \{\subseteq, <, c\}$, where $\subseteq$ and $<$ are binary relation symbols and c is a unary function symbol, satisfying the following: T is a subset of $2^{<\omega}$ satisfying that $(T, \subseteq)$ is a tree (recall Definition 2.1), $N \leq \omega$ and $<$ is the usual linear order on N, and $c : N \to T$ is an injective function such that $m < n < N$ implies $|c(m)| < |c(n)|$. We use c_n to denote $c(n)$ and call this the *n-th coding node* in T.

Convention 3.2. The length of c_n shall be denoted by l_n. When necessary to avoid confusion between more than one tree, the n-th coding node of a tree T will be denoted as c_n^T, and its length as $l_n^T = |c_n^T|$.

Definition 3.3. A graph G with vertices enumerated as $\langle v_n : n < N \rangle$ is *represented* by a tree T with coding nodes $\langle c_n : n < N \rangle$ if and only if for each pair $i < n < N$, $v_n \mathrel{\mathrm{E}} v_i \longleftrightarrow c_n(l_i) = 1$. We will often simply say that T *codes* G.

The next step is to determine which tree configurations code triangles, for those are the configurations that must be omitted from any tree coding a triangle-free graph. Notice that if v_0, v_1, v_2 are the vertices of some triangle and c_0, c_1, c_2 are coding nodes coding the vertices v_0, v_1, v_2 and the edge relationships between them, where $|c_0| < |c_1| < |c_2|$, then it must be the case that $c_1(|c_0|) = c_2(|c_0|) = c_2(|c_1|) = 1$. Moreover, notice that this is the only way a triangle can be coded by nodes in a tree.

Now we present a criterion which, when satisfied, guarantees that any node t in the tree may be extended to a coding node without coding a triangle with any coding nodes of length less than $|t|$.

Definition 3.4 (Triangle-Free Criterion)**.** Let $T \subseteq 2^{<\omega}$ be a tree with coding nodes $\langle c_n : n < N \rangle$, where $N \leq \omega$. *T satisfies the Triangle-Free Criterion (TFC)* if the following holds: For each $t \in T$, if $l_n < |t|$ and $t(l_i) = c_n(l_i) = 1$ for some $i < n$, then $t(l_n) = 0$.

In words, a tree T with coding nodes $\langle c_n : n < N \rangle$ satisfies the TFC if for each $n < N$, whenever a node u in T has the same length as coding node c_n, and u and c_n both have passing number 1 at the level of a coding node c_i for some $i < n$, then $u^\frown 1$ must not be in T. In particular, the TFC implies that if c_n has passing number 1 at c_i for any $i < n$, then c_n cannot split; that is, $c_n{}^\frown 1$ must not be in T.

Remark 3.5. The point of the TFC is as follows: Whenever a finite tree T satisfies the TFC, then any maximal node of T may be extended to a new coding node without coding a triangle with the coding nodes in T.

The next proposition provides a characterization of tree representations of triangle-free graphs.

Proposition 3.6 (Triangle-Free Tree Representation)**.** *Let $T \subseteq 2^{<\omega}$ be a tree with coding nodes $\langle c_n : n < N \rangle$ coding a countable graph* G *with vertices $\langle v_n : n < N \rangle$, where $N \leq \omega$. Assume that the coding nodes in T are dense in T, meaning that for each $t \in T$, there is some coding node $c_n \in T$ such that $t \subseteq c_n$. Then the following are equivalent:*

1. G *is triangle-free.*
2. T *satisfies the Triangle-Free Criterion.*

Proof. Note that if N is finite, then the coding nodes in T being dense in T implies that every maximal node in T is a coding node; in this case, the maximal nodes in T have different lengths.

Suppose (2) fails. Then there are $i < j < N$ and $t \in T$ with length greater than l_j such that $t(l_i) = c_j(l_i) = 1$ and $t(l_j) = 1$. Since every node in T extends to a coding node, there is a $k > j$ such that $c_k \supseteq t$. Then c_k has passing number 1 at both c_i and c_j. Thus, the coding nodes c_i, c_j, c_k code that the vertices $\{v_i, v_j, v_k\}$ have edges between each pair, implying G contains a triangle. Therefore, (1) fails.

Conversely, suppose that (1) fails. Then G contains a triangle, so there are $i < j < k < N$ such that the vertices v_i, v_j, v_k have edges between each pair. Since the coding nodes c_i, c_j, c_k code these edges, it is the case that $c_j(l_i) = c_k(l_i) = c_k(l_j) = 1$. Hence, the nodes c_i, c_j, c_k witness the failure of the TFC. $\square$

Definition 3.7 (Parallel 1's)**.** For two nodes $s, t \in 2^{<\omega}$, we say that s and t *have parallel* 1*'s* if there is some $l < \min(|s|, |t|)$ such that $s(l) = t(l) = 1$.

Definition 3.8. Let T be a tree with coding nodes $\langle c_n : n < N\rangle$ such that, above the stem of T, splitting in T occurs only at the levels of coding nodes. Then T satisfies the *Splitting Criterion* if for each $n < N$ and each non-maximal t in T with $|t| = |c_n|$, t splits in T if and only if t and c_n have no parallel 1's.

Notice that whenever a tree T with coding nodes satisfies the Splitting Criterion, each coding node which is not solely a sequence of 0's will not split in T. Thus, the Splitting Criterion produces maximal splitting subject to ensuring that no nodes can be extended to code a triangle, while simultaneously reducing the number of similarity types of trees under consideration later for the big Ramsey degrees, if we require each coding node to have at least one passing number of 1.

Next, strong triangle-free trees are defined. These trees provide the intuition and the main structural properties of their skewed variant defined in Section 4.

Definition 3.9 (Strong triangle-free tree)**.** A *strong triangle-free tree* is a tree with coding nodes, $(T, N; \subseteq, <, c)$ such that for each $n < N$, the length of the n-th coding node c_n is $l_n = n + 1$ and

1. If $N = \omega$, then T has no maximal nodes. If $N < \omega$, then all maximal nodes of T have the same length, which is l_{N-1}.
2. $\mathrm{stem}(T)$ is the empty sequence $\langle\rangle$.
3. $c_0 = \langle 1\rangle$, and for each $0 < n < N$, $c_n(l_{n-1}) = 1$.
4. For each $n < N$, the sequence of length l_n consisting of all 0's, denoted 0^{l_n}, is a node in T.
5. T satisfies the Splitting Criterion.

T is a *strong triangle-free tree densely coding* $\mathcal{H}_3$ if T is an infinite strong triangle-free tree and the set of coding nodes is dense in T.

Strong triangle-free trees can be defined more generally than we choose to present here, for instance, by relaxing conditions (2) and (3), leaving off the restriction that $l_n = n + 1$, and letting c_0 be any node. The notion of strong subtree of a given strong triangle-free tree can also be made precise, and the collection of such trees end up forming a space somewhat similar to the Milliken space of strong trees. However, as Milliken-style theorems are impossible to prove for strong triangle-free trees, as will be shown in Example 3.18, we restrict here to a simpler presentation with the aim of building the reader's understanding of the essential structure of strong triangle-free trees, as the strong coding trees defined in the next section are skewed and slightly relaxed versions of trees in Definition 3.9.

We now set up to present a method for constructing strong triangle-free trees densely coding $\mathcal{H}_3$. Let $\mathcal{K}_3$ denote the Fraïssé class of all triangle-free countable graphs. Given a graph H and a subset V_0 of the vertices of H, the notation $\mathrm{H}|V_0$ denotes the induced subgraph of H on the vertices in V_0. In [15], Henson proved that a countable graph H is universal for $\mathcal{K}_3$ if and only if H satisfies the following property.

(A_3) (i) H does not admit any triangles,
(ii) If V_0, V_1 are disjoint finite sets of vertices of H and $\mathrm{H}|V_0$ does not admit an edge, then there is another vertex which is connected in H to every member of V_0 and to no member of V_1.

Henson used this property to construct a universal triangle-free graph $\mathcal{H}_3$ in [15], as well as universal graphs for each Fraïssé class of countable graphs omitting k-cliques, as the analogues of the Rado graph for countable k-clique free graphs. The following property $(A_3)'$ is a reformulation of Henson's property (A_3).

$(A_3)'$ (i) H does not admit any triangles.
(ii) Let $\langle v_n : n < \omega\rangle$ enumerate the vertices of H, and let $\langle F_i : i < \omega\rangle$ be any enumeration of the finite subsets of ω such that for each $i < \omega$, $\max(F_i) < i$ and each finite set appears infinitely many times in the enumeration. Then there is a strictly increasing sequence $\langle n_i : i < \omega\rangle$ such that for each $i < \omega$, if $\mathrm{H}|\{v_m : m \in F_i\}$ has no edges, then for all $m < i$, $v_{n_i} \, E \, v_m \longleftrightarrow m \in F_i$.

It is straightforward to check the following fact.

Fact 3.10. *Let* H *be a countably infinite graph. Then* H *is universal for $\mathcal{K}_3$ if and only if $(A_3)'$ holds.*

The following tree re-formulation of property $(A_3)'$ will be used to build trees with coding nodes which code $\mathcal{H}_3$. Let $T \subseteq 2^{<\omega}$ be a tree with coding nodes $\langle c_n : n < \omega\rangle$. We say that T *satisfies property* $(A_3)^{\mathtt{tree}}$ if the following holds:

$(A_3)^{\mathtt{tree}}$ (i) T satisfies the Triangle-Free Criterion,

(ii) Let $\langle F_i : i < \omega\rangle$ be any enumeration of finite subsets of ω such that for each $i < \omega$, $\max(F_i) < i$, and each finite subset of ω appears as F_i for infinitely many indices i. For each $i < \omega$, if for all pairs $j < k$ in F_i it is the case that $c_k(l_j) = 0$, then there is some $n \geq i$ such that for all $m < i$, $c_n(l_m) = 1$ if and only if $m \in F_i$.

Fact 3.11. *A tree T with coding nodes $\langle c_n : n < \omega\rangle$ codes $\mathcal{H}_3$ if and only if T satisfies $(A_3)^{\mathtt{tree}}$.*

Remark 3.12. Any strong triangle-free tree in which the coding nodes are dense automatically satisfies $(A_3)^{\mathtt{tree}}$, and hence codes $\mathcal{H}_3$.

The next lemma shows that any finite strong triangle-free tree is able to be extended to a tree satisfying $(A_3)^{\mathtt{tree}}$.

Lemma 3.13. *Let T be a finite strong triangle-free tree with coding nodes $\langle c_n : n < N\rangle$, where $N < \omega$. Given any $F \subseteq N-1$ for which the set $\{c_n : n \in F\}$ codes no edges, there is a maximal node $t \in T$ such that for all $n < N-1$,*

$$t(l_n) = 1 \quad \longleftrightarrow \quad n \in F. \tag{8}$$

Proof. The proof is by induction on N over all strong triangle-free trees with N coding nodes. For $N \leq 1$, the lemma trivially holds but is not very instructive, so we shall start with the case $N = 2$. Let T be a strong triangle-free tree with coding nodes $\{c_0, c_1\}$. By (2) of Definition 3.9, the stem of T is the empty sequence, so both $\langle 0\rangle$ and $\langle 1\rangle$ are in T. By (3) of Definition 3.9, $c_0 = \langle 1\rangle$, and $c_1(l_0) = 1$. By the Splitting Criterion, c_0 does not split in T but $\langle 0\rangle$ does, so $\langle 0, 0\rangle$, $\langle 0, 1\rangle$, and $\langle 1, 0\rangle$ are in T while $\langle 1, 1\rangle$ is not in T. Note that $c_1 = \langle 0, 1\rangle$, since it must be that $l_1 = 2$ and $c_1(l_0) = 1$, and $\langle 1, 1\rangle$ is not in T. The only non-empty $F \subseteq 1$ is $F = \{0\}$. The coding node c_1 satisfies that $c_1(l_n) = 1$ if and only if $n \in \{0\}$. For $F = \emptyset$, both the nodes $t = \langle 0, 0\rangle$ and $t = \langle 1, 0\rangle$ satisfy that for all $n < 1$, $t(l_n) = 1$ if and only if $n \in F$.

Now assume that the lemma holds for all $N' < N$, where $N \geq 3$. Let T be a strong triangle-free tree with N coding nodes. Let F be a subset of $N-1$ such that $\{c_n : n \in F\}$ codes no edges. By the induction hypothesis, there is a node t in T of length l_{N-2} such that for all $n < N-2$, $t(l_n) = 1$ if and only if $n \in F$. If $N-2 \notin F$, then as $t^\frown 0$ is guaranteed to be in T by the Splitting Criterion, the node $t' = t^\frown 0$ in T satisfies that for all $n < N-1$, $t'(l_n) = 1$ if and only if $n \in F$. Now suppose $N-2 \in F$. We claim that $t^\frown 1$ is in T. By the Splitting Criterion, if $t^\frown 1$ is not in T, then it must be the case that t and c_{N-2} have a parallel 1. So there is some $i < N-2$ such that $t(l_i) = c_{N-2}(l_i) = 1$. As t codes edges only with those vertices with indexes $n < N-2$ which are in $F \setminus \{N-2\}$, it follows that $i \in F$. But then $\{c_i, c_{N-2}\}$ codes an edge, contradicting the assumption on F. Therefore, t and c_{N-2} do not have any parallel 1's, and hence $t^\frown 1$ is in T. Letting $t' = t^\frown 1$, we see that for all $n < N-1$, $t(l_n) = 1$ if and only if $n \in F$. □

We now present a method for constructing strong triangle-free trees densely coding $\mathcal{H}_3$. Here and throughout the paper, 0^n denotes the sequence of length n consisting of all 0's.

Theorem 3.14 (Strong Triangle-Free Tree $\mathbb{S}$ Densely Coding $\mathcal{H}_3$)**.** *Let $\langle F_i : i < \omega\rangle$ be any sequence enumerating the finite subsets of ω so that each finite set appears infinitely often. Assume that for each $i < \omega$, $F_i \subseteq i-1$ and $F_{3i} = F_{3i+2} = \emptyset$. Then there is a strong triangle-free tree $\mathbb{S}$ which satisfies property $(A_3)^{\mathtt{tree}}$ and densely codes $\mathcal{H}_3$. Moreover, this property is satisfied specifically by the coding node c_{4i+j} meeting requirement F_{3i+j}, for each $i < \omega$ and $j \leq 2$.*

Proof. Let $\langle F_i : i < \omega\rangle$ satisfy the hypotheses. Enumerate the nodes in $2^{<\omega}$ as $\langle u_i : i < \omega\rangle$ in such a manner that $i < k$ implies $|u_i| \leq |u_k|$. Then $u_0 = \emptyset$, $|u_1| = 1$, and for all $i \geq 2$, $|u_i| < i$. We will build a strong triangle-free tree $\mathbb{S} \subseteq 2^{<\omega}$ with coding nodes $c_n \in \mathbb{S} \cap 2^{n+1}$ densely coding $\mathcal{H}_3$ satisfying the following properties:

(i) $c_0 = \langle 1\rangle$, and for each $n < \omega$, $l_n := |c_n| = n+1$ and $c_{n+1}(l_n) = 1$.

(ii) For $n = 4i + j$, where $j \leq 2$, c_n *satisfies requirement* F_{3i+j}, meaning that if $\{c_k : k \in F_{3i+j}\}$ codes no edges, then for all $k < n-1$, $c_n(l_k) = 1$ if and only if $k \in F_{3i+j}$.

(iii) For $n = 4i + 3$, if u_i is in $\mathbb{S} \cap 2^{\leq n}$, then c_n is a coding node extending u_i. If u_i is not in $\mathbb{S}$, then $c_n = 0^n{}^\frown 1$.

As in Lemma 3.13, the first two coding nodes of $\mathbb{S}$ are completely determined by the definition of strong triangle-free tree. Thus, $c_0 = \langle 1 \rangle$, $c_1 = \langle 0, 1 \rangle$, and the tree $\mathbb{S}$ up to height 2 consists of the nodes $\{\emptyset, \langle 0 \rangle, \langle 1 \rangle, \langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle\}$. Denote this tree as $\mathbb{S}_2$. Since $F_0 = F_1 = \emptyset$, c_0 and c_1 trivially satisfy requirements F_0 and F_1, respectively. It is simple to check that $\mathbb{S}_2$ is a strong triangle-free tree, and that (i) - (iii) are satisfied.

For the general construction step, suppose $n \geq 2$, $\mathbb{S}_n \subseteq 2^{\leq n}$ has been constructed, and coding nodes $\langle c_i : i < n \rangle$ have been chosen so that $\mathbb{S}_n$ is a strong triangle-free tree satisfying (i) - (iii). Extend each maximal node in $\mathbb{S}_n$ to length $n+1$ according to the Splitting Criterion. Thus, for each $s \in \mathbb{S}_n \cap 2^n$, $s^\frown 0$ is in $\mathbb{S}_{n+1}$, and $s^\frown 1$ is in $\mathbb{S}_{n+1}$ if and only if s has no parallel 1's with c_{n-1}. Now we choose c_n so that (i) - (iii) hold. There are three cases.

Case 1. Either $n = 4i$ and $i \geq 1$, or $n = 4i + 2$ and $i < \omega$. Let n' denote $3i$ if $n = 4i$, and let n' denote $3i + 2$ if $n = 4i + 2$. Then $F_{n'} = \emptyset$, so let $c_n = 0^n{}^\frown 1$.

Case 2. $n = 4i+1$ and $1 \leq i < \omega$. If for all pairs of integers $k < m$ in F_{3i+1} it is the case that $c_m(l_k) = 0$, then take c_n to be a maximal node in $\mathbb{S}_{n+1}$ such that for all $k < n-1$, $c_n(l_k) = 1$ if and only if $k \in F_{3i+1}$, and $c_n(l_{n-1}) = 1$. Otherwise, let $c_n = 0^n{}^\frown 1$.

Case 3. $n = 4i + 3$ and $i < \omega$. Recall that $|u_i| \leq i$, so $|u_i| \leq n - 3$. If u_i is in $\mathbb{S}_i$, then take c_n to be the maximal node in $\mathbb{S}_{n+1}$ which is u_i extended by all 0's until its last entry, which is 1. Precisely, letting $q = n - |u_i|$, set $c_n = u_i{}^\frown 0^q{}^\frown 1$. If u_i is not in $\mathbb{S}_i$, let $c_n = 0^n{}^\frown 1$.

(i) - (iii) hold automatically by the choices of c_n in Cases 1 - 3. What is left is to check is that such nodes in Cases 1 - 3 actually exist in $\mathbb{S}_{n+1}$. The node $0^n{}^\frown 1$ is in $\mathbb{S}_{n+1}$, as it has no parallel 1's with c_{n-1}. Thus, in Case 1 and the second halves of Cases 2 and 3, the node we declared to be c_n is indeed in $\mathbb{S}_{n+1}$.

In Case 2 where $n = 4i + 1$ with $i \geq 1$, suppose that $F_{3i+1} \neq \emptyset$ and for all pairs $k < m$ of integers in F_{3i+1}, $c_m(l_k) = 0$. Since $\max(F_{3i+1}) \leq 3i - 1 \leq n - 3$ and since by the induction hypothesis, $\mathbb{S}_{n-1}$ is a strong triangle-free tree, Lemma 3.13 implies that there is a node $t \in \mathbb{S}_{n-1}$ such that for each $k < n-1$, $s(l_k) = 1$ if and only if $k \in F_{3i+1}$. Note that $t^\frown 0$ and c_{n-1} have no parallel 1's, since $c_{n-1} = 0^{n-1}{}^\frown 1$. Thus, by the Splitting Criterion, $t^\frown 0^\frown 1$ is in $\mathbb{S}_{n+1}$, and this node satisfies our choice of c_n.

In Case 3 when $n = 4i + 3$, if $u_i \in \mathbb{S}_i$, then by the Splitting Criterion, also $u_i{}^\frown 0^q$ is in $\mathbb{S}_n$, where $q = n - |u_i|$. Since $n - 1 = 4i + 2$, $c_{n-1} = 0^{n-1}{}^\frown 1$; so $u_i{}^\frown 0^q$ has no parallel 1's with c_{n-1}. Thus, by the Splitting Criterion, $u_i{}^\frown 0^q{}^\frown 1$ is in $\mathbb{S}_{n+1}$.

Let $\mathbb{S} = \bigcup_{n<\omega} \mathbb{S}_n$. By the construction, $\mathbb{S}$ is an infinite strong triangle-free tree with coding nodes $\langle c_n : n < \omega \rangle$. (ii) implies that $\mathbb{S}$ satisfies $(A_3)^{\mathtt{tree}}$ and hence codes $\mathcal{H}_3$. By (iii), the coding nodes are dense in $\mathbb{S}$. $\square$

Example 3.15 (A Strong Triangle-Free Tree)**.** Presented here is a concrete example of the first six steps of constructing a strong triangle-free tree densely coding $\mathcal{H}_3$. In the construction of Theorem 3.14, $F_0 = F_1 = F_2 = \emptyset$. The coding nodes $c_0 = \langle 1 \rangle$ and $c_1 = \langle 0, 1 \rangle$ are determined by the definition of strong triange-free tree. The coding node c_2 we choose to be $\langle 0, 0, 1 \rangle$. (It could also have been chosen to be $\langle 1, 0, 1 \rangle$.) Since u_0 is the empty sequence, c_3 can be any sequence which has last entry 1; in this example we let $c_3 = \langle 1, 0, 0, 1 \rangle$. $F_3 = \emptyset$, so $c_4 = \langle 0, 0, 0, 0, 1 \rangle$. Suppose $F_4 = \{0, 2\}$. Then we may take $c_5 = \langle 0, 1, 0, 1, 0, 1 \rangle$ to code edges between the vertex v_5 and the vertices v_0 and v_2; we also make v_5 have an edge with v_4. Notice that having chosen the coding node c_n, each maximal node $s \in \mathbb{S}_{n+1}$ splits in $\mathbb{S}_{n+2}$ if and only if $s(i) + c_n(i) \leq 1$ for all $i \leq n$. See Figure 3. The graph on the left with vertices $\{v_0, \ldots, v_5\}$ is being coded by the coding nodes $\{c_0, \ldots, c_5\}$. The tree and the graph are intended to continue growing upwards to the infinite tree $\mathbb{S}$ coding the graph $\mathcal{H}_3$.

Remark 3.16. We have set up the definition of strong triangle-free tree so that no coding node in a strong triangle-free tree splits. The purpose of not allowing coding nodes to split is to simplify later work by reducing the number of different isomorphism types of trees coding a given finite triangle-free graph. The purposes of the density of the coding nodes and the Splitting Criterion are to saturate the trees with as many

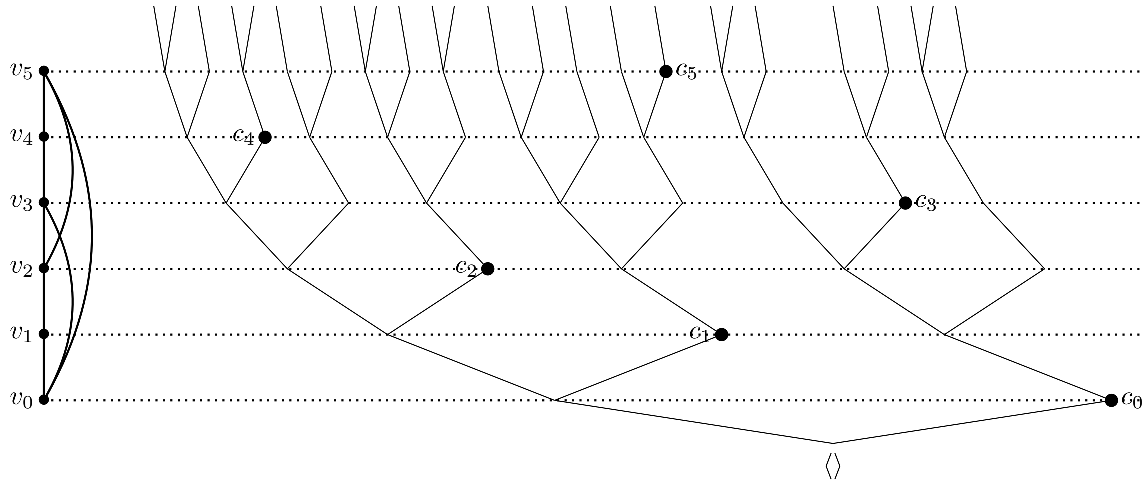

FIGURE 3. A strong triangle-free tree $\mathbb{S}$ densely coding $\mathcal{H}_3$

extensions as possible coding vertices without coding any triangles, so as to allow for thinning to subtrees which still can code $\mathcal{H}_3$, setting the stage for later Ramsey-theoretic results.

Remark 3.17. Given a strong triangle-free tree T densely coding $\mathcal{H}_3$, the collection of all strong triangle-free subtrees S of T densely coding $\mathcal{H}_3$ forms an interesting space of trees. The author has proved Halpern-Läuchli-style theorems for such trees, provided that the stem is fixed. This was the author's first approach toward the main theorem of this paper, and these proofs formed the strategy for the proofs in later sections. Oddly enough, though, having stems splitting at the same levels as coding nodes presents an obstacle to fully developing Ramsey theory on strong triangle-free trees. One may create a bad coloring using interference between stems and coding nodes as seen below. Such a bad coloring on coding nodes prevents the transition from cone-homogeneity to homogeneity on a strong triangle-free subtree with dense coding nodes.

Example 3.18 (A bad coloring)**.** Given a strong triangle-free tree $\mathbb{S}$ with coding nodes $\langle c_n : n < \omega\rangle$ dense in $\mathbb{S}$, let $s_i = 0^i$, for each $i < \omega$. Note that each s_i splits in $\mathbb{S}$ and that $|c_n| = |s_{n+1}|$, for each $n < \omega$. Color all coding nodes c_n extending $s_0{}^\frown 1$, which is exactly $\langle 1\rangle$, blue. Let k be given and suppose for each $i \le k$, we have colored all coding nodes extending $s_i{}^\frown 1$. The coding node c_k extends $s_i{}^\frown 1$ for some $i \le k$, so it has already been assigned a color. If c_k is blue, color every coding node in $\mathbb{S}$ extending $s_{k+1}{}^\frown 1$ red; if c_k is red, color every coding node in $\mathbb{S}$ extending $s_{k+1}{}^\frown 1$ blue. This produces a red-blue coloring of the coding nodes such that any subtree S of $\mathbb{S}$ with coding nodes dense in S and satisfying the Splitting Criterion (which would be the natural definition of infinite strong triangle-free subtree) has coding nodes of both colors: For given a coding node c of S, the node $0^{|c|}$ is a splitting node in S, and all coding nodes in S extending $0^{|c|}{}^\frown 1$ have color different from the color of c.

Since this example precludes a satisfactory Ramsey theory of strong triangle-free trees coding $\mathcal{H}_3$, instead of presenting those Ramsey-theoretic results on strong triangle-free trees which were obtained, we immediately move on to the skew version of strong triangle-free trees. Their full Ramsey theory will be developed in the rest of the article.

4. Strong coding trees

This section introduces the main tool for our investigation of the big Ramsey degrees for the universal triangle-free graph, namely *strong coding trees*. Essentially, strong coding trees are simply stretched versions of strong triangle-free trees, so that all the coding structure is preserved while removing any entanglements between coding nodes and splitting nodes, as seen in Example 3.18, which could prevent Ramsey theorems. The collection of all subtrees of a strong coding tree T which are isomorphic to T, partially ordered by a relation defined later in this section, will be seen, by the end of Section 6, to form a space of trees coding $\mathcal{H}_3$ with many similarities to the Milliken space of strong trees [21].

4.1. **Definitions and notation.** The following terminology and notation will be used throughout. Recall that by a *tree*, we mean exactly a subset $T \subseteq 2^{<\omega}$ which is closed under meets and is a union of level sets; that is, $s, t \in T$ and $|t| \geq |s|$ imply that $t \restriction |s|$ is also a member of T. Further, recall Definition 3.1 of a *tree with coding nodes*. Let $T \subseteq 2^{<\omega}$ be a tree with coding nodes $\langle c_n^T : n < N\rangle$, where $N \leq \omega$, and let l_n^T denote $|c_n^T|$. $\widehat{T}$ denotes the collection of all initial segments of nodes in T; thus, $\widehat{T} = \{t \restriction n : t \in T \text{ and } n \leq |t|\}$. A node $s \in T$ is called a *splitting node* if both $s^\frown 0$ and $s^\frown 1$ are in $\widehat{T}$; equivalently, s is a splitting node in T if there are nodes $s_0, s_1 \in T$ such that $s_0 \supseteq s^\frown 0$ and $s_1 \supseteq s^\frown 1$. Given t in a tree T, the *level* of T of length $|t|$ is the set of all $s \in T$ such that $|s| = |t|$. By our definition of tree, this is exactly the set of $s \restriction |t|$ such that $s \in T$ and $|s| \geq |t|$. T is *skew* if each level of T has exactly one of either a coding node or a splitting node. A skew tree T is *strongly skew* if additionally for each splitting node $s \in T$, every $t \in T$ such that $|t| > |s|$ and $t \not\supset s$ also satisfies $t(|s|) = 0$; that is, the passing number of any node passing by, but not extending, a splitting node is 0. The set of *levels* of a skew tree $T \subseteq 2^{<\omega}$, denoted L^T, is the set of those $l < \omega$ such that T has either a splitting or a coding node of length l. Let $\langle d_m^T : m < M\rangle$ enumerate the collection of all coding and splitting nodes of T in increasing order of length. The nodes d_m^T will be called the *critical nodes* of T. Note that $N \leq M$, and $M = \omega$ if and only if $N = \omega$. For each $m < M$, the *m-th level of* T is

$$\mathrm{Lev}_T(m) = \{s \in \widehat{T} : |s| = |d_m^T|\}. \tag{9}$$

Then for any strongly skew tree T,

$$T = \bigcup_{m<M} \mathrm{Lev}_T(m). \tag{10}$$

Let m_n denote the integer such that $c_n^T \in \mathrm{Lev}_T(m_n)$. Then $d_{m_n}^T = c_n^T$, and the critical node d_m^T is a splitting node if and only if $m \neq m_n$ for any n. For each $0 < n < N$, the *n-th interval* of T is $\bigcup\{\mathrm{Lev}_T(m) : m_{n-1} < m \leq m_n\}$. The 0-*th interval* of T is defined to be $\bigcup_{m \leq m_0} \mathrm{Lev}_T(m)$. Thus, the 0-th interval of T is the set of those nodes in T with lengths in $[0, l_0^T]$, and for $0 < n < N$, the n-th interval of T is the set of those nodes in T with lengths in $(l_{n-1}^T, l_n^T]$.

The next definition provides notation for the set of exactly those nodes just above the $(n-1)$-st coding node which will split in the n-th interval of T. Define

$$\mathrm{Spl}(T, 0) = \{t \in \widehat{T} : |t| = |\,\mathrm{stem}(T)| + 1 \text{ and } \exists m < m_0 \text{ such that } d_m^T \supseteq t\}. \tag{11}$$

For $n \geq 1$, define

$$\mathrm{Spl}(T, n) = \{t \in \widehat{T} : |t| = l_{n-1} + 1 \text{ and } \exists m \in (m_{n-1}, m_n) \text{ such that } d_m^T \supseteq t\}. \tag{12}$$

Thus, $\mathrm{Spl}(T, n)$ is the set of nodes in $\widehat{T}$ of length just one above the length of c_{n-1} (or the stem of T if $n = 0$) which extend to a splitting node in the n-th interval of T. The lengths of the nodes in $\mathrm{Spl}(T, n)$ were chosen to so that they provide information about passing numbers at c_{n-1}^T. For $t \in \mathrm{Spl}(T, n)$, let $\mathrm{spl}_T(t)$ denote the minimal extension of t which splits in T.

Given a node s in T for which there is an $i < |s|$ such that $s \restriction i$ is a splitting node in T, the *splitting predecessor of* t *in* T, denoted $\mathrm{splitpred}_T(s)$, is the proper initial segment $u \subset s$ of maximum length such that both $u^\frown 0$ and $u^\frown 1$ are in $\widehat{T}$. Thus, $\mathrm{splitpred}_T(s)$ is the longest splitting node in T which is a proper initial segment of s. When the tree T is clear from the context, the subscripts and superscripts of T will be dropped.

4.2. **Definition and construction of strong coding trees.** Now we present a new tool for representing the universal triangle-free graph, namely strong coding trees. The following Parallel 1's Criterion is a central concept, ensuring that a finite subtree of a strong coding tree T can be extended inside T so that the criterion $(A_3)^{\texttt{tree}}$ can be met.

Definition 4.1 (Parallel 1's Criterion)**.** Let $T \subseteq 2^{<\omega}$ be a strongly skew tree with coding nodes $\langle c_n : n < N\rangle$. We say that T satisfies the *Parallel 1's Criterion* if the following hold: Given any set of two or more nodes $\{t_i : i < \tilde{i}\}$ in T and some l such that $t_i(l) = 1$ for all $i < \tilde{i}$,

(1) There is a coding node c_n in T such that for all $i < \tilde{i}$, $l_n < |t_i|$ and $t_i(l_n) = 1$; we say that c_n *witnesses* the parallel 1's of $\{t_i : i < \tilde{i}\}$.

(2) Letting l' be least such that $t_i(l')=1$ for all $i<\tilde{i}$, and letting n be least such that c_n witnesses the parallel 1's of the set of nodes $\{t_i : i<\tilde{i}\}$, then T has no splitting nodes and no coding nodes of lengths strictly between l' and l_n.

We say that a set of nodes $\{t_i : i<\tilde{i}\}$ has a new set of parallel 1's at l if $t_i(l)=1$ for all $i<\tilde{i}$, and l is least such that this occurs. Thus, the Parallel 1's Criterion says that any new set of parallel 1's must occur at a level l which is above the last splitting node in T in the interval $(l_{n-1}, l_n]$ containing l, and that c_n must witness this set of parallel 1's.

Definition 4.2 (Splitting Criterion for Skew Trees)**.** A strongly skew tree T with coding nodes $\langle c_n : n<N\rangle$ satisfies the *Splitting Criterion for Skew Trees* if the following hold: For each $1\le n<N$ and each $s\in\widehat{T}$ of length $l_{n-1}+1$, s is in $\mathrm{Spl}(T,n)$ if and only if s and $c_n \restriction (l_{n-1}+1)$ have no parallel 1's. For each $s\in\widehat{T}$ of length $|\mathrm{stem}(T)|+1$, s is in $\mathrm{Spl}(T,0)$ if and only if $s=\mathrm{stem}(T)^\frown 0$.

Notice that any tree with coding nodes satisfying the Splitting Criterion for Skew Trees also satisfies the Triangle-Free Criterion (Definition 3.4), and hence will not code any triangles.

Now we arrive at the main structural concept for coding copies of $\mathcal{H}_3$. This extends the idea of Milliken's strong trees - branching as much as possible whenever one split occurs - to skew trees with the additional property that they can code omissions of triangles.

Definition 4.3 (Strong coding tree)**.** A tree $T\subseteq 2^{<\omega}$ with coding nodes $\langle c_n : n<\omega\rangle$ is a *strong coding tree* if T is strongly skew, for each node $t\in T$, the node $0^{|t|}$ is also in T, and the following hold:

(1) The coding nodes of T are dense in T.
(2) For each $n\ge 1$, $c_n(l_{n-1})=1$.
(3) T satisfies the Parallel 1's Criterion.
(4) T satisfies the Splitting Criterion for Skew Trees.
(5) c_0 extends $\mathrm{stem}(T)^\frown 1$ and does not split.
(6) Given $n<\omega$, $s\in\mathrm{Spl}(T,n)$, and $i<2$, there is exactly one extension $s_i\supseteq \mathrm{spl}(s)^\frown i$ of length l_n in T, and its unique immediate extension in $\widehat{T}$ is $s_i{}^\frown i$.
(7) For each $n<\omega$, each node t in $\widehat{T}$ of length $l_{n-1}+1$ which is not in $\mathrm{Spl}(T,n)$ has exactly one extension of length l_n in T, say t_*, and its unique immediate extension in $\widehat{T}$ is $t_*{}^\frown 0$. Here, l_{-1} denotes the length of $\mathrm{stem}(T)$.

An example of a strong coding tree is presented in Figure 4. One should notice that upon "zipping up" the splits occurring in the intervals between coding nodes in $\mathbb{T}$ to the next coding node level, one recovers $\mathbb{S}$. The existence of strong coding trees will be proved in Theorem 4.6.

Recall that $\langle d_m : m<\omega\rangle$ enumerates the set of all critical nodes (coding nodes and splitting nodes) in T in order of strictly increasing length.

Definition 4.4 (Finite strong coding tree)**.** Given a strong coding tree T, by an *initial segment* or *initial subtree of* T we mean the first m levels of T, for some $m<\omega$. We shall use the notation

$$r_m(T)=\bigcup_{k<m}\mathrm{Lev}_T(k). \tag{13}$$

A tree with coding nodes is a *finite strong coding tree* if and only if it is equal to some $r_{m+1}(T)$ where d_m is a coding node or else $m=0$.

Thus, finite strong coding trees are exactly the finite trees with coding nodes $\langle c_n : n<N\rangle$, where $N<\omega$, which have all maximal nodes of the length of its longest coding node and satisfy (2) - (7) of Definition 4.3 for all $n<N$.

The next lemma extends the ideas of Lemma 3.13 to the setting of finite strong coding trees.

Lemma 4.5. *Let A be any finite strong coding tree with coding nodes $\langle c_n : n<N\rangle$, where $N<\omega$. Let A^+ denote the nodes of length $l_{N-1}+1$ extending the maximal nodes in A as determined by (6) and (7) in Definition 4.3. Then given any $F\subseteq N$ such that $\{c_n : n\in F\}$ codes no edges, there is a $t\in A^+$ such that for all $n<N$,*

$$t(l_n)=1 \quad\longleftrightarrow\quad n\in F. \tag{14}$$

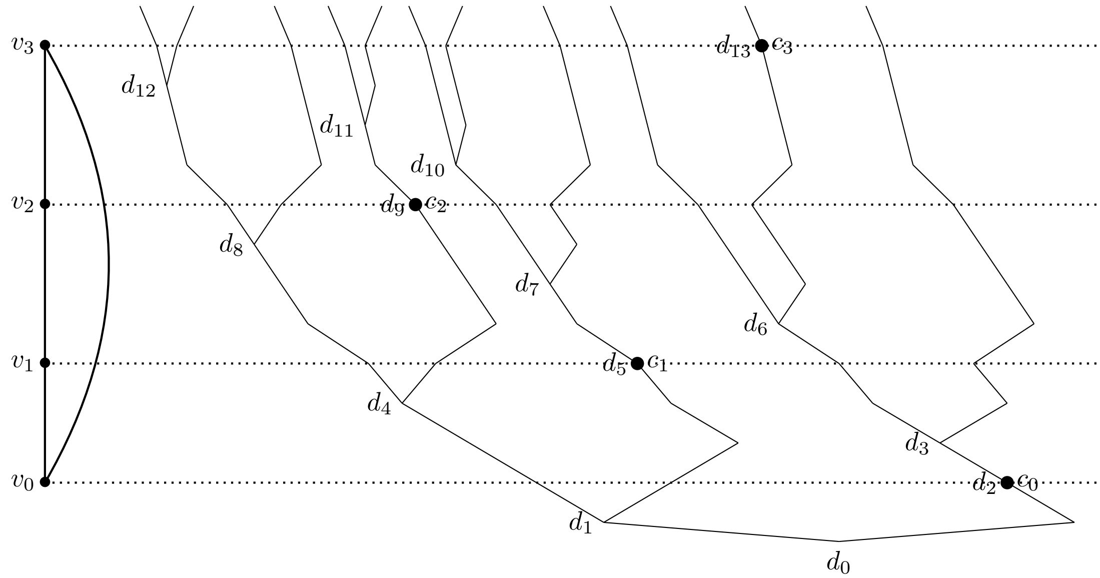

FIGURE 4. A strong coding tree $\mathbb{T}$

Proof. The proof is by induction on N over all finite strong strong coding trees with N coding nodes. For $N = 0$, $A = \emptyset$, the lemma vacuously holds. For $N = 1$, it follows from the definition of finite strong coding tree that A has critical nodes $d_0 = \text{stem}(A)$, d_1 which is a splitting node extending $d_0{}^\frown 0$, and $d_2 = c_0$ which extends $d_0{}^\frown 1$. Thus, A^+ has three nodes, $t_0 \supset d_0{}^\frown 0$ with passing number 0 at c_0; $t_1 \supset d_0{}^\frown 1$ with passing number 1 at c_0; and $t_2 = c_0{}^\frown 0$ which of course has passing number 0 at c_0. Both of the nodes t_0 and t_2 satisfy equation (14) if $F = \emptyset$, and t_1 satisfies (14) if $F = \{0\}$.

Now suppose that $N \geq 2$ and the lemma holds for $N-1$. Let A be a finite strong coding tree with coding nodes $\langle c_n : n < N\rangle$. Let F be a subset of N such that $\{c_n : n \in F\}$ codes no edges, and let m be the index such that $d_{m-1} = c_{N-2}$. By the induction hypothesis, there is a node u in $(r_m(A))^+$ such that for all $n < N-1$, $u(l_n) = 1$ if and only if $n \in F$. If $N-1 \notin F$, by (6) and (7) of the definition of strong coding tree there is an extension $t \supset u$ in A^+ with passing number 0 at c_{N-1}, and this t satisfies (14) for F.

If $N-1 \in F$, it suffices to show that $u \in \text{Spl}(A, N-1)$, for then there will be a $t \supset u$ in A^+ with passing number 1 at c_{N-1}, and this t will satisfy (14). By the Splitting Criterion for Skew Trees, if $u \notin \text{Spl}(A, N-1)$, then u and $c_{N-1} \restriction (l_{N-2}+1)$ must have a parallel 1. Then by the Parallel 1's Criterion, there is some $i \leq N-2$ such that $u(l_i) = c_{N-1}(l_i) = 1$. Since u codes edges only with those vertices with indexes less than $N-1$ in F, it follows that i must be in F. But then $\{c_i, c_{N-1}\}$ is a subset of F coding an edge, contradicting the assumption on F. Therefore, u is in $\text{Spl}(A, N-1)$. $\square$

We now present a flexible method for constructing a strong coding tree $\mathbb{T}$. This should be thought of as a stretched and skewed version of the strong triangle-free tree $\mathbb{S}$ which was constructed in Theorem 3.14. The passing numbers at the coding nodes in $\mathbb{T}$ code edges and non-edges exactly as the passing numbers of the coding nodes in $\mathbb{S}$. In particular, given the same enumeration $\langle F_i : i < \omega\rangle$ of the finite subsets of ω, $\mathcal{H}_3$ is coded in the same order by both $\mathbb{S}$ and $\mathbb{T}$.

The strong coding tree $\mathbb{T}$ which we construct will be *regular*: For each n, nodes in $\text{Spl}(\mathbb{T}, n)$ extend to splitting nodes in the n-th interval of $\mathbb{T}$ from lexicographically least to largest. Regularity is not necessary for achieving the main theorems of this article. However, as any strong coding tree contains a subtree which is a regular strong coding tree, it does no harm to only work with regular trees.

Theorem 4.6. *Let $\langle F_i : i < \omega\rangle$ be any sequence enumerating the finite subsets of ω so that each finite set appears cofinally often. Assume further that for each $i < \omega$, $F_i \subseteq i-1$ and $F_{3i} = F_{3i+2} = \emptyset$. Then there is a strong coding tree $\mathbb{T}$ which densely codes $\mathcal{H}_3$, where for each $i < \omega$ and $j \leq 2$, the coding node c_{4i+j} meets requirement F_{3i+j}.*

Proof. Let $\langle F_i : i < \omega\rangle$ satisfy the hypotheses, and let $\langle u_i : i < \omega\rangle$ be an enumeration of all the nodes in $2^{<\omega}$ in such a way that each $|u_i| \le i$. We construct a strong coding tree $\mathbb{T} \subseteq 2^{<\omega}$ with coding nodes $\langle c_n : n < \omega\rangle$ and lengths $l_n = |c_n|$ so that for each $n < \omega$, $r_{m_n+1}(\mathbb{T}) := \bigcup\{\mathrm{Lev}_{\mathbb{T}}(i) : i \le m_n\}$ is a finite strong coding tree and $\mathrm{Lev}_{\mathbb{T}}(m_n + 1)$ satisfies (6) and (7) of the definition of strong coding tree, where m_n is the index such that the m_n-th critical node d_{m_n} is equal to the n-th coding node c_n, and the following properties are satisfied:

(i) For $n = 4i + j$, $j \le 2$, c_n meets requirement F_{3i+j}.
(ii) For $n = 4i + 3$, if u_i is in $r_{m_{n-3}+2}(\mathbb{T})$, then c_n is a coding node extending u_i. Otherwise, $c_n = 0^{l_{n-1}-1}{}^\frown\langle 1, 1\rangle^\frown 0^{q_n}$ where $q_n = l_n - (l_{n-1} + 1)$.

To begin, define $\mathrm{Lev}_{\mathbb{T}}(0) = \{\langle\rangle\}$. Then the minimum length splitting node in $\mathbb{T}$ is $\langle\rangle$, and we label it d_0. Let $\mathrm{Lev}_{\mathbb{T}}(1) = \{\langle 0\rangle, \langle 1\rangle\}$. To satisfy (5) of Definition 4.3, c_0 is going to extend $\langle 1\rangle$, so in order to satisfy (4), it must be the case that $\mathrm{Spl}(\mathbb{T}, 0) = \{\langle 0\rangle\}$. Take the splitting node d_1 to be $\langle 0\rangle$. Let $\mathrm{Lev}_{\mathbb{T}}(2) = \{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 0\rangle\}$, and define $c_0 = \langle 1, 0\rangle$. Then $l_0 = 2$, $d_2 = c_0$, and

$$r_{m_0+1}(\mathbb{T}) = \bigcup\{\mathrm{Lev}_{\mathbb{T}}(i) : i \le 2\} \tag{15}$$

is a finite strong coding tree satisfying (i) and (ii). The next level of $\mathbb{T}$ must satisfy (6) and (7). Extend $\langle 0, 0\rangle$ to $\langle 0, 0, 0\rangle$, extend $\langle 0, 1\rangle$ to $\langle 0, 1, 1\rangle$, and extend $\langle 1, 0\rangle$ to $\langle 1, 0, 0\rangle$, and let these compose $\mathrm{Lev}_{\mathbb{T}}(3)$.

For the sake of clarity, the next few levels of $\mathbb{T}$ up to the level of c_1 will be constructed concretely. To satisfy (2), the next coding node c_1 must extend $\langle 0, 1, 1\rangle$, since this is the only node in $\mathrm{Lev}_{\mathbb{T}}(3)$ which has passing number 1 at c_0. The knowledge that c_1 will extend $\langle 0, 1, 1\rangle$ along with the Splitting Criterion for Skew Trees determine that $\mathrm{Spl}(\mathbb{T}, 1) = \{\langle 0, 0, 0\rangle, \langle 1, 0, 0\rangle\}$, since these are the nodes in $\mathrm{Lev}_{\mathbb{T}}(3)$ which have no parallel 1's with $\langle 0, 1, 1\rangle$. As we are building $\mathbb{T}$ to be regular, $\langle 1, 0, 0\rangle$ is first in $\mathrm{Spl}(\mathbb{T}, 1)$ to be extended to a splitting node. Let $d_3 = \langle 1, 0, 0\rangle$, and let $\mathrm{Lev}_{\mathbb{T}}(4) = \{\langle 0, 0, 0, 0\rangle, \langle 0, 1, 1, 0\rangle, \langle 1, 0, 0, 0\rangle, \langle 1, 0, 0, 1\rangle\}$, so that $\mathbb{T}_4$ is strongly skew. Next, let $d_4 = \langle 0, 0, 0, 0\rangle$ as this node should split since it is the only extension of $\langle 0, 0, 0\rangle$ in $\mathrm{Lev}_{\mathbb{T}}(4)$. Let

$$\mathrm{Lev}_{\mathbb{T}}(5) = \{\langle 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 1\rangle, \langle 0, 1, 1, 0, 0\rangle, \langle 1, 0, 0, 0, 0\rangle, \langle 1, 0, 0, 1, 0\rangle\}. \tag{16}$$

Let $c_1 = \langle 0, 1, 1, 0, 0\rangle$, as this is the only extension of $\langle 0, 1, 1\rangle$ in $\mathrm{Lev}_{\mathbb{T}}(5)$. Thus, $d_5 = c_1$, $l_1 = 5$, $\mathrm{spl}_{\mathbb{T}}(\langle 1, 0, 0\rangle) = \langle 1, 0, 0\rangle$ and $\mathrm{spl}_{\mathbb{T}}(\langle 0, 0, 0\rangle) = \langle 0, 0, 0, 0\rangle$. Moreover, $r_6(\mathbb{T})$ is a regular, finite strong coding tree satisfying requirements (i) - (ii). The next level of $\mathbb{T}$ is determined by (6) and (7), so let

$$\mathrm{Lev}_{\mathbb{T}}(6) = \{\langle 0, 0, 0, 0, 0, 0\rangle, \langle 0, 0, 0, 0, 1, 1\rangle, \langle 0, 1, 1, 0, 0, 0\rangle, \langle 1, 0, 0, 0, 0, 0\rangle, \langle 1, 0, 0, 1, 0, 1\rangle\}. \tag{17}$$

This constructs the tree $r_7(\mathbb{T})$, which is $\mathbb{T}$ up to the level of $l_1+1 = 6$. Notice that the second lexicographically least node in $\mathrm{Lev}_{\mathbb{T}}(l_1 + 1)$ is $\langle 0, 0, 0, 0, 1, 1\rangle = 0^{(l_1-1)}{}^\frown\langle 1, 1\rangle$.

Suppose $r_{m_n-1} + 2(\mathbb{T})$ has been constructed so that $r_{m_{n-1}+1}(\mathbb{T})$ is a finite strong coding tree satisfying (i) and (ii) and such that $\mathrm{Lev}_{\mathbb{T}}(m_{n-1} + 1)$ satisfies (6) and (7) of Definition 4.3, where m_{n-1} is the index such that $d_{m_{n-1}} = c_{n-1}$. As part of the induction hypothesis, suppose also that the second lexicographically least node in $\mathrm{Lev}_{\mathbb{T}}(m_{n-1} + 1)$ is $0^{(m_{n-1}-1)}{}^\frown\langle 1, 1\rangle$, this being true in the base case of $r_{m_1+2}(\mathbb{T})$. Enumerate the members of $\mathrm{Lev}_{\mathbb{T}}(m_{n-1} + 1)$ in decreasing lexicographical order as $\langle s_k : k < K\rangle$. At this stage, we need to know which node s_k will be extended to the next coding node c_n as this determines the set $\mathrm{Spl}(\mathbb{T}, n)$. We will show how to choose k_* in the three cases below, so that extending s_{k_*} to c_n will meet requirements (i) and (ii). Once k_* is chosen, $\mathrm{Spl}(\mathbb{T}, n)$ is the set $\{s_k : k \in K_{sp}\}$, where K_{sp} is the set of those $k < K$ such that for all $i < n$, $s_k(l_i) + s_{k_*}(l_i) \le 1$, that is, s_k and s_{k_*} have no parallel 1's at or below l_{n-1}. Then let $c_n = s_{k^*_n}{}^\frown 0^{|K_{sp}|}$, and extend all nodes in $\{s_k : k < K\}$ according to (6) and (7) in the definition of strong coding tree. We point out that l_n will equal $l_{n-1} + |K_{sp}| + 1$.

There are three cases to consider regarding which $k < K$ should be k_*.

Case 1. $n = 4i$ or $n = 4i + 2$ for some $i < \omega$. Let n' denote $3i$ if $n = 4i$ and $3i + 2$ if $n = 4i + 2$. In this case, $F_{n'} = \emptyset$. Let $k_* = K - 2$. Since s_{K-1} is the lexicographic least member of $\mathrm{Lev}_{\mathbb{T}}(m_{n-1} + 1)$, s_{K-1} must be $0^{l_{n-1}+1}$. Hence, s_{K-2} being next lexicographic largest implies that $s_{K-2} = 0^{(l_{n-1}-1)}{}^\frown\langle 1, 1\rangle$. Let $k_* = K - 2$. Then any extension of s_{k_*} to a coding node will have passing number 1 at c_{n-1} and passing number 0 at c_i for all $i < n - 1$.

Case 2. $n = 4i + 1$ for some $1 \le i < \omega$. If there is a pair $k < m$ of integers in F_{3i+1} such that $c_m(l_k) = 1$, then again take k_* to be $K - 2$. Otherwise, $c_m(l_k) = 0$ for all pairs $k < m$ in F_{3i+1}. Note that $i \ge 1$ implies that $\max(F_{3i+1}) \leq 3i-1 \leq n-3$. Since by the induction hypothesis $r_{m_{n-2}+1}(\mathbb{T})$ is a finite strong coding tree, Lemma 4.5 implies there is some $t \in \mathrm{Lev}_{\mathbb{T}}(m_{n-3}+2)$ such that $t(l_j) = 1$ if and only if $j \in F_{3i+1}$. Let t' be the node in $2^{<\omega}$ of length $l_{n-2}+1$ which extends t by all 0's. By our construction, this node is in $r_{m_{n-2}+2}(\mathbb{T})$. Since, by Case 1, c_{n-1} is the node of length l_{n-1} extending $0^{l_{n-2}-1}{}^\frown\langle 1,1\rangle$ by all 0's, one sees that $t' \restriction (l_{n-2}+1)$ and $c_{n-1} \restriction (l_{n-2}+1)$ have no parallel 1's. Thus, $t' \restriction (l_{n-2}+1)$ is in $\mathrm{Spl}(\mathbb{T}, n-1)$. Let k_* be the index in K such that s_{k_*} is the rightmost extension of t' in $\mathrm{Lev}_{\mathbb{T}}(m_{n-1}+1)$.

Case 3. $n = 4i+3$ for some $i < \omega$. If $u_i \notin r_{m_{n-3}+2}(\mathbb{T})$, then let $k_* = K-2$. Otherwise, $u_i \in r_{m_{n-3}+2}(\mathbb{T})$. Let u' be the leftmost extension of u_i in $r_{m_{n-2}+2}(\mathbb{T})$ of length $l_{n-2}+1$. In particular, $u'(l_{n-2}-1) = u'(l_{n-2}) = 0$. As in Case 2, c_{n-1} is the node of length l_{n-1} such that for all $l < l_{n-1}$, $c_{n-1}(l) = 1$ if and only if $l \in \{l_{n-2}-1, l_{n-2}\}$. Thus, u' and $c_{n-1} \restriction (l_{n-2}+1)$ have no parallel 1's, so by the induction hypothesis, $u' \in \mathrm{Spl}(\mathbb{T}, n-1)$. Hence, there is an extension $u'' \supseteq u'$ in $r_{m_{n-1}+2}(\mathbb{T})$ such that $u''(l_{n-1}) = 1$. Let k_* be the index of the node u''.

To finish the construction of $\mathbb{T}$ up to level l_n+1, let $l_n = l_{n-1} + |K_{sp}| + 1$. For each $k \notin K_{sp}$, extend s_k via all 0's to length l_n+1. Note in each of the three cases, k_* is not in K_{sp}, since s_{k_*} has passing number 1 at c_{n-1}. Thus, c_n is the extension of s_{k_*} by all 0's to length l_n, and its immediate extension, or passing number by itself, is 0. Enumerate K_{sp} as $\langle k_i : i < |K_{sp}|\rangle$ so that $s_{k_i} >_{\mathrm{lex}} s_{k_{i+1}}$ for each i. Let $\mathrm{spl}(s_{k_i}) = s_{k_i}{}^\frown 0^i$; in particular, $\mathrm{spl}(s_{k_0}) = s_{k_0}$. For each $i < |K_{sp}|$, letting $p_i = |K_{sp}| - i$, $s_{k_i}{}^\frown 0^{|K_{sp}|}$ and $\mathrm{spl}(s_{k_i})^\frown 1^\frown 0^{p_i-2}{}^\frown 1$ are the two extensions of s_{k_i} in $\mathrm{Lev}_{\mathbb{T}}(l_n+1)$. This constructs $\mathrm{Lev}_{\mathbb{T}}(l_n+1)$. Notice that for each $j < 2$, the $t \in \mathrm{Lev}_{\mathbb{T}}(l_n+1)$ extending $\mathrm{spl}(s_{k_i})^\frown j$ has passing number $t(l_n) = j$.

Let $\mathbb{T} = \bigcup_{i<\omega} \mathrm{Lev}_{\mathbb{T}}(i)$. Then $\mathbb{T}$ is a strong coding tree because each initial segment $r_{m_n+1}(\mathbb{T})$, $n < \omega$, is a finite strong coding tree, and the coding nodes are dense in $\mathbb{T}$. □

Fact 4.7. *Any strong coding tree is a perfect tree.*

Proof. Let t be any node in T, and and let j be minimal such that $l_j \geq |t|$. Extend t leftmost in T to the node of length l_j, and label this t'. Let $s = 0^{l_j}$. By density of coding nodes in T, there is a coding node c_k in T extending s, with $k \geq j+2$. Extending t' leftmost in T to length $l_{k-1}+1$ produces a node t'' in $\widehat{T}$ which has no parallel 1's with $c_k \restriction (l_{k-1}+1)$. Thus, $t'' \in \mathrm{Spl}(T, k)$, so t'' extends to a splitting node in T before reaching the level of c_k. □

In particular, it follows from the definition of strong coding tree that in any strong coding tree T, for any $n < \omega$, the node $0^{l_{n-1}}$ will split in T before the level l_n.

4.3. **The space $(\mathcal{T}(T), \leq, r)$ of strong coding trees.** The space of subtrees of a given strong coding tree, equipped with a strong partial ordering, will form the fundamental structure allowing for the Ramsey theorems in latter sections. It turns out that not every subtree of a given strong coding tree T can be extended within T to form another strong coding tree. The notion of valid subtree provides conditions when a finite subtree can be extended in any desired manner within T. Some lemmas guaranteeing that finite valid subtrees of a given strong coding tree T can be extended to any desired configuration within T are presented at the end of this subsection. These lemmas will be very useful in subsequent sections. Those familiar with topological Ramsey spaces will notice the influence of [33] in our chosen style of presentation, the idea being that the space of strong coding trees has a similar character to the topological Ramsey space of Milliken's infinite strong trees, though background in [33] is not necessary for understanding this article.

To begin, we define a strong notion of isomorphism between meet-closed sets by augmenting Sauer's notion of strong similarity type from [31] to fit the present setting. Given a subset $S \subseteq 2^{<\omega}$, recall that the *meet closure of* S, denoted $S^\wedge$, is the set of all meets of pairs of nodes in S. In this definition s and t may be equal, so $S^\wedge$ contains S. We say that S is *meet-closed* if $S = S^\wedge$. Note that each tree is meet-closed, but there are meet-closed sets which are not trees, as Definition 2.1 of tree applies throughout this paper.

Definition 4.8 ([31])**.** $S \subseteq 2^{<\omega}$ is an *antichain* if $s \subseteq t$ implies $s = t$, for all $s, t \in S$. A set $S \subseteq 2^{<\omega}$ is *transversal* if $|s| = |t|$ implies $s = t$ for all $s, t \in S$. A set $D \subseteq 2^{<\omega}$ is *diagonal* if D is an antichain with $D^\wedge$ being transversal. A diagonal set D is *strongly diagonal* if additionally for any $s, t, u \in D$ with $s \neq t$, if $|s \wedge t| < |u|$ and $s \wedge t \not\subseteq u$, then $u(|s \wedge t|) = 0$.

It follows that the meet closure of any antichain of coding nodes in a strong coding tree is strongly diagonal. In fact, strong coding trees were designed with this property in mind.

The following augments Sauer's Definition 3.1 in [31] to the setting of trees with coding nodes. The lexicographic order on $2^{<\omega}$ between two nodes $s,t\in 2^{<\omega}$, with neither extending the other, is defined by $s<_{\mathrm{lex}} t$ if and only if $s\supseteq (s\wedge t)^\frown 0$ and $t\supseteq (s\wedge t)^\frown 1$. It is important to note that in a given strong coding tree T, each node s at the level of a coding node c_n in T has exactly one immediate extension in $\widehat{T}$. This is the unique node s^+ of length l_n+1 in $\widehat{T}$ such that $s^+\supset s$. This fact is used in (7) of the following definition.

Definition 4.9. Let $S,T\subseteq 2^{<\omega}$ be meet-closed subsets of a fixed strong coding tree $\mathbb{T}$. The function $f:S\to T$ is a *strong similarity* of S to T if for all nodes $s,t,u,v\in S$, the following hold:

(1) f is a bijection.
(2) f preserves lexicographic order: $s<_{\mathrm{lex}} t$ if and only if $f(s)<_{\mathrm{lex}} f(t)$.
(3) f preserves initial segments: $s\wedge t\subseteq u\wedge v$ if and only if $f(s)\wedge f(t)\subseteq f(u)\wedge f(v)$.
(4) f preserves meets: $f(s\wedge t)=f(s)\wedge f(t)$.
(5) f preserves relative lengths: $|s\wedge t|<|u\wedge v|$ if and only if $|f(s)\wedge f(t)|<|f(u)\wedge f(v)|$.
(6) f preserves coding nodes: f maps the set of coding nodes in S onto the set of coding nodes in T.
(7) f preserves passing numbers at coding nodes: If c is a coding node in S and u is a node in S with $|u|\ge|c|$, then $(f(u))^+(|f(c)|)=u^+(|c|)$; in words, the passing number of the immediate successor of $f(u)$ at $f(c)$ equals the passing number of the immediate successor of u at c.

In all cases above, it may be that $s=t$ and $u=v$ so that (3) implies $s\subseteq u$ if and only if $f(s)\subseteq f(u)$, etc. It follows from (4) that $s\in S$ is a splitting node in S if and only if $f(s)$ is a splitting node in T. We say that S and T are *strongly similar* if there is a strong similarity of S to T, and in this case write $S\overset{s}{\sim}T$. If $T'\subseteq T$ and f is a strong similarity of S to T', then f is a *strong similarity embedding* of S into T, and T' is a *strong similarity copy* of S in T. For $A\subseteq T$, let $\mathrm{Sim}^s_T(A)$ denote the set of all subsets of T which are strongly similar to A. The notion of strong similarity is relevant for all meet-closed subsets of a strong coding tree, including subsets which form trees. Note that if A is a meet-closed set which is not a tree and $S=\{u\restriction|v|:u,v\in A \text{ and } |u|\ge|v|\}$ is its induced tree, technically A and S are not strongly similar. This distinction will present no difficulties.

Not only are strong coding trees perfect, but the ones constructed in the manner of Theorem 4.6, and hence any tree with the same strong similarity type, also have the following useful property.

Fact 4.10. *Let $\mathbb{T}$ be constructed in the manner of Theorem 4.6, and let T be a strong coding tree which is strongly similar to T. Then for each even integer $n<\omega$, each node in T of length l_n splits in T before the level of c_{n+2}.*

Proof. Given a node t in T at the level of c_n, if t does not already split before the level of c_{n+1}, then its only extension to length $l_{n+1}+1$ has passing number 0 at c_{n+1}; call this extension t'. Now since $n+2$ is even, the coding node c_{n+2} has passing number 0 at all c_i, $i<n+1$, and passing number 1 at c_{n+1}. Thus, t' and $c_{n+2}\restriction(l_{n+1}+1)$ have no parallel 1's, so t' splits before reaching the level of c_{n+2}. $\square$

Depending on how a finite subtree A of a strong coding tree T sits inside T, it may be impossible to extend A inside of T to another strong coding tree. As a simple example, the set of nodes $A=\{\langle\rangle,\langle 0,0,0,0\rangle,\langle 1,0,0,1\rangle\}$ in $\mathbb{T}$ is strongly similar to $r_2(\mathbb{T})$. However A cannot be extended in $\mathbb{T}$ to a strong coding tree strongly similar to $\mathbb{T}$ with $\langle 0,0,0,0\rangle$ being a splitting node. The reasons are as follows. Any such extension A' in $\mathbb{T}$ must have nodes extending $\langle 0,0,0,0,0,0\rangle$, $\langle 0,0,0,0,1,1\rangle$, and $\langle 1,0,0,1,0,1\rangle\}$. The nodes $\langle 0,0,0,0,1,1\rangle$ and $\langle 1,0,0,1,0,1\rangle$ have parallel 1's, so the next coding node must witness them. In order to be strongly similar to $r_3(\mathbb{T})$, $\langle 0,0,0,0,1,1\rangle$ must be extended to the next coding node in A', and by the Triangle-Free Criterion, any such node is immediately succeeded by a 0, so it cannot witness the new parallel 1's, thus failing to satisfy the Parallel 1's Criterion.

Another potential problem is the following. Let T be a strong coding tree and take m such that d^T_m is a splitting node, $d^T_{m+2}=c^T_n$ is a coding node, and $|d^T_{m-2}|>l_{n-1}$, where $n\ge 3$. So, d^T_m is a splitting node with at least two splitting nodes preceding it in T and at least one splitting node proceeding it before the next coding node in T. It follows by the structure of strong coding trees that there are at least two maximal nodes in $r_{m+1}(T)$ which have no parallel 1's but which are pre-determined to passing c^T_n with passing number 1, as their only extensions of length l_n+1 in $\widehat{T}$ both have passing number 1 at c^T_n. It follows that any strong coding subtree S of T with the same initial segment as T up to level m, i.e. $r_{m+1}(S)=r_{m+1}(T)$, is necessarily going to have $r_{m+2}(S) = r_{m+2}(T)$; for if the splitting node d^S_{m+1} is not equal to d^T_{m+1}, then the pre-determined new parallel 1's appear in $r_{m+2}(S)$ before the splitting node d^S_{m+1}, implying S violates the Parallel 1's Criterion. Thus, if $r_{m+2}(S)$ is a finite strong coding tree end-extending $r_{m+1}(T)$ into T and strongly similar to $r_{m+2}(T)$, then $r_{m+2}(S)$ must actually equal $r_{m+2}(T)$. Clearly this is not what we want.

Definition 4.11. Let $X = \{x_i : i < \tilde{i}\}$ be a level set of two or more nodes in $\widehat{T}$, and let l be their length. We say that X has *no pre-determined new sets of parallel* 1*'s in* T if either X contains a coding node, or else for any $l_n > l$, there are extensions $y_i \supseteq x_i$ of length l_n such that the following holds: For each $I \subseteq \tilde{i}$ of size at least two, if there is an $l' < l_n$ such that $y_i(l') = 1$ for all $i \in I$, then there is an $l'' < l$ such that $y_i(l'') = 1$ for all $i \in I$.

It in order to determine whether a level set of nodes $X = \{x_i : i < \tilde{i}\}$ of length l, not containing a coding node, has pre-determined new sets of parallel 1's in T, it suffices to extend the nodes in X leftmost in $\widehat{T}$ to nodes $y_i \supseteq x_i$ of length $l_n + 1$, where c_n is the minimal coding node in T of length greater than l. If there is an $l' < l$ such that $\{i < \tilde{i} : x_i(l') = 1\}$ contains the set $\{i < \tilde{i} : y_i(l_n) = 1\}$, then X has no pre-determined new sets of parallel 1's. Note that any level set in $\widehat{T}$ of length $l_n + 1$ for some $n < \omega$ is valid in T.

Definition 4.12. A subtree A, finite or infinite, of a strong coding tree T is *valid* in T if each level set in A has no pre-determined new sets of parallel 1's in T.

The main point is that valid subtrees are safe to work with: They can always be extended within the ambient strong coding tree to any desired strong similarity type. This will be seen clearly in the lemmas at the end of the section. Note that all finite strong coding trees are valid, as their maximal level contains a coding node.

We now come to the definition of the space of strong coding subtrees of a fixed strong coding tree. The partial ordering $\le$ is defined on the collection of all strong coding trees as follows: For strong coding trees S and T,

$$S \le T \quad \longleftrightarrow \quad S \text{ is a valid subtree of } T \text{ and } S \overset{s}{\sim} T. \tag{18}$$

Definition 4.13 (The space $(\mathcal{T}(T), \le, r)$)**.** Let T be any strong coding tree. Define $\mathcal{T}(T)$ to be the collection of all strong coding trees S such that $S \le T$. As previously defined, for $m < \omega$, $r_m(S)$ denotes $\bigcup_{i<m} \mathrm{Lev}_S(m)$, the initial subtree of S containing the first m critical nodes of S. The restriction map r is formally a map from $\omega \times \mathcal{T}(T)$ which on input (m, S) produces $r_m(S)$. Let $\mathcal{A}_m(T)$ denote $\{r_m(S) : S \in \mathcal{T}(T)\}$, and let $\mathcal{A}(T) = \bigcup_{m<\omega} \mathcal{A}_m(T)$, the collection of all *finite approximations* to members of $\mathcal{T}(T)$.

For $A \in \mathcal{A}_m(T)$ and $S \in \mathcal{T}(T)$ with A valid in S, define

$$[A, S] = \{U \le S : r_m(U) = A\}, \tag{19}$$

and define

$$r_{m+1}[A, S] = \{B \in \mathcal{A}_{m+1} : r_m(B) = A \text{ and } B \text{ is valid in } S\}. \tag{20}$$

Techniques for building valid subtrees of a given strong coding tree are now developed. The next lemma provides a means for extending a particular maximal node s in a finite subtree A of a strong coding tree T to a particular extension t in T, and extending the rest of the maximal nodes in A to the length of t, without introducing new sets of parallel 1's. Let $\{s_i : i < \tilde{i}\}$ be some level set of nodes in a strong coding tree T. We say that a level set of extensions $\{t_i : i < \tilde{i}\}$, where each $t_i \supseteq s_i$, *adds no new sets of parallel* 1*'s over* $\{s_i : i < \tilde{i}\}$ if whenever $l < |t_0|$ and the set $I_l := \{i < \tilde{i} : t_i(l) = 1\}$ has cardinality at least 2, then there is an $l' < |s_0|$ such that $\{i < \tilde{i} : s_i(l') = 0\} = I_l$.

Lemma 4.14. *Suppose T is a strong coding tree and $\{s_i : i < \tilde{i}\}$ is a set of two or more nodes in $\widehat{T}$ of length $l_k + 1$. Let $n_* > k$, let l_* denote l_{n_*}, and let t_0 be any extension of s_0 in $\widehat{T}$ of length $l_* + 1$. For each $0 < i < \tilde{i}$, let t_i denote the leftmost extension of s_i in $\widehat{T}$ of length $l_* + 1$. Then the set $\{t_i : i < \tilde{i}\}$ adds no new sets of parallel* 1*'s over $\{s_i : i < \tilde{i}\}$.*

Proof. Assume the hypotheses, and suppose that there is some $l < l_*$ such that the set $I_l = \{i < \tilde{i} : t_i(l) = 1\}$ has more than one member. Then by the Parallel 1's Criterion, there is an $n \le n_*$ such that $t_i(l_n) = 1$ for all $i < \tilde{i}$. Since for each $0 < i < \tilde{i}$, t_i is the leftmost extension of s_i, by (6) and (7) in the definition of strong coding tree, the passing number of t_i at l_j is 0, for all $k < j \le n_*$. It follows that any n such that c_n witnesses the parallel 1's in $\{t_i : i \in I_l\}$ must be less than or equal to k. $\square$

In fact, any sets of parallel 1's from the set $\{t_i : i < \tilde{i}\}$ constructed in the preceding lemma occur at a level below l.

Given a set of nodes S in a strong coding tree, the *tree induced by* S is the set of nodes $\{s \restriction |v| : s \in S,\ v \in S^\wedge\}$. For a finite tree A, we shall use the notation $\max(A)$ in a slightly non-standard way.

Notation 4.15. Given a finite tree A, $\max(A)$ denotes the set of terminal nodes in A which have the maximal length of any node in A. Thus,

$$\max(A) = \{t \in A : t = l_A\}, \tag{21}$$

where $l_A = \max\{|s| : s \in A\}$. Note in particular that $\max(A)$ is a level set.

The following lemma is immediate from finitely many applications of Lemma 4.14, using the fact that maximal nodes of valid subtrees can be extended leftmost to any length without adding any new sets of parallel 1's.

Lemma 4.16. *Let A be a finite valid subtree of any strong coding tree T and let l be the length of the nodes in* $\max(A)$. *Let* $\mathrm{Spl}(u)$ *be any nonempty level subset of* $\max(A)$, *and let Z be any subset of* $\max(A) \setminus \mathrm{Spl}(u)$. *Then given any enumeration $\{z_i : i < \tilde{i}\}$ of* $\mathrm{Spl}(u)$ *and $l' \ge l$, there is an $l_* > l'$ and extensions $s_i^0, s_i^1 \supset z_i$ for all $i < \tilde{i}$, and $s_z \supset z$ for all $z \in Z$, each of length l_*, such that, letting*

$$X = \{s_i^j : s \in \mathrm{Spl}(u),\ j \in \{0,1\}\} \cup \{s_z : z \in Z\}, \tag{22}$$

and B be the tree induced by $A \cup X$, the following hold:

1. *The splitting in B above A occurs in the order of the enumeration of* $\mathrm{Spl}(u)$. *Thus, for $i < i' < \tilde{i}$, $|s_i^0 \wedge s_i^1| < |s_{i'}^0 \wedge s_{i'}^1|$.*
2. *B has no new sets of parallel 1's over A.*

Convention 4.17. When working within a fixed strong coding tree T, the passing numbers at coding nodes c_n^T are completely determined by T. Thus, for a finite subset A of T such that l_A equals l_n^T for some $n < \omega$, then saying that A *satisfies the Parallel 1's Criterion* implies that the extension $A \cup \{s^+ : s \in \max(A)\}$ satisfies the Parallel 1's Criterion.

Lemma 4.18 shows that given a valid subtree of a strong coding tree T, any of its maximal nodes can be extended to a coding node c_k^T in T while the rest of the maximal nodes can be extended to length l_k^T so that their passing numbers are anything desired, subject only to the Triangle-Free Criterion. Recall that any node u in T at the level of a coding node c_k^T has a unique immediate extension u^+ of length $l_k^T + 1$ in $\widehat{T}$; so there is no ambiguity to consider $u^+(l_k^T)$ to be the passing number of u at c_k, even though technically u is not defined on input l_k^T.

Lemma 4.18 (Passing Number Choice Extension Lemma)**.** *Let T be a strong coding tree and A be any finite valid subtree of T. Let l_A denote the length of the members of* $\max(A)$ *and let A^+ denote the set of all members of $\widehat{T}$ of length $l_A + 1$ which extend some member of* $\max(A)$. *List the nodes of A^+ as s_i, $i < \tilde{i}$. Fix any $d < \tilde{i}$. For each $i \ne d$, if s_i and s_d have no parallel 1's, fix any $\varepsilon_i \in \{0,1\}$; if s_i and s_d have parallel 1's, let $\varepsilon_i = 0$. In particular, $\varepsilon_d = 0$.*

Then for each $j < \omega$, there is a coding node c_k with $k \ge j$ extending s_d and extensions $u_i \supseteq s_i$, $i \in \tilde{i} \setminus \{d\}$, of length l_k such that the passing number of u_i at c_k is ε_i for each $i \in \tilde{i} \setminus \{d\}$. Furthermore, the nodes u_i can be chosen so that any new parallel 1's among $\{u_i : i < \tilde{i}\}$ which were not witnessed in A are witnessed by c_k, and their first instances take place in the k-th interval of T. In particular, if $A \cup \{s_i : i < \tilde{i}\}$ satisfies the Parallel 1's Criterion, then $A \cup \{u_i : i < \tilde{t}\}$ also satisfies the Parallel 1's Criterion, where $u_d = c_k$.

Proof. Assume the hypotheses of the lemma. Let j' be such that the nodes $\{s_i : i < \tilde{i}\}$ are in the j'-th interval of T. For each $i < \tilde{i}$, let t_i be the leftmost extension of s_i of length $l_{j'} + 1$. Since A is a valid subtree of T, no new sets of parallel 1's are acquired by $\{t_i : i < \tilde{i}\}$. Let $j < \omega$ be given and take $k \ge \max(j, j'+1)$ minimal such that $c_k \supseteq t_d$, and let $u_d = c_k$. Such a k exists since the coding nodes are dense in T. For each $i \ne d$, extend t_i via its leftmost extension to the level of $l_{k-1} + 1$, and label it t_i'. By Lemma 4.14, for

$i \neq d$, no new sets of parallel 1's are acquired by $\{t'_i : i \in \tilde{i} \setminus \{d\}\} \cup \{u_d \restriction (l_{k-1}+1)\}$. For each $i \neq d$ for which $\varepsilon_i = 0$, let u_i be the leftmost extension of t'_i of length $l_k + 1$. For $i < \tilde{i}$ such that $\varepsilon_i = 1$, let u_i be the rightmost extension of t'_i to length $l_k + 1$. Note that for each $i < \tilde{i}$, the passing number of of u_i at c_k is ε_i.

For any $I \subseteq \tilde{i}$ of size at least two, if there is some l such that $u_i(l) = 1$ for all $i \in I$, and the least l for which this holds is greater than l_A, then it must be that $u_i(l_k) = 1$ for each $i \in I$, since no new sets of parallel 1's are acquired among $\{u_i : i < \tilde{i}\}$ below $l_{k-1} + 1$. Thus, the set $\{u_i : i < \tilde{i}\}$ satisfies the lemma. If A satisfies the Parallel 1's Criterion, then it is clear that $A \cup \{u_i : i < \tilde{i}\}$ also satisfies the Parallel 1's Criterion, since all the new parallel 1's are witnessed by the coding node $u_d = c_k$. □

The final lemma of this section combines the previous two, to show that any finite valid subtree of a strong coding tree can be extended to another valid subtree with any prescribed strong similarity type.

Lemma 4.19. *Let A be a finite valid subtree of any strong coding tree T, and let l_A be the length of the nodes in $\max(A)$. Fix any member $u \in \max(A)^+$. Let $\mathrm{Spl}(u)$ be any set of nodes $s \in \max(A)^+$ which have no parallel 1's with u, and let Z denote $\max(A)^+ \setminus (\mathrm{Spl}(u) \cup \{u\})$. Let $l \geq l_A$ be given. Then there is an $l_* > l$ and extensions $u_* \supset u$, $s^0_*, s^1_* \supset s$ for all $s \in \mathrm{Spl}(u)$, and $s_* \supset s$ for all $s \in Z$, each of length l_*, such that, letting*

$$X = \{u_*\} \cup \{s^i_* : s \in \mathrm{Spl}(u),\ i \in \{0,1\}\} \cup \{s_* : s \in Z\}, \tag{23}$$

and B be the tree induced by $A \cup X$, the following hold:

1. *u_* is a coding node.*
2. *For each $s \in \mathrm{Spl}(t)$ and $i \in \{0,1\}$, the passing number of s^i_* at u_* is i.*
3. *For each $s \in Z$, the passing number of s_* at u_* is 0.*
4. *Splitting among the extensions of the $s \in \mathrm{Spl}(u)$ occurs in reverse lexicographic order: For s and t in $\mathrm{Spl}(u)$, $|s^0_* \wedge s^1_*| < |t^0_* \wedge t^1_*|$ if and only if $s_* >_{\mathrm{lex}} t_*$.*
5. *There are no new sets of parallel 1's among the nodes in X until they pass the level of the longest splitting node in B below u_*.*

In particular, if A satisfies the Parallel 1's Criterion, then so does B.

Proof. Since A is valid in T, apply Lemma 4.16 to extend $\max(A)$ to have splitting nodes in the desired order without adding any new sets of parallel 1's. Then apply Lemma 4.18 to extend to a level with a coding node and passing numbers as prescribed. □

It follows from Lemma 4.19 that whenever A is a finite strong coding tree which is valid in some strong coding tree T and strongly similar to $r_m(T)$, then $r_{m+1}[A, T]$ is infinite. In particular, A can be extended to a strong coding tree S such that $S \leq T$.

Remark 4.20. It is straightforward to check that the space $(\mathcal{T}(T), \leq, r)$ of strong coding trees satisfies Axioms **A.1**, **A.2**, and **A.3(1)** of Todorcevic's axioms in Chapter 5 of [33] guaranteeing a topological Ramsey space. On the other hand, **A.3(2)** does not hold, and **A.4**, the pigeonhole principle, holds in a modified form where the finite subtree being extended is a valid subtree of the strong coding tree, as will follow from Theorem 6.3. It remains open what sort of infinitary Ramsey theory in the vein of [22] holds in $(\mathcal{T}(T), \leq, r)$, in terms of its Ellentuck topology.

5. Halpern-Lauchli-style Theorems for strong coding trees

The Ramsey theory content for strong coding trees begins in this section. The ultimate goal is to obtain a Ramsey theorem for colorings of strictly similar (Definition 8.3) copies of any given finite antichain of coding nodes, as these are the structures which will code finite triangle-free graphs. This is accomplished in Theorem 8.9. As a mid-point theorems, we will prove Milliken-style theorems (Theorems **??** and 6.3) for finite trees satisfying some strong versions of the Parallel 1's Criterion. Just as the Halpern-Läuchli Theorem forms the core content of Milliken's Theorem in the setting of strong trees, so too in the setting of strong coding trees, Halpern-Läuchli-style theorems are proved first and then applied to obtain Milliken-style theorems in later sections.

The main and only theorem of this section is Theorem 5.2. This general theorem encompasses colorings of two different types of level set extensions of a fixed finite tree: The level set either contains a splitting node (Case (a)) or a coding node (Case (b)). In Case (a), we obtain a direct analogue of the Halpern-Läuchli Theorem. In Case (b), we obtain a weaker version of the Halpern-Läuchli Theorem, which is later strengthened to the direct analogue in Lemma 6.8.

The structure of the proof follows the basic outline of Harrington's proof of the Halpern-Läuchli Theorem, as outlined to the author by Laver. The reader wishing to read that proof as a warm-up is referred to Section 2 of [3]. In the setting of strong coding trees, new considerations arise, and new forcings have to be established to achieve the result. The main reasons that new forcings are needed are firstly, that there are two types of nodes, coding and splitting nodes, and secondly, that the extensions achieving homogeneity must be extendible to a strong coding tree valid inside the ambient tree. This second property necessitates that the extensions be valid and satisfy the Parallel 1's Criterion, and is responsible for the strong definition of the partial ordering on the forcing. The former is responsible for there being Cases (a) and (b). The forcings will consist of conditions which are finite functions with images which are certain level sets of a given tree strong coding tree T, but the partial ordering will be stronger than the partial ordering of subtree as branches added will have some dependence between them, so these are not simply Cohen forcings.

Remark 5.1. Although the proof uses the set-theoretic technique of forcing, the whole construction takes place in the original model of ZFC, not in some generic extension. The forcing should be thought of as conducting an unbounded search for a finite object, namely the finite set of nodes of a prescribed form where homogeneity is attained. Thus, the result and its proof hold using only the standard axioms of mathematics.

The following terminology and notation will be used throughout. Let T be a strong coding tree. Given finite subtrees U, V of T, we write $U \sqsubseteq V$ to mean that there is some k such that $U = \bigcup_{m<k} \mathrm{Lev}_U(m) = \bigcup_{m<k} \mathrm{Lev}_V(m)$, and we say that V *extends* U, or that U is an *initial subtree of* V. We write $U \sqsubset V$ if U is a proper initial subtree of V. Recall from Definition 4.13 that $S \le T$ means that S is a valid subtree of T which is strongly similar to T, and hence also a strong coding tree. Given a finite strong coding tree B, $[B, T]$ denotes the set of all $S \le T$ such that S extends B. A set $X \subseteq \widehat{T}$ is a *level set* if all nodes in X have the same length. For level sets X, Y we shall also say that Y *extends* X if X and Y have the same number of nodes and each node in X is extended by a unique node in Y. For level sets $Y = \{y_i : i \le d\}$ and $X = \{x_i : i \le d\}$ with $y_i \supseteq x_i$ for each $i \le d$, we say that Y *has no new sets of parallel* 1*'s over* X if for each $I \subseteq d+1$ for which there is an l such that $y_i(l) = 1$ for each $i \in I$, then there is an l' such that $x_i(l') = 1$ for each $i \in I$. For any tree $U \subseteq \widehat{T}$ and any $l < \omega$, let $U \restriction l$ denote the set of $s \in \widehat{U}$ such that $|s| = l$. A set of two or more nodes $\{x_i : i \in I\}$ in $\widehat{T}$ is said to have *first parallel* 1*'s at level* l if l is least such that $x_i(l) = 1$ for all $i \in I$.

For each $s \in \widehat{T}$, if $i \in \{0, 1\}$ and $s^\frown i$ is in $\widehat{T}$, then we say that $s^\frown i$ is an *immediate extension of* s in T. Thus, splitting nodes in T have two immediate extensions in T, and non-splitting nodes, including every node at the level of a coding node, have exactly one immediate extension in T. For a non-splitting node s in T, we let s^+ denote the immediate extension of s in T. Given a finite subtree A of T, let l_A denote the maximum of the lengths of members of A, and let $\max(A)$ denote the set of all nodes in A with length l_A. Let A^+ denote the set of immediate extensions in $\widehat{T}$ of the members of $\max(A)$:

$$A^+ = \{s^\frown i : s \in \max(A),\ i \in \{0, 1\},\ \text{and } s^\frown i \in \widehat{T}\}. \tag{24}$$

Note that A^+ is a level set of nodes of length $l_A + 1$.

We now provide the set-up for the two cases before stating the theorem.

The Set-up for Theorem 5.2. Let $\mathbb{T}$ be a fixed strong coding tree, and let $T \le \mathbb{T}$ be given. Let A be a finite valid subtree of T satisfying the Parallel 1's Criterion. It is fine for A to have terminal nodes at different levels, indeed, we need to allow this for the intended applications later. Without loss of generality and to simplify the presentation of the proof, assume that 0^{l_A} is in A. Let A_e be a subset of A^+ containing 0^{l_A+1} and of size at least two. Let C be a finite valid subtree of T containing A such that C satisfies the Parallel 1's Criterion and the collection of all nodes in C not in A, denoted $C \setminus A$, forms a level set extending A_e. Assume moreover that 0^{l_C} is the node in C extending 0^{l_A+1}, where l_C is the length of the nodes in $C \setminus A$. The two cases are the following:

Case (a). $C \setminus A$ contains a splitting node.

In Case (a), define $\mathrm{Ext}_T(A,C)$ to be the collection of all level sets $X \subseteq T$ extending A_e such that $A\cup X \overset{s}{\sim} C$ and $A \cup X$ is valid in T. We point out that $A\cup X$ being valid in T is equivalent to X having no pre-determined new parallel 1's. It will turn out to be necessary to require this of X, and the extensions for which the coloring is relevant will have this property anyway.

Case (b). $C \setminus A$ contains a coding node.

In Case (b), define $\mathrm{Ext}_T(A,C)$ to be the collection of all level sets $X \subseteq T$ extending A_e such that $A\cup X \overset{s}{\sim} C$. Since X contains a coding node, $A\cup X$ is automatically valid in T. Recalling (7) of Definition 4.9, $A\cup X \overset{s}{\sim} C$ implies that, letting $f: A\cup X \to C$ be the strong similarity map, for each $x\in X$ the passing number of x^+ at the coding node in X equals the passing number of $(f(x))^+$ at the coding node in $C\setminus A$. Given any $X \in \mathrm{Ext}_T(A,C)$, let $\mathrm{Ext}_T(A,C;X)$ denote the set of $Y\in \mathrm{Ext}_T(A,C)$ such that Y extends X.

In both cases, $A\cup X \overset{s}{\sim} C$ implies that $A\cup X$ satisfies the Parallel 1's Criterion.

Theorem 5.2. *Let $T\le \mathbb{T}$ be any strong coding tree and let B be a finite strong coding tree valid in T. Let $A \sqsubset C$ be finite valid subtrees of T such that both A and C satisfy the Parallel 1's Criterion, A is a subtree of B, $C\setminus A$ is a level set of size at least two, and $0^{l_C}\in C$. Further, assume that the nodes in $C\setminus A$ extend nodes in $\max(A)\cap\max(B)$. Let A_e denote the set of nodes in A^+ which are extended to nodes in $C\setminus A$.*

In Case (a), given any coloring $h:\mathrm{Ext}_T(A,C)\to 2$, there is a strong coding tree $S\in[B,T]$ such that h is monochromatic on $\mathrm{Ext}_S(A,C)$.

In Case (b), suppose $X\in \mathrm{Ext}_T(A,C)$ and m_0 are given for which there is a $B'\in r_{m_0}[B,T]$ with $X\subseteq \max(B')$. Then for any coloring $h:\mathrm{Ext}_T(A,C)\to 2$ there is a strong coding tree $S\in[r_{m_0-1}(B'),T]$ such that h is monochromatic on $\mathrm{Ext}_S(A,C;X)$.

Proof. Let T, A, A_e, B, C be given satisfying the hypotheses of either Case (a) or (b), and let $h:\mathrm{Ext}_T(A,C)\to 2$ be a given coloring. Let $d+1$ equal the number of nodes in A_e. List the nodes of A_e as $s_0,\dots,s_d$, letting s_d denote the node of A_e that is extended to the critical node in $C\setminus A$: a splitting node in Case (a) and a coding node in Case (b). For each $i\le d$, let t_i denote the node in $\max(C)$ which extends s_i. In particular, t_d denotes the splitting or coding node in $\max(C)$. Let i_0 denote the integer such that s_{i_0} is the node of A_e which is a sequence of 0's. Then t_{i_0} is the sequence of all 0's in $C\setminus A$. Notice that i_0 can equal d only if we are in Case (a) and moreover the splitting node in $C\setminus A$ is a sequence of 0's. In Case (b), the following notation will be used: For each $i\le d$, t_i^+ denotes the member in $\max(C)^+$ extending t_i. Let I_0 denote the set of all $i<d$ such that $t_i^+(|t_d|)=0$ and let I_1 denote the set of all $i<d$ such that $t_i^+(|t_d|)=1$.

Let L denote the collection of all $l<\omega$ such that there is a member of $\mathrm{Ext}_T(A,C)$ with maximal nodes of length l. L is infinite since B is valid in T. In Case (a), L is exactly the set of all $l<\omega$ for which there is a splitting node of length l extending s_d, and in Case (b), L is exactly the set of all $l<\omega$ for which there is a coding node of length l extending s_d, as this follows from the validity of B in T and Lemma 4.18. For each $i\in(d+1)\setminus\{i_0\}$, let $T_i=\{t\in T: t\supseteq s_i\}$; let $T_{i_0}=\{t\in T: t\supseteq s_{i_0}$ and $t\in 0^{<\omega}\}$, the collection of all leftmost nodes in T extending s_{i_0}.

Let $\kappa=\beth_{2d}$. The following forcing notion $\mathbb{P}$ adds κ many paths through T_i, for each $i\in d\setminus\{i_0\}$, and one path through T_d. If $i_0\ne d$, then $\mathbb{P}$ will add one path through T_{i_0}, though allowing κ many ordinals to label this path in order to simplify notation.

<u>Case (a)</u>. $\mathbb{P}$ is the set of conditions p such that p is a function of the form

$$p:(d\times\vec{\delta}_p)\cup\{d\}\to T\restriction l_p,$$

where $\vec{\delta}_p\in[\kappa]^{<\omega}$ and $l_p\in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d of length l_p;
(ii) For each $i<d$, $\{p(i,\delta):\delta\in\vec{\delta}_p\}\subseteq T_i\restriction l_p$; and
(iii) $\{p(i,\delta):(i,\delta)\in d\times\vec{\delta}_p\}\cup\{p(d)\}$ has no pre-determined new parallel 1's.

<u>Case (b)</u>. $\mathbb{P}$ is the set of conditions p such that p is a function of the form

$$p:(d\times\vec{\delta}_p)\cup\{d\}\to T\restriction l_p,$$

where $\vec{\delta}_p\in[\kappa]^{<\omega}$ and $l_p\in L$, such that

(i) $p(d)$ is *the* coding node extending s_d of length l_p;
(ii) For each $i < d$, $\{p(i,\delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction l_p$.
(iii) For each $\delta \in \vec{\delta}_p$, $j \in \{0,1\}$, and $i \in I_j$, the immediate extension of $p(i,\delta)$ in T is j; that is, the passing number of $(p(i,\delta))^+$ at $p(d)$ is j.

In both Cases (a) and (b), the partial ordering on $\mathbb{P}$ is defined as follows: $q \le p$ if and only if $l_q \ge l_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, and

(i) $q(d) \supseteq p(d)$, and $q(i,\delta) \supseteq p(i,\delta)$ for each $(i,\delta) \in d \times \vec{\delta}_p$;
(ii) The set $\{q(i,\delta) : (i,\delta) \in d \times \vec{\delta}_p\} \cup \{q(d)\}$ has no new sets of parallel 1's over $\{p(i,\delta) : (i,\delta) \in d \times \vec{\delta}_p\} \cup \{p(d)\}$.

Given $p \in \mathbb{P}$, we shall use $\mathrm{ran}(p)$ to denote the *range of* p, $\{p(i,\delta) : (i,\delta) \in d \times \vec{\delta}_p\} \cup \{p(d)\}$. If p, q are members of $\mathbb{P}$, we shall use the abbreviation *q has no new parallel 1's over p* to mean that $\mathrm{ran}(q)$ has no new sets of parallel 1's over $\mathrm{ran}(p)$.

The proof of the theorem proceeds in three parts. Part I proves that $\mathbb{P}$ is an atomless partial order. Part II proves Lemma 5.3 which is the main tool for building fusion sequences while preserving homogeneity. This is applied in Part III to build the tree S which is valid in T and such that $\mathrm{Ext}_S(A, C)$ is homogeneous for h in Case (a), and $\mathrm{Ext}_S(A, C; X)$ is homogeneous for h in Case (b). For the first two parts of the proof, we present a general proof, indicating the steps at which the two cases require different approaches. Part III will require the cases to be handled separately.

Part I. $\mathbb{P}$ is an atomless partial ordering.

Claim 1. *$(\mathbb{P}, \le)$ is a partial ordering.*

Proof. The order $\le$ on $\mathbb{P}$ is clearly reflexive and antisymmetric. Transitivity follows from the fact that the requirement (ii) is a transitive property. If $p \ge q$ and $q \ge r$, then $\vec{\delta}_p \subseteq \vec{\delta}_q \subseteq \vec{\delta}_r$ and $l_p \le l_q \le l_r$. Since r has no new sets of parallel 1's over q and q has no new sets of parallel 1's over p, it follows that r has no new sets of parallel 1's over p. Thus, $p \ge r$. □

We show that $\mathbb{P}$ is atomless by proving the following stronger claim.

Claim 2. *For each $p \in \mathbb{P}$ and $l > l_p$, there are $q, r \in \mathbb{P}$ with $l_q, l_r > l$ such that $q, r < p$ and q and r are incompatible.*

Proof. Let $p \in \mathbb{P}$ and $l > l_p$ be given, and let $\vec{\delta}_r = \vec{\delta}_q = \vec{\delta}_p$.

In Case (a), take $q(d)$ and $r(d)$ to be incomparable splitting nodes in T extending $p(d)$ to some lengths greater than l. Such splitting nodes exist since strong coding trees are perfect. Let $l_q = |q(d)|$ and $l_r = |r(d)|$. For each $(i,\delta) \in d \times \vec{\delta}_p$, let $q(i,\delta)$ be the leftmost extension (in T) of $p(i,\delta)$ to length l_q, and let $r(i,\delta)$ be the leftmost extension of $p(i,\delta)$ to length l_r. Then q and r have no pre-determined new parallel 1's, since $\mathrm{ran}(p)$ has no pre-determined new parallel 1's and all nodes except $q(d)$ and $r(d)$ are leftmost extensions in T of members of $\mathrm{ran}(p)$; so q and r are members of $\mathbb{P}$. By Lemma 4.14, both q and r have no new parallel 1's over p, so $q, r \le p$. Since neither of $q(d)$ and $r(d)$ extends the other, q and r are incompatible.

In Case (b), let s be a splitting node in T of length greater than l extending $p(d)$. Let c^T_k be the least coding node in T above s. Let s_0, s_1 extend $s^\frown 0, s^\frown 1$ leftmost in T to the level of c^T_k, respectively. For each $(i,\delta) \in d \times \vec{\delta}_p$, let $p'(i,\delta)$ be the leftmost extension in T of $p(i,\delta)$ of length l^T_k. By Lemma 4.18, there are $q(d) \supseteq s_0$ and $q(i,\delta) \supseteq p'(i,\delta)$, $(i,\delta) \in d \times \vec{\delta}_p$, such that

(1) $q(d)$ is a coding node;
(2) q has no new parallel 1's over p;
(3) For each $j < 2$, $i \in I_j$ if and only if the immediate extension of $q(i,\delta)$ is j.

Then $q \in \mathbb{P}$ and $q \le p$. Likewise by Lemma 4.18, we may extend $\{p'(i,\delta) : (i,\delta) \in d \times \vec{\delta}_p\} \cup \{s_1\}$ to $\{r(i,\delta) : (i,\delta) \in d \times \vec{\delta}_p\} \cup \{r(d)\}$ to form a condition $r \in \mathbb{P}$ extending p. Since the coding nodes $q(d)$ and $r(d)$ are incomparable, q and r are incompatible conditions in $\mathbb{P}$. □

From now on, whenever ambiguity will not arise by doing so, we will refer to the splitting node in Case (a) and the coding node in Case (b) simply as the *critical node*.

Let $\dot{b}_d$ be a $\mathbb{P}$-name for the generic path through T_d; that is, $\dot{b}_d = \{\langle p(d), p\rangle : p \in \mathbb{P}\}$. Note that for each $p \in \mathbb{P}$, p forces that $\dot{b}_d \restriction l_p = p(d)$. By Claim 2, it is dense to force a critical node in $\dot{b}_d$ above any given level in T, so $\mathbf{1}_{\mathbb{P}}$ forces that the set of levels of critical nodes in $\dot{b}_d$ is infinite. Thus, given any generic filter G for $\mathbb{P}$, $\dot{b}_d^G = \{p(d) : p \in G\}$ is a cofinal path of critical nodes in T_d. Let $\dot{L}_d$ be a $\mathbb{P}$-name for the set of lengths of critical nodes in $\dot{b}_d$. Note that $\mathbf{1}_{\mathbb{P}} \Vdash \dot{L}_d \subseteq L$. Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on $\dot{L}_d$. For each $i < d$ and $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ be a $\mathbb{P}$-name for the α-th generic branch through T_i; that is, $\dot{b}_{i,\alpha} = \{\langle p(i,\alpha), p\rangle : p \in \mathbb{P} \text{ and } \alpha \in \vec{\delta}_p\}$. For $i < d$ and for any condition $p \in \mathbb{P}$ and $\alpha \in \vec{\delta}_p$, p forces that $\dot{b}_{i,\alpha} \restriction l_p = p(i,\alpha)$.

For ease of notation, we shall write sets $\{\alpha_i : i < d\}$ in $[\kappa]^d$ as vectors $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1}\rangle$ in strictly increasing order. For $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1}\rangle \in [\kappa]^d$, rather than writing out $\langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}}, \dot{b}_d\rangle$ each time we wish to refer to these generic branches, we shall simply

$$\text{let } \dot{b}_{\vec{\alpha}} \text{ denote } \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}}, \dot{b}_d\rangle, \tag{25}$$

since the branch $\dot{b}_d$ being unique causes no ambiguity. For any $l < \omega$,

$$\text{let } \dot{b}_{\vec{\alpha}} \restriction l \text{ denote } \{\dot{b}_{i,\alpha_i} \restriction l : i < d\} \cup \{\dot{b}_d \restriction l\}. \tag{26}$$

Using the abbreviations just defined, h is a coloring on sets of nodes of the form $\dot{b}_{\vec{\alpha}} \restriction l$ whenever this is forced to be a member of $\mathrm{Ext}_T(A, C)$.

Part II. The goal now is to prove Claims 3 and 4 and Lemma 5.3. To sum up, they secure that there are infinite pairwise disjoint sets $K_i \subseteq \kappa$ for $i < d$, and a set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ which are compatible, have the same images in T, and such that for some fixed $\varepsilon^* \in \{0, 1\}$, for each $\vec{\alpha} \in \prod_{i<d} K_i'$, $p_{\vec{\alpha}}$ forces $h(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon^*$ for ultrafilter many $l \in \dot{L}_d$. Moreover, we will find nodes t_i^*, $i \le d$, such that for each $\vec{\alpha} \in \prod_{i<d} K_i$, $p_{\vec{\alpha}}(i, \alpha_i) = t_i^*$. Lemma 5.3 will enable fusion processes for constructing S with one color on $\mathrm{Ext}_S(A, C)$ in Part III. There are no differences between the arguments for Cases (a) and (b) in Part II.

For each $\vec{\alpha} \in [\kappa]^d$, choose a condition $p_{\vec{\alpha}} \in \mathbb{P}$ such that

(1) $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$.
(2) $\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\} \cup \{p(d)\} \in \mathrm{Ext}_T(A, C)$.
(3) $p_{\vec{\alpha}} \Vdash$ "There is an $\varepsilon \in 2$ such that $h(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon$ for $\dot{\mathcal{U}}$ many l in $\dot{L}_d$."
(4) $p_{\vec{\alpha}}$ decides a value for ε, call it $\varepsilon_{\vec{\alpha}}$.
(5) $h(\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\} \cup \{p(d)\}) = \varepsilon_{\vec{\alpha}}$.

Properties (1) - (5) can be guaranteed as follows. Recall that for $i \le d$, t_i denotes the member of $C \setminus A$ extending s_i. For each $\vec{\alpha} \in [\kappa]^d$, let

$$p^0_{\vec{\alpha}} = \{\langle (i, \delta), t_i\rangle : i < d,\ \delta \in \vec{\alpha}\} \cup \{\langle d, t_d\rangle\}.$$

Then $p^0_{\vec{\alpha}}$ is a condition in $\mathbb{P}$, $\vec{\delta}_{p^0_{\vec{\alpha}}} = \vec{\alpha}$, so (1) holds. Further, $\{p^0_{\vec{\alpha}}(i, \alpha_i) : i < d\} \cup \{p^0_{\vec{\alpha}}(d)\}$ is a member of $\mathrm{Ext}_T(A, C)$ since it is exactly $C \setminus A$. It is important to note that for any $p \le p^0_{\vec{\alpha}}$, $\{p(i, \alpha_i) : i < d\} \cup \{p(d)\}$ is also a member of $\mathrm{Ext}_T(A, C)$, as this follows from the fact that $\{p(i, \delta) : (i, \delta) \in d \times \vec{\delta}_{p^0_{\vec{\alpha}}}\} \cup \{p(d)\}$ has no new sets of parallel 1's over the image of $p^0_{\vec{\alpha}}$. Thus (2) holds for any $p \le p^0_{\vec{\alpha}}$. Take an extension $p^1_{\vec{\alpha}} \le p^0_{\vec{\alpha}}$ which forces $h(\dot{b}_{\vec{\alpha}} \restriction l)$ to be the same value for $\dot{\mathcal{U}}$ many $l \in \dot{L}_d$, giving (3). For Property (4), since $\mathbb{P}$ is a forcing notion, there is a $p^2_{\vec{\alpha}} \le p^1_{\vec{\alpha}}$ deciding a value $\varepsilon_{\vec{\alpha}}$ for which $p^2_{\vec{\alpha}}$ forces that $h(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many l in $\dot{L}_d$. By extending $p^2_{\vec{\alpha}}$ if necessary, we may take $p^3_{\vec{\alpha}} \le p^2_{\vec{\alpha}}$ such that $p^3_{\vec{\alpha}}$ decides $h(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$, for some $l \in \dot{L}$ such that l is greater than or equal to the lengths of the nodes in $p^2_{\vec{\alpha}}$, and such that $l_{p^3_{\vec{\alpha}}} \ge l$. Let $p_{\vec{\alpha}}$ be $p^3_{\vec{\alpha}}$ truncated to level l. Then $p_{\vec{\alpha}} \le p^2_{\vec{\alpha}}$ so it still satisfies (1) through (4). Since $\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\} \cup \{p_{\vec{\alpha}}(d)\}$ is what $p_{\vec{\alpha}}$ forces $\dot{b}_{\vec{\alpha}} \restriction l$ to be, it follows that $p_{\vec{\alpha}}$ forces $h(\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\} \cup \{p_{\vec{\alpha}}(d)\}) = \varepsilon_{\vec{\alpha}}$, so (5) holds.

Now we prepare for an application of the Erdős-Rado Theorem (recall Theorem 2.4). We are assuming $\kappa = \beth_{2d}$, which is at least $\beth_{2d-1}(\aleph_0)^+$, so that $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$. Given two sets of ordinals J, K we shall write $J < K$ if every member of J is less than every member of K. Let $D_e = \{0, 2, \dots, 2d-2\}$ and $D_o = \{1, 3, \dots, 2d-1\}$, the sets of even and odd integers less than $2d$, respectively. Let $\mathcal{I}$ denote the collection of all functions $\iota : 2d \to 2d$ such that $\iota \restriction D_e$ and $\iota \restriction D_o$ are strictly increasing sequences and $\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \dots < \{\iota(2d-2), \iota(2d-1)\}$. Thus, each ι codes two strictly increasing sequences

$\iota \restriction D_e$ and $\iota \restriction D_o$, each of length d. For $\vec{\theta} \in [\kappa]^{2d}$, $\iota(\vec{\theta})$ determines the pair of sequences of ordinals $(\theta_{\iota(0)}, \theta_{\iota(2)}, \dots, \theta_{\iota(2d-2))}), (\theta_{\iota(1)}, \theta_{\iota(3)}, \dots, \theta_{\iota(2d-1)})$, both of which are members of $[\kappa]^d$. Denote these as $\iota_e(\vec{\theta})$ and $\iota_o(\vec{\theta})$, respectively. To ease notation, let $\vec{\delta}_{\vec{\alpha}}$ denote $\vec{\delta}_{p_{\vec{\alpha}}}$, $k_{\vec{\alpha}}$ denote $|\vec{\delta}_{\vec{\alpha}}|$, and let $l_{\vec{\alpha}}$ denote $l_{p_{\vec{\alpha}}}$. Let $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}} \rangle$ denote the enumeration of $\vec{\delta}_{\vec{\alpha}}$ in increasing order.

Define a coloring f on $[\kappa]^{2d}$ into countably many colors as follows: Given $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, to reduce the number of subscripts, letting $\vec{\alpha}$ denote $\iota_e(\vec{\theta})$ and $\vec{\beta}$ denote $\iota_o(\vec{\theta})$, define

$$\begin{aligned} f(\iota, \vec{\theta}) = {} & \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, p_{\vec{\alpha}}(d), \langle \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d \rangle, \\ & \langle \langle i, j \rangle : i < d,\ j < k_{\vec{\alpha}},\ \text{and } \delta_{\vec{\alpha}}(j) = \alpha_i \rangle, \langle \langle j, k \rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \rangle \rangle. \end{aligned} \tag{27}$$

Let $f(\vec{\theta})$ be the sequence $\langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I} \rangle$, where $\mathcal{I}$ is given some fixed ordering. Since the range of f is countable, apply the Erdős-Rado Theorem to obtain a subset $K \subseteq \kappa$ of cardinality $\aleph_1$ which is homogeneous for f. Take $K' \subseteq K$ such that between each two members of K' there is a member of K and $\min(K') > \min(K)$. Take subsets $K_i \subseteq K'$ such that $K_0 < \dots < K_{d-1}$ and each $|K_i| = \aleph_0$.

Claim 3. *There are $\varepsilon^* \in 2$, $k^* \in \omega$, t_d, and $\langle t_{i,j} : j < k^* \rangle$, $i < d$, such that for all $\vec{\alpha} \in \prod_{i<d} K_i$ and each $i < d$, $\varepsilon_{\vec{\alpha}} = \varepsilon^*$, $k_{\vec{\alpha}} = k^*$, $p_{\vec{\alpha}}(d) = t_d$, and $\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle = \langle t_{i,j} : j < k^* \rangle$.*

Proof. Let ι be the member in $\mathcal{I}$ which is the identity function on $2d$. For any pair $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, there are $\vec{\theta}, \vec{\theta}' \in [K]^{2d}$ such that $\vec{\alpha} = \iota_e(\vec{\theta})$ and $\vec{\beta} = \iota_e(\vec{\theta}')$. Since $f(\iota, \vec{\theta}) = f(\iota, \vec{\theta}')$, it follows that $\varepsilon_{\vec{\alpha}} = \varepsilon_{\vec{\beta}}$, $k_{\vec{\alpha}} = k_{\vec{\beta}}$, $p_{\vec{\alpha}}(d) = p_{\vec{\beta}}(d)$, and $\langle \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d \rangle = \langle \langle p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j)) : j < k_{\vec{\beta}} \rangle : i < d \rangle$. Thus, define ε^*, k^*, t_d, $\langle \langle t_{i,j} : j < k^* \rangle : i < d \rangle$ to be $\varepsilon_{\vec{\alpha}}$, $k_{\vec{\alpha}}$, $p_{\vec{\alpha}}(d)$, $\langle \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d \rangle$ for any $\vec{\alpha} \in \prod_{i<d} K_i$. $\square$

Let l^* denote the length of t_d. Then all the nodes $t_{i,j}$, $i < d$, $j < k^*$, also have length l^*.

Claim 4. *Given any $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, if $j, k < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)$, then $j = k$.*

Proof. Let $\vec{\alpha}, \vec{\beta}$ be members of $\prod_{i<d} K_i$ and suppose that $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)$ for some $j, k < k^*$. For each $i < d$, let ρ_i be the relation from among $\{<, =, >\}$ such that $\alpha_i \, \rho_i \, \beta_i$. Let ι be the member of $\mathcal{I}$ such that for each $\vec{\gamma} \in [K]^d$ and each $i < d$, $\theta_{\iota(2i)} \ \rho_i \ \theta_{\iota(2i+1)}$. Then there is a $\vec{\theta} \in [K']^{2d}$ such that $\iota_e(\vec{\theta}) = \vec{\alpha}$ and $\iota_o(\vec{\theta}) = \vec{\beta}$. Since between any two members of K' there is a member of K, there is a $\vec{\gamma} \in [K]^d$ such that for each $i < d$, $\alpha_i \, \rho_i \, \gamma_i$ and $\gamma_i \, \rho_i \, \beta_i$, and furthermore, for each $i < d-1$, $\{\alpha_i, \beta_i, \gamma_i\} < \{\alpha_{i+1}, \beta_{i+1}, \gamma_{i+1}\}$. Given that $\alpha_i \, \rho_i \, \gamma_i$ and $\gamma_i \, \rho_i \, \beta_i$ for each $i < d$, there are $\vec{\mu}, \vec{\nu} \in [K]^{2d}$ such that $\iota_e(\vec{\mu}) = \vec{\alpha}$, $\iota_o(\vec{\mu}) = \vec{\gamma}$, $\iota_e(\vec{\nu}) = \vec{\gamma}$, and $\iota_o(\vec{\nu}) = \vec{\beta}$. Since $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)$, the pair $\langle j, k \rangle$ is in the last sequence in $f(\iota, \vec{\theta})$. Since $f(\iota, \vec{\mu}) = f(\iota, \vec{\nu}) = f(\iota, \vec{\theta})$, also $\langle j, k \rangle$ is in the last sequence in $f(\iota, \vec{\mu})$ and $f(\iota, \vec{\nu})$. It follows that $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\gamma}}(k)$ and $\delta_{\vec{\gamma}}(j) = \delta_{\vec{\beta}}(k)$. Hence, $\delta_{\vec{\gamma}}(j) = \delta_{\vec{\gamma}}(k)$, and therefore j must equal k. $\square$

For any $\vec{\alpha} \in \prod_{i<d} K_i$ and any $\iota \in \mathcal{I}$, there is a $\vec{\theta} \in [K]^{2d}$ such that $\vec{\alpha} = \iota_o(\vec{\theta})$. By homogeneity of f and by the first sequence in the second line of equation (27), there is a strictly increasing sequence $\langle j_i : i < d \rangle$ of members of k^* such that for each $\vec{\alpha} \in \prod_{i<d} K_i$, $\delta_{\vec{\alpha}}(j_i) = \alpha_i$. For each $i < d$, let t_i^* denote t_{i,j_i}. Then for each $i < d$ and each $\vec{\alpha} \in \prod_{i<d} K_i$,

$$p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j_i)) = t_{i,j_i} = t_i^*. \tag{28}$$

Let t_d^* denote t_d.

Lemma 5.3. *For any finite subset $\vec{J} \subseteq \prod_{i<d} K_i$, the set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is compatible. Moreover, $p_{\vec{J}} := \bigcup \{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is a member of $\mathbb{P}$ which is below each $p_{\vec{\alpha}}$, $\vec{\alpha} \in \vec{J}$.*

Proof. For any $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, whenver $j, k < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)$, then $j = k$, by Claim 4. It then follows from Claim 3 that for each $i < d$,

$$p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) = t_{i,j} = p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j)) = p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(k)). \tag{29}$$

Thus, for each $\vec{\alpha}, \vec{\beta} \in \vec{J}$ and each $\delta \in \vec{\delta}_{\vec{\alpha}} \cap \vec{\delta}_{\vec{\beta}}$, for all $i < d$,

$$p_{\vec{\alpha}}(i, \delta) = p_{\vec{\beta}}(i, \delta). \tag{30}$$

Thus, $p_{\vec{J}} : \bigcup\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is a function. Let $\vec{\delta}_{\vec{J}} = \bigcup\{\vec{\delta}_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$. For each $\delta \in \vec{\delta}_{\vec{J}}$ and $i < d$, $p_{\vec{J}}(i, \delta)$ is defined, and it is exactly $p_{\vec{\alpha}}(i, \delta)$, for any $\vec{\alpha} \in \vec{J}$ such that $\delta \in \vec{\delta}_{\vec{\alpha}}$. Thus, $p_{\vec{J}}$ is a member of $\mathbb{P}$, and $p_{\vec{J}} \le p_{\vec{\alpha}}$ for each $\vec{\alpha} \in \vec{J}$. □

We conclude this section with a general claim which will be useful in Part III.

Claim 5. *If $\beta \in \bigcup_{i<d} K_i$, $\vec{\alpha} \in \prod_{i<d} K_i$, and $\beta \notin \vec{\alpha}$, then β is not a member of $\vec{\delta}_{\vec{\alpha}}$.*

Proof. Suppose toward a contradiction that $\beta \in \vec{\delta}_{\vec{\alpha}}$. Then there is a $j < k^*$ such that $\beta = \delta_{\vec{\alpha}}(j)$. Let i be such that $\beta \in K_i$. Since $\beta \neq \alpha_i = \delta_{\vec{\alpha}}(j_i)$, it must be that $j \neq j_i$. However, letting $\vec{\beta}$ be any member of $\prod_{i<d} K_i$ with $\beta_i = \beta$, then $\beta = \delta_{\vec{\beta}}(j_i) = \delta_{\vec{\alpha}}(j)$, so Claim 4 implies that $j_i = j$, a contradiction. □

Part III. In this last part of the proof, we build a strong coding tree S valid in T on which the coloring h is homogeneous. Cases (a) and (b) are now handled separately.

Part III Case (a). Recall that $\{s_i : i \le d\}$ enumerates the members of A_e, which is a subset of B^+. Let s_d^- denote $s_d \restriction l_A$, and let $i_d \in \{0, 1\}$ be such that $s_d = s_d^- {}^\frown i_d$. Let m' be the integer such that $B \in \mathcal{A}_{m'}(T)$. Let σ denote the strong similarity map from B onto $r_{m'}(\mathbb{T})$, and let $M = \{m_j : j < \omega\}$ be the strictly increasing enumeration of those $m > m'$ such that the splitting node in $\max(r_m(\mathbb{T}))$ extends $\sigma(s_d^-)^\frown i_d$. We will find $U_{m_0} \in r_{m_0}[B, T]$ and in general, $U_{m_{j+1}} \in r_{m_{j+1}}[U_{m_j}, T]$ so that for each $j < \omega$, h takes color ε^* on $\mathrm{Ext}_{U_{m_j}}(A, C)$. Then setting $S = \bigcup_{j<\omega} U_{m_j}$ will yield S to be a member of $[B, T]$ for which $\mathrm{Ext}_S(A, C)$ is homogeneous for h, with color ε^*.

First extend each node in B^+ to level l^* as follows. Recall that for each $i \le d$, $t_i^* \supseteq t_i$, so the set $\{t_i^* : i \le d\}$ extends A_e. For each node u in $B^+ \setminus A_e$, let u^* denote its leftmost extension in $T \restriction l^*$. Then the set

$$U^* = \{t_i^* : i \le d\} \cup \{u^* : u \in B^+ \setminus A_e\} \tag{31}$$

extends each member of B^+ to a unique node. Furthermore, by the choice of $p^0_{\vec{\alpha}}$ for each $\alpha \in [K]^d$ and the definition of the partial ordering on $\mathbb{P}$, it follows that the set $\{t_i^* : i \le d\}$ has no new sets of parallel 1's over A_e. Since the nodes u^* are leftmost extensions of members of $B^+ \setminus A_e$ and B is valid in T, it follows from Lemma 4.14 that U^* has no new sets of parallel 1's over B. Furthermore, U^* has no pre-determined new sets of parallel 1's, by (iii) in the definition of the partial ordering $\mathbb{P}$ for Case (a). Thus, $B \cup U^*$ satisfies the Parallel 1's Criterion and is valid in T. If $m_0 = m' + 1$, then let $U_{m'+1} = B \cup U^*$ and extend $U_{m'+1}$ to a member $U_{m_1-1} \in r_{m_1-1}[U_{m'+1}, T]$. If $m_0 > m' + 1$, apply Lemma 4.19 to extend above U^* to construct a member $U_{m_0-1} \in r_{m_0-1}[B, T]$. In this case, $\max(r_{m'+1}(U_{m_0}))$ is not U^*, but rather $\max(r_{m'+1}(U_{m_0}))$ extends U^*.

Assume $j < \omega$ and we have constructed U_{m_j-1} so that every member of $\mathrm{Ext}_{U_{m_j-1}}(A, C)$ is colored ε^* by h. Fix some $Y_j \in r_{m_j}[U_{m_j-1}, T]$ and let V_j denote $\max(Y_j)$. The nodes in V_j will not be in the tree S we are constructing; rather, we will extend the nodes in V_j to construct $U_{m_j} \in r_{m_j}[U_{m_j-1}, T]$.

We now start to construct a condition q which will satisfy Claim 9. Let $q(d)$ denote the splitting node in V_j and let $l_q = |q(d)|$. For each $i < d$ for which s_i and s_d do not have parallel 1's, let Z_i denote the set of all $v \in T_i \cap V_j$ such that v and $q(d)$ have no parallel 1's. For each $i < d$ for which s_i and s_d do have parallel 1's, let $Z_i = T_i \cap V_j$. For each $i < d$, take a set $J_i \subseteq K_i$ of cardinality $|Z_i|$ and label the members of Z_i as $\{z_\alpha : \alpha \in J_i\}$. Notice that each member of $\mathrm{Ext}_T(A, C)$ above V_j extends some set $\{z_{\alpha_i} : i < d\} \cup \{q(d)\}$, where each $\alpha_i \in J_i$. Let $\vec{J}$ denote the set of those $\langle \alpha_0, \dots, \alpha_{d-1} \rangle \in \prod_{i<d} J_i$ such that the set $\{z_{\alpha_i} : i < d\} \cup \{q(d)\}$ is in $\mathrm{Ext}_T(A, C)$. Notice that for each $i < d$, $J_i = \{\alpha_i : \vec{\alpha} \in \vec{J}\}$, since each node in Z_i is in some member of $\mathrm{Ext}_T(A, C)$: Extending all the other t_j^* $(j \neq i)$ via their leftmost extensions in T to length l_q, along with $q(d)$, constructs a member of $\mathrm{Ext}_T(A, C)$. By Lemma 5.3, the set $\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is compatible. The fact that $p_{\vec{J}}$ is a condition in $\mathbb{P}$ will be used to make the construction of q very precise.

Let $\vec{\delta}_q = \bigcup\{\vec{\delta}_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$. For each $i < d$ and $\alpha \in J_i$, define $q(i, \alpha) = z_\alpha$. Notice that for each $\vec{\alpha} \in \vec{J}$ and $i < d$,

$$q(i, \alpha_i) \supseteq t_i^* = p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{J}}(i, \alpha_i), \tag{32}$$

and

$$q(d) \supseteq t_d^* = p_{\vec{\alpha}}(d) = p_{\vec{J}}(d). \tag{33}$$

For each $i < d$ and $\gamma \in \vec{\delta}_q \setminus J_i$, there is at least one $\vec{\alpha} \in \vec{J}$ and some $k < k^*$ such that $\delta_{\vec{\alpha}}(k) = \gamma$. Let $q(i,\gamma)$ be the leftmost extension of $p_{\vec{J}}(i,\gamma)$ in T of length l_q. Define

$$q = \{q(d)\} \cup \{\langle (i,\delta), q(i,\delta)\rangle : i < d,\ \delta \in \vec{\delta}_q\}. \tag{34}$$

Claim 6. *For all $\vec{\alpha} \in \vec{J}$, $q \le p_{\vec{\alpha}}$.*

Proof. Given $\vec{\alpha} \in \vec{J}$, it follows from the definition of q that $\vec{\delta}_q \supseteq \vec{\delta}_{\vec{\alpha}}$, $q(d) \supseteq p_{\vec{\alpha}}(d)$, and for each pair $(i,\gamma) \in d \times \vec{\delta}_{\vec{\alpha}}$, $q(i,\gamma) \supseteq p_{\vec{\alpha}}(i,\gamma)$. So it only remains to show that q has no new sets of parallel 1's over $p_{\vec{\alpha}}$. It follows from Claim 5 that $\vec{\delta}_{\vec{\alpha}} \cap \bigcup_{i<d} K_i = \vec{\alpha}$. Hence, for each $i < d$ and $\gamma \in \vec{\delta}_{\vec{\alpha}} \setminus \{\alpha_i\}$, $q(i,\gamma)$ is the leftmost extension of $p_{\vec{\alpha}}(i,\gamma)$. Since $\vec{\alpha}$ is in $\vec{J}$, $\{q(i,\alpha_i) : i < d\} \cup \{q(d)\}$ is in $\mathrm{Ext}_T(A,C)$ by definition of $\vec{J}$. This implies that $\{q(i,\alpha_i) : i < d\} \cup \{q(d)\}$ has no new parallel 1's over A, as this set union A must be strongly similar to C which satisfies the Parallel 1's Criterion, and since the critical node in $C \setminus A$ is a splitting node, $C \setminus A$ has no new parallel 1's over A. It follows that $\{q(i,\delta) : (i,\delta) \in d \times \delta \in \vec{\delta}_{\vec{\alpha}}\} \cup \{q(d)\}$ has no new parallel 1's over $\{p_{\vec{\alpha}}(i,\delta) : (i,\delta) \in d \times \delta \in \vec{\delta}_{\vec{\alpha}}\} \cup \{p_{\vec{\alpha}}(d)\}$. Therefore, $q \le p_{\vec{\alpha}}$. □

Remark 5.4. Notice that we did not prove that $q \le p_{\vec{J}}$. That will be blatantly false for all large enough j, as the union of the sets Z_i, $i < d$, composed from V_j will have many new sets of parallel 1's over $p_{\vec{J}}$. This is one fundamental difference between the forcings being used for this theorem and the forcings adding κ many Cohen reals used in Harrington's proof of the Halpern-Läuchli Theorem.

To construct U_{m_j}, take an $r \le q$ in $\mathbb{P}$ which decides some l_j in $\dot{L}_d$ for which $h(\dot{b}_{\vec{\alpha}} \restriction l_j) = \varepsilon^*$, for all $\vec{\alpha} \in \vec{J}$. This is possible since for all $\vec{\alpha} \in \vec{J}$, $p_{\vec{\alpha}}$ forces $h(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon^*$ for $\dot{\mathcal{U}}$ many $l \in \dot{L}_d$. Without loss of generality, we may assume that the nodes in the image of r have length l_j. Notice that since r forces $\dot{b}_{\vec{\alpha}} \restriction l_j = \{r(i,\alpha_i) : i < d\} \cup \{r(d)\}$ for each $\vec{\alpha} \in \vec{J}$, and since the coloring h is defined in the ground model, it is simply true in the ground model that $h(\{r(i,\alpha_i) : i < d\} \cup \{r(d)\}) = \varepsilon^*$ for each $\vec{\alpha} \in \vec{J}$. Extend the splitting node $q(d)$ in V_j to $r(d)$. For each $i < d$ and $\alpha_i \in J_i$, extend $q(i,\alpha_i)$ to $r(i,\alpha_i)$. Let V_j^- denote $V_j \setminus (\{q(i,\alpha_i) : i < d,\ \alpha_i \in J_i\} \cup \{q(d)\})$. For each node v in V_j^-, let v^* be the leftmost extension of v in $T \restriction l_j$. Let

$$U_{m_j} = U_{m_j - 1} \cup \{r(d)\} \cup \{r(i,\alpha_i) : i < d,\ \alpha_i \in J_i\} \cup \{v^* : v \in V_j^-\}. \tag{35}$$

Claim 7. *$U_{m_j} \in r_{m_j}[U_{m_j-1}, T]$ and every $X \in \mathrm{Ext}_{U_{m_j}}(A,C)$ with $\max(X) \subseteq \max(U_{m_j})$ satisfies $h(X) = \varepsilon^*$.*

Proof. Recall that $U_{m_j-1} \sqsubset Y_j$ are both valid in T. Since $r \le q$, it follows that $\{r(i,\delta) : (i,\delta) \in d \times \vec{\delta}_q\} \cup \{r(d)\}$ has no new sets of parallel 1's over $\{q(i,\delta) : (i,\delta) \in d \times \vec{\delta}_q\} \cup \{q(d)\}$, which is a subset of V_j. All other nodes in $\max(U_{m_j})$ are leftmost extensions of nodes in V_j. Thus, $\max(U_{m_j})$ extends V_j and has no new sets of parallel 1's over V_j, so $U_{m_j} \stackrel{s}{\sim} r_{m_j}(\mathbb{T})$. Further, $\max(U_{m_j})$ has no pre-determined new parallel 1's since $r \in \mathbb{P}$. It follows that $U_{m_j} \in r_{m_j}[U_{m_j-1}, T]$.

For each $X \in \mathrm{Ext}_{U_{m_j}}(A,C)$ with $X \subseteq \max(U_{m_j})$, the truncation $A \cup \{x \restriction l_q : x \in X\}$ is a member of $\mathrm{Ext}_{Y_j}(A,C)$. Thus, there corresponds a sequence $\vec{\alpha} \in \vec{J}$ such that $\{x \restriction l_q : x \in X\} = \{q(i,\alpha_i) : i < d\} \cup \{q(d)\}$. Then $X = \{r(i,\alpha_i) : i < d\} \cup \{r(d)\}$, which has h-color ε^*. □

Let $S = \bigcup_{j<\omega} U_{m_j}$. For each $X \in \mathrm{Ext}_S(A,C)$, there corresponds a $j < \omega$ such that $X \in \mathrm{Ext}_{U_{m_j}}(A,C)$ and $X \subseteq \max(U_{m_j})$. By Claim 10, $h(X) = \varepsilon^*$. Thus, $S \in [B,T]$ and satisfies the theorem. This concludes the proof of the theorem for Case (a).

Part III Case (b). Let $X \in \mathrm{Ext}_T(A,C)$ and m_0 be given such that there is a $B' \in r_{m_0}[B,T]$ with $X \subseteq \max(B')$. Let U_{m_0-1} denote $r_{m_0-1}(B')$. We will build an $S \in [U_{m_0-1}, T]$ such that every member of $\mathrm{Ext}_S(A,C;X)$ has the same h-color. Let $n_{B'}$ be the index such that $c^T_{n_{B'}}$ is the coding node in $\max(B')$. Label the members of X as x_i, $i \le d$, so that each $x_i \supseteq s_i$. For Case (b), back in Part II, when choosing the $p_{\vec{\alpha}}$, $\vec{\alpha} \in [\kappa]^d$, first define

$$p^0_{\vec{\alpha}} = \{\langle (i,\delta), x_i\rangle : i < d,\ \delta \in \vec{\alpha}\} \cup \{\langle d, x_d\rangle\}, \tag{36}$$

so that each node t_i^* will extend x_i, for $i \leq d$. Then choose $p^k_{\vec{\alpha}}$, $1 \leq k \leq 3$, as before, with the additional requirement that $p_{\vec{\alpha}}(d) = c^T_n$ for some $n \geq n_{B'} + 3$. Everything else in Part II remains the same.

We will build $U_{m_0} \in r_{m_0}[U_{m_0-1}, T]$ so that its maximal members extend $\max(B')$, and hence each member of X is extended uniquely in $\max(U_{m_0})$. Let V_0 denote $\max(B')$. Let V_0^l and V_0^r denote those members v of V_0 such that the immediate extension of v is 0 or 1, respectively. For each $v \in V_0^r \setminus X$, v has no parallel 1's with x_d, so the Passing Number Choice Lemma 4.18 guarantees that there is a member v^* extending v to length $l^* := |t_d^*| \geq l^T_{n+3}$ such that v^* has immediate successor 1 in T. For each $v \in V_0^l \setminus X$, take v^* to be the leftmost extension of v of length l^*. Let

$$V^* = \{t_i^* : i \leq d\} \cup \{v^* : v \in V_0 \setminus X\}. \tag{37}$$

Claim 8. *$U_{m_0-1} \cup V^*$ is a member of $r_{m_0}[U_{m_0-1}, T]$.*

Proof. By the construction, V^* extends V_0, and for each $z \in V^*$, the passing number of z at t_d^* is equal to the passing number of $z \restriction l_{B'}$ at c_n^T. Thus, it will follow that $U_{m_0-1} \cup V^* \overset{s}{\sim} B'$ once we prove that $U_{m_0-1} \cup V^*$ satisfies the Parallel 1's Criterion.

Let Y be any subset of V^* for which there is an l such that $y(l) = 1$ for all $y \in Y$. Since for each $\vec{\alpha} \in [K]^d$, $p_{\vec{\alpha}} \leq p^0_{\vec{\alpha}}$, it follows that $\{t_i^* : i \leq d\}$ has no new sets of parallel 1's over X. It follows that if $Y \subseteq \{t_i^* : i \leq d\}$, then the parallel 1's of Y are either witnessed in U_{m_0-1} or else are witnessed by the coding node in X, and hence by t_d^*. In particular, the parallel 1's of Y are witnessed in $U_{m_0-1} \cup V^*$.

If Y contains v^* for some $v \in V_0^l \setminus X$, then there must be an $l' < |x_d|$ where this set of parallel 1's is first witnessed, as v^* is the leftmost extension of v in $T \restriction l^*$ and therefore any coding node of T where v^* has passing number 1 must have length less than $|x_d|$. Since U_{m_0-1} satisfies the Parallel 1's Criterion, the set of parallel 1's in Y is witnessed by a coding node in U_{m_0-1}.

Now suppose that $Y \subseteq \{v^* : v \in V_0^r \setminus X\} \cup \{t_i^* : i \leq d\}$. If $Y \cap \{t_i^* : i \leq d\}$ is contained in $\{t_i^* : i \in I_1\}$, then t_d^* witnesses the parallel 1's in Y. Otherwise, there is some $t_i^* \in Y$ with $i \in I_0$. Note that t_i^* has immediate extension 0 at t_d^*, and so in the interval in T with t_d^*, t_i^* takes the leftmost path; also $t_i^*(|x_d|) = 0$. By the construction in the proof of Lemma 4.18, all v^* for $v \in V_0^r$ extend v leftmost until the interval of T containing the coding node t_d^*. Hence, any parallel 1's between such v^* and t_i^* must occur at a level below $|x_d|$. Thus, the parallel 1's in Y must first appear in U_{m_0-1}, and hence be witnessed by some coding node in U_{m_0-1}.

Therefore, $U_{m_0-1} \cup V^*$ satisfies the Parallel 1's Criterion, and hence $U_{m_0-1} \cup V^* \in r_{m_0}[U_{m_0-1}, T]$. $\square$

Define $U_{m_0} = U_{m_0-1} \cup V^*$. Let $M = \{m_j : j < \omega\}$ enumerate the set of $m \geq m_0$ such that the coding node $c^{\mathbb{T}}_m \supseteq c^{\mathbb{T}}_{m_0}$. By strong similarity of T with $\mathbb{T}$, for any $S \in [U_{m_0}, T]$, the coding node c^S_m will extend t_d^* if and only if $m \in M$. Take any $U_{m_1-1} \in r_{m_1-1}[U_{m_0}, T]$. Notice that $\{t_i^* : i \leq d\}$ is the only member of $\mathrm{Ext}_{U_{m_1-1}}(A, C; X)$, and it has h-color ε^*.

Assume now that $1 \leq j < \omega$ and we have constructed U_{m_j-1} so that every member of $\mathrm{Ext}_{U_{m_j-1}}(A, C; X)$ is colored ε^* by h. Fix some $Y_j \in r_{m_j}[U_{m_j-1}, T]$. Let V_j denote $\max(Y_j)$. The nodes in V_j will not be in the tree S we are constructing; rather, we will construct $U_{m_j} \in r_{m_j}[U_{m_j-1}, T]$ so that $\max(U_{m_j})$ extends V_j. Let $q(d)$ denote the coding node in V_j and let $l_q = |q(d)|$. Recall that for $k \in \{0, 1\}$, I_k denotes the set of $i < d$ for which t_i^* has passing number k at t_d^*. For each $k \in \{0, 1\}$ and each $i \in I_k$, let Z_i be the set of nodes z in $T_i \cap V_j$ such that z has passing number k at $q(d)$.

We now construct a condition q similarly, but not exactly, as in Case (a). For each $i < d$, let J_i be a subset of K_i with the same size as Z_i. For each $i < d$, label the nodes in Z_i as $\{z_\alpha : \alpha \in J_i\}$. Let $\vec{J}$ denote the set of those $\langle \alpha_0, \dots, \alpha_{d-1} \rangle \in \prod_{i<d} J_i$ such that the set $\{z_{\alpha_i} : i < d\} \cup \{q(d)\}$ is in $\mathrm{Ext}_T(A, C)$. Notice that for each $i < d$ and $\vec{\alpha} \in \vec{J}$, $z_{\alpha_i} \supseteq t_i^* = p_{\vec{\alpha}}(i, \alpha_i)$, and $q(d) \supseteq t_d^* = p_{\vec{\alpha}}(d)$. Furthermore, for each $i < d$ and $\delta \in J_i$, there is an $\vec{\alpha} \in \vec{J}$ such that $\alpha_i = \delta$. Let $\vec{\delta}_q = \bigcup \{\vec{\delta}_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$. For each pair $(i, \gamma) \in d \times \vec{\delta}_q$ with $\gamma \in J_i$, define $q(i, \gamma) = z_\gamma$. For each pair $(i, \gamma) \in d \times \vec{\delta}_q$ with $\gamma \in \vec{\delta}_q \setminus J_i$, there is at least one $\vec{\alpha} \in \vec{J}$ and some $k < k^*$ such that $\delta_{\vec{\alpha}}(k) = \gamma$. By Lemma 5.3, $p_{\vec{\beta}}(i, \gamma) = p_{\vec{\alpha}}(i, \gamma) = t^*_{i,k}$, for any $\vec{\beta} \in \vec{J}$ for which $\gamma \in \vec{\delta}_{\vec{\beta}}$. For $i \in I_0$, let $q(i, \gamma)$ be the leftmost extension of $t^*_{i,k}$ in T to length l_q. This will have passing number 0 at $q(d)$, and any parallel 1's between this node and any other nodes in V_j must be witnessed at or below t_d^*. For $i \in I_1$, let $q(i, \gamma)$ be the extension of $t^*_{i,k}$ as in Lemma 4.18: extend $t^*_{i,k}$ leftmost in T until the interval of T containing $q(d)$; in that interval, extend to the next splitting node and take the right branch of length l_q.

Let this node be $q(i,\gamma)$. This has passing number 1 at $q(d)$, and any parallel 1's between $q(i,\gamma)$ and another node must be either witnessed by $q(d)$ or else at or below t_d^*. Define

$$q = \{q(d)\} \cup \{\langle (i,\delta), q(i,\delta)\rangle : i < d,\ \delta \in \vec{\delta}_q\}. \tag{38}$$

By the construction, q is a member of $\mathbb{P}$.

Claim 9. *For each $\vec{\alpha} \in \vec{J}$, $q \le p_{\vec{\alpha}}$.*

Proof. Let n denote the index such that $c_n^T = q(d)$. It suffices to show that for each $\vec{\alpha} \in \vec{J}$, q has no new sets of parallel 1's over $p_{\vec{\alpha}}$, since by construction, we have that $q(i,\delta) \supseteq p_{\vec{\alpha}}(i,\delta)$ for all $(i,\delta) \in d \times \vec{\delta}_{\vec{\alpha}}$.

Let $\vec{\alpha} \in \vec{J}$ be given, and let Y be any subset of $\{q(i,\delta) : (i,\delta) \in d \times \vec{\delta}_{\vec{\alpha}}\}$ of size at least 2 for which for some l, $y(l) = 1$ for all $y \in Y$. If $Y \subseteq \{q(i,\alpha_i) : i < d\} \cup \{q(d)\}$, then Y has no new parallel 1's over X, since $\vec{\alpha} \in \vec{J}$ implies that $\{q(i,\alpha_i) : i < d\} \cup \{q(d)\}$ is in $\mathrm{Ext}_T(A,C;X)$. Since $\{p(i,\alpha_i) : i < d\} \cup \{p(d)\}$ extends X and Y consists of extensions of members of $\{p(i,\alpha_i) : i < d\} \cup \{p(d)\}$, it follows that Y has no new parallel 1's over $\{p(i,\alpha_i) : i < d\} \cup \{p(d)\}$.

Now suppose Y contains some $q(i,\delta)$, where $\delta \in \vec{\delta}_{\vec{\alpha}} \setminus \{\alpha_i\}$. Recall that by Claim 5, $\vec{\delta}_{\vec{\alpha}} \cap (\bigcup_{i<d} K_i) = \vec{\alpha}$; so in particular, $\delta \notin \bigcup_{i<d} J_i$. By construction of q, if $i \in I_0$, then $q(i,\delta)$ has no new parallel 1's above l^* with any other $q(j,\gamma)$, $(j,\gamma) \in d \times \vec{\delta}_{\vec{\alpha}}$. If $i \in I_1$, it follows from the construction of q that any parallel 1's $q(i,\delta)$ has with another member of $\mathrm{ran}(q)$ below l_{n-1}^T is witnessed below l^*. Further, any parallel 1's $q(i,\delta)$ has in the interval $(l_{n-1}^T, l_n^T]$ are witnessed by the coding node $q(d)$. Thus, any new sets of parallel 1's in Y occurring above length l^* must be witnessed by $q(d)$. Therefore, q has no new parallel 1's over $p_{\vec{\alpha}}$, and hence, $q \le p_{\vec{\alpha}}$. $\square$

To construct U_{m_j}, we will extend each node in V_j uniquely in such a manner so that these extensions along with U_{m_j-1} form a member of $r_{m_j}[U_{m_j-1}, T]$. It suffices to find some V^* extending V_j such that the coding node in V^* extends the coding node in V_j, the passing number of each v^* in V^* extending some v in V_j is the same as the passing number of v in V_j, and no new sets of parallel 1's occur in V^* over V_j. Then $U_{m_j-1} \cup V^*$ will be strongly similar to $r_{m_j}(\mathbb{T})$ and hence a member of $r_{m_j}[U_{m_j-1}, T]$.

Take an $r \le q$ in $\mathbb{P}$ which decides some l_j in $\dot{L}_d$ such that $h(\dot{b}_{\vec{\alpha}} \restriction l_j) = \varepsilon^*$ for all $\vec{\alpha} \in \vec{J}$, and such that there are at least two coding nodes in T of lengths between l_q and l_r. Without loss of generality, we may assume that the nodes in the image of r have length l_j. Extend the coding node $q(d)$ in V_j to $r(d)$. For each $i < d$ and $\delta \in J_i$, extend $q(i,\delta)$ to $r(i,\delta)$. Let V_j^l and V_j^r denote the set of those $v \in V_j$ with passing number 0 and 1, respectively, at $q(d)$. Extend these nodes according to the construction of Lemma 4.18 as follows: For each node v in $V_j^l \setminus (\{q(i,\delta) : i < d,\, \delta \in J_i\} \cup \{q(d)\})$, let v^* be the leftmost extension of v in $T \restriction l_j$. For each node v in $V_j^r \setminus (\{q(i,\delta) : i < d,\, \delta \in J_i\} \cup \{q(d)\})$, extend v leftmost to v' of length $l_{n(r)-1}^T$, and then let v^* be the right extension of $\mathrm{splitpred}_T(v')$ to length l_r, where $n(r)$ is the index such that $c_{n(r)}^T = r(d)$. Then each member of V_j^l has passing number 0 at $r(d)$ and each member of V_j^r has passing number 1 at $r(d)$. Let V_j^- denote $V_j \setminus (\{q(i,\delta) : i < d,\, \delta \in J_i\} \cup \{q(d)\})$, and define

$$V^* = \{r(d)\} \cup \{r(i,\alpha_i) : i < d,\ \alpha_i \in J_i\} \cup \{v^* : v \in V_j^-\} \tag{39}$$

and

$$U_{m_j} = U_{m_j-1} \cup V^*. \tag{40}$$

Claim 10. *U_{m_j} is a member of $r_{m_j}[U_{m_j-1}, T]$, and $h(Y) = \varepsilon^*$ for each $Y \in \mathrm{Ext}_{U_{m_j}}(A,C;X)$.*

Proof. By the construction of V^*, for each $v \in V_j$, its extension v^* in V^* has the same passing number at $r(d)$ as v does at $q(d)$. Since $r \le q$, all parallel 1's in $\{r(i,\delta) : i < d,\ \delta \in J_i\} \cup \{r(d)\}$ are already witnessed in V_j. Each v in $V_j^l \setminus (\{q(i,\delta) : i < d,\, \delta \in J_i\} \cup \{q(d)\})$ has extension v^* which has no new parallel 1's with any other member of V^* above l_q. Any set $Y \subseteq V_j^r \cup \{q(i,\delta) : i < d,\ \delta \in J_i\} \cup \{q(d)\}$ cannot have new parallel 1's in the interval $(l^*, l_{n(r)-1}]$, since for each $v \in V_j^r \setminus (\{q(i,\delta) : i < d,\, \delta \in J_i\} \cup \{q(d)\})$, $v^* \restriction l_{n(r)-1}$ is the leftmost extension of v in T of length $l_{n(r)-1}$. In the interval $(l^*, l_{n(r)-1}]$, Lemma 4.14 implies the only new sets of of parallel 1's in Y must be witnessed by $r(d)$.

Thus, any sets of parallel 1's among V^* are already witnessed in V_j. Therefore, $U_{m_j-1} \cup V^*$ satisfies the Parallel 1's Criterion and is strongly similar to Y_j, and hence is in $r_{m_j}[U_{m_j-1}, T]$.

Now suppose $Z \subseteq V^*$ is a member of $\mathrm{Ext}_{U_{m_j}}(A, C; X)$. Then $Z \restriction l_q$ is in $\mathrm{Ext}_T(A, C; X)$, so Z extends $\{q(i, \alpha_i) : i < d\} \cup \{q(d)\}$ for some $\vec{\alpha} \in \vec{J}$. Thus, $Z = \{r(i, \alpha_i) : i < d\} \cup \{r(d)\}$ for that $\vec{\alpha}$, and r forces that $h(Z) = \varepsilon^*$. Since h and Z are finite, they are in the ground model, so $h(Z)$ simply equals ε^*. □

To finish the proof of the theorem for Case (b), Define $S = \bigcup_{j<\omega} U_{m_j}$. Then $S \in [B', T]$, and for each $Z \in \mathrm{Ext}_S(A, C; X)$, there is a $j < \omega$ such that $Z \in \mathrm{Ext}_{U_{m_j}}(A, C)$ and each member of $\max(U_{m_j})$ extending X has h-color ε^*.

This concludes the proof of the theorem. □

6. Ramsey Theorem for finite trees satisfying the Strict Parallel 1's Criterion

Our first Ramsey theorem for colorings of finite subtrees of a strong coding tree appears in this section. Theorem 6.3, proves that for any finite coloring of the copies of a given finite tree satisfying the Strict Parallel 1's Criterion (Definition 6.1) in a strong coding tree T, there is a strong coding tree $S \le T$ in which all strictly similar (Definition 6.2) copies have the same color.

Let A be a subtree of a strong coding tree T. Given $l < \omega$, define

$$A_{l,1} = \{t \restriction (l+1) : t \in A,\ |t| \ge l+1, \text{ and } t(l) = 1\}. \tag{41}$$

We say that l is a *minimal level of a new set of parallel* 1*'s in* A if the set $A_{l,1}$ has at least two distinct members, and for each $l' < l$, the set $\{s \in A_{l,1} : s(l') = 1\}$ has cardinality strictly smaller than $|A_{l,1}|$.

Definition 6.1 (Strict Parallel 1's Criterion)**.** A subtree A of a strong coding tree satisfies the *Strict Parallel* 1*'s Criterion* if A satisfies the Parallel 1's Criterion and additionally, the following hold: For each l which is the minimal length of a set of new parallel 1's in A,

1. The critical node in A with minimal length greater than or equal to l is a coding node in A, say c;
2. There are no terminal nodes in A in the interval $[l, |c|)$ (c can be terminal in A);
3. $A_{l,1} = \{t \restriction (l+1) : t \in A_{|c|,1}\}$.

Thus a tree A satisfies the Strict Parallel 1's Criterion if it satisfies the Parallel 1's Criterion and moreover, each new set of parallel 1's in A is witnessed by a coding node in A before any other new set of parallel 1's, critical node, or terminal node in A appears.

Definition 6.2 (Strictly Similar)**.** Given A, B subtrees of a strong coding tree, we say that A and B are *strictly similar* if A and B are strongly similar and both satisfy the Strict Parallel 1's Criterion.

Theorem 6.3. *Let T be a strong coding tree and let A be a finite subtree of T satisfying the Strict Parallel 1's Criterion. Then for any coloring of all strictly similar copies of A in T into finitely many colors, there is a strong coding subtree $S \le T$ such that all strictly similar copies of A in S have the same color.*

Theorem 6.3 will be proved via four lemmas and then doing an induction argument. Recall that Case (b) of Theorem 5.2 only showed that, when $C \setminus A$ contains a coding node and $X \in \mathrm{Ext}_T(A, C)$, there is some $S \le T$ which is homogeneous for all members of $\mathrm{Ext}_S(A, C; X)$. This is weaker than the direct analogue of the statement proved for Case (a) in Theorem 5.2, and this disparity is addressed by the following. Lemma 6.7 will build a fusion sequence to obtain an $S \le T$ which is end-homogeneous on $\mathrm{Ext}_S(A, C)$, using Case (b) of Theorem 5.2. Lemma 6.8 will use a new forcing and many arguments from the proof of Theorem 5.2 obtain an analogue of Case (a) when $C \setminus A$ contains a coding node. The only difference is that this analogue holds for $\mathrm{Ext}^{SP}_S(A, C)$, rather than $\mathrm{Ext}_S(A, C)$, which is why Theorem 6.3 requires the Strict Parallel 1's Criterion. The last two lemmas involve fusion to construct subtrees which have one color on $\mathrm{Ext}^{SP}_S(A', C)$, for each A' strictly similar to A, for the two cases: $C \setminus A$ contains a coding node, and $C \setminus A$ contains a splitting node. The theorem then follows by induction and an application of Ramsey's Theorem.

The following basic assumption, similar to Case (b) of Theorem 5.2, will be used in much of this section.

Assumption 6.4. Let $A \subseteq C$ be fixed non-empty finite subtrees of a strong coding tree T such that A and C satisfy the Strict Parallel 1's Criterion. Let A_e be a subset of A^+, and assume that A_e and $C \setminus A$ are level sets, and that $C \setminus A$ extends A_e, contains a coding node, and contains the sequence 0^{l_C}. Let $d+1 = |A_e|$ and list the nodes of A_e as $\langle s_i : i \le d\rangle$, and the nodes of $C \setminus A$ as $\langle t_i : i \le d\rangle$ so that each t_i extends s_i and t_d is the coding node in $C \setminus A$. For $k \in \{0, 1\}$, let I_k denote the set of $i \le d$ such that the immediate extension of t_i in T is k. Since $C \setminus A$ contains a coding node, the immediate successors of the t_i are well-defined in T.

As usual, when we talk about the parallel 1's of $C \setminus A$, we are taking into account the passing numbers of the members of $(C \setminus A)^+$ at the coding node t_d. Recall that values of the immediate successors of the t_i, $i \le d$, are considered when determining whether or not a level set Y is in $\mathrm{Ext}_T(A, C)$, this being defined as in Case (b) of the previous section. We hold to the convention that for $Y \in \mathrm{Ext}_T(A, C)$, the nodes in Y are labeled y_i, $i \le d$, where $y_i \supseteq s_i$ for each i. In particular, y_d is the coding node in Y. Define

$$\mathrm{Ext}_T^{SP}(A, C) = \{Y \in \mathrm{Ext}_T(A, C) : A \cup Y \text{ satisfies the Strict Parallel 1's Criterion}\}. \tag{42}$$

Recall the definition of $\mathrm{splitpred}_T(x)$ from Subsection 4.1. We point out that if the parallel 1's in $C \setminus A$ are already witnessed in A, then $\mathrm{Ext}_T^{SP}(A, C)$ is equal to $\mathrm{Ext}_T(A, C)$. If there are parallel 1's in $C \setminus A$ not witnessed in A, then $Y \in \mathrm{Ext}_T^{SP}(A, C)$ if and only if $Y \in \mathrm{Ext}_T(A, C)$ and additionally for the minimal l such that $\{i < d : y_i(l) = 1\} = I_1$, $A \cup \{\mathrm{splitpred}_T(y_i \restriction l) : i \in I_1\} \cup \{y_i \restriction l : i \in I_0\}$ satisfies the Parallel 1's Criterion. Now we define the notion of minimal pre-extension of A to a copy of C. This will be used in the next lemma to obtain a strong form of end-homogeneity for the case when $\max(C)$ has a coding node.

Definition 6.5 (Minimal pre-extension of A to a copy of C)**.** Let $X = \{x_i : i \le d\}$ be any level set extending A_e such that $x_i \supseteq s_i$ for each $i \le d$ and such that the length l of the nodes in X is the length of some coding node in T. We say that X is a *minimal pre-extension in T of A to a copy of C* if

(i) $\{i \le d : x_i^+(l) = 1\} = I_1$, where x_i^+ denotes the immediate extension of x_i in $\widehat{T}$; and
(ii) $A \cup \{\mathrm{splitpred}_T(x_i) : i \in I_1\} \cup \{x_i : i \in I_0\}$ satisfies the Parallel 1's Criterion.

We will simply call such an X a *minimal pre-extension* when T, A, and C are clear. Minimal pre-extensions are exactly the level sets in T which can be extended to a member of $\mathrm{Ext}_T^{SP}(A, C)$. For X any minimal pre-extension, define $\mathrm{Ext}_T(A, C; X)$ to be the set of all $Y \in \mathrm{Ext}_T(A, C)$ such that Y extends X. Then

$$\mathrm{Ext}_T^{SP}(A, C) = \bigcup \{\mathrm{Ext}_T(A, C; X) : X \text{ is a minimal pre}-\text{extension}\}, \tag{43}$$

Definition 6.6. A coloring on $\mathrm{Ext}_T^{SP}(A, C)$ is *end-homogeneous* if for each minimal pre-extension X of A to a copy of C, every member of $\mathrm{Ext}_T(A, C; X)$ has the same color.

Lemma 6.7 (End-homogeneity)**.** *Assume 6.4, and let k be minimal such that* $\max(A) \subseteq r_k(T)$*. Then for any coloring h of* $\mathrm{Ext}_T(A, C)$ *into two colors, there is a $T' \in [r_k(T), T]$ such that h is end-homogeneous on* $\mathrm{Ext}_{T'}^{SP}(A, C)$*.*

Proof. Let $(n_i)_{i<\omega}$ enumerate those integers greater than k such that there is a minimal pre-extension of A to a copy of C from among the maximal nodes in $r_{n_i}(T)$. Each of these $r_{n_i}(T)$ contains a coding node in its maximal level, though there may be minimal pre-extensions contained in $\max(r_{n_i}(T))$ not containing that coding node.

Let T_{-1} denote T. Suppose that $j < \omega$ and T_{j-1} are given so that the coloring h is homogeneous on $\mathrm{Ext}_{T_{j-1}}(A, C; X)$ for each minimal pre-extension X in $r_{n_j-1}(T_{j-1})$. Let U_{j-1} denote $r_{n_j-1}(T_{j-1})$. Enumerate the collection of all minimal pre-extensions of A to C from among $\max(r_{n_j}(T_{j-1}))$ as $X_0, \dots, X_q$. We will do an inductive argument over $p \le q$ to obtain a $T_j \in [U_{j-1}, T_{j-1}]$ such that $\max(r_{n_j}(T_j))$ extends $\max(r_{n_j}(T_{j-1}))$ and $\mathrm{Ext}_{T_j}(A, C; Z)$ is homogeneous for each minimal pre-extension Z in $\max(r_{n_j}(T_{j-1}))$.

Suppose $p \le q$ and for all $i < p$, there are strong coding trees S_i such that $S_0 \in [U_{j-1}, T_{j-1}]$, and for all $i' < i < p$, $S_i \in [U_{j-1}, S_{i'}]$ and h is homogeneous on $\mathrm{Ext}_{S_i}(A, C; X_i)$. Let l denote the length of the nodes in $\max(r_{n_j}(T_{j-1}))$. Note that X_p is contained in $r_{n_j}(S_{p-1}) \restriction l$, though l does not have to be the length of any node in S_{p-1}. The point is that the set of nodes Y_p in $\max(r_{n_j}(S_{p-1}))$ extending X_p is again a minimal pre-extension. Extend the nodes in Y_p to some $Z_p \in \mathrm{Ext}_{S_{p-1}}(A, C; Y_p)$, and let l' denote the length of the nodes in Z_p. Note that Z_p has no new sets of parallel 1's over $A \cup Y_p$. Let W_p consist of the nodes in Z_p along with the leftmost extensions of the nodes in $\max(r_{n_j}(S_{p-1})) \setminus Y_p$ to the length l' in S_{p-1}.

Let S'_{p-1} be a strong coding tree in $[U_{j-1}, S_{p-1}]$ such that $\max(r_{n_j}(S'_{p-1}))$ extends W_p. Such an S'_{p-1} exists by Lemma 4.19, since W_p has exactly the same set of new parallel 1's over $r_{n_{j-1}}(S_{p-1})$ as does $\max(r_{n_j}(S_{p-1}))$. Apply Case (b) of Theorem 5.2 to obtain a strong coding tree $S_p \in [U_{j-1}, S'_{p-1}]$ such that the coloring on $\mathrm{Ext}_{S_p}(A, C; Z_p)$ is homogeneous. At the end of this process, let $T_j = S_q$. Note that for each minimal pre-extension $Z \subseteq \max(r_{n_j}(T_j))$, there is a unique $p \le q$ such that Z extends X_p, since each node in $\max(r_{n_j}(T_j))$ is a unique extension of one node in $\max(r_{n_j}(T_{j-1}))$, and hence $\mathrm{Ext}_{T_j}(A, C; Z)$ is homogeneous.

Having chosen each T_j as above, let $T' = \bigcup_{j<\omega} r_{n_j}(T_j)$. Then T' is a strong coding tree which is in $[r_k(T), T]$, and for each minimal pre-extension Z in T', $\mathrm{Ext}_{T'}(A, C; Z)$ is homogeneous for h. Therefore, h is end-homogeneous on $\mathrm{Ext}^{SP}_{T'}(A, C)$. $\square$

The next lemma provides a means for uniformizing the end-homogeneity from the previous lemma to obtain one color for all members of $\mathrm{Ext}^{SP}_{S}(A, C)$. This will yield almost the full analogue of Case (a) of Theorem 5.2 for Case (b), when the level sets being colored contain a coding node, the difference being the restriction to strictly similar extensions rather than just strongly similar extensions. The arguments are often similar to those of Case (a) of Theorem 5.2, but sufficiently different to warrant a proof.

Lemma 6.8. *Assume 6.4, and suppose that B is a finite strong coding tree valid in T and A is a subtree of B such that $\max(A) \subseteq \max(B)$. Suppose that h is end-homogeneous on $\mathrm{Ext}^{SP}_{T}(A, C)$. Then there is an $S \in [B, T]$ such that h is homogeneous on $\mathrm{Ext}^{SP}_{S}(A, C)$.*

Proof. Given any $U \in [B, T]$, let $\mathrm{MPE}_U(A, C)$ denote the set of all minimal pre-extensions of A to a copy of C in U. Without loss of generality, we may assume that the nodes in $C \setminus A$ occur in an interval of T strictly above the interval of T containing B. This presents no obstacle to the application, as the goal is to find some $S \in [B, T]$ for which h takes the same value on every extension in $\mathrm{Ext}_U(A, C)$ extending some member of $\mathrm{MPE}_S(A, C)$, and we can take the first level of S above B to be in the interval of T strictly above B since B is valid in T.

Enumerate the nodes of A_e as $\{s_i : i \le d\}$, letting i_0 be the index such that s_{i_0} is a sequence of all 0's. In the notation of Assumption 6.4, i_0 is a member of I_0. Each member Y of $\mathrm{MPE}_T(A, C)$ will be enumerated as $\{y_i : i \le d\}$ so that $y_i \supseteq s_i$ for each $i \le d$. Given $Y \in \mathrm{MPE}_T(A, C)$, define the notation

$$\mathrm{splitpred}_T(Y) = \{y_i : i \in I_0\} \cup \{\mathrm{splitpred}_T(y_i) : i \in I_1\}. \tag{44}$$

Since C satisfies the Strict Parallel 1's Criterion, $C \setminus A$ is in $\mathrm{MPE}_T(A, C)$. Let C^- denote $\mathrm{splitpred}_T(C \setminus A)$. Since we are assuming that $C \setminus A$ is contained in an interval of T above the interval containing $\max(A)$, each node of C^- extends one node of A_e. For any $U \in [B, T]$, define $X \in \mathrm{Ext}_U(A, C^-)$ if and only if $X = \mathrm{splitpred}_U(Y)$ for some $Y \in \mathrm{MPE}_U(A, C)$. Equivalently, $X \in \mathrm{Ext}_U(A, C^-)$ if and only if the following three conditions hold:

1. X extends A_e; label the nodes in X as $\{x_i : i \le d\}$ so that $x_i \supseteq s_i$.
2. There is a coding node c in U such that $|c| = |x_{i_0}|$; for each $i \in I_0$, the passing number of x_i at c is 0; and for each $i \in I_1$, $x_i = \mathrm{splitpred}_U(y_i)$ for some $y_i \supseteq s_i$ in U of length $|c|$ such that the passing number of y_i at c is 1.
3. The set $A \cup X$ satisfies the Parallel 1's Criterion.

Thus, X is a member of $\mathrm{Ext}_U(A, C^-)$ if and only if $\{x_i : i \in I_0\}$ along with the rightmost paths extending $\{x_i : i \in I_1\}$ to length $|x_{i_0}|$ forms a minimal pre-extension of A to a copy of C in U. Note that condition (3) implies that X has no new sets of parallel 1's over A, since X contains no coding node.

By assumption, the coloring h on $\mathrm{Ext}^{SP}_{T}(A, C)$ is end-homogeneous. Thus, it induces a coloring on $\mathrm{MPE}_T(A, C)$, by giving $Y \in \mathrm{MPE}_T(A, C)$ the h-color that all members of $\mathrm{Ext}_T(A, C; Y)$ have. This further induces a coloring h' on $\mathrm{Ext}_T(A, C^-)$, since a set of nodes X in T is in $\mathrm{Ext}_T(A, C^-)$ if and only if $X = \mathrm{splitpred}_T(Y)$ for some $Y \in \mathrm{MPE}_T(A, C)$. Define $h'(\mathrm{splitpred}_T(Y))$ to be the color of h on $\mathrm{Ext}_T(A, C; Y)$.

Let L denote the collection of all $l < \omega$ such that there is a member of $\mathrm{Ext}_T(A, C^-)$ with maximal nodes of length l. For each $i \in (d+1) \setminus \{i_0\}$, let $T_i = \{t \in T : t \supseteq s_i\}$. Let $T_{i_0} = \{t \in T \cap 0^{<\omega} : t \supseteq s_{i_0}\}$, the collection of all leftmost nodes in T extending s_{i_0}. Let $\kappa = \beth_{2d+2}$. The following forcing notion $\mathbb{Q}$ will add κ many paths through each T_i, $i \in (d+1) \setminus \{i_0\}$ and one path through T_{i_0}. The present case is handled similarly to Case (a) of Theorem 5.2, so much of the current proof refers back to the proof of Theorem 5.2.

We now define a new forcing. Let $\mathbb{Q}$ be the set of conditions p such that p is a function of the form

$$p : (d+1) \times \vec{\delta}_p \to T,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$, $l_p \in L$, and there is some some coding node $c^T_{n(p)}$ in T such that $l^T_{n(p)} = l_p$, and

(i) For each $(i, \delta) \in (d+1) \times \vec{\delta}_p$, $p(i, \delta) \in T_i$ and $l^T_{n(p)-1} < |p(i, \delta)| \le l_p$; and

(ii) (α) If $i \in I_1$, then $p(i, \delta) = \mathrm{splitpred}_T(y)$ for some $y \in T_i \restriction l_p$ which has immediate extension 1 in T.

(β) If $i \in I_0$, then $p(i,\delta) \in T_i \restriction l_p$ and has immediate extension 0 in T.

It follows from the definition that for $p \in \mathbb{Q}$, the range of p, $\mathrm{ran}(p) := \{p(i,\delta) : (i,\delta) \in (d+1) \times \vec{\delta}_p\}$, has no pre-determined new sets of parallel 1's. Furthermore, all nodes in $\mathrm{ran}(p)$ are contained in the $n(p)$-th interval of T. We point out that $\mathrm{ran}(p)$ may or may not contain a coding node. If it does, then that coding node must appear as $p(i,\delta)$ for some $i \in I_0$.

The partial ordering on $\mathbb{Q}$ is defined as follows: $q \le p$ if and only if $l_q \ge l_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$,

(i) $q(i,\delta) \supseteq p(i,\delta)$ for each $(i,\delta) \in (d+1) \times \vec{\delta}_p$; and

(ii) $\{q(i,\delta) : (i,\delta) \in (d+1) \times \vec{\delta}_p\}$ has no new sets of parallel 1's over $\mathrm{ran}(p)$.

It is routine to show that Claims 1 and 2 in the proof of Theorem 5.2 also hold for $(\mathbb{Q}, \le)$. That is, $(\mathbb{Q}, \le)$ is an atomless partial order, and any condition in $\mathbb{Q}$ can be extended by two incompatible conditions of length greater than any given $l < \omega$.

Let $\dot{\mathcal{U}}$ be a $\mathbb{Q}$-name for a non-principal ultrafilter on L. For each $i \le d$ and $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ be a $\mathbb{Q}$-name for the α-th generic branch through T_i; that is, $\dot{b}_{i,\alpha} = \{\langle p(i,\alpha), p\rangle : p \in \mathbb{Q} \text{ and } \alpha \in \vec{\delta}_p\}$. For any condition $p \in \mathbb{Q}$, for $(i,\alpha) \in I_0 \times \vec{\delta}_p$, p forces that $\dot{b}_{i,\alpha} \restriction l_p = p(i,\alpha)$. For $(i,\alpha) \in I_1 \times \vec{\delta}_p$, p forces that $\mathrm{splitpred}_T(\dot{b}_{i,\alpha} \restriction l_p) = p(i,\alpha)$. For $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_d\rangle \in [\kappa]^{d+1}$,

$$\text{let } \dot{b}_{\vec{\alpha}} \text{ denote } \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d,\alpha_d}\rangle. \tag{45}$$

For $l \in L$, we shall use the abbreviation

$$\dot{b}_{\vec{\alpha}} \restriction l \text{ to denote } \mathrm{splitpred}_T(\dot{b}_{\vec{\alpha}} \restriction l), \tag{46}$$

which is exactly $\{\dot{b}_{i,\alpha_i} \restriction l : i \in I_0\} \cup \{\mathrm{splitpred}_T(\dot{b}_{i,\alpha_i} \restriction l) : i \in I_1\}$.

Similarly to Part II of the proof of Theorem 5.2, we will find infinite pairwise disjoint sets $K_i \subseteq \kappa$, $i \le d$, such that $K_0 < K_1 < \dots K_d$, and conditions $p_{\vec{\alpha}}$, $\vec{\alpha} \in \prod_{i \le d} K_i$, such that these conditions are pairwise compatible, have the same images in T, and force the same color ε^* for $h'(\dot{b}_{\vec{\alpha}} \restriction l)$ for $\dot{\mathcal{U}}$ many levels l in L. Moreover, the nodes $\{t_i^* : i \le d\}$ obtained from the application of the Erdős-Rado Theorem for this setting will extend $\{s_i : i \le d\}$ and form a member of $\mathrm{Ext}_T(A, C^-)$. The arguments are mostly similar to those in Part II of Theorem 5.2, so we only fill in the details for arguments which are necessarily different.

Part II. For each $\vec{\alpha} \in [\kappa]^{d+1}$, choose a condition $p_{\vec{\alpha}} \in \mathbb{Q}$ such that

(1) $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$.
(2) $\{p_{\vec{\alpha}}(i,\alpha_i) : i \le d\} \in \mathrm{Ext}_T(A, C^-)$.
(3) $p_{\vec{\alpha}} \Vdash$ "There is an $\varepsilon \in 2$ such that $h(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon$ for $\dot{\mathcal{U}}$ many l in $\dot{L}_d$."
(4) $p_{\vec{\alpha}}$ decides a value for ε, call it $\varepsilon_{\vec{\alpha}}$.
(5) $h(\{p_{\vec{\alpha}}(i,\alpha_i) : i \le d\}) = \varepsilon_{\vec{\alpha}}$.

Properties (1) - (5) can be guaranteed as follows. For each $i \le d$, let t_i denote the member of C^- which extends s_i. For each $\vec{\alpha} \in [\kappa]^{d+1}$, let

$$p^0_{\vec{\alpha}} = \{\langle (i,\delta), t_i\rangle : i \le d,\ \delta \in \vec{\alpha}\}.$$

Then $p^0_{\vec{\alpha}}$ is a condition in $\mathbb{P}$ and $\vec{\delta}_{p^0_{\vec{\alpha}}} = \vec{\alpha}$, so (1) holds. Further, $\mathrm{ran}(p^0_{\vec{\alpha}})$ is a member of $\mathrm{Ext}_T(A, C^-)$ since it is exactly C^-. Note that for any $p \le p^0_{\vec{\alpha}}$, $\{p(i,\alpha_i) : i \le d\}$ is also a member of $\mathrm{Ext}_T(A, C^-)$, so (2) holds for any $p \le p^0_{\vec{\alpha}}$. Take an extension $p^1_{\vec{\alpha}} \le p^0_{\vec{\alpha}}$ which forces $h'(\dot{b}_{\vec{\alpha}} \restriction l)$ to be the same value for $\dot{\mathcal{U}}$ many $l \in \dot{L}_d$, and then take $p^2_{\vec{\alpha}} \le p^1_{\vec{\alpha}}$ deciding a value $\varepsilon_{\vec{\alpha}}$ for which $p^2_{\vec{\alpha}}$ forces that $h'(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many l in $\dot{L}_d$. This satisfies (3) and (4). Take $p_{\vec{\alpha}} \le p^2_{\vec{\alpha}}$ which decides $h'(\dot{b}_{\vec{\alpha}} \restriction l_{p_{\vec{\alpha}}}) = \varepsilon_{\vec{\alpha}}$. Then $p_{\vec{\alpha}}$ satisfies (1) through (5), since $p_{\vec{\alpha}}$ forces $h'(\{p_{\vec{\alpha}}(i,\alpha_i) : i \le d\}) = \varepsilon_{\vec{\alpha}}$.

We are assuming $\kappa = \beth_{2d+2}$. Let $D_e = \{0, 2, \dots, 2d\}$ and $D_o = \{1, 3, \dots, 2d+1\}$, the sets of even and odd integers less than $2d+2$, respectively. Let $\mathcal{I}$ denote the collection of all functions $\iota : (2d+2) \to (2d+2)$ such that $\iota \restriction D_e$ and $\iota \restriction D_o$ are strictly increasing sequences and $\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \dots < \{\iota(2d), \iota(2d+1)\}$. For $\vec{\theta} \in [\kappa]^{2d+2}$, $\iota(\vec{\theta})$ determines the pair of sequences of ordinals $(\theta_{\iota(0)}, \theta_{\iota(2)}, \dots, \theta_{\iota(2d)})), (\theta_{\iota(1)}, \theta_{\iota(3)}, \dots, \theta_{\iota(2d+1)})$, both of which are members of $[\kappa]^{d+1}$. Denote these as $\iota_e(\vec{\theta})$ and $\iota_o(\vec{\theta})$, respectively. Let $\vec{\delta}_{\vec{\alpha}}$ denote $\vec{\delta}_{p_{\vec{\alpha}}}$, $k_{\vec{\alpha}}$ denote $|\vec{\delta}_{\vec{\alpha}}|$, and let $l_{\vec{\alpha}}$ denote $l_{p_{\vec{\alpha}}}$. Let $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}}\rangle$ denote the enumeration of $\vec{\delta}_{\vec{\alpha}}$ in increasing order.

Define a coloring f on $[\kappa]^{2d+2}$ into countably many colors as follows: Given $\vec{\theta} \in [\kappa]^{2d+2}$ and $\iota \in \mathcal{I}$, to reduce the number of subscripts, letting $\vec{\alpha}$ denote $\iota_e(\vec{\theta})$ and $\vec{\beta}$ denote $\iota_o(\vec{\theta})$, define

$$\begin{aligned} f(\iota, \vec{\theta}) = \ & \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, \langle \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i \leq d \rangle, \\ & \langle \langle i, j \rangle : i \leq d, \ j < k_{\vec{\alpha}}, \text{ and } \delta_{\vec{\alpha}}(j) = \alpha_i \rangle, \langle \langle j, k \rangle : j < k_{\vec{\alpha}}, \ k < k_{\vec{\beta}}, \ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \rangle \rangle. \end{aligned} \tag{47}$$

Let $f(\vec{\theta})$ be the sequence $\langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I} \rangle$, where $\mathcal{I}$ is given some fixed ordering. By the Erdős-Rado Theorem, there is a subset $K \subseteq \kappa$ of cardinality $\aleph_1$ which is homogeneous for f. Take $K' \subseteq K$ such that between each two members of K' there is a member of K and $\min(K') > \min(K)$. Then take subsets $K_i \subseteq K'$ such that $K_0 < \cdots < K_d$ and each $|K_i| = \aleph_0$. The following three claims and lemma are direct analogues of Claims 3, 4, and 5, and Lemma 5.3. Their proofs follow by simply making the correct notational substitutions, and so are omitted.

Claim 11. *There are $\varepsilon^* \in 2$, $k^* \in \omega$, and $\langle t_{i,j} : j < k^* \rangle$, $i \leq d$, such that for all $\vec{\alpha} \in \prod_{i \leq d} K_i$ and each $i \leq d$, $\varepsilon_{\vec{\alpha}} = \varepsilon^*$, $k_{\vec{\alpha}} = k^*$, and $\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle = \langle t_{i,j} : j < k^* \rangle$.*

Let $l^* = |t_{i_0}|$. Then for each $i \in I_0$, the nodes $t_{i,j}$, $j < k^*$, have length l^*; and for each $i \in I_1$, the nodes $t_{i,j}$, $j < k^*$, have length in the interval (l^T_{n-1}, l^T_n), where n is the index of the coding node in T of length l^*.

Claim 12. *Given any $\vec{\alpha}, \vec{\beta} \in \prod_{i \leq d} K_i$, if $j, k < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)$, then $j = k$.*

For any $\vec{\alpha} \in \prod_{i \leq d} K_i$ and any $\iota \in \mathcal{I}$, there is a $\vec{\theta} \in [K]^{2d+2}$ such that $\vec{\alpha} = \iota_o(\vec{\theta})$. By homogeneity of f, there is a strictly increasing sequence $\langle j_i : i \leq d \rangle$ of members of k^* such that for each $\vec{\alpha} \in \prod_{i \leq d} K_i$, $\delta_{\vec{\alpha}}(j_i) = \alpha_i$. For each $i \leq d$, let t^*_i denote t_{i,j_i}. Then for each $i \leq d$ and each $\vec{\alpha} \in \prod_{i \leq d} K_i$,

$$p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j_i)) = t_{i,j_i} = t^*_i. \tag{48}$$

Lemma 6.9. *For any finite subset $\vec{J} \subseteq \prod_{i \leq d} K_i$, the set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is compatible. Moreover, $p_{\vec{J}} := \bigcup \{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is a member of $\mathbb{P}$ which is below each $p_{\vec{\alpha}}$, $\vec{\alpha} \in \vec{J}$.*

Claim 13. *If $\beta \in \bigcup_{i \leq d} K_i$, $\vec{\alpha} \in \prod_{i \leq d} K_i$, and $\beta \notin \vec{\alpha}$, then β is not a member of $\vec{\delta}_{\vec{\alpha}}$.*

<u>Part III</u>. Let $(n_j)_{j<\omega}$ denote the set of indices for which there is an $X \in \mathrm{MPE}_T(A, C)$ with $X = \max(V)$ for some V of $r_{n_j}[B, T]$. For $i \in I_0$, let $u^*_i = t^*_i$. For $i \in I_1$, let u^*_i be the leftmost extension of t^*_i in $T \restriction l^*$. Note that $\{u^*_i : i \leq d\}$ has no new sets of parallel 1's over A_e. Extend each node u in $\max(B) \setminus A_e$ to its leftmost extension in $T \restriction l^*$ and label that extension u^*. Let

$$U^* = \{u^*_i : i \leq d\} \cup \{u^* : u \in \max(r_k(T)) \setminus A_e\}. \tag{49}$$

Thus, U^* extends $\max(B)$, all sets of parallel 1's in U^* are already witnessed in B since B is valid in T, and U^* has no new pre-determined parallel 1's.

Suppose that $j < \omega$ and for all $i < j$, there have been chosen $S_i \in r_{n_i}[B, T]$ such that h' is constant of value ε^* on $\mathrm{Ext}_{S_i}(A, C^-)$, and for $i < i' < j$, $S_i \sqsubset S_{i'}$. Let k_B be the integer such that $B = r_{k_B}(B)$, and let e be the index such that l^T_{e-1} is greater than the length of the maximal nodes in B. For $j = 0$, take V_0 to be any member of $r_{n_0}[B, T]$ such that the nodes in $\max(r_{k_B+1}(V_0))$ extend the nodes in U^* and have length greater than l^T_e. This is possible by Lemma 4.19. For $j \geq 1$, take $V_j \in r_{n_j}[B, T]$ such that $V_j \sqsupset S_{j-1}$. Let X denote $\max(V_j)$. Then the nodes in $\mathrm{splitpred}_T(X)$ extend the nodes in U^*, and moreover, extend the nodes in $\max(S_{j-1})$ if $j \geq 1$. By the definition of n_j, the set of nodes X contains a coding node. For each $i \in I_0$, let Y_i denote the set of all $t \in T_i \cap X$ which have immediate extension 0 in T. For each $i \in I_1$, let Y_i denote the set of all splitting nodes in $T_i \cap \mathrm{splitpred}_T(X)$. For each $i \leq d$, let J_i be a subset of K_i of size $|Y_i|$, and enumerate the members of Y_i as $q(i, \delta)$, $\delta \in J_i$. Let $\vec{J}$ denote the set of $\vec{\alpha} \in \prod_{i \leq d} J_i$ such that the set $\{q(i, \alpha_i) : i \leq d\}$ has no new sets of parallel 1's over A. Thus, the set of $\{q(i, \alpha_i) : i \leq d\}$, $\vec{\alpha} \in \vec{J}$, is exactly the collection of sets of nodes in $\mathrm{splitpred}_T(X)$ which are members of $\mathrm{Ext}_T(A, C^-)$. Moreover, for each $\vec{\alpha} \in \vec{J}$ and all $i \leq d$,

$$q(i, \alpha_i) \supseteq t^*_i = p_{\vec{\alpha}}(i, \alpha_i). \tag{50}$$

To complete the construction of the desired $q \in \mathbb{Q}$ for which $q \leq p_{\vec{\alpha}}$ for all $\vec{\alpha} \in \vec{J}$, let $\vec{\delta}_q = \bigcup \{\vec{\delta}_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$. For each pair (i, γ) with $\gamma \in \vec{\delta}_q \setminus J_i$, there is at least one $\vec{\alpha} \in \vec{J}$ and some $j < k^*$ such that $\gamma = \delta_{\vec{\alpha}}(j)$. As in Case (a) of Theorem 5.2, for any other $\vec{\beta} \in \vec{J}$ for which $\gamma \in \vec{\delta}_{\vec{\beta}}$, it follows that $p_{\vec{\beta}}(i,\gamma) = p_{\vec{\alpha}}(i,\gamma) = t^*_{i,j}$ and $\delta_{\vec{\beta}}(j) = \gamma$. If $i \in I_0$, let $q(i,\gamma)$ be the leftmost extension of $t^*_{i,j}$ in $T \restriction l^{V_j}_{n_j}$. If $i \in I_1$, let $q(i,\gamma)$ be the leftmost extension of $t^*_{i,j}$ to a splitting node in T in the interval $(l^{V_j}_{n_j-1}, l^{V_j}_{n_j}]$. Such a splitting node must exist because of the construction of U^*. Precisely, let c^X denote the coding node in X. Note that $c^X \restriction l_B$ must have no parallel 1's with any $s_{i'}$, $i' \in I_1$, since X contains a member of $\mathrm{MPE}_T(A,C)$. If c^X does not extend $t^*_{i'}$ for any $i' \leq d$, then $c^X \restriction l^*$ is the leftmost extension in T of $c^X \restriction l_B$, which implies that $c^X \restriction l^*$ has no parallel 1's with $t^*_{i,j}$. Thus, $q(i,\gamma)$, being the leftmost extension of $t^*_{i,j}$, has no parallel 1's with c^X. If c^X extends some $t^*_{i',j'}$, then $c^X \restriction l_B = s_{i'}$. For c^X to be a node in a member of $\mathrm{MPE}_T(A,C)$, $c^X \restriction l_B$ must not have parallel 1's with any s_i, $i \in I_1$. In particular, i' must be in I_0, and $t^*_{i,j}$ has no parallel 1's with $t^*_{i',j'}$, because s_i and $s_{i'}$ have no parallel 1's and by the definition of the partial ordering on $\mathbb{Q}$, since $t^*_{i,j}$ and $t^*_{i',j'}$ are in $\mathrm{ran}(p_{\vec{\alpha}})$ for any $\vec{\alpha} \in [K']^{d+1}$, and $p_{\vec{\alpha}} \leq p^0_{\vec{\alpha}}$. Thus, the leftmost extension $q(i,\gamma)$ of $t^*_{i,j}$ has no parallel 1's with c^X. Therefore, $q(i,\gamma)$ is well-defined. Define

$$q = \bigcup_{i \leq d} \{\langle (i,\alpha), q(i,\alpha)\rangle : \alpha \in \vec{\delta}_q\}. \tag{51}$$

By a proof similar to that of Claim 9, it follows that $q \leq p_{\vec{\alpha}}$, for each $\vec{\alpha} \in \vec{J}$.

Take an $r \leq q$ in $\mathbb{P}$ which decides some l_j in L which is strictly greater than the length of the next coding node above the coding node c^X in X, and such that for all $\vec{\alpha} \in \vec{J}$, $h'(\dot{b}_{\vec{\alpha}} \restriction l_j) = \varepsilon^*$. Without loss of generality, we may assume that the maximal nodes in r have length l_j. If $c^X = q(i',\alpha')$ for some $i' \in I_0$ and $\alpha' \in J_{i'}$, then let c_r denote $r(i',\alpha')$; otherwise, let c_r denote the leftmost extension of c^X in T of length l_j. Let Z_0 denote those nodes in $\mathrm{splitpred}_T(X) \setminus Y_0$ which have length equal to c^X; in particular, Z_0 is the set of nodes in X which are not splitting nodes in $\mathrm{splitpred}_T(X)$ and are also not in Y_0. For each $z \in Z_0$, let s_z denote the leftmost extension of z in T to length l_j. Let Z_1 denote the set of all splitting nodes in $\mathrm{splitpred}_T(X) \setminus Y_1$. For each $z \in Z_1$, let s_z denote the splitting predecessor in T of the leftmost extension of z in T to length l_j. This splitting predecessor exists in T for the following reason: If z is a splitting node in $\mathrm{splitpred}_T(X)$, then z has no parallel 1's with c^X, and so the leftmost extension of z to any length has no parallel 1's with any extension of c^X. In particular, the set $\{s_z : z \in Z_0 \cup Z_1\}$ has no new sets of parallel 1's over $\mathrm{splitpred}_T(X)$.

Let

$$Z^- = \{q(i,\alpha) : i \leq d,\ \alpha \in J_i\} \cup \{s_z : z \in Z_0 \cup Z_1\}. \tag{52}$$

Let Z^* denote the extensions in T of all members of Z^- to length l_j. Let j^- denote the index such that the maximal coding node in V_j below c^X is $c^{V_j}_{n_{j^-}}$. Note that Z^* has no new sets of parallel 1's over $\mathrm{splitpred}_T(X)$; furthermore, the tree induced by $r_{n_{j^-}}(V_j) \cup Z^*$ is strongly similar to V_j, except possibly for the coding node being in the wrong place. Using Lemma 4.19, extend the nodes in Z^* to obtain some $S_j \in r_{n_j}[r_{n_{j^-}}(V_j), T]$ where $\max(S_j)$ extends Z^*. Then every member of $\mathrm{Ext}_{S_j}(A, C^-)$ has the same h' color ε^*, by the choice of r, since each minimal pre-extension in $\mathrm{MPE}_{S_j}(A,C)$ extends some member of $\mathrm{Ext}_{S_j}(A, C-)$ which extends members in $\mathrm{ran}(r)$ and so have h'-color ε^*.

Let $S = \bigcup_{j<\omega} S_j$. Then S is a strong coding tree in $[B,T]$. Let $Y \in \mathrm{Ext}^{SP}_S(A,C)$. Then there is some $X \in \mathrm{MPE}_S(A,C)$ such that Y extends X. Since $\mathrm{splitpred}_S(X)$ is in $\mathrm{Ext}_{S_j}(A, C^-)$ for some $j < \omega$, $\mathrm{splitpred}_S(X)$ has h' color ε^*. Thus, Y has h-color ε^*. □

Recall that given a tree A, $\mathrm{Sim}^s_T(A)$ denote the set of all subtrees A' of T which are strongly similar to A.

Lemma 6.10. *Assume 6.4. Then there is a strong coding subtree $S \leq T$ such that for each $A' \in \mathrm{Sim}^s_S(A)$, h is homogeneous on $\mathrm{Ext}^{SP}_S(A',C)$.*

Proof. Let $(k_i)_{i<\omega}$ be the sequence of integers such that $r_{k_i}(T)$ contains a strictly similar copy of A which is valid in $r_{k_i}(T)$ and such that $\max(A) \subseteq \max(r_{k_i}(T))$. Let $k_{-1} = 0$, $T_{-1} = T$, and $U_{-1} = r_0(T)$.

Suppose $i < \omega$, and $U_{i-1} \overset{s}{\sim} r_{k_{i-1}}(T)$ and T_{i-1} are given satisfying that for each $A' \in \mathrm{Sim}^s_{U_{i-1}}(A)$ valid in U_{i-1} with $\max(A) \subseteq \max(U_{i-1})$, h is homogeneous on $\mathrm{Ext}^{SP}_{U_{i-1}}(A',C)$. Let U_i be in $r_{k_i}[U_{i-1}, T_{i-1}]$. Enumerate the set of all $A' \in \mathrm{Sim}^s_{U_i}(A)$ which are valid in U_i and have $\max(A') \subseteq \max(U_i)$ as $\langle A_0, \dots, A_n\rangle$. Apply Lemma 6.7 to obtain $R_0 \in [U_i, T_{i-1}]$ which is end-homogeneous for $\mathrm{Ext}^{SP}_{R_0}(A_0,C)$. Then apply Lemma

6.8 to obtain $R'_0 \in [U_i, R_0]$ such that $\mathrm{Ext}^{SP}_{R'_0}(A_0, C)$ is homogeneous for h. Given R'_j for $j < n$, apply Lemma 6.7 to obtain a $R_{j+1} \in [U_i, R'_j]$ which is end-homogeneous for $\mathrm{Ext}^{SP}_{R_{j+1}}(A_{j+1}, C)$. Then apply Lemma 6.8 to obtain $R'_{j+1} \in [U_i, R_{j+1}]$ such that $\mathrm{Ext}^{SP}_{R'_{j+1}}(A_{j+1}, C)$ is homogeneous for c. Let $T_i = R'_n$.

Let $U = \bigcup_{i<\omega} U_i$. Then $U \leq T$ and h has the same color on $\mathrm{Ext}^{SP}_U(A, C)$ for each $A' \in \mathrm{Sim}^s_U(A)$ which is valid in U. Finally, take $S \leq U$. Then for each $k < \omega$, $r_k(S)$ is valid in U, so in particular, each $A' \in \mathrm{Sim}^s_S(A)$ is valid in U. Hence, h is homogeneous on $\mathrm{Ext}^{SP}_S(A', C)$. □

A similar lemma holds for the setting of Case (a) in Theorem 5.2. Since the critical node is a splitting node in this case, we do not need to restrict to Strict Parallel 1's Criterion copies of A in T.

Lemma 6.11. *Let T be a strong coding tree and let A, C, h be as in Case (a) of Theorem 5.2. Then there is a strong coding tree $S \leq T$ such that for each $A' \in \mathrm{Sim}^s_S(A)$, $\mathrm{Ext}_S(A', C)$ is homogeneous for h.*

Proof. Similarly to the fusion argument in proof of Lemma 6.10 but applying Case (a) of Theorem 5.2 in place of Lemmas 6.7 and 6.8, one builds a strong coding tree $S \leq T$ such that for each copy A' of A in S, $\mathrm{Ext}_S(A', C)$ is homogeneous for h. □

Proof of Theorem 6.3. The proof is by induction on the number of critical nodes. Suppose first that A consists of a single node. Then such a node must be a splitting node in $0^{<\omega} \cap T$, so $\mathrm{Sim}^s_T(A)$ is the infinite set of all splitting nodes in $0^{<\omega} \cap T$. Let h be any finite coloring on $\mathrm{Sim}^s_T(A)$. By Ramsey's Theorem, infinitely many members of $\mathrm{Sim}^s_T(A)$ must have the same h color, so there is a subtree $S \leq T$ for which all its nodes in $S \cap 0^{<\omega}$ have the same h color. Such an $S \leq T$ exists by the definition of strong coding tree, since T is strongly skew, perfect, and the coding nodes are dense in T.

Now assume that $n \geq 1$ and the theorem holds for each finite tree B with n or less critical nodes such that B satisfies the Strict Parallel 1's Criterion and $\max(B)$ contains a node which is a sequence of all 0's. Let C be a finite tree with $n+1$ critical nodes containing a maximal node in $0^{<\omega}$, and suppose h maps $\mathrm{Sim}^s_T(C)$ into finitely many colors. Let d denote the maximal critical node in C and let $B = \{t \in C : |t| < |d|\}$. Apply Lemma 6.10 or 6.11, depending on whether d is a coding or splitting node, to obtain $T' \leq T$ so that for each $V \in \mathrm{Sim}^s_{T'}(B)$, the set $\mathrm{Ext}^{SP}_{T'}(V, C)$ is homogeneous for h. Define g on $\mathrm{Sim}^s_{T'}(B)$ by letting $g(V)$ be the value of h on $V \cup X$ for any $X \in \mathrm{Ext}^{SP}_{T'}(V, C)$. By the induction hypothesis, there is an $S \leq T'$ such that g is homogeneous on $\mathrm{Sim}^{SP}_S(B)$. It follows that h is homogeneous on $\mathrm{Sim}^{SP}_S(C)$.

To finish, let A be any tree satisfying the Strict Parallel 1's Criterion where $\max(A)$ does not contain a member of $0^{<\omega}$, and let g be a finite coloring of $\mathrm{Sim}^s_T(A)$. Let l_A denote the longest length of nodes in A, and let C be the tree induced by $A \cup \{0^{l_A}\}$. Then there is a one-to-one correspondence between members of $\mathrm{Sim}^s_T(A)$ and $\mathrm{Sim}^s_T(C)$; say $\varphi : \mathrm{Sim}^s_T(A) \to \mathrm{Sim}^s_T(C)$ by definining $\varphi(A')$ to be the member of $\mathrm{Sim}^s_T(C)$ which is the tree induced by adding the node $0^{l_{A'}}$ to A'. For $C' \in \mathrm{Sim}^s_T(C)$, define $h(C') = g(\varphi^{-1}(C'))$. Take $S \leq T$ homogeneous for h. Then S is homogeneous for g on $\mathrm{Sim}^s_S(A)$. □

7. Incremental strong coding trees

This section develops the notion of incremental new sets of parallel 1's, and the related concepts of Incremental Parallel 1's Criterion, incremental strong coding subtrees, and sets of witnessing coding nodes. The main lemma, Lemma 7.5, will be instrumental in attaining the Ramsey theorem in the next section. This will be a Ramsey theorem for finite colorings of strictly similar copies of any given finite subtree of a strong coding tree. The work in this section sets the stage for the removal of the requirement of any form of Parallel 1's Criterion on the finite tree whose copies are being colored.

Definition 7.1 (Incremental parallel 1's)**.** Let Z be a finite subtree of a strong coding tree T, and let $\langle l_j : j < \tilde{j}\rangle$ list in increasing order the minimal lengths of new parallel 1's in Z. We say that Z has *incremental new sets of parallel* 1*'s*, or simply *incremental parallel* 1*'s*, if the following holds. For each $j < \tilde{j}$ for which

$$Z_{l_j,1} := \{z \restriction (l_{j+1}) : z \in Z,\ |z| > l_j,\ \text{and } z(l_j) = 1\} \tag{53}$$

has size at least three, letting m denote the length of the longest critical node in Z below l_j, for each proper subset $Y \subsetneq Z_{l_j,1}$ of cardinality at least two, there is a $j' < j$ such that $l_{j'} > m$, $Y_{l_{j'},1} := \{y \restriction (l_{j'}+1) : y \in Y$ and $y(l_{j'}) = 1\}$ has the same size as Y, and $Y_{l_{j'},1} = Z_{l_{j'},1}$.

We shall say that an infinite tree S has *incremental new parallel* 1*'s* if for each $l < \omega$, the initial subtree $S \restriction l$ of S has incremental new parallel 1's.

Definition 7.2 (Incremental Parallel 1's Criterion)**.** Let Z be a subtree of a strong coding tree T. We say that Z satisfies the *Incremental Parallel* 1*'s Criterion* if Z has incremental new parallel 1's and satisfies the Parallel 1's Criterion.

Thus, to satisfy the Incremental Parallel 1's Criterion, a tree must have a coding node witnessing *each* of its new sets of parallel 1's, and these are occuring incrementally. Note that any strong coding tree does not satisfy the Incremental Parallel 1's Criterion. In the next section, we will be interested in extending finite trees A to trees E which satisfy the Incremental Parallel 1's Criterion, for such E automatically satisfy the Strict Parallel 1's Criterion, so the Ramsey theorems from the previous section can be applied.

The next definition of an incremental strong coding tree will be vital to finding bounds for the big Ramsey degrees in $\mathcal{H}_3$.

Definition 7.3 (Incremental Strong Coding Tree)**.** A strong coding tree T is called *incremental* if it satisfies the following. Let n be any integer for which there are at least three distinct nodes in $T \restriction (|c^T_n| + 1)$ which have passing number 1 at c^T_n, and list the set of those nodes as $\langle t_i : i < \tilde{i} \rangle$. Let m denote the length of the maximal splitting node in T below c^T_n. Let $\mathcal{P}$ denote the collection of all proper subsets $P \subseteq \tilde{i}$ of size at least two, and let $\tilde{k} = |\mathcal{P}|$. Then there is an ordering $\langle P_k : k < \tilde{k} \rangle$ of $\mathcal{P}$ and a strictly increasing sequence $\langle p_k : k < \tilde{k} \rangle$ such that

(i) $m < p_0$ and $p_{\tilde{k}-1} < |c^T_n|$;
(ii) $k < k' < \tilde{k}$ implies $P_k \not\supseteq P_{k'}$; and
(iii) For each $k < \tilde{k}$, p_k is minimal such that $\{i < \tilde{i} : t_i(p_k) = 1\} = P_k$.

The main lemma of this section shows that given a strong coding tree T, there is an incremental strong coding subtree $S \le T$ and moreover, a set $W \subseteq T$ of coding nodes disjoint from S such that each new set of parallel 1's in S is witnessed by a coding node in W. This set-up is what will allow for the definition and use of envelopes in the next section, as it will ensure that subtrees from S can be enhanced with witnessing coding nodes from W so that their union satisfies the Strict Parallel 1's Criterion. This will allow application of Theorem 6.3 to obtain upper bounds on the finite big Ramsey degrees in the universal triangle-free graph.

Definition 7.4 (Incrementally witnessed parallel 1's)**.** Let $S \le T$ be an incremental strong coding tree. We say that the sets of parallel 1's in S are *incrementally witnessed in* T if the following hold. For each $n < \omega$, given $\mathcal{P}$, $\langle P_k : k < \tilde{k} \rangle$, and $\langle p_k : k < \tilde{k} \rangle$ satisfying Definition 7.3, there is a coding node $w_{n,k}$ in T satisfying

(1) $|d^S_{m_n-1}| < |w^\wedge_{n,0}| < p_0 \le |w_{n,0}| < |w^\wedge_{n,1}| < p_1 \le |w_{n,1}| < \cdots < |w^\wedge_{n,\tilde{k}-1}| < p_{\tilde{k}-1} \le |w_{n,\tilde{k}-1}| < |c^S_n|$.
(2) $w_{n,k}$ witnesses the parallel 1's in $S_{p_k,1}$; that is, for all $z \in S \restriction (p_k + 1)$, $z(|w_k|) = 1$ if and only if $z(p_k) = 1$.

The main lemma of this section shows that given a strong coding tree T, there is an incremental strong coding subtree $S \le T$ and moreover, a set $W \subseteq T$ of coding nodes disjoint from S such that each new set of parallel 1's in S is witnessed by a coding node in W. This set-up is what will allow for the definition and use of envelopes in the next section, as it will ensure that subtrees from S can be enhanced with witnessing coding nodes from W so that their union satisfies the Strict Parallel 1's Criterion. This will allow application of Theorem 6.3 to obtain upper bounds on the finite big Ramsey degrees in the universal triangle-free graph.

Lemma 7.5. *Let T be a strong coding tree. Then there is an incremental strong coding tree $S \le T$ and a set of coding nodes $W \subseteq T$ such that each new set of parallel* 1*'s in S is incrementally witnessed in T by a coding node in W.*

Proof. Let $\langle d^T_m : m < \omega \rangle$ denote the critical nodes in T in order of increasing length. Let $\langle m_n : n < \omega \rangle$ denote the indices such that $d^T_{m_n} = c^T_n$, so the m_n-th critical node in T is the n-th coding node in T. Let S_0 be a valid subtree of T which is strongly similar to $r_{m_0+1}(T)$. Since $r_{m_0+1}(T)$ has only one node with passing number 1 at c^T_0, there is nothing to do; vacuously S_0 has incremental new sets of parallel 1's and these are vacuously witnessed in T.

Suppose now that $n \ge 1$ and we have chosen $S_{n-1} \overset{s}{\sim} r_{k_{n-1}+1}(T)$ valid in T so that S_{n-1} is incremental and has its new sets of parallel 1's incrementally witnessed in T. Take some $S'_{n-1} \in r_{k_n}[S_{n-1}, T]$, so S'_{n-1} is valid in T. There is a one-to-one correspondence between the nodes in $\max(r_{k_n+1}(T))$ and $\max(r_{k_n}(T))^+$, and hence also between $\max(r_{k_n+1}(T))$ and $\max(S'_{n-1})^+$. Let $\varphi : \max(r_{k_n+1}(T)) \rightarrow \max(S'_{n-1})^+$ be the lexicographic order preserving bijection. Let $\langle t_i : i < \tilde{i}\rangle$ be the lexicographically increasing enumeration of those nodes in $\max(r_{k_n+1}(T))$ which have passing number 1 at c^T_n. Let $s_i = \varphi(t_i)$. Then $\{s_i : i < \tilde{i}\}$ is the set of nodes which must extend to have passing number 1 at the next coding node in S, c^S_n. If $\tilde{i} \le 2$, there is nothing to do; extend to some $S_n \in r_{k_n+1}[S'_{n-1}, T]$.

Otherwise, $\tilde{i} \ge 3$. List all subsets of $\tilde{i}$ of size at least two as $\langle P_k : k < \tilde{k}\rangle$ in any manner so long as the following is satisfied: For each $k < k' < \tilde{k}$, $P_k \not\supseteq P_{k'}$. Let X_0 denote $\max(S'_{n-1})^+$. Given $k < \tilde{k}$ and X_k, let $w^\wedge_{n,k}$ be some splitting node in T in $0^{<\omega}$ with length above the lengths in X_k. Extend all nodes in X_k leftmost in T to length $|w^\wedge_{n,k}| + 1$, and let Y_k denote the level set of these extensions. Apply Lemma 4.19 to extend the nodes in $Y_k \cup \{w^\wedge_{n,k}{}^\frown 1\}$ to a level set Z_k in T such that the following hold:

1. The extension of $w^\wedge_{n,k}{}^\frown 1$ is a coding node, label it $w_{n,k}$;
2. Enumerating $Z_k \setminus \{w_{n,k}\}$ as $\{z_i : i < \tilde{i}\}$ so that for each $z_i \supseteq s_i$, then for each $i < \tilde{i}$, the immediate extension of z_i in T is 1 if and only if $i \in P_k$.
3. The only possible set of new parallel 1's in Z_k over $S'_{n-1} \cup X_k$ is $\{z_i : i \in P_k\}$.

If $k < \tilde{k} - 1$, let $X_{k+1} = Z_k$ and continue the procedure. Upon obtaining $Z_{\tilde{k}-1}$, apply Lemma 4.19 to obtain an $S_n \in r_{m_n+1}[S'_{n-1}, T]$ such that $\max(S_n)$ extends $Z_{\tilde{k}-1}$.

To finish, let $S = \bigcup_{n<\omega} S_n$. Then $S \le T$, S is incremental, and the sets of parallel 1's in S are strongly incrementally witnessed in T. Let $W = \{w_{n,k} : n < \omega,\ k < \tilde{k}_n\}$, where $\tilde{k}_n$ is the number of subsets of $S_{l_n,1}$ of size at least two. $\square$

8. Ramsey theorem for strict similarity types

The strongest Ramsey theorem proved so far is Theorem 6.3, a Milliken-style theorem for colorings of finite trees satisfying the Strict Parallel 1's Criterion. In this section we obtain a general Ramsey theorem for all *strictly similar* copies (Definition 8.3) of any finite tree for which the maximal nodes are exactly the coding nodes forming an antichain. This involves a new notion of envelope for strongly diagonal subsets of strong coding trees, the main property being that any envelope satisfies the Strict Parallel 1's Criterion. Then applying Theorem 6.3, Lemma 7.5, and envelopes, we obtain Theorem 8.9, the main Ramsey theorem for strong coding trees in this paper.

Recall from Definition 4.8 that a strongly diagonal subset of $2^{<\omega}$ is an antichain Z such that its meet closure forms a transversal with the property that for any splitting node $s \in Z^\wedge$, all nodes in $Z^\wedge$ of length greater than $|s|$, except for those nodes extending s, have passing number 0 at s. It is a byproduct of the definition of strong coding trees that any subset of a strong coding tree forming an antichain is in fact strongly diagonal. Henceforth, we shall use the term *antichain of coding nodes*, or simply *antichain*, to refer to strongly diagonal sets of coding nodes in a strong coding tree. If Z is an antichain, then by the *tree induced by* Z we mean the set

$$\{z \restriction |u| : z \in Z \text{ and } u \in Z^\wedge\}. \tag{54}$$

We say that an antichain satisfies the Parallel 1's Criterion (Strict Parallel 1's Criterion) if and only if the tree it induces satisfies the Parallel 1's Criterion (Strict Parallel 1's Criterion).

Let Z be an antichain of coding nodes. Enumerate the nodes in Z in order of increasing length as $\langle z_i : i < \tilde{i}\rangle$. For each $l < |z_{\tilde{i}-1}|$, let

$$I^Z_l = \{i < \tilde{i} : |z_i| > l \text{ and } z_i(l) = 1\}, \tag{55}$$

and define

$$Z_{l,1} = \{z_i \restriction (l+1) : i \in I^Z_l\}. \tag{56}$$

Thus, $Z_{l,1}$ is the collection of all $z_i \restriction (l+1)$ which have passing number 1 at level l. Given l such that $|Z_{l,1}| \ge 2$, we say that the set of parallel 1's at level l is *witnessed by the coding node z_j in Z* if $z_i(|z_j|) = 1$ for each $i \in I^Z_l$, and either $|z_j| \le l$ or else both $|z_j| > l$ and Z has no splitting nodes and no coding nodes of length in $[l, |z_j|]$. A level l is *the minimal level of a new set of parallel 1's in Z* if $|I^Z_l| \ge 2$ and whenever

$l' < l$ and $I^Z_{l'} \subseteq I^Z_l$, then $|I^Z_{l'}| < |I^Z_l|$. It follows that if there are two or more members of Z extending some $0^l{}^\frown 1$, then l is the minimal level of a new set of parallel 1's, namely of I^Z_l.

Definition 8.1. Given Z an antichain of coding nodes, if l is the minimal level of a new set of parallel 1's in Z, *the admissible interval for* I^Z_l is the interval $[l, l^*]$, where $l^* > l$ is maximal satisfying the following:

(1) $Z^\wedge$ has no splitting node and no coding node of length in (l, l^*).
(2) Each $l' \in (l, l^*]$ is not the minimal level of a new set of parallel 1's in Z.

If l is the minimal level of a new set of parallel 1's in Z, we say that the set of parallel 1's indexed by I^Z_l is *minimally witnessed in* Z if, letting $k < \tilde{i}$ be minimal such that $|z_k| \geq l$, $|z_k|$ is in the admissible interval $[l, l^*]$ and z_k witnesses the parallel 1's in I^Z_l; that is, $\{i < \tilde{i} : z_i(|z_k|) = 1\} = I^Z_l$. Note that z_k is in the interval $[l, l^*]$ if and only if either $|z_k| = l$ or $|z_k| = l^*$. Otherwise, we say that I^Z_l is *not minimally witnessed in* Z.

The following fact is immediate from the previous definition.

Fact 8.2. *If all new sets of parallel* 1*'s are minimally witnessed in an antichain* Z*, then the tree induced by* Z *satisfies the Strict Parallel* 1*'s Criterion.*

Definition 8.3 (Strict similarity type)**.** Given Z a finite antichain of coding nodes in some strong coding tree T, list the minimal levels of new sets of parallel 1's in Z which are not minimally witnessed in Z in increasing order as $\langle l_j : j < \tilde{j} \rangle$. Enumerate all nodes in $Z^\wedge$ as $\langle u^Z_m : m < \tilde{m} \rangle$ in order of increasing length. Thus, each u^Z_m is either a splitting node in $Z^\wedge$ or else a coding node z_i for some $i < \tilde{i}$. The sequence

$$\langle \langle l_j : j < \tilde{j} \rangle, \langle I^Z_{l_j} : j < \tilde{j} \rangle, \langle |u^Z_m| : m < \tilde{m} \rangle \rangle \tag{57}$$

is *the strict similarity sequence of* Z.

Let Y be another finite antichain in T, and let

$$\langle \langle p_j : j < \tilde{k} \rangle, \langle I^Y_{p_j} : j < \tilde{k} \rangle, \langle |u^Y_m| : m < \tilde{n} \rangle \rangle \tag{58}$$

be its strict similarity sequence. We say that Y and Z have the same *strict similarity type* or are *strictly similar*, written $Y \stackrel{ss}{\sim} Z$, if

(1) $Y^\wedge$ and $Z^\wedge$ are strongly similar;
(2) $\tilde{j} = \tilde{k}$ and $\tilde{m} = \tilde{n}$;
(3) For each $j < \tilde{j}$, $I^Y_{n_j} = I^Z_{l_j}$; and
(4) The function $\varphi : \{p_j : j < \tilde{j}\} \cup \{|u^Y_m| : m < \tilde{m}\} \to \{l_j : j < \tilde{j}\} \cup \{|u^Z_m| : m < \tilde{m}\}$, defined by $\varphi(p_j) = l_j$ and $\varphi(u^Y_m) = u^Z_m$, is an order preserving bijection between these two linearly ordered sets of natural numbers.

Define

$$\mathrm{Sim}^{ss}_T(Z) = \{Y \subseteq T : Y \stackrel{ss}{\sim} Z\}. \tag{59}$$

Note that for two antichains $Y \stackrel{ss}{\sim} Z$, the map $f : Y \to Z$ by $f(y_i) = z_i$ for each $i < \tilde{i}$ induces a strong similarity map from $Y^\wedge$ onto $Z^\wedge$ by defining $f(y_i \wedge y_j) = z_i \wedge z_j$ for each pair $i, j < \tilde{i}$. Then $f(u^Y_m) = u^Z_m$ for each $m < \tilde{m}$. Further, by (3) and (4) of Definition 8.3, this map preserves the order in which minimal sets of parallel 1's appear, relative to all other minimal sets of parallel 1's and the nodes in $Y^\wedge$ and $Z^\wedge$.

The definition of strictly similar in Definition 8.3 extends Definition 6.2 to finite sets which do not necessarily satisfy the Parallel 1's Criterion. When Z is an antichain such that its induced tree satisfies the Incremental Parallel 1's Criterion, then Definitions 6.2 and 8.3 coincide, and further, for such Z, these coincide with the notion of strongly similar.

Fact 8.4. *Let* T *be a strong coding tree, and* A *and* B *be subsets of* T*. Suppose* A *satisfies the Incremental Parallel* 1*'s Criterion. Then* $B \stackrel{s}{\sim} A$ *if and only if* $B \stackrel{ss}{\sim} A$*.*

The following notion of envelope is defined in terms of structure without regard to an ambient strong coding tree. In any given strong coding tree T, there will certainly be finite subtrees of T which have no envelope in T. This poses no problem to our intended application, as by the work done in the previous section, inside a given strong coding tree T, there will be an incremental strong coding tree S along with a set of witnessing coding nodes $W \subseteq T$ so that each finite antichain in S has an envelope consisting of nodes from W. Thus, envelopes of antichains in S will exist in T.

Definition 8.5 (Envelopes)**.** Let Z be a finite antichain of coding nodes and let $\langle\langle l_j : j < \tilde{j}\rangle, \langle I_{l_j} : j < \tilde{j}\rangle, \langle |u_m| : m < \tilde{m}\rangle\rangle$ be the strict similarity sequence of Z. A finite set $E(Z)$ is an *envelope* of Z if $E(Z) = Z \cup W$ is an antichain of coding nodes, where $W = \{w_j : j < \tilde{j}\}$, such that the following hold: For each $j < \tilde{j}$,

(1) w_j is in the admissible interval of l_j; that is, $l_j \le |w_j| \le l_j^*$;
(2) $I_{|w_j|} = I_{l_j}$;
(3) w_j has no parallel 1's with any member of $Z \cup (W \setminus \{w_j\})$; and
(4) $l_{j-1}^* < |w_j^\wedge| < l_j$ and there is no member of $(Z \cup W)^\wedge$ with length in $(|w_j^\wedge|, |w_j|)$.

The set W is called the set of *witnessing coding nodes*, since they minimally witness all parallel 1's in Z not minimally witnessed by any coding node in Z. The next fact follows immediately from the definitions.

Fact 8.6. *Let S be any strongly incremental strong coding tree and Z be any antichain in S. Then any envelope E of Z satisfies the Incremental Parallel 1's Criterion, and hence also the Strict Parallel 1's Criterion.*

Lemma 8.7. *Let Y and Z be strictly similar antichains. Then any envelope of Y is strictly similar to any envelope of Z; in particular, any two envelopes of Y are strictly similar.*

Proof. Let $Y = \{y_i : i < \tilde{i}\}$ and $Z = \{z_i : i < \tilde{i}\}$ be the enumerations of Y and Z, respectively, in order of increasing length. Let

$$\langle\langle p_j : j < \tilde{j}\rangle, \langle I^Y_{p_j} : j < \tilde{j}\rangle, \langle |u^Y_m| : m < \tilde{m}\rangle\rangle \tag{60}$$

and

$$\langle\langle l_j : j < \tilde{j}\rangle, \langle I^Z_{l_j} : j < \tilde{j}\rangle, \langle |u^Z_m| : m < \tilde{m}\rangle\rangle \tag{61}$$

be their strict similarity sequences, respectively. Let $E = Y \cup V$ and $F = Z \cup W$ be any envelopes of Y and Z, respectively. Enumerate the nodes in V and W in order of increasing length as $\{v_j : j < \tilde{j}\}$ and $\{w_j : j < \tilde{j}\}$, respectively. Note that $|E| = |F| = \tilde{i} + \tilde{j}$, since exactly $\tilde{j}$ many coding nodes are added to make envelopes of Y and Z. Let $\tilde{k} = \tilde{i} + \tilde{j}$, and let $\{e_k : k < \tilde{k}\}$ and $\{f_k : k < \tilde{k}\}$ be the enumerations of E and F in order of increasing length, respectively. For each $j < \tilde{j}$, let k_j be the index in $\tilde{k}$ such that $e_{k_j} = v_j$ and $f_{k_j} = w_j$. For $k < \tilde{k}$, let $E(k)$ denote the tree induced by E restricted to those nodes of length less than or equal to $|e_k|$; precisely, $E(k) = \{e \restriction |t| : e, t \in E^\wedge \text{ and } |t| \le \min(|e|, |e_k|)\}$. Likewise for F.

If $\tilde{j} = 0$, then $E = Y$ and $F = Z$, so $E \overset{s}{\sim} F$ follows from $E \overset{ss}{\sim} F$. Suppose now that $\tilde{j} \ge 1$. It must be the case that $p_0 > |u_0^Y|$, since u_0^Y is the stem of the tree induced by Y, and Y does not have any sets of parallel 1's below its stem. Likewise, $l_0 > |u_0^Z|$. Let m_0 be the least integer below $\tilde{m}$ such that $|u^Y_{m_0}| > p_0$. Then the admissible interval $[p_0, p_0^*]$ is contained in the interval $(|u^Y_{m_0-1}|, |u^Y_{m_0}|)$, and moreover,

$$|u^Y_{m_0-1}| < |v_0^\wedge| < p_0 \le |v_0| \le p_0^*, \tag{62}$$

by the definition of envelope. Since $Y \overset{ss}{\sim} Z$, it follows that the admissible interval $[l_0, l_0^*]$ is contained in $(|u^Z_{m_0-1}|, |u^Z_{m_0}|)$ and

$$|u^Z_{m_0-1}| < |w_0^\wedge| < l_0 \le |w_0| \le l_0^*. \tag{63}$$

Thus, $E(k_0 - 1)$ is exactly the tree induced by Y restricted below $|u^Z_{m_0-1}|$, which is strongly similar to the tree induced by Z restricted below $|u^Z_{m_0-1}|$, this being exactly $F(k_0 - 1)$.

Now suppose that $j < \tilde{j}$ and $E(k_j - 1) \overset{s}{\sim} F(k_j - 1)$. Let m_j be the least integer below $\tilde{m}$ such that $|u^Y_{m_j}| > p_j$. Then the only nodes in $E^\wedge$ in the interval $(|u^Y_{m_j-1}|, |u^Y_{m_j}|)$ are $v_j^\wedge$ and v_j. Likewise, the only nodes in $F^\wedge$ in the interval $(|u^Z_{m_j-1}|, |u^Z_{m_j}|)$ are $w_j^\wedge$ and w_j. Extend the strong similarity map $g : E(k_j - 1) \to F(k_j - 1)$ to the map $g^* : E(k_j) \to F(k_j)$ as follows: Define $g^* = g$ on $E(k_j - 1)$, $g^*(v_j^\wedge) = w_j^\wedge$, and $g^*(v_j) = (w_j)$. If the sequence of 0's of length $|v_j|$ is in E, then define g^* of that node to be the sequence of 0's of length $|w_j|$. For each node s in $E(k_j)$ of length $|v_j|$ besides v_j itself, s extends a unique maximal node s^- in $E(k_j - 1)$; define $g^*(s)$ to be the unique node in $F(k_j)$ of length $|w_j|$ extending $g(s^-)$. Note that each node t in $E(k_j)$

of length $|v_j^\wedge|$, besides $v_j^\wedge$ itself, is equal to $s \restriction |v_j^\wedge|$ for some unique s as above; define $g^*(t)$ to be $g^*(s) \restriction |w_j^\wedge|$. As the only new set of parallel 1's in Y in this interval is I_j^Y, which is equal to I_j^Z, and as

$$\max(l_{j-1}^*, |u_{m_j-1}^Y|) < |v_j^\wedge| < p_j \leq |v_j| \leq p_j^*, \tag{64}$$

and similarly for w_j, and v_j, w_j witness the parallel 1's indexed by I_j^Y, I_j^Z, respectively, it follows that g^* is a strong similarity map from $E(k_j)$ to $F(k_j)$.

If $j < \tilde{j} - 1$, noting that the only nodes in the tree induced by E with length in the interval $(|v_j|, |v_{j+1}^\wedge|)$ are in the tree induced by Y, and likewise, all nodes in the tree induced by F in the interval $(|w_j|, |w_{j+1}^\wedge|)$ are in the tree induced by Z, it follows that $E(k_{j+1} - 1)$ is strongly similar to $F(k_{j+1} - 1)$. Then the induction continues.

To finish, when $j = \tilde{j} - 1$, all nodes in the tree induced by E in the interval $(|v_{\tilde{j}-1}|, |y_{\tilde{i}-1}|]$ are in fact nodes in $Y^\wedge$. Likewise, all nodes in the tree induced by F in the interval $(|w_{\tilde{j}-1}|, |z_{\tilde{i}-1}|]$ are in $Z^\wedge$. Further, all sets of parallel 1's in E and F in these intervals are already witnessed at or below $|v_{\tilde{j}-1}|$ and $|w_{\tilde{j}-1}|$, respectively. Thus, the strict similarity between Y and Z induces an extension of the strong similarity between $E(k_{\tilde{j}-1})$ and $E(k_{\tilde{j}-1})$ to a strong similarity between $E^\wedge$ and $F^\wedge$. □

Lemma 8.8. *Let S be a strongly incremental strong coding tree, a subtree of T. Let Z be a finite antichain of coding nodes in S, and let E be any envelope of Z in T. Enumerate the nodes in Z and E in order of increasing length as $\langle z_i : i < \tilde{i}\rangle$ and $\langle e_k : k < \tilde{k}\rangle$, respectively. Then whenever $F \overset{s}{\sim} E$, the subset $F \restriction Z := \{f_{k_i} : i < \tilde{i}\}$ of F is strictly similar to Z, where $\langle f_k : k < \tilde{k}\rangle$ enumerates the nodes in F in order of increasing length and for each $i < \tilde{i}$, k_i is the index such that $e_{k_i} = z_i$.*

Proof. Recall that $F \overset{s}{\sim} E$ implies $F \overset{ss}{\sim} E$ and that E and hence F satisfy the Incremental Parallel 1's Criterion, since E is an envelope of a diagonal subset of an incremental strong coding tree. Let $\iota_{Z,F} : Z \to F$ be the injective map defined via $\iota_{Z,F}(z_i) = f_{k_i}$, for each $i < \tilde{i}$, and let $F \restriction Z$ denote $\{f_{k_i} : i < \tilde{i}\}$, the image of $\iota_{Z,F}$. Then $F \restriction Z$ is a subset of F which we claim is strictly similar to Z.

Since F and E satisfy the Incremental Parallel 1's Criterion, the strong similarity map $g : E \to F$ satisfies that for each $j < \tilde{k}$, the sets of new parallel 1's at level of the j-th coding node are equal:

$$\{k < \tilde{k} : e_k(|e_j|) = 1\} = \{k < \tilde{k} : g(e_k)(|g(e_j)|) = 1\} = \{k < \tilde{k} : f_k(|f_j|) = 1\}. \tag{65}$$

Since $\iota_{Z,F}$ is the restriction of g to Z, $\iota_{Z,F}$ also takes each new set of parallel 1's in Z to the corresponding set of new parallel 1's in $F \restriction Z$, with the same set of indices. Thus, $\iota_{Z,F}$ witnesses that $F \restriction Z$ is strictly similar to Z. □

Theorem 8.9 (Ramsey Theorem for Strict Similarity Types)**.** *Let Z be a finite antichain of coding nodes in a strong coding tree T, and let h color of all subsets of T which are strictly similar to Z into finitely many colors. Then there is an incremental strong coding tree $S \leq T$ such that all subsets of S strictly similar to Z have the same h color.*

Proof. First, note that there is an envelope E of a copy of Z in T: By Lemma 7.5, there is a strongly incremental strong coding tree $U \leq T$ and a set of coding nodes $V \subseteq T$ such that each $Y \subseteq U$ which is strictly similar to Z has an envelope in T by adding nodes from V. Since U is strongly similar to T, there is subset Y of U which is strictly similar to Z. Let E be any envelope of Y in T, using witnessing coding nodes from V.

By Lemma 8.7, all envelopes of copies of Z are strictly similar. Define a coloring h^* on $\mathrm{Sim}_T^{ss}(E)$ as follows: For each $F \in \mathrm{Sim}_T^{ss}(E)$, define $h^*(F) = h(F \restriction Z)$, where $F \restriction Z$ is the subset of F provided by Lemma 8.8. The set $F \restriction Z$ is strictly similar to Z, so the coloring h^* is well-defined. Since envelopes satisfy the Strict Parallel 1's Criterion, Theorem 6.3 yields a strong coding tree $T' \leq T$ such that $\mathrm{Sim}_{T'}^{ss}(E)$ is homogeneous for h^*. Lemma 7.5 implies there is an incremental strong coding tree $S \leq T'$ and a set of coding nodes $W \subseteq T'$ such that each $Y \subseteq S$ which is strictly similar to Z has an envelope F in T'. Thus, $h(Y) = h^*(F)$. Therefore, h takes only one color on the set of all $Y \subseteq S$ which are strictly similar to Z. □

Remark 8.10. If Z is not incremental, then S will have no strictly similar copies of Z, since every antichain in S is strongly incremental. Thus, non-incremental antichains will not contribute to the big Ramsey degrees.

Remark 8.11. The definition of envelope can be extended to handle any finite subset of a strong coding tree, where maximal nodes can be any nodes in a strong coding tree rather than just coding nodes. This is accomplished using the same definition of strict similarity type, accounting for all minimal new sets of parallel 1's, and then letting envelopes consist of adding new coding nodes as before to witness these sets of parallel 1's in their admissible intervals. Then Theorem 8.9 extends to a Ramsey theorem for strict similarity types of any finite subset of a strong coding tree. However, as the main result of this paper only needs Theorem 8.9, in order to avoid unnecessary length, we do not present the full generality here.

9. The universal triangle-free graph has finite big Ramsey degrees

The main theorem of this paper, Theorem 9.2, will now be proved: The universal triangle-free homogeneous graph $\mathcal{H}_3$ has finite big Ramsey degrees. This result will follow from Theorem 8.9, which is the Ramsey Theorem for Strict Similarity Types, along with Lemma 9.1, which shows that any strong coding tree contains an infinite strongly diagonal set of coding nodes which code the universal triangle-free graph.

Recall from the discussion in the previous section that in a strong coding tree, a set of coding nodes is strongly diagonal if and only if it is an antichain. Given an antichain D of coding nodes from a strong coding tree, its meet closure, $D^\wedge$ has at most one node of any given length. Let L_D denote the set of all lengths of nodes $t \in D^\wedge$ such that t is not the splitting predecessor of any coding node in D. Define

$$D^* = \bigcup \{t \restriction l : t \in D^\wedge \setminus D \text{ and } l \in L_D\}. \tag{66}$$

Then $(D^*, \subseteq)$ is a tree.

For a strong coding tree T, let $(T, \subseteq)$ be the reduct of $(T, \omega; \subseteq, <, c)$. Then $(T, \subseteq)$ is simply the tree structure of T, disregarding the difference between coding nodes and non-coding nodes. We say that two trees $(T, \subseteq)$ and $(S, \subseteq)$ are *strongly similar trees* if they satisfy Definition 3.1 in [31]. This is the the same as the modification of Definition 4.9 leaving out (6) and changing (7) to apply to passing numbers of *all* nodes in the trees. When we say that two finite trees are strongly similar trees, we will be implying that when extending the two trees to include the immediate extensions of their maximal nodes, the two extensions are still strongly similar. Thus, strong similarity of finite trees implies passing numbers of their immediate extensions are preserved.

Lemma 9.1. *Let $T \le \mathbb{T}$ be a strong coding tree. Then there is an infinite antichain of coding nodes $D \subseteq T$ which code $\mathcal{H}_3$ in exactly the same way that $\mathbb{T}$ does: $c^D_n(l^D_i) = c^{\mathbb{T}}_n(l^{\mathbb{T}}_i)$, for all $i < n < \omega$. Moreover, $(D^*, \subseteq)$ and $(\mathbb{T}, \subseteq)$ are strongly similar trees.*

Proof. To simplify the indexing of the construction, we will construct a subtree $\mathbb{D} \subseteq \mathbb{T}$ such that $\mathbb{D}$ the set of coding nodes in $\mathbb{D}$ form an antichain satisfying the lemma. Then, since T is strongly similar to $\mathbb{T}$, letting $\varphi : \mathbb{T} \to T$ be the strong similarity map between $\mathbb{T}$ and T, the image of φ on the coding nodes of $\mathbb{D}$ will yield an antichain of coding nodes $D \subseteq T$ satisfying the lemma.

We will construct $\mathbb{D}$ so that for each n, the node of length $l^{\mathbb{D}}_n + 1$ which is going to be extended to the next coding node $c^{\mathbb{D}}_{n+1}$ will split at a level lower than any of the other nodes of length $l^{\mathbb{D}}_{n+1}$ split in $\mathbb{D}$. Above that, the splitting will be regular in the interval until the next coding node. Recall that for each $i < \omega$, $\mathbb{T}$ has either a coding node or a splitting node of length i. To avoid some superscripts, let $l_n = |c^{\mathbb{T}}_n|$ and $k_n = |c^{\mathbb{D}}_n|$. Let j_n be the index such that $c^{\mathbb{D}}_n = c^{\mathbb{T}}_{j_n}$, so that k_n equals l_{j_n}. The set of nodes in $\mathbb{D} \setminus \{c^{\mathbb{D}}_n\}$ of length k_n shall be indexed as $\{d_t : t \in \mathbb{T} \restriction l_n\}$.

Define $d_{\langle\rangle} = \langle\rangle$ and let $\mathrm{Lev}_{\mathbb{D}}(0) = \{d_{\langle\rangle}\}$. As the node $\langle\rangle$ splits in $\mathbb{T}$, so the node $d_{\langle\rangle}$ will split in $\mathbb{D}$. Extend $\langle 1\rangle$ to a splitting node in $\mathbb{T}$ and label this extension $v_{\langle 1\rangle}$. Let $a_{\langle 0\rangle}$ be the leftmost node in $\mathbb{T}$ of length $|v_{\langle 1\rangle}| + 1$, let $a_{\langle 1\rangle} = v_{\langle 1\rangle}{}^\frown 0$, and $u_{\langle 1\rangle} = v_{\langle 1\rangle}{}^\frown 1$. Extend $a_{\langle 0\rangle}$ to the shortest splitting node containing it in $\mathbb{T} \cap 0^{<\omega}$; label this $d_{\langle 0\rangle}$. Let $d_{\langle 1\rangle}$ be the leftmost extension of $a_{\langle 1\rangle}$ in $\mathbb{T}$ of length $|d_{\langle 0\rangle}|$, and let $u'_{\langle 1\rangle}$ be the leftmost extension of $u_{\langle 1\rangle}$ in $\mathbb{T}$ of length $|d_{\langle 0\rangle}|$. Apply Lemma 4.19 to extend $d_{\langle 0\rangle}{}^\frown 0$, $d_{\langle 0\rangle}{}^\frown 1$, $d_{\langle 1\rangle}{}^\frown 0$, and $u'_{\langle 1\rangle}{}^\frown 0$ to nodes $d_{\langle 0,0\rangle}$, $d_{\langle 0,1\rangle}$, $d_{\langle 1,0\rangle}$, and $c^{\mathbb{D}}_0$, respectively, so that the tree induced by these nodes satisfy the Parallel 1's Criterion, $c^{\mathbb{D}}_0$ is a coding node, and the immediate extension of $d_{\langle i_0,i_1\rangle}$ in $\mathbb{T}$ is i_1, for all $\langle i_0, i_1\rangle$ in $\mathrm{Lev}_{\mathbb{T}}(2)$. Let $k_0 = |c^{\mathbb{D}}_0|$, and notice that we have constructed $\mathbb{D} \restriction (\le k_0)$ satisfying the lemma.

For the induction step, suppose $n \ge 1$ and we have constructed $\mathbb{D} \restriction (\le k_{n-1})$ satisfying the lemma. Then by the induction hypothesis, there is a strong similarity map of the trees $\varphi : \mathbb{T} \restriction (\le l_{n-1}) \to \mathbb{D}^* \restriction (\le k_{n-1})$,

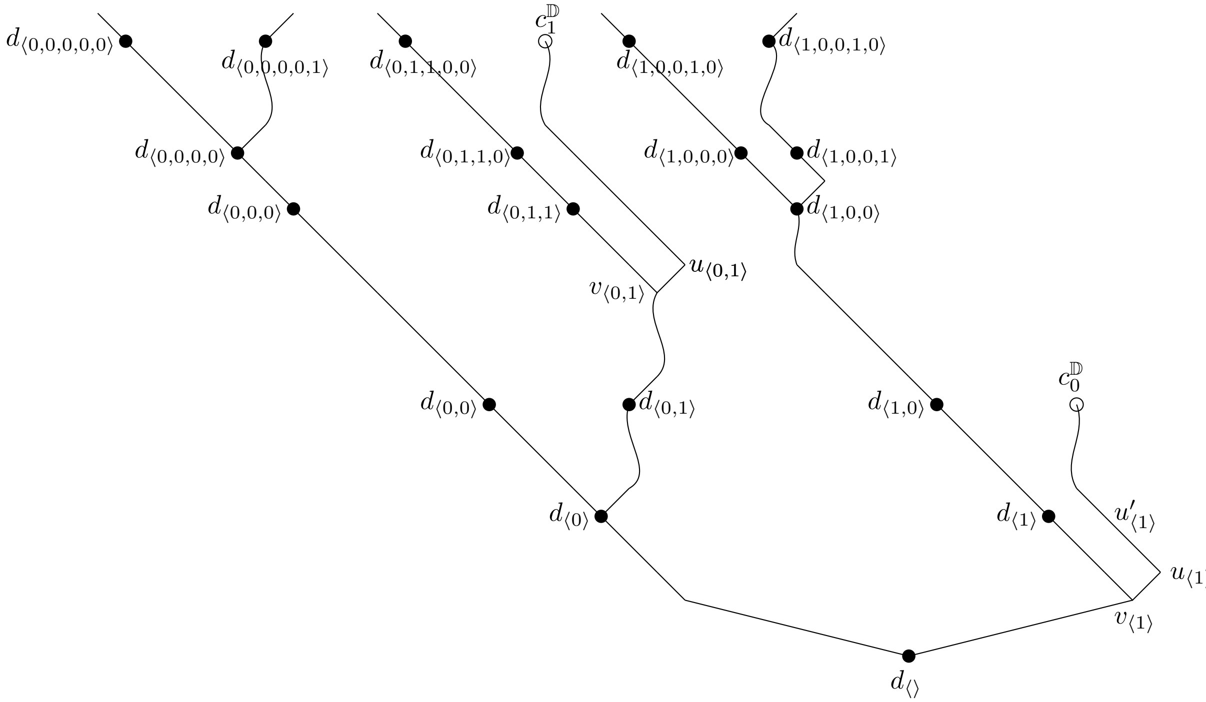

FIGURE 5. The construction of $\mathbb{D}$

where for each $t \in \mathbb{T} \restriction l_{n-1}$, $d_t = \varphi(t)$. Let s denote the node in $\mathbb{T} \restriction l_{n-1}$ which extends to the coding node $c_n^{\mathbb{T}}$. Let v_s be a splitting node in $\mathbb{T}$ extending d_s. Let $u_s = v_s{}^\frown 1$ and extend all nodes d_t, $t \in (\mathbb{T} \restriction l_{n-1}) \setminus \{s\}$, leftmost to length $|u_s|$ and label these d'_t. Extend $v_s{}^\frown 0$ leftmost to length $|u_s|$ and label it d'_s. Let $X = \{d'_t : t \in \mathbb{T} \restriction l_{n-1}\} \cup \{u_s\}$ and let $\mathrm{Spl}(u_s)$ be the set of all nodes in X which have no parallel 1's with u_s. Apply Lemma 4.19 to obtain a coding node $c_n^{\mathbb{D}}$ extending u_s and nodes d_w, $w \in \mathbb{T} \restriction l_n$, so that, letting $k_n = |c_n^{\mathbb{D}}|$ and

$$\mathbb{D} \restriction k_n = \{d_m : m \in \mathbb{T} \restriction l_n\} \cup \{c_n^{\mathbb{D}}\}, \tag{67}$$

the following hold. $\mathbb{D} \restriction (\leq k_n)$ satisfies the Parallel 1's Criterion, and $\mathbb{D}^* \restriction (\leq k_n)$ is strongly similar as a tree to $\mathbb{T} \restriction (\leq l_n)$. Thus, the coding nodes in $\mathbb{D} \restriction (\leq k_n)$ code exactly the same graph as the coding nodes in $\mathbb{T} \restriction (\leq l_n)$.

Let $\mathbb{D} = \bigcup_{n<\omega} \mathbb{D} \restriction (\leq k_n)$. Then the set of coding nodes in $\mathbb{D}$ forms an antichain of maximal nodes in $\mathbb{D}$. Further, the tree generated by the the meet closure of the set $\{c_n^{\mathbb{D}} : n < \omega\}$ is exactly $\mathbb{D}$, and $\mathbb{D}^*$ and $\mathbb{T}$ are strongly similar as trees. By the construction, for each pair $i < n < \omega$, $c_n^{\mathbb{D}}(k_i) = c_n^{\mathbb{T}}(l_i)$; hence they code $\mathcal{H}_3$ in the same order.

To finish, let ψ be the strong similarity map from $\mathbb{T}$ to S. Letting D be the ψ-image of $\{c_n^{\mathbb{D}} : n < \omega\}$, we obtain an antichain of coding nodes in S such that D^* and $\mathbb{D}^*$ are strongly similar trees, and hence D^* is strongly similar as a tree to $\mathbb{T}$. Thus, the antichain of coding nodes D codes $\mathcal{H}_3$ and satisfies the lemma. □

The filled-in nodes in the graphic form the tree $\mathbb{D}^*$. The coding nodes are exactly the maximal nodes of $\mathbb{D}$ and form an antichain. Notice that the collection of nodes $\{d_t : t \in \mathbb{T} \restriction (\leq 2)\}$, which are exactly the filled-in nodes in the figure, forms a tree strongly similar to $\mathbb{T} \restriction 2$. The bent lines indicate that the next node was chosen either to be least such that it was a critical node or according to Lemma 4.19.

Main Theorem 9.2. *The universal triangle-free graph has finite big Ramsey degrees.*

Proof. Let G be a finite triangle-free graph, and let f be a coloring of all the copies of G in $\mathcal{H}_3$ into finitely many colors. By Theorem 4.6, there is a strong coding tree $\mathbb{T}$ in which the coding nodes code $\mathcal{H}_3$. Let

$\mathcal{A}$ denote the set of all antichains of coding nodes of $\mathbb{T}$ which code a copy of G. For each $Y \in \mathcal{A}$, let $h(Y) = f(\mathrm{G}')$, where G′ is the copy of G coded by the coding nodes in Y. Then h is a finite coloring on $\mathcal{A}$.

Let $n(\mathrm{G})$ be the number of different strict similarity types of incremental strongly diagonal subsets of $\mathbb{T}$ coding G, and let $\{Z_i : i < n(\mathrm{G})\}$ be a set of one representative from each of these different strict similarity types. Successively apply Theorem 8.9 to obtain incremental strong coding trees $\mathbb{T} \geq T_0 \geq \cdots \geq T_{n(\mathrm{G})-1}$ so that for each $i < n(\mathrm{G})$, h is takes only one color on $\mathrm{Sim}^s_{T_i}(Z_i)$. Let $S = T_{n(\mathrm{G})-1}$.

By Lemma 9.1 there is a strongly diagonal subtree $D \subseteq S$ which also codes $\mathcal{H}_3$. Then every set of coding nodes in D coding G is automatically strongly diagonal and incremental. Therefore, every copy of G in the copy of $\mathcal{H}_3$ coded by the coding nodes in D is coded by an incremental strongly diagonal set. Thus, the number of strict similarity types of incremental strongly diagonal subsets of $\mathbb{T}$ coding G provides an upper bound for the big Ramsey degree of G in $\mathcal{H}_3$. □

10. Concluding Remarks

The number of strict similarity types of antichains of coding nodes in a strong coding tree which code a given finite graph G is bounded by the number of subtrees of the binary tree of height $2(|\mathrm{G}| + 1)$, times the number of ways to choose incremental sets of new parallel 1's between any successive levels of the tree. We leave it as an open problem to determine this recursive function precisely.

Although we have not yet proved the lower bounds to obtain the precise big Ramsey degrees $T(\mathrm{G}, \mathcal{K}_3)$ for finite triangle-free graphs inside the universal triangle-free graph, we conjecture that they will be equal to the number of strict similarity types of strongly incremental antichains coding G. We further conjecture that once found, the lower bounds will satisfy the conditions needed for Zucker's work in [37] to apply. If so, then $\mathcal{H}_3$ would admit a big Ramsey structure and any big Ramsey flow will be a universal completion flow, and any two universal completion flows will be universal. We refer the interested reader to Theorem 1.6 in [37] and surrounding comments.

The author is currently working to extend the techniques developed here to prove that for each $k > 3$, the universal k-clique-free homogeneous graph $\mathcal{H}_k$ has finite big Ramsey degrees. Preliminary analyses indicate that the methodology created in this paper is robust enough to apply, with modifications, to a large class of Fraïssé limits of Fraïssé classes of relational structures omitting some irreducible substructure.

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