DEPENDENT CHOICE, PROPERNESS, AND GENERIC ABSOLUTENESS

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ABSTRACT. We observe that Dependent Choice is a sufficient choice principle for developing the basic theory of proper forcing, and for deriving generic absoluteness for the Chang model in the presence of large cardinals. We also investigate some basic consequences of the Proper Forcing Axiom in ZF, and formulate a natural question about the generic absoluteness of the Proper Forcing Axiom in ZF + DC and ZFC.

1. INTRODUCTION

The Axiom of Choice is a staple of modern mathematics which requires little introduction nowadays. However, as the reader is probably aware, the Axiom of Choice also has some controversial implications, especially in analysis. Weakening the Axiom of Choice to one of its countable variants, the *Principle of Dependent Choice*, is enough to let most of classical analysis through, while not enough to prove the existence of 'pathological' subsets of the real line (e.g., non-measurable sets or sets without the Baire property).

In this work we observe that Dependent Choice is enough—in fact the minimum needed—of the Axiom of Choice to develop the theory of proper forcing. We also show that it is enough to provide us with generic absoluteness for the Chang model, in the preserve of large cardinals, at least assuming our forcings preserve Dependent Choice.[1]

We also show that assuming the Proper Forcing Axiom a stronger version of Dependent Choice must hold, and $2^{\aleph_0} = \aleph_2$. We do not know how to prove that the Proper Forcing Axiom *does not* imply the Axiom of Choice,[2] but we do suggest a possible way to prove such a statement assuming some generic absoluteness of the Proper Forcing Axiom (together with reasonable large cardinal assumptions).

1.1. **The basic definitions.** In our paper we redefine the notion of $H(\kappa)$ to work in a context where the Axiom of Choice may fail. We then use this definition when discussing proper forcing. In our paper a notion of forcing is a partially ordered set $\mathbb{P}$ with a maximum denoted by $1_\mathbb{P}$. If $p, q \in \mathbb{P}$ we say that q *extends* p, or that it is a *stronger condition*, if $q \leq p$. If two conditions have a common extension we say that

[1]Monro showed in [7] that Dependent Choice is not always preserved by forcing in ZF.

[2]Without assuming the consistency, with ZF+DC, of the existence of a Reinhardt cardinal with a supercompact cardinal below it.

they are *compatible*. A forcing $\mathbb{P}$ is *weakly homogeneous* if for every two conditions p and q there is an automorphism π such that πp and q are compatible. If $\mathbb{P}$ is weakly homogeneous, then for every $p \in \mathbb{P}$ and every statement φ in the forcing language for $\mathbb{P}$, $p \Vdash \varphi$ if and only if $\Vdash \varphi$. Also, if $\mathbb{P}$ and $\mathbb{Q}$ are weakly homogeneous forcing notions, then so is $\mathbb{P} \times \mathbb{Q}$.

We denote $\mathbb{P}$-names by $\dot{x}$, and if x is a set in the ground model, we denote the canonical name for x by $\check{x}$. If $\dot{x}$ and $\dot{y}$ are $\mathbb{P}$-names, we say that $\dot{y}$ *appears in* $\dot{x}$ if there is some condition p such that $\langle p, \dot{y} \rangle \in \dot{x}$. We then recursively define

$$\dot{x} \restriction p = \{\langle p_*, \dot{y} \restriction p \rangle \mid \dot{y} \text{ appears in } \dot{x},\ p_* \leq p,\ p_* \Vdash \dot{y} \in \dot{x}\}.$$

Not assuming the Axiom of Choice, the terminology, definitions and notations surrounding cardinals become vague. We consider *cardinals* in the broader sense: if x can be well-ordered, then $|x|$ is the least ordinal that has a bijection with x; otherwise, $|x|$ is the *Scott cardinal* of x:

$$|x| = \{y \mid \exists f \colon x \to y \text{ a bijection}\} \cap V_\alpha,$$

for the least α where the intersection is non-empty. Greek letters (with the notable exception of φ and π) will usually denote ordinals, and specifically $\aleph$ numbers when we assume that they denote infinite cardinals.

We write $|x| \leq |y|$ to denote that there is an injection from x into y, and $|x| \leq^* |y|$ to denote that x is empty or else there is a surjection from y onto x. The notation $|x| < |y|$ denotes the statement $|x| \leq |y|$ and $|y| \nleq |x|$, and $|x| <^* |y|$ is defined from $\leq^*$ similarly.[3]

Finally, for a set x, the *Hartogs number of* x is $\aleph(x) = \min\{\kappa \mid \kappa \nleq |x|\}$ and the *Lindenbaum number of* x is $\aleph^*(x) = \min\{\kappa \mid \kappa \nleq^* |x|\}$.

2. Hereditary sets

One of the key constructions in set theory is structures of the form $H(\kappa)$, where κ is a cardinal. These are usually[4] defined as $\{x \mid |\operatorname{tcl}(x)| < \kappa\}$. Assuming that κ is a regular cardinal, these sets are models of ZFC^-, namely all the axioms of ZFC except the Power Set Axiom (which can hold for "small sets" when κ is sufficiently large). Of course, in almost all the definitions we have we can replace $H(\kappa)$ with V_α for "sufficiently nice α", meaning one where enough of Replacement is satisfied. But this requires more care in noting what these Replacement axioms are which we are using, and virtually nobody wants that. Not when a nice alternative like $H(\kappa)$ is available.

In the absence of the Axiom of Choice, however, we want to have $H(\kappa)$ reflect the fact that choice fails. This cannot be done when taking the same definition at face value, since it implies that if $x \in H(\kappa)$ then x can be well-ordered. This means that we need to modify the definition of $H(\kappa)$.

Definition 2.1. Given a set x, $H(x)$ is the class $\{y \mid |x| \nless^* |\operatorname{tcl}(y)|\}$. We denote by $H(<x)$ the union of the classes $H(y)$ such that $|y| <^* |x|$.

Note that for an infinite cardinal κ, $H(\kappa) = H(<\kappa^+)$ and if κ is a regular limit cardinal, then $H(<\kappa) = H(\kappa)$.

Proposition 2.2. *Let $\kappa > \omega$ be an uncountable cardinal. The following properties hold:*

    (1) $H(\kappa)$ *is a transitive set of height κ.*

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[3]Note that $|x| < |y|$ is equivalent to $|x| \leq |y|$ and $|x| \neq |y|$, as the Cantor–Bernstein theorem is provable in ZF. For $\leq^*$, however, the dual-Cantor–Bernstein theorem is not provable without choice, and so $|x| <^* |y|$ is in fact a stronger statement than $|x| \leq^* |y|$ and $|x| \neq |y|$.

[4]Assuming choice, that is.

(2) $H(\kappa)$ *is a model of* ZF^-, *namely* ZF *without the power set axiom.*
(3) $H(\kappa) = V_\kappa$ *if and only if* κ *is a strong limit cardinal, namely* $\kappa \not\leq^* |V_\alpha|$ *for all* $\alpha < \kappa$. *If in addition* κ *is regular, then* $H(\kappa) \models \mathsf{ZF}_2$.

Proof. Clearly $H(\kappa)$ is a transitive class. We want to argue that $H(\kappa) \subseteq V_\kappa$, which means it is a set. For this we need the following claim: for a set x the following are equivalent:

(a) the von Neumann rank of x is α, and
(b) $\alpha = \min\{\beta \mid \mathrm{tcl}(x) \cap V_{\beta+1} \setminus V_\beta = \varnothing\}$.

In other words, if x has rank α, then its transitive closure contains elements of any rank below α.

We prove this equivalence by $\in$-induction on x. Suppose that $\mathrm{rank}(x) = \alpha$. Then for all $\beta < \alpha$ there is some $y \in x$ such that $\alpha > \mathrm{rank}(y) \geq \beta$. By the induction hypothesis, $\mathrm{tcl}(y) \cap V_{\gamma+1} \setminus V_\gamma \neq \varnothing$ for all $\gamma < \beta$, and therefore $\mathrm{tcl}(x) \cap V_{\gamma+1} \setminus V_\gamma \neq \varnothing$; moreover $y \in \mathrm{tcl}(x) \cap V_{\beta+1} \setminus V_\beta$, so indeed α is the minimal ordinal satisfying (b). On the other hand, assuming (b) it is clear that $\mathrm{rank}(x) \geq \alpha$. But if $\mathrm{rank}(x) > \alpha$, then for some $y \in x$ the rank of y is at least α, in which case the above argument shows that $\mathrm{tcl}(x) \cap V_{\alpha+1} \setminus V_\alpha \neq \varnothing$, so α is not the minimal ordinal satisfying (b).

Using this equivalence we argue that if $\mathrm{rank}(x) > \kappa$, then $x \notin H(\kappa)$. This is clear, since $\mathrm{tcl}(x)$ can be mapped onto κ by mapping each element to its rank. So indeed $H(\kappa) \subseteq V_\kappa$, and so it is a transitive set.

For the second part, the transitivity of $H(\kappa)$ and the fact that $\kappa > \omega$ provide us with all the axioms except Replacement. But note that if $f \colon x \to y$, and y can be mapped onto κ, then x can be mapped onto κ. So in fact $H(\kappa)$ satisfies the second-order variant of Replacement.

Finally, if κ is a strong limit cardinal, then by definition $V_\alpha \in H(\kappa)$ for all $\alpha < \kappa$, and equality follows. If κ is also regular, then by the work of Blass–Dimitriou–Löwe in [2], this implies that $V_\kappa \models \mathsf{ZF}_2$. $\square$

Remark 2.3. It is interesting to note that the requirement that κ is an $\aleph$ is crucial for the proof that $H(\kappa)$ is even a set. The definition can be used with any cardinal x, but if x is not well-ordered, then $|x| \not\prec^* \kappa$ for any ordinal κ, meaning that $\mathrm{Ord} \subseteq H(x)$. In the definition, we could have artificially cropped $H(x)$ when x is not well-orderable, in order to ensure that $H(x)$ remains a set, but we want to somehow reflect the structure of the universe in a natural way and this cropping seems artificial and unjustified by the mathematics at this point.

Proposition 2.4. *The class* $\{H(<\kappa) \mid \kappa \text{ is an } \aleph\text{-number}\}$ *is a continuous filtration of the universe.*

Proof. Trivially this class is a continuous sequence of transitive set. If x is any set, then taking $\kappa = \aleph^*(\mathrm{tcl}(x))$ implies that $x \in H(\kappa) = H(<\kappa^+)$. $\square$

Remark 2.5. It is consistent that $\aleph^*(\mathbb{R}) = \kappa$ is a limit cardinal (e.g. assuming determinacy) in which case $\mathbb{R} \in H(\kappa)$ but $\mathbb{R} \notin H(<\kappa)$.

3. Dependent Choice

Definition 3.1. Let κ be an $\aleph$ number. We denote by DC_κ the following statement: Every κ-closed tree without maximal elements has a chain of order type κ. We use $\mathsf{DC}_{<\kappa}$ to abbreviate $\forall \lambda < \kappa, \mathsf{DC}_\lambda$, and for $\kappa = \aleph_0$ we simply write DC.[5]

Among the basic consequences of DC we have the Axiom of Choice for countable families of non-empty sets, the countability of every countable union of countable sets, the regularity of ω_1, and many others.

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[5]DC has many known equivalents [5, Form 43].

Lemma 3.2. *Every well-orderable tree has a branch. In particular, if T is a κ-closed well-orderable tree without maximal elements, then T has a branch of order type at least κ.*

Proof. Enumerate T and proceed by induction. $\qquad\square$

Theorem 3.3. *The following are equivalent:*
1. DC.
2. *The Löwenheim–Skolem theorem for countable languages: every structure in a countable language has a countable elementary submodel.*
3. *For every $\alpha \geq \omega$ and every countable $A \subseteq V_\alpha$ there is a countable elementary submodel M of V_α such that $A \subseteq M$.*
4. *For every $\alpha \geq \omega$ there is a countable elementary submodel $M \prec \langle V_\alpha, \in \rangle$.*

Proof. The proof that (1) implies (2) is the usual proof of the Löwenheim–Skolem theorem, noting that it does not use more than DC. The implications (2) $\implies$ (3) and (3) $\implies$ (4) are easy and trivial, respectively. The last implication is obtained by noting that if DC fails, then there is some α such that DC fails in V_α. Taking a countable elementary submodel M there is some $T \in M$ such that T is a counterexample to DC. However, by elementarity $T \cap M$ is a subtree of T which is countable and without maximal elements and therefore contains an infinite branch, which is a contradiction. $\qquad\square$

Of course, in the above equivalence, we can replace V_α by $H(\aleph_\alpha)$, or by any filtration of the universe.

Part 1. CCC, properness, and forcing axioms without choice

4. Proper Forcing

Definition 4.1. Let $\mathbb{P}$ be a notion of forcing. We say that $\mathbb{P}$ is *proper* if for any sufficiently large κ,[6] and every countable elementary submodel $M \prec (H(\kappa), \in, \mathbb{P})$, if $p \in \mathbb{P} \cap M$, then p has an extension which is M-generic. Namely, there is some $q \leq p$ such that for any open dense $D \in M$, $D \cap M$ is predense below q.

Without choice, however, properness can be a bit quirky.

Proposition 4.2. *The following are equivalent:*
1. DC.
2. $\mathrm{Col}(\omega, \omega_1)$ *is not proper.*
3. *There exists a forcing which is not proper.*

Proof. The implication from (1) to (2) is the usual proof in ZFC that a proper forcing cannot collapse ω_1. The implication from (2) to (3) is trivial. Finally, if DC fails, then the definition of properness holds vacuously by the definition of "sufficiently large κ", thus making every forcing proper. $\qquad\square$

Remark 4.3. We could have replaced "sufficiently large" by explicitly requiring κ to be any fixed cardinal such that $H(\kappa)$ has both the power set of $\mathbb{P}$ and the basic tools needed for defining the forcing relation. One could also define a hierarchy saying that $\mathbb{P}$ is "κ-proper" if every countable elementary submodel $M \prec H(\kappa)$ such that $\mathbb{P} \in M$, etc., and then define proper as "κ-proper for all κ". However, we feel that the above is a good definition as it is flexible enough to allow for a certain degree of freedom in choosing κ and as it provides us with this nice characterization of DC.

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[6]I.e., for a tail of κ.

On the other hand, if one decides that it is best to require something explicit about κ, or stick to the "κ-properness" definition, then the failure of DC at $H(\omega_1)$ (e.g. if there is a Dedekind-finite set of reals) will imply that every forcing is proper, whereas this will not be the case if DC first fails at some point above $H(\omega_1)$. So we would be getting an odd equivalence of "every forcing is proper" given that choice of 'properness'; avoiding this oddness is a good motivation for picking the inherently ambiguous definition of properness as we did above.

Proposition 4.4. *Every σ-closed forcing is proper.*

Proof. If M is a countable elementary submodel of some $H(\kappa)$ with $\mathbb{P} \in M$, then enumerate the dense open sets in M as $\{D_n \mid n \in \omega\}$ and enumerate $\mathbb{P} \cap M$ as $\{p_n \mid n \in \omega\}$, and for any $p \in \mathbb{P} \cap M$, recursively construct a decreasing sequence such that $q_0 = p$ and q_{n+1} is the least p_k such that $p_k \in D_n$ and $p_k \leq q_n$. Since $\mathbb{P}$ is σ-closed, there is some lower bound q of all the q_n's and it is clear that q is an M-generic condition. $\square$

One ambition might be to prove that every ccc forcing is proper, but without the Axiom of Choice chain conditions are tricky to even formulate. Mirna Džamonja suggested in private communication to the authors to use this as a way to formulate ccc, at least in the presence of DC, by saying that every condition is M-generic for any countable elementary submodel M. Of course, under this definition it is trivial that every ccc forcing is proper.

Theorem 4.5. *If $\mathbb{P}$ is proper and DC holds, then $\Vdash$ DC.*

Proof. Let $\dot{T}$ be a $\mathbb{P}$-name for a tree of height ω without maximal nodes and let $p \in \mathbb{P}$. Let M be a countable elementary submodel of some sufficiently large $H(\kappa)$ such that $\dot{T}, p, \mathbb{P} \in M$.

Let q be an M-generic condition extending p and let $\dot{T}_* = M \cap \dot{T}$. For each $n < \omega$, let D_n be the collection of conditions $p \in \mathbb{P}$ for which there is some $\mathbb{P}$-name $\dot{t}$ such that $p \Vdash$ "$\dot{t}$ is a node in $\dot{T}$ of level n". For every n, D_n is a dense subset of $\mathbb{P}$ in M. Hence, $D_n \cap M$ is predense below q for each n, and as a consequence q forces that $\dot{T}_*$ is a subtree of $\dot{T}$ of height ω. By a similar argument, we also have that q forces $\dot{T}_*$ not to have any maximal nodes. But then, since of course $\dot{T}_*$ is forced to be countable, q forces that $\dot{T}_*$ has a branch. We have shown that every condition in $\mathbb{P}$ can be extended to a condition adding an branch through $\dot{T}$, which means that $\Vdash$ DC. $\square$

This shows that one cannot violate DC with a proper forcing, which is somewhat similar to the fact you cannot violate AC with any forcing. We will next argue that in fact ZF + DC is a good base theory for working with proper forcing.

Theorem 4.6.   (1) *Two-step iterations of proper forcings are proper.*
   (2) *Countable support iterations of proper forcings are proper.*

Proof. If DC fails, then this holds vacuously. If DC holds, the usual proofs as given in the context of ZFC also work here, noting that all relevant countable choices in the ZFC proofs can equally be made in the present context; for a reference see [1]. However, the following caveat is needed here.

In ZFC we usually define the iteration $\mathbb{P} * \dot{\mathbb{Q}}$ as a partial order on $\langle p, \dot{q} \rangle$, where $\dot{q}$ is a $\mathbb{P}$-name such that $\Vdash \dot{q} \in \dot{\mathbb{Q}}$. We could replace this with $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$, and while for a two-step iteration this is entirely inconsequential, for a countable support iteration this matters a lot.

Moving from $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$ to $\Vdash \dot{q} \in \dot{\mathbb{Q}}$ usually requires the Axiom of Choice. But we can circumvent this as follows: if $\dot{q}$ is a name which appears in $\dot{\mathbb{Q}}$ and $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$, then define $\dot{q}_*$ as $\bigcup\{1_\mathbb{Q} \restriction p' \mid p' \perp p\} \cup \dot{q} \restriction p$, and easily $\Vdash \dot{q}_* \in \dot{\mathbb{Q}}$ and $p \Vdash \dot{q}_* = \dot{q}$. $\square$

5. The Proper Forcing Axiom

Definition 5.1. The Proper Forcing Axiom (PFA) states that if $\mathbb{P}$ is proper, then for every collection of dense open sets $\mathcal{D} = \{D_\alpha \mid \alpha < \omega_1\}$ there exists a $\mathcal{D}$-generic filter.

Proposition 5.2. PFA *implies* DC_{ω_1}.

Proof. Let T be a σ-closed tree of height ω_1 and no maximal elements. Then T is proper. Consider $D_\alpha = T \setminus T \restriction \alpha$, for $\alpha < \omega_1$. Then D_α is a dense open set, so by PFA there is a generic filter meeting each D_α, which is a cofinal branch in T. $\square$

Proposition 5.3. PFA *implies that* $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ *and that there is a well-order of* $H(\aleph_2)$ *of length* ω_2 *definable over* $H(\aleph_2)$ *from parameters.*

Proof. First note that by DC_{ω_1}, we immediately have that $\aleph_1 \leq 2^{\aleph_0}$, and applying PFA to the binary tree $2^{<\omega}$ we get that $\aleph_1 < 2^{\aleph_0}$. Moreover, by DC_{ω_1} we can choose a ladder system on ω_1, i.e., a sequence $\langle C_\alpha \mid \alpha < \omega_1$ is a limit ordinal$\rangle$ such that each C_α is a cofinal subset of α of order type ω. It then also follows that there is a sequence $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ of pairwise disjoint stationary subsets of ω_1.

Next we want to point out that Moore's proof that the Mapping Reflection Principle implies that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ in [8] can be repeated under PFA even without choice. The following changes with respect to the proof as presented in Moore's paper are necessary:

(1) Require that $\mathrm{cf}(\theta) > \omega_1$ in the statement of MRP (Definition 2.3).
(2) Replace $H((2^{\omega_1})^+)$ with $H(\aleph^*(2^{\omega_1}))$ in Lemma 4.4.
(3) Note that the existence of a ladder and DC_{ω_1} are enough for the proofs of Proposition 4.2 and Theorem 4.3 to go through. $\square$

Thus, PFA implies $H(\omega_2) \models \mathsf{AC}$ and hence that all the richness of the usual ZFC combinatorics is available to this structure.

Remark 5.4. One could argue that the correct formulation of forcing axioms (and PFA specifically) should require that $<2^{\aleph_0}$ dense open sets admit a filter meeting them all, and add that $\aleph_1 \neq 2^{\aleph_0}$ to avoid trivialities. In that case in models of $\mathsf{ZF} + \mathsf{DC} + (|x| < 2^{\aleph_0} \to |x| \leq \aleph_0)$, e.g. in Solovay's model or models of AD, this type of axiom holds vacuously since by DC every countable family of dense open sets admits a filter meeting them, and every collection of strictly less than $2^{\aleph_0}$ sets is necessarily countable. We feel that this vacuity is a good reason not to adopt such definition.

Of course, we can replace $<2^{\aleph_0}$ by $<^* 2^{\aleph_0}$, namely require that there is a surjective map from the continuum onto the family of dense open sets but there is no surjection in the other direction. Since there is a surjection from the continuum onto ω_1, and we require that $2^{\aleph_0} \neq \aleph_1$, there is no surjection in the other direction. Therefore formulating PFA this way implies every collection of $\aleph_1$ dense open sets admits a generic filter. This is enough to prove DC_{ω_1}, and the above arguments would then imply that $2^{\aleph_0} = \aleph_2$.

Even weakening PFA to Martin's Axiom (assuming DC and defining ccc as "every condition is M-generic for every countable model M" as proposed to us by Dža-monja), the corresponding forcing axiom applied to $\mathrm{Add}(\omega, x)$ would imply that if there is a surjection from the continuum onto x, then either there is a surjection from x onto the reals or there is an injection from x into the continuum. And in particular $\aleph_1 < 2^{\aleph_0}$.

We finish this section by pointing out the following consistency result.

Proposition 5.5. *If κ is a supercompact cardinal,[7] then the standard iteration for forcing* PFA *in* ZFC *using lottery sums forces* PFA *in* ZF + DC. $\hfill\square$

6. Does PFA imply the Axiom of Choice?

Probably not. In fact, if $\delta < \kappa$ are such that δ is supercompact and κ is a Reinhardt cardinal, then forcing PFA via a forcing $\mathbb{P} \subseteq V_\delta$ as in Proposition 5.5 will preserve the Reinhardtness of κ.

We would of course have liked to produce a model of PFA without the Axiom of Choice starting from a more reasonable assumption (in the region of supercompact cardinals). However, this seems to be a much more difficult task than one would originally believe it to be. We believe that outlining this difficulty is constructive, and raises some interesting questions about the absoluteness of PFA even between models of ZFC.

The "obvious construction" would be to start with a model of ZFC + PFA, then construct a symmetric extension 'à la the first model Cohen', using $\mathrm{Add}(\omega_2, \omega_2)$ as our forcing, thus obtaining an intermediate model which is closed under ω_1-sequences. Using the König–Yoshinobu Theorem from [6], the full generic extension must satisfy PFA, so the intermediate model *should too.*

However, when getting down to brass tacks, one sees that the question is this: If $\mathbb{P}$ is a proper forcing, is $\mathbb{P}$ also proper in a large model which agrees on the same ω_1-sequences?

This leads us to a very interesting question, which even in the context of ZFC has no obvious answer:

Question 6.1. Suppose that $\mathbb{P}$ is an ω_2-directed-closed forcing and $\Vdash$ PFA. Does that mean that PFA holds in the ground model?

Of course, to make our example, we would need to argue with a proof in a model of ZF + DC$_{\omega_1}$, rather than a model of ZFC. But we are optimistic that a proof in ZFC will also work—mutatis mutandis—in ZF + DC$_{\omega_1}$.

When approaching the above question, it seems that an easy way to solve it would be to prove the following statement: if $\mathbb{Q}$ is an ω_2-directed-closed forcing and $\mathbb{P}$ is proper, then $\Vdash_{\mathbb{Q}}$ "$\check{\mathbb{P}}$ is proper". However in a private correspondence, Yasuo Yoshinobu sent us a proof of the following ZFC theorem, showing that this version of the question admits a negative answer.

Theorem 6.2 (Yoshinobu; ZFC). *Given a regular cardinal $\kappa > \omega$, there is a forcing of the form $\mathrm{Add}(\omega, 1) * \mathrm{Col}(\omega_1, 2^\kappa) * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is a ccc forcing, which collapses ω_1 after forcing with $\mathrm{Add}(\kappa, 1)$.*

Part 2. **Generic Absoluteness without Choice**

7. Motivation and limitations

Definition 7.1. Let κ be an ordinal, the κ-*Chang model* denoted by $\mathcal{C}_\kappa$, is the $\subseteq$-minimal model of ZF containing all the ordinals and closed under κ-sequences. This model can always be constructed as $L(\mathcal{P}_{\kappa^+}(\mathrm{Ord}))$. In the case where $\kappa = \omega$, we omit ω from the terminology and notation, and refer to it as the Chang model.

In this section we will prove that under large cardinal assumptions, the theory of the Chang model is generically absolute under forcings which preserve DC. There is a foundational motivation for this in that most of real analysis lives inside $L(\mathbb{R})$ and can be carried out in ZF + DC. This means that ZF + DC is a very natural meta-theory for analysis. This makes it natural to enquire whether or not full AC

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[7]See section 8 for supercompactness without choice.

is an overkill when it comes to deriving generic absoluteness for $L(\mathbb{R})$ or even for the Chang model modulo large cardinals,[8] in the same way that, say, CH or ¬CH are indeed overkills when deriving generic absoluteness for these inner models with ZFC as base theory.

It is important to realize that non-trivial instances of generic absoluteness for the Chang model may be blocked unless we restrict our considerations to forcing notions preserving DC:

Proposition 7.2. *The Chang model cannot have generic absoluteness for its Σ_2 theory in* ZF*, even in the presence of large cardinals.*

Proof. Suppose that V is a model of ZFC and κ is a supercompact cardinal (or extendible, or any other not-yet inconsistent large cardinal). Repeat a Feferman–Levy style construction taking the symmetric collapse of $\mathrm{Col}(\omega_1, <\aleph_{\omega_1})$ (see § 4 in [3] for details).

This provides us with a model M of ZF + DC where κ is still a supercompact cardinal in the sense of Woodin, as defined below,[9] and $M \models \mathrm{cf}(\omega_2) = \omega_1$; moreover any countable sequence of ordinals in M belongs to the ground model V. Therefore the Chang model $\mathcal{C}$ in M is the same as those of V.

Working in M, let G be an M-generic filter for $\mathrm{Col}(\omega, \omega_1)$. Then in $M[G]$ we have that $\omega_1^{M[G]} = \omega_2^M$ and therefore $M[G] \models \mathrm{cf}(\omega_1) = \omega$. This fact is reflected to $\mathcal{C}$ since it is witnessed by a countable sequence of ordinals, despite the fact that $\mathrm{Col}(\omega, \omega_1)$ has size $\aleph_1$ (without using any choice!) and thus is strictly less than κ. However, $\mathrm{cf}(\omega_1) = \omega$ is a Σ_2 statement over the Chang model. $\square$

Remark 7.3. The above proof shows of course that if κ is, say, a supercompact cardinal in the ZFC model V, then there is a symmetric extension M satisfying DC in which κ remains supercompact and such that Σ_2-generic absoluteness for the Chang models fails between M and the $\mathrm{Col}(\omega, \omega_1)$-extension of M (which kills DC). Hence, there is no hope for proving Σ_2-generic absoluteness for the Chang model in general over the base theory ZF+DC unless we restrict to forcings preserving DC.

Question 7.4. Can generic absoluteness for $L(\mathbb{R})$ fail in the presence of large cardinals without the requirement that DC is preserved? Moreover, is it at all possible in the presence of $\mathbb{R}^{\#}$ that ω_1 is singular in $L(\mathbb{R})$?

8. Supercompactness

Definition 8.1 (Woodin, Definition 220 in [9])**.** For an ordinal α we say that κ is V_α-*supercompact* if there exists some $\beta > \alpha$ and an elementary embedding $j\colon V_\beta \to N$ such that:

   (1) N is a transitive set and $N^{V_\alpha} \subseteq N$,
   (2) the critical point of j is κ (in particular j is non-trivial),
   (3) $\alpha < j(\kappa)$.

If κ is V_α-supercompact for all α, we say that it is a *supercompact* cardinal.

Assuming the Axiom of Choice holds, this definition is of course equivalent to the standard definition by deriving fine and normal measures from the elementary embedding.

Given an infinite regular cardinal λ and a non-empty set X, $\mathrm{Col}(\lambda, <X)$ is the forcing, ordered by reverse inclusion, of all functions p such that $|\mathrm{dom}\, p| < \lambda$,

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[8]Note that $L(\mathbb{R})$ is a definable inner model of the Chang model.

[9]The fact the elementary embeddings from the ground model lift follows from Theorem 4.9 in [4]. While the proof there does nothing to prove the needed closure properties of these lifted embeddings, this is not hard to verify by hand.

$\operatorname{dom} p \subseteq \lambda \times X$, and $p(\alpha, x) \in x$ for all $\alpha < \lambda$ and $x \in X$. It is straightforward to see that $\operatorname{Col}(\omega, <X)$ is weakly homogeneous and that forcing with $\operatorname{Col}(\omega, <X)$ adds surjections from ω onto all members of X.

The following technical lemma, Lemma 8.2, is a rendering of [9], Theorem 226. This lemma will be crucially used in the proof of Lemma 9.1 (and of Lemma 9.3).

Lemma 8.2 (Woodin; $\mathsf{ZF} + \mathsf{DC}$). *Suppose κ is a supercompact cardinal and $\mathbb{P} \in V_\kappa$ is a forcing notion preserving DC. There is then a strictly increasing sequence $\langle \kappa_\xi \mid \xi < \kappa \rangle$ of strongly inaccessible cardinals below κ, together with a sequence $\langle \mathcal{Q}_\xi \mid 1 \le \xi \le \kappa \rangle$ of forcing notions and a sequence $\langle \dot{\mathcal{Q}}_\xi^{\mathbb{P}} \mid 1 \le \xi \le \kappa \rangle$ of $\mathbb{P}$-names for forcing notions, satisfying the following conditions.*

(1) *κ_0 is such that $\mathbb{P} \in V_{\kappa_0}$ and such that*
- *$\operatorname{Col}(\omega_1, <V_{\kappa_0})$ forces $\mathsf{DC}_{<\kappa_0}$ over V_κ, and*
- *$\mathbb{P}$ forces that $\operatorname{Col}(\omega_1, <V_{\kappa_0}[\dot{G}_\mathbb{P}])$ forces $\mathsf{DC}_{<\kappa_0}$ over $V_\kappa[\dot{G}_\mathbb{P}]$.*

Also,
- *$\mathcal{Q}_1 = \operatorname{Col}(\omega_1, <V_{\kappa_0})$, and*
- *$\dot{\mathcal{Q}}_1^{\mathbb{P}}$ is a $\mathbb{P}$-name for $\operatorname{Col}(\omega_1, <V_{\kappa_0}[\dot{G}_\mathbb{P}])$.*

(2) *For every $\xi > 0$, letting $\gamma_\xi = \sup\{\kappa_{\xi'} \mid \xi' < \xi\}$, κ_ξ is such that*
- *$\mathcal{Q}_\xi$ forces that $\operatorname{Col}(\gamma_\xi, <V_{\kappa_\xi}[\dot{G}_{\mathcal{Q}_\xi}])$ forces $\mathsf{DC}_{<\kappa_\xi}$ over V_κ, and*
- *$\mathbb{P} * \dot{\mathcal{Q}}_\xi^{\mathbb{P}}$ forces that $\operatorname{Col}(\gamma_\xi, <V_{\kappa_\xi}[\dot{G}_{\mathbb{P}*\dot{\mathcal{Q}}_\xi^{\mathbb{P}}}])$ forces $\mathsf{DC}_{<\kappa_\xi}$ over $V_\kappa[\dot{G}_\mathbb{P}]$.*

Also,
- *$\mathcal{Q}_{\xi+1} = \mathcal{Q}_\xi * \operatorname{Col}(\gamma_\xi, <V_{\kappa_\xi}[\dot{G}_{\mathcal{Q}_\xi}])$, and*
- *$\dot{\mathcal{Q}}_{\xi+1}^{\mathbb{P}}$ is a $\mathbb{P}$-name for $\dot{\mathcal{Q}}_\xi^{\mathbb{P}} * \operatorname{Col}(\gamma_\xi, <V_{\kappa_\xi}[\dot{G}_{\dot{\mathcal{Q}}_\xi^{\mathbb{P}}}])$.*

(3) *For every limit ξ, $\mathcal{Q}_\xi$ is the Easton limit of $\langle \mathcal{Q}_{\xi'} \mid \xi' < \xi \rangle$ and $\dot{\mathcal{Q}}_\xi^{\mathbb{P}}$ is a $\mathbb{P}$-name for the Easton limit of $\langle \dot{\mathcal{Q}}_{\xi'} \mid \xi' < \xi \rangle$.*

(4) *For every $\bar{\delta} < \kappa$ such that $\mathbb{P} \in V_{\bar{\delta}}$, if $\bar{\delta}$ is supercompact in V (in $V^\mathbb{P}$), then $\bar{\delta}$ remains supercompact after forcing with $\mathcal{Q}_\kappa$ over V (after forcing with $\mathbb{P} * \dot{\mathcal{Q}}_\kappa^{\mathbb{P}}$).*

(5) *If $H \subseteq \mathcal{Q}_\kappa$ is V-generic, then $V_\kappa[H] \models \mathsf{ZFC}$.*

(6) *If $G \subseteq \mathbb{P}$ is V-generic and $H \subseteq \mathcal{Q}_\kappa^{\mathbb{P}}$ is $V[G]$-generic, then $V_\kappa[G][H] \models \mathsf{ZFC}$.*

(7) *$\mathcal{Q}_\kappa$ is weakly homogeneous.*

Proof. Except for (7), the proof of all conclusions is the same as in the proof of Theorem 226 in [9]. As to conclusion (7), suppose q_0 and q_1 are two $\mathcal{Q}_\kappa$-conditions. We want to show that there is an automorphism $\pi \colon \mathcal{Q}_\kappa \to \mathcal{Q}_\kappa$ such that πq_0 and q_1 are compatible. We may assume without any loss of generality that q_0 and q_1 have the same support $E \subseteq \kappa$ and that for every $\xi \in E$, the trivial condition forces $d_\xi := \operatorname{dom}(q_0(\xi)) = \operatorname{dom}(q_1(\xi))$.

Let d'_ξ (for $\xi \in E$) be a uniformly defined sequence such that each d'_ξ is a name for a subset of $V_{\kappa_\xi}[\dot{G}_{\mathcal{Q}_\xi}]$ disjoint from d_ξ and such that there is a sequence $\langle \dot{d}oth_\xi \mid \xi \in \sigma \rangle$ of names for bijections

$$\dot{h}_\xi \colon V_{\kappa_\xi}[\dot{G}_{\mathcal{Q}_\xi}] \to V_{\kappa_\xi}[\dot{G}_{\mathcal{Q}_\xi}]$$

mapping d'_ξ onto d_ξ. One way to find $\langle d'_\xi \mid \xi \in \sigma \rangle$ and $\langle \dot{h}_\xi \mid \xi \in \sigma \rangle$ is to let d'_ξ be a canonically chosen name for $d_\xi \times \{d_\xi\}$ and let $\dot{h}_\xi$ be a canonically chosen name for the function on $V_{\kappa_\xi}[\dot{G}_{\mathcal{Q}_\xi}]$ which is the identity on $V_{\kappa_\xi}[\dot{G}_{\mathcal{Q}_\xi}] \setminus (d_\xi \cup d'_\xi)$ and which, for $x \in d_\xi$, sends x to $\langle x, d_\xi \rangle$ and $\langle x, d_\xi \rangle$ to x. Let now $\pi \colon \mathcal{Q}_\kappa \to \mathcal{Q}_\kappa$ be the function sending $q \in \mathcal{Q}_\kappa$ to the condition with the same support as q such that for every $\xi \in \operatorname{supp}(q)$, $(\pi q)(\xi)$ is forced to be $q(\xi)$ unless $\xi \in E$, in which case $(\pi q)(\xi)$ is forced to be the condition in the relevant Levy collapsing whose domain is the set of $\langle \alpha, x \rangle$ such that $\langle \alpha, \dot{h}_\xi(x) \rangle \in \operatorname{dom}(q(\xi))$ and such that for every such $\langle \alpha, x \rangle$,

$$((\pi q)(\xi))(i, x) = (q(\xi))(i, \dot{h}_\xi(x)).$$

But then π is an automorphism of $\mathcal{Q}_\kappa$ with the desired property. $\qquad\square$

9. Generic absoluteness for the Chang model

Lemma 9.1 and its companion, Lemma 9.3, are the key ingredients in the proof of Theorem 9.4.

Lemma 9.1. *(ZF + DC) Suppose $\delta < \kappa$ are supercompact cardinals, $\mathbb{P} \in V_\delta$ is a forcing notion preserving DC, $G \subseteq \mathbb{P}$ is a V-generic filter, and $\langle \mathcal{Q}_\xi \mid \xi \leq \kappa \rangle$ is the iteration given by Lemma 8.2. Then there is, in some outer model, a $V[G]$-generic filter $K \subseteq \mathcal{Q}_\kappa \times \mathrm{Col}(\omega, <\delta)$, together with an elementary embedding*

$$j \colon \mathcal{C}^{V_\kappa[G]} \to \mathcal{C}^{V_\kappa[K]}$$

Proof. Let $G \subseteq \mathbb{P}$ be a V-generic filter and let $H \subseteq \mathcal{Q}_\kappa^{\mathbb{P}}$ be a $V[G]$-generic filter. By Lemma 8.2 (4) and (5), δ remains supercompact of δ in $V_\kappa[G][H]$ and $V_\kappa[G][H] \models$ ZFC. Hence, by a classical ZFC result of Woodin there is, in some outer model N, a $V_\kappa[G][H]$-generic filter $J \subseteq \mathrm{Col}(\omega, <\delta)$, for which there is an elementary embedding

$$j \colon \mathcal{C}^{V_\kappa[G][H]} \to \mathcal{C}^{V_\kappa[G][H][J]}.$$

Claim 9.2. *For every limit ordinal $\alpha \leq \kappa$, $\mathcal{Q}_\alpha$ and $\mathcal{Q}_\alpha^{\mathbb{P}}$ are forcing-equivalent in $V[G]$.*

Proof. It suffices to prove, by induction on α, for $\alpha < \kappa$ a nonzero limit ordinal, that $\mathcal{Q}_\alpha$ and $\mathcal{Q}_\alpha^{\mathbb{P}}$ are forcing-equivalent in $V[G]$, and for this it is enough to prove that if $\xi < \kappa$, and $H_\xi \subseteq \mathcal{Q}_\xi$ and $H_\xi' \subseteq \mathcal{Q}_\xi^{\mathbb{P}}$ are $V[G]$-generic, such that $V[G][H_\xi] = V[G][H_\xi']$, then the following holds by a uniform procedure (in ξ).

(1) If $H_{\xi+1} \subseteq \mathcal{Q}_{\xi+1}$ is $V[G]$-generic such that $H_{\xi+1} \cap \mathcal{Q}_\xi = H_\xi$, then there is a bijection $f_\xi \colon V_{\kappa_\xi}[H_\xi] \to V_{\kappa_\xi}[G][H_\xi']$ in $V[G][H_{\xi+1}]$.

(2) If $H_{\xi+1}' \subseteq \mathcal{Q}_{\xi+1}'$ is $V[G]$-generic such that $H_{\xi+1}' \cap \mathcal{Q}_\xi = H_\xi'$, then there is a bijection $f_\xi' \colon V_{\kappa_\xi}[H_\xi] \to V_{\kappa_\xi}[G][H_\xi']$ in $V[G][H_{\xi+1}']$.

Now, the equality $V[G][H_\xi] = V[G][H_\xi']$ gives us in particular that $V_{\kappa_\xi}[H_\xi] \subseteq V_{\kappa_\xi}[G][H_\xi']$. But then we can find f_ξ if we are in the situation (1) (resp., f_ξ' if we are in situation (2)) by Cantor–Bernstein's theorem, since $V_{\kappa_\xi}[G][H_\xi] = V_{\kappa_\xi}[G][H_\xi']$ and since we have one-to-one functions $i_\xi, i_\xi' \colon V_{\kappa_\xi}[G][H_\xi] \to V_{\kappa_\xi}[H_\xi]$ in $V[G][H_{\xi+1}]$ and $V[G][H_{\xi+1}']$, respectively, defined by letting $i_\xi(x)$ be the $\leq_\xi$-first $\mathbb{P}$-name $\dot{x} \in V_{\kappa_\xi}[H_\xi]$ such that $\dot{x}_G = x$, where $\leq_\xi$ is a well-order of $V_{\kappa_\xi}[H_\xi]$ canonically definable from $H_{\xi+1}$ and, similarly, letting $i_\xi'(x)$ be the $\leq_\xi'$-first $\mathbb{P}$-name $\dot{x} \in V_{\kappa_\xi}[H_\xi]$ such that $\dot{x}_G = x$, where $\leq_\xi'$ is a well-order of $V_{\kappa_\xi}[H_\xi]$ canonically definable from $H_{\xi+1}'$. Finally, the above description is obviously uniform in ξ. $\qquad\square$

The above claim finishes the proof since then there is some $V[G]$-generic filter $K \subseteq \mathcal{Q}_\kappa \times \mathrm{Col}(\omega, <\delta)$ such that $V[G][K] = V[G][H][J]$ and since $\mathcal{C}^{V_\kappa[G][H]} = \mathcal{C}^{V_\kappa[G]}$ as $\mathcal{Q}_\kappa^{\mathbb{P}}$ is σ-closed in the DC-model $V[G]$ (which follows immediately from the definition of $\mathcal{Q}_\kappa^{\mathbb{P}}$). $\qquad\square$

Similarly, we can prove the following lemma.

Lemma 9.3 (ZF + DC). *Suppose $\delta < \kappa$ are supercompact cardinals, $\mathbb{P} \in V_\delta$ is a forcing notion preserving DC, and $\langle \mathcal{Q}_\xi \mid \xi \leq \kappa \rangle$ is the iteration given by Lemma 8.2. Then there is, in some outer model, a V-generic $K \subseteq \mathcal{Q}_\kappa \times \mathrm{Col}(\omega, <\delta)$, together with an elementary embedding $j \colon \mathcal{C}^{V_\kappa} \to \mathcal{C}^{V_\kappa[K]}$.*

The main theorem in this section is the following.

Theorem 9.4 ($\mathsf{ZF} + \mathsf{DC}$). *Suppose for every closed and unbounded Π_2-definable class C of ordinals there is a supercompact cardinal $\kappa \in C$. Then, for every set-forcing $\mathbb{P}$ and every V-generic filter $G \subseteq \mathbb{P}$, if $V[G] \models \mathsf{DC}$, then the structures $(\mathcal{C}^V; \in, r)_{r \in \mathbb{R}^V}$ and $(\mathcal{C}^{V[G]}; \in, r)_{r \in \mathbb{R}^V}$ have the same Σ_2-theory.*

Proof. Suppose, towards a contradiction, that $\mathbb{P}$ forces that $(\mathcal{C}^V; \in, r)_{r \in \mathbb{R}^V}$ and $(\mathcal{C}^{V[\dot{G}]}; \in, r)_{r \in \mathbb{R}^V}$ disagree on the truth value of some sentence $\exists x \forall y \varphi(x, y)$, where φ is a restricted formula. By our large cardinal assumption we may then fix a supercompact cardinal κ such that

$$(\mathcal{C}^{V_\kappa}; \in, r)_{r \in \mathbb{R}^V} \models \exists x \forall y \varphi(x, y) \iff (\mathcal{C}^{V_\kappa[G]}; \in, r)_{r \in \mathbb{R}^V} \models \neg \exists x \forall y \varphi(x, y)$$

and such that there is a supercompact cardinal $\delta < \kappa$ such that $\mathbb{P} \in V_\delta$.

We show that $(\mathcal{C}^{V_\kappa}; \in, r)_{r \in \mathbb{R}^V}$ and $(\mathcal{C}^{V_\kappa[G]}; \in, r)_{r \in \mathbb{R}^V}$ are elementarily equivalent, which will be a contradiction.

By Lemma 9.1 and Lemma 9.3 we know that there are, in some outer model, $K, K' \subseteq \mathcal{Q}_\kappa \times \mathrm{Col}(\omega, <\delta)$ which are V-generic and $V[G]$-generic, respectively, for which there are elementary embeddings

$$j \colon \mathcal{C}^{V_\kappa} \to \mathcal{C}^{V_\kappa[K]}$$

and

$$j' \colon \mathcal{C}^{V_\kappa[G]} \to \mathcal{C}^{V_\kappa[J']}$$

(where $\langle \mathcal{Q}_\xi \mid \xi \leq \kappa \rangle$ is the iteration given by Lemma 8.2). But then, by the weak homogeneity of $\mathcal{Q}_\kappa \times \mathrm{Col}(\omega, <V_\delta)$—where the weak homogeneity of $\mathcal{Q}_\kappa$ is given by Lemma 8.2 (7)—, the theories of $(\mathcal{C}^{V_\kappa}; \in, r)_{r \in \mathbb{R}^V}$ and $(\mathcal{C}^{V_\kappa[G]}; \in, r)_{r \in \mathbb{R}^V}$ are the same. $\square$

Using a similar argument one can prove the following.

Theorem 9.5 ($\mathsf{ZF} + \mathsf{DC}$). *Suppose there is, for every closed and unbounded class of ordinals C, a supercompact cardinal κ such that $\kappa \in C$. Then, for every set-forcing $\mathbb{P}$ and every V-generic filter $G \subseteq \mathbb{P}$, if $V[G] \models \mathsf{DC}$, then the structures $(\mathcal{C}^V; \in, r)_{r \in \mathbb{R}^V}$ and $(\mathcal{C}^{V[G]}; \in, r)_{r \in \mathbb{R}^V}$ are elementarily equivalent.*

Note the second order character of the hypothesis of Theorem 9.5. This hypothesis can of course be taken to be an infinite scheme asserting that for every formula $\Theta(x)$ with parameters defining a closed and unbounded class of ordinals there is some supercompact cardinal κ such that $\Theta(\kappa)$. Many reasonable large cardinal assumption could be used here in place of this. Nevertheless, it is not clear to us whether the existence of one supercompact cardinal suffices to yield the conclusion.

The following version of Theorem 9.4 for $L(\mathbb{R})$ can be derived by the same argument as in the proof of Theorem 9.4 together with the fact that, under the existence of a supercompact cardinal, $\mathbb{R}^\#$ exists in any set-forcing extension, and together with Woodin's classical result that the Axiom of Determinacy holds in $L(\mathbb{R})$ assuming ZFC and the existence of infinitely many Woodin cardinals below a measurable cardinal.

Theorem 9.6 ($\mathsf{ZF} + \mathsf{DC}$). *Suppose there are two supercompact cardinals. Then, for every set-forcing $\mathbb{P}$ and every V-generic filter $G \subseteq \mathbb{P}$-generic filter G over V, if $V[G] \models \mathsf{DC}$, then*

(1) *the structures $(L(\mathbb{R})^V; \in, r)_{r \in \mathbb{R}^V}$ and $(L(\mathbb{R})^{V[G]}; \in, r)_{r \in \mathbb{R}^V}$ are elementarily equivalent, and*

(2) *the Axiom of Determinacy holds in $L(\mathbb{R})^{V[G]}$.*

Acknowledgements

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