THE MINIMAL SIZE OF INFINITE MAXIMAL ANTICHAINS IN DIRECT PRODUCTS OF PARTIAL ORDERS

Miloš S. Kurilić

Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
e-mail: milos@dmi.uns.ac.rs

Abstract

For a partial order $\mathbb{P}$ having infinite antichains by $\mathfrak{a}(\mathbb{P})$ we denote the minimal cardinality of an infinite maximal antichain in $\mathbb{P}$ and investigate how does this cardinal invariant of posets behave in finite products. In particular we show that $\min\{\mathfrak{a}(\mathbb{P}), \mathfrak{p}(\operatorname{sq}\mathbb{P})\} \leq \mathfrak{a}(\mathbb{P}^n) \leq \mathfrak{a}(\mathbb{P})$, for all $n \in \mathbb{N}$, where $\mathfrak{p}(\operatorname{sq}\mathbb{P})$ is the minimal size of a centered family without a lower bound in the separative quotient of the poset $\mathbb{P}$, or $\mathfrak{p}(\operatorname{sq}\mathbb{P}) = \infty$, if there is no such family. So we have $\mathfrak{a}(\mathbb{P} \times \mathbb{P}) = \mathfrak{a}(\mathbb{P})$ whenever $\mathfrak{p}(\operatorname{sq}\mathbb{P}) \geq \mathfrak{a}(\mathbb{P})$ and we show that, in addition, this equality holds for all infinite Boolean algebras of size $\leq \omega_1$ (without zero), all reversed trees, all atomic posets and, in particular, for all posets of the form $\langle \mathcal{C}, \subset \rangle$, where $\mathcal{C}$ is a family of nonempty closed sets in a compact T_1-space containing all singletons. As a by-product we obtain the following combinatorial statement: If X is an infinite set and $\{A_i \times B_i : i \in I\}$ an infinite partition of the square X^2, then at least one of the families $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ contains an infinite partition of X.

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1 Introduction

For a partial order $\mathbb{P}$ having infinite antichains, the cardinal invariant $\mathfrak{a}(\mathbb{P})$ is defined as the minimal size of an infinite maximal antichain in $\mathbb{P}$ and can be regarded as a generalization of the invariant of the continuum $\mathfrak{a} := \mathfrak{a}((P(\omega)/\operatorname{Fin})^+)$ (the *almost-disjointness number*). More about the other "small cardinals" mentioned here, $\mathfrak{p}$ (the *pseudointersection number*), $\mathfrak{t}$ (the *tower number*), $\mathfrak{h}$ (the *distributivity number*) and $\mathfrak{b}$ (the *unbounding number*) can be found in [1, 2].

Is there a partial order $\mathbb{P}$ such that $\mathfrak{a}(\mathbb{P} \times \mathbb{P}) < \mathfrak{a}(\mathbb{P})$? Unfortunately, although the author spent lot of time on this topic, a reader interested in this question will not find an answer in the present paper which, in fact, shows that the equality

$$\mathfrak{a}(\mathbb{P} \times \mathbb{P}) = \mathfrak{a}(\mathbb{P}) \tag{1}$$

holds over a large class of partial orders.

While, by a theorem of Kurepa [6], the square of a reversed Suslin tree has uncountable antichains and, hence, it is consistent that the cellularity of a product is larger than the cellularity of its factors, the inequality $\mathfrak{a}(\mathbb{P}\times\mathbb{P}) \leq \mathfrak{a}(\mathbb{P})$ holds for each poset $\mathbb{P}$ having infinite antichains (if A is a maximal antichain in $\mathbb{P}$, then $A\times A$ is a maximal antichain in $\mathbb{P}\times\mathbb{P}$). In fact, we even do not know *is it consistent* that the inequality is strong for some poset. Concerning the last question we note that, as far as we know, it is not clear what is going on with the poset $(P(\omega)/\,\mathrm{Fin})^+$. Namely, in [10] Spinas defined the small cardinals $\mathfrak{a}_\lambda := \mathfrak{a}(((P(\omega)/\,\mathrm{Fin})^+)^\lambda)$, for $\lambda \leq \omega$, proved that $\mathfrak{b} \leq \mathfrak{a}_\omega \leq \ldots \leq \mathfrak{a}_2 \leq \mathfrak{a}_1 = \mathfrak{a}$ and showed the consistency of $\mathfrak{b} < \mathfrak{a}_n$, for all $n \in \mathbb{N}$. It is known (see [1, 2, 7]) that $\omega_1 \leq \mathfrak{p} = \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}$, thus in models of $\mathfrak{b} = \mathfrak{a}$ we have $\mathfrak{a}_\lambda = \mathfrak{a}$, for all $\lambda \leq \omega$, and, in particular, this holds under CH or MA (implying that $\mathfrak{p} = \mathfrak{c}$). As far as we know it is not known whether the inequality $\mathfrak{a}_2 < \mathfrak{a}$ is consistent with ZFC. We remark that the consistency of $\mathfrak{h}_2 < \mathfrak{h}$ is proved by Shelah and Spinas (see [8] and [9]).

The equality (1) holds over several important classes of posets. For example, if $\mathbb{B}$ is an infinite complete Boolean algebra, then, clearly, $\mathfrak{a}(\mathbb{B}^+) = \omega$ and, hence, $\mathfrak{a}((\mathbb{B}^+)^n) = \mathfrak{a}(\mathbb{B}^+)$, for all $n \in \mathbb{N}$. The same holds for each infinite Boolean algebra $\mathbb{B}$ of size $\leq \omega_1$ (see Corollary 4.2). In Section 3 we show that, in particular, $\mathfrak{a}(\mathbb{P}\times\mathbb{P}) \geq \min\{\mathfrak{a}(\mathbb{P}), \mathfrak{p}(\mathrm{sq}\,\mathbb{P})\}$, which implies that the equality (1) holds for each poset $\mathbb{P}$ satisfying $\mathfrak{p}(\mathrm{sq}\,\mathbb{P}) \geq \mathfrak{a}(\mathbb{P})$. Some consequences are given in Section 4; for example, the equality (1) holds if $\mathbb{P} = \langle \mathcal{C}, \subset\rangle$, where $\mathcal{C}$ is a collection of nonempty closed sets in a compact T_1 space containing all singletons. In Sections 5 and 6 we show that the equality (1) holds for all reversed trees and all atomic posets.

2 Preliminaries

If $\mathbb{P} = \langle P, \leq\rangle$ is a partial order, then two elements p and q of P are called *compatible* iff there is $r \in P$ such that $r \leq p$ and $r \leq q$; otherwise p and q are called *incompatible* and we write $p \perp q$. A set $D \subset P$ is called *dense in* $\mathbb{P}$ iff for each $p \in P$ there is $q \in D$ satisfying $q \leq p$. A set is dense in a Boolean algebra $\mathbb{B}$ iff it is dense in the poset $\mathbb{B}^+ = \langle B \setminus \{0_\mathbb{B}\}, \leq_\mathbb{B}\rangle$.

Separative quotient A partial order $\mathbb{P} = \langle P, \leq\rangle$ is called *separative* iff for each $p, q \in P$ satisfying $p \not\leq q$ there is $r \in P$ such that $r \leq p$ and $r \perp q$. The *separative modification* of $\mathbb{P}$ is the pre-order $\mathrm{sm}\,\mathbb{P} = \langle P, \leq^*\rangle$, where $p \leq^* q$ iff $\forall r \leq p\ \exists s \leq r\ s \leq q$. The *separative quotient* of $\mathbb{P}$ is the separative partial order $\mathrm{sq}\,\mathbb{P} = \langle P/{=^*}, \trianglelefteq\rangle$, where $p =^* q \Leftrightarrow p \leq^* q \wedge q \leq^* p$ and $[p] \trianglelefteq [q] \Leftrightarrow p \leq^* q$. A proof of the following well known facts can be found in [4].

Fact 2.1 *If $\mathbb{P} = \langle P, \leq \rangle$ is a non-separative partial order, then the corresponding quotient mapping $h : \mathbb{P} \to \operatorname{sq}\mathbb{P}$, given by $h(p) = [p]$, is an epimorphism such that for all $p, q \in P$ we have*

$$p \perp_{\mathbb{P}} q \Leftrightarrow h(p) \perp_{\operatorname{sq}\mathbb{P}} h(q). \tag{2}$$

The poset $\operatorname{sq}\mathbb{P}$ is, up to isomorphism, the unique separative partial ordering $\mathbb{P}'$ such that there is an epimorphism $h : \mathbb{P} \to \mathbb{P}'$ satisfying (2).

A poset $\mathbb{P}$ is separative iff $\operatorname{sq}\mathbb{P} \cong \mathbb{P}$ iff $\mathbb{P}$ is isomorphic to a dense set of some Boolean algebra, iff $\mathbb{P}$ is isomorphic to a dense set of some (unique up to isomorphism) complete Boolean algebra (the Boolean completion of $\mathbb{P}$, denoted by $\operatorname{ro}\mathbb{P}$).

Atomic and atomless posets If $\mathbb{P} = \langle P, \leq \rangle$ is a partial order, an element p of P is an *atom* iff there are no $q, r \leq p$ such that $q \perp r$. If $\operatorname{At}(\mathbb{P})$ denotes the set of all atoms of $\mathbb{P}$, then $\mathbb{P}$ is called: *atomless* iff $\operatorname{At}(\mathbb{P}) = \emptyset$, *atomic* iff $\operatorname{At}(\mathbb{P})$ is a dense subset of $\mathbb{P}$.

Fact 2.2 *Let $\mathbb{P} = \langle P, \leq \rangle$ be a partial order and $p \in P$. Then*

(a) If $p \in \operatorname{At}(\mathbb{P})$, then $p \downarrow \subset [p] \cap \operatorname{At}(\mathbb{P})$;

(b) $p \in \operatorname{At}(\mathbb{P})$ iff $[p] \in \operatorname{At}(\operatorname{sq}\mathbb{P})$;

(c) $\mathbb{P}$ is atomless iff $\operatorname{sq}\mathbb{P}$ is atomless;

(d) $\mathbb{P}$ is atomic iff $\operatorname{sq}\mathbb{P}$ is atomic;

(e) $\mathbb{P}$ is separative and atomic with κ atoms iff $\mathbb{P}$ is isomorphic to a suborder of $P(\kappa)^+$ containing all singletons;

Proof. (a) Let $q \leq p \in \operatorname{At}(\mathbb{P})$. Then $q \leq^* p$ and for each $r \leq p$ there is $s \leq q, r$, which means that $p \leq^* q$. So $p =^* q$, that is $q \in [p]$. It is clear that $q \in \operatorname{At}(\mathbb{P})$.

(b) If p is not an atom in $\mathbb{P}$, then there are $q, r \leq p$ such that $q \perp r$ and, by Fact 2.1, $[q], [r] \trianglelefteq [p]$ and $[q] \perp_{\operatorname{sq}\mathbb{P}} [r]$; thus $[p]$ is not an atom in $\operatorname{sq}\mathbb{P}$. Conversely, if $[p]$ is not an atom in $\operatorname{sq}\mathbb{P}$, then there are $[q], [r] \trianglelefteq [p]$ such that $[q] \perp_{\operatorname{sq}\mathbb{P}} [r]$. Since $q, r \leq^* p$ there are $s, t \in P$ such that

$$s \leq q, p \quad \text{and} \quad t \leq r, p \tag{3}$$

and, hence, $s, t \leq p$. Suppose that $s \not\perp_{\mathbb{P}} t$. Then, by (2), $[s] \not\perp_{\operatorname{sq}\mathbb{P}} [t]$ and by (3) we have $[s] \trianglelefteq [q]$ and $[t] \trianglelefteq [r]$, which is impossible because $[q] \perp_{\operatorname{sq}\mathbb{P}} [r]$. Thus $s \perp_{\mathbb{P}} t$ and p is not an atom of $\mathbb{P}$. Clearly, (c) follows from (b).

(d) If $\mathbb{P}$ is atomic, $[p] \in \operatorname{sq}\mathbb{P}$ and $a \in \operatorname{At}(\mathbb{P})$, where $a \leq p$, then $[a] \trianglelefteq [p]$ and, by (b), $[a] \in \operatorname{At}(\operatorname{sq}\mathbb{P})$. Thus the set $\operatorname{At}(\operatorname{sq}\mathbb{P})$ is a dense in $\operatorname{sq}\mathbb{P}$.

If $\operatorname{sq}\mathbb{P}$ is atomic, $p \in \mathbb{P}$ and $[a] \in \operatorname{At}(\operatorname{sq}\mathbb{P})$, where $[a] \trianglelefteq [p]$, then $a \leq^* p$ and, hence, there is $b \leq a, p$. Since by (b) we have $a \in \operatorname{At}(\mathbb{P})$, by (a) we obtain $b \in \operatorname{At}(\mathbb{P})$. Thus the set $\operatorname{At}(\mathbb{P})$ is a dense in $\mathbb{P}$.

(e) Let $\mathrm{At}(\mathbb{P}) = \{a_\alpha : \alpha \in \kappa\}$ be an enumeration. Since the set $\mathrm{At}(\mathbb{P})$ is dense in $\mathbb{P}$, the function $f : \mathbb{P} \to P(\kappa)$ defined by $f(p) = \{\alpha \in \kappa : a_\alpha \leq p\}$ maps $\mathbb{P}$ into $P(\kappa)^+$ and, clearly, $p \leq q$ implies $f(p) \subset f(q)$. If $\neg p \leq q$, then, since P is separative, there is $r \leq p$ such that $r \perp q$ and there is $\alpha \in \kappa$, such that $a_\alpha \leq r$, which implies $\neg a_\alpha \leq q$. So, $\alpha \in f(p) \setminus f(q)$ and, hence, $\neg f(p) \subset f(q)$. Thus, for each $p, q \in P$ we have $p \leq q$ iff $f(p) \subset f(q)$, that is f is an embedding of $\mathbb{P}$ into $P(\kappa)^+$ and, clearly, $f(a_\alpha) = \{\alpha\}$, for each $\alpha \in \kappa$.

If $\mathbb{P}$ is a suborder of $P(\kappa)^+$ and $[\kappa]^1 \subset P$, then P is dense in $P(\kappa)$ and, by Fact 2.1, $\mathbb{P}$ is separative. Clearly, $\mathrm{At}(\mathbb{P}) = [\kappa]^1$ is dense in $\mathbb{P}$ so $\mathbb{P}$ is atomic. □

Centered families If $\mathbb{P} = \langle P, \leq \rangle$ is a partial order, a family $C \subset P$ is called *centered* iff each finite nonempty subset $K \subset C$ has a lower bound in P. A centered family $C \subset P$ is called a *maximal centered family* iff for each centered family $C' \subset P$ satisfying $C \subset C'$ we have $C' = C$. An easy application of Zorn's lemma shows that each centered family is contained in some maximal centered family.

If a poset $\mathbb{P}$ contains a centered family without a lower bound we define

$$\mathfrak{p}(\mathbb{P}) := \min\{|C| : C \subset P \text{ is a centered family without a lower bound}\}$$

and, since each finite centered family has a lower bound, we have $\mathfrak{p}(\mathbb{P}) \geq \omega$ and, hence, $|P| \geq \omega$. Otherwise, we define $\mathfrak{p}(\mathbb{P}) = \infty$.

Fact 2.3 *Let $\mathbb{P} = \langle P, \leq \rangle$ be a partial order. Then*

(a) If the poset $\mathbb{P}$ is not atomic, then $\mathfrak{p}(\mathbb{P}) < \infty$ and $\mathfrak{p}(\mathrm{sq}\,\mathbb{P}) < \infty$;

(b) A finite set $K \subset P$ has a lower bound in $\mathbb{P}$ iff K has a lower bound in $\mathrm{sm}\,\mathbb{P}$ iff $h[K]$ has a lower bound in $\mathrm{sq}\,\mathbb{P}$;

(c) $\mathfrak{p}(\mathbb{P}) \leq \mathfrak{p}(\mathrm{sq}\,\mathbb{P})$.

Proof. (a) If the poset $\mathbb{P}$ is not atomic, then there is $p \in P$ such that

$$\forall q \leq p \ \ \exists r, s \leq q \ \ r \perp s. \tag{4}$$

Let C be a maximal centered family in $\mathbb{P}$ such that $p \in C$. Suppose that C has a lower bound q. Then $q \leq p$ and, by (4), there are $r, s \leq q$ such that $r \perp s$, which implies $r < q$ and, hence, $r \notin C$. If $K \in [C]^{<\omega}$, then $r \leq \{r\} \cup K$, thus $C \cup \{r\}$ is a centered family larger than C, which is impossible. So C is a centered family in $\mathbb{P}$ without a lower bound, which implies that $\mathfrak{p}(\mathbb{P}) < \infty$. By Fact 2.2(d) the poset $\mathrm{sq}\,\mathbb{P}$ is not atomic as well and, by the previous consideration, $\mathfrak{p}(\mathrm{sq}\,\mathbb{P}) < \infty$.

(b) We show that for $K = \{a_1, \ldots, a_n\} \subset P$ the following three conditions are equivalent: (i) $\exists c \in P \ \ c \leq \{a_i : i \leq n\}$, (ii) $\exists d \in P \ \ d \leq^* \{a_i : i \leq n\}$

and (iii) $\exists [d] \in P/=^* \; [d] \trianglelefteq \{[a_i] : i \leq n\}$. Since $x \leq y$ implies $x \leq^* y$, which implies $[x] \trianglelefteq [y]$ we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (ii) is true since the relation $\trianglelefteq$ is well-defined. For a proof of (ii) $\Rightarrow$ (i) we assume that $d \in P$ and

$$\forall i \leq n \;\; \forall c \leq d \;\; \exists e \leq c \;\; e \leq a_i. \tag{5}$$

By recursion we define $c_0, \dots, c_n \in P$ such that $c_0 = d$ and $c_i \leq c_{i-1}, a_i$, for $i > 0$. Suppose that the sequence $c_0, \dots, c_{i-1}$ satisfies the conditions. By (5), for i and $c = c_{i-1}$ there is $c_i \leq c_{i-1}, a_i$, and the recursion works. So $c_n \leq c_{n-1} \leq \dots \leq c_1$ and for each $i > 0$ we have $c_i \leq a_i$, which implies that $c_n \leq a_i$ for all $i > 0$.

(c) Suppose that $\kappa := \mathfrak{p}(\operatorname{sq}\mathbb{P}) < \mathfrak{p}(\mathbb{P})$ and that $\{[a_\alpha] : \alpha < \kappa\}$ is a centered family without a lower bound in the poset $\operatorname{sq}\mathbb{P}$. By (b), $\{a_\alpha : \alpha < \kappa\}$ is a centered family in $\mathbb{P}$ and, since $\kappa < \mathfrak{p}(\mathbb{P})$, it has a lower bound in $\mathbb{P}$, say a. So, for each $\alpha < \kappa$ we have $a \leq a_\alpha$ and, hence, $[a] \trianglelefteq [a_\alpha]$, which is impossible. □

Antichains If $\mathbb{P} = \langle P, \leq \rangle$ is a partial order, a set $A \subset P$ is called an *antichain in* $\mathbb{P}$ iff $a \perp b$, for different $a, b \in A$. By Zorn's lemma each antichain is contained in a maximal one. Let $\mathcal{A}(\mathbb{P}) := \{|A| : A \text{ is a maximal antichain in } \mathbb{P}\}$ and $\operatorname{cc}(\mathbb{P}) := \min\{\kappa : \text{ each antichain in } \mathbb{P} \text{ is of size } < \kappa\}$. By a theorem of Tarski, $\operatorname{cc}(\mathbb{P})$ is a finite or an uncountable regular cardinal (see [5], p. 245).

If $\mathbb{P}$ contains infinite antichains, let us define $\mathfrak{a}(\mathbb{P}) := \min(\mathcal{A}(\mathbb{P}) \setminus \omega)$, that is

$$\mathfrak{a}(\mathbb{P}) := \min\{|A| : A \subset P \text{ is an infinite maximal antichain in } \mathbb{P}\}.$$

Fact 2.4 *Let $\mathbb{P} = \langle P, \leq \rangle$ be a partial order and A a nonempty subset of P. Then*

(a) The following conditions are equivalent: (i) A is a (maximal) antichain in $\mathbb{P}$, (ii) A is a (maximal) antichain in $\operatorname{sm}\mathbb{P}$, (iii) $\{[a] : a \in A\}$ is a (maximal) antichain in $\operatorname{sq}\mathbb{P}$ and $a \neq^* b$, for different $a, b \in A$;

(b) $\mathcal{A}(\mathbb{P}) = \mathcal{A}(\operatorname{sq}\mathbb{P})$. Thus $\operatorname{cc}(\mathbb{P}) = \operatorname{cc}(\operatorname{sq}\mathbb{P})$ and, if $\mathbb{P}$ contains infinite antichains, $\mathfrak{a}(\mathbb{P}) = \mathfrak{a}(\operatorname{sq}\mathbb{P})$;

(c) If $\mathbb{P}$ does not contain infinite antichains, then $\mathbb{P}$ is atomic;

(d) The following conditions are equivalent: (i) $\mathbb{P}$ does not contain infinite antichains, (ii) $|\operatorname{sq}\mathbb{P}| < \omega$. Then $\mathfrak{p}(\operatorname{sq}\mathbb{P}) = \infty$.

Proof. (a) The equivalence (i) $\Leftrightarrow$ (ii) is true since by Fact 2.3(b) for $a, b \in P$ we have:

$$a \perp_{\mathbb{P}} b \;\Leftrightarrow\; a \perp_{\operatorname{sm}\mathbb{P}} b \;\Leftrightarrow\; [a] \perp_{\operatorname{sq}\mathbb{P}} [b]. \tag{6}$$

(ii) $\Rightarrow$ (iii). If A is an antichain in $\operatorname{sm}\mathbb{P}$, then, by (6), for different $a, b \in A$ we have $[a] \perp_{\operatorname{sq}\mathbb{P}} [b]$ and, hence, $\{[a] : a \in A\}$ is an antichain in $\operatorname{sq}\mathbb{P}$. In addition, $a =^* b$ would imply $a \leq^* a, b$ and hence $a \not\perp_{\operatorname{sm}\mathbb{P}} b$, which is not true. Thus

$a \neq^* b$. If, in addition, A is a maximal antichain in $\mathrm{sm}\,\mathbb{P}$, and $[p] \in P/=^*$, then, by the maximality of A there is $a \in A$ such that $a \not\perp_{\mathrm{sm}\,\mathbb{P}} p$, which by (6) implies $[a] \not\perp_{\mathrm{sq}\,\mathbb{P}} [p]$ and, hence, $\{[a] : a \in A\}$ is a maximal antichain in $\mathrm{sq}\,\mathbb{P}$.

(iii) $\Rightarrow$ (i). If (iii) holds, then for different $a, b \in A$ we have $[a] \neq [b]$ and, since $\{[a] : a \in A\}$ is an antichain in $\mathrm{sq}\,\mathbb{P}$ we have $[a] \perp_{\mathrm{sq}\,\mathbb{P}} [b]$, which, by (6) implies $a \perp_{\mathbb{P}} b$. So, A is an antichain in $\mathbb{P}$. If, in addition, $\{[a] : a \in A\}$ is a maximal antichain in $\mathrm{sq}\,\mathbb{P}$, then for $p \in P$ we have $[p] \in P/=^*$ and, by the maximality, there is $a \in A$ such that $[a] \not\perp_{\mathrm{sq}\,\mathbb{P}} [p]$, which by (6) implies $a \not\perp_{\mathbb{P}} p$. Thus A is a maximal antichain in $\mathbb{P}$.

(b) By (a), if $\kappa \in \mathcal{A}(\mathbb{P})$ and A is a maximal antichain in $\mathbb{P}$, where $|A| = \kappa$, then $\{[a] : a \in A\}$ is a maximal antichain in $\mathrm{sq}\,\mathbb{P}$ of size κ and, hence, $\kappa \in \mathcal{A}(\mathrm{sq}\,\mathbb{P})$. Conversely, if $\kappa \in \mathcal{A}(\mathrm{sq}\,\mathbb{P})$ and $\mathcal{A}$ is a maximal antichain in $\mathrm{sq}\,\mathbb{P}$, where $|\mathcal{A}| = \kappa$, then the set $A \subset P$ obtained by picking exactly one element from each element of $\mathcal{A}$ is a maximal antichain in $\mathbb{P}$ of size κ, thus $\kappa \in \mathcal{A}(\mathbb{P})$.

(c) If $\mathbb{P}$ is not atomic, then there is $p \in P$ such that $p \downarrow \cap \mathrm{At}(\mathbb{P}) = \emptyset$. Since $p \notin \mathrm{At}(\mathbb{P})$, there are $q_0, r_0 \leq p$ such that $q_0 \perp r_0$. Since $r_0 \notin \mathrm{At}(\mathbb{P})$, there are $q_1, r_1 \leq r_0$ such that $q_1 \perp r_1$ etc. Now $\{q_n : n \in \omega\}$ is an infinite antichain in $\mathbb{P}$.

(d) If $\mathrm{cc}(\mathbb{P}) < \omega$, then by (c) the poset $\mathbb{P}$ is atomic and by Fact 2.2(d), the poset $\mathrm{sq}\,\mathbb{P}$ is atomic and, clearly separative. So, by Fact 2.2(e) if $|\mathrm{At}(\mathrm{sq}\,\mathbb{P})| = \kappa$, then $\mathrm{sq}\,\mathbb{P}$ is isomorphic to a suborder of $P(\kappa)^+$ containing all singletons, which implies that $\kappa \in \mathcal{A}(\mathrm{sq}\,\mathbb{P})$. By (b) we have $\kappa \in \mathcal{A}(\mathbb{P})$ and, since $\mathrm{cc}(\mathbb{P}) < \omega$ we have $\kappa < \omega$. So $|\mathrm{sq}\,\mathbb{P}| \leq |P(\kappa)^+| < \omega$. Conversely, if $|\mathrm{sq}\,\mathbb{P}| < \omega$, then each antichain in $\mathrm{sq}\,\mathbb{P}$, and, by (b), in $\mathbb{P}$ is finite. Also, since each finite centered family has a lower bound, we have $\mathfrak{p}(\mathrm{sq}\,\mathbb{P}) = \infty$. □

Example 2.5 By Fact 2.3(c) we have $\mathfrak{p}(\mathbb{P}) \leq \mathfrak{p}(\mathrm{sq}\,\mathbb{P}) \leq \infty$ and here we give simple examples showing that everything is possible.

$\mathfrak{p}(\mathbb{P}) = \mathfrak{p}(\mathrm{sq}\,\mathbb{P}) = \infty$ holds, if $\mathbb{P}$ is the ordinal ω. A topological characterization of separative posets satisfying $\mathfrak{p}(\mathbb{P}) = \infty$ is given in Theorem 4.3.

$\mathfrak{p}(\mathbb{P}) < \mathfrak{p}(\mathrm{sq}\,\mathbb{P}) = \infty$ holds, if $\mathbb{P}$ is the reversed ordinal ω, in notation ω^*.

$\mathfrak{p}(\mathbb{P}) = \mathfrak{p}(\mathrm{sq}\,\mathbb{P}) < \infty$ holds, if $\mathbb{P}$ is the reversed binary tree, $\langle {}^{<\omega}2, \supset \rangle$.

$\mathfrak{p}(\mathbb{P}) < \mathfrak{p}(\mathrm{sq}\,\mathbb{P}) < \infty$ holds, if $\mathbb{P} = \langle P(\omega) \setminus \mathrm{Fin}, \subset \rangle = \langle [\omega]^\omega, \subset \rangle$. This poset is atomless, non-separative and the Fréchet filter witnesses that $\mathfrak{p}(\mathbb{P}) = \omega$. Also we have $\mathrm{sq}\,\mathbb{P} = (P(\omega)/\mathrm{Fin})^+$ and the cardinal $\mathfrak{p}(\mathrm{sq}\,\mathbb{P}) = \mathfrak{p}$ (the pseudointersection number) is uncountable.

The classes of posets which are relevant for this paper and some of their simple representatives are described in Figure 1.

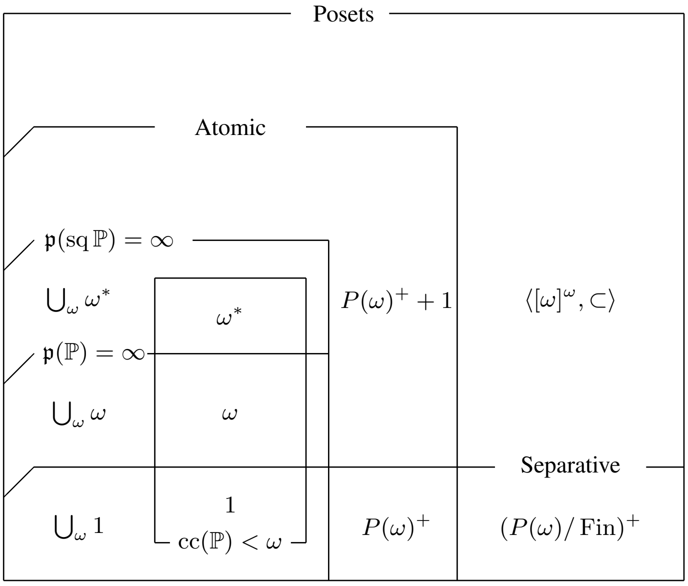

Figure 1: Relevant classes of posets

Direct products Direct product of partial orders $\mathbb{P}_i = \langle P_i, \leq_i\rangle$, $i \in I$, is the poset $\langle \prod_{i\in I} P_i, \leq\rangle$, where $\langle p_i : i \in I\rangle \leq \langle q_i : i \in I\rangle$ iff $p_i \leq_i q_i$, for all $i \in I$.

Fact 2.6 *If $\mathbb{P}_i$, $i \in I$, and $\mathbb{P}$ are partial orderings, then*

(a) $\mathrm{sq}\left(\prod_{i\in I}\mathbb{P}_i\right) \cong \prod_{i\in I}\mathrm{sq}\,\mathbb{P}_i$

(b) $\mathfrak{p}\left(\prod_{i\in I}\mathbb{P}_i\right) = \min\left\{\mathfrak{p}(\mathbb{P}_i) : i \in I\right\}$*;*

(c) $\mathfrak{p}(\mathbb{P}^\kappa) = \mathfrak{p}(\mathbb{P})$ *and* $\mathfrak{p}(\mathrm{sq}(\mathbb{P}^\kappa)) = \mathfrak{p}(\mathrm{sq}(\mathbb{P}))$*, for each cardinal* κ*.*

If, in addition, at least one of the partial orders $\mathbb{P}_i$ has infinite antichains then

(d) $\mathfrak{a}\left(\prod_{i\in I}\mathbb{P}_i\right) \leq \min\left\{\prod_{i\in I}\kappa_i : \langle\kappa_i : i \in I\rangle \in \left(\prod_{i\in I}\mathcal{A}(\mathbb{P}_i)\right) \setminus {}^I\omega\right\}$*;*

(e) $\mathfrak{a}\left(\prod_{i\in I}\mathrm{sq}\,\mathbb{P}_i\right) = \mathfrak{a}\left(\prod_{i\in I}\mathbb{P}_i\right)$.

Proof. (b) Suppose that $\kappa := \mathfrak{p}(\prod_{i\in I}\mathbb{P}_i) < \min\{\mathfrak{p}(\mathbb{P}_i) : i \in I\}$. Then $\kappa < \infty$ and, hence, in $\prod_{i\in I}\mathbb{P}_i$ there is a κ-sized centered family without a lower bound, say $C = \{p_\alpha : \alpha \in \kappa\}$. Then, for $i \in I$, $C_i = \{p_\alpha(i) : \alpha \in \kappa\}$ is a centered family in $\mathbb{P}_i$ and, by the assumption, it has a lower bound, say q_i. But then $\langle q_i : i \in I\rangle$ is a lower bound for C in $\prod_{i\in I}\mathbb{P}_i$, which is impossible.

Suppose that $\kappa := \min\{\mathfrak{p}(\mathbb{P}_i) : i \in I\} < \mathfrak{p}(\prod_{i\in I}\mathbb{P}_i)$. Then $\kappa < \infty$ and $\kappa = \mathfrak{p}(\mathbb{P}_{i_0})$, for some $i_0 \in I$, so, in $\mathbb{P}_{i_0}$ there is a κ-sized centered family without a lower bound, say C. Now, for a fixed $p \in \prod_{i\in I\setminus\{i_0\}} \mathbb{P}_i$, the set $C \times \{p\}$ is a centered family in $\prod_{i\in I} \mathbb{P}_i$ which does not have a lower bound, which implies that $\mathfrak{p}(\prod_{i\in I} \mathbb{P}_i) \leq \kappa$ and we obtain a contradiction.

(c) By (b) we have $\mathfrak{p}(\mathbb{P}^\kappa) = \mathfrak{p}(\mathbb{P})$. By (a) and (b) we have $\mathfrak{p}(\mathrm{sq}(\mathbb{P}^\kappa)) = \mathfrak{p}(\mathrm{sq}(\mathbb{P})^\kappa) = \mathfrak{p}(\mathrm{sq}(\mathbb{P}))$.

(d) Let $\prod_{i\in I} \kappa_i'$ be the right hand side of the inequality in (d) and, for $i \in I$, let A_i be a maximal antichain in $\mathbb{P}_i$ such that $|A_i| = \kappa_i'$. It is easy to see that $\prod_{i\in I} A_i$ is a maximal antichain in $\prod_{i\in I} \mathbb{P}_i$. Thus $\mathfrak{a}(\prod_{i\in I} \mathbb{P}_i) \leq |\prod_{i\in I} A_i| = \prod_{i\in I} \kappa_i'$.

(e) By (a) and Fact 2.4(b) we have $\mathfrak{a}(\prod_{i\in I} \mathrm{sq}(\mathbb{P}_i)) = \mathfrak{a}(\mathrm{sq}(\prod_{i\in I} \mathbb{P}_i)) = \mathfrak{a}(\prod_{i\in I} \mathbb{P}_i)$ □

Remark 2.7 We make some comments concerning Fact 2.6(d).

The assumption that at least one of the posets $\mathbb{P}_i$ has infinite antichains is necessary since, otherwise, the product $\prod_{i\leq n} \mathbb{P}_i$ would not contain infinite antichains. (By Ramsey's theorem, if $\mathbb{P} \times \mathbb{Q}$ contains infinite antichains, then $\mathbb{P}$ or $\mathbb{Q}$ has that property. An induction shows that the same holds for all finite products).

If $\mathbb{P}$ and $\mathbb{Q}$ are partial orders having infinite antichains, then $\mathfrak{a}(\mathbb{P} \times \mathbb{Q}) \leq \max\{\mathfrak{a}(\mathbb{P}), \mathfrak{a}(\mathbb{Q})\}$ and if, in addition, $\mathbb{P}$ and $\mathbb{Q}$ do not have finite maximal antichains, this bound is, in general, the best possible. Namely, if $\mathbb{A}_\kappa$ denotes the antichain of size κ, then $\mathfrak{a}(\mathbb{A}_\omega \times \mathbb{A}_{\omega_1}) = \omega_1 = \max\{\mathfrak{a}(\mathbb{A}_\omega), \mathfrak{a}(\mathbb{A}_{\omega_1})\}$.

If each of the posets $\mathbb{P}_i$, $i \leq n$, contains both a finite maximal antichain and an infinite antichain, then, by Fact 2.6(d), $\mathfrak{a}(\prod_{i\leq n} \mathbb{P}_i) \leq \min\{\mathfrak{a}(\mathbb{P}_i) : i \leq n\}$.

3 Finite products: bounds on $\mathfrak{a}(\prod_{i\leq n} \mathbb{P}_i)$

In this section we prove the following statement giving bounds on $\mathfrak{a}(\prod_{i\leq n} \mathbb{P}_i)$.

Theorem 3.1 *If $\mathbb{P}_i$, $i \in \{1, \dots, n\}$, are partial orders having infinite antichains, then*

$$\begin{aligned}
\omega &\leq \min(\{\mathfrak{a}(\mathbb{P}_i) : i \leq n\} \cup \{\mathfrak{p}(\mathrm{sq}\,\mathbb{P}_i) : i \leq n\}) \\
&\leq \mathfrak{a}(\textstyle\prod_{i\leq n} \mathbb{P}_i) \\
&\leq \min\{\textstyle\prod_{i\leq n} \kappa_i : \langle \kappa_i : i \leq n\rangle \in (\prod_{i\leq n} \mathcal{A}(\mathbb{P}_i)) \setminus {}^n\omega\} \\
&\leq \max\{\mathfrak{a}(\mathbb{P}_i) : i \leq n\}
\end{aligned}$$

A proof of Theorem 3.1 is given at the end of the section. First we recall some definitions and facts and prove some auxiliary statements, which will be used in the rest of the paper. For sets $K, H \subset \omega$ we will write $K < H$ iff $k < h$, for each $k \in K$ and $h \in H$. $K \sqsubseteq H$ will denote that $K \subset H$ and $H \cap [0, \max(K)] = K$.

We remind the reader that a family of finite sets $\mathcal{F} = \{K_i : i \in I\}$ is called a Δ-*system* iff there is a finite (possibly empty) set R (the *root*) such that $R \subset K_i$, for all $i \in I$, and $K_{i_1} \cap K_{i_2} = R$, for different $i_1, i_2 \in I$. For uncountable families of finite sets we have the following statement (Δ-System Lemma, see [5], p. 49):

Theorem 3.2 (Šanin) *Each uncountable family of finite sets contains an uncountable Δ-system.*

Fact 3.3 *(Folklore) If $\{K_i : i \in \omega\} \subset [\omega]^{<\omega}$, where $K_i \neq K_j$, for different $i, j \in \omega$, then there is $M \in [\omega]^\omega$ satisfying (a) or (b), where*

(a) There is $R \in [\omega]^{<\omega}$ such that $R \subsetneq K_i$, for all $i \in M$, and

$$\forall i, j \in M \ \ (i < j \Rightarrow R < (K_i \setminus R) < (K_j \setminus R)), \tag{7}$$

(b) There is $X \in [\omega]^\omega$ such that

$$\forall K \sqsubseteq X \ \ \exists n \in \omega \ \ \forall i \in M \setminus n \ \ K \sqsubseteq K_i. \tag{8}$$

Proof. In the Cantor space, 2^ω, the corresponding sequence of characteristic functions, $\langle \chi_{K_i} : i \in \omega \rangle$, has a subsequence $\langle \chi_{K_i} : i \in I \rangle$ converging to some $f \in 2^\omega$.

If $f = \chi_R$, for some $R \in [\omega]^{<\omega}$, and $m_0 := \max R + 1$, then, since the sets of the form $B_m := \{g \in 2^\omega : g \upharpoonright m = \chi_R \upharpoonright m\}$, $m \geq m_0$, form a neighborhood base at the point χ_R, for each $m \geq m_0$ there is $k_m \in \omega$ such that for each $i \in I \setminus k_m$ we have $\chi_{K_i} \in B_m$, that is $K_i \cap [0, m) = R$. In addition, since the sets K_i, $i \in I$, are different and $P(m)$ is a finite set, there is $l_m \in \omega$ such that $K_i \cap [m, \infty) \neq \emptyset$, for all $i \in I \setminus l_m$. So, for $i \geq \max\{k_m, l_m\}$ both conditions are satisfied and, thus,

$$\forall m \geq m_0 \ \ \exists n \in \omega \ \ \forall i \in I \setminus n \ \ \Big(K_i \cap [0, m) = R \ \wedge \ K_i \cap [m, \infty) \neq \emptyset\Big). \tag{9}$$

By recursion we define a sequence $\langle i_j : j \in \omega \rangle$ in I and a sequence $\langle m_j : j \in \omega \rangle$ in ω, such that $m_0 := \max R + 1$ and that for all $j, j' \in \omega$ we have:

(i) $K_{i_j} \cap [0, m_j) = R$ and $K_{i_j} \cap [m_j, \infty) \neq \emptyset$,

(ii) If $j < j'$, then $i_j < i_{j'}$, $m_j < m_{j'}$ and $R < (K_{i_j} \setminus R) < (K_{i_{j'}} \setminus R)$.

First, by (9) we take i_0 such that $K_{i_0} \cap [0, m_0) = R$ and $K_{i_0} \cap [m_0, \infty) \neq \emptyset$.

If $j' > 0$ and if $\langle i_j : j < j' \rangle$ and $\langle m_j : j < j' \rangle$ are sequences satisfying (i) and (ii), then we define $m_{j'} := \max K_{i_{j'-1}} + 1$. Then, by (ii), $\bigcup_{j<j'} K_{i_j} \subset [0, m_{j'})$, by (i) we have $m_j < m_{j'}$, for all $j < j'$, and using (9) we pick $i_{j'} > i_{j'-1}$ such that $K_{i_{j'}} \cap [0, m_{j'}) = R$ and $K_{i_{j'}} \cap [m_{j'}, \infty) \neq \emptyset$. The recursion works.

By (i) and (ii), $M := \{i_j : j \in \omega\}$ is a set satisfying (a).

If $f = \chi_X$, where $X \in [\omega]^\omega$, then the sets $B_K := \{g \in 2^\omega : g \upharpoonright [0, \max K] = \chi_X \upharpoonright [0, \max K]\}$, $K \sqsubseteq X$, form form a neighborhood base at the point χ_X of

2^ω. So, for each $K \sqsubseteq X$ there is $n \in \omega$ such that for each $i \in I \setminus n$ we have $\chi_{K_i} \in B_K$, that is $\chi_{K_i} \upharpoonright [0, \max K] = \chi_X \upharpoonright [0, \max K]\}$, which means that $K_i \cap [0, \max K] = K$, that is $K \sqsubseteq K_i$. So defining $M := I$ we obtain (b). □

Lemma 3.4 *(Maximal centered subsequences) If $\mathbb{Q} = \langle Q, \leq \rangle$ is a partial order, κ a cardinal, $\langle q_\alpha : \alpha < \kappa \rangle \in {}^\kappa Q$ and*

$$\mathcal{S} := \Big\{ S \in P(\kappa) \setminus \{\emptyset\} : \{q_\alpha : \alpha \in S\} \text{ is centered in } \mathbb{Q} \Big\}, \tag{10}$$

then $\langle \mathcal{S}, \subset \rangle$ is a downwards closed suborder of the poset $\langle P(\kappa) \setminus \{\emptyset\}, \subset \rangle$ containing all singletons $\{\alpha\}$, $\alpha \in \kappa$, and each $S_0 \in \mathcal{S}$ is contained in some maximal element of the poset $\langle \mathcal{S}, \subset \rangle$.

Proof. Let $\mathcal{S}_{S_0} := \{S \in \mathcal{S} : S_0 \subset S\}$, let $\mathcal{L}$ be a nonempty chain in the poset $\Pi := \langle \mathcal{S}_{S_0}, \subset \rangle$ and $S' = \bigcup_{S \in \mathcal{L}} S$. Then $S_0 \subset S' \subset \kappa$. If $K \in [S']^{<\omega} \setminus \{\emptyset\}$, then for each $\alpha \in K$ there is $S_\alpha \in \mathcal{L}$ such that $\alpha \in S_\alpha$, and, since $\mathcal{L}$ is a chain, there is $\alpha_0 \in K$ such that $K \subset S_{\alpha_0}$. Since $S_{\alpha_0} \in \mathcal{S}$ there is $c \in Q$ such that $c \leq q_\alpha$, for all $\alpha \in K$, thus $S' \in \mathcal{S}$ and, consequently, $S' \in \mathcal{S}_{S_0}$ and S' is an upper bound for $\mathcal{L}$. By Zorn's lemma there is a maximal element of $\mathcal{S}_{S_0}$, say S^*. If $S \in \mathcal{S}$ and $S^* \subset S$, then $S_0 \subset S$, which implies $S \in \mathcal{S}_{S_0}$ and, by the maximality of S^* we obtain $S = S^*$. Thus S^* is a maximal element of $\mathcal{S}$ containing S_0. □

Lemma 3.5 *(Infinite centered subsequences left or right) If $\mathbb{P}$ and $\mathbb{Q}$ are partial orders and $A = \{\langle p_\alpha, q_\alpha \rangle : \alpha < \kappa\}$ a maximal antichain in $\mathbb{P} \times \mathbb{Q}$, where $\kappa \geq \omega$, then* (A) *or* (B) *holds, where*

(A) *There is an infinite set $S \subset \kappa$ such that $\{p_\alpha : \alpha \in S\}$ is centered in $\mathbb{P}$,*

(B) *There is an infinite set $S \subset \kappa$ such that $\{q_\alpha : \alpha \in S\}$ is centered in $\mathbb{Q}$.*

Proof. Suppose that (B) is not true. Then the set $\mathcal{S}$, defined by (10), is a family of finite nonempty subsets of κ and, by Lemma 3.4, the family $\mathcal{S}_{max}$ of maximal elements of the order $\langle \mathcal{S}, \subset \rangle$ is a covering of κ, which implies that $|\mathcal{S}_{max}| = \kappa$. Let $\mathcal{S}_{max} = \{K_i : i < \kappa\}$ be an enumeration. By the maximality of K_i's we have

$$\forall i < \kappa \;\; \exists c \in Q \;\; c \leq \{q_\alpha : \alpha \in K_i\}, \tag{11}$$

$$\forall i < \kappa \;\; \forall \beta \in \kappa \setminus K_i \;\; \neg \exists c \in Q \;\; c \leq \{q_\beta\} \cup \{q_\alpha : \alpha \in K_i\}, \tag{12}$$

$$\forall \{i, j\} \in [\kappa]^2 \;\; (K_i \setminus K_j \neq \emptyset \;\wedge\; K_j \setminus K_i \neq \emptyset). \tag{13}$$

Suppose that there is $i \in \kappa$ such that $\{p_\alpha : \alpha \in K_i\}$ is not a maximal antichain in $\mathbb{P}$. Then there is $\langle a, c \rangle \in P \times Q$ such that $a \perp \{p_\alpha : \alpha \in K_i\}$ and $c \leq \{q_\alpha : \alpha \in K_i\}$

and, by the maximality of A, there is $\beta \in \kappa$ such that $p_\beta \not\perp a$, which implies $\beta \notin K_i$, and that $q_\beta \not\perp c$; thus, there is $c' \leq \{q_\beta\} \cup \{q_\alpha : \alpha \in K_i\}$ which is impossible by (12). Thus

$$\forall i \in \kappa \ \ (\{p_\alpha : \alpha \in K_i\} \text{ is a maximal antichain in } \mathbb{P}). \tag{14}$$

We show that there are an infinite set $M \subset \kappa$ and a finite set $R \subset \kappa$ such that

$$\forall i \in M \ \ R \subsetneq K_i, \tag{15}$$

$$\forall \{i,j\} \in [M]^2 \ \ (K_i \setminus R) \cap (K_j \setminus R) = \emptyset. \tag{16}$$

If $\kappa > \omega$, then by Theorem 3.2 there is a Δ-system $\{K_i : i \in M\} \subset \mathcal{S}_{\max}$ with a root R, where $|M| > \omega$. So, by (13), conditions (15) and (16) are satisfied.

If $\kappa = \omega$, then $\mathcal{S}_{max} = \{K_i : i < \omega\} \subset [\omega]^{<\omega}$. By (13), the assumptions of Fact 3.3 are satisfied. Suppose that there are sets $M, X \in [\omega]^\omega$ satisfying (8). Then for $K \in [X]^{<\omega}$, then there is $K' \sqsubseteq X$ such that $K \subset K'$ and, by (8), there is $i \in M$ such that $K' \sqsubseteq K_i$. So, since $\{q_n : n \in K_i\}$ is a centered set in $\mathbb{Q}$ and $K \subset K_i$, $\{q_n : n \in K\}$ is centered too. Thus X is an infinite set satisfying (B), which contradicts our assumption. So, by Fact 3.3, there are sets $M \in [\omega]^\omega$ and $R \in [\omega]^{<\omega}$ satisfying (15) and such that for all $i, j \in M$ satisfying $i < j$ we have $R < (K_i \setminus R) < (K_j \setminus R)$, which implies (16).

By recursion, for $k \in \omega$ we define $i_k \in M$, $\alpha_k \in \kappa$ and $c_k \in P$ such that for all $k, l \in \omega$ we have:

(i) $k \neq l \Rightarrow i_k \neq i_l$,
(ii) $\alpha_k \in K_{i_k} \setminus R$,
(iii) $c_k \leq p_{\alpha_0}, \dots, p_{\alpha_k}$,
(iv) $c_k \perp \{p_\alpha : \alpha \in R\}$.

Let $i_0 \in M$. By (15) there is $\alpha_0 \in K_{i_0} \setminus R$ and by (14) $c_0 := p_{\alpha_0} \perp \{p_\alpha : \alpha \in R\}$.

Suppose that the sequence $\langle \langle i_j, \alpha_j, c_j \rangle : j \leq k \rangle$ satisfies (i)-(iv). We choose $i_{k+1} \in M \setminus \{i_j : j \leq k\}$. By (14) $\{p_\alpha : \alpha \in K_{i_{k+1}} \setminus R\} \cup \{p_\alpha : \alpha \in R\}$ is a maximal antichain in $\mathbb{P}$ and by (iv) there are $\alpha_{k+1} \in K_{i_{k+1}} \setminus R$ and $c_{k+1} \in P$ such that $c_{k+1} \leq c_k, p_{\alpha_{k+1}}$, which by (iii) and (iv) implies $c_{k+1} \leq p_{\alpha_0}, \dots, p_{\alpha_k}, p_{\alpha_{k+1}}$ and $c_{k+1} \perp \{p_\alpha : \alpha \in R\}$. So, the sequence $\langle \langle i_j, \alpha_j, c_j \rangle : j \leq k+1 \rangle$ satisfies conditions (i)-(iv) and the recursion works.

Now $S := \{\alpha_k : k \in \omega\} \subset \kappa$, by (i), (ii) and (16) we have $|S| = \omega$ and by (iii), condition (A) of the lemma is satisfied. □

Proof of Theorem 3.1. The first inequality is evident and the third and the fourth follow from Fact 2.6(d). In order to prove the second inequality for $n = 2$ suppose that, on the contrary, there are posets $\mathbb{P}$ and $\mathbb{Q}$ having infinite antichains and such that

$$\kappa := \mathfrak{a}(\mathbb{P} \times \mathbb{Q}) < \min\{\mathfrak{a}(\mathbb{P}), \mathfrak{a}(\mathbb{Q}), \mathfrak{p}(\operatorname{sq}\mathbb{P}), \mathfrak{p}(\operatorname{sq}\mathbb{Q})\} \tag{17}$$

and that $A = \{\langle p_\alpha, q_\alpha\rangle : \alpha \in \kappa\}$ is a κ-sized maximal antichain in the poset $\mathbb{P}\times\mathbb{Q}$, that is,

$$\forall\{\alpha,\beta\} \in [\kappa]^2 \ \Big(p_\alpha \perp p_\beta \ \vee \ q_\alpha \perp q_\beta\Big), \tag{18}$$

$$\forall p \in P \ \ \forall q \in Q \ \ \exists \alpha \in \kappa \ \ \Big(p \not\perp p_\alpha \ \wedge \ q \not\perp q_\alpha\Big). \tag{19}$$

W.l.o.g. we suppose that $S_0 \subset \kappa$ is an infinite set satisfying condition (B) of Lemma 3.5. By Lemma 3.4, the poset $\langle \mathcal{S}, \subset\rangle$ contains a maximal element S such that $S_0 \subset S$. Thus we have

$$\forall K \in [S]^{<\omega}\setminus\{\emptyset\} \ \ \exists c \in Q \ \ c \leq \{q_\alpha : \alpha \in K\}, \tag{20}$$

$$\forall \beta \in \kappa \setminus S \ \ \exists K \in [S]^{<\omega}\setminus\{\emptyset\} \ \ \neg\exists c \in Q \ \ c \leq \{q_\beta\} \cup \{q_\alpha : \alpha \in K\}. \tag{21}$$

By (20) and since $x \leq y$ implies $x \leq^* y$, $\mathcal{B} := \{[q_\alpha] : \alpha \in S\}$ is a centered family in the separative quotient $\mathrm{sq}\,\mathbb{Q}$; so, since by (17) we have $|S| \leq \kappa < \mathfrak{p}(\mathrm{sq}\,\mathbb{Q})$, the family $\mathcal{B}$ has a lower bound in $\mathrm{sq}\,\mathbb{Q}$. In other words there is $q \in Q$ such that

$$\forall \alpha \in S \ \ q \leq^* q_\alpha. \tag{22}$$

By (20), for different $\alpha, \beta \in S$ we have $q_\alpha \not\perp q_\beta$ and, by (18), $p_\alpha \perp p_\beta$; thus $\{p_\alpha : \alpha \in S\}$ is an infinite antichain in $\mathbb{P}$ of size $\leq \kappa$. So, by (17), there is $p \in P$ such that

$$\forall \alpha \in S \ \ p \perp p_\alpha. \tag{23}$$

By the maximality of A there is $\beta \in \kappa$ such that $\langle p_\beta, q_\beta\rangle \not\perp \langle p, q\rangle$. Consequently $p \not\perp p_\beta$, which by (23) implies $\beta \in \kappa \setminus S$ so, by (21) there is $K \in [S]^{<\omega}\setminus\{\emptyset\}$ such that

$$\neg\exists c \in Q \ \ c \leq \{q_\beta\} \cup \{q_\alpha : \alpha \in K\}. \tag{24}$$

Now, by (22), $q \leq^* \{q_\alpha : \alpha \in K\}$ and, since $\langle p_\beta, q_\beta\rangle \not\perp \langle p, q\rangle$, there is $q' \in Q$ such that $q' \leq q_\beta, q$, which implies that $q' \leq^* \{q_\beta\} \cup \{q_\alpha : \alpha \in K\}$. By Fact 2.3(b) there is $c \in Q$ such that $c \leq \{q_\beta\} \cup \{q_\alpha : \alpha \in K\}$, which contradicts (24).

So the second inequality is true for $n = 2$ and now we assume that it is true for n and that $\mathbb{P}_i$, $i \leq n+1$ are partial orders with infinite antichains. By Fact 2.6 (a) and (b) we have $\mathfrak{p}(\mathrm{sq}(\prod_{i=1}^n \mathbb{P}_i)) = \mathfrak{p}(\prod_{i=1}^n \mathrm{sq}\,\mathbb{P}_i) = \min\{\mathfrak{p}(\mathrm{sq}\,\mathbb{P}_i) : i \leq n\} \geq \min(\{\mathfrak{a}(\mathbb{P}_i) : i \leq n\} \cup \{\mathfrak{p}(\mathrm{sq}\,\mathbb{P}_i) : i \leq n\})$, which, together with the induction hypothesis, implies

$$\begin{array}{rl} & \mathfrak{a}(\prod_{i=1}^{n+1} \mathbb{P}_i) \\ = & \mathfrak{a}((\prod_{i=1}^{n} \mathbb{P}_i) \times \mathbb{P}_{n+1}) \\ \geq & \min\{\mathfrak{a}(\prod_{i=1}^{n} \mathbb{P}_i), \mathfrak{a}(\mathbb{P}_{n+1}), \mathfrak{p}(\mathrm{sq}\prod_{i=1}^{n} \mathbb{P}_i), \mathfrak{p}(\mathrm{sq}\,\mathbb{P}_{n+1})\} \\ \geq & \min\{\min(\{\mathfrak{a}(\mathbb{P}_i) : i \leq n\} \cup \{\mathfrak{p}(\mathrm{sq}\,\mathbb{P}_i) : i \leq n\}), \mathfrak{a}(\mathbb{P}_{n+1}), \mathfrak{p}(\mathrm{sq}\,\mathbb{P}_{n+1})\} \\ = & \min(\{\mathfrak{a}(\mathbb{P}_i) : i \leq n+1\} \cup \{\mathfrak{p}(\mathrm{sq}\,\mathbb{P}_i) : i \leq n+1\}) \end{array}$$

and the second inequality is true for $n + 1$. $\Box$

4 Finite powers

In the rest of the paper for a partial order $\mathbb{P}$ we consider the cardinal invariants $\mathfrak{a}(\mathbb{P}^n)$, for $n \in \mathbb{N}$. The following simple examples show that the relevant invariants $\mathfrak{a}(\mathbb{P})$ and $\mathfrak{p}(\mathrm{sq}\,\mathbb{P})$ are, in general, unrelated: if $\mathbb{P}$ is a disjoint union of ω_1-many copies of the reversed tree ${}^{<\omega}2$, then $\mathfrak{a}(\mathbb{P}) = \omega_1 > \omega = \mathfrak{p}(\mathrm{sq}\,\mathbb{P})$; if $\mathbb{P}$ is a disjoint union of ω copies of the reversed tree ${}^{<\omega_1}2$, then $\mathfrak{a}(\mathbb{P}) = \omega < \omega_1 = \mathfrak{p}(\mathrm{sq}\,\mathbb{P})$.

Theorem 4.1 *If $\mathbb{P}$ is a partial order having infinite antichains, then*
(a) $\min\{\mathfrak{a}(\mathbb{P}), \mathfrak{p}(\mathrm{sq}\,\mathbb{P})\} \leq \mathfrak{a}(\mathbb{P}^n) \leq \mathfrak{a}(\mathbb{P})$*, for all* $n \in \mathbb{N}$*;*
(b) If $\mathfrak{p}(\mathrm{sq}\,\mathbb{P}) \geq \mathfrak{a}(\mathbb{P})$*, then* $\mathfrak{a}(\mathbb{P}^n) = \mathfrak{a}(\mathbb{P})$*, for all* $n \in \mathbb{N}$*;*
(c) If $\mathfrak{p}(\mathrm{sq}\,\mathbb{P}) < \mathfrak{a}(\mathbb{P})$*, then* $\mathfrak{p}(\mathrm{sq}\,\mathbb{P}) \leq \mathfrak{a}(\mathbb{P}^n) \leq \mathfrak{a}(\mathbb{P})$*, for all* $n \in \mathbb{N}$*. If, in addition,* $\mathbb{P}$ *contains a finite maximal antichain, then for all* $n \in \mathbb{N}$ *we have*

$$\mathfrak{p}(\mathrm{sq}\,\mathbb{P}) \leq \mathfrak{a}(\mathbb{P}^{n+1}) \leq \mathfrak{a}(\mathbb{P}^n) \leq \mathfrak{a}(\mathbb{P}).$$

(d) If $\mathcal{K}$ *be a class of posets such that* $\mathrm{sq}\,\mathbb{P} \in \mathcal{K}$*, for each* $\mathbb{P} \in \mathcal{K}$*, then the following conditions are equivalent:*
(i) $\mathfrak{a}(\mathbb{P} \times \mathbb{P}) = \mathfrak{a}(\mathbb{P})$*, for each* $\mathbb{P} \in \mathcal{K}$ *having infinite antichains;*
(ii) $\mathfrak{a}(\mathbb{P} \times \mathbb{P}) = \mathfrak{a}(\mathbb{P})$*, for each separative* $\mathbb{P} \in \mathcal{K}$ *having infinite antichains.*

Proof. Statement (a) follows from Theorem 3.1 and statement (b) follows from (a).
(c) Suppose that $\mathbb{P}$ contains a maximal antichain of size $k \in \omega$. Then, by Theorem 3.1, $\mathfrak{a}(\mathbb{P}^{n+1}) = \mathfrak{a}(\mathbb{P}^n \times \mathbb{P}) \leq \mathfrak{a}(\mathbb{P}^n)k = \mathfrak{a}(\mathbb{P}^n)$.
(d) Suppose that (ii) is true and that $\mathbb{P} \in \mathcal{K}$, where $\mathrm{cc}(\mathbb{P}) > \omega$. Then $\mathrm{sq}\,\mathbb{P} \in \mathcal{K}$, by Fact2.4(b) we have $\mathrm{cc}(\mathrm{sq}\,\mathbb{P}) > \omega$ and, by (ii), $\mathfrak{a}(\mathrm{sq}\,\mathbb{P} \times \mathrm{sq}\,\mathbb{P}) = \mathfrak{a}(\mathrm{sq}\,\mathbb{P})$. By Fact 2.6(e) we have $\mathfrak{a}(\mathrm{sq}\,\mathbb{P} \times \mathrm{sq}\,\mathbb{P}) = \mathfrak{a}(\mathbb{P} \times \mathbb{P})$ and, by Fact 2.4(b), $\mathfrak{a}(\mathrm{sq}\,\mathbb{P}) = \mathfrak{a}(\mathbb{P})$. Thus $\mathfrak{a}(\mathbb{P} \times \mathbb{P}) = \mathfrak{a}(\mathbb{P})$. $\Box$

Corrolary 4.2 The equality $\mathfrak{a}((\mathbb{B}^+)^n) = \mathfrak{a}(\mathbb{B}^+)$ holds for each infinite Boolean algebra $\mathbb{B}$ of size $\leq \omega_1$ and each $n \in \mathbb{N}$.

Proof. If $\mathfrak{p}(\mathbb{B}^+) = \omega_1$, we apply Theorem 4.1(b); if $\mathfrak{p}(\mathbb{B}^+) = \omega$, then it is easy to construct a countable maximal antichain in $\mathbb{B}^+$ so $\mathfrak{a}(\mathbb{B}^+) = \omega = \mathfrak{a}((\mathbb{B}^+)^n)$. $\Box$

By Theorem 4.1(b) the equality $\mathfrak{a}(\mathbb{P}^n) = \mathfrak{a}(\mathbb{P})$ holds for all $n \in \mathbb{N}$, if, in particular, $\mathbb{P}$ is a poset satisfying $\mathfrak{p}(\mathrm{sq}\,\mathbb{P}) = \infty$. The following theorem is a topological characterization of separative posets with that property.

Theorem 4.3 *If $\mathbb{P}$ is a separative poset then $\mathfrak{p}(\mathbb{P}) = \infty$ iff $\mathbb{P} \cong \langle \mathcal{C}, \subset \rangle$, for a collection $\mathcal{C}$ of nonempty closed sets in a compact T_1 space containing all singletons.*

Proof. Let $\mathfrak{p}(\mathbb{P}) = \infty$. By Fact 2.3(a) the poset $\mathbb{P}$ is atomic, and, by Fact 2.2(e), we can assume that $\mathbb{P} = \langle \mathcal{D}, \subset \rangle$, where $[X]^1 \subset \mathcal{D} \subset P(X) \setminus \{\emptyset\}$ and $X = \mathrm{At}(\mathbb{P})$. If X is a finite set, then the discrete topology on X works. Otherwise, if $\mathcal{P} := \{X \setminus D : D \in \mathcal{D}\}$ and $x, y \in X$, where $x \neq y$, then $X \setminus \{x\}, X \setminus \{y\} \in \mathcal{P}$ and, hence $\bigcup \mathcal{P} = X$, which means that $\mathcal{P}$ is a subbase for some topology $\mathcal{O}$ on X and $\mathcal{D} \subset \mathcal{F} \setminus \{\emptyset\}$, where $\mathcal{F}$ is the corresponding family of closed sets. Since the family $\mathcal{D}$ contains all singletons the space $\langle X, \mathcal{O} \rangle$ is a T_1-space. If $X = \bigcup_{i \in I} X \setminus D_i$ is an open cover of X (by the sets from the subbase $\mathcal{P}$) then $\bigcap_{i \in I} D_i = \emptyset$ and, hence, $\{D_i : i \in I\}$ is not a centered family in $\mathbb{P}$, which means that there is a finite set $K \subset I$ such that $\bigcap_{i \in K} D_i = \emptyset$. Thus $X = \bigcup_{i \in K} X \setminus D_i$ is a finite subcover of the initial cover and, by the Alexander theorem (see [3], p. 221), the space $\langle X, \mathcal{O} \rangle$ is compact.

Let $\langle X, \mathcal{O} \rangle$ be a compact T_1 space, $\mathcal{F}$ the corresponding family of closed sets and $[X]^1 \subset \mathcal{C} \subset \mathcal{F} \setminus \{\emptyset\}$. If $\mathcal{C}' \subset \mathcal{C}$ is a centered family in the poset $\langle \mathcal{C}, \subset \rangle$, then $\mathcal{C}'$ is a family of closed sets with the finite intersection property and, by compactness, there is $x \in \bigcap \mathcal{C}'$. Thus $\{x\}$ is a lower bound for $\mathcal{C}'$ in the poset $\langle \mathcal{C}, \subset \rangle$. □

Corrolary 4.4 *If $\langle X, \mathcal{O} \rangle$ is an infinite compact T_1-space, $\mathcal{F}$ the corresponding family of closed sets, $[X]^1 \subset \mathcal{C} \subset \mathcal{F} \setminus \{\emptyset\}$ and $\mathbb{P} = \langle \mathcal{C}, \subset \rangle$, then $\mathfrak{a}(\mathbb{P}^n) = \mathfrak{a}(\mathbb{P})$, for all $n \in \mathbb{N}$.*

Example 4.5 Let $\mathbb{P} = \langle \mathcal{C}, \subset \rangle$ be the poset defined in Corollary 4.4.

If the space $\langle X, \mathcal{O} \rangle$ is a continuum (a connected compact Hausdorff space), then, by the Sierpiński theorem (see [3], p. 358), the set X can not be partitioned into ω-many closed sets and, hence, $\mathfrak{a}(\mathbb{P}^n) = \mathfrak{a}(\mathbb{P}) > \omega$, for all $n \in \mathbb{N}$.

If the space $\langle X, \mathcal{O} \rangle$ is the Cantor cube, 2^ω, and $\mathcal{C} = \mathcal{F} \setminus \{\emptyset\}$, then we have $\mathfrak{a}(\mathbb{P}^n) = \mathfrak{a}(\mathbb{P}) = \omega$, for all $n \in \mathbb{N}$, because the basic clopen sets $B_0, B_{10}, B_{110}, \ldots$ and the singleton $\{\langle 1, 1, 1, \ldots \rangle\}$ form a partition if 2^ω.

5 Reversed trees

In this section $\mathcal{T} = \langle T, \leq \rangle$ will be a *reversed tree*, which means that for each $t \in T$ the set $(t, \infty) := \{s \in T : t < s\}$ is either empty, or a reversed well order. The following theorem is the main statement of this section.

Theorem 5.1 *For each reversed tree $\mathcal{T}$ containing infinite antichains we have*

$$\mathfrak{a}(\mathcal{T} \times \mathcal{T}) = \mathfrak{a}(\mathcal{T}).$$

A proof of the theorem will be given at the end of the section. We start with some conventions and facts which will be used in the proof. First, if $\mathcal{T} = \langle T, \leq \rangle$ is a reversed tree and α an ordinal, by L_α we will denote the α-th level of $\mathcal{T}$ and by $\mathrm{Min}(\mathcal{T})$ the set of all minimal elements of $\mathcal{T}$.

Fact 5.2 *If $\mathcal{T} = \langle T, \leq \rangle$ is a reversed tree, then for all $x, y, z \in T$ we have*
(a) $x \not\perp y \Leftrightarrow x \not\parallel y \Leftrightarrow x \leq y \vee y \leq x$
(b) $x \not\perp y \wedge y \leq z \Rightarrow x \not\perp z$;
(c) $x < y \Rightarrow \exists p \in T\ (x \leq p < y \wedge (p, y) = \emptyset)$.
If S is a nonempty subset of T, then
(d) $\langle S, \leq\restriction S^2 \rangle$ is a reversed tree,
(e) Each element of S is contained in some maximal element of S.

Fact 5.3 *A reversed tree $\mathcal{T} = \langle T, \leq \rangle$ is separative iff each $t \in T \setminus \mathrm{Min}(T)$ has at least two immediate predecessors.*

Proof. ($\Rightarrow$) Let $\mathcal{T}$ be separative and $t \in T \setminus \mathrm{Min}(T)$. By Fact 5.2(c) t has an immediate predecessor, say $p < t$. Since $t \not\leq p$ there is $r \leq t$ such that $r \perp p$ and, clearly, $r < t$. By Fact 5.2(c) there is $q \in T$ such that $r \leq q < t$ and q is an immediate predecessor of t. Now, since $r \perp p$ we have $p \neq q$.

($\Leftarrow$) Suppose that each $t \in T \setminus \mathrm{Min}(T)$ has at least two immediate predecessors. If $t \not\leq s$ then, clearly $t \neq s$. If $t \perp s$, we are done. Otherwise we have $t > s$ and, by Fact 5.2(c), there is $q \in T$ such that $s \leq q < t$ and q is an immediate predecessor of t. By the assumption t has one more immediate predecessor, say $q' < t$ and, clearly, $q' \perp s$. □

Fact 5.4 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a reversed tree. Then*
(a) $p \leq^* q \Leftrightarrow p \leq q \vee (q < p \wedge (-\infty, p] \setminus (-\infty, q] \subset (q, \infty))$;
(b) $p =^* q \Leftrightarrow p = q \vee (q < p \wedge (-\infty, p] \setminus (-\infty, q] \subset (q, \infty))$
$\vee (p < q \wedge (-\infty, q] \setminus (-\infty, p] \subset (p, \infty))$;
(c) $p \leq q \Rightarrow p \leq^* q \Rightarrow p \not\perp q$;
(d) Each equivalence class $C \in T/=^*$ has a maximum;
(e) $\mathrm{sq}\,\mathcal{T} \cong \langle T_1, \leq\restriction T_1 \rangle$, where $T_1 = \{\max(C) : C \in T/=^*\}$. Thus $\mathrm{sq}\,\mathcal{T}$ is a separative reversed tree.

Proof. (a) In a reversed tree we have $p \leq^* q$ iff each $r \leq p$ is comparable with q. So, if $p \leq^* q$ and $p \not\leq q$ then $q < p$ and, if $r \leq p$ and $r \not\leq q$ we have $r > q$.

Let $q < p$, $(-\infty, p] \setminus (-\infty, q] \subset (q, \infty))$ and $r \leq p$. Then either $r \leq q$, or, by the assumption, $r > q$; so r is comparable with q. Thus $p \leq^* q$.

The statements (b) and (c) follow from (a).

(d) Suppose that p and q are different maximal elements of C. Then $p =^* q$ and, by (b), $p < q$ or $q < p$, which is impossible, by the maximality of p and q.

(e) Clearly $\mathrm{sq}\,\mathcal{T} = \langle T/=^*, \trianglelefteq\rangle$, where $C_1 \trianglelefteq C_2$ iff $p \leq^* q$, for some $p \in C_1$ and $q \in C_2$. It is evident that the mapping $f : T/=^* \to T_1$, defined by $f(C) = \max(C)$, for all $C \in T/=^*$, is a bijection. For a proof that f is an isomorphism, we take $C_1, C_2 \in T/=^*$, define $p := \max(C_1)$ and $q := \max(C_2) = q$, notice that $C_1 \trianglelefteq C_2$ iff $p \leq^* q$ and show that $p \leq^* q \Leftrightarrow p \leq q$. The implication "$\Leftarrow$" follows from (c). On the other hand, if $p \leq^* q$, then, by (c) again, $p \leq q$ (and we are done) or $q < p$. If $q < p$ then $q \leq^* p$ and, hence, $p =^* q$, which means that $C_1 = C_2$ and $p = q$. So $p \leq q$ again. □

Proposition 5.5 *For each infinite separative reversed tree $\mathcal{T}$ we have*

$$\mathfrak{a}(\mathcal{T} \times \mathcal{T}) = \mathfrak{a}(\mathcal{T}).$$

Proof. The separativity of $\mathcal{T}$ implies that $\mathcal{T}$ has infinite antichains.

If $\mathfrak{a}(\mathcal{T}) = \omega$ and A is a maximal antichain in $\mathcal{T}$ of size ω, then $A \times A$ is a maximal antichain in $\mathcal{T} \times \mathcal{T}$ of the same size and, hence, $\mathfrak{a}(\mathcal{T} \times \mathcal{T}) = \omega$.

So, in the sequel we assume that $\mathfrak{a}(\mathcal{T}) = \lambda > \omega$. Since the set of maximal elements of $\mathcal{T}$, L_0, is a maximal antichain in $\mathcal{T}$, we have $|L_0| < \omega$ or $|L_0| = \lambda$.

If $|L_0| = \lambda$, say $L_0 = \{r_\xi : \xi < \lambda\}$, suppose that $A = \{\langle a_\alpha, b_\alpha\rangle : \alpha < \mu\}$ is a maximal antichain in $\mathcal{T} \times \mathcal{T}$, where $\mu < \lambda$. Then for each $\xi < \lambda$ there is $\alpha < \mu$ such that $\langle a_\alpha, b_\alpha\rangle \not\perp \langle r_\xi, r_\xi\rangle$. Thus there are $\alpha < \mu$ and different $\xi, \xi' < \lambda$ such that $\langle a_\alpha, b_\alpha\rangle \not\perp \langle r_\xi, r_\xi\rangle, \langle r_{\xi'}, r_{\xi'}\rangle$, which implies that a_α is compatible with r_ξ and $r_{\xi'}$, which is impossible. So $\mathfrak{a}(\mathcal{T} \times \mathcal{T}) = \lambda = \mathfrak{a}(\mathcal{T})$.

In the sequel we consider the remaining case when $\mathcal{T}$ is an infinite separative reversed tree satisfying $L_0 = \{r_1, \ldots, r_k\}$, where $k \in \mathbb{N}$, and $\mathfrak{a}(\mathcal{T}) = \lambda > \omega$. Let $T_{<\omega} = \bigcup_{n\in\omega} L_n$, $S_\lambda = \{t \in T : t$ has λ immediate predecessors$\}$ and

$$K = \{x \in T_{<\omega} : (x, \infty) \cap S_\lambda = \emptyset\}.$$

Claim 5.6 *(a) K is an upwards closed subtree of $\mathcal{T}$ and $L_0 \subset K$.*

(b) Each centered subset C of K has a lower bound in $\mathcal{T}$. In addition, if C is an infinite centered subset of K, it has λ-many lower bounds belonging to L_ω.

(c) If $x \notin K$, then

- either there is $y \in K \cap S_\lambda$ such that $x \leq u$, where u is one of λ-many immediate predecessors of y,

- or the set $\{y_n : n \in \omega\}$, where $(x, \infty] \cap L_n = \{y_n\}$, for $n \in \omega$, is a branch in K and $x \leq u$, where u is one of λ-many lower bounds of $\{y_n : n \in \omega\}$ in L_ω.

Proof. (b) If C is a centered subset of K, then C is a chain. So, if $|C| < \omega$, then $\min(C)$ is a lower bound of C.

If C is an infinite chain, then w.l.o.g. we assume that it is a maximal centered subset of K. So, since K is an upwards closed subset of T, there is $j_0 \leq k$ such that $C = \{y_n : n \in \omega\}$, where $r_{j_0} = y_0 > y_1 > y_2 > \dots$ and y_{n+1} is an immediate predecessor of y_n (in $\mathcal{T}$), thus $y_n \in L_n$, for all $n \in \omega$. For $n \in \omega$ we have $y_n > y_{n+1} \in K$ and, hence, the set $A_n \subset L_{n+1}$ of all immediate predecessors of y_n different from y_{n+1} is non-empty (since $\mathcal{T}$ is separative) and of size $< \lambda$ (because $y_n \notin S_\lambda$).

Suppose that some of the sets A_n is infinite and let n_0 be the minimal such n. Then the set

$$A' = \{r_j : j \neq j_0\} \cup \bigcup_{n \leq n_0} A_n \cup \{y_{n_0+1}\}$$

is of size $< \lambda$ and we show that it is a maximal antichain in $\mathcal{T}$. If $x \in T$, then there is $j \leq k$ such that $x \leq r_j$ and, if $j \neq j_0$, then x is compatible with r_j. If $x \leq r_{j_0} = y_0$, then, if $x \leq y_{n_0+1}$, we are done. Otherwise, let n_1 be the minimal $n \leq n_0 + 1$ such that $x \not\leq y_n$. Then $n_1 \geq 1$, $x \leq y_{n_1-1}$ and $x \neq y_{n_1}$ and, hence, x is comparable with some element of A_{n_1-1}. So, $|A_n| < \omega$, for all $n \in \omega$.

Let A_ω be the set of lower bounds of C belonging to L_ω. We show that the set

$$A'' = \{r_j : j \neq j_0\} \cup \bigcup_{n \in \omega} A_n \cup A_\omega$$

is a maximal antichain in $\mathcal{T}$. If $x \in T$, then there is $j \leq k$ such that $x \leq r_j$ and, if $j \neq j_0$, then x is compatible with r_j. If $x \leq r_{j_0} = y_0$, then we have two cases. First, if $x \leq y_n$, for all $n \in \omega$, then x is of height $\geq \omega$ and, hence, $x \leq z$, for some $z \in L_\omega$. Now, since $[x, \infty)$ is a linearly ordered subset of T, z is comparable with y_n and, hence, $z < y_n$, for all $n \in \omega$, which implies that $z \in A_\omega$; so x is compatible with an element of $A_\omega \subset A''$. Otherwise, let n_0 be the minimal element n of ω such that $x \not\leq y_n$. Then $n_0 \geq 1$, $x \leq y_{n_0-1}$ and $x \neq y_{n_0}$ and, hence, x is comparable with some element of A_{n_0-1}. Thus A'' is a maximal antichain in $\mathcal{T}$ and, since $1 \leq |A_n| < \omega$, for all $n \in \omega$, we have $|A_\omega| = \lambda$.

(c) Let $x \in T \setminus K$. If $x < y$ for some $y \in K \cap S_\lambda$ we are done. Otherwise we have $x \notin T_{<\omega}$, because $x \in T_{<\omega}$ would imply that the set $(x, \infty) \cap S_\lambda$ is non-empty and, hence, its element of the minimal height would be an element of $K \cap S_\lambda$ above x. Let $(x, \infty] \cap L_n = \{y_n\}$, for $n \in \omega$, and suppose that $y_n \notin K$, for some $n \in \omega$. Then $(y_n, \infty) \cap S_\lambda \neq \emptyset$ and if k_0 is the minimal $k < n$ such that $y_k \in S_\lambda$, then $y_k \in K \cap S_\lambda$ and $x < y_k$, which is impossible. Thus $\{y_n : n \in \omega\}$ is a branch in K, by (b) it has λ-many lower bounds belonging to L_ω and, clearly, x is less than or equal to some of them. □

Towards a contradiction let us suppose that $A = \{\langle a_\alpha, b_\alpha \rangle : \alpha < \mu\}$ is a maximal antichain in $\mathcal{T} \times \mathcal{T}$, where $\omega \leq \mu < \lambda$.

Claim 5.7 *$a_\alpha, b_\alpha \in K$, for all $\alpha < \mu$.*

Proof. Suppose that $a_{\alpha'} \not\in K$, for some $\alpha' < \mu$. Then, by Claim 5.6(c) we have the following two cases.

1. For some $y \in K \cap S_\lambda$, where $\{u_\xi : \xi < \lambda\}$ is the set of immediate predecessors of y, there is $\xi < \lambda$ such that $a_{\alpha'} \leq u_\xi$. By the maximality of A, for each $\xi < \lambda$ there is $\alpha_\xi < \mu$ such that $\langle a_{\alpha_\xi}, b_{\alpha_\xi}\rangle$ is compatible with $\langle u_\xi, b_{\alpha'}\rangle$ and, hence, there are $\alpha < \mu$ and different $\xi, \xi' < \lambda$ such that $\langle a_\alpha, b_\alpha\rangle$ is compatible with $\langle u_\xi, b_{\alpha'}\rangle$ and $\langle u_{\xi'}, b_{\alpha'}\rangle$. But this means that a_α is compatible with u_ξ and $u_{\xi'}$, which implies that $a_\alpha \geq y$ and, hence, $a_\alpha > a_{\alpha'}$. Consequently, $\alpha \neq \alpha'$, $a_\alpha \not\perp a_{\alpha'}$ and $b_\alpha \not\perp b_{\alpha'}$ and, hence, $\langle a_\alpha, b_\alpha\rangle$ is compatible with $\langle a_{\alpha'}, b_{\alpha'}\rangle$, which is impossible, since A is an antichain.

2. The set $\{y_n : n \in \omega\}$, where $(a_{\alpha'}, \infty] \cap L_n = \{y_n\}$, for $n \in \omega$, is a branch in K, $\{u_\xi : \xi < \lambda\}$ is the set of its lower bounds in L_ω and $a_{\alpha'} \leq u_\xi$, for some $\xi < \lambda$. As in the first case we obtain $\alpha < \mu$ and different $\xi, \xi' < \lambda$ such that $\langle a_\alpha, b_\alpha\rangle$ is compatible with $\langle u_\xi, b_{\alpha'}\rangle$ and $\langle u_{\xi'}, b_{\alpha'}\rangle$, which implies that $a_\alpha > u_\xi, u_{\xi'}$. Since $(u_\xi, \infty) = \{y_n : n \in \omega\}$ we have $a_\alpha > a_{\alpha'}$ and obtain a contradiction as above.

Thus $a_\alpha \in K$, for all $\alpha < \mu$ and, similarly, $b_\alpha \in K$, for all $\alpha < \mu$. □

W.l.o.g. we suppose that $S_0 \subset \mu$ is an infinite set satisfying condition (B) of Lemma 3.5. By Lemma 3.4, the poset $\langle \mathcal{S}, \subset\rangle$ contains a maximal element S such that $S_0 \subset S$. Since $\mathcal{T}$ is a reversed tree, this means that S is a maximal subset of μ such that $\{b_\alpha : \alpha \in S\}$ is a chain in $\mathcal{T}$. Thus by Claim 5.7 and the maximality of S we have

$$\{b_\alpha : \alpha \in S\} \text{ is a chain in } K, \tag{25}$$

$$\forall \beta \in \mu \setminus S \;\; \exists \alpha \in S \;\; b_\beta \perp b_\alpha. \tag{26}$$

By (25) and Claim 5.6(b), there is $b \in T$ such that

$$\forall \alpha \in S \;\; b \leq b_\alpha. \tag{27}$$

By (25), for different $\alpha, \alpha' \in S$ we have $b_\alpha \not\perp b_{\alpha'}$ and, since A is an antichain, $a_\alpha \perp a_{\alpha'}$. Thus $\{a_\alpha : \alpha \in S\}$ is an antichain in $\mathcal{T}$ of size $\leq \mu < \lambda$ and, since $\mathfrak{a}(\mathcal{T}) = \lambda$, there is $a \in T$ such that

$$\forall \alpha \in S \;\; a \perp a_\alpha. \tag{28}$$

By the maximality of A there is $\beta < \mu$ such that $\langle a_\beta, b_\beta\rangle \not\perp \langle a, b\rangle$. Consequently $a \not\perp a_\beta$, which by (28) implies $\beta \in \mu \setminus S$ and, by (26), there is $\alpha \in S$ such that $b_\beta \perp b_\alpha$. But $b_\beta \not\perp b$ and, by (27), $b \leq b_\alpha$, which, by Fact 5.2(b) implies $b_\beta \not\perp b_\alpha$ and we have a contradiction. □

Proof of Theorem 5.1. By Fact 5.4(e) the class of trees is closed under separative quotients; so the statement follows from Theorem 4.1(d) and Proposition 5.5. □

6 Atomic posets

Theorem 6.1 *For each atomic poset $\mathbb{P}$ containing infinite antichains we have*

$$\mathfrak{a}(\mathbb{P} \times \mathbb{P}) = \mathfrak{a}(\mathbb{P}).$$

Proof. By Fact 2.2(d), the separative quotient of an atomic poset is separative, so, by Theorem 4.1(d), w.l.o.g. we suppose that $\mathbb{P}$ is an separative atomic poset having infinite antichains and that $|\operatorname{At}(\mathbb{P})| = \mu$. In addition, by Fact 2.2(e) we can assume that $\mathbb{P} = \langle \mathcal{P}, \subset \rangle$, where

$$[\mu]^1 \subset \mathcal{P} \subset P(\mu)^+. \tag{29}$$

Since $\operatorname{cc}(\mathbb{P}) > \omega$ we have $\mu \geq \omega$.

If $\mathfrak{a}(\mathbb{P}) = \omega$, then, by Theorem 4.1(a), $\mathfrak{a}(\mathbb{P} \times \mathbb{P}) = \omega$, and the proof is over.

If $\mathfrak{a}(\mathbb{P}) = \lambda > \omega$, then, since $[\mu]^1$ is a maximal antichain in $\mathbb{P}$ we have $\lambda \leq \mu$. Suppose that $\mathfrak{a}(\mathbb{P} \times \mathbb{P}) = \kappa < \lambda$ and that $A = \{\langle p_\alpha, q_\alpha \rangle : \alpha \in \kappa\}$ is a maximal antichain in $\mathbb{P}$. According to Lemma 3.5 w.l.o.g. we assume that there is an infinite set $S_0 \subset \kappa$ such that $\{q_\alpha : \alpha \in S_0\}$ is a centered family in $\mathbb{P}$ and by Lemma 3.4 there is a maximal element S of the poset $\langle \mathcal{S}, \subset \rangle$ such that $S_0 \subset S$, which by (29) implies

$$\forall K \in [S]^{<\omega} \setminus \{\emptyset\} \ \ \textstyle\bigcap_{\alpha \in K} q_\alpha \neq \emptyset, \tag{30}$$

$$\forall \beta \in \kappa \setminus S \ \ \exists K \in [S]^{<\omega} \setminus \{\emptyset\} \ \ q_\beta \cap \textstyle\bigcap_{\alpha \in K} q_\alpha = \emptyset. \tag{31}$$

Since $\{q_\alpha : \alpha \in S\}$ is a centered family in $\mathbb{P}$, the set $\{p_\alpha : \alpha \in S\}$ is an infinite antichain in $\mathbb{P}$ and, since $|S| \leq \kappa < \lambda = \mathfrak{a}(\mathbb{P})$, it is not maximal, which by (29) means that there is $\xi \in \mu \setminus \bigcup_{\alpha \in S} p_\alpha$. Now $K := \{\alpha \in \kappa : \xi \in p_\alpha\} \subset \kappa \setminus S$ and we show that $|K| < \omega$ and that

$$B := \{q_\alpha : \alpha \in K\}$$

is a finite partition of μ. First, if $\zeta \in \mu$, then by (29) we have $\langle \{\xi\}, \{\zeta\} \rangle \in \mathcal{P}^2$ and, by the maximality of A, there is $\alpha \in \kappa$ such that $\langle \{\xi\}, \{\zeta\} \rangle \not\perp \langle p_\alpha, q_\alpha \rangle$ and, hence, $\xi \in p_\alpha$, which implies $\alpha \in K$, and $\zeta \in q_\alpha$. So, $\mu = \bigcup_{\alpha \in K} q_\alpha$. Second, if $\alpha_1, \alpha_2 \in K$ and $\alpha_1 \neq \alpha_2$, then $\xi \in p_{\alpha_1} \cap p_{\alpha_2}$ and, since A is an antichain, $q_{\alpha_1} \cap q_{\alpha_2} = \emptyset$ so B is a partition of μ, which by (29) implies that B is a maximal antichain in $\mathbb{P}$. Since $|B| \leq \kappa < \lambda = \mathfrak{a}(\mathbb{P})$ we have $|B| < \omega$.

By (30) the family $\{q_\alpha : \alpha \in S\} \subset P(\mu)$ has the finite intersection property and, hence, there is an ultrafilter $\mathcal{U}$ on μ such that $\{q_\alpha : \alpha \in S\} \subset \mathcal{U}$. Since B is a finite partition of μ there is $\beta \in K$ such that $q_\beta \in \mathcal{U}$. But $\beta \in \kappa \setminus S$ and, by (31), there is a nonempty finite set $K_1 \subset S$ such that $q_\beta \cap \bigcap_{\alpha \in K_1} q_\alpha = \emptyset$, which is impossible. $\square$

A slight variation of the proof of Theorem 6.1 gives a proof of the following statement which is of purely combinatorial nature and, perhaps, of wider interest.

Theorem 6.2 *If X is an infinite set and $\{A_i \times B_i : i \in I\}$ an infinite partition of the square $X \times X$, (where $A_i, B_i \subset X$, for $i \in I$), then at least one of the families $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ contains an infinite partition of the set X.*

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