

Inner Models from Extended Logics*

Juliette Kennedy[†]
Helsinki

Menachem Magidor[‡]
Jerusalem

Jouko Väänänen[§]
Helsinki and Amsterdam

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Abstract

If we replace first order logic by second order logic in the original definition of Gödel's inner model L , we obtain HOD ([36]). In this paper we consider inner models that arise if we replace first order logic by a logic that has some, but not all, of the strength of second order logic. Typical examples are the extensions of first order logic by generalized quantifiers, such as the Magidor-Malitz quantifier ([26]), the cofinality quantifier ([38]), stationary logic ([5]) or the Härtig-quantifier ([14, 15]). Our first set of results show that both L and HOD manifest some amount of *formalism freeness* in the sense that they are not very sensitive to the choice of the underlying logic. Our second set of results shows that the cofinality quantifier, stationary logic, and the Härtig-quantifier give rise to new robust inner models between L and HOD. We show, among other things, that assuming a proper class of Woodin cardinals the regular cardinals of V are Mahlo in the inner model arising from the cofinality quantifier and the theory of that model is

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(set) forcing absolute and independent of the cofinality in question. Respectively, assuming a proper class of measurable Woodin cardinals the regular cardinals of V are measurable in the inner model arising from stationary logic and the theory of that model is (set) forcing absolute.

1 Introduction

Inner models, together with the forcing method, are the basic building blocks used by set theorists to prove relative consistency results on the one hand and to try to chart the “true” universe of set theory V on the other hand.

The first and best known, also the smallest of the inner models is Gödel’s L , the universe of constructible sets. An important landmark among the largest inner models is the universe of hereditarily ordinal definable sets HOD , also introduced by Gödel¹. In between the two extremes there is a variety of inner models arising from enhancing Gödel’s L by normal ultrafilters on measurable cardinals, or in a more general case extenders, something that L certainly does not have itself.

We propose a construction of inner models which arise not from adding normal ultrafilters, or extenders, to L , but by changing the underlying construction of L . We show that the new inner models have similar forcing absoluteness properties as $L(\mathbb{R})$, but at the same time they satisfy the Axiom of Choice.

Gödel’s hierarchy of constructible sets is defined by reference to first order definability. Sets on a higher level are the first order definable sets of elements of

¹Gödel introduced HOD in his 1946 *Remarks before the Princeton Bicentennial conference on problems in mathematics* [13]. The lecture was given during a session on computability organized by Alfred Tarski, and in it Gödel asks whether notions of definability and provability can be isolated in the set-theoretic formalism, which admit a form of robustness similar to that exhibited by the notion of general recursiveness: “Tarski has stressed in his lecture the great importance (and I think justly) of the concept of general recursiveness (or Turing computability). It seems to me that this importance is largely due to the fact that with this concept one has succeeded in giving an absolute definition of an interesting epistemological notion, i.e. one not depending on the formalism chosen. In all other cases treated previously, such as definability or demonstrability, one has been able to define them only relative to a given language, and for each individual language it is not clear that the one thus obtained is not the one looked for. For the concept of computability however... the situation is different... This, I think, should encourage one to expect the same thing to be possible also in other cases (such as demonstrability or definability).”

Gödel contemplates the idea that constructibility might be a suitable analog of the notion of general recursiveness. Gödel also considers the same for HOD , and predicts the consistency of the axiom $V = HOD + 2^{\aleph_0} > \aleph_1$ (proved later by McAloon [32]). See [17] for a development of Gödel’s proposal in a “formalism free” direction.

lower levels. The inner model L enjoys strong forcing absoluteness: truth in L cannot be changed by forcing, in fact not by any method of extending the universe without adding new ordinals. Accordingly, it is usually possible to settle in L , one way or other, any set theoretical question which is otherwise independent of ZFC. However, the problem with L is that it cannot have large cardinals on the level of the Erdős cardinal $\kappa(\omega_1)$ or higher. To remedy this, a variety of inner models, most notably the smallest inner model L^μ with a measurable cardinal, have been introduced (see e.g. [42]).

We investigate the question to what extent is it essential that first order definability is used in the construction of Gödel's L . In particular, what would be the effect of changing first order logic to a stronger logic? In fact there are two precedents: Scott and Myhill [36] showed that if first order definability is replaced by second order definability the all-encompassing class HOD of hereditarily ordinal definable sets is obtained. The inner model L is thus sensitive to the definability concept used in its construction. The inner model HOD has consistently even supercompact cardinals [33]. However, HOD does not solve any of the central independent statements of set theory; in particular, it does not solve the Continuum Hypothesis or the Souslin Hypothesis [32].

A second precedent is provided by Chang [6] in which first order definability was replaced by definability in the infinitary language $\mathcal{L}_{\omega_1\omega_1}$, obtaining what came to be known as the *Chang model*. Kunen [20] showed that the Chang model fails to satisfy the Axiom of Choice, if the existence of uncountably many measurable cardinals is assumed. We remark that the inner model $L(\mathbb{R})$ arises in the same way if $\mathcal{L}_{\omega_1\omega}$ is used instead of $\mathcal{L}_{\omega_1\omega_1}$. Either way, the resulting inner model fails to satisfy the Axiom of Choice if enough large cardinals are assumed. This puts these inner models in a different category. On the other hand, the importance of both the Chang model and $L(\mathbb{R})$ is accentuated by the result of Woodin [47] that under large cardinal assumptions the first order theory of the Chang model, as well as of $L(\mathbb{R})$, is absolute under set forcing. So there would be reasons to expect that these inner models would solve several independent statements of set theory, e.g. the CH. However, the failure of the Axiom of Choice in these inner models dims the light such “solutions” would shed on CH. For example, assuming large cardinals, the model $L(\mathbb{R})$ satisfies the statement “Every uncountable set of reals contains a perfect subset”, which under AC would be equivalent to CH . On the other hand, large cardinals imply that there is in $L(\mathbb{R})$ a surjection from \mathbb{R} onto ω_2 , which under AC would imply $\neg CH$.

In this paper we define analogs of the constructible hierarchy by replacing first order logic in Gödel's construction by any one of a number of logics. The inner

models HOD , $L(\mathbb{R})$ and the Chang model are special cases, obtained by replacing first order definability by definability in \mathcal{L}^2 , $\mathcal{L}_{\omega_1\omega}$ and $\mathcal{L}_{\omega_1\omega_1}$, respectively. Our main focus is on extensions of first order logic by generalized quantifiers in the sense of Mostowski [34] and Lindström [24]. We obtain new inner models which are L -like in that they are models of ZFC and their theory is absolute under set forcing, and at the same time these inner models contain large cardinals, or inner models with large cardinals.

The resulting inner models enable us to make distinctions in set theory that were previously unknown. However, we also think of the arising inner models as a tool to learn more about extended logics. As it turns out, for many non-equivalent logics the inner model is the same. In particular for many non-elementary logics the inner model is the same as for first order logic. We may think that such logics have some albeit distant similarity to first order logic. On the other hand, some other logics give rise to the inner model HOD . We may say that they bear some resemblance to second order logic.

Our main results can be summarized as follows:

- (A) For the logics $\mathcal{L}(Q_\alpha)$ we obtain just L , for any choice of α . If $0^\#$ exists the same is true of the Magidor-Malitz logics $\mathcal{L}(Q_\alpha^{\text{MM}})$.
- (B) If $0^\#$ exists the cofinality quantifier logic $\mathcal{L}(Q_\omega^{\text{cf}})$ yields a proper extension C^* of L . But $C^* \neq HOD$, if there are uncountably many measurable cardinals.
- (C) If there is a proper class of Woodin cardinals, then regular cardinals $> \aleph_1$ are Mahlo and indiscernible in C^* , and the theory of C^* is invariant under (set) forcing.
- (D) The Dodd-Jensen Core Model is contained in C^* . If there is an inner model with a measurable cardinal, then such an inner model is also contained in C^* .
- (E) If there is a Woodin cardinal and a measurable cardinal above it, then CH is true in the version $C^*(x)$ of C^* , obtained by allowing a real parameter x , for a cone of reals x .
- (F) If there is a proper class of measurable Woodin cardinals, then then regular cardinals are measurable in $C(\text{aa})$, the version of L obtained by using stationary logic $\mathcal{L}(\text{aa})$, and the theory of $C(\text{aa})$ is invariant under (set) forcing.

2 Basic concepts

We define an analogue of the constructible hierarchy of Gödel by replacing first order logic in the construction by an arbitrary logic \mathcal{L}^* . We think of logics in the sense of Lindström [25], Mostowski [35], Barwise [4], and the collection [2]. What is essential is that a logic \mathcal{L}^* has two components i.e. $\mathcal{L}^* = (S^*, T^*)$, where S^* is the class of sentences of \mathcal{L}^* and T^* is the truth predicate of \mathcal{L}^* . We usually write $\varphi \in \mathcal{L}^*$ for $\varphi \in F^*$ and $\mathcal{M} \models \varphi$ for $T^*(\mathcal{M}, \varphi)$. The classes F^* and T^* may be defined with parameters, as in the case of $\mathcal{L}_{\kappa\lambda}$, where κ and λ can be treated as parameters. A logic \mathcal{L}^* is a *sublogic* of another logic \mathcal{L}^+ , $\mathcal{L}^* \leq \mathcal{L}^+$, if for every $\varphi \in F^*$ there is $\varphi^+ \in F^+$ such that for all \mathcal{M} : $\mathcal{M} \models \varphi \iff \mathcal{M} \models \varphi^+$. We assume that our logics have first order logic as sublogic.

Example 2.1. 1. *First order logic* $\mathcal{L}_{\omega\omega}$ (or FO) is the logic (S^*, T^*) , where S^* is the set of first order sentences and T^* is the usual truth definition for first order sentences.

2. *Infinitary logic* $\mathcal{L}_{\kappa\lambda}$, where κ and $\lambda \leq \kappa$ are regular cardinals, is the logic (S^*, T^*) , where S^* consists of the sentences built inductively from conjunctions and disjunctions of length $< \kappa$ of sentences of $\mathcal{L}_{\kappa\lambda}$, and homogeneous strings of existential and universal quantifiers of length $< \lambda$ in front of formulas of $\mathcal{L}_{\kappa\lambda}$. The class T^* is defined in the obvious way. We allow also the case that κ or λ is ∞ . We use $\mathcal{L}_{\kappa\lambda}^\omega$ to denote that class of formulae of $\mathcal{L}_{\kappa\lambda}$ with only finitely many free variables.
3. The logic $\mathcal{L}(Q)$ with a generalized quantifier Q is the logic (S^*, T^*) , where S^* is obtained by adding the new quantifier Q to first order logic. The exact syntax depends on the type of Q , (see our examples below). The class T^* is defined by first fixing the *defining model class* \mathcal{K}_Q of Q and then defining T^* by induction on formulas:

$$\mathcal{M} \models Qx_1, \dots, x_n \varphi(x_1, \dots, x_n, \vec{b}) \iff$$

$$(M, \{(a_1, \dots, a_n) \in M^n : \mathcal{M} \models \varphi(a_1, \dots, a_n, \vec{b})\}) \in \mathcal{K}_Q.$$

Thought of in this way, the defining model class of the existential quantifier is the class $\mathcal{K}_\exists = \{(M, A) : \emptyset \neq A \subseteq M\}$, and the defining model class of the universal quantifier is the class $\mathcal{K}_\forall = \{(M, A) : A = M\}$. Noting that the generalisations of \exists with defining class $\{(M, A) : A \subseteq M, |A| \geq n\}$,

where n is fixed, are definable in first order logic, Mostowski [34] introduced the generalisations Q_α of \exists with defining class

$$\mathcal{K}_{Q_\alpha} = \{(M, A) : A \subseteq M, |A| \geq \aleph_\alpha\}.$$

Many other generalized quantifiers are known today in the literature and we will introduce some important ones later.

4. *Second order logic* \mathcal{L}^2 is the logic (S^*, T^*) , where S^* is obtained from first order logic by adding variables for n -ary relations for all n and allowing existential and universal quantification over the new variables. The class T^* is defined by the obvious induction. In this inductive definition of T^* the second order variables range over all relations of the domain (and not only e.g. over definable relations).

We now define the main new concept of this paper:

Definition 2.2. Suppose \mathcal{L}^* is a logic. If M is a set, let $\text{Def}_{\mathcal{L}^*}(M)$ denote the set of all sets of the form $X = \{a \in M : (M, \in) \models \varphi(a, \vec{b})\}$, where $\varphi(x, \vec{y})$ is an arbitrary formula of the logic \mathcal{L}^* and $\vec{b} \in M$. We define a hierarchy (L'_α) of *sets constructible using \mathcal{L}^** as follows:

$$\begin{aligned} L'_0 &= \emptyset \\ L'_{\alpha+1} &= \text{Def}_{\mathcal{L}^*}(L'_\alpha) \\ L'_\nu &= \bigcup_{\alpha < \nu} L'_\alpha \text{ for limit } \nu \end{aligned}$$

We use $C(\mathcal{L}^*)$ to denote the class $\bigcup_\alpha L'_\alpha$.

Thus a typical set in $L'_{\alpha+1}$ has the form

$$X = \{a \in L'_\alpha : (L'_\alpha, \in) \models \varphi(a, \vec{b})\} \tag{1}$$

where $\varphi(x, \vec{y})$ is a formula of \mathcal{L}^* and $\vec{b} \in L'_\alpha$. It is important to note that $\varphi(x, \vec{y})$ is a formula of \mathcal{L}^* *in the sense of V* , not in the sense of $C(\mathcal{L}^*)$. Also, note that $(L'_\alpha, \in) \models \varphi(a, \vec{b})$ refers to T^* in the sense of V , not in the sense of $C(\mathcal{L}^*)$.

By definition, $C(\mathcal{L}_{\omega\omega}) = L$. Myhill-Scott [36] showed that $C(\mathcal{L}^2) = \text{HOD}$ (See Theorem 10.1 below). Chang [6] considered $C(\mathcal{L}_{\omega_1\omega_1})$ and pointed out that this is the smallest transitive model of ZFC containing all ordinals and closed under countable sequences. Kunen [20] showed that $C(\mathcal{L}_{\omega_1\omega_1})$ fails to satisfy the Axiom of Choice, if we assume the existence of uncountably many measurable cardinals (see Theorem 5.8 below). Sureson [43, 44] investigated a Covering Lemma for $C(\mathcal{L}_{\omega_1\omega_1})$.

Proposition 2.3. *For any \mathcal{L}^* the class $C(\mathcal{L}^*)$ is a transitive model of ZF containing all the ordinals.*

Proof. As in the usual proof of ZF in L , which works here too. \square

We cannot continue and follow the usual proof of AC in L , because the *syntax* of \mathcal{L}^* may introduce sets into $C(\mathcal{L}^*)$ without introducing a well-ordering for them (See Theorem 2.10). To overcome this difficulty, we introduce the following concept (we limit ourselves to logics in which every formula has only finitely many free variables):

Definition 2.4. A logic \mathcal{L}^* is *adequate to truth in itself*² if for all finite vocabularies K there is function $\varphi \mapsto \ulcorner \varphi \urcorner$ from all formulas $\varphi(x_1, \dots, x_n) \in \mathcal{L}^*$ in the vocabulary K into ω , and a formula $\text{Sat}_{\mathcal{L}^*}(x, y, z)$ in \mathcal{L}^* such that:

1. The function $\varphi \mapsto \ulcorner \varphi \urcorner$ is one to one and has a recursive range.
2. For all admissible sets M , formulas φ of \mathcal{L}^* in the vocabulary K , structures $\mathcal{N} \in M$ in the vocabulary K , and $a_1, \dots, a_n \in N$ the following conditions are equivalent:
 - (a) $M \models \text{Sat}_{\mathcal{L}^*}(\mathcal{N}, \ulcorner \varphi \urcorner, \langle a_1, \dots, a_n \rangle)$
 - (b) $\mathcal{N} \models \varphi(a_1, \dots, a_n)$.

We may admit ordinal parameters in this definition.

Example 2.5. First order logic $\mathcal{L}_{\omega\omega}$ and the logic $\mathcal{L}(Q_\alpha)$ are adequate to truth in themselves. Also second order logic is adequate to truth in itself. Infinitary logics are for obvious reasons not adequate to truth in themselves, but there is a more general notion which applies better for them (see [10, 46]). In infinitary logic what accounts as a formula depends on set theory. For example, in the case of $\mathcal{L}_{\omega_1\omega}$ the formulas essentially code in their syntax all reals.

Proposition 2.6. *If \mathcal{L}^* is adequate to truth in itself, there are formulas $\Phi_{\mathcal{L}^*}(x)$ and $\Psi_{\mathcal{L}^*}(x, y)$ of \mathcal{L}^* in the vocabulary $\{\in\}$ such that if M is an admissible set and $\alpha = M \cap \text{On}$, then:*

1. $\{a \in M : (M, \in) \models \Phi_{\mathcal{L}^*}(a)\} = L'_\alpha \cap M$.

²This is a special case of a concept with the same name in [10].

2. $\{(a, b) \in M \times M : (M, \in) \models \Psi_{\mathcal{L}^*}(a, b)\}$ is a well-order $<'_\alpha$ the field of which is $L'_\alpha \cap M$.

It is important to note that the formulas $\Phi_{\mathcal{L}^*}(x)$ and $\Psi_{\mathcal{L}^*}(x, y)$ are in the extended logic \mathcal{L}^* , not necessarily in first order logic.

Recall that we have defined the logic \mathcal{L}^* as a pair (S^*, T^*) . We can use the set-theoretical predicates S^* and T^* to write “ $(M, \in) \models \Phi(a)$ ” and “ $(M, \in) \models \Psi_{\mathcal{L}^*}(x, y)$ ” of Proposition 2.6 as formulas $\tilde{\Phi}_{\mathcal{L}^*}(M, x)$ and $\tilde{\Psi}_{\mathcal{L}^*}(M, x, y)$ of the first order language of set theory, such that for all M with $\alpha = M \cap \text{On}$ and $a, b \in M$:

1. $\tilde{\Phi}_{\mathcal{L}^*}(M, a) \leftrightarrow [(M, \in) \models \Phi_{\mathcal{L}^*}(a)] \leftrightarrow a \in L'_\alpha$.
2. $\tilde{\Psi}_{\mathcal{L}^*}(M, a, b) \leftrightarrow [(M, \in) \models \Psi_{\mathcal{L}^*}(a, b)] \leftrightarrow a <'_\alpha b$.

Proposition 2.7. *If \mathcal{L}^* is adequate to truth in itself, then $C(\mathcal{L}^*)$ satisfies the Axiom of Choice.*

Proof. Let us fix α and show that there is a well-order of L'_α in $C(\mathcal{L}^*)$. Let $\kappa = |\alpha|^+$. Then $\Psi(x, y)$ defines on L'_κ a well-order $<'_\kappa$ of L'_κ . The relation $<'_\kappa$ is in $L'_{\kappa+1} \subseteq C(\mathcal{L}^*)$ by the definition of $C(\mathcal{L}^*)$. \square

There need not be a first order definable well-order of the class $C(\mathcal{L}^*)$ (see Theorem 6.5 for an example) although there always is in V a definable relation which well-orders $C(\mathcal{L}^*)$. Of course, in this case $V \neq C(\mathcal{L}^*)$.

Note that trivially

$$\mathcal{L}^* \leq \mathcal{L}^+ \text{ implies } C(\mathcal{L}^*) \subseteq C(\mathcal{L}^+).$$

Thus varying the logic \mathcal{L}^* we get a whole hierarchy inner models $C(\mathcal{L}^*)$. Many questions can be asked about these inner models. For example we can ask: (1) do we get all the known inner models in this way, (2) under which conditions do these inner models satisfy GCH, (3) do inner models obtained in this way have other characterisations (such as L , HOD and $C(\mathcal{L}_{\omega_1\omega_1})$ have), etc.

A set a is *ordinal definable* if there is a formula $\varphi(x, y_1, \dots, y_n)$ and ordinals $\alpha_1, \dots, \alpha_n$ such that

$$\forall x (x \in a \iff \varphi(x, \alpha_1, \dots, \alpha_n)). \quad (2)$$

A set a is *hereditarily ordinal definable* if a itself and also every element of $\text{TC}(a)$ is ordinal definable. When we look at the construction of $C(\mathcal{L}^*)$ we can observe that sets in $C(\mathcal{L}^*)$ are always hereditarily ordinal definable when the formulas of \mathcal{L}^* are finite (more generally, the formulas may be hereditarily ordinal definable):

Proposition 2.8. *If \mathcal{L}^* is any logic such that the formulas F^* and S^* do not contain parameters (except hereditarily ordinal definable ones) and in addition every formula of \mathcal{L}^* (i.e. element of the class F^*) is a finite string of symbols (or more generally hereditarily ordinal definable), then every set in $C(\mathcal{L}^*)$ is hereditarily ordinal definable.*

Proof. Recall the construction of the successor stage of $C(\mathcal{L}^*)$: $X \in L'_{\alpha+1}$ if and only if for some $\varphi(x, \vec{y}) \in \mathcal{L}^*$ and some $\vec{b} \in L'_\alpha$

$$X = \{x \in L'_\alpha : (L'_\alpha, \in) \models \varphi(x, \vec{b})\}.$$

Now we can note that

$$X = \{x \in L'_\alpha : S_{\mathcal{L}^*}((L'_\alpha, \in), \varphi(x, \vec{b}))\}.$$

Thus if L'_α is ordinal definable, then so is X . Moreover,

$$\begin{aligned} \forall z(z \in L'_{\alpha+1} &\iff \exists \varphi(x, \vec{u})(F^*(\varphi(x, \vec{u})) \wedge \\ \forall y(y \in z &\iff y \in L'_\alpha \wedge S^*((L'_\alpha, \in), \varphi(x, \vec{u}))), \end{aligned}$$

or in short

$$\forall z(z \in L'_{\alpha+1} \iff \psi(z, L'_\alpha)),$$

where $\psi(z, w)$ is a first order formula in the language of set theory. When we compare this with (2) we see that if L'_α is ordinal definable and if the (first order) set-theoretical formulas F^* and S^* have no parameters, then also $L'_{\alpha+1}$ is ordinal definable. It follows that the class $\langle L'_\alpha : \alpha \in \text{On} \rangle$ is ordinal definable, whence $\langle L'_\alpha : \alpha < \nu \rangle$, and thereby also L'_ν , is in HOD for all limit ν . \square

Thus, unless the formulas of the logic \mathcal{L}^* are syntactically complex (as happens in the case of infinitary logics like $\mathcal{L}_{\omega_1\omega}$ and $\mathcal{L}_{\omega_1\omega_1}$), the hereditarily ordinal definable sets form a firm ceiling for the inner models $C(\mathcal{L}^*)$.

Theorem 2.9. $C(\mathcal{L}_{\infty\omega}) = V$.

Proof. Let $(L'_\alpha)_\alpha$ be the hierarchy behind $C(\mathcal{L}_{\infty\omega})$. We show $V_\alpha \subseteq C(\mathcal{L}_{\infty\omega})$ by induction on α . For any set a let the formulas $\theta_a(x)$ of set theory be defined by the following induction:

$$\theta_a(x) = \bigwedge_{b \in a} \exists y(y E x \wedge \theta_b(y)) \wedge \forall y(y E x \rightarrow \bigvee_{b \in a} \theta_b(y)).$$

Note that in any transitive set M containing a :

$$(M, \in) \models \forall x(\theta_a(x) \iff x = a).$$

Let us assume $V_\alpha \subseteq C(\mathcal{L}_{\infty\omega})$, or more exactly, $V_\alpha \in L'_\beta$. Let $X \subseteq V_\alpha$. Then

$$X = \{a \in L'_\beta : L'_\beta \models a \in V_\alpha \wedge \bigvee_{b \in X} \theta_b(a)\} \in L'_{\beta+1}.$$

□

Note that the proof actually shows $C(\mathcal{L}_{\infty\omega}^\omega) = V$.

Theorem 2.10. $C(\mathcal{L}_{\omega_1\omega}^\omega) = L(\mathbb{R})$.

Proof. Let $(L'_\alpha)_\alpha$ be the hierarchy behind $C(\mathcal{L}_{\omega_1\omega})$. We first show $L(\mathbb{R}) \subseteq C(\mathcal{L}_{\omega_1\omega})$. Since $C(\mathcal{L}_{\omega_1\omega})$ is clearly a transitive model of ZF it suffices to show that $\mathbb{R} \subseteq C(\mathcal{L}_{\omega_1\omega})$. Let $X \subseteq \omega$. Let $\varphi_n(x)$ be a formula of set theory which defines the natural number n in the obvious way. Then

$$X = \{a \in L'_\alpha : L'_\alpha \models a \in \omega \wedge \bigvee_{n \in X} \varphi_n(a)\} \in L'_{\alpha+1}.$$

Next we show $C(\mathcal{L}_{\omega_1\omega}) \subseteq L(\mathbb{R})$. We prove by induction on α that $L'_\alpha \subseteq L(\mathbb{R})$. Suppose this has been proved for α and $L'_\alpha \in L_\beta(\mathbb{R})$. Suppose $X \in L'_{\alpha+1}$. This means that there is a formula $\varphi(x, \vec{y})$ of $\mathcal{L}_{\omega_1\omega}$ and a finite sequence $\vec{b} \in L'_\alpha$ such that

$$X = \{a \in L'_\alpha : L'_\alpha \models \varphi(a, \vec{b})\}.$$

It is possible to write a first order formula Φ of set theory such that

$$X = \{a \in L_\beta(\mathbb{R}) : L_\beta(\mathbb{R}) \models \Phi(a, L'_\alpha, \varphi, \vec{b})\}.$$

Since there is a canonical coding of formulas of $\mathcal{L}_{\omega_1\omega}$ by reals we can consider φ as a real parameter. Thus $X \in L_{\beta+1}(\mathbb{R})$. □

Theorem 2.11. $C(\mathcal{L}_{\omega_1\omega}) = C(\mathcal{L}_{\omega_1\omega_1})$ (= Chang model).

Proof. The model $C(\mathcal{L}_{\omega_1\omega})$ is closed under countable sequences, for if $a_n \in C(\mathcal{L}_{\omega_1\omega})$ for $n < \omega$, then the $\mathcal{L}_{\omega_1\omega}$ -formula

$$\forall y(y \in x \leftrightarrow \bigvee_n y = \langle n, a_n \rangle).$$

defines the sequence $\langle a_n : n < \omega \rangle$. Since the Chang model is the smallest transitive model of ZF closed under countable sequences, the claim follows. □

We can construe the inner model L^μ as a model of the form $C(\mathcal{L}^*)$ as follows (See also Theorem 8.3):

Definition 2.12. Suppose U is a normal ultrafilter on κ . We define a generalised quantifier Q_U^κ as follows:

$$\begin{aligned} \mathcal{M} \models Q_\kappa^U wxyv\theta(w)\varphi(x,y)\psi(v) &\iff \\ \exists \pi : (S, R) \cong (\kappa, <) \wedge \pi'' A \in U, \end{aligned}$$

where

$$\begin{aligned} S &= \{a \in M : \mathcal{M} \models \theta(a)\} \\ R &= \{(a, b) \in M^2 : \mathcal{M} \models \varphi(a, b)\} \\ A &= \{a \in M : \mathcal{M} \models \psi(a)\} \end{aligned}$$

Theorem 2.13. $C(Q_\kappa^U) = L^U$.

Proof. Let (L'_α) be the hierarchy that defines $C(Q_\kappa^U)$. We prove for all a : $L'_\alpha = L_\alpha^U$. We use induction on α . Suppose the claim is true up to α . Suppose $X \in L'_{\alpha+1}$, e.g.

$$X = \{a \in L'_\alpha : (L'_\alpha, \in) \models \varphi(a, \vec{b})\},$$

where $\varphi(x, \vec{y}) \in FO(Q_\kappa^U)$ and $\vec{b} \in L'_\alpha$. We show $X \in L_\alpha^U$. To prove this we use induction on $\varphi(x, \vec{y})$. Suppose

$$X = \{a \in L'_\alpha : (L'_\alpha, \in) \models Q_\kappa^U wxyv\theta(z, a, \vec{b})\varphi(x, y, a, \vec{b})\psi(v, a, \vec{b})\}$$

and the claim has been proved for θ , φ and ψ . Let

$$Y_a = \{c \in L'_\alpha : (L'_\alpha, \in) \models \theta(c, a, \vec{b})\},$$

$$R_a = \{(c, d) \in L_\alpha'^2 : (L'_\alpha, \in) \models \varphi(c, d, a, \vec{b})\},$$

and

$$S_a = \{c \in L'_\alpha : (L'_\alpha, \in) \models \psi(c, a, \vec{b})\}.$$

Thus

$$X = \{a \in L'_\alpha : \exists \pi : (Y_a, R_a) \cong (\kappa, <) \wedge \pi'' S_a \in U\}.$$

But now

$$X = \{a \in L_\alpha^U : \exists \pi : (Y_a, R_a) \cong (\kappa, <) \wedge \pi'' S_a \in U \cap L^U\},$$

so $X \in L^U$.

Claim 2: For all $a: L_\alpha^U \in C(Q_\kappa^U)$. We use induction on α . It suffices to prove for all $\alpha: U \cap L_\alpha^U \in C(Q_\kappa^U)$. Suppose the claim is true up to α . We show $U \cap L_{\alpha+1}^U \in C(Q_\kappa^U)$. Now

$$\begin{aligned} U \cap L_{\alpha+1}^U &= U \cap \text{Def}(L_\alpha^U, \in, U \cap L_\alpha^U) \\ &= \{X \subseteq L_\alpha^U : X \in U \wedge X \in \text{Def}(L_\alpha^U, \in, U \cap L_\alpha^U)\} \end{aligned}$$

□

3 Absolute logics

The concept of an absolute logic tries to capture what it is in $\mathcal{L}_{\omega\omega}$ that makes it “first order”. Is it possible that logics that are “first order” in the same way as $\mathcal{L}_{\omega\omega}$ is turn out to be substitutable with $\mathcal{L}_{\omega\omega}$ in the definition of the constructible hierarchy?

Barwise writes in [3, pp. 311-312]:

“Imagine a logician κ using T as his metatheory for defining the basic notions of a particular logic \mathcal{L}^* . When is it reasonable for us, as outsiders looking on, to call \mathcal{L}^* a “first order” logic? If the words “first order” have any intuitive content it is that the truth or falsity of $\mathcal{M} \models^* \varphi$ should depend only on φ and \mathcal{M} , not on what subsets of M may or may not exist in κ ’s model of his set theory T . In other words, the relation \models^* should be absolute for models of T . What about the predicate $\varphi \in \mathcal{L}^*$ of φ ? To keep from ruling out $L_{\omega_1\omega}$ (the predicate $\varphi \in L_{\omega_1\omega}$ is not absolute since the notion of countable is not absolute) we demand only that the notion of \mathcal{L}^* -sentence be persistent for models of T : i.e. that if $\varphi \in \mathcal{L}^*$ holds in κ ’s model of T then it should hold in any end extension of it.”

Using this idea as a guideline, Barwise [3] introduced the concept of an absolute logic:

Definition 3.1. Suppose A is any class and T is any theory in the language of set theory. A logic \mathcal{L}^* is T -absolute if there are a Σ_1 -predicate $F_{\mathcal{L}^*}(x)$, a Σ_1 -predicate $S_{\mathcal{L}^*}(x, y)$, and a Π_1 -predicate $P_{\mathcal{L}^*}(x, y)$ such that $\varphi \in \mathcal{L}^* \iff F_{\mathcal{L}^*}(\varphi), M \models$

$\varphi \iff S_{\mathcal{L}^*}(M, \varphi)$ and $T \vdash \forall x \forall y (F_{\mathcal{L}^*}(x) \rightarrow (S_{\mathcal{L}^*}(x, y) \leftrightarrow P_{\mathcal{L}^*}(x, y)))$. If parameters from a class A are allowed, we say that \mathcal{L}^* is *absolute with parameters from A* .

Note that the stronger T is, the weaker the notion of T -absoluteness is. Barwise [3] calls KP-absolute logics *strictly* absolute.

As a consequence of Theorems 2.9 and 2.10 absolute logics may be very strong from the point of view of the inner model construction. However, this is so only because of the potentially complex syntax of the absolute logics, as is the case with $\mathcal{L}_{\omega_1\omega}$. Accordingly we introduce the following notion:

Definition 3.2. A absolute logic \mathcal{L}^* has *T -absolute syntax* if there is a Π_1 -predicate $G_{\mathcal{L}^*}(x)$ such that $T \vdash \forall x (F_{\mathcal{L}^*}(x) \leftrightarrow G_{\mathcal{L}^*}(x))$. With may allow parameters, as in Definition 3.1.

In other words, “absolute syntax” means that the class of \mathcal{L}^* -formulas has a Δ_1^T -definition. Obviously, $\mathcal{L}_{\omega_1\omega}$ does not satisfy this condition. On the other hand, many absolute logics, such as $\mathcal{L}_{\omega\omega}$, $\mathcal{L}(Q_0)$, weak second order logic, \mathcal{L}_{HYP} , etc have absolute syntax.

The original definition of absolute logics does not allow parameters. Still there are many logics that are absolute apart from dependence on a parameter. In our context it turns out that we can and should allow parameters.

The **cardinality quantifier** Q_α is defined as follows:

$$\mathcal{M} \models Q_\alpha x \varphi(x, \vec{b}) \iff |\{a \in M : \mathcal{M} \models \varphi(a, \vec{b})\}| \geq \aleph_\alpha.$$

A slightly stronger quantifier is

$$\mathcal{M} \models Q_\alpha^E x, y \varphi(x, y, \vec{c}) \iff \{(a, b) \in M^2 : \mathcal{M} \models \varphi(a, b, \vec{c})\} \text{ is an}$$

equivalence relation with $\geq \aleph_\alpha$ classes.

Example 3.3. 1. $\mathcal{L}_{\infty\omega}$ is KP-absolute [3].

2. $\mathcal{L}(Q_\alpha)$ is ZFC-absolute with ω_α as parameter.

3. $\mathcal{L}(Q_\alpha^E)$ is ZFC-absolute with ω_α as parameter.

Theorem 3.4. Suppose \mathcal{L}^* is ZFC+V=L-absolute with parameters from L , and the syntax of \mathcal{L}^* is (ZFC+V=L)-absolute with parameters from L . Then $C(\mathcal{L}^*) = L$.

Proof. We use induction on α to prove that $L'_\alpha \subseteq L$. We suppose $L'_\alpha \subseteq L$. Suppose ZFC_n is a finite part of ZFC so that \mathcal{L}^* is $\text{ZFC}_n + V = L$ -absolute. Then $L'_\alpha \in L_\gamma$ for some γ such that $L_\gamma \models \text{ZFC}_n$. We show that $L'_{\alpha+1} \subseteq L_{\gamma+1}$. Suppose $X \in L'_{\alpha+1}$. Then X is of the form

$$X = \{a \in L'_\alpha : (L'_\alpha, \in) \models \varphi(a, \vec{b})\},$$

where $\varphi(x, \vec{y}) \in \mathcal{L}^*$ and $\vec{b} \in L'_\alpha$. W.l.o.g., $\varphi(x, \vec{y}) \in L_\gamma$. By the definition of absoluteness,

$$X = \{a \in L_\gamma : (L_\gamma, \in) \models a \in L'_\alpha \wedge F_{\mathcal{L}^*}(\varphi(x, \vec{y})) \wedge \mathcal{S}_{\mathcal{L}^*}(L'_\alpha, \varphi(a, \vec{b}))\}.$$

Hence $X \in L'_{\gamma+1}$. This also shows that $\langle L'_\alpha : \alpha < \nu \rangle \in L$, and thereby $L'_\nu \in L$, for limit ordinals ν . \square

A consequence of the Theorem 3.4 is the following:

Conclusion: The constructible hierarchy L is unaffected if first order logic is enriched in the construction of L by any of the following, simultaneously or separately:

- Recursive infinite conjunctions $\bigwedge_{n=0}^{\infty} \varphi_n$ and disjunctions $\bigvee_{n=0}^{\infty} \varphi_n$.
- Cardinality quantifiers Q_α , $\alpha \in On$.
- Equivalence quantifiers Q_α^E , $\alpha \in On$.
- Well-ordering quantifier

$$\mathcal{M} \models Wx, y\varphi(x, y) \iff$$

$$\{(a, b) \in M^2 : \mathcal{M} \models \varphi(a, b)\} \text{ is a well-ordering.}$$

- Recursive game quantifiers

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \dots \bigwedge_{n=0}^{\infty} \varphi_n(x_0, y_0, \dots, x_n, y_n),$$

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \dots \bigvee_{n=0}^{\infty} \varphi_n(x_0, y_0, \dots, x_n, y_n).$$

- Magidor-Malitz quantifiers at \aleph_0

$$\begin{aligned} \mathcal{M} \models Q_0^{\text{MM},n} x_1, \dots, x_n \varphi(x_1, \dots, x_n) &\iff \\ \exists X \subseteq M (|X| \geq \aleph_0 \wedge \forall a_1, \dots, a_n \in X : \mathcal{M} \models \varphi(a_1, \dots, a_n)). \end{aligned}$$

Thus Gödel's $L = C(\mathcal{L}_{\omega\omega})$ exhibits some robustness with respect to the choice of the logic.

4 The Magidor-Malitz quantifier

The Magidor-Malitz quantifier [26] extends Q_1 by allowing us to say that there is an uncountable set such that, not only every *element* of the set satisfies a given formula $\varphi(x)$, but even any *pair of elements* from the set satisfy a given formula $\psi(x, y)$. One can express with the Magidor-Malitz quantifier much more than with Q_1 , e.g. the existence of a long branch or of a long antichain in a tree, but this quantifier is still axiomatizable if one assumes \diamond . On the other hand, the price we pay for the increased expressive power is that it is consistent, relative to the consistency of ZF, that Magidor-Malitz logic is very badly incompact [1]. We show that while it is consistent, relative to the consistency of ZF, that the Magidor-Malitz logic generates an inner model different from L , if we assume 0^\sharp , the inner model collapses to L . This is a bit surprising, because the existence of 0^\sharp implies that L is very “slim”, in the sense that it is not something that an a priori bigger inner model would collapse to. The key to this riddle is that under 0^\sharp the Magidor-Malitz logic itself loses its “sharpness” and becomes in a sense absolute between V and L .

Definition 4.1. The Magidor-Malitz quantifier in dimension n is the following:

$$\begin{aligned} \mathcal{M} \models Q_\alpha^{\text{MM},n} x_1, \dots, x_n \varphi(x_1, \dots, x_n) &\iff \\ \exists X \subseteq M (|X| \geq \aleph_\alpha \wedge \forall a_1, \dots, a_n \in X : \mathcal{M} \models \varphi(a_1, \dots, a_n)). \end{aligned}$$

The original Magidor-Malitz quantifier had dimension 2 and $\alpha = 1$:

$$\begin{aligned} \mathcal{M} \models Q_1^{\text{MM}} x_1, x_2 \varphi(x_1, x_2) &\iff \\ \exists X \subseteq M (|X| \geq \aleph_1 \wedge \forall a_1, a_2 \in X : \mathcal{M} \models \varphi(a_1, a_2)). \end{aligned}$$

The logics $\mathcal{L}(Q_\kappa^{\text{MM}, < \omega})$ and $\mathcal{L}(Q_\kappa^{\text{MM}, n})$ are adequate to truth in themselves (recall Definition 2.4), with κ as a parameter.

Note that putting $n = 1$ gives us Q_1 :

$$\begin{aligned} \mathcal{M} \models Q_1 x \varphi(x) &\iff \\ \exists X \subseteq M (|X| \geq \aleph_1 \wedge \forall a \in X : \mathcal{M} \models \varphi(a)). \end{aligned}$$

We have already noted that for $\alpha = 0$ this quantifier is absolute because the non-existence of X can be reduced to the well-foundedness of a certain definable tree.

Theorem 4.2. *If 0^\sharp exists, then $C(Q_\alpha^{MM, < \omega}) = L$.*

Proof. We treat only the case $n = 2, \alpha = 1$. The general case is treated similarly, using induction on n . The proof hinges on the following lemma:

Lemma 4.3. *Suppose 0^\sharp exists and $A \in L$, $A \subseteq [\eta]^2$. If there is an uncountable B such that $[B]^2 \subseteq A$, then there is such a set B in L .*

Proof. Let us first see how the Lemma helps us to prove the theorem. We will use induction on α to prove that $L'_\alpha \subseteq L$. We suppose $L'_\alpha \subseteq L$, and hence $L'_\alpha \in L_\gamma$ for some canonical indiscernible γ . We show that $L'_{\alpha+1} \subseteq L_{\gamma+1}$. Suppose $X \in L'_{\alpha+1}$. Then X is of the form

$$X = \{a \in L'_\alpha : (L'_\alpha, \epsilon) \models \varphi(a, \vec{b})\},$$

where $\varphi(x, \vec{y}) \in \mathcal{L}(Q_1^{MM})$ and $\vec{b} \in L'_{\alpha+1}$. For simplicity we suppress the mention of \vec{b} . Since we can use induction on φ , the only interesting case is

$$X = \{a \in L'_\alpha : \exists Y (|Y| \geq \aleph_1 \wedge \forall x, y \in Y : (L'_\alpha, \epsilon) \models \psi(x, y, a))\},$$

where we already have

$$A = \{\{c, d\} \in [L'_\alpha]^2 : (L'_\alpha, \epsilon) \models \psi(c, d, a)\} \in L.$$

Now the Lemma implies

$$X = \{a \in L'_\alpha : \exists Y \in L (|Y| \geq \aleph_1 \wedge \forall x, y \in Y : (L'_\alpha, \epsilon) \models \psi(x, y, a))\}.$$

Since $L_\gamma \prec L$, we have

$$X = \{a \in L'_\alpha : \exists Y \in L_\gamma (|Y| \geq \aleph_1 \wedge \forall x, y \in Y : (L'_\alpha, \epsilon) \models \psi(x, y, a))\}.$$

Finally,

$$\begin{aligned} X &= \{a \in L_\gamma : (L_\gamma, \epsilon) \models a \in L'_\alpha \wedge \\ &\exists z (\exists f : (\aleph_1)^V \xrightarrow{1-1} z) \wedge \forall x, y \in z \psi(x, y, a)^{(L'_\alpha, \epsilon)}\}. \end{aligned}$$

□

Now we prove the lemma. W.l.o.g. the set B of the lemma satisfies $|B| = \aleph_1$, say $B = \{\delta_i : i < \omega_1\}$ in increasing order. Let I be the canonical closed unbounded class of indiscernibles for L . Let $\delta_i = \tau_i(\alpha_0^i, \dots, \alpha_{k_i}^i)$, where $\alpha_0^i, \dots, \alpha_{k_i}^i \in I$. W.l.o.g., τ_i is a fixed term τ . Thus also k_i is a fixed number k . By the Δ -lemma, by thinning I if necessary, we may assume that the finite sets $\{\alpha_0^i, \dots, \alpha_k^i\}$, $i < \omega_1$, form a Δ -system with a root $\{\alpha_0, \dots, \alpha_n\}$ and leaves $\{\beta_0^i, \dots, \beta_k^i\}$, $i < \omega_1$. W.l.o.g., the mapping $i \mapsto \beta_0^i$ is strictly increasing in i . Let $\gamma_0 = \sup\{\beta_0^i : i < \omega_1\}$. W.l.o.g., the mapping $i \mapsto \beta_1^i$ is also strictly increasing in i . Let $\gamma'_1 = \sup\{\beta_1^i : i < \omega_1\}$. It may happen that $\gamma_1 = \gamma_0$. Then we continue to β_2^i, β_3^i , etc until we get $\gamma'_{k_0} = \sup\{\beta_{k_0}^i : i < \omega_1\} > \gamma_0$. Then we let $\gamma_1 = \gamma'_{k_0}$. We continue in this way until we have $\gamma_0 < \dots < \gamma_{k_s-1}$, all limit points of I .

Recall that whenever γ is a limit point of the set I there is a natural L -ultrafilter $\mathcal{U}_\gamma \subseteq L$ on γ , namely $A \in \mathcal{U}_\gamma \iff \exists \delta < \gamma ((I \setminus \delta) \cap \gamma \subseteq A)$. Recall also the following property of the L -ultrafilters \mathcal{U}_γ :

- **Rowbottom Property:** Suppose $\gamma_1 < \dots < \gamma_n$ are limits of indiscernibles and $\mathcal{U}_{\gamma_1}, \dots, \mathcal{U}_{\gamma_n}$ are the corresponding L -ultrafilters. Suppose $C \subseteq [\gamma_1]^{n_1} \times \dots \times [\gamma_n]^{n_n}$, where $C \in L$. Then there are $B_1 \in \mathcal{U}_{\gamma_1}, \dots, B_n \in \mathcal{U}_{\gamma_n}$ such that

$$[B_1]^{n_1} \times \dots \times [B_n]^{n_n} \subseteq C \text{ or } [B_1]^{n_1} \times \dots \times [B_n]^{n_n} \cap C = \emptyset. \quad (3)$$

We apply this to the ordinals $\gamma_1, \dots, \gamma_n$ and to a set C of sequences

$$(\zeta_0^0, \dots, \zeta_{k_0-1}^0, \eta_{k_s-1}^0, \dots, \eta_{k_0-1}^0, \dots, \zeta_{k_s-1}^s, \dots, \zeta_{k_s-1}^s, \eta_{k_s-1}^s, \dots, \eta_{k_s-1}^s) \quad (4)$$

such that

$$\left\{ \begin{aligned} &\tau(\alpha_0, \dots, \alpha_n, \zeta_0^0, \dots, \zeta_{k_0-1}^0, \dots, \zeta_{k_s-1}^s, \dots, \zeta_{k_s-1}^s), \\ &\tau(\alpha_0, \dots, \alpha_n, \eta_0^0, \dots, \eta_{k_0-1}^0, \dots, \eta_{k_s-1}^s, \dots, \eta_{k_s-1}^s) \end{aligned} \right\} \in A \quad (5)$$

Since $A \in L$, also $C \in L$. Note that

$$C \subseteq [\gamma_1]^{2k_0} \times \dots \times [\gamma_s]^{2k_s}$$

By the Rowbottom Property there are $B_0 \in \mathcal{U}_{\gamma_0}, \dots, B_s \in \mathcal{U}_{\gamma_s}$ such that

$$[B_1]^{2k_0} \times \dots \times [B_s]^{2k_s} \subseteq C \text{ or } [B_1]^{2k_0} \times \dots \times [B_s]^{2k_s} \cap C = \emptyset. \quad (6)$$

Claim: $[B_1]^{2k_0} \times \dots \times [B_s]^{2k_s} \subseteq C$.

To prove the claim suppose $[B_1]^{2k_0} \times \dots \times [B_s]^{2k_s} \cap C = \emptyset$. Since $B_j \in \mathcal{U}_{\gamma_j}$, there is $\xi_j < \gamma_j$ such that $(I \setminus \xi_j) \cap \gamma_j \subseteq B_j$. We can now find $i_1, i_2 < \omega_1$ such that in the sequence

$$\beta_0^{i_1}, \dots, \beta_{k_0-1}^{i_1}, \dots, \beta_{k_s-1}^{i_1}, \dots, \beta_{k_s-1}^{i_2}, l \in \{1, 2\},$$

where

$$\beta_0^{i_1}, \dots, \beta_{k_0-1}^{i_1} < \gamma_0 \text{ and } \beta_{k_j-1}^{i_1}, \dots, \beta_{k_j-1}^{i_2} < \gamma_j \text{ for all } j,$$

we actually have

$$\xi_0 < \beta_0^{i_1}, \dots, \beta_{k_0-1}^{i_1} < \gamma_0 \text{ and for all } j: \xi_j < \beta_{k_j-1}^{i_1}, \dots, \beta_{k_j-1}^{i_2} < \gamma_j, l \in \{1, 2\}.$$

Then since

$$\tau(\alpha_0, \dots, \alpha_n, \beta_0^{i_1}, \dots, \beta_{k_0-1}^{i_1}, \dots, \beta_{k_s-1}^{i_1}, \dots, \beta_{k_s-1}^{i_2}) \in B,$$

and $[B]^2 \subseteq A$, we have

$$\{\tau(\alpha_0, \dots, \alpha_n, \beta_0^{i_1}, \dots, \beta_{k_0-1}^{i_1}, \dots, \beta_{k_s-1}^{i_1}, \dots, \beta_{k_s-1}^{i_2}), \\ \tau(\alpha_0, \dots, \alpha_n, \beta_0^{i_2}, \dots, \beta_{k_0-1}^{i_2}, \dots, \beta_{k_s-1}^{i_2}, \dots, \beta_{k_s-1}^{i_1})\} \in A$$

Hence

$$(\beta_0^{l_1}, \dots, \beta_{k_0-1}^{l_1}, \beta_0^{l_2}, \dots, \beta_{k_0-1}^{l_2}, \dots, \beta_{k_s-1}^{l_1}, \dots, \beta_{k_s-1}^{l_1}, \beta_{k_s-1}^{l_2}, \dots, \beta_{k_s-1}^{l_2}) \in C \quad (7)$$

contrary to the assumption $[B_1]^{2k_0} \times \dots \times [B_s]^{2k_s} \cap C = \emptyset$. We have proved the claim.

Now we define

$$B^* = \{\tau(\alpha_0, \dots, \alpha_n, \zeta_0^0, \dots, \zeta_{k_0-1}^0, \dots, \zeta_{k_s-1}^s, \dots, \zeta_{k_s-1}^s) : \\ (\zeta_0^0, \dots, \zeta_{k_0-1}^0) \in B_0^{k_0}, \dots, (\zeta_{k_s-1}^s, \dots, \zeta_{k_s-1}^s) \in B_s^{k_s}\}. \quad (8)$$

Then $B^* \in L$, $|B^*| = \aleph_1$ and $[B^*]^2 \subseteq A$. \square

What if we do not assume $0^\#$? We show that if we start from L and use forcing we can get a model in which $C(Q_{\omega_1}^{MM,2}) \neq L$.

Theorem 4.4. *If $\text{Con}(ZF)$, then $\text{Con}(ZFC + C(Q_{\omega_1}^{MM,2}) \neq L)$.*

Proof. Assume $V = L$. Jensen and Johnsbråten [16] define a sequence T_n of Souslin trees in L and a CCC forcing notion \mathbb{P} which forces the set a of n such that \tilde{T}_n is Souslin to be non-constructible. But $a \in C(Q_{\omega_1}^{MM,2})$. So we are done. \square

This result can be strengthened in a number of ways. In [1] an ω_1 -sequence of Souslin trees is constructed from \diamond giving rise to forcing extensions in which $\mathcal{L}(Q_{\omega_1}^{\text{MM},2})$ can express some ostensibly second order properties, and $C(Q_{\omega_1}^{\text{MM},2})$ is very different from L .

There are several stronger versions of $Q_{\kappa}^{\text{MM},<\omega}$, for example

$$Q_{\kappa}^{\text{MR}}x_1, x_2, x_3\psi(x_1, x_2, x_3) \iff$$

$$\exists \mathcal{X}(\forall X_1, X_2 \in \mathcal{X})(\forall x_1, x_2 \in X_1)(\forall x_3 \in X_2)\psi(x_1, x_2, x_3, \vec{y}),$$

where X_1, X_2 range over sets of size κ and \mathcal{X} ranges over families of size κ of sets of size κ ([29]). The above is actually just one of the various forms of similar quantifiers that $\mathcal{L}(Q_{\kappa}^{\text{MR}})$ has. The logic $\mathcal{L}(Q_{\aleph_1}^{\text{MR}})$ is still countably compact assuming \diamond . We do not know whether $0^\#$ implies $C(Q_{\kappa}^{\text{MR}}) = L$.

5 The Cofinality Quantifier

The cofinality quantifier of Shelah [38] says that a given linear order has cofinality κ . Its main importance lies in the fact that it satisfies the compactness theorem irrespective of the cardinality of the vocabulary. Such logics are called *fully compact*. This logic has also a natural complete axiomatization, provably in ZFC.

The cofinality quantifier Q_{κ}^{cf} for a regular κ is defined as follows:

$$\mathcal{M} \models Q_{\kappa}^{\text{cf}}xy\varphi(x, y, \vec{a}) \iff \{(c, d) : \mathcal{M} \models \varphi(c, d, \vec{a})\}$$

is a linear order of cofinality κ .

We will denote by C_{κ}^* the inner model $C(Q_{\kappa}^{\text{cf}})$. Note that C_{κ}^* need not compute cofinality κ correctly, it just knows which ordinals have cofinality κ in V . The model knows this as if the model had an oracle for exactly this but nothing else. Thus while many more ordinals may have cofinality κ in V than in C_{κ}^* , still the property of an ordinal having cofinality κ in V is recognised in C_{κ}^* in the sense that for all β and $A, R \in C_{\kappa}^*$:

- $\{\alpha < \beta : \text{cf}^V(\alpha) = \kappa\} \in C_{\kappa}^*$
- $\{\alpha < \beta : \text{cf}^V(\alpha) \neq \kappa\} \in C_{\kappa}^*$
- $\{\alpha < \beta : \text{cf}^V(\alpha) = \kappa \iff \text{cf}^{C_{\kappa}^*}(\alpha) = \kappa\} \in C_{\kappa}^*$

- $\{a \in A : \{(b, c) : (a, b, c) \in R\}$ is a linear order on A with cofinality (in V) equal to $\kappa\} \in C_\kappa^*$.

Let On_κ be the class of ordinals of cofinality κ . It is easy to see that

$$C_\kappa^* = L(\text{On}_\kappa),$$

where $L(\text{On}_\kappa)$ is L defined in the expanded language $\{\in, \text{On}_\kappa\}$.

We use C^* to denote C_ω^* . Our results show that the inner models C_κ^* all resemble C^* in many ways, and accordingly we indeed focus mostly on C^* .

The following related quantifier turn out to be useful, too:

$$\mathcal{M} \models Q_{<\kappa}^{\text{cf}} xy\varphi(x, y, \vec{a}) \iff \{(c, d) : \mathcal{M} \models \varphi(c, d, \vec{a})\} \\ \text{is a linear order of cofinality } < \kappa.$$

We use $C_{\kappa, \lambda}^*$ to denote $C(Q_\kappa^{\text{cf}}, Q_\lambda^{\text{cf}})$ and $C_{<\kappa}^*$ to denote $C(Q_{<\kappa}^{\text{cf}})$. Respectively, $C_{\leq\kappa}^*$ denotes $C(Q_{\leq\kappa}^{\text{cf}})$.

The logics $C_{\kappa, \lambda}^*$ and $C_{<\kappa}^*$ are adequate to truth in themselves (recall Definition 2.4), with κ, λ as parameters, whence these inner models satisfy AC.

We can translate the formulas $\Phi_{\mathcal{L}(Q_\kappa^{\text{cf}})}(x)$ and $\Psi_{\mathcal{L}(Q_\kappa^{\text{cf}})}(x, y)$ into $\hat{\Phi}_{\mathcal{L}(Q_\kappa^{\text{cf}})}(x, \kappa)$ and $\hat{\Psi}_{\mathcal{L}(Q_\kappa^{\text{cf}})}(x, y, \kappa)$ in the first order language of set theory by systematically replacing

$$Q_\kappa^{\text{cf}} xy\varphi(x, y, \vec{a})$$

by the canonical set-theoretic formula saying the same thing. Then for all M with $\alpha = M \cap \text{On}$ and $a, b \in M$:

1. $\hat{\Phi}_{\mathcal{L}(Q_\kappa^{\text{cf}})}(a, \kappa) \leftrightarrow [(M, \in) \models \Phi_{\mathcal{L}(Q_\kappa^{\text{cf}})}(a)] \leftrightarrow a \in C_\kappa^*$.
2. $\hat{\Psi}_{\mathcal{L}(Q_\kappa^{\text{cf}})}(a, b, \kappa) \leftrightarrow [(M, \in) \models \Psi_{\mathcal{L}(Q_\kappa^{\text{cf}})}(a, b)] \leftrightarrow a <'_\alpha b$.

Lemma 5.1. *If M_1 and M_2 are two transitive models of ZFC such that for all α :*

$$M_1 \models \text{cf}(\alpha) = \kappa \iff M_2 \models \text{cf}(\alpha) = \kappa,$$

then

$$(C_\kappa^*)^{M_1} = (C_\kappa^*)^{M_2}.$$

Proof. Let (L'_α) be the hierarchy defining $(C_\kappa^*)^{M_1}$ and (L''_α) be the hierarchy defining $(C_\kappa^*)^{M_2}$. By induction, $L'_\alpha = L''_\alpha$ for all α . \square

By letting $M_2 = V$ in Proposition 5.1 we get

Corollary. *Suppose M is a transitive model of ZFC such that for all α :*

$$\text{cf}(\alpha) = \kappa \iff M \models \text{cf}(\alpha) = \kappa,$$

then

$$(C_\kappa^*)^M = C_\kappa^*.$$

This is a useful criterion. Note that $(C_\kappa^*)^M \neq C_\kappa^*$ is a perfectly possible situation: In Theorem 6.2 below we construct a model M in which CH is false in C^* . So $(C^*)^M \neq L$. Thus in M it is true that $(C^*)^L \neq C^*$. $(C_\kappa^*)^M \neq C_\kappa^*$ also if $\kappa = \omega$, $V = L^\mu$ and $M = C^*$ (see the below Theorem 5.14). In this respect C_κ^* resembles HOD. There are other respects in which C_κ^* resembles L .

Lemma 5.2. *Suppose (L'_α) is the hierarchy forming C_κ^* . Then for $\alpha < \kappa$ we have $L'_\alpha = L_\alpha$.*

We can relativize C^* to a set X of ordinals as follows. Let us define a new generalized quantifier as follows:

$$\mathcal{M} \models Q_X xy\varphi(x, y, \vec{a}) \iff \{(c, d) : \mathcal{M} \models \varphi(c, d, \vec{a})\} \\ \text{is a well-order of type } \in X.$$

We define $C^*(X)$ as $C(Q_\omega^{\text{cf}}, Q_X)$. Of course, $C^*(X) = L(\text{On}_\omega, X)$.

We prove a stronger form of the next Proposition in the next Theorem, but we include this here for completeness:

Proposition 5.3. *If 0^\sharp exists, then $0^\sharp \in C(Q_\kappa^{\text{cf}})$.*

Proof. Let I be the canonical set of indiscernibles obtained from 0^\sharp . Let us first prove that ordinals ξ which are regular cardinals in L and have cofinality $> \omega$ in V are in I . Suppose $\xi \notin I$. Note that $\xi > \min(I)$. Let δ be the largest element of $I \cap \xi$. Let $\lambda_1 < \lambda_2 < \dots$ be an infinite sequence of elements of I above ξ . Let

$$\tau_n(x_1, \dots, x_{k_n}), n < \omega,$$

be a list of all the Skolem terms of the language of set theory relative to the theory $\text{ZFC} + V = L$. If $\alpha < \xi$, then

$$\alpha = \tau_{n_\alpha}(\gamma_1, \dots, \gamma_{m_n}, \lambda_1, \dots, \lambda_{l_n})$$

for some $\gamma_1, \dots, \gamma_{m_n} \in I \cap \delta$ and some $l_n < \omega$. Let us fix n for a moment and consider the set

$$A_n = \{\tau_{n\alpha}(\beta_1, \dots, \beta_{m_n}, \lambda_1, \dots, \lambda_{l_n}) : \beta_1, \dots, \beta_n < \delta\}.$$

Note that $A_n \in L$ and $|A_n|^L \leq |\delta|^L < \xi$, because ξ is a cardinal in L . Let $\eta_n = \sup(A_n)$. Since ξ is regular in L , $\eta_n < \xi$. Since ξ has cofinality $> \omega$ in V , $\eta = \sup_n \eta_n < \xi$. But we have now proved that every $\alpha < \xi$ is below η , a contradiction. So we may conclude that necessarily $\xi \in I$.

Suppose now $\kappa = \omega$. Let

$$X = \{\xi \in L'_{\aleph_\omega} : (L'_{\aleph_\omega}, \in) \models \text{“}\xi \text{ is a regular cardinal in } L \text{”} \wedge \neg Q_\kappa^{cf} xy(x \in y \wedge y \in \xi)\}$$

Now X is an infinite subset of I and $X \in C(Q_\kappa^{cf})$. Hence $0^\sharp \in C(Q_\kappa^{cf})$:

$$0^\sharp = \{\ulcorner \varphi(x_1, \dots, x_n) \urcorner : (L_{\aleph_\omega}, \in) \models \varphi(\gamma_1, \dots, \gamma_n) \text{ for some } \gamma_1 < \dots < \gamma_n \text{ in } X\}.$$

If $\kappa = \aleph_\alpha > \omega$, then we use

$$X = \{\xi \in L'_{\aleph_{\alpha+\omega}} : (L'_{\aleph_{\alpha+\omega}}, \in) \models \text{“}\xi \text{ is a regular cardinal in } L \text{”} \wedge Q_\kappa^{cf} xy(x \in y \wedge y \in \xi)\}$$

and argue as above that $0^\sharp \in C(Q_\kappa^{cf})$. \square

More generally, the above argument shows that $x^\sharp \in C^*(x)$ for any $x \in C^*$ such that x^\sharp exists. Hence $C^* \neq L(x)$ whenever x is a set of ordinals such that x^\sharp exists in V (see Theorem 5.4).

Theorem 5.4. *Exactly one of the following always holds:*

1. C^* is closed under sharps, (equivalently, x^\sharp exists for all $x \subset On$ such that $x \in C^*$).
2. C^* is not closed under sharps and moreover $C^* = L(x)$ for some set $x \subset On$. (Equivalently, there is $x \subset On$ such that $x \in C^*$ but x^\sharp does not exist.)

Proof. Suppose (1) does not hold. Suppose $a \subseteq \lambda$, $\lambda > \omega_1$, such that $a \in C^*$ but a^\sharp does not exist. Let $S = \{\alpha < \lambda^+ : \text{cf}^V(\alpha) = \omega\}$. We show that $C^* = L(a, S)$. Trivially, $C^* \supseteq L(a, S)$. For $C^* \subseteq L(a, S)$ it is enough to show that one can detect in $L(a, S)$ whether a given $\delta \in On$ has cofinality ω (in V) or not. If $\text{cf}(\delta) = \omega$, and $c \subseteq \delta$ is a cofinal ω -sequence in δ , then the Covering Theorem for $L(a)$ gives a set $b \in L(a)$ such that $c \subseteq b \subset \lambda$, $\sup(c) = \sup(b)$ and $|b| = \lambda$. The order type of b is in S . Hence whether δ has cofinality ω or not can be detected in $L(a, S)$. \square

Corollary. *If x^\sharp does not exist for some $x \in C^*$, then there is λ such that $C^* \models 2^\kappa = \kappa^+$ for all $\kappa \geq \lambda$.*

Theorem 5.5. *The Dodd-Jensen Core model is contained in C^* .*

Proof. Let K be the Dodd-Jensen Core model of C^* . We show that K is the core model of V . Assume otherwise and let M_0 be the minimal Dodd-Jensen mouse missing from K . (Minimality here means in the canonical pre-well ordering of mice.) Let κ_0 be the cardinal of M_0 on which M_0 has the M_0 normal measure. Denote this normal measure by U_0 . Note that $M_0 = J_\alpha^{U_0}$ for some α . $J_\alpha[U_0]$ is the Jensen J -hierarchy of constructibility from U_0 , where $J_\alpha[U_0] = \bigcup_{\beta < \omega\alpha} S_\beta^{U_0}$, where $S_\beta^{U_0}$ is the finer S -hierarchy.

Let ξ_0 be $(\kappa_0^+)^{M_0}$. (If $(\kappa_0^+)^{M_0}$ does not exist in M_0 put $\xi_0 = M \cap ON = \omega\alpha$.) Let $\delta = \text{cf}^V(\xi_0)$.

For an ordinal β let M_β be the β 'th iterated ultrapower of M_0 where for $\beta \leq \gamma$ let $j_{\beta,\gamma} : M_\beta \rightarrow M_\gamma$ be the canonical ultrapower embedding. $j_{\beta,\gamma}$ is a Σ_0 -embedding. Let $\kappa_\beta = j_{0\beta}(\kappa_0)$, $U_\beta = j_{0\beta}(U_0)$, $\xi_\beta = j_{0\beta}(\xi_0)$. (In case $(\kappa_0^+)^{M_0}$ does not exist we put $\xi_\beta = M_\beta \cap ON$.) κ_β is the critical point of $j_{\beta,\gamma}$ for $\beta < \gamma$. For a limit β and $A \in M_\beta$, $A \subseteq \kappa_\beta$ $A \in U_\beta$ iff $\kappa_\gamma \in A$ for large enough $\gamma < \beta$.

Claim. For every β $\xi_\beta = \sup j''_{0\beta}(\xi_0)$. Hence $\text{cf}^V(\xi_\beta) = \delta$.

Proof. Every $\eta < \xi_\beta$ is of the form $j_{0\beta}(f)(\kappa_{\gamma_0}, \dots, \kappa_{\gamma_{n-1}})$ for some $\gamma_0 < \gamma_1 \dots < \gamma_{n-1} < \beta$ and for some $f \in M_0$, $f : \kappa_0^n \rightarrow \xi_0$. By definition of ξ_0 there is $\rho < \xi_0$ such that $f(\alpha_0, \dots, \alpha_{n-1}) < \rho$ for every $\langle \alpha_0, \dots, \alpha_{n-1} \rangle \in \kappa_0^n$. Hence it follows that every value of $j_{0\beta}(f)$ is bounded by $j_{0\beta}(\rho)$. So $\eta < j_{0\beta}(\rho)$, which proves the claim. \square

The usual proof of GCH in $L[U]$ shows that $\kappa_\beta^{\kappa_\beta} \cap M_\beta \subseteq J_{\xi_\beta}^{U_\beta}$ and that $J_{\xi_\beta}^{U_\beta}$ is the increasing union of δ members of M_β , each one having cardinality κ_β in M_β .

Claim. Let $\kappa_0 < \eta < \kappa_\beta$ be such that $M_\beta \models \eta$ is regular, then either there is $\gamma < \beta$ such that $\eta = \kappa_\gamma$ or $\text{cf}^V(\eta) = \delta$.

Proof. By induction on β . The claim is vacuously true for $\beta = 0$. For β limit $\kappa_\beta = \sup\{\kappa_\gamma \mid \gamma < \beta\}$. Hence there is $\alpha < \beta$ such that $\eta < \kappa_\alpha$. $j_{\alpha,\beta}(\eta) = \eta$ so $M_\alpha \models \eta$ is regular. So the claim in this case follows from the induction assumption.

We are left with the case that $\beta = \alpha + 1$. If $\eta \leq \kappa_\alpha$ the claim follows from the inductive assumption for α like in the limit case. So we are left with the case $\kappa_\alpha < \eta < \kappa_\beta$. M_β is the ultrapower of M_α by U_α , so η is represented in this ultrapower

by a function $f \in M_\alpha$ whose domain is κ_α . By the assumption $\eta < \kappa_\beta = j_{\alpha\beta}(\kappa_\alpha)$ we can assume that for every $\rho < \kappa_\alpha$ $f(\rho) < \kappa_\alpha$. By the assumption $\kappa_\alpha < \eta$ we can assume that $\rho < f(\rho)$ for every $\rho < \kappa_\alpha$ and by the assumption that η is regular in M_β we can assume that $f(\rho)$ is regular in M_α for every $\rho < \kappa_\alpha$. In order to simplify notation put $M = M_\alpha$, $\kappa = \kappa_\alpha$, $U = U_\alpha$, and $\xi = \xi_\alpha$.

In order to show that $\text{cf}^V(\eta) = \delta$ we shall define (in V) a sequence $\langle g_\nu | \nu < \delta \rangle$ of functions in $\kappa^\kappa \cap M$ such that :

1. The sequence is increasing modulo U .
2. For every $\rho < \kappa$, $g_\nu(\rho) < f(\rho)$.
3. The ordinals represented by these functions in the ultrapower of M by U are cofinal in η .

By the definition of ξ and the previous claim we can represent $\kappa^\kappa \cap M$ as an increasing union $\bigcup_{\psi < \delta} F_\psi$ where for every $\psi < \delta$ $F_\psi \in M$ and F_ψ has cardinality κ in M . For $\psi < \delta$ fix an enumeration in M of $\langle h_\rho^\psi | \rho < \kappa \rangle$ of the set $G_\psi = \{h \in F_\psi | \forall \rho < \kappa (h(\rho) < f(\rho))\}$. Let $f_\psi \in \kappa^\kappa$ be defined by $f_\psi(\rho) = \sup(\{h_\mu^\psi(\rho) | \mu < \rho\})$. Clearly $f_\psi \in M$, f_ψ bounds all the functions in G_ψ modulo U . Also because for all $\rho < \kappa$ and $h \in G_\psi$ $h(\rho) < f(\rho)$ we get that $f_\psi(\rho) < f(\rho)$. (Recall that $f(\rho) > \rho$, $f(\rho)$ is regular in M and $f_\psi(\rho)$ is the sup of a set in M whose cardinality in M is ρ .)

Define g_ν by induction on $\nu < \delta$. By induction we shall also define an increasing sequence $\langle \psi_\nu | \nu < \delta \rangle$ such that $\psi_\nu < \delta$ and $g_\nu \in G_{\psi_\nu}$. Given $\langle \psi_\mu | \mu < \nu \rangle$ let σ be their sup. Let g_ν be f_σ and let ψ_ν be the minimal member of $\delta - \sigma$ such that $f_\sigma \in G_{\psi_\nu}$. The induction assumptions on g_μ, ψ_μ for $\mu < \nu$ and the properties of f_σ yields that g_ν and ψ_ν also satisfy the required inductive assumption.

The fact that the sequence of ordinals represented by $\langle g_\nu | \nu < \delta \rangle$ in the ultrapower of M by U is cofinal in η follows from the fact that every ordinal below η is represented by some function h which is bounded everywhere by f , hence it belongs to G_ψ for some $\psi < \delta$. There is ν such that $\psi < \psi_\nu$. Then $g_{\nu+1}$ will bound h modulo U . \square

The minimality of M_0 (hence the minimality of the equivalent M_β) implies that for every β $\mathcal{P}(\kappa_\beta) \cap K = \mathcal{P}(\kappa_\beta) \cap M_\beta$. It follows that $\rho \leq \kappa_\beta$ is regular in K iff it is regular in M_β . In particular for every β , κ_β is regular in K since it is regular in M_β .

Claim. Let λ be a regular cardinal greater than $\max(|M_0|, \delta)$ then there there is $D \in C^*$, $D \subseteq E = \{\kappa_\beta | \beta < \lambda\}$ which is cofinal in λ .

Proof. Note that $\lambda > |M_0|$ implies that the set $E = \{\kappa_\beta \mid \beta < \lambda\}$ is a club in λ . Let S_0^λ be the set of ordinals in λ whose cofinality (in V) is ω . Obviously both $E - S_0^\lambda$ and $E - S_0^\lambda$ are unbounded in λ . Let C be the set of the ordinals of $\lambda - \kappa_0$ which are regular in K . By definition of C^* and K both S_0^λ and C are in C^* . Also $E \subseteq C$ since κ_β is regular in M_β , hence regular in K .

If $\delta \neq \omega$ then we can take $D = C - S_0^\lambda$ which by claim 5 It is a subset of E which is unbounded in λ . If $\delta = \omega$ then similarly we can take $D = C \cap S_0^\lambda$. In both cases $D \in C^*$. \square

Pick λ, E as in the Claim above and let $D \subseteq \lambda$ be the witness to the claim. It is well known that for every $X \in M_\lambda$ $X \in U_\lambda$ iff $X \subseteq \lambda$ and X contains a final segment of E . Since U_λ is an ultrafilter on λ in M_λ we get that for $X \in M_\lambda$, $X \subseteq \lambda$ $X \in U_\beta$ iff X contains a final segment of D . Let F_D be the filter on λ generated by final segments of D . $D \in C^*$ implies that $L(F_D) \subseteq C^*$. $M_\lambda = J_\alpha^{U_\lambda}$ for some ordinal α . But since $U_\lambda = F_D \cap M_\lambda$ we get that $M_\lambda = J_\alpha^{F_D}$. It implies that $M_\lambda \in C^*$. Namely C^* contains an iterate of the mouse M_0 . But then by standard Dodd-Jensen Core model techniques $M_0 \in C^*$, which is clearly a contradiction. \square

Theorem 5.6. *Suppose an inner model with a measurable cardinal exists. Then C^* contains some inner model L^ν for a measurable cardinal.*

Proof. This is as the proof of Theorem 5.5. Suppose L^μ exists, but does not exist in C^* . Let κ_0 be the cardinal of $M_0 = L^\mu$ on which L^μ has the normal measure. Denote this normal measure by U_0 . Let ξ_0 be $(\kappa_0^+)^{M_0}$. Let $\delta = \text{cf}^V(\xi_0)$.

For an ordinal β let M_β be the β 'th iterated ultrapower of M_0 where for $\beta \leq \gamma$ let $j_{\beta,\gamma} : M_\beta \rightarrow M_\gamma$ be the canonical ultrapower embedding. $j_{\beta,\gamma}$ is a Σ_0 -embedding. Let $\kappa_\beta = j_{0\beta}(\kappa_0)$, $U_\beta = j_{0\beta}(U_0)$, $\xi_\beta = j_{0\beta}(\xi_0)$. κ_β is the critical point of $j_{\beta,\gamma}$ for $\beta < \gamma$. For a limit β and $A \in M_\beta$, $A \subseteq \kappa_\beta$ $A \in U_\beta$ iff $\kappa_\gamma \in A$ for large enough $\gamma < \beta$.

Claim. For every β $\xi_\beta = \sup j''_{0\beta}(\xi_0)$. Hence $\text{cf}^V(\xi_\beta) = \delta$.

Proof. Every $\eta < \xi_\beta$ is of the form $j_{0\beta}(f)(\kappa_{\gamma_0}, \dots, \kappa_{\gamma_{n-1}})$ for some $\gamma_0 < \gamma_1 \dots < \gamma_{n-1} < \beta$ and for some $f \in M_0$, $f : \kappa_0^n \rightarrow \xi_0$. By definition of ξ_0 there is $\rho < \xi_0$ such that $f(\alpha_0, \dots, \alpha_{n-1}) < \rho$ for every $\langle \alpha_0, \dots, \alpha_{n-1} \rangle \in \kappa_0^n$. Hence it follows that every value of $j_{0\beta}(f)$ is bounded by $j_{0\beta}(\rho)$. So $\eta < j_{0\beta}(\rho)$, which proves the claim. \square

The usual proof of GCH in $L[U]$ shows that $\kappa_\beta^{\kappa_\beta} \cap M_\beta \subseteq J_{\xi_\beta}^{U_\beta}$ and that $J_{\xi_\beta}^{U_\beta}$ is the increasing union of δ members of M_β , each one having cardinality κ_β in M_β .

Claim. Let $\kappa_0 < \eta < \kappa_\beta$ be such that $M_\beta \models \eta$ is regular, then either there is $\gamma < \beta$ such that $\eta = \kappa_\gamma$ or $\text{cf}^V(\eta) = \delta$.

Proof. By induction on β . The claim is vacuously true for $\beta = 0$. For β limit $\kappa_\beta = \sup\{\kappa_\gamma \mid \gamma < \beta\}$. Hence there is $\alpha < \beta$ such that $\eta < \kappa_\alpha$. $j_{\alpha\beta}(\eta) = \eta$ so $M_\alpha \models \eta$ is regular. So the claim in this case follows from the induction assumption.

We are left with the case that $\beta = \alpha + 1$. If $\eta \leq \kappa_\alpha$ the claim follows from the inductive assumption for α like in the limit case. So we are left with the case $\kappa_\alpha < \eta < \kappa_\beta$. M_β is the ultrapower of M_α by U_α , so η is represented in this ultrapower by a function $f \in M_\alpha$ whose domain is κ_α . By the assumption $\eta < \kappa_\beta = j_{\alpha\beta}(\kappa_\alpha)$ we can assume that for every $\rho < \kappa_\alpha$ $f(\rho) < \kappa_\alpha$. By the assumption $\kappa_\alpha < \eta$ we can assume that $\rho < f(\rho)$ for every $\rho < \kappa_\alpha$ and by the assumption that η is regular in M_β we can assume that $f(\rho)$ is regular in M_α for every $\rho < \kappa_\alpha$. In order to simplify notation put $M = M_\alpha$, $\kappa = \kappa_\alpha$, $U = U_\alpha$, and $\xi = \xi_\alpha$.

In order to show that $\text{cf}^V(\eta) = \delta$ we shall define (in V) a sequence $\langle g_\nu \mid \nu < \delta \rangle$ of functions in $\kappa^\kappa \cap M$ such that :

1. The sequence is increasing modulo U .
2. For every $\rho < \kappa$, $g_\nu(\rho) < f(\rho)$.
3. The ordinals represented by these functions in the ultrapower of M by U are cofinal in η .

By the definition of ξ and the previous claim we can represent $\kappa^\kappa \cap M$ as an increasing union $\bigcup_{\psi < \delta} F_\psi$ where for every $\psi < \delta$ $F_\psi \in M$ and F_ψ has cardinality κ in M . For $\psi < \delta$ fix an enumeration in M of $\langle h_\rho^\psi \mid \rho < \kappa \rangle$ of the set $G_\psi = \{h \in F_\psi \mid \forall \rho < \kappa (h(\rho) < f(\rho))\}$. Let $f_\psi \in \kappa^\kappa$ be defined by $f_\psi(\rho) = \sup(\{h_\mu^\psi(\rho) \mid \mu < \rho\})$. Clearly $f_\psi \in M$, f_ψ bounds all the functions in G_ψ modulo U . Also because for all $\rho < \kappa$ and $h \in G_\psi$ $h(\rho) < f(\rho)$ we get that $f_\psi(\rho) < f(\rho)$. (Recall that $f(\rho) > \rho$, $f(\rho)$ is regular in M and $f_\psi(\rho)$ is the sup of a set in M whose cardinality in M is ρ .)

Define g_ν by induction on $\nu < \delta$. By induction we shall also define an increasing sequence $\langle \psi_\nu \mid \nu < \delta \rangle$ such that $\psi_\nu < \delta$ and $g_\nu \in G_{\psi_\nu}$. Given $\langle \psi_\mu \mid \mu < \nu \rangle$ let σ be their sup. Let g_ν be f_σ and let ψ_ν be the minimal member of $\delta - \sigma$ such that $f_\sigma \in G_{\psi_\nu}$. The induction assumptions on g_μ, ψ_μ for $\mu < \nu$ and the properties of f_σ yields that g_ν and ψ_ν also satisfy the required inductive assumption.

The fact that the sequence of ordinals represented by $\langle g_\nu \mid \nu < \delta \rangle$ in the ultrapower of M by U is cofinal in η follows from the fact that every ordinal below

η is represented by some function h which is bounded everywhere by f , hence it belongs to G_ψ for some $\psi < \delta$. There is ν such that $\psi < \psi_\nu$. Then $g_{\nu+1}$ will bound h modulo U . \square

We know already that $K \subseteq C^*$. For every β $\mathcal{P}(\kappa_\beta) \cap K = \mathcal{P}(\kappa_\beta) \cap M_\beta$. It follows that $\rho \leq \kappa_\beta$ is regular in K iff it is regular in M_β . In particular for every β , κ_β is regular in K since it is regular in M_β .

Claim. Let λ be a regular cardinal greater than $\max(|M_0|, \delta)$ then there there is $D \in C^*$, $D \subseteq E = \{\kappa_\beta | \beta < \lambda\}$ which is cofinal in λ .

Proof of the Claim: Note that $\lambda > |M_0|$ implies that the set $E = \{\kappa_\beta | \beta < \lambda\}$ is a club in λ . Let S_0^λ be the set of ordinals in λ whose cofinality (in V) is ω . Obviously both $E - S_0^\lambda$ and $E - S_0^\lambda$ are unbounded in λ . Let C be the set of the ordinals of $\lambda - \kappa_0$ which are regular in K . By definition of C^* and K both S_0^λ and C are in C^* . Also $E \subseteq C$ since κ_β is regular in M_β , hence regular in K .

If $\delta \neq \omega$ then we can take $D = C - S_0^\lambda$ which by claim 5 It is a subset of E which is unbounded in λ . If $\delta = \omega$ then similarly we can take $D = C \cap S_0^\lambda$. In both cases $D \in C^*$. The Claim is proved.

Pick λ, E as in the Claim above and let $D \subseteq \lambda$ be the witness to the claim. It is well known that for every $X \in M_\lambda$ $X \in U_\lambda$ iff $X \subseteq \lambda$ and X contains a final segment of E . Since U_λ is an ultrafilter on λ in M_λ we get that for $X \in M_\lambda$, $X \subseteq \lambda$ $X \in U_\beta$ iff X contains a final segment of D . Let F_D be the filter on λ generated by final segments of D . $D \in C^*$ implies that $L(F_D) \subseteq C^*$. $M_\lambda = J_\alpha^{U_\lambda}$ for some ordinal α . But since $U_\lambda = F_D \cap M_\lambda$ we get that $M_\lambda = J_\alpha^{F_D}$. It implies that $M_\lambda \in C^*$. Namely C^* contains an iterate of M_0 . Hence C^* contains an inner model with a measurable cardinal. \square

Later (Theorem 5.14) we will show that if L^μ exists, then $(C^*)^{L^\mu}$ can be obtained by adding to the ω^2 th iterate of L^μ the sequence $\{\kappa_{\omega \cdot n} : n < \omega\}$.

In the presence of large cardinals, even with just uncountably many measurable cardinals, we can separate C^* from both L and HOD. We first observe that in the special case that $V = C^*$, there cannot exist even a single measurable cardinal. The proof is similar to Scott's proof that measurable cardinals violate $V = L$:

Theorem 5.7. *If there is a measurable cardinal κ , then $V \neq C_\lambda^*$ for all $\lambda < \kappa$.*

Proof. Suppose $V = C_\lambda^*$ but $\kappa > \lambda$ is a measurable cardinal. Let $i : V \rightarrow M$ with critical point κ and $M^\kappa \subseteq M$. Now $(C_\lambda^*)^M = (C_\lambda^*)^V = V$, whence $M = V$. This contradicts Kunen's result [19] that there cannot be a non-trivial $i : V \rightarrow V$. \square

Kunen [20] proved that if there are uncountably many measurable cardinals, then AC fails in Chang's model $C(\mathcal{L}_{\omega_1\omega_1})$. Recall that Chang's model contains C^* and C^* does satisfy AC.

Theorem 5.8. *If $\langle \kappa_n : n < \omega \rangle$ is any sequence of measurable cardinals (in V) $> \lambda$, then $\langle \kappa_n : n < \omega \rangle \notin C_\lambda^*$ and $C_\lambda^* \neq \text{HOD}$.*

Proof. We proceed as in Kunen's proof ([20]) that the AC fails in the Chang model, if there are uncountably many measurable cardinals, except that we only use infinitely many measurable cardinals. Suppose κ_n , $n < \omega$, are measurable $> \lambda$. Let $\mu = \sup_n \kappa_n$. Let \prec be the first well-order of μ^ω in C_λ^* in the canonical well-order of C_λ^* . Suppose $\langle \kappa_n : n < \omega \rangle \in C_\lambda^*$. Then it is number η for some η in the well-order \prec . By [20, Lemma 2] there are only finitely many measurable cardinals ξ such that η is moved by the ultrapower embedding of a normal ultrafilter on ξ . Let n be such that the ultrapower embedding $j : V \rightarrow M$ by the normal ultrafilter on κ_n does not move η . Since $\kappa_n > \lambda$, $(C_\lambda^*)^M = C_\lambda^*$. Since μ is a strong limit cardinal $> \lambda$, $j(\mu) = \mu$. Since the construction of C_λ^* proceeds in M exactly as it does in V , $j(\prec)$ is also in M the first well-ordering of μ^ω that appears in C_λ^* . Hence $j(\prec) = \prec$. Since $j(\eta) = \eta$, the sequence $\langle \kappa_n : n < \omega \rangle$ is fixed by j . But this contradicts the fact that j moves κ_n .

If the κ_n are the first ω measurable cardinals above λ , then the sequence $\langle \kappa_n : n < \omega \rangle$ is in HOD and hence $C_\lambda^* \neq \text{HOD}$. □

Definition 5.9. The *weak Chang model* is the model $C_{\omega_1}^\omega = C(\mathcal{L}_{\omega_1\omega_1}^\omega)$.

The weak Chang model clearly contains C^* . It is a potentially interesting intermediate model between $L(\mathbb{R})$ and the (full) Chang model. If there is a measurable Woodin cardinal, then the Chang model satisfies AD, whence the weak model cannot satisfy AC, as it contains $C(\mathcal{L}_{\omega_1\omega}^\omega)$ and hence all the reals, and the even bigger (full) Chang model cannot contain a well-ordering of all reals.

Theorem 5.10. 1. *If $V = L^\mu$, then $C_{\omega_1}^\omega \neq L(\mathbb{R})$.*

2. *If V is the inner model for ω_1 measurable cardinals, then $C_{\omega_1}^\omega \neq \text{Chang model}$.*

Proof. For (1), suppose $V = L^\mu$, where μ is a normal measure on κ . By Theorem 5.6 there is in C^* , hence in $C_{\omega_1}^\omega$, an inner model L^ν with a measurable cardinal δ . But κ is in L^μ the smallest ordinal which is measurable in an inner model.

Hence $\kappa \leq \delta$. If $C_{\omega_1}^\omega = L(\mathbb{R})$, then there is an $A \subseteq \omega_1$ such that $C_{\omega_1}^\omega = L(A)$. But by [37] there cannot be an inner model with a measurable cardinal δ in $L(A)$, where $A \subseteq \delta$.

For (2), we commence by noting that in the inner model for ω_1 measurable cardinals there is a Σ_3^1 -well-order of \mathbb{R} [40]. By means of this well-order we can well-order the formulas of $\mathcal{L}_{\omega_1\omega_1}^\omega$ in the Chang model. In this way we can define a well-order of $C_{\omega_1}^\omega$ in the Chang model. However, since we assume uncountably many measurable cardinals, the Chang model does not satisfy AC [20] (see also Theorem 5.8). Hence it must be that $C_{\omega_1}^\omega \neq \text{Chang model}$. \square

If there is a Woodin cardinal, then $C^* \neq V$ in the strong sense that \aleph_1 is a large cardinal in C^* . So not only are there countable sequences of measurable cardinals which are not in C^* but there are even reals which are not in C^* :

Theorem 5.11. *If there is a Woodin cardinal, then ω_1 is (strongly) Mahlo in C^* .*

Proof. To prove that ω_1 is strongly inaccessible in C^* suppose $\alpha < \aleph_1$ and

$$f : \omega_1 \rightarrow (2^\alpha)^{C^*}$$

is one-one. Let λ be Woodin, $\mathbb{Q}_{<\lambda}$ the countable stationary tower forcing and G generic for this forcing. In $V[G]$ there is $j : V \rightarrow M$ such that $V[G] \models M^\omega \subset M$ and $j(\omega_1) = \lambda$. Thus

$$j(f) : \lambda \rightarrow ((2^\alpha)^{C^*})^M.$$

Let $a = j(f)(\omega_1^V)$. If $a \in V$, then $j(a) = a$, whence, as a i.e. $j(a)$ is in the range of $j(f)$, $a = f(\delta)$ for some $\delta < \omega_1$. Then

$$a = j(a) = j(f)(j(\delta)) = j(f)(\delta),$$

contradicting the fact that $a = j(f)(\omega_1)$. Hence $a \notin V$. However,

$$(C^*)^M = (C_{<\lambda}^*)^V,$$

since by general properties of this forcing, an ordinal has cofinality ω in M iff it has cofinality $< \lambda$ in V . Hence $a \in C_{<\lambda}^* \subseteq V$, a contradiction.

To see that ω_1 is Mahlo in C^* , suppose D is a club on ω_1^V , $D \in C^*$. Let j and M be as above. Then $j(D)$ is a club on λ in $(C^*)^M$. Since ω_1^V is the critical point of j , $j(D) \cap \omega_1^V = D$. Since $j(D)$ is closed, $\omega_1^V \in j(D)$. \square

Remark. In the previous theorem we can replace the assumption of a Woodin cardinal by MM^{++} (see Definition 7.9).

For cardinals $\kappa > \omega_1$ we have an even better result:

Theorem 5.12. *Suppose there is a Woodin cardinal λ . Then every regular cardinal κ such that $\omega_1 < \kappa < \lambda$ is weakly compact in C^* .*

Proof. Suppose λ is a Woodin cardinal, $\kappa > \omega_1$ is regular and $\kappa < \lambda$. To prove that κ is strongly inaccessible in C^* we use the “ $\leq \omega$ -closed” stationary tower forcing from [12, Section 1]. With this forcing, cofinality ω is not changed, whence $(C^*)^M = C^*$, so the proof of Theorem 5.11 can be repeated mutatis mutandis. Thus we need only prove the tree property. Let the forcing, j and M be as above with $j(\kappa) = \lambda$. Suppose T is a κ -tree in C^* . Then $j(T)$ is a λ -tree in $(C^*)^M = C^*$. We may assume $j(T \upharpoonright \kappa) = T \upharpoonright \kappa$. Let $t \in j(T)$ be of height κ and $b = \{u \in j(T) : u < t\} = \{u \in T : u < t\}$. Now b is a κ -branch of T in C^* . \square

As a further application of ω -closed stationary tower forcing we extend the above result as follows:

Theorem 5.13. *If there is a proper class of Woodin cardinals, then the regular cardinals $\geq \aleph_2$ are indiscernible³ in C^* .*

Proof. We use the ω -closed stationary tower forcing of [12]. Let us first prove an auxiliary claim:

Claim 1: If $\lambda_1 < \dots < \lambda_k$ and $\bar{\lambda}_1 < \dots < \bar{\lambda}_k$ are Woodin cardinals, and $\beta_1, \dots, \beta_l < \min(\lambda_1, \bar{\lambda}_1)$, then

$$C^* \models \Phi(\beta_1, \dots, \beta_l, \lambda_1, \dots, \lambda_k) \leftrightarrow \Phi(\beta_1, \dots, \beta_l, \bar{\lambda}_1, \dots, \bar{\lambda}_k)$$

for all formulas $\Phi(x_1, \dots, x_l, y_1, \dots, y_k)$ of set theory.

To prove Claim 1, assume w.l.o.g. $\bar{\lambda}_1 > \lambda_1$. We use induction on k . The case $k = 0$ is clear. Let us then assume the claim for $k - 1$. Let G be generic for the $\leq \omega$ -closed stationary tower forcing of [12, Section 1] with the generic embedding

$$j : V \rightarrow M, M^\omega \subseteq M, j(\lambda_1) = \bar{\lambda}_1, j(\bar{\lambda}_i) = \bar{\lambda}_i \text{ for } i > 1.$$

³The cardinals are indiscernible even if the quantifier Q_ω^{cf} is added to the language of set theory.

A special feature of the ω -closed stationary tower forcing of [12] is that it does not introduce new ordinals of cofinality ω . Thus

$$C^{*V} = C^{*V[G]} = C^{*M}.$$

Suppose now

$$C^* \models \Phi(\beta_1, \dots, \beta_l, \lambda_1, \lambda_2, \dots, \lambda_k).$$

By the induction hypothesis, applied to $\lambda_2, \dots, \lambda_k$ and $\bar{\lambda}_2, \dots, \bar{\lambda}_k$,

$$C^* \models \Phi(\beta_1, \dots, \beta_l, \lambda_1, \bar{\lambda}_2, \dots, \bar{\lambda}_k).$$

Since j is an elementary embedding,

$$C^* \models \Phi(\beta_1, \dots, \beta_l, \bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_k).$$

Claim 1 is proved.

Claim 2: If $\lambda_1 < \dots < \lambda_k$ are Woodin cardinals, $\kappa_1 < \dots < \kappa_k$ are regular cardinals $> \aleph_1$, $\lambda_1 > \max(\kappa_1, \dots, \kappa_k)$, and $\beta_1, \dots, \beta_l < \kappa_1$, then

$$C^* \models \Phi(\beta_1, \dots, \beta_l, \kappa_1, \dots, \kappa_k) \leftrightarrow \Phi(\beta_1, \dots, \beta_l, \lambda_1, \dots, \lambda_k)$$

for all formulas $\Phi(x_1, \dots, x_l, y_1, \dots, y_k)$ of set theory.

We use induction on k to prove the claim. The case $k = 0$ is clear. Let us assume the claim for $k - 1$. Using ω -stationary tower forcing we can find

$$j : V \rightarrow M, M^\omega \subseteq M, j(\kappa_1) = \lambda_1, j(\lambda_i) = \lambda_i \text{ for } i > 1.$$

Now we use the Claim to prove the theorem. Suppose now

$$C^* \models \Phi(\beta_1, \dots, \beta_l, \kappa_1, \kappa_2, \dots, \kappa_k).$$

By the induction hypothesis, applied to $\kappa_2, \dots, \kappa_k$ and $\lambda_2, \dots, \lambda_k$,

$$C^* \models \Phi(\beta_1, \dots, \beta_l, \kappa_1, \lambda_2, \dots, \lambda_k).$$

Since j is an elementary embedding,

$$C^* \models \Phi(\beta_1, \dots, \beta_l, \lambda_1, \lambda_2, \dots, \lambda_k).$$

Claim 2 is proved.

The claim of the theorem follows immediately from Claim 2. □

Note that we cannot extend Theorem 5.13 to \aleph_1 , for \aleph_1 has the following property, recognizable in C^* , which no other uncountable cardinal has: it has uncountable cofinality but all of its (limit) elements have countable cofinality.

Theorem 5.14. *If $V = L^\mu$, then C^* is exactly the inner model $M_{\omega^2}[E]$, where M_{ω^2} is the ω^2 th iterate of V and $E = \{\kappa_{\omega \cdot n} : n < \omega\}$.*

Proof. In $M_{\omega^2}[E]$ we can detect which ordinals have cofinality ω in V : They are the ordinals with cofinality in $E \cup \{\sup(E)\}$, and ordinals which have cofinality ω already in $M_{\omega^2}[E]$. Hence every level of the construction of C^* is in $M_{\omega^2}[E]$ and hence $C^* \subseteq M_{\omega^2}[E]$. On the other hand, we can define $M_{\omega^2}[E]$ inside C^* as follows: The set E is the set of ordinals $< \kappa_{\omega^2}$ which have cofinality ω in V but are regular in the core model. But the core model is included in C^* by Theorem 5.5. Thus $E \in C^*$. The measure $i_{0\omega^2}(\mu)$ on κ_{ω^2} can be defined from E as follows: Let μ' be defined for $X \subseteq \kappa_{\omega^2}$ by

$$\mu'(X) = 1 \text{ if and only if } \exists \alpha \in E \forall \beta \in E (\alpha < \beta \rightarrow \beta \in X).$$

By [18, Th. 5.8 (ii)] $L^{\mu'} = M_{\omega^2}$ and $i_{0\omega^2}(\mu) = \mu' \cap M_{\omega^2}$. Hence $M_{\omega^2}[E] \subseteq C^*$. \square

The situation is similar with the inner model for two measurable cardinals: To get C^* we first iterate the first measurable ω^2 times, then the second ω^2 times, and in the end take two Prikry sequences.

We now prove the important property of C^* that its truth is invariant under (set) forcing. We have to assume large cardinals because conceivably C^* could satisfy $V = L$ but in a (set) forcing extension C^* would violate $V = L$ (see Section 6 below).

Theorem 5.15. *Suppose there is a proper class of Woodin cardinals. Suppose \mathbb{P} is a forcing notion and $G \subseteq \mathbb{P}$ is generic. Then*

$$Th((C^*)^V) = Th((C^*)^{V[G]}).$$

Moreover, the theory $Th(C^)$ is independent of the cofinality used, and forcing does not change the reals of these models.*

Proof. Let G be \mathbb{P} -generic. Let us choose a Woodin cardinal $\lambda > |\mathbb{P}|$. Let H_1 be generic for the countable stationary tower forcing $\mathbb{Q}_{<\lambda}$. In $V[H_1]$ there is a

generic embedding $j_1 : V \rightarrow M_1$ such that $V[H_1] \models M_1^\omega \subseteq M_1$ and $j(\omega_1) = \lambda$. Hence $(C^*)^{V[H_1]} = (C^*)^{M_1}$ and

$$j_1 : (C^*)^V \rightarrow (C^*)^{M_1} = (C^*)^{V[H_1]} = (C^*_{<\lambda})^V.$$

and therefore by elementarity $\text{Th}((C^*)^V) = \text{Th}((C^*_{<\lambda})^V)$.

Since $|\mathbb{P}| < \lambda$, λ is still Woodin in $V[G]$. Let H_2 be generic for the countable stationary tower forcing $\mathbb{Q}_{<\lambda}$ over $V[G]$. Let $j_2 : V[G] \rightarrow M_2$ be the generic embedding. Now $V[G, H_2] \models M_2^\omega \subseteq M_2$ and $j_2(\omega_1) = \lambda$. Hence

$$j_2 : (C^*)^{V[G]} \rightarrow (C^*)^{M_2} = (C^*)^{V[G, H_2]} = (C^*_{<\lambda})^{V[G]} = (C^*_{<\lambda})^V.$$

and therefore by elementarity $(C^*)^V \equiv (C^*_{<\lambda})^V \equiv (C^*)^{V[G]}$. □

We may ask, for which λ and μ is $C^*_{<\lambda} = C^*_{<\mu}$? Observations:

- If $V = L^\mu$, then $C^* = C^*_{<\lambda}$ for all λ . This follows from Theorem 5.14.
- It is possible that $C^* \neq C^*_{<\omega_2}$. Let us use the $\leq \omega$ -closed stationary tower forcing of [12, Section 1] to map ω_3 to λ . In this model V_1 the inner model C^* is preserved. It is easy to see that in the extension the set A of ordinals below λ of cofinality ω_1 is not in V . If $C^* = C^*_{<\omega_2}$, then A is in $C^*_{<\omega_2}$. We are done.
- It is possible that C^* changes. Extend the previous model V_1 to V_2 by collapsing ω_1 to ω . Then $(C^*)^{V_2} = (C^*_{<\omega_2})^{V_1} \neq (C^*)^{V_1}$. So C^* has changed.
- Question: Does a Woodin cardinal imply $C^*_{<\omega_2} \neq C^*_{\omega, \omega_1}$?

We do not know whether the CH is true or false in C^* . Forcing absoluteness of the theory of C^* under the hypothesis of large cardinals implies, however, that large cardinals decide the CH in C^* . This is in sharp contrast to V itself where we know that large cardinals do not decide CH [23]. We can at the moment only prove that the size of the continuum of C^* is at most ω_2^V . In the presence of a Woodin cardinal this tells us nothing, as then ω_2^V is (strongly) Mahlo in C^* (Theorem 5.11). So the below result is only interesting when we do not assume the existence of Woodin cardinals. However, we show later that in the presence of large cardinals there is a cone of reals x such that the relativized version of C^* , $C^*(x)$, satisfies CH .

Theorem 5.16. $|\mathcal{P}(\omega) \cap C^*| \leq \aleph_2$.

Proof. We use the notation of Definition 2.2. Suppose $a \subseteq \omega$ and $a \in L'_\xi$ for some ξ . Let $\mu > \xi$ be a sufficiently large cardinal. We build an increasing elementary chain $(M_\alpha)_{\alpha < \omega_1}$ such that

1. $a \in M_0$ and $M_0 \models a \in C^*$.
2. $|M_\alpha| \leq \omega$.
3. $M_\alpha \prec H(\mu)$.
4. $M_\gamma = \bigcup_{\alpha < \gamma} M_\alpha$, if $\gamma = \cup \gamma$.
5. If $\beta \in M_\alpha$ and $\text{cf}^V(\beta) = \omega$, then $M_{\alpha+1}$ contains an ω -sequence from $H(\mu)$, cofinal in β .
6. If $\beta \in M_\alpha$ and $\text{cf}^V(\beta) > \omega$ then for unboundedly many $\gamma < \omega_1$ there is $\rho \in M_{\gamma+1}$ with

$$\sup\left(\bigcup_{\xi < \gamma} (M_\xi \cap \beta)\right) < \rho < \beta.$$

Let M be $\bigcup_{\alpha < \omega_1} M_\alpha$, N the transitive collapse of M , and ζ the ordinal $N \cap On$. Note that $|N| \leq \omega_1$, whence $\zeta < \omega_2$. By construction, an ordinal in N has cofinality ω in V if and only if it has cofinality ω in N . Thus $(L'_\xi)^N = L'_\xi$ for all $\xi < \zeta$. Since $N \models a \in C^*$, we have $a \in L'_\zeta$. The claim follows. \square

The proof of Theorem 5.16 gives the following more general result:

Theorem 5.17. *Let κ be a regular cardinal and δ an ordinal. Then*

$$|\mathcal{P}(\delta) \cap C_\kappa^*| \leq (|\delta| \cdot \kappa^+)^+.$$

Corollary. *If $\delta \geq \kappa^+$ is a cardinal in C_κ^* and $\lambda = |\delta|^+$, then $C_\kappa^* \models 2^\delta \leq \lambda$.*

Corollary. *Suppose $V = C^*$. Then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for $\alpha \geq 1$, and $2^{\aleph_0} = \aleph_1$ or $2^{\aleph_0} = \aleph_2$.*

Theorem 5.18. *Suppose $E \subseteq \aleph_2$ is stationary. Then $\diamond_{\aleph_2^V}(E)$ holds in C^* .*

Proof. The proof is as the standard proof of $\diamond_{\aleph_2}(E)$ in L , with a small necessary patch. We construct a sequence $s = \{(S_\alpha, D_\alpha) : \alpha < \aleph_2^V\}$ taking always for limit α the pair (S_α, D_α) to be the least $(S, D) \in L'_{\aleph_2^V}$ in the well-order (see Proposition 2.6)

$$R = \{(a, b) \in (L'_{\aleph_2^V})^2 : L'_{\aleph_2^V} \models \Psi_{\mathcal{L}(Q_{\aleph_2^V}^{\text{cf}})}(a, b)\}$$

such that $S \subseteq \alpha$, $D \subseteq \alpha$ a club, and $S \cap \beta \neq S_\beta$ for $\beta \in D$, if any exists, and $S_\alpha = D_\alpha = \alpha$ otherwise. Note that $s \in C^*$. We show that the sequence s is a diamond sequence in C^* . Suppose it is not and $(S, D) \in C^*$ is a counter-example, $S \subseteq \aleph_2^V$ and $D \subseteq \aleph_2^V$ club such that $S \cap \beta \neq S_\beta$ for all $\beta \in D$. As in the proof of Theorem 5.16 we can construct $M \prec H(\mu)$ such that $|M| = \aleph_1^V$, the order-type of $M \cap \aleph_2^V$ is in E , $\{s, (S, D)\} \subset M$, and if N is the transitive collapse of M , with ordinal $\delta \in E$, then $\{s \upharpoonright \delta, (S \cap \delta, D \cap \delta)\} \subset N$ and $(L'_\xi)^N = L'_\xi$ for all $\xi < \delta$. Because of the way M is constructed, the well-order R restricted to L'_δ is defined in M on $L'_{\aleph_2^V}$ by the same formula $\hat{\Psi}_{\mathcal{L}(Q_{\aleph_2^V}^{\text{cf}})}(x, y)$ as R is defined on $L'_{\aleph_2^V}$ in $H(\mu)$. Since $S \cap \delta \in L'_{\aleph_2^V}$ and $S \cap \beta \neq S_\beta$ for $\beta \in D \cap \delta$, we may assume, w.l.o.g., that $(S, D) \in L'_{\aleph_2^V}$. Furthermore, we may assume, w.l.o.g., that (S, D) is the R -least counter-example to s being a diamond sequence. Thus the pair $(S \cap \delta, D \cap \delta)$ is the R -least (S', D') such that $S' \subseteq \delta$, $D' \subseteq \delta$ a club, and $S' \cap \beta \neq S'_\beta$ for $\beta \in D'$. It follows that $(S', D') = (S_\delta, D_\delta)$ and, since $\delta \in D$, a contradiction. \square

The proof generalises easily to a proof of:

Theorem 5.19. *Suppose λ is regular, $\mu = \lambda^+$ and $E \subseteq \mu$ is stationary. Then $\diamond_\mu(E)$ holds in C^* .*

A problem in using condensation type arguments, such as we used in the proofs of Theorem 5.16 and Theorem 5.18 above, is the non-absoluteness of C^* . There is no reason to believe that $(C^*)^{C^*} = C^*$ in general (see Theorem 6.2). Moreover, we prove in Theorem 6.6 the consistency of C^* failing to satisfy CH, relative to the consistency of an inaccessible cardinal.

We now prove a result which seems to lend support to the idea that C^* satisfies CH, at least assuming large cardinals. Let \leq_T be the Turing-reducibility relation between reals. The *cone* of a real x is the set of all reals y with $x \leq_T y$. A set of reals is called a *cone* if it is the cone of some real. Suppose A is a projective set of reals closed under Turing-equivalence. If we assume PD, then by a result of D. Martin [31] there is a cone which is included in A or is disjoint from A .

Theorem 5.20. *If there are infinitely many Woodin cardinals and a measurable cardinal above them, then there is a cone of reals x such that $C^*(x)$ satisfies the Continuum Hypothesis.*

Proof. We first observe that if two reals x and y are Turing-equivalent, then $C^*(x) = C^*(y)$. Hence the set

$$\{y \subseteq \omega : C^*(y) \models CH\} \quad (9)$$

is closed under Turing-equivalence, and therefore amenable to the above mentioned result by Martin on cones. We just have to show that

(I) The set (9) is projective.

(II) For every real x there is a real y such that $x \leq_T y$ and y is in the set (9).

We start with a proof of (I): Suppose N is a well-founded model of ZFC^- and N thinks that $\lambda \in M$ is a Woodin cardinal. We say that N is **iterable**, if all countable iterations of forming generic ultrapowers of N by stationary tower forcing at λ are well-founded. We use the notation $L'_\alpha(y)$ for the levels of the construction of $C^*(y)$.

Lemma 5.21. *Suppose there is a Woodin cardinal and a measurable cardinal above it. The following conditions are equivalent:*

(i) $C^*(y) \models CH$.

(ii) *There is $\alpha < \omega_2^V$ such that $L'_\alpha(y) \models CH$ and if $\alpha < \beta < \omega_2^V$, then we have $\mathcal{P}(\omega)^{L'_\alpha(y)} = \mathcal{P}(\omega)^{L'_\beta(y)}$.*

(iii) *There is a countable iterable structure M with a Woodin cardinal such that $y \in M$, $M \models \exists \alpha ("L'_\alpha(y) \models CH")$ and for all countable iterable structures N with a Woodin cardinal such that $y \in N$: $\mathcal{P}(\omega)^{(C^*)^N} \subseteq \mathcal{P}(\omega)^{(C^*)^M}$.*

Proof. (i) implies (ii) by the proof of Theorem 5.16. Assume then (ii). To prove (iii), let M be an iterable structure of cardinality \aleph_1 containing $L'_\alpha(y)$, all reals of C^* , and a Woodin cardinal λ . Next we form an iteration sequence of length ω_1^V . Let $M_0 \prec M$ be countable containing $L'_\alpha(y)$, all reals of C^* , and λ . Let $M_{\gamma+1}$ be a generic ultrapower of the stationary tower on the image of λ in M_γ . For limit $\alpha \leq \omega_1^V$, the model M_α is the direct limit of the M_β , $\beta < \alpha$. By the iterability condition each M_α is well-founded. Let $\pi_{\alpha\beta}$ be the canonical embedding $M_\alpha \rightarrow$

M_β . Let N_α be the Mostowski collapse of M_α and $j_\alpha : N_\alpha \rightarrow M_\alpha$ the inverse of the collapsing function. We have the directed system:

$$\begin{array}{ccccccc}
& & \pi_{01} & & & \pi_{\gamma\gamma+1} & \\
M_0 & \rightarrow & M_1 & \dots & \rightarrow & \dots & M_\gamma & \rightarrow & M_{\gamma+1} & \dots & \rightarrow & M_{\omega_1} \\
j_0 \uparrow & & j_1 \uparrow & & & & j_\gamma \uparrow & & j_{\gamma+1} \uparrow & & & \uparrow j_{\omega_1} \\
N_0 & \rightarrow & N_1 & \dots & \rightarrow & \dots & N_\gamma & \rightarrow & N_{\gamma+1} & \dots & \rightarrow & N_{\omega_1} \\
& & \sigma_{01} & & & & \sigma_{\gamma\gamma+1} & & & & &
\end{array}$$

Let η be such that $j_0(\eta) = \omega_1^V (\in M_0)$. Since $\pi_{0\gamma}(\omega_1^V)$ is extended in each step of this iteration of length ω_1 of countable models, $\sigma_{0\omega_1}(\eta) = \omega_1^V$. Suppose $\delta \in N_{\omega_1}$ has uncountable cofinality in N_{ω_1} . Since the only possibilities for the cofinality (in V) for an ordinal $< \alpha$ are ω and ω_1 , the only possibilities for the cofinality for an ordinal in N_0 are ω and η . Suppose $\delta = \sigma_{\gamma\omega_1}(\bar{\delta})$. Since δ has uncountable cofinality in N_{ω_1} , the cofinality of $\bar{\delta}$ in N_γ is $\omega_1^{N_\gamma}$. Because ω_1 is moved at each step of the iteration, and the iteration has length ω_1^V , the cofinality of δ is ω_1^V . We have shown that N_{ω_1} is correct about cofinality ω . Thus the $C^*(y)$ -model built inside N_{ω_1} , i.e. $(C^*(y))^{N_{\omega_1}}$, will be the real $C^*(y)$ up to the ordinals that N_{ω_1} has. Let $j_0(\bar{L}'_\alpha(y)) = L'_\alpha(y)$, $K = \sigma_{0\omega_1}(\bar{L}'_\alpha(y))$, $j_{\omega_1}(K) = \pi_{0\omega_1}(L'_\alpha(y))$. It follows that $K = L'_\beta(y)$ for some β . Note that $a \in L'_\beta$ and β is the smallest β for which $a \in L'_\beta(y)$. Hence $\beta = \alpha$ and $L'_\beta(y) = L'_\alpha(y)$. It follows that the reals of $C^*(y)$ can be expressed as the increasing union of countable sets, hence its cardinality is $\leq \aleph_1$. \square

Claim (I) follows immediately from the above lemma.

Proof of (II): Fix x . Let \mathbb{P} be the standard forcing which, in $C^*(x)$, forces a subset A of $\omega_1^{C^*(x)}$, such that A codes, via the canonical pairing function in $C^*(x)$, an onto mapping $\omega_1^{C^*(x)} \rightarrow \mathcal{P}(\omega) \cap C^*(x)$. Let $A \in V$ be \mathbb{P} -generic over $C^*(x)$. Note that \mathbb{P} does not add any new reals. Now we code A by a real by means of almost disjoint forcing. Let Z_α , $\alpha < \omega_1^{C^*(x)}$, be a sequence in $C^*(x)$ of almost disjoint subsets of ω . Let \mathbb{Q} be the standard CCC-forcing, known from [30], for adding a real y' such that for all $\alpha < \omega_1^{C^*(x)}$:

$$|z_\alpha \cap y'| \geq \omega \iff \alpha \in A.$$

Let $y = x \oplus y'$. Of course, $x \leq_T y$. Now

$$C^*(x) \subseteq C^*(x)[A] \subseteq C^*(y).$$

By the definition of A , $C^*(x)[A] \models CH$. The forcing \mathbb{Q} is of cardinality \aleph_1 in $C^*(x)[A]$, hence $C^*(y) \models CH$. We have proved (2). \square

Assuming large cardinals, the set of reals of C^* seems like an interesting countable Σ_3^1 -set with a Σ_3^1 -well-ordering. It might be interesting to have a better understanding of this set. This set is contained in the reals of the so called M_1^\sharp , the smallest inner model for a Woodin cardinal (M. Magidor and R. Schindler, unpublished).

6 Consistency results about C^*

We define a version of Namba forcing that we call *modified Namba forcing* and then use this to prove consistency results about C^* .

Suppose $S = \{\lambda_n : n < \omega\}$ is a sequence of regular cardinals $> \omega_1$ such that every λ_n occurs infinitely many times in the sequence. Let $\langle B_n : n < \omega \rangle$ be a partition of ω . The forcing \mathcal{P} is defined as follows: Conditions are trees T with ω levels, consisting of finite sequences of ordinals, defined as follows: If $(\alpha_0, \dots, \alpha_i) \in T$, let

$$\text{Suc}_T((\alpha_0, \dots, \alpha_i)) = \{\beta : (\alpha_0, \dots, \alpha_i, \beta) \in T\}.$$

The forcing \mathcal{P} consists of trees, called S -trees, such that if $(\alpha_0, \dots, \alpha_i) \in T$ and $i \in B_n$, then

1. $|\text{Suc}_T((\alpha_0, \dots, \alpha_{i-1}))| \in \{1, \lambda_n\}$,
2. For every n there are $\alpha_i, \dots, \alpha_k$ such that $k \in B_n$ and $|\text{Suc}_T((\alpha_0, \dots, \alpha_k))| = \lambda_n$.

If $|\text{Suc}_T((\alpha_0, \dots, \alpha_{i-1}))| = \lambda_n$, we call $(\alpha_0, \dots, \alpha_{i-1})$ a *splitting point* of T . Otherwise $(\alpha_0, \dots, \alpha_{i-1})$ is a *non-splitting point* of T . The stem $\text{stem}(T)$ of T is the maximal (finite) initial segment that consists of non-splitting points. If $s = (\alpha_0, \dots, \alpha_i) \in T$, then

$$T_s = \{(\alpha_0, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_n) \in T : i \leq n < \omega\}.$$

A condition T *extends* another condition T' , $T' \leq T$, if $T' \subseteq T$. If $\langle T_n : n < \omega \rangle$ is a generic sequence of conditions, then the stems of the trees T_n form a sequence $\langle \alpha_n : n < \omega \rangle$ such that $\langle \alpha_i : i \in B_n \rangle$ is cofinal in λ_n . Thus in the generic extension $\text{cf}(\lambda_n) = \omega$ for all $n < \omega$.

We shall now prove that no other regular cardinals get cofinality ω .

Proposition 6.1. *Suppose $\kappa \notin S$ is regular. Then $\mathcal{P} \Vdash \text{cf}(\kappa) \neq \omega$.*

Proof. Let us first prove that if τ is a name for an ordinal, then for all $T \in \mathcal{P}$ there is $T^* \leq T$ such that $\text{stem}(T^*) = \text{stem}(T)$ and if $T^{**} \leq T^*$ decides which ordinal β is, and $s = \text{stem}(T^{**})$, then T_s^* decides β . Suppose T is given and the length of its stem is l . Let us look at the level $l + 1$ of T . Let $l \in B_n$. If there are λ_n nodes on this level such that the claim holds for these extensions of T we can take a fusion and this is the desired T^* . Otherwise there are λ_n nodes on this level such that the claim is true also for this extension of T . Let us call a node T' of T *good* if the claim is true when T is taken to be T' . Suppose first there are λ_n good nodes. W.l.o.g. they all have the same stem length k . Again we get the desired T^* by fusion. Suppose then there are not λ_n many good nodes. So there must be λ_n bad nodes. We repeat this process on the level $k + 1$. Suppose the process does not end. We get $T' \leq T$ consisting of bad nodes. Since T forces that τ is an ordinal, there is $T'' \leq T'$ such that T'' decides which ordinal τ is. We get a contradiction: the node of the stem of T'' , which is also a node of T' , cannot be a bad one.

Suppose now $\langle \beta_n : n < \omega \rangle$ is a name for an ω -sequence of ordinals below κ , and $T \in \mathcal{P}$ forces this. We construct $T^* \leq T$ and an ordinal $\delta < \kappa$ such that T^* forces the sequence $\langle \beta_n : n < \omega \rangle$ to be bounded below κ by δ . For each n we have a function f_n defined on extensions T^* of T with the same stem deciding β_n to be $f_n(T^*)$. Let us call T *good for* $\langle \beta_n : n < \omega \rangle$ if for all infinite branches f through T and all n there is k such that f_n restricted to stem length k is decided by T . It follows from the above that there is $T^* \leq T$ with the same stem as T and good for $\langle \beta_n : n < \omega \rangle$.

Without loss of generality, T itself is good for $\langle \beta_n : n < \omega \rangle$. Fix $\delta < \kappa$. We consider the following game G_δ . During the game the players determine an infinite branch through T . If the game has reached node t on height k with $k + 1 \in B_n$ we consider two cases:

Case 1: $\kappa > \lambda_n$. Bad moves by giving a successor of t .

Case 2: $\kappa < \lambda_n$. First Bad plays a subset A of successors of t such that $|A| < \lambda_n$. Then Good moves a successor not in the set.

Good player loses this game if at some stage of the game a member of the sequence β_n is forced to go above δ .

Main Claim: There is $\delta < \kappa$ such that Bad does not win G_δ (hence Good wins).

Proof. Assume the contrary, i.e. that Bad wins for all $\delta < \kappa$. Let τ_δ be a strategy for Bad for any given $\delta < \kappa$. We define a subtree of T as follows. Suppose we have reached a node t . We assume inductively that there were δ and strategy τ_δ that took us to this node. Suppose $\text{len}(t) = k$ and $k + 1 \in B_n$.

Case 1: $\kappa > \lambda_n$. Bad moves by giving a successor of t . Take all of them arising from different ordinals δ . Their number is less than κ as $\lambda_n < \kappa$.

Case 2: $\kappa < \lambda_n$. First Bad plays a subset A of successors of t such that $|A| < \lambda_n$. Then Good moves. There are at most κ many subsets A that Bad can play. As $\kappa < \lambda_n$ and λ_n is regular, their union is of cardinality $< \lambda_n$. Let us pick one that is not in the union.

In this way we get a subtree of cardinality $< \kappa$, as κ is regular. The nodes of this tree determine values for the cofinal sequence $\langle \beta_n : n < \omega \rangle$. Since there are $< \kappa$ values, they are bounded in κ by some $\delta < \kappa$. Let us now play G_δ so that Bad plays τ_δ . The play will stay in the subtree. Since we do not exceed δ , Good wins, a contradiction. □

Now we return to the main part of the proof. By the Main Claim there is δ such that Good wins G_δ . Let us look at the subtree of all plays of G_δ where Good plays her winning strategy. A subtree T^* of T is generated and T^* forces the sequence $\langle \beta_n : n < \omega \rangle$ to be bounded by δ . □

The above modified Namba forcing permits us to carry out the following basic construction: Suppose $V = L$. Let us add a Cohen real r . We can code this real with the above modified Namba forcing so that in the end for all $n < \omega$:

$$\text{cf}^V(\aleph_{n+2}^L) = \omega \iff n \in r.$$

Thus in the extension $r \in C^*$.

Theorem 6.2. *Con(ZF) implies $\text{Con}((C^*)^{C^*}) \neq C^*$.*

Proof. We start with $V = L$. We add a Cohen real a . In the extension $C^* = L$, for cofinalities have not changed, so to decide whether $\text{cf}(a) = \omega$ or not it suffices to decide this in L . With modified Namba forcing we can change—as above—the cofinality of \aleph_{n+2}^L to ω according to whether $n \in a$ or $n \notin a$. In the extension $C^* = L(a)$, for cofinality ω has only changed from L to the extent that the cofinalities of \aleph_n^L may have changed, but this we know by looking at a . Thus $(C^*)^{C^*} = (C^*)^{L(a)} = L$, while $C^* \neq L$. Thus $(C^*)^{C^*} \neq C^*$. □

We now prepare ourselves to iterating this construction in order to code more sets into C^* .

Definition 6.3 (Shelah). A forcing notion \mathcal{P} satisfies the *S-condition* if player II has a super strategy (defined below) in the following game G :

1. There are two players I and II and ω moves.
2. In the start of the game player I plays a tree T_0 of finite height and a function $f : T_0 \rightarrow \mathcal{P}$ such that for all $t, t' \in T_0$: $t <_{T_0} t' \Rightarrow f(t') <_{\mathcal{P}} f(t)$.
3. Then II decides what the successors of the top nodes are and extends f .
4. Player I extends the tree with non-splitting nodes of finite height and extends f .
5. Then II decides what the successors of the top nodes are and extends f .
6. etc, etc

Player II wins if the resulting tree T is an S -tree, and for every S -subtree T^* of T there is a condition $B^* \in \mathcal{P}$ such that

$B^* \Vdash$ “The f -image of some branch through T^* is included in the generic set”.

A *super strategy* of II is a winning strategy in which the moves depend only on the predecessors in T of the current node, as well as on their f -images.

By [39, Theorem 3.6] (see also [7, 2.1]), forcing with the S -condition does not collapse \aleph_1 .

Lemma 6.4. *Modified Namba forcing satisfies the S -condition.*

Proof. Suppose the game has progressed to the following:

1. A tree T has been constructed, as well as $f : T \rightarrow \mathcal{P}$.
2. Player I has played a non-splitting end-extension T' of T .

Suppose η is a maximal node in T' . We are in stage n . Now II adds λ_n extensions to η . Let E denote these extensions. Let B be the S -tree $f(\eta)$. Find a node ρ in B which is a splitting node and splits into λ_n nodes. Let g map the elements of E 1-1 to successors of ρ in B . Now we extend f to E by letting the image of $e \in E$ be the subtree B_η of B consisting of ρ and the predecessors of ρ extended by first $g(e)$ and then the subtree of B above $g(e)$.

We show that this is a super strategy. Suppose T is a tree resulting from II playing the above strategy. Let T^* be any S -subtree of T . We construct an S -tree

$B^* \in \mathcal{P}$ as follows. Let B^* be the union of all the stems of the trees B_η , where η is a splitting point of T^* . Clearly, B^* is an S -tree. To see that

$B^* \Vdash$ “The f -image of some branch through T^* is included in the generic set”,

let G be a generic containing B^* . This generic is a branch γ through B^* . In view of the definition of B^* , there is a branch β through T^* such that $f''\beta = \gamma$. \square

Theorem 6.5. *Suppose $V = L$ and κ is a cardinal of cofinality $> \omega$. There is a forcing notion \mathbb{P} which forces $C^* \models 2^\omega = \kappa$ and preserves cardinals between L and C^* .*

Proof. Suppose $V = L$. Let us add κ Cohen reals $\{r_\alpha : \alpha < \kappa\}$. We code these reals with revised countable support iterated modified Namba forcing so that in the end we have a forcing extension in which for $\alpha < \kappa$ and $n < \omega$:

$$\text{cf}^V(\aleph_{\omega \cdot \alpha + n + 2}^L) = \omega \iff n \in r_\alpha.$$

Thus in the extension $r_\alpha \in C^*$ for all $\alpha < \kappa$. We can now note that in the extension $C^* = L[\{r_\alpha : \alpha < \kappa\}]$. First of all, each r_α is in C^* . This gives “ \supseteq ”. For the other direction, we note that whether an ordinal has cofinality ω in V can be completely computed from the set $\{r_\alpha : \alpha < \kappa\}$. \square

Note that the above theorem gives a model in which $C^* \models 2^\omega = \aleph_3$, but then in the extension $|\aleph_3^{C^*}| = \aleph_1$.

Theorem 6.6. *It is consistent, relative to the consistency of an inaccessible cardinal, that $V = C^*$ and $2^{\aleph_0} = \aleph_2$.*

Proof. Start with an inaccessible κ and $V = L$. Iterate over κ with revised countable support adding Cohen reals and coding generic sets using modified Namba forcing. The iteration satisfies the S -condition, hence \aleph_1 is preserved, and κ is the new \aleph_2 . In the extension $2^{\aleph_0} = \aleph_2$ and $V = C^*$. \square

7 Stationary logic

Stationary logic is the extension of first order logic by the following second order quantifier:

Definition 7.1. $\mathcal{M} \models \text{aas}\varphi(s) \iff \{A \in [M]^{\leq\omega} : \mathcal{M} \models \varphi(A)\}$ contains a club of countable subsets of M . (I.e. almost all countable subsets A of M satisfy $\varphi(A$.) We denote $\neg\text{aas}\neg\varphi$ by $\text{stat } s\varphi$.

Some examples of the expressive power of stationary logic are the following: We can express “ $\varphi(\cdot)$ is countable” with $\text{aas}\forall y(\varphi(y) \rightarrow s(y))$. If we have a linear order $\varphi(\cdot, \cdot)$, we can express the fact that it has cofinality ω with $\text{aas}\forall x\exists y(\varphi(x, y) \wedge s(y))$. We can express the fact that $\varphi(\cdot, \cdot)$ is \aleph_1 -like with $\forall x\text{aas}\forall y(\varphi(y, x) \rightarrow s(y))$. Finally, we can express the fact that an \aleph_1 -like linear order $\varphi(\cdot, \cdot)$ contains a closed unbounded subset (i.e. a copy of ω_1) with $\text{aas}(\text{sup}(s) \in \text{dom}(\varphi))$.

Moreover, the set $\{\alpha < \kappa : \text{cf}(\alpha) = \omega\}$ is $\mathcal{L}(\text{aa})$ -definable on $(\kappa, <)$ by means of $\text{aas}(\text{sup}(s) = \alpha)$. The property of a set $A \subseteq \{\alpha < \kappa : \text{cf}(\alpha) = \omega\}$ of being stationary is definable in $\mathcal{L}(\text{aa})$ by means of $\text{stat } s(\text{sup}(s) \in A)$.

The logic $\mathcal{L}(\text{aa})$ is countably compact, axiomatizable, and has the Löwenheim-Skolem property down to ω_1 . It has the Löwenheim-Skolem-Tarski property down to ω_1 , assuming PFA^{++} . Suppose A is a stationary subset of a regular $\kappa > \omega$ such that $\forall\alpha \in A(\text{cf}(\alpha) = \omega)$. The ω -club filter $\mathcal{F}^\omega(A)$ is the set of subsets of A which are unbounded in κ and which are closed under limits of increasing ω -sequences whenever the limit is in A . Note that $\mathcal{F}^\omega(A)$ is $< \kappa$ -closed. The property of $B \subseteq \kappa$ of belonging to $\mathcal{F}^\omega(A)$ is definable from A in $\mathcal{L}(\text{aa})$ by means of $\text{aas}(\text{sup}(s) \in A \rightarrow \text{sup}(s) \in B)$.

We denote $C(\mathcal{L}(\text{aa}))$ by

$$C(\text{aa}).$$

The logic $\mathcal{L}(\text{aa})$ is adequate to truth in itself. Hence $C(\text{aa})$ satisfies AC.

Note that $\{\alpha < \kappa : \text{cf}^V(\alpha) = \omega\} \in C(\text{aa})$. The property of $A \subseteq \{\alpha < \kappa : \text{cf}^V(\alpha) = \omega\}$ of being V -stationary is definable in $C(\text{aa})$. If $A \in C(\text{aa})$, then $(\mathcal{F}^\omega(A))^V \cap C(\text{aa}) \in C(\text{aa})$.

We now introduce a useful auxiliary concept⁴:

Definition 7.2 ([9]). A first order structure \mathcal{M} is *club-determined* if

$$\mathcal{M} \models \forall \vec{s} \forall \vec{x} [\text{aat}\vec{\varphi}(\vec{x}, \vec{s}, \vec{t}) \vee \text{aat}\neg\vec{\varphi}(\vec{x}, \vec{s}, \vec{t})],$$

where $\varphi(\vec{x}, \vec{s}, \vec{t})$ is any formula in $\mathcal{L}(\text{aa})$.

⁴In [9] the name “finitely determinate” is used.

On a club-determined structure the quantifier stat (“stationarily many”) and aa (“club many”) coincide on definable sets. The truth of $\text{aat}\varphi(\vec{x}, \vec{s}, \vec{t})$ in a structure \mathcal{M} can be written in the form of a two-person perfect information zero-sum game $G(\varphi, \mathcal{M}, \vec{x}, \vec{s})$, resembling the so-called club-game. A structure \mathcal{M} is club-determined if and only the game $G(\varphi, \mathcal{M}, \vec{x}, \vec{s})$ is determined for all formulas φ and all parameters \vec{s}, \vec{x} .

There are several results in [9] suggesting that club-determined structures have a ‘better’ model theory than arbitrary structures. For a start, every consistent first order theory has a club-determined model. Moreover, every uncountable model has an $\mathcal{L}(\text{aa})$ -elementary submodel of cardinality \aleph_1 , while for arbitrary structures this cannot be proved in ZFC (but it follows from MM^{++} , see Definition 7.9).

Definition 7.3. We say that the inner model $C(\text{aa})$ is *club-determined* if every level L'_α is.

Intuitively speaking, if $C(\text{aa})$ is club-determined, its definition is more robust—the quantifier aa is more lax—than it would be otherwise, and in consequence, $C(\text{aa})$ is a little easier to compute.

Theorem 7.4. *If there are a proper class of measurable Woodin cardinals, then $C(\text{aa})$ is club-determined.*

Proof. Suppose δ is measurable Woodin. We denote the countable stationary tower forcing for δ by $Q_{<\delta}$. A set p is *self-generic* for $Q_{<\delta}$ if for all dense $D \subseteq Q_{<\delta}$ there is $S \in p \cap D$ such that $p \cap \text{Supp}(S) \in S$. Let

$$T = \{p : p \prec V_{\delta+2}, |p| = \omega_1, \text{otp}(p \cap \delta) = \omega_1, p \text{ is self-generic for } Q_{<\delta}\}.$$

Claim: T is stationary, i.e. every algebra on $V_{\delta+2}$ has a subalgebra which belongs to T .

Proof of the Claim: Fix an algebra on $V_{\delta+2}$. We define a sequence of length ω_1 of countable structures. During the construction we dovetail taking care of three things. The first is the order type ω_1 , the second is the self-genericity, and the third is the building of a subalgebra.

Let \mathcal{U} be a normal ultrafilter on δ . Let $M_0 \prec V_{\delta+2}$ be countable, $\mathcal{U} \in M_0$. Note that \mathcal{U} is still a normal ultrafilter in $V_{\delta+2}$. We construct $M_1 \subseteq V_{\delta+2}$ such that $M_0 \prec M_1$ and M_1 is a δ -end-extension of M_0 (i.e. $M_1 \cap \delta = M_0 \cap \delta$). Without loss of generality we add a well-ordering to $V_{\delta+2}$. The set $M_0 \cap \mathcal{U}$ is countable,

hence $\bigcap(M_0 \cap \mathcal{U}) \neq \emptyset$. Let $\eta_0 = \min(\bigcap(M_0 \cap \mathcal{U}))$. We let M_1 be the Skolem-hull of $M_0 \cup \{\eta_0\}$ inside $V_{\delta+2}$. This is a δ -extension of M_0 . η_0 is $\min(M_1 \setminus M_0)$, by the normality of \mathcal{U} . We repeat this process ω_1 times. In the final model the order type of intersection with δ will have order type ω_1 .

When we now define M_2 from M_1 , we take care of self-genericity. Fix some $D \in M_1$, $D \subseteq Q_{<\delta}$, D dense, to be taken care of at this stage. Now we use the fact that δ is Woodin. Since δ is Woodin, there are unboundedly many cardinals $\rho < \delta$ such that D is ρ -semi-proper i.e. for every countable elementary substructure N of $V_{\delta+1}$, $\rho, D \in N$, there is a δ -end-extension of N that catches the anti-chain D . By elementarity, M_0 knows that there are unboundedly many such ρ . Therefore we can choose such $\rho_0 > \eta_0$ and find an extension M_2 of M_1 which is a ρ_0 -end-extension of M_1 and which catches the anti-chain D . We iterate this ω_1 times making sure that every dense set is taken care of.

Let $M' = \bigcup_{\alpha < \omega_1} M_\alpha$. Now $M' \cap \delta$ has order type ω_1 and M' is self-generic. In addition, we can make sure that cofinally many stages are closed under the chosen algebra. This proves the Claim.

We force with the set \mathcal{P} of stationary subsets of T . Let H be a generic ultrafilter for \mathcal{P} . Let N be the generic ultrapower $Ult(V, H)$. The model N is not well-founded (we identify the well-founded part with its transitive collapse). But it is well-founded up to and including $\delta + 2$. In this embedding $j(\omega_1)$ becomes δ . Because of self-genericity, we have generated a generic filter for $Q_{<\delta}$. Namely, consider

$$G = \{S \in Q_{<\delta} : \{p \in T : p \cap \text{Supp}(S) \in S\} \in H\}.$$

This is a generic filter for $Q_{<\delta}$, by self-genericity. The generic ultrapower $M = Ult(V, G)$ is well-founded. M is embedded into N , which is not well-founded. Typical member of M is the equivalence class of a function g , $\text{dom}(g)$ countable $X \in V_\delta$, into V . We lift this function to a function g^* defined by $g^*(p) = g(p \cap X)$. This defines an embedding of M into N . It is the identity on δ .

Claim: Let $S \subseteq \delta$ in M . If M thinks that S is stationary, then $V[G]$ thinks that S is stationary.

Proof of the Claim: Let τ be a name for club disjoint from S . Suppose some condition r in $Q_{<\delta}$ forces τ to be stationary in the inner model M but non-stationary in $V[G]$. We make a change to the definition of T as follows: We assume in the definition of T that $p \cap \text{Supp}(r) \in r$ and call the resulting set T_r . Same argument as above shows that T_r is stationary. We can think of T_r as a condition of the Woodin tower $P_{<\delta^*}$ (rather than $Q_{<\delta^*}$). We force with $P_{<\delta^*}$ below T_r getting a generic G^* .

We get generic ultrapower $M^* = Ult(V, G^*)$. From G^* we can now define a set \bar{G} as follows. Essentially, \bar{G} is the restriction of G^* to $Q_{<\delta}$, $\bar{G} = G^* \cap Q_{<\delta}$. Since we force below T_r , and by the definition of self-genericity, the set \bar{G} is generic and $V[G^*]$ contains $V[\bar{G}]$. Let us denote $\bar{M} = Ult(V, \bar{G})$. Now we have a canonical embedding $\pi : \bar{M} \rightarrow M^*$. The mapping π is identity on $\delta + 1$. In $V[G^*]$ the model M^* contains (by properties of stationary tower forcing) every bounded subset of the Woodin cardinal δ^* . Therefore every subset of δ in $V[G^*]$ belongs to M^* . Note that \bar{M} thinks S is stationary, because $r \in <\bar{G}$. As $S \subseteq \delta$ and π identity on $\delta + 1$, we have $\pi(S) = S$. By elementarity M^* thinks it (i.e. S) is stationary. But M^* and $V[G^*]$ agree about subsets of δ . Therefore $V[G^*]$ thinks S is stationary. Therefore $V[\bar{G}]$ thinks S is stationary, because $V[\bar{G}] \subseteq V[G^*]$. This contradicts the assumption about the condition r that it forces S to be non-stationary.

Now back to the proof of Theorem 7.4. Suppose $C(aa)$ is not club-determined. Let α be the least such that L'_α is not club-determined. First we collapse α to \aleph_1 with countable conditions. The forcing is countable closed, hence preserves $C(aa)$ and its construction. Therefore, after the collapse, L'_α is still the least counter-example, and now in the extension $\alpha < \omega_2$. We work from this on in the extension.

Pick δ measurable Woodin. We force with the usual $Q_{<\delta}$. We get G, M and $j : V \rightarrow M$. We claim that in the hierarchies $C(aa)^M, C(aa)^{V[G]}, C(aa_{<\delta})^V$ the first point where we get failure of finite determinacy is the same in all cases.

Suppose we have failure of finite determinacy at L'_α and the size of α , and hence of L'_α is \aleph_1 . M thinks that size of $j(\alpha)$ is ω_1^M i.e. δ . Hence M has a function h that maps one-to-one onto $(L'_{j(\alpha)})^M \rightarrow \delta$. In $L'_{j(\alpha)}$ we can define $S \subseteq (\mathcal{P}_{\omega_1}(L'_{j(\alpha)}))^M$ in the aa -logic over $L'_{j(\alpha)}$, and it is stationary co-stationary in M . Let us now look at $S^* = \{s^* : s \in S\}$, where $s^* = \{\eta < \delta : h^{-1}(\eta) \in s\}$. M thinks that the set S^* is an aa -definable stationary subset of δ (i.e. ω_1). Therefore $V[G]$ thinks that S^* is stationary. $V[G]$ knows about h , so $V[G]$ knows that we have failure of finite determinacy. Note that a club (of countable subsets of some set) in M is still a club in $V[G]$. The hierarchy L' up to $j(\alpha)$ in the sense of $V[G]$ is the same as in the sense of M . Thus we have a failure of finite determinacy in V for the $aa_{<\delta}$ -logic and it happens in the same stage. Why? The forcing $Q_{<\delta}$ preserves δ a cardinal, and as it is of size δ , it does not collapse any cardinal $\geq \delta$. So V also thinks that the size of $j(\alpha)$ is δ . Let us look at the construction of $C(aa_{<\delta})$ from the point of V . Up to the point $j(\alpha)$ we have in $V[G]$ finite determinacy. The aa -hierarchy in $V[G]$ up to $j(\alpha)$ is the same as the $aa_{<\delta}$ -hierarchy up to $j(\alpha)$ in V . For suppose at some point we get a different

answer. We translate it to the question whether some subset of δ is stationary or not. By the above claim, the answer is the same in V and $V[G]$.

Now, as under MM. $j(\alpha)$ is a definable stage. We know a priori, even though the embedding j is defined only after we know G , it is the first offending stage and we know what the offending set is. Now we can translate the offending set in V to a subset of δ . We ask whether ω_1 is in the set or not. We put it into the generic, and get a contradiction. So we cannot have an offending set. \square

Lemma 7.5. *Suppose δ is a regular cardinal, \mathbb{P} is a forcing notion such that $|\mathbb{P}| = \delta$, and G is \mathbb{P} -generic. If δ is still a regular cardinal in $V[G]$, then for all $N \in V$, the set $(\mathcal{P}_{<\delta}(N))^V$ is stationary in $(\mathcal{P}_{<\delta}(N))^{V[G]}$.*

Proof. Without loss of generality, N is an ordinal β . Suppose τ is a forcing term for an algebra on β . Let μ be a big enough regular cardinal. We build in V a chain $M_\alpha, \alpha < \delta$, of elementary substructures of H_μ^V in such a way that $\mathbb{P}, \tau, \beta \in M_0$, $|M_\alpha|^V < \delta$, $M_\nu = \bigcup_{\alpha < \nu} M_\alpha$ for limit ν , and $\mathbb{P} \subseteq \bigcup_{\alpha < \delta} M_\alpha$. Let G be \mathbb{P} -generic. Since δ is regular in $V[G]$, we can construct, in $V[G]$, an ordinal $\gamma < \delta$ such that if $D \subseteq \mathbb{P}$ is a dense set in M_γ , then $D \cap G \cap M_\gamma \neq \emptyset$. Now $M_\gamma \cap \beta \in V$ is closed under that value of τ in $V[G]$. \square

Lemma 7.6. *Suppose $C(aa)$ is club-determined, δ is Woodin, \mathbb{P} is the countable stationary tower, $G \subseteq \mathbb{P}$ is generic and M is the associated generic ultrapower. Then $C(aa)^M = C(aa_{<\delta})^V$.*

Proof. Recall that $j(\omega_1) = \delta$ and $V[G] \models M^\omega \subseteq M$. Let (L'_α) be the hierarchy generating $C(aa)^M$ and (L''_α) the hierarchy generating $C(aa_{<\delta})^V$. We prove by induction on α that $(L'_\alpha)^M = (L''_\alpha)^V$. We show now that for all α we have $(L'_\alpha)^M = L''_\alpha$. Let us assume this holds up to α and we then consider $(L'_{\alpha+1})^M = L''_{\alpha+1}$. Let $N = L''_\alpha (= L'_\alpha)^M \in V$. We prove by induction on the formula $\varphi(P_0, \dots, P_n)$ that for countable sets P_0, \dots, P_n in M :

$$(N \models \varphi(P_0, \dots, P_n))^M \iff (N \models \varphi(P_0, \dots, P_n))^V,$$

when the aa-quantifier is interpreted as $aa_{\leq \aleph_0}$ in M and as $aa_{<\delta}$ in V . Note that the proof of Theorem 7.4 shows that also $C(aa_{<\delta})$ is club-determined. Suppose $\varphi(P_0, \dots, P_n)$ is a formula with second order variables P_0, \dots, P_n . We show that the following are equivalent for any subsets P_1, \dots, P_n of N in V , which are countable in M :

- (i) M satisfies “The set B of countable subsets P_0 of N satisfying the formula $\varphi(P_0, P_1, \dots, P_n)$ in N is stationary”.
- (ii) V satisfies “The set C of subsets P_0 of N of cardinality $< \delta$ satisfying the formula $\varphi(P_0, P_1, \dots, P_n)$ in N is stationary”.

Assume first (ii). By club-determinacy the set C contains a club H of sets of cardinality $< \delta$ in V . If (i) is false, then, by club-determinacy, the set B is disjoint from a club K of countable sets in M . The set K is still club in $V[G]$. By Lemma 7.5, $H \cap B \neq \emptyset$, contradicting the Induction Hypothesis.

Assume then (i). By club-determinacy the set B contains a club K of countable sets in M . The set K is still club in $V[G]$. If (ii) is false, then, again by club-determinacy, the set C is disjoint from a club H of sets of cardinality $< \delta$ in V . By Lemma 7.5, $H \cap B \neq \emptyset$, contradicting the Induction Hypothesis. □

Theorem 7.7. *Suppose there are a proper class of measurable Woodin cardinals. Then the first order theory of $C(aa)$ is (set) forcing absolute.*

Proof. Suppose \mathbb{P} is a forcing notion and δ is a Woodin cardinal $> |\mathbb{P}|$. Let $j : V \rightarrow M$ be the associated elementary embedding. By Lemma 7.6 we can argue

$$C(aa) \equiv (C(aa))^M = (C(aa_{<\delta}))^V.$$

On the other hand, let $H \subseteq \mathbb{P}$ be generic over V . Then δ is still Woodin, so we have the associated elementary embedding $j' : V[H] \rightarrow M'$. By Lemma 7.6 we can again argue

$$(C(aa))^{V[H]} \equiv (C(aa))^{M'} = (C(aa_{<\delta}))^{V[H]}.$$

Finally, we may observe that $(C(aa_{<\delta}))^{V[H]} = (C(aa_{<\delta}))^V$. Hence

$$(C(aa))^{V[H]} \equiv (C(aa))^V.$$

□

Theorem 7.8. *Suppose there are a proper class of measurable Woodin cardinals. Then every regular $\kappa \geq \aleph_1$ is measurable in $C(aa)$.*

Proof. The basic case is ω_1 . The above proof shows that ω_1 is measurable in $C(aa)$. For an arbitrary regular cardinal we first collapse it to ω_1 and then use the basic case. □

Corollary. *Suppose there is a supercompact cardinal. Then every regular $\kappa \geq \aleph_1$ is measurable in $C(aa)$.*

Proof. Suppose κ is supercompact. Let α be the least such that L'_α is not club-determined. Since $V_\kappa \prec_2 V$, we can assume $\alpha < \kappa$. Now we can proceed as above to show that ω_1 is measurable in $C(aa)$. If there is a regular cardinal λ which is not measurable in $C(aa)$, there is one, again by $V_\kappa \prec_2 V$, below κ . If we collapse λ to ω_1 , κ remains supercompact. So we can argue as above to prove that λ is measurable in $C(aa)$. \square

Now we shall proceed to proving that the conclusion of Theorem 7.8 is also a consequence of a strong form of Martin's Maximum.

Definition 7.9 (Martin's Maximum⁺⁺). (MM⁺⁺) is the statement that for every stationary set preserving forcing \mathbb{P} , any sequence $\langle D_\alpha : \alpha < \omega_1 \rangle$ of dense open subsets of \mathbb{P} and any \mathbb{P} -terms τ_α , $\alpha < \omega_1$, such that

$$\mathbb{P} \vdash \tau_\alpha \text{ is a stationary subset of } \omega_1$$

there is a filter $F \subseteq \mathbb{P}$ which meets every D_α and for which each set $\{\xi : \exists p \in F(p \Vdash \xi \in \tau_\alpha)\}$ is stationary in ω_1 . The ordinary Martin's Maximum MM is MM⁺⁺ without the terms τ_α .

Theorem 7.10. *Assume MM⁺⁺. Then every regular $\kappa \geq \aleph_1$ is measurable in $C(aa)$.*

We first show that if MM⁺⁺ is assumed, then \aleph_1^V is measurable in $C(aa)$. Then Theorem 7.10 follows from Lemma 7.13 below.

Lemma 7.11. *Suppose κ is a regular cardinal $> \omega$ which is not measurable in $C(aa)$. Suppose $A \in C(aa)$ is a V -stationary set of ω -cofinal ordinals $< \kappa$. Then A can be split into two V -stationary subsets, both in $C(aa)$.*

Proof. Otherwise there is $A \subseteq \kappa$ such that $A \in C(aa)$, $\forall \alpha \in A(\text{cf}^V(\alpha) = \omega)$, A is stationary on κ in V , and A cannot be split into two sets A_0 and A_1 , both in $C(aa)$ and both stationary in V . Consider $\mathcal{U} = \mathcal{F}^\omega(A) \cap C(aa)$. We already know that $\mathcal{U} \in C(aa)$. Since $\mathcal{F}^\omega(\kappa)$ is $< \kappa$ -complete in V , \mathcal{U} is $< \kappa$ -complete in $C(aa)$. By our assumption about A , \mathcal{U} is a $< \kappa$ -complete ultrafilter on κ in $C(aa)$, contrary to the assumption. \square

Lemma 7.12. *Suppose κ is a regular cardinal $> \omega$ which is not measurable in $C(\text{aa})$. Suppose $2^\lambda < \kappa$. Then there is a set $\{A_\alpha : \alpha < \lambda\} \in C(\text{aa})$ of disjoint V -stationary subsets of $\{\alpha < \kappa : \text{cf}(\alpha) = \omega\}$.*

Proof. We construct in $C(\text{aa})$ a tree $\{A_s : s \in {}^\lambda 2\}$ of V -stationary subsets of $\{\alpha < \kappa : \text{cf}(\alpha) = \omega\}$ as follows. Let $A_\emptyset = \{\alpha < \kappa : \text{cf}(\alpha) = \omega\}$. If A_s is defined and is V -stationary, use Lemma 7.11 to get A_{s0} and A_{s1} . Otherwise $A_{s0} = A_{s1} = A_s$. For limit ν we let $A_{s|\nu} = \bigcap_{\alpha < \nu} A_{s|\alpha}$. Clearly, $A_\emptyset = \bigcup_{s: \lambda \rightarrow 2} A_s$. Choose $s : \lambda \rightarrow 2$ such that A_s is V -stationary (recall that $2^\lambda < \kappa$). Now each $A_{s|\alpha}$, $\alpha < \lambda$, is V -stationary. Hence both $A_{s|\alpha \cap s(\alpha)}$ and $A_{s|\alpha \cap (1-s(\alpha))}$ are V -stationary. Now the family $\{A_{s|\alpha \cap (1-s(\alpha))} : \alpha < \lambda\}$ is a family of λ disjoint V -stationary sets. The whole construction can be carried out in $C(\text{aa})$. \square

Lemma 7.13. *If MM holds and there is a non- $C(\text{aa})$ real number, then every regular $\kappa > \aleph_1$ is measurable in $C(\text{aa})$.*

Proof. Suppose $X \subseteq \omega$ but $\kappa > \aleph_1$ is not measurable in $C(\text{aa})$. As in the proof of Lemma 7.12, we can construct disjoint V -stationary subsets $\langle A_n : n < \omega \rangle \in C(\text{aa})$ of $\{\alpha < \kappa : \text{cf}(\alpha) = \omega\}$. We now use the following result:

Proposition 7.14 (Foreman-Magidor-Shelah 1988). *Assume MM. Suppose κ is a regular cardinal $> \aleph_1$ and $\langle A_n : n < \omega \rangle$ are disjoint stationary subsets of $\{\alpha < \kappa : \text{cf}(\alpha) = \omega\}$. Then for all $X \subseteq \omega$ there is a $\beta < \kappa$ such that*

$$X = \{n \in \omega : A_n \cap \beta \text{ is stationary in } \beta\}.$$

This Proposition gives a $\beta < \kappa$ such that

$$X = \{n \in \omega : A_n \cap \beta \text{ is } V\text{-stationary in } \beta\}.$$

But then $X \in C(\text{aa})$ and Lemma 7.13 is proved. \square

For the proof of Theorem 7.10, that is, for the proof that \aleph_1^V is measurable in $C(\text{aa})$, we will show that every subset of \aleph_1^V in $C(\text{aa})$ contains a club in $C(\text{aa})$ or is disjoint from a club in $C(\text{aa})$. Here is first a rough outline of the proof. Suppose there is $S \subseteq \omega_1$ in $C(\text{aa})$ which is bi-stationary in V . Pick S so that it is the first such in the canonical construction of $C(\text{aa})$. We embed V into a generic ultrapower M by an embedding which moves ω_1 to ω_2 . In this embedding $C(\text{aa})$ becomes, from the perspective of V , the inner model⁵ $C(\text{aa}_{\leq \aleph_1})$, at least up to

⁵The quantifier $\text{aa}_{\leq \aleph_1}$ is defined like aa by reference to a club of sets of size $\leq \aleph_1$ rather than a club of countable sets.

levels below ω_2^M . We further embed M into another generic ultrapower N by an embedding which is the identity on ω_2^V . Since ω_1^V is countable in M it is either in $j(S)$ or in $\omega_2^V \setminus j(S)$. We argue that in the first case $j(S)$ contains a club and in the second case $\omega_2^V \setminus j(S)$ contains a club. Both cases contradict the fact that $j(S)$ is bi-stationary in N .

Lemma 7.15. *MM⁺⁺ implies that every subset of ω_1^V which is in $C(\text{aa})$ is already in L'_α for some $\alpha < \omega_2^V$.*

Proof. Suppose $X \subseteq \omega_1^V$ and $X \in L'_\alpha$. Let \mathbb{P} be the Levy collapse of $|\alpha|$ to \aleph_1^V . This forcing is countably closed so it does not change $C(\text{aa})$, so

$$\mathbb{P} \Vdash \exists \alpha (\check{X} \in L'_\alpha \wedge |\alpha| \leq \aleph_1).$$

By MM⁺⁺ we can eliminate the forcing. We use the “++” of MM⁺⁺ to keep \aleph_1 definable sets, forced by \mathbb{P} to be stationary, also stationary in the application of MM⁺⁺. In this way we obtain

$$\exists \alpha (X \in L'_\alpha \wedge |\alpha| \leq \aleph_1).$$

□

Let \mathbb{P} be the po-set of stationary sets $\subseteq \omega_1$ ordered by \supseteq . Let $G \subseteq \mathbb{P}$ be generic. Consider the generic ultrapower V^{ω_1}/G . Martin’s Maximum implies that NS_{ω_1} is \aleph_2 -saturated [11]. By \aleph_2 -saturation this model is well-founded. Let $V^{\omega_1}/G \cong M$, where M is transitive. Let $j : V \rightarrow M$ (in $V[G]$) be an elementary embedding such that the critical point of j is ω_1 . It is easy to see that $j(\omega_1^V)(= \omega_1^M) = \omega_2^V$.

Let $(L'_\alpha)_{\alpha \in \text{On}}$ denote the hierarchy used to build $C(\text{aa})$, and $(L''_\alpha)_{\alpha \in \text{On}}$ the hierarchy used to build $C(\text{aa}_{\leq \aleph_1})$. We now show that $C(\text{aa})$ inside the inner model M of $V[G]$ looks like $C(\text{aa}_{\leq \aleph_1})$ in V , at least for levels up to ω_2^M .

Lemma 7.16. *Suppose \mathbb{P} is a notion of forcing with \aleph_2 -c.c. and G is \mathbb{P} -generic. Suppose δ is an ordinal such that $|\delta|^V = \aleph_2$. Then the set $\{x \subseteq \delta : x \text{ is countable in } V\}$ contains a club in $V[G]$ of sets in V .*

Lemma 7.17. *If a set of countable subsets of ω_2 in M is stationary in M , it is a stationary set of subsets of ω_2 of size $\leq \aleph_1$ in V .*

Proof. Thus X_b is a stationary set of countable sets in M but Y_b is non-stationary in V as a set of sets of size $\leq \aleph_1$. Without loss of generality we may assume that X_b is a set of countable subsets of α in M . There is a one-one function $f \in M$ such that $M \models f : \alpha \rightarrow \omega_1 (= \omega_2^V)$. The function f translates the set X_b of countable subsets of α to a stationary set S of countable subsets of ω_1^M (i.e. of ω_2^V). We will now produce a model N , possibly non-well-founded, and an elementary embedding $k : M \rightarrow N$ in $V[G]$. We construct N so that $k \upharpoonright \omega_2^M = id$ and $\mathcal{P}(\omega_2^V) \subseteq N$. This will yield a contradiction as follows: S is stationary in M , hence $k(S)$ is stationary in N . But k is the identity on ω_2 . So $k(S) = S$. Thus S is stationary in N . But the club set in V that misses S is in N , a contradiction.

We now construct N . Let us fix some large $\kappa > \alpha$. Let E be the set of subsets p of κ of size ω_1 such that $p \cap \omega_2$ has order type ω_1 . We consider the set \mathcal{S} of stationary subsets of E . Chang's Conjecture, a consequence of MM by [11], gives non-trivial clubs on E , for it gives for every algebra on κ a club of $p \in E$ which are closed under the algebra. Let $G^* \subseteq \mathcal{S}$ be generic. We consider the generic ultrapower V^E/G^* . We collapse the well-founded part of this possibly non-well-founded model. Denote by N the resulting model. The cardinal κ is in N , for any ordinal $\alpha < \kappa$ is represented in N by the function $C_\alpha(p) = \text{otp}(p \cap \alpha)$ for $p \in E$. Such elements form an initial segment of N and are well-founded. We show that G^* generates a generic for \mathbb{P} as follows:

$$\forall A \subseteq \omega_1 (A \in G \iff \{p \in E : p \cap \omega_1 \in A\} \in G^*).$$

It can be shown that this G is generic for \mathbb{P} . Without loss of generality we can assume that this was our original G . Working in $V[G^*]$ we now define the embedding $k : M \rightarrow N$. Take an element $[f]$ of M , where $f : \omega_1 \rightarrow V$. We lift f to E by $\bar{f}(p) = f(p \cap \omega_1)$. If $f \sim g$, then $\bar{f} \sim \bar{g}$. Finally we define $k([f]) = [\bar{f}]$.

The mapping k is the identity on ω_2 : Let $\alpha < \omega_2$. There is a well-ordering of ω of order-type α in M . But k is the identity on ω , hence the identity on this well-ordering. Thus $k(\alpha) = \alpha$. We can note that $k(\omega_2^V) = \omega_2^V$ as follows: By elementarity $k(\omega_2^V) = k(\omega_1^M) = \omega_1^N$. Let us check that $\omega_1^N = \omega_2^V$. A typical countable ordinal in N is represented by a function $f : E \rightarrow \omega_1$. By the Pressing Down Lemma there is $\alpha < \omega_2$ such that $f \sim_{G^*} g$, where $g(p) = \text{otp}(p \cap \alpha)$. □

Lemma 7.18. *MM implies that $(L'_\alpha)^M = L''_\alpha$ for $\alpha < \omega_2^M$.*

Proof. We show now that for all $\alpha < \omega_2^M$ we have $(L'_\alpha)^M = L''_\alpha$. Let us assume this holds up to α and we then consider $(L'_{\alpha+1})^M = L''_{\alpha+1}$. Let $N = L''_\alpha (=$

$L'_\alpha)^M \in V$. We prove by induction on the formula $\varphi(P_0, \dots, P_n)$ that for countable sets P_0, \dots, P_n in M :

$$(N \models \varphi(P_0, \dots, P_n))^M \iff (N \models \varphi(P_0, \dots, P_n))^V,$$

when the aa -quantifier is interpreted as $aa_{\leq \aleph_0}$ in M and as $aa_{\leq \aleph_1}$ in V . Suppose $\varphi(P_0, \dots, P_n)$ is a formula with second order variables P_0, \dots, P_n . For simplicity we assume $n = 0$. We show that the following are equivalent:

- (i) M satisfies “The set B of countable subsets P of N satisfying $\varphi(P)$ in N is stationary”.
- (ii) V satisfies “The set C of subsets P of N of cardinality \aleph_1 satisfying $\varphi(P)$ in N is stationary”.

Assume first (ii). Suppose $g : \omega_1 \rightarrow N$ is onto. Then $\{\alpha < \omega_2 : g^{\alpha} \in C\}$ is stationary. Since our forcing is \aleph_2 -c.c., this set is stationary also in $V[G]$, hence in M . Note that in M the sets g^{α} are countable. By induction hypothesis this implies (i).

Assume then (i). By Lemma 7.17 the set B is a stationary set of sets of size \aleph_2 in $V[G]$. By Lemma 7.16 there is in $V[G]$ a club $E \subseteq V$ of subsets of ω_2 . Now $E \cap B$ is stationary. By Induction Hypothesis, $E \cap B \subseteq C$. Hence (ii) follows. \square

The above essentially proves:

Theorem 7.19. *Assume MM^{++} . Then $C(aa)$ is club-determined.*

Finishing the proof of Theorem 7.10: We show that in the construction of $C(aa)$ we can never get a stationary co-stationary set $S \subseteq \omega_1^V$. Suppose we do. Consider the first time you generate such a set. The embedding j moves this to a stage that M thinks is the first stage where such a set is constructed in $C(aa)$. This set $j(S)$ is now a subset of ω_2^V . The old ω_1 is a countable ordinal in M . It either belongs to $j(S)$ or to its complement. Suppose it is in the set. In this case the original set must contain a club. If it is in $j(S)$, then the complement of the original set S contains a club. In either case we obtain a contradiction. \square

8 Variants of stationary logic

There are several variants of stationary logic. The earliest version of stationary logic is based on the following quantifier introduced in [38]:

If \mathcal{A} is a linear order, let $H(\mathcal{A})$ be the set of all initial segments of \mathcal{A} , and $D(\mathcal{A})$ the filter generated by closed unbounded sets of initial segments.

Definition 8.1. $\mathcal{M} \models Q^{St}xyz\varphi(x, \vec{a})\psi(y, z, \vec{a})$ if and only if (M_0, R_0) , where $M_0 = \{b \in M : \mathcal{M} \models \varphi(b, \vec{a})\}$ and $R_0 = \{(b, c) \in M : \mathcal{M} \models \psi(b, c, \vec{a})\}$ is an \aleph_1 -like linear order and the set \mathcal{I} of initial segments of (M_0, R_0) with an R_0 -supremum in M_0 is stationary in the set \mathcal{D} of all (countable) initial segments of M_0 in the following sense: If $\mathcal{J} \subseteq \mathcal{D}$ is unbounded in \mathcal{D} (i.e. $\forall x \in \mathcal{D} \exists y \in \mathcal{J} (x \subseteq y)$) and σ -closed in \mathcal{D} (i.e. if $x_0 \subseteq x_1 \subseteq \dots$ in \mathcal{J} , then $\bigcup_n x_n \in \mathcal{J}$), then $\mathcal{J} \cap \mathcal{I} \neq \emptyset$.

The logic $\mathcal{L}(Q^{St})$, a sublogic of $\mathcal{L}(aa)$, is recursively axiomatizable and \aleph_0 -compact [38]. We call this logic *Shelah's stationary logic*, and denote $C(\mathcal{L}(Q^{St}))$ by $C(aa^-)$. For example, we can say in the logic $\mathcal{L}(Q^{St})$ that a formula $\varphi(x)$ defines a stationary (in V) subset of ω_1 in a transitive model M containing ω_1 as an element as follows:

$$M \models \forall x(\varphi(x) \rightarrow x \in \omega_1) \wedge Q^{St}xyz\varphi(x)(\varphi(y) \wedge \varphi(z) \wedge y \in z).$$

Hence

$$C(aa^-) \cap \text{NS}_{\omega_1} \in C(aa^-)$$

and this in fact suffices to characterise $C(aa^-)$ completely.

Lemma 8.2. $C(aa^-) = L[D]$, where $D = C(aa^-) \cap \text{NS}_{\omega_1}$. In particular, $C(aa^-) \subseteq C(aa)$.

Theorem 8.3. If there is a Woodin cardinal, then $C(aa^-) = L[D]$, where $D = C(aa^-) \cap \text{NS}_{\omega_1}$ is an ultrafilter in $C(aa^-)$. In particular, $C(aa^-) \models GCH$.

Proof. Suppose λ is Woodin. Let \mathbb{P} be the (countable) stationary tower forcing such that if G is \mathbb{P} -generic then in $V[G]$ there is an elementary embedding $j : V \rightarrow M$ such that $j(\omega_1) = \lambda$. Let \mathcal{F} be the club filter on \aleph_1 . Thus $C(aa^-) = L(\mathcal{F})$. Let us try to compute $L(\mathcal{F}^M)$ in M , where \mathcal{F}^M is now the club filter on $j(\omega_1) = \lambda$ in M . We show that $L(\mathcal{F}^M) = L(\mathcal{U})^V$, where \mathcal{U} is the club filter on λ in V . For this end we first prove:

Main claim: Suppose $S \subseteq \lambda$, $S \in M \cap V$ and S is stationary in M . Then $V[G] \models "S \text{ is stationary}"$.

Suppose D is a maximal anti-chain in \mathbb{P} . We say that a set Q catches D if there is $q \in D \cap Q$ such that $Q \cap \cup q \in q$. Suppose η is an inaccessible cardinal

$< \lambda$. We say that D is *semi-proper below* η if the following holds: Let $sp(D)$ be the set of $P \in \mathcal{P}_{\omega_1}(V_{\eta+1})$ such that there is $Q \leq_{end} P$ below η (i.e. $Q \supseteq P$ and $Q \cap V_\eta = P$), Q is closed under the functions $f : V_\eta \rightarrow V_\eta$ that P knows about i.e. $f \in P$ and Q catches D . D is *semi-proper below* η if the set $sp(D)$ contains a club of $\mathcal{P}_{\omega_1}(V_{\lambda+1})$.

Lemma 8.4. [22] *For every maximal anti-chain D in \mathbb{P} there are unboundedly many inaccessible $\eta < \lambda$ such that D is semi-proper below η .*

Suppose S is not stationary in $V[G]$. Then there is a \mathbb{P} -term τ for a club disjoint from S . For every $\alpha < \lambda$ consider the maximal anti-chain D_α of conditions which force an ordinal $> \alpha$ into the club. Take an elementary substructure N_0 of a large $H(\kappa)$ such that $\langle D_\alpha : \alpha < \lambda \rangle$ and other relevant elements of the proof are in N_0 , and $\delta = \sup(N_0 \cap On) \in S$, where $\delta < \lambda$. We find a condition which forces δ into the club τ . This contradicts the assumption that τ is disjoint from S . Let T be the set of $P \in \mathcal{P}_{\omega_1}(V_\delta)$ such that for all $\alpha \in P$, P catches D_α .

Claim: $T \in \mathbb{P}$.

Let A be an algebra on V_δ . Assume the functions of the algebra are closed under composition. We find an element of T in this algebra. Take P , a countable elementary substructure of a large $H(\kappa)$ containing all relevant elements of the proof. Choose $\alpha_0 \in P$. There is $\eta_0 > \alpha_0$, $\eta_0 \in P$, such that D_{α_0} is semi-proper below η_0 . Hence there is an end-extension P_1 of P below η_0 , closed under the relevant functions. Choose α_1 and choose $\eta_1 > \alpha_1$ so that D_{α_1} is semi-proper below η_1 . Dovetailing in this way one finally gets P' in the club A such that all ordinals α of P' , P' catches D_α . Claim is proved and $T \in \mathbb{P}$.

Claim: $T \Vdash \tau$ is unbounded below δ , hence $T \Vdash \delta \in \tau$.

Suppose $T' \leq T$ and T' forces that some $\alpha < \delta$ is such that $T' \Vdash \tau \cap \delta \subseteq \alpha$. W.l.o.g. α is a member of every element in T' . Each set in T' catches D_α below δ , i.e.

$$\forall P \in T' \exists q_P \in D_\alpha \cap P (P \cap (\cup q_P) \in q_P).$$

By Fodor's Lemma we can find $T'' \leq T'$ such that q_P is the same q for all $P \in T''$. Now $T'' \leq q$ and $q \in D_\alpha$. By the definition of D_α , q forces some ordinal above α to be in τ , a contradiction.

The Main Claim is proved. Now $L(\mathcal{F}^M) = L(\mathcal{U})^V$ follows by induction on the construction of the inner model, as membership in \mathcal{U} is checked only for subsets of λ which are in M , and therefore the Main Claim applies.

Next we show that $\mathcal{F}' = \mathcal{F} \cap C(\text{aa}^-)$ measures every set in $C(\text{aa}^-)$. Let us consider $L(\mathcal{F}')^V$ and assume \mathcal{F}' does *not* measure every subset of ω_1^V . Take a minimal set $B \subseteq \omega_1$ in $C(\text{aa}^-)$ such that $B \notin \mathcal{F}'$ and $\omega_1^V \setminus B \notin \mathcal{F}'$. Thus B is bstationary in V . Suppose G_1 is \mathbb{P} -generic such that $B \in G_1$. Suppose G_2 is \mathbb{P} -generic such that $\omega_1 \setminus B \in G_2$. We obtain elementary embeddings $j_1 : V \rightarrow M_1$ and $j_2 : V \rightarrow M_2$ such that

$$\begin{aligned}\omega_1 &\in j_1(B) \\ \omega_1 &\in j_2(\omega_1 \setminus B) \\ j_1 : (L(\mathcal{F}))^V &\rightarrow (L(\mathcal{F}^{M_1}))^{M_1} = L(\mathcal{U}) \\ j_2 : (L(\mathcal{F}))^V &\rightarrow (L(\mathcal{F}^{M_2}))^{M_2} = L(\mathcal{U}).\end{aligned}$$

Now by elementarity $j_1(B)$ is in M_1 the first subset of μ not measured by \mathcal{F}^{M_1} and $j_2(B)$ is in M_2 the first subset of μ not measured by \mathcal{F}^{M_2} . By (4) above, both $j_1(B)$ and $j_2(B)$ is the first subset of μ not measured by \mathcal{U} . Hence $j_1(B) = j_2(B)$. This contradicts the fact that $\omega_1 \in j_1(B) \cap \mu \setminus j_2(B)$. \square

Theorem 8.5. *If there is a proper class of Woodin cardinals, then for all set forcings P and generic sets $G \subseteq P$*

$$\text{Th}(C(\text{aa}^-)^V) = \text{Th}(C(\text{aa}^-)^{V[G]}).$$

Proof. Let G be P -generic. Let us choose a Woodin cardinal $\lambda > |P|$. Let \mathcal{U} be the club-filter on λ in V . Let H_1 be generic for the countable stationary tower forcing and $j_1 : V \rightarrow M_1$ the generic embedding with $j_1(\omega_1) = \lambda$. As in the proof of Theorem 8.3,

$$j_1 : C(\text{aa}^-)^V \rightarrow C(\text{aa}^-)^{M_1} = L(\mathcal{U})^V.$$

and therefore by elementarity $C(\text{aa}^-)^V \equiv L(\mathcal{U})^V$.

Since $|P| < \lambda$, λ is still Woodin in $V[G]$. Let H_2 be generic for the countable stationary tower forcing over $V[G]$ and $j_2 : V[G] \rightarrow M_2$ the generic embedding with $j_2(\omega_1) = \lambda$. Hence, as in the proof of Theorem 8.3,

$$j_2 : C(\text{aa}^-)^{V[G]} \rightarrow C(\text{aa}^-)^{M_2} = L(\mathcal{U})^{V[G]} = L(\mathcal{U})^V$$

and therefore by elementarity $C(\text{aa}^-)^{V[G]} \equiv L(\mathcal{U})^V$. In the end, $C(\text{aa}^-)^V \equiv C(\text{aa}^-)^{V[G]}$. \square

Proposition 8.6. *If $0^\#$ exists, then $0^\# \in C(\text{aa}^-)$.*

Proof. Assume $0^\#$. A first order formula $\varphi(x_1, \dots, x_n)$ holds in L for an increasing sequence of indiscernibles below ω_1^V if and only if there is a club C of ordinals $< \omega_1^V$ such that every increasing sequence $a_1 < \dots < a_n$ from C satisfies $\varphi(a_1, \dots, a_n)$ in L . Similarly, $\varphi(x_1, \dots, x_n)$ does not hold in L for an increasing sequence of indiscernibles below ω_1^V if and only if there is a club of ordinals $a_1 < \omega_1$ such that there is a club of ordinals a_2 with $a_1 < a_2 < \omega_1$ such that \dots such that there is a club of ordinals a_n with $a_{n-1} < a_n < \omega_1$ satisfying $\neg\varphi(a_1, \dots, a_n)$. From this it follows that $0^\# \in C(\text{aa}^-)$. \square

Theorem 8.7. *It is consistent relative to the consistency of ZFC that*

$$C^* \not\subseteq C(\text{aa}^-) \wedge C(\text{aa}^-) \not\subseteq C^*.$$

Proof. First we add two Cohen reals r_0 and r_1 , to obtain V_1 . Now we use modified Namba forcing (see Section 6) to make $\text{cf}(\aleph_{n+1}^L) = \omega$ if and only if $n \in r_0$. This forcing satisfies the S -condition (see Section 6), and therefore, since we have CH, will not—by [11]—kill the stationarity of any stationary subset of ω_1 . The argument is essentially the same as for Namba forcing. Let the extension of V_1 by \mathbb{P} be V_2 . In V_2 we have $C(\text{aa}^-) = L$ because we have not changed stationary subsets of ω_1 . But $V_2 \models r_0 \in C^*$.

Let S_n , $n < \omega$, be in L a definable sequence of disjoint stationary subsets of ω_1 such that $\bigcup_n S_n = \omega_1$. Working in V_2 , we use the canonical forcing notion which kills the stationarity of S_n if and only if $n \in r_1$. Let the resulting model be V_3 . The cofinalities of ordinals are the same in V_0 and V_3 , whence $(C^*)^{V_0}$ is the same as $(C^*)^{V_3}$. Thus $V_3 \models r_1 \in C(\text{aa}^-) \setminus C^*$. Now we argue that $V_3 \models C(\text{aa}^-) = L(r_1)$. First of all, $L(r_1) \subseteq C(\text{aa}^-)$ by the construction of V_3 . Next we prove by induction on the construction of $C(\text{aa}^-)$ as a hierarchy L'_α , $\alpha \in On$, that $L'_\alpha \subseteq L(r_1)$. When we consider $L'_{\alpha+1}$ and assume $L'_\alpha \subseteq L(r_1)$, we have to decide whether a subset of ω_1 , constructible from r_1 , is stationary or not. But this information is written into r_1 . Thus $L'_{\alpha+1} \subseteq L(r_1)$. \square

The logics $\mathcal{L}(Q_\omega^{\text{cf}})$, giving rise to C^* , and $\mathcal{L}(\text{aa}^-)$, giving rise to $C(\text{aa}^-)$, are two important logics, both introduced by Shelah. Since $\mathcal{L}(Q_\omega^{\text{cf}})$ is *fully compact*, $\mathcal{L}(\text{aa}^-)$ cannot be a sub-logic of it. On the other hand, it is well-known and easy to show that $\mathcal{L}(Q_\omega^{\text{cf}})$ is a sub-logic of $\mathcal{L}(\text{aa})$. Therefore it is interesting to note the following corollary:

Corollary. *It is consistent, relative to the consistency of ZFC, that $\mathcal{L}(Q_\omega^{\text{cf}}) \not\subseteq \mathcal{L}(Q^{\text{St}})$ and hence $\mathcal{L}(Q^{\text{St}}) \neq \mathcal{L}(\text{aa})$.*

We do not know whether it is consistent that $\mathcal{L}(Q_\omega^{\text{cf}}) \subseteq \mathcal{L}(\text{aa}^-)$ or that $\mathcal{L}(\text{aa}^-) = \mathcal{L}(\text{aa})$.

A modification of $C(\text{aa}^-)$ is the following $C(\text{aa}^0)$:

Definition 8.8. $\mathcal{M} \models Q^{St,0}xyzu\varphi(x, y, \vec{a})\psi(u, \vec{a})$ if and only if $M_0 = \{(b, c) \in M : \mathcal{M} \models \varphi(b, c, \vec{a})\}$ is a linear order of cofinality ω_1 and every club of initial segments has one with supremum in $R_0 = \{b \in M : \mathcal{M} \models \psi(b, \vec{a})\}$. The inner model $C(\text{aa}^0)$ is defined as $C(\mathcal{L}(Q^{St,0}))$.

Proposition 8.9. *If there is a Woodin cardinal, then $C(\text{aa}^0) \models \aleph_1^V$ is a measurable cardinal.*

Proof. The proof of this is—*mutatis mutandis*—as in the proof for $C(\text{aa}^-)$. □

Proposition 8.10. *If 0^\dagger exists, then $0^\dagger \in C(\text{aa}^0)$.*

Proof. Assume 0^\dagger . There is a club class of indiscernibles for the inner model $L(U)$ where U is (in $L(U)$) a normal measure on an ordinal δ . Let us choose as indiscernible α above δ of V -cofinality ω_1^V . We can define 0^\dagger as follows: An increasing sequence of indiscernibles satisfies a given formula $\varphi(x_1, \dots, x_n)$ if and only if there is a club C of ordinals below α such that every increasing sequence $a_1 < \dots < a_n$ from C satisfies $\varphi(a_1, \dots, a_n)$ in $L(U)$. Similarly, $\varphi(x_1, \dots, x_n)$ does not hold in $L(U)$ for an increasing sequence of indiscernibles below α if and only if there is a club of ordinals $a_1 < \alpha$ such that there is a club of ordinals a_2 with $a_1 < a_2 < \alpha$ such that \dots such that there is a club of ordinals a_n with $a_{n-1} < a_n < \alpha$ satisfying $\neg\varphi(a_1, \dots, a_n)$. From this it follows that $0^\dagger \in C(\text{aa}^0)$. □

Corollary. *If there is a Woodin cardinal, then $C(\text{aa}^-) \neq C(\text{aa}^0)$. Then also the logics $\mathcal{L}(Q^{St})$ and $\mathcal{L}(Q^{St,0})$ are non-equivalent.*

Proof. If there is a Woodin cardinal, then 0^\dagger exists and $C(\text{aa}^-)$ does not contain 0^\dagger by Theorem 8.3, while $C(\text{aa}^0)$ does contain by the above Proposition. □

Note that it is probably possible to prove the non-equivalence of $\mathcal{L}(Q^{St})$ and $\mathcal{L}(Q^{St,0})$ in ZFC with a model theoretic argument using the exact definitions of the logics and by choosing the structures very carefully. But the non-equivalence result given by the above Corollary is quite robust in the sense it is not at all sensitive to the exact definitions of the logics as long as the central separating feature, manifested in structures of the form $(\alpha, <)$, is respected.

9 The H\"artig-quantifier

The H\"artig-quantifier $Ixy\varphi(x, \vec{z})\psi(y, \vec{z}) \iff |\{a : \varphi(a, \vec{z})\}| = |\{a : \psi(a, \vec{z})\}|$ is an interesting case to study, because if $V = L$, then $\mathcal{L}(I)$ and \mathcal{L}^2 are Δ -equivalent (in the sense of [28]) but on the other hand $\mathcal{L}(I)$ can be quite weak, for example its decision problem may be Δ_3^1 and its L\"owenheim-number can be $< 2^\omega$ ([45]).

Let us denote $C(\mathcal{L}(I))$ by $C(I)$. The logic $L(I)$ is adequate to truth in itself, hence $C(I)$ satisfies AC. Note that by forcing with cardinal-preserving forcing one preserves $C(I)$. Therefore $C(I)$ is neither provably contained nor provably contains C^* or $C(\text{aa})$.

Note also that if 0^\sharp exists, then $0^\sharp \in C(I)$ because we can use the quantifier I to say of an ordinal $\alpha < \gamma$ that it is a cardinal, as follows:

$$(L'_\gamma, \in) \models \forall z(z \in \alpha \rightarrow \neg Ixy(x \in z)(x \in \alpha)).$$

Since uncountable cardinals belong to the canonical set of indiscernibles, we can define an infinite set of indiscernibles and hence 0^\sharp .

In fact a stronger result holds:

Theorem 9.1. *The Jensen-Dodd Core Model is included in $C(I)$. If L^μ exists, then $L^\mu \subseteq C(I)$ for some ν . If $V = L^\mu$, then $V = C(I)$.*

Proof. We consider mice and the core model K in the sense of [8]. Suppose M_0 is a mouse. So M_0 is of the form $L_{\alpha_0}^{U_0}$, where U_0 is a normal measure on some β_0 in M . Let us consider the iterated ultrapowers $M_\xi = L_{\alpha_\xi}^{U_\xi}$, U_ξ a normal measure of β_ξ in M_ξ , $\xi \leq \lambda$, with commuting embeddings $i_{\alpha\beta} : M_\alpha \rightarrow M_\beta$. Let λ be a limit cardinal $> |M_0|^+$ of uncountable cofinality. Then $U_\lambda \subseteq \mathcal{C}_\lambda$, where \mathcal{C}_λ is the filter generated on λ by the sets $\{\mu < \lambda : \mu_0 \leq \mu \text{ a cardinal}\}$, $\mu_0 < \lambda$. Note that $L_{\alpha_\lambda}^{U_\lambda} \cap \mathcal{C}_\lambda = U_\lambda$. Consider the stage $L'_{\alpha_{\lambda+1}}$ in the construction of $C(I)$ in Definition 2.2. At this stage we can define \mathcal{C}_λ and hence M_λ . Thus $M_\lambda \subseteq C(I)$. Let X be the Skolem Hull of $\beta_0 \cup \{\beta_\lambda, U_\lambda\}$ in M_λ , and $\pi : N \rightarrow X$ the Mostowski collapse of X . Then $N = M_0$. Thus $M_0 \subseteq C(I)$. We have proved $K \subseteq C(I)$.

The other claims are proved similarly. \square

Theorem 9.2. *If it is consistent to have a supercompact cardinal, then it is consistent to have a supercompact cardinal and $C(I) \neq \text{HOD}$*

Proof. Start with supercompact λ and GCH. Add a Cohen real. $C(I)$ remains the same. Now code the Cohen real to the $2^\kappa = \kappa^+$ predicate by adding subsets

to cardinals without losing the supercompactness, making sure that $V = \text{HOD}$. Then $C(I) \neq \text{HOD} = V$, as $C(I)$ is still the ground model. \square

Definition 9.3. We say that an inner model M satisfies $\text{Cov}(M)$, if for every uncountable sets of ordinals there is a superset of the same cardinality in M .

Theorem 9.4. *If there is no inner model with a measurable cardinal, then we have $\text{Cov}(C(I))$, but this does not follow from $\neg 0^\dagger$.*

Proof. Suppose there are no inner models with a measurable cardinal. We know that $K \subseteq C(I)$. By the Dodd-Jensen Covering Lemma, $\text{Cov}(K)$ holds. Hence $\text{Cov}(C(I))$ holds. On the other hand, suppose there is a measurable cardinal κ in an inner model. Let $V = L[D]$, where D is a normal measure on κ . Then $C(I) = V$. Let us apply Prikry forcing to κ , obtaining $V[G]$. In $V[G]$ we have $C(I) = L[D]$ because no cardinals have been collapsed. The condition $\text{Cov}(C(I))$ fails in $V[G]$ because the Prikry-sequence cannot be covered by a set in $C(I)$ ($= L[D]$). So $\text{Cov}(C(I))$ does not follow from $\neg 0^\dagger$. \square

We do not know whether it is consistent that $C(I)$ contains a supercompact cardinal. However, consider the following modification:

$$I'xy\varphi(x, \vec{z})\psi(y, \vec{z}) \iff |\{a : \varphi(a, \vec{z})\}| = 2^{|\{a : \psi(a, \vec{z})\}|}.$$

Note that $C(I) \subseteq C(I')$. With the method of [33] it is possible to prove:

Proposition 9.5. *It is consistent, relative to the consistency of a supercompact cardinal, that $C(I')$ contains a supercompact cardinal.*

In Theorem 5.17 we proved a very weak form of the Generalized Continuum Hypothesis for C^* . The proof was based on special model theoretic properties of the logic $\mathcal{L}(Q_\omega^{\text{cf}})$. We now isolate a general formulation of this property, which will then help us prove a similar result for $C(I)$,

A logic \mathcal{L}^* satisfies $\text{LST}(\kappa)$ if every model in a finite vocabulary has an \mathcal{L}^* -elementary submodel of size $< \kappa$. First order logic and $\mathcal{L}(Q_0)$ satisfy $\text{LST}(\aleph_1)$. The logics $\mathcal{L}(Q_1)$, $\mathcal{L}(Q_\omega^{\text{cf}})$, and $\mathcal{L}(Q_1^{\text{MM}})$ satisfy $\text{LST}(\aleph_2)$, as does $\mathcal{L}(\text{aa})$ if MM^{++} is assumed. The logic $\mathcal{L}(I)$ can consistently satisfy $\text{LST}(\kappa)$, where κ is the first weakly inaccessible, relative to the consistency of a super compact cardinal. It is also consistent, again relative to the consistency of a super compact cardinal, that $\mathcal{L}(I)$ satisfies $\text{LST}(2^\omega)$ [41]. On the other hand, if $\mathcal{L}(I)$ satisfies $\text{LST}(\kappa)$ for some κ then κ is at least as big as the first weakly inaccessible cardinal, the Singular Cardinals Hypothesis holds above κ , and Projective Determinacy is true [27].

Proposition 9.6. *If $\mathcal{L}(I)$ satisfies $LST(\kappa)$ and $\delta < \kappa$ is a cardinal of $C(I)$, then $C(I) \models 2^\delta \leq \kappa$.*

Proof. Suppose $\delta < \kappa$ is a cardinal in $C(I)$. We show $2^\delta \leq \kappa$ in $C(I)$. We use the notation of Definition 2.2. Suppose $a \subseteq \delta$ and $a \in L'_\xi$ for some ξ . Let $\mu > \xi$ be a sufficiently large cardinal. Let $M \prec_{\mathcal{L}(I)} H(\mu)$ such that $|M| < \kappa$, and let N be the transitive collapse of M with $\pi : M \cong N$ the canonical isomorphism. Let $\zeta = N \cap On$. Note that $|N| < \kappa$, whence $\zeta < \kappa$. If $\alpha = \pi(\beta) \in N$ is a real cardinal, then clearly it is a cardinal in the sense of N . Conversely, suppose $\alpha = \pi(\beta) \in N$ is not a real cardinal. Then there is $\gamma = \pi(\nu) < \alpha$ such that (in V) $|\alpha| = |\gamma|$. Now we use $\mathcal{L}(I)$.

$$\begin{aligned}
|\alpha| = |\gamma| &\Rightarrow N \models Ixy(x \in \alpha)(y \in \gamma) \\
&\Rightarrow M \models Ixy(x \in \beta)(y \in \nu) \\
&\Rightarrow H(\mu) \models Ixy(x \in \beta)(y \in \nu) \\
&\Rightarrow |\beta| = |\nu| \\
&\Rightarrow \text{“}\beta \text{ is not a cardinal number”} \\
&\Rightarrow H(\mu) \models \text{“}\beta \text{ is not a cardinal number”} \\
&\Rightarrow M \models \text{“}\beta \text{ is not a cardinal number”} \\
&\Rightarrow N \models \text{“}\alpha \text{ is not a cardinal number”}
\end{aligned}$$

Thus $(L'_\xi)^N = L'_\xi$ for all $\xi < \zeta$. Since $N \models a \in C(I)$, we have $a \in L'_\zeta$. The claim follows. \square

Proposition 9.7. *If $\mathcal{L}(I)$ satisfies $LST(\kappa)$ and $E \subseteq \kappa$ is stationary, then $C(I)$ satisfies $\diamond_\kappa(E)$.*

Proof. This is proved as Proposition 9.6 and Theorem 5.18. To avoid repetition of the proof of Theorem 5.18 we give only an outline. We construct a sequence $s = \{(S_\alpha, D_\alpha) : \alpha < \kappa\} \in C(I)$ taking always for limit α the pair (S_α, D_α) to be the least $(S, D) \in L'_\kappa$ in the canonical well-order of $C(I)$ such that $S \subseteq \alpha$, $D \subseteq \alpha$ a club, and $S \cap \beta \neq S_\beta$ for $\beta \in D$. Suppose s is not a diamond sequence and $(S, D) \in L'_\kappa$ is a minimal counter-example. We can construct $M \prec H(\mu)$ such that $|M| < \kappa$, the order-type of $M \cap \kappa$ is in E , $\{s, (S, D)\} \subset M$, and if N is the transitive collapse of M , with ordinal $\delta \in E$, then $\{s \upharpoonright \delta, (S \cap \delta, D \cap \delta)\} \subset N$ and $(L'_\xi)^N = L'_\xi$ for all $\xi < \delta$. Thus the pair $(S \cap \delta, D \cap \delta)$ is the minimal (S', D')

such that $S' \subseteq \delta$, $D' \subseteq \delta$ a club, and $S' \cap \beta \neq S'_\beta$ for $\beta \in D'$. It follows that $(S', D') = (S_\delta, D_\delta)$ and, since $\delta \in D$, a contradiction. \square

10 Higher order logics

The basic result about higher order logics, proved in [36], is that they give rise to the inner model HOD of hereditarily ordinal definable sets. In this section we show that this result enjoys some robustness, i.e. ostensibly much weaker logics than second order logic still give rise to HOD.

Theorem 10.1 (Myhill-Scott [36]). $C(\mathcal{L}^2) = \text{HOD}$.

Proof. We give the proof for completeness. We show $\text{HOD} \subseteq C(\mathcal{L}^2)$. Let $X \in \text{HOD}$. There is a first order $\varphi(x, \vec{y})$ and ordinals $\vec{\beta}$ such that for all a

$$a \in X \iff \varphi(a, \vec{\beta}).$$

By Levy Reflection there is an α such that $X \subseteq V_\alpha$ and for all $a \in V_\alpha$

$$a \in X \iff V_\alpha \models \varphi(a, \vec{\beta}).$$

Since we proceed by induction, we may assume $X \subseteq C(\mathcal{L}^2)$. Let γ be such that $X \subseteq L'_\gamma$. We can choose γ so big that $|L'_\gamma| \geq |V_\alpha|$. We show now that $X \in L'_{\gamma+1}$. We give a second order formula $\Phi(x, y, \vec{z})$ such that

$$X = \{a \in L'_\gamma : L'_\gamma \models \Phi(a, \alpha, \vec{\beta})\}.$$

We know

$$X = \{a \in L'_\gamma : V_\alpha \models \varphi(a, \vec{\beta})\}.$$

Intuitively, X is the set of $a \in L'_\gamma$ such that in L'_γ some $(M, E, a^*, \alpha^*, \vec{\beta}^*) \cong (V_\alpha, \in, a, \alpha, \vec{\beta})$ satisfies $\varphi(a^*, \vec{\beta}^*)$. Let $\theta(x, y, \vec{z})$ be a second order formula of the vocabulary $\{E\}$ such that for any $M, E \subseteq M^2$ and $a^*, \alpha^*, \vec{\beta}^* \in M$: $(M, E) \models \theta(a^*, \alpha^*, \vec{\beta}^*)$ iff there are an isomorphism $\pi : (M, E) \cong (V_\delta, \in)$ such that $\pi : (\alpha^*, E) \cong (\delta, \in)$, and $(V_\delta, \in) \models \varphi(\pi(a^*), \pi(\vec{\beta}^*))$.

We conclude $X \in L'_{\gamma+1}$ by proving the:

Claim The following are equivalent for $a \in L'_\gamma$:

(1) $a \in X$.

(2) $L'_\gamma \models \exists M, E(\text{TC}(\{a\}) \cup \alpha + 1 \cup \vec{\beta} \cup \{\vec{\beta}\} \subseteq M \wedge (M, E) \models \theta(a, \alpha, \vec{\beta}))$.

(1) \rightarrow (2) : Suppose $a \in X$. Thus $V_\alpha \models \varphi(a, \vec{\beta})$. Let $M \subseteq L'_\gamma$ and $E \subseteq M^2$ such that $\alpha + 1, \text{TC}(a), \vec{\beta} \in M$ and there is an isomorphism

$$f : (V_\alpha, \in, \alpha, a, \vec{\beta}) \cong (M, E, \alpha^*, a^*, \vec{\beta}^*).$$

We can assume $\alpha^* = \alpha$, $a^* = a$ and $\vec{\beta}^* = \vec{\beta}$ by doing a partial Mostowski collapse for (M, E) . So then $(M, E) \models \varphi(a, \vec{\beta})$, whence $(M, E) \models \theta(a, \alpha, \vec{\beta})$. We have proved (2).

(2) \rightarrow (1) : Suppose $M \subseteq L'_\gamma$ and $E \subseteq M^2$ such that $\text{TC}(\{a\}) \cup \alpha + 1 \cup \vec{\beta} \cup \{\vec{\beta}\} \subseteq M$ and $(M, E) \models \theta(a, \alpha, \vec{\beta})$. We may assume $E \upharpoonright \text{TC}(\{a\}) \cup \alpha + 1 \cup \vec{\beta} \cup \{\vec{\beta}\} = \in \upharpoonright \text{TC}(\{a\}) \cup \alpha + 1 \cup \vec{\beta} \cup \{\vec{\beta}\}$. There is an isomorphism $\pi : (M, E) \cong (V_\alpha, \in)$ such that $(V_\alpha, \in) \models \varphi(\pi(a), \pi(\vec{\beta}))$. But $\pi(a) = a$ and $\pi(\vec{\beta}) = \vec{\beta}$. So in the end $(V_\alpha, \in) \models \varphi(a, \vec{\beta})$. We have proved (1). □

In second order logic \mathcal{L}^2 one can quantify over arbitrary subsets of the domain. A more general logic is obtained as follows:

Definition 10.2. Let F be any class function on cardinal numbers. The logic $\mathcal{L}^{2,F}$ is like \mathcal{L}^2 except that the second order quantifiers range over a domain M over subsets of M of cardinality $\leq \kappa$ whenever $F(\kappa) \leq |M|$.

Examples of possible functions are $F(\kappa) = 0, \kappa, \kappa^+, 2^\kappa, \aleph_\kappa, \beth_\kappa$, etc. Note that $\mathcal{L}^2 = \mathcal{L}^{2,F}$ whenever $F(\kappa) \leq \kappa$ for all κ . The logic $\mathcal{L}^{2,F}$ is weaker the bigger values $F(\kappa)$ takes on. For example, if $F(\kappa) = 2^{2^\kappa}$, the second order variables of $\mathcal{L}^{2,F}$ range over “tiny” subsets of the universe. Philosophically second order logic is famously marred by the difficulty of imagining how a universally quantified variable could possibly range over *all* subsets of an infinite domain. If the universally quantified variable ranges only over “tiny” size subsets, one can conceivably think that there is some coding device which uses the elements of the domain to code all the “tiny” subsets.

Inspection of the proof of Theorem 10.1 reveals that actually the following more general fact holds:

Theorem 10.3. For all $F: C(\mathcal{L}^{2,F}) = \text{HOD}$.

Let \mathcal{L}^2_κ denote the modification of \mathcal{L}^2 in which the second order variables range over subsets (relations, functions, etc) of cardinality at most κ .

Theorem 10.4. *Suppose 0^\sharp exists. Then $0^\sharp \in C(\mathcal{L}_\kappa^2)$*

Proof. As in the proof of Theorem 5.3. □

A consequence of Theorem 10.3 is the following:

Conclusion: The second order constructible hierarchy $C(\mathcal{L}^2) = \text{HOD}$ is unaffected if second order logic is modified in any of the following ways:

- Extended in any way to a logic definable with hereditarily ordinal definable parameters. This includes third order logic, fourth order logic, etc.
- Weakened by allowing second order quantification in domain M only over subsets X such that $2^{|X|} \leq |M|$.
- Weakened by allowing second order quantification in domain M only over subsets X such that $2^{2^{|X|}} \leq |M|$.
- Any combination of the above.

Thus Gödel's $\text{HOD} = C(\mathcal{L}^2)$ has a some robustness as to the choice of the logic \mathcal{L}^2 . It is the common feature of the logics that yield HOD that they are able to express quantification over all subsets of some part of the universe the size of which is not a priori bounded. We can perhaps say, that this is the essential feature of second order logic that results in $C(\mathcal{L}^2)$ being HOD. What is left out are logics in which one can quantify over, say all countable subsets. Let us call this logic $\mathcal{L}_{\aleph_0}^2$. Consistently⁶, $C(\mathcal{L}_{\aleph_0}^2) \neq \text{HOD}$. Many would call a logic such as $\mathcal{L}_{\aleph_0}^2$ second order.

Let Σ_n^1 denote the fragment of second order logic in which the formulas have, if in prenex normal form with second order quantifiers preceding all first order quantifiers, only n second order quantifier alternations, the first second order quantifier being existential. Note that trivially $C(\Sigma_n^1) = C(\Pi_n^1)$. The Myhill-Scott proof shows that $C(\Sigma_n^1) = \text{HOD}$ for $n \geq 2$. What about $C(\Sigma_1^1)$? We write

$$\text{HOD}_1 =_{\text{df}} C(\Sigma_1^1).$$

Note that for all β and $A \in \text{HOD}_1$:

- $\{\alpha < \beta : \text{cf}^V(\alpha) = \omega\} \in \text{HOD}_1$

⁶Assume $V = L$ and add a Cohen subset X of ω_1 . Now code X into HOD with countably closed forcing using [32]. In the resulting model $C(\mathcal{L}_{\aleph_0}^2) = L \neq \text{HOD}$.

- $\{(a, b) \in A^2 : |a|^V \leq |b|^V\} \in \text{HOD}_1$
- $\{\alpha < \beta : \alpha \text{ cardinal in } V\} \in \text{HOD}_1$
- $\{(\alpha_0, \alpha_1) \in \beta^2 : |\alpha_0|^V \leq (2^{|\alpha_1|})^V\} \in \text{HOD}_1$
- $\{\alpha < \beta : (2^{|\alpha|})^V = (|\alpha|^+)^V\} \in \text{HOD}_1$

These examples show that HOD_1 contains most if not all of the inner models considered above. In particular we have:

Lemma 10.5. 1. $C^* \subseteq \text{HOD}_1$.

2. $C(Q_1^{MM, < \omega}) \subseteq \text{HOD}_1$

3. $C(I) \subseteq \text{HOD}_1$.

4. If 0^\sharp exists, then $0^\sharp \in \text{HOD}_1$

Naturally, $\text{HOD}_1 = \text{HOD}$ is consistent, since we only need to assume $V = L$. So we focus on $\text{HOD}_1 \neq \text{HOD}$.

Theorem 10.6. *It is consistent, relative to the consistency of infinitely many weakly compact cardinals that for some λ :*

$$\{\kappa < \lambda : \kappa \text{ weakly compact (in } V)\} \notin \text{HOD}_1,$$

and, moreover, $\text{HOD}_1 = L \neq \text{HOD}$.

Proof. Let us assume $V = L$ and κ_n , $n < \omega$, are weakly compact cardinals in increasing order. Let $\lambda = \sup_n \kappa_n$. We proceed as in [21, §3]. As a preliminary forcing \mathbb{P} we add by reverse Easton iteration a Cohen subset C_α for each inaccessible $\alpha < \lambda$ (including ω), preserving the weak compactness of each κ_n . W.l.o.g. $\min(C_\alpha) > \kappa_n$ for $\alpha > \kappa_n$. Let V_1 denote the extension by \mathbb{P} . Let \mathbb{Q} be the forcing in V_1 which adds a κ_n -Souslin tree T_n to κ_n , $n \in C_\omega$, by homogeneous trees as conditions, killing the weak compactness of κ_n for (and only for) $n \in C_\omega$. W.l.o.g. the tree T_{n+1} consists of sequences $s \in 2^{< \kappa_{n+1}}$ with $\min(s) > \kappa_n$. Let V_2 denote the extension of V_1 by \mathbb{Q} . Let \mathbb{S} force in V_2 a branch through T_n , $n \in C_\omega$, by initial segments, restoring the weak compactness of κ_n , $n \in C_\omega$. The crucial observation now is that V_1 and V_3 are forcing extensions of V obtained by using isomorphic forcing notions. Moreover these isomorphic forcing notions are homogeneous, so

both in V_1 and V_3 we have $\text{HOD} = L$. However, in V_2 the non-constructible real C_ω is in HOD as

$$V_2 \models \forall n < \omega (n \in C_\omega \leftrightarrow \kappa_n \text{ is not weakly compact}).$$

We show now that, in contrast,

$$V_2 \models \text{HOD}_1 = L.$$

We use induction on α to prove in V_2 that $L'_\alpha \subseteq L$. Suppose this holds up to α and we have $X \in L'_{\alpha+1}$. Thus for some Σ^1_1 -formula $\varphi(x, y)$ and some $a \in L'_\alpha$ we have

$$X = \{b \in L'_\alpha : L'_\alpha \models \varphi(b, a)\}.$$

Since $L'_\alpha \subseteq L$, without loss of generality, for some β , some first order formula $\varphi'(x, y, Y)$ with a new unary predicate symbol Y , and some $a' \in L_\beta$:

$$X = \{b \in L_\beta : \exists Y \subseteq \beta ((L_\beta, Y) \models \varphi'(b, a', Y))\}.$$

It suffices to prove now for a given $b \in L_\beta$:

$$V_1 \models \exists Y \subseteq \beta ((L_\beta, Y) \models \varphi'(b, a', Y)) \iff$$

$$V_2 \models \exists Y \subseteq \beta ((L_\beta, Y) \models \varphi'(b, a', Y))$$

The direction from left to right is trivial because $V_1 \subseteq V_2$. So let us assume the right hand side of the equivalence holds. Then

$$V_3 \models \exists Y \subseteq \beta ((L_\beta, Y) \models \varphi'(b, a', Y)),$$

as $V_2 \subseteq V_3$. But V_3 and V_1 are extensions of V by isomorphic homogeneous forcing notions. So

$$V_1 \models \exists Y \subseteq \beta ((L_\beta, Y) \models \varphi'(b, a', Y))$$

follows. □

The above proof works also with “weakly compact” replaced by other large cardinal properties, e.g. “measurable” or “supercompact”. We can start, for example, with a ω supercompact cardinals, code each one of them into cardinal exponentiation, detectible by means of HOD_1 , above all of them, without losing their supercompactness or introducing new supercompact cardinals, and then proceed

as in the proof of Theorem 10.6. Note that we can also start with a supercompact cardinal and code, using the method of [33], every set into cardinal exponentiation, detectible by means of HOD_1 , without losing the supercompact cardinal. In the final model there is a super compact cardinal while $V = \text{HOD}_1$.

We shall now prove an analogue of Theorem 10.6 without assuming any large cardinals. Let $\mathbb{C}(\kappa)$ be Cohen forcing for adding a subset for a regular cardinal κ . Let $R(\kappa)$ be the statement that there is a bounded subset $A \subseteq \kappa$ and a set $C \subseteq \kappa$ which is $\mathbb{C}(\kappa)$ -generic over $L[A]$, such that $\mathcal{P}(\kappa) \subseteq L[A, C]$.

Theorem 10.7. *It is consistent, relative to the consistency of ZFC that:*

$$\{n < \omega : R(\aleph_n)\} \notin \text{HOD}_1,$$

and, moreover, $\text{HOD}_1 = L \neq \text{HOD}$.

Proof. The proof is very much like the proof of Theorem 10.6 so we only indicate the necessary modifications. Let us assume $V = L$. As a preliminary forcing \mathbb{C} we apply Cohen forcing $\mathbb{C}_n = \mathbb{C}(\aleph_n)$ for each \aleph_n (including \aleph_0) adding a Cohen subset $C_n \subseteq \aleph_n \setminus \{0, 1\}$. W.l.o.g. $\min(C_{n+1}) > \aleph_n$. Let V_1 denote the extension. Let \mathbb{P} be the product forcing in V_1 which adds a non-reflecting stationary set A_n to \aleph_n , $n \notin C_0$, by means of:

$$\mathbb{P}_n = \{p : \gamma \rightarrow 2 : \aleph_{n-1} < \gamma < \aleph_n, \forall \alpha < \gamma (\text{cf}(\alpha) > \omega \rightarrow \{\beta < \alpha : p(\beta) = 0\} \text{ is non-stationary in } \alpha)\}.$$

Let us note that \mathbb{P}_n is strategically \aleph_{n-1} -closed, for the second player can play systematically at limits in such a way that during the game a club is left out of $\{\beta : p(\beta) = 0\}$. Let V_2 denote the extension of V_1 by \mathbb{P} . Now

$$V_2 \models C_0 = \{n < \omega : R(\aleph_n)\},$$

for if $n \in C_0$, then $R(\aleph_n)$ holds in V_2 by construction, and on the other hand, if $n \notin C_0$, then $R(\aleph_n)$ fails in V_2 because one can show with a back-and-forth argument that with \mathbb{P} and \mathbb{C} as above, we always have $V^{\mathbb{P}} \neq V^{\mathbb{C}}$.

Let \mathbb{Q} force in V_2 a club into A_n , $n \in C_\omega$, by closed initial segments with a last element. The crucial observation now is that $\mathbb{P}_n \star \mathbb{Q}_n$ is the same forcing as \mathbb{C}_n . To see this, it suffices to find a dense \aleph_{n-1} -closed subset of $\mathbb{P}_n \star \mathbb{Q}_n$ of cardinality \aleph_n . Let D consist of pairs $(p, A) \in \mathbb{P}_n \star \mathbb{C}_n$ such that $\text{dom}(p) = \max(A) + 1$ and $\forall \beta \in A (p(\beta) = 1)$. This set is clearly \aleph_{n-1} -closed. □

Proposition 10.8. *If 0^\sharp exists, then $0^\sharp \in C(\Delta_1^1)$, hence $C(\Delta_1^1) \neq L$.*

Proof. As Proposition 8.6. □

11 Semantic extensions of ZFC

For another kind of application of extended logics in set theory we consider the following concept:

Definition 11.1. Suppose \mathcal{L}^* is an abstract logic. We use $\text{ZFC}(\mathcal{L}^*)$ to denote the usual ZFC-axioms in the vocabulary $\{\in\}$ with the modification that the formula $\varphi(x, \vec{y})$ in the Schema of Separation

$$\forall x \forall x_1 \dots \forall x_n \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge \varphi(z, \vec{x})))$$

and the formula $\psi(u, z, \vec{x})$ in the Schema of Replacement

$$\begin{aligned} \forall x \forall x_1 \dots \forall x_n (\forall u \forall z \forall z' ((u \in x \wedge \psi(u, z, \vec{x}) \wedge \psi(u, z', \vec{x})) \rightarrow z = z') \\ \rightarrow \exists y \forall z (z \in y \leftrightarrow \exists u (u \in x \wedge \psi(u, z, \vec{x})))) \end{aligned}$$

is allowed to be taken from \mathcal{L}^* .

The concept of a a model (M, E) , $E \subseteq M \times M$, satisfying the axioms $\text{ZFC}(\mathcal{L}^*)$ is obviously well-defined. Note that $\text{ZFC}(\mathcal{L}^*)$ is at least as strong as ZFC in the sense that every model of $\text{ZFC}(\mathcal{L}^*)$ is, a fortiori, a model of ZFC.

The class of (set) models of ZFC is, of course, immensely rich, ZFC being a first order theory. If ZFC is consistent, we have countable models, uncountable models, well-founded models, non-well-founded models etc. We now ask the question, what can we say about the models of $\text{ZFC}(\mathcal{L}^*)$ for various logics \mathcal{L}^* ? Almost by definition, the inner model $C(\mathcal{L}^*)$ is a class model of $\text{ZFC}(\mathcal{L}^*)$:

$$C(\mathcal{L}^*) \models \text{ZFC}(\mathcal{L}^*).$$

But $\text{ZFC}(\mathcal{L}^*)$ can very well have other models.

Theorem 11.2. *A model of ZFC is a model of $\text{ZFC}(\mathcal{L}(Q_0))$ if and only if it is an ω -model.*

Proof. Suppose first (M, E) is an ω -model of ZFC. Then we can eliminate Q_0 in (M, E) : Given a first order formula $\varphi(x, \vec{a})$ with some parameters \vec{a} there is, by the Axiom of Choice, either a one-one function from

$$\{b \in M : (M, E) \models \varphi(b, \vec{a})\} \tag{10}$$

onto a natural number of (M, E) or onto an ordinal of (M, E) which is infinite in (M, E) . Since (M, E) is an ω -model, these two alternatives correspond exactly

to (10) being finite (in V) or infinite (in V). So Q_0 has, in (M, E) , a first order definition. For the converse, suppose (M, E) is a model of $ZFC(\mathcal{L}(Q_0))$ but some element a in $\omega^{(M, E)}$ has infinitely many predecessors in V . By using the Schema of Separation, applied to $\mathcal{L}(Q_0)$, we can define the set $B \in M$ of elements a in $\omega^{(M, E)}$ that have infinitely many predecessors in V . Hence we can take the smallest element of B in (M, E) . This is clearly a contradiction. \square

In similar way one can show that a model of ZFC is a model of $ZFC(\mathcal{L}(Q_1))$ if and only if its set of ordinals is \aleph_1 -like or it has an \aleph_1 -like cardinal.

Theorem 11.3. *A model of ZFC is a model of $ZFC(\mathcal{L}(Q_0^{MM}))$ if and only if it is well-founded.*

Proof. Suppose first (M, E) is a well-founded model of ZFC. Then we can eliminate Q_0^{MM} in (M, E) because it is absolute: The existence of an infinite set X such that every pair from the set satisfies a given first-order formula can be written as the non-well-foundedness of a relation in M and non-well-foundedness is an absolute property in transitive models. For the converse, suppose (M, E) is a model of $ZFC(\mathcal{L}(Q_0^{MM}))$. Since Q_0 is definable from Q_0^{MM} we can assume (M, E) is an ω -model and $\omega^{(M, E)} = \omega$. Suppose some ordinal a in (M, E) is non-well-founded. To reach a contradiction it suffices to show that the set of such a is $\mathcal{L}(Q_0^{MM})$ -definable in (M, E) . Let $\varphi(x, y, z)$ be the first order formula of the language of set theory which says:

- $x = \langle x_1, x_2 \rangle, y = \langle y_1, y_2 \rangle$
- $x_1, x_2 < \omega, y_1, y_2 < a$
- $x_1 \neq x_2$
- $x_1 < x_2 \rightarrow y_2 < y_1$.

Let us first check that $Q_0^{MM}xy\varphi(x, y, a)$ holds in (M, E) . Let (a_n) be a decreasing sequence (in V) of elements of a . Let X be the set of pairs $\langle n, a_n \rangle$, where $n < \omega$. By construction, any pair $\langle x, y \rangle$ in $[X]^2$ satisfies $\varphi(x, y, a)$. Thus $Q_0^{MM}xy\varphi(x, y, a)$ holds in (M, E) . For the converse, suppose $Q_0^{MM}xy\varphi(x, y, b)$ holds in (M, E) . Let Y be an infinite set such that every $\langle x, y \rangle$ in $[Y]^2$ satisfies $\varphi(x, y, b)$. Every two pairs in Y have a different natural number as the first component. So we can choose pairs from Y where the first components increase. But then the second components decrease and b has to be non-well-founded. \square

Theorem 11.4. *A structure is a model of $ZFC(\mathcal{L}_{\omega_1\omega})$ if and only if it is isomorphic to a transitive a model M of ZFC such that $M^\omega \subseteq M$.*

Proof. Suppose first M is a transitive a model of ZFC such that $M^\omega \subseteq M$. Then we can eliminate $\mathcal{L}_{\omega_1\omega}$ because the semantics of $\mathcal{L}_{\omega_1\omega}$ is absolute in transitive models and the assumption $M^\omega \subseteq M$ guarantees that all the $\mathcal{L}_{\omega_1\omega}$ -formulas of the language of set theory are elements of M . For the converse, suppose (M, E) is a model of $ZFC(\mathcal{L}_{\omega_1\omega})$. Since Q_0 is definable in $\mathcal{L}_{\omega_1\omega}$, we may assume that (M, E) is an ω -model and $\omega^{(M,E)} = \omega$. Suppose (a_n) is a sequence (in V) of elements of M . Let

$$\varphi(x, y, u_0, u_1, \dots, z_0, z_1, \dots)$$

be the $\mathcal{L}_{\omega_1\omega}$ -formula

$$\bigwedge_n (x = u_n \rightarrow y = z_n).$$

Note that (M, E) satisfies

$$\forall x \in \omega \exists y \varphi(x, y, 0, 1, \dots, a_0, a_1, \dots).$$

If we apply the Schema of Replacement of $ZFC(\mathcal{L}_{\omega_1\omega})$, we get an element b of M which has all the a_n as its elements. By a similar application of the Schema of Separation we get $\{a_n : n \in \omega\} \in M$. Thus M is closed under ω -sequences and in particular it is well-founded. \square

By a similar argument one can see that the only model of the class size theory $ZFC(\mathcal{L}_{\infty\omega})$ is the class size model V itself. This somewhat extreme example shows that by going far enough along this line eventually gives everything. One can also remark that the class of models of $ZFC(\mathcal{L}_{\omega_1\omega_1})$ is exactly the same as the class of models of $ZFC(\mathcal{L}_{\omega_1\omega})$. This is because in transitive models M such that $M^\omega \subseteq M$ also the truth of $\mathcal{L}_{\omega_1\omega_1}$ -sentences is absolute. So despite their otherwise huge difference, the logics $\mathcal{L}_{\omega_1\omega}$ and $\mathcal{L}_{\omega_1\omega_1}$ do not differ in the current context.

Second order logic is again an interesting case. Note that $ZFC(L^2)$ is by no means the same as the so-called second order ZFC, or ZFC^2 as it is denoted. We have not changed the Separation and Replacement Schemas into a second order form, we have just allowed second order formulas to be used in the schemas instead of first order formulas. So, although the models of ZFC^2 are, up to isomorphism, of the form V_κ , and are therefore, a fortiori, also models of $ZFC(L^2)$, we shall see below that models of $ZFC(L^2)$ need not be of that form.

Theorem 11.5. *Assume $V = L$. A structure is a model of $ZFC(L^2)$ if and only if it is isomorphic to a model M of ZFC of the form L_κ where κ is inaccessible.*

Proof. First of all, if $V = L$ and $L_\kappa \models ZFC$, where κ is inaccessible, then trivially $L_\kappa \models ZFC(L^2)$. For the converse, suppose $(M, E) \models ZFC(L^2)$. Because Q_0^{MM} is definable in L^2 , we may assume (M, E) is a transitive model (M, \in) .

We first observe that the model M satisfies $V = L$. To this end, suppose $\alpha \in M$ and $x \in M$ is a subset of α . Let β be minimal β such that $x \in L_\beta$. There is a binary relation on α , second order definable over M , with order type β . By the second order Schema of Separation this relation is in the model M . So $M \models "x \in L_\beta"$. Hence $M \models V = L$. Let $M = L_\alpha$. It is easy to see that α has to be an inaccessible cardinal. □

Note that if $0^\#$ exists, then $0^\#$ is in every transitive model of $ZFC(L^2)$. If there is an inaccessible cardinal and we add a Cohen real, then $ZFC(L^2)$ has a transitive model M which is not of the form V_α . This is a consequence of the homogeneity of Cohen forcing.

12 Open Questions

This topic abounds in open questions. We mention here the most obvious:

1. Can C^* contain measurable cardinals? Note that there are no measurable cardinals if $V = C^*$ (Theorem 5.7).
2. Does C^* satisfy CH, if V has large cardinals? Note that if there are large cardinals then the relativized version $C^*(x)$ of C^* satisfies CH for a cone of reals x (Theorem 5.20).
3. Can $C(I)$ contain a supercompact cardinal? Note that the answer is positive if a slight modification of $C(I)$ is used (Proposition 9.5).

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Juliette Kennedy
 Department of Mathematics and Statistics
 University of Helsinki

Menachem Magidor
 Department of Mathematics
 Hebrew University Jerusalem

Jouko Väänänen
 Department of Mathematics and Statistics
 University of Helsinki
 and
 Institute for Logic, Language and Computation
 University of Amsterdam