ON PROPERTIES OF COMPACTA THAT DO NOT REFLECT IN SMALL CONTINUOUS IMAGES

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Abstract. Assuming that there is a stationary set in ω_2 of ordinals of countable cofinality that does not reflect, we prove that there exists a compact space which is not Corson compact and whose all continuous images of weight $\leq \omega_1$ are Eberlein compacta. We also prove that under Martin's axiom countable functional tightness does not reflect in small continuous images of compacta.

1. Introduction

There is a significant amount of research related to properties of structures that reflect in substructures of smaller cardinality, see e.g. Bagaria, Magidor, Sakai [2], Koszmider [9, 10], Fuchino and Rinot [5], Tall [18]. Reflection phenomena in topology are usually studied following the following pattern:

Problem 1.1. *Does a topological space X has a property (P) provided all its subspaces of small cardinality have property (P)?*

Tall [18] gives a survey on results and problems of this type. Recently Tkachuk [19] and Tkachuk and Tkachenko [20] have investigated which topological properties reflect in small continuous images which, in particular, amounts to asking the following kind of questions.

Problem 1.2. *Does a topological space X has property (P) provided every continuous image of X of weight $\leq \omega_1$ has property (P)?*

Eberlein compacta and Corson compacta are two well-studied classes of compact spaces related to functional analysis, see the next section. Answering two questions of type 1.2 posed in [20], we show in this note that it is relatively consistent that neither Eberlein compactness nor Corson compactness reflects in continuous images of weight $\leq \omega_1$. In fact, assuming that there is a stationary set $S \subseteq \omega_2$ of ordinals of countable cofinality such that $S \cap \alpha$ is stationary in no $\alpha < \omega_2$, we construct a compact space K of weight ω_2 which simultaneously answers in the negative both the questions: K is not Corson compact while all its images of weight at most ω_1 are Eberlein compacta that can be embedded into a

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Hilbert space. In addition, our space K gives partial negative answers to problems posed by Jardón and Tkachuk ([6], Questions 4.13-15) on the reflection of type 1.1 for Corson compacta and related classes. The construction of the space is given in section 3 and uses the familiar idea of a ladder system associated to the set $S \subseteq \omega_2$; see, for instance, Ciesielski and Pol [4] where a construction of this type was used to solve a problem on the structure of $C(K)$ spaces.

In the final section of this note we give a partial negative answer to another problem from [20]: we show, assuming a weak version of Martin's axiom, that countable functional tightness does not reflect in small continuous images of compact spaces.

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2. Preliminaries

All the spaces we consider are assumed to be Hausdorff. Given a topological space X, $w(X)$ denotes its topological weight, i.e. the minimal size of a base in X. Recall that a family $\mathcal{V}$ of nonempty open subsets of X is a *π-base* if every nonempty open set in X contains some $V \in \mathcal{V}$.

Our examples will be constructed from some Boolean algebras. If $\mathfrak{A}$ is a Boolean algebra then we write $\mathrm{ult}(\mathfrak{A})$ for its Stone space (of all ultrafilters on $\mathfrak{A}$). We write $\widehat{a} = \{x \in \mathrm{ult}(\mathfrak{A}) : a \in x\}$ for $a \in \mathfrak{A}$. Recall that sets $\widehat{a}$ form a base for the topology on $\mathrm{ult}(\mathfrak{A})$.

We shall use the following result which is a very particular case of the Mardešić factorization theorem [14]. We enclose the sketch of a direct argument.

Theorem 2.1. *Let $\mathfrak{A}$ be any Boolean algebra. If L is a continuous image of* $\mathrm{ult}(\mathfrak{A})$ *and $w(L) \le \omega_1$ then there is a subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$ such that $|\mathfrak{B}| \le \omega_1$ and L is a continuous image of* $\mathrm{ult}(\mathfrak{B})$.

Proof. Note that L has a base $\mathcal{U}$ of cardinality $\le \omega_1$ such that every $U \in \mathcal{U}$ is F_σ. Let $f : \mathrm{ult}(\mathfrak{A}) \to L$ be a continuous surjection. For every $U \in \mathcal{U}$, the set $f^{-1}(U)$ is of type F_σ so it can be written as a union of countably many sets of the form $\widehat{a}$, $a \in \mathfrak{A}$.

It follows that there is $\mathfrak{B} \subseteq \mathfrak{A}$ of size at most ω_1 such that, writing $\pi : \mathrm{ult}(\mathfrak{A}) \to \mathrm{ult}(\mathfrak{B})$ for the natural projection, we have $f(x) = f(y)$ whenever $x, y \in \mathrm{ult}(\mathfrak{A})$ and $\pi(x) = \pi(y)$. Hence we can write $f = f' \circ \pi$, where $f' : \mathrm{ult}(\mathfrak{B}) \to L$. It follows that f' is continuous and the proof is complete. □

A compact space K is said to be *Eberlein compact* if it is homeomorphic to a weakly compact subset of some Banach space; equivalently, by the classical Amir-Lindenstrauss theorem, K is Eberlein compact if it can be embedded into

$$c_0(\kappa) = \{x \in \mathbb{R}^\kappa : \{\alpha : |x_\alpha| \ge \varepsilon\} \text{ is finite for every } \varepsilon > 0\},$$

for some κ. Here $c_0(\kappa)$ is equipped with the topology inherited from $\mathbb{R}^\kappa$ (this topology agrees on bounded sets with the weak topology of the Banach space $c_0(\kappa)$).

In particular, if $n \in \omega$ then every compact subset of

$$\sigma_n(\kappa) = \{x \in 2^\kappa : |\{\alpha : |x_\alpha \neq 0\}| \leq n\},$$

is Eberlein compact. In fact it is uniform Eberlein compact in the sense that it can be embedded as a weakly compact subspace of a Hilbert space (note that $\sigma_n(\kappa)$ is a bounded subset of $l_2(\kappa)$).

A compact space K is said to be *Corson compact* if there is κ such that K is homeomorphic to a subset of the $\Sigma-$product of real lines

$$\Sigma(\mathbb{R}^\kappa) = \{x \in \mathbb{R}^\kappa : |\{\alpha : x_\alpha \neq 0\}| \leq \omega\}.$$

Since $c_0(\kappa) \subseteq \Sigma(\mathbb{R}^\kappa)$, the class of Corson compacta contains (properly) the class of Eberlein compacta. Negrepontis [15] and Kalenda [7] offer extensive surveys on Eberlein and Corson compacta and related classes. We only recall here that both uniform Eberlein compacta and Corson compacta are stable under continuous images, see e.g. [15], 6.26 and [7], p. 2.

A family $\mathcal{F}$ in a Boolean algebra is said to be *centred* if $a_1 \cap a_2 \cap \ldots a_k \neq 0$ for every natural number k and every $a_i \in \mathcal{F}$. We shall use the following standard fact.

Lemma 2.2. *For a Boolean algebra $\mathfrak{A}$ the following are equivalent*

(i) $\mathrm{ult}(\mathfrak{A})$ *is Corson compact;*

(ii) there is a family $\mathcal{G} \subseteq \mathfrak{A}$ generating $\mathfrak{A}$ and such that every centred subfamily of $\mathcal{G}$ is countable.

Proof. $(i) \to (ii)$. Since $\mathrm{ult}(\mathfrak{A})$ is Corson compact and zerodimensional, $\mathrm{ult}(\mathfrak{A})$ is homeomorphic to a compact space K contained in $\Sigma(2^\kappa)$ for some κ. The algebra of clopen subsets of K is generated by the family $\mathcal{C} = \{C_\alpha : \alpha < \kappa\}$, where $C_\alpha = \{x \in K : x_\alpha = 1\}$. Every centred subfamily of $\mathcal{C}$ is countable by the definition of $\Sigma(2^\kappa)$.

$(ii) \to (i)$. Take $f : \mathrm{ult}(\mathfrak{A}) \to 2^{\mathcal{G}}$, where $f(x)(G) = 1$ if $G \in x$ and $= 0$ otherwise. Then f is continuous, and $f[\mathrm{ult}(\mathfrak{A})] \subseteq \Sigma(2^{\mathcal{G}})$ since every ultrafilter on $\mathfrak{A}$ contains at most countably many generators from $\mathcal{G}$. Moreover, f is injective since $\mathcal{G}$ generates $\mathfrak{A}$. □

3. On Eberlein and Corson compacta

Let γ be a limit ordinal. A set $F \subseteq \gamma$ is said to be *closed* if it is closed in the interval topology defined on ordinals smaller that γ. Such a set F is unbounded in γ if for every $\beta < \gamma$ there is $\alpha \in F$ such that $\beta < \alpha$. A set $S \subseteq \gamma$ is *stationary* if $S \cap F \neq \emptyset$ for every closed and unbounded $F \subseteq \gamma$.

It is not difficult to check that the set $S_\omega = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$ is stationary in ω_2. However, such a set reflects in the sense that, for instance, $S_\omega \cap \omega_1$ is stationary in ω_1. We shall work assuming the following.

Axiom 3.1. There is a stationary set $S \subseteq \omega_2$ such that

(a) $\mathrm{cf}(\alpha) = \omega$ for every $\alpha \in S$;
(b) $S \cap \beta$ is not stationary in β for every $\beta < \omega_2$ with $\mathrm{cf}(\beta) = \omega_1$.

Note that in 3.1(b) we can say that $S \cap \beta$ is not stationary in β for every limit $\beta < \omega_2$ because if $\mathrm{cf}(\beta) = \omega$ then β is a limit of a sequence of successor ordinals.

Basic information on 3.1 can be found in Jech [13]; recall that 3.1 follows from Jensen's principle $\square_{\omega_1}$ ([13], Lemma 23.6) and hence it holds true in the constructible universe ([13], Theorem 27.1). In fact one cannot deny 3.1 and prove the consistency of the the statement *every stationary set* $S \subseteq \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$ *reflect at some* $\gamma < \omega_2$ without assuming the existence of large cardinals, see Magidor [12] and [13], page 697.

Construction 3.2. Throughout this section we consider the space $K = \mathrm{ult}(\mathfrak{A})$, where the Boolean algebra $\mathfrak{A}$ is defined as follows.

Fix a set $S \subseteq \omega_2$ as in 3.1. For every $\alpha \in S$ we pick an increasing sequence $(p_n(\alpha))_{n<\omega}$ of ordinals such that $p_n(\alpha) \to \alpha$. Put

$$A_\alpha = \{p_n(\alpha) : n < \omega\}, \quad \text{and} \quad X = \bigcup_{\alpha \in S} A_\alpha.$$

Finally, let $\mathfrak{A}$ be the algebra of subsets of X generated by finite subsets of X together with the family $\{A_\alpha : \alpha \in S\}$.

We shall prove that $K = \mathrm{ult}(\mathfrak{A})$ is not Corson compact because S is stationary in ω_2 while the absence of stationary reflection for S implies that $\mathrm{ult}(\mathfrak{B})$ is Eberlein compact for every small subalgebra $\mathfrak{B}$ of $\mathfrak{A}$.

Lemma 3.3. *If* $\mathfrak{A}$ *is the algebra defined in 3.2 then the space* $\mathrm{ult}(\mathfrak{A})$ *is not Corson compact*

Proof. Suppose that there is a family $\mathcal{G} \subseteq \mathfrak{A}$ as in Lemma 2.2(ii). Note that every $A \in \mathfrak{A}$ is either countable or co-countable in X. The family $\mathcal{G}_0 = \{G \in \mathcal{G} : |X \setminus G| \le \omega\}$ is centred so it is at most countable. Hence, replacing every $G \in \mathcal{G}_0$ by its complement, we may assume that every $G \in \mathcal{G}$ is countable.

Let $\mathcal{G}_1 = \{G \in \mathcal{G} : |G| = \omega\}$. Note that every $G \in \mathcal{G}_1$ is, modulo a finite set, a finite union of sets A_α.

For every $\alpha \in S$ there must be $G_\alpha \in \mathcal{G}_1$ such that $|A_\alpha \cap G_\alpha| = \omega$. Indeed, otherwise A_α would be almost disjoint from every $G \in \mathcal{G}_1$ so would not be in the algebra generated by $\mathcal{G}$. Note that the function $\alpha \to G_\alpha$ is finite-to-one.

It follows that for every $\alpha \in S$ there is $\varphi(\alpha) < \alpha$ such that $\varphi(\alpha) \in G_\alpha$. By the pressing down lemma, there is ξ such that the set $\{\alpha \in S : \varphi(\alpha) = \xi\}$ is stationary. It follows that $\{G \in \mathcal{G}_1 : \xi \in G\}$ is of cardinality ω_2, and this is a contradiction. □

The second part of the argument is based on the following auxiliary result which is stated in a slightly stronger form suitable for inductive argument.

Lemma 3.4. *For every* β, γ *such that* $\beta < \gamma < \omega_2$ *there is a family*

$$\mathcal{B}(\beta, \gamma) = \{B_\alpha : \alpha \in S \cap (\beta, \gamma)\},$$

such that

(i) $B_\alpha \subseteq A_\alpha \setminus \beta$ *and* $|A_\alpha \setminus B_\alpha| < \omega$ *for every* $\alpha \in S \cap (\beta, \gamma)$*;*
(ii) $B_\alpha \cap B_{\alpha'} = \emptyset$ *whenever* $\alpha, \alpha' \in S \cap (\beta, \gamma)$ *and* $\alpha \neq \alpha'$*.*

Proof. We prove the assertion by induction on γ.

For the successor step $\gamma \to \gamma + 1$ there is nothing to prove in case $\gamma \notin S$. Suppose $\gamma \in S$ and take $\mathcal{B}(\beta, \gamma)$ satisfying (i) and (ii). Then $B_\alpha \cap A_\gamma$ is finite for every $\alpha < \gamma$, $\alpha \in S$ and therefore

$$\{B_\alpha \setminus A_\gamma : \alpha \in (\beta, \gamma) \cap S\} \cup \{A_\gamma \setminus \beta\},$$

is the required family for the interval (β, γ).

Suppose that γ is a limit ordinal. Then $S \cap \gamma$ is not stationary in γ so there is a closed unbounded set $C \subseteq \gamma$ such that $C \cap S = \emptyset$. In other words, $S \cap \gamma$ is contained in a set $\gamma \setminus C$ which is open and hence is a union of disjoint subintervals.

Fix $\beta < \gamma$. If $\xi, \eta \in C$, $\beta < \xi < \eta$ and $(\xi, \eta) \cap C = \emptyset$ then we can apply the inductive assumption to $S \cap (\xi, \eta)$ and get the required family $\mathcal{B}(\xi, \eta)$. The union of families obtained in this way is clearly the family that satisfies (i) and (ii). □

Lemma 3.5. *For every algebra* $\mathfrak{B} \subseteq \mathfrak{A}$*, where* $\mathfrak{A}$ *is as in 3.2, if* $|\mathfrak{B}| \leq \omega_1$ *then the space* $\mathrm{ult}(\mathfrak{B})$ *is uniform Eberlein compact.*

Proof. Let $\gamma < \omega_2$ and let $\mathfrak{B}_\gamma$ be a subalgebra of $\mathfrak{A}$ generated by all finite sets in X and the family $\{A_\alpha : \alpha < \beta\}$. It follows directly from Lemma 3.4 that $\mathfrak{B}_\gamma$ has a generating family $\mathcal{G}$ such that are no three different elements in $\mathcal{G}$ having nonempty intersection. Then the space $\mathrm{ult}(\mathfrak{B}_\gamma)$ can be embedded into $\sigma_2(2^\gamma)$ (as in Lemma 2.2) so it is uniform Eberlein compact.

Now every subalgebra $\mathfrak{A} \subseteq \mathfrak{A}$ of size $\leq \omega_1$ is included in $\mathfrak{B}_\gamma$ for some $\gamma < \omega_2$. Hence $\mathrm{ult}(\mathfrak{B})$ is a continuous image of $\mathrm{ult}(\mathfrak{B}_\gamma)$ and thus it is uniform Eberlein compact as well. □

The following answers simultaneously, subject to our set-theoretic assumption , Questions 4 and 5 in [20].

Theorem 3.6. *Assume 3.1. There is a scattered compact space* K *with the third derivative empty such that*

(i) K *is not Corson compact (in fact it is not* ω_2-*Corson compact in the sense of* [7]*);*
(ii) *If* L *is a continuous image of* K *and* $w(L) \leq \omega_1$ *then* L *is uniform Eberlein compact.*

Proof. We take $K = \mathrm{ult}(\mathfrak{A})$, where $\mathfrak{A}$ is the algebra defined above in 3.2. Since K is a Stone space of an algebra generated by an almost disjoint family, it is clear that $K^{(3)} = \emptyset$. Indeed, every ultrafilter $x \in \mathrm{ult}(\mathfrak{A})$ is either principal or there is a unique α such that $A_\alpha \in X$ or else $x \in K^{(2)}$ is the unique ultrafilter containing all $X \setminus A_\alpha$.

Then K is not Corson compact by Lemma 3.3. If L is a continuous images of K and $w(L) \leq \omega_1$ then L is uniform Eberlein compact by Theorem 2.1, Lemma 3.5 and the fact that uniform Eberlein compacta are stable under continuous images. □

As we mentioned in the introduction, the space K from Theorem 3.6 settles in the negative some reflection problems of type 1.1.

Lemma 3.7. *Let (P) be a property of compact space that is stable under taking closed subspaces. If K is a compact space and all continuous images of weight $\le \omega_1$ have property (P) then all closed subsets L of K of cardinality $\le \omega_1$ have property (P).*

Proof. Take a closed subspace $L \subseteq K$ with $|L| \le \omega_1$. Then there is a family $\mathcal{F}$ of continuous functions $K \to [0,1]$ such that $|\mathcal{F}| \le \omega_1$ and $\mathcal{F}$ distinguishes points of L. Let $g : K \to [0,1]^{\mathcal{F}}$ be the diagonal mapping, i.e. $g(x)(f) = f(x)$ for $f \in \mathcal{F}$. Then $\widetilde{K} = g[K] \subseteq [0,1]^{\mathcal{F}}$ so $w(\widetilde{K}) \le |\mathcal{F}| \le \omega_1$ and hence $\widetilde{K}$ has property (P). It follows that $\widetilde{L} = g[L] \subseteq \widetilde{K}$ also has property (P), and $\widetilde{L}$ is homeomorphic to L. □

Corollary 3.8. *Assume 3.1 and take the space K as in Theorem 3.6. Then K is not Corson compact while for every $Y \subseteq K$, if $|Y| \le \omega_1$ then $\overline{Y}$ is uniform Eberlein compact.*

Proof. Recall that X was defined as the union of all the sets A_α, $\alpha \in S$. If $Y \subseteq X$ and $|Y| \le \omega_1$ then $Y \subseteq X \cap \gamma$ for some $\gamma < \omega_2$ and this easily implies that $|\overline{Y}| \le \omega_1$. If $Y \subseteq K^{(1)}$ then $\overline{Y} \subseteq Y \cup \{\infty\}$, where ∞ is the only point in $K^{(2)}$.

We conclude that $|\overline{Y}| \le \omega_1$ for every $Y \subseteq K$ with $|Y| \le \omega_1$ and the assertion follows from Lemma 3.7 □

The corollary above gives partial negative answers to problems posed by Jardón and Tkachuk ([6], Questions 4.13-15) if we assume the continuum hypothesis together with 3.1, so for instance if we are in the constructible universe.

4. On countable functional tightness

Definition 4.1. For a topological space X and a cardinal number κ we write $\tau_0(X) \le \kappa$ if every function $f : X \to \mathbb{R}$ is continuous provided $f_{|Y} : Y \to \mathbb{R}$ is continuous for every subspace $Y \subseteq X$ with $|Y| \le \kappa$. The corresponding cardinal number $\tau_0(X)$ is called the *functional tightness* of the space X.

Recall that $\tau(X)$, the tightness of a space X is defined so that for every $A \subseteq X$ and every $x \in \overline{A}$ there is $B \subseteq A$ such that $|B| \le \tau(X)$ and $x \in \overline{B}$. The following fact can be found in [1].

Lemma 4.2. *The functional tightness $\tau_0(X)$ does not exceed the density of X for every space X. In particular, $\tau_0(X) \le \tau(X)$.*

Tkachuk (see Theorem 2.11 in [19]) proved that if K is a compact space of uncountable tightness then K has a continuous image L of uncountable tightness with $w(L) = \omega_1$.

Recall that $\tau_0(2^\kappa) = \omega$ if and only if there are no measurable cardinals $\le \kappa$, see Uspenskii [21], cf. [16]. Using this theorem it is noted in [20] that if there are measurable cardinals then the countable functional tightness does not reflect in small continuous images of compacta.

Let λ be the Lebesgue measure on $[0,1]$. Write $\mathcal{N}$ for the ideal of λ-null sets. Recall that the assertion $\mathrm{cov}(\mathcal{N}) > \omega_1$ means that $[0,1]$ cannot be covered by ω_1-many sets from $\mathcal{N}$.

We shall work in the measure algebra $\mathfrak{A}$ of the Lebesgue measure on $[0,1]$; the corresponding measure on $\mathfrak{A}$ is still denoted by λ. The following is an immediate consequence of a result due to Kamburelis [8], Lemma 3.1; see also [3], Theorem 4.4.

Theorem 4.3. *If* $\mathrm{cov}(\mathcal{N}) > \omega_1$ *then every continuous image of* $\mathrm{ult}(\mathfrak{A})$ *of* π*-weight* $\leq \omega_1$ *is separable.*

We can now give a (partial) negative solution to Question 4.3 from [20].

Theorem 4.4. *Assuming* $\mathrm{cov}(\mathcal{N}) > \omega_1$, *there is a compact space* S *with* $\tau_0(S) > \omega$, *such that* $\tau_0(L) = \omega$ *for every continuous image* L *of* S *of weight* ω_1.

Our result is based on the construction described in the following lemma.

Lemma 4.5. *Let* $(s_n)_n$ *be a pairwise disjoint sequence in* $\mathfrak{A}^+$. *Let*

$$\mathcal{F} = \{a \in \mathfrak{A} : \lim_n \lambda(a \cap s_n)/\lambda(s_n) = 1\},$$

$$F = \{x \in \mathrm{ult}(\mathfrak{A}) : \mathcal{F} \subseteq x\}.$$

Then

(i) $\mathcal{F}$ *is a non-principal filter in* $\mathfrak{A}$;
(ii) F *is a closed subset of* $\mathrm{ult}(\mathfrak{A})$ *with empty interior;*
(iii) for every countable $Y \subseteq \mathrm{ult}(\mathfrak{A}) \setminus F$ *we have* $\overline{Y} \cap F = \emptyset$.

Proof. Part (i) follows by standard calculations and part (ii) is a direct consequence of (i). We shall check (iii). Let $Y = \{y_n : n \in \omega\} \subseteq \mathrm{ult}(\mathfrak{A}) \setminus F$. For every n we have $y_n \notin F$ so there is $a_0^n \in y_n$ such that $-a_0^n \in \mathcal{F}$. Then we choose a decreasing sequence $(a_k^n)_k$ such that

(a) $a_k^n \leq a_0^n$, $a_k^n \in y_n$ for every k
(b) $\lim_k \lambda(a_k^n) = 0$.

The following fact can be proved by a standard diagonalization (cf. [11]).

Claim. There is a function $g : \omega \to \omega$ such that writing $a_g := \bigcup_{n\in\omega} a^n_{g(n)}$, we have $\widehat{a_g} \cap F = \emptyset$.

Using Claim we get $Y \subseteq \widehat{a_g}$ and it follows that $\overline{Y} \cap F = \emptyset$. □

Proof. (of Theorem 4.4). Let S be the Stone space of the measure algebra $\mathfrak{A}$. Take the set $F \subseteq S$ from Lemma 4.5. Then condition (iii) implies that the function $\chi_F : S \to \mathbb{R}$ is continuous on every countable subspace of S. But χ_F is clearly not continuous because the interior of $\mathcal{F}$ is empty. Hence $\tau_0(S) > 0$.

Let now L be a continuous image of S such that $w(L) \leq \omega_1$. Then L is separable by Theorem 4.3 and $\tau_0(L) = \omega$ by Lemma 4.2, so the proof is complete. □

Remark 4.6. We enclose some remarks concerning Theorem 4.4

(1) Lemma 4.4 originates in Kunen [11]; see Plebanek [17] for other applications.
(2) In fact, under $\mathsf{M}A(\omega_1)$ one can check that in the proof we actually get $\tau_0(S) > \omega_1$, since under $\mathsf{M}A(\omega_1)$ one can strengthen (iii) of Lemma 4.5 to saying that $\overline{Y} \cap F = \emptyset$ for every $Y \subseteq S \setminus F$ with $|Y| \le \omega_1$.
(3) The proof of 4.4 says a bit more, that $\tau_0(L) = \omega$ whenever L is a continuous image of S having a π-base of cardinality $\le \omega_1$.

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