An Observation related to the method of Lemke-Hobson

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Introduction

Even though Nash´s theorem asserts the existence of equilibrium points for the mixed extension of a normal game, it does not tell us how to find them. Even in the case of two matrices or two person games many algorithms have been proposed by Vorobev[13], Kuhn[3] and Mangasarian[5] to determine all equilibrium pairs, they are more of theoretical interest than for actual computation. The algorithm proposed by Lemke and Howson[4] seems until now to be one of the most effective for finding an equilibrium pair.

In the case of n-person games there have been some attempts in order to get the computation of an equilibrium point, let us mention the work of Rosenmüller[11], Sobel[12], Wilson[14] and Garcia, Lemke and Luethi[2]. This last reference is relationed with the simplicial approximation of equilibrium points. This suggested the constructive, combinatorial approach of Scarf for finding, among other things, approximations to fixed-points of continuous mappings. However, the effective computation even in simple cases and low dimension is still open and very intricate questions which have to be attacked and solved in the future.

This paper deals with a new approach based on the rather effective algorithm Lemke and Howson to general n-person games. The definition of general n-person games is going to be accordingly in the next pages.

A new approach

We assume that the readers shall know the basic definitions about n-person games given in Burger[1] and the material in chapter VII of Parthasarathy – Raghavan[10] regarding the bi-matrix case. Here we follow the notation presented in the last book.

At a first step, we consider the problem of a 3-person non-cooperative game to extend the method of Lemke and Howson. Let A_1, A_2, A_3 payoff matrices of dimension $m\times n, n\times s$ and $s\times m$ respectively; x^1 and $\bar{x}^1$ vectors in R^m, x^2 and $\bar{x}^2$ vectors in R^n and x^3 and $\bar{x}^3$ vectors in R^s. By $A_i > 0$ we mean that all entries in A_i are positive and by $x \geq 0$ that all entries in x are nonnegative and let $e = (1,1,\ldots,1)$ be appropriated vector. A 3-person non-cooperative game is given by $\Gamma = (X_i; B_i, i = 1,2,3)$, where $X_i = \{x^i \in R^{n_i} : x^i \geq 0, (x^i, e) = 1\}$ and the payoff functions $B_i(x^1, x^2, x^3) = x^i A_i x^{i+1}$. (We take $i+1=1$ if $i=3$)

Definition 1: A *point* $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ *is an equilibrium point of the 3-person game if:*

$$\begin{aligned} &\bar{x}^1 A_1 \bar{x}^2 \geq x^1 A_1 \bar{x}^2 \qquad \forall x^1 \geq 0 \qquad (x^1, e) = 1 \\ &\bar{x}^2 A_2 \bar{x}^3 \geq x^2 A_2 \bar{x}^3 \qquad \forall x^2 \geq 0 \qquad (x^2, e) = 1 \\ &\bar{x}^3 A_3 \bar{x}^1 \geq x^3 A_3 \bar{x}^1 \qquad \forall x^3 \geq 0 \qquad (x^3, e) = 1 \end{aligned} \tag{1}$$

Theorem 1: *A point* $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ *is an equilibrium point of the 3-person game if and only if for some scalars* p_1, p_2 *and* p_3 *which satisfies* $A_1\bar{x}^2 \leq p_1 e, A_2\bar{x}^3 \leq p_2 e$ *and* $A_3\bar{x}^1 \leq p_3 e$, *respectively, we have:*

$$\begin{aligned} &(\bar{x}^1, A_1\bar{x}^2 + A_3^t\bar{x}^3) = p_1 + p_3 \\ &(\bar{x}^2, A_2\bar{x}^3 + A_1^t\bar{x}^1) = p_2 + p_1 \end{aligned} \tag{2}$$

$$\left(\bar{x}^{3}, A_{3}\bar{x}^{1} + A_{2}^{t}\bar{x}^{2}\right) = p_{3} + p_{2}$$

Proof: $A_1\bar{x}^2 \le p_1 e$ implies that $x^1 A_1 \bar{x}^2 \le p_1$ for all x^1, by the same reason we have that $x^3 A_3 \bar{x}^1 \le p_3$ and $x^2 A_2 \bar{x}^3 \le p_2$ for all x^3 and x^2, respectively. Define $p_1 = \bar{x}^1 A_1 \bar{x}^2$, $p_2 = A_2 \bar{x}^3$ and $p_3 = \bar{x}^3 A_3 \bar{x}^1$, then we obtain (2).

On the other hand we have $p_1 + p_3 = \left(\bar{x}^1, A_1\bar{x}^2 + A_3^t\bar{x}^3\right) \le p_1 + \bar{x}^1 A_3^t \bar{x}^3$, then $p_3 \le \bar{x}^1 A_3^t \bar{x}^3$. But by hypothesis $A_3\bar{x}^1 \le p_3 e$, so $\bar{x}^3 A_3 \bar{x}^1 \le p_3$, this means that $p_3 = \bar{x}^1 A_3^t \bar{x}^3$, then $x^3 A_3 \bar{x}^1 \le p_3 = \bar{x}^3 A_3 \bar{x}^1$ and (1) is valid. □

Theorem 2: A *point* $\left(\bar{x}^1, \bar{x}^2, \bar{x}^3\right)$ *is an equilibrium point of the 3-person game if and only if* $\left(\bar{x}^1, \bar{x}^2, \bar{x}^3, p_1, p_2, p_3\right)$ *is a solution to the problem:*

$$max\left\{\left(x^1, A_1x^2 + A_3^t x^3\right) + \left(x^2, A_2x^3 + A_1^t x^1\right) + \left(x^3, A_3x^1 + A_2^t x^2\right) - 2\left(p_1 + p_2 + p_3\right)\right\}$$

subject to:

$$A_1\bar{x}^2 \le p_1 e; A_2\bar{x}^3 \le p_2 e; A_3\bar{x}^1 \le p_3 e; x^1, x^2, x^3 \ge 0$$
$$\left(x^1, e\right) = \left(x^2, e\right) = \left(x^3, e\right) = 1$$

Proof: if $\left(\bar{x}^1, \bar{x}^2, \bar{x}^3\right)$ is an equilibrium point, using definition 2 we see that it is a solution to the maximization problem. On the other hand, if $\left(\bar{x}^1, \bar{x}^2, \bar{x}^3, p_1, p_2, p_3\right)$ is a solution to the maximization problem, because the objective function is nonpositive and reaches the maximum when it is zero we obtain $\bar{x}^1 A_1 \bar{x}^2 = p_1; \bar{x}^2 A_2 \bar{x}^3 = p_2$ and $\bar{x}^3 A_3 \bar{x}^1 = p_3$, therefore it is an equilibrium point. □

Let us consider the convex sets:

$$S_1 = \left\{\left(x^1, p_3\right): A_3x^1 - p_3 e \le 0, x^1 \ge 0, \left(x^1, e\right) = 1\right\}$$
$$S_2 = \left\{\left(x^2, p_1\right): A_1x^2 - p_1 e \le 0, x^2 \ge 0, \left(x^2, e\right) = 1\right\}$$
$$S_3 = \left\{\left(x^3, p_2\right): A_2x^3 - p_2 e \le 0, x^3 \ge 0, \left(x^3, e\right) = 1\right\}$$

Definition 2: $\left(\bar{x}^1, \bar{x}^2, \bar{x}^3, p_1, p_2, p_3\right)$ *is called an extreme equilibrium point if* $\left(\bar{x}^1, p_3\right)$ *is an extreme point of* S_1, $\left(\bar{x}^2, p_1\right)$ *is an extreme point of* S_2 *and* $\left(\bar{x}^3, p_2\right)$ *is an extreme point of* S_3. *Furthermore,* $\left(\bar{x}^1, \bar{x}^2, \bar{x}^3\right)$ *is an equilibrium point for* A_1, A_2, A_3.

Theorem 3: A*ny equilibrium point of a 3-person game is convex combination of some extreme equilibrium points.*

From here we can describe all extreme points of S_1, S_2 and S_3 as those points which satisfy

$$\left(x^1, A_1x^2 + A_3^t x^3\right) + \left(x^2, A_2x^3 + A_1^t x^1\right) + \left(x^3, A_3x^1 + A_2^t x^2\right) - 2\left(p_1 + p_2 + p_3\right) = 0$$

N-person game model

Let $A_1, A_2, ..., A_n$ payoff matrices of dimension $m_1 \times m_2, m_2 \times m_3, ..., m_n \times m_1$ respectively; x^i and $\overline{x}^i$ vectors in $R^{m_i}, i = 1,...,n$. A n-person non-cooperative game is given by $\Gamma = (X_i; B_i, i = 1,...,n)$, where $X_i = \{x^i \in R^{m_i} : x^i \geq 0, (x^i, e) = 1\}$ and the payoff functions $B_i(x^1,...,x^n) = x^i A_i x^{i+1}$. (We take $i + 1 = 1$ if $i = n$)

Definition 3: A *point* $(\overline{x}^1, \overline{x}^2, ..., \overline{x}^n)$ *is an equilibrium point of the n-person game if:*

$$\overline{x}^i A_i \overline{x}^{i+1} \geq x^i A_i \overline{x}^{i+1} \qquad \forall x^i \geq 0 \qquad (x^i, e) = 1 \qquad i = 1,...,n$$

Theorem 4: *A point* $(\overline{x}^1, \overline{x}^2, ..., \overline{x}^n)$ *is an equilibrium point of the n-person game if and only if for some scalars* $p_1, p_2, ..., p_n$ *which satisfies* $A_i \overline{x}^{i+1} \leq p_i e, i = 1,...,n$ *, respectively, we have:*

$$(\overline{x}^i, A_i \overline{x}^{i+1} + A_{i-1}^t \overline{x}^{i-1}) = p_i + p_{i-1}$$

Conclusion

We have just extended this method for general n-person non-cooperative games having cycles in the interaction among players. Even if this is a relevant approach result, we are far for solving all the n-person games. We would like to emphasize that we may get similar results in games with rational payoff functions studied by Marchi[6] in a possible approach for studying and computing friendly equilibriums points (Marchi[7]), even if this seems natural, it

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