THE κ^+-ANTICHAIN PROPERTY FOR $(\kappa, 1)$-SIMPLIFIED MORASSES

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Abstract. We discuss Shelah, Väänänen and Veličković's recently introduced κ^+-antichain property for $(\kappa, 1)$-simplified morasses. We give a streamlined characterization of the property, and show how the property can be destroyed by forcing and hence that it is consistent that no $(\omega, 1)$-simplified morasses have the property. We briefly touch on the combination of the antichain property with complete amalgamation systems.

Preliminaries

We start by fixing some standard notation and reminding the reader of the definition of $(\kappa, 1)$-simplified morasses, in order to be able to introduce an extremely interesting partial order on κ^+, compatible with the usual ordering of the ordinals, recently isolated by Shelah, Väänänen and Veličković ([5]).

Notation 1.1. For a set X and cardinal κ, $[X]^\kappa = \{Y \subseteq X \mid \overline{\overline{Y}} = \kappa\}$.

Notation 1.2. If $\tau < \theta$ are ordinals the set of order preserving functions from τ to θ, $\{f \mid f : \tau \longrightarrow_{o.p.} \theta\}$, is denoted $(\theta)^\tau$.

Notation 1.3. The *strong supremum* of a set of ordinals X, $\mathrm{ssup}(X)$, is the least γ such that $X \subseteq \gamma$. So $\mathrm{ssup}(X)$ is $\max(X) + 1$ if X has a maximal element and $\sup(X)$ otherwise.

Notation 1.4. If κ is a regular cardinal then $\mathrm{Add}(\kappa, 1)$ is the usual forcing to add a single Cohen subset of κ: the set of partial functions of size $<\kappa$ from κ to 2 ordered by reverse inclusion.

Definition 1.5. If $\tau < \theta$ are ordinals then $\mathcal{F} = \{\mathrm{id}, h\} \subseteq (\theta)^\tau$ is an *amalgamation pair* if there is some $\sigma < \tau$ such that

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- $h \restriction \sigma = \mathrm{id}$,
- for all ξ such that $\sigma + \xi < \tau$ we have $h(\sigma + \xi) = \tau + \xi$, and
- $\tau \cup h``\tau$ is an initial segment of θ.

In this case we say that σ is the *splitting point* of $\mathcal{F}$. We say $\mathcal{F}$ is *exact* if $\theta = \tau \cup h``\tau$ and *almost exact* if $\theta = (\tau \cup h``\tau) + 1$.

Let κ be a regular cardinal.

Definition 1.6. ([6]) $\mathcal{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa \rangle \rangle$ is a $(\kappa, 1)$-*simplified morass* if

- $\langle \theta_\alpha \mid i < \kappa \rangle \in (\kappa)^\kappa$ and $\theta_\kappa = \kappa^+$
- for each $\alpha \leq \beta \leq \kappa$ one has that $\mathcal{F}_{\alpha\beta} \subseteq \{ f \mid f : \theta_\alpha \longrightarrow_{o.p.} \theta_\beta \}$ and
  - for each $\alpha \leq \kappa$, $\quad \mathcal{F}_{\alpha\alpha} = \{\mathrm{id}\}$
  - for each $\alpha < \kappa$,
    $$\mathcal{F}_{\alpha\alpha+1} \text{ is a singleton or an amalgamation pair}$$
  - for each $\alpha \leq \beta \leq \gamma \leq \kappa$,
    $$\mathcal{F}_{\alpha\gamma} = \{ g \cdot f \mid f \in \mathcal{F}_{\alpha\beta} \,\&\, g \in \mathcal{F}_{\beta\gamma} \}$$
  - for each $\alpha \leq \beta < \kappa$, $\quad \overline{\overline{\mathcal{F}_{\alpha\beta}}} < \kappa$
- if $\varepsilon \leq \kappa$ is a limit ordinal then $\mathcal{M}$ is *directed at* ε:

  if α, $\beta < \varepsilon$, $e_\alpha \in \mathcal{F}_{\alpha\varepsilon}$ and $e_\beta \in \mathcal{F}_{\beta\varepsilon}$ there are $\gamma \in [\alpha \cup \beta, \varepsilon)$, $g \in \mathcal{F}_{\gamma\varepsilon}$, $f_\alpha \in \mathcal{F}_{\alpha\gamma}$ and $f_\beta \in \mathcal{F}_{\beta\gamma}$ such that $e_\alpha = g \cdot f_\alpha$ and $e_\beta = g \cdot f_\beta$
- $\bigcup \{ f``\theta_\alpha \mid \alpha < \kappa \,\&\, f \in \mathcal{F}_{\alpha\kappa} \} = \kappa^+$.

We say $\mathcal{M}$ is *neat* if $\theta_\beta = \bigcup \{ f``\theta_\alpha \mid f \in \mathcal{F}_{\alpha\beta} \}$ for all $\alpha < \beta < \kappa$ (and $\theta_0 = 1$).

From now on let $\mathcal{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa \rangle \rangle$ be a $(\kappa, 1)$-simplified morass.

Recall Stanley's important lemma.

Lemma 1.7. ([6], Stanley). *If $\alpha \leq \beta \leq \kappa$, ξ_0, $\xi_1 < \theta_\alpha$ and $\xi < \theta_\beta$, f_0, $f_1 \in \mathcal{F}_{\alpha\beta}$, and $f(\xi_0) = f(\xi_1) = \xi$, then $\xi_0 = \xi_1$ and $f_0 \restriction \xi_0 = f_1 \restriction \xi_0$.*

This motivates the following two pieces of notation.

Notation 1.8. (*cf.* [2], [3]) For $\alpha \leq \kappa$ write ξ_α for the unique ξ' such that there is some $f \in \mathcal{F}_{\alpha\kappa}$ with $f(\xi') = \xi$, and write ψ^α_ξ for $f \restriction \xi_\alpha$. (In [3] the extended notation, $\psi_{(\alpha,\xi_\alpha),(\kappa,\xi)}$, was also used for ψ^α_ξ.)

Notation 1.9. If $\alpha \leq \beta \leq \kappa$, $\xi < \theta_\alpha$ and $f \in \mathcal{F}_{\alpha\beta}$ then $(\alpha, \xi) \triangleleft (\beta, f(\xi))$.

By Stanley's lemma $\triangleleft$ is a (collection of) tree(s).

It is also useful to have a name for the function that marks at which level below κ two elements of κ^+ separate (*i.e.*, *branch apart*).

Notation 1.10. ([2]) If ξ, $\zeta < \kappa$ then $b(\xi, \zeta) =$ the least $\alpha + 1$ such that $\xi_{\alpha+1} \neq \zeta_{\alpha+1}$.

2. $\prec$ AND THE κ^+-ANTICHAIN PROPERTY

We continue to let $\mathcal{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa \rangle \rangle$ be a $(\kappa, 1)$-simplified morass.

Shelah, Väänänen and Veličković ([5]) isolated a very interesting partial order, compatible with the usual ordering of the ordinals, on κ^+.

Definition 2.1. ([5]) For ξ, $\zeta < \kappa^+$ define $\xi \prec \zeta$ if and only if $\xi_\alpha \leq \zeta_\alpha$ for all $\alpha \leq \kappa$ for which both ξ_α and ζ_α are defined.

Note that $\xi_\kappa = \xi$ for all $\xi < \kappa^+$, so $\xi \prec \zeta$ implies $\xi < \zeta$.

Definition 2.2. $\mathcal{M}$ has the κ^+-*antichain property* if for every $X \in [\kappa^+]^{\kappa^+}$ there are ξ, $\zeta \in X$ such that $\xi \prec \zeta$.

Shelah, Väänänen and Veličković showed in [5] that the usual forcing for adding an $(\omega_1, 1)$-simplified morass (as in [6], [10]) actually adds one with the ω_2-antichain property. Their proof, in fact, applies just as well for arbitrary regular κ in place of ω_1.

Shelah, Väänänen and Veličković's original definition of the antichain property (which they gave in the case $\kappa = \omega_1$) in fact talks about an order induced by $\prec$ on sequences of ordinals.

Definition 2.3. $\mathcal{M}$ has the *SVV-κ^+-antichain property* if for every for every $\delta < \kappa$ and $X \in [(\kappa^+)^\delta]^{\kappa^+}$ there are s, $t \in X$ such that $\mathrm{dom}(s) = \mathrm{dom}(t) = \delta$ and for all $\gamma < \delta$ we have $s(\gamma) \prec t(\gamma)$

Note 2.4. $\mathcal{M}$ has the *SVV-κ^+-antichain property* if and only if for every $X \in [(\kappa^+)^{<\kappa}]^{\kappa^+}$ there are s, $t \in X$ such that $\mathrm{dom}(s) = \mathrm{dom}(t)$ and for all $\tau \in \mathrm{dom}(s)$ we have $s(\tau) \prec t(\tau)$.

Proof. Immediate $\star_{2.4}$

Proposition 2.5. *If $\mathcal{M}$ has the SVV-κ^+-antichain property it has the κ^+-antichain property.*

Proof. We prove the contrapositive. If $X \in [\kappa^+]^{\kappa^+}$ and $\delta < \kappa$, by thinning if necessary we may assume that if ξ, $\zeta \in X$ and $\xi < \zeta$ then $\xi + \delta < \zeta$. For $\xi \in X$ let $s^\xi \in (\kappa^+)^{<\kappa}$ be the function given by $s^\xi(\gamma) = \xi + \gamma$. If X is an antichain in $\prec$ then $\{\, s^\xi \mid \xi \in X \,\}$ is an antichain in the product ordering. $\star_{2.5}$

In [5] the SVV-ω_2-antichain property is always proved, mentioned or used in the context of CH holding. The culmination of the next few results, in the case $\kappa = \omega_1$, indicates that this is not mere coincidence.

Notation 2.6. For each $\alpha < \kappa$ and $\xi < \theta_\alpha$ let

$$B_{(\alpha,\xi)} = \{\, f(\xi) \mid f \in \mathcal{F}_{\alpha\kappa} \,\}.$$

As a memnomic, $B_{(\alpha,\xi)}$ is the *blossom* above (α, ξ).

Notation 2.7. Let

$$F = \{\, \xi < \kappa^+ \mid \forall \alpha < \kappa \, (\xi_\alpha \text{ is defined } \longrightarrow \overline{\overline{B_{(\alpha,\xi_\alpha)}}} = \kappa^+) \,\}.$$

So F is the set of elements of κ^+ all of whose predecessors in $\lhd$ have a maximal cardinality collection of blossom above them.

Lemma 2.8. *F is cobounded in κ^+,* i.e, $\overline{\overline{\kappa^+ \setminus F}} \leq \kappa$.

Proof. For each $\xi \in \kappa^+ \setminus F$ let $\alpha(\xi)$ be the least α such that $\overline{\overline{B_{(\alpha,\xi_\alpha)}}} < \kappa^+$. As $\mathcal{M}$ is a $(\kappa, 1)$-simplified morass we have that $\theta_\beta < \kappa$ for all $\beta < \kappa$. Hence $\overline{\overline{\{\, (\alpha(\xi), \xi_{\alpha(\xi)}) \mid \xi \in \kappa^+ \setminus F \,\}}} \leq \kappa$. Thus we have that $\kappa^+ \setminus F \subseteq \bigcup \{\, B_{(\alpha(\xi),\xi_{\alpha(\xi)})} \mid \xi \in \kappa^+ \setminus F \,\}$. However the latter is a union of κ many sets of size κ. $\star_{2.8}$

Lemma 2.9. *If $\xi \in F$ there are unboundedly many $\alpha < \kappa$ such that $\mathcal{F}_{\alpha\alpha+1}$ is an amalgamation pair and $\sigma_\alpha \leq \xi_\alpha = \xi_{\alpha+1}$.*

Proof. Let $\xi \in F$. Suppose, towards a contradiction that the lemma is false for ξ. Let $\beta < \kappa$ be least such that for all $\alpha \in [\beta, \kappa)$ it is not the case that $\mathcal{F}_{\alpha\alpha+1}$ is an amalgamation pair and $\sigma_\alpha \leq \xi_\alpha = \xi_{\alpha+1}$. Then $B_{(\beta,\xi_\beta)} \subseteq \xi + 1$, contradicting the assumption that $\xi \in F$ (and hence $\overline{\overline{B_{(\beta,\xi_\beta)}}} = \kappa^+$). $\star_{2.9}$

Corollary 2.10. *There are antichains of size κ in $\prec\restriction \kappa \times \kappa$. There are antichains of order-type $\kappa + 1$ in $\prec$.*

Proof. Let $\xi \in F \setminus \kappa$. Let $H_\xi = \{\alpha < \kappa \mid \mathcal{F}_{\alpha\alpha+1}$ is an amalgamation pair and $\sigma_\alpha \leq \xi_\alpha = \xi_{\alpha+1}\}$. For each $\alpha \in H_\xi$ we have $\mathcal{F}_{\alpha\alpha+1} = \{\mathrm{id}, h_\alpha\}$. Then $\{h_\alpha(\xi_\alpha) \mid \alpha \in H_\xi\}$ is an antichain in $\prec$, and we still have an antichain if we adjoin ξ to this set. $\star_{2.10}$

Proposition 2.11. *If $\kappa < 2^{<\kappa}$ then no $(\kappa, 1)$-simplified morass $\mathcal{M}$ has the SVV-κ^+-antichain property.*

Proof. Let $\delta < \kappa$ be such that $\kappa^+ \leq \kappa < 2^{<\delta}$. By Corollary (2.10) choose a $\prec$-antichain $\{\xi^i \mid i < \delta\}$ of size δ with $\xi^i < \xi^j$ for $i < j$.

Now let X consist of all distinct increasing sequences of length δ from $\{\xi^i \mid i < \delta\}$. Suppose that s, $t \in X$. Let $\gamma < \delta$ be least such that $s(\gamma) \neq t(\gamma)$. We have that there are distinct $i, j < \delta$ such that $s(\gamma) = \xi^i$ and $t(\gamma) = \xi^j$. As $\{\xi^i \mid i < \delta\}$ is a $\prec$-antichain we have $s(\gamma) \not\prec t(\gamma)$ and $t(\gamma) \not\prec s(\gamma)$. Thus the SVV-$\kappa^+$-antichain property fails. $\star_{2.11}$

Consequently the following result, together with Proposition (2.5), shows that the property defined in Definition (2.2) is a streamlined equivalent of the SVV-κ^+-antichain property when the latter does not simply always fail for cardinal arithmetic reasons. This explains taking Definition (2.2) as our official definition of *the* κ^+-antichain property.

Proposition 2.12. *If $2^{<\kappa} = \kappa$ and $\mathcal{M}$ has the κ^+-antichain property then $\mathcal{M}$ has the SVV-κ^+-antichain property.*

Proof. Again we prove the contrapositive.

Suppose that $X \in [(\kappa^+)^\delta]^{\kappa^+}$. As $2^{<\kappa} = \kappa$, after thinning if necessary, we can suppose that we can enumerate X as $\langle s^i \mid i < \kappa^+\rangle$ with there being some $\rho < \kappa$ such that $s^i \restriction \rho = s^j \restriction \rho$ and $\mathrm{ssup}(\mathrm{rge}(s^i)) < s^j(\rho)$ for $i < j < \kappa^+$.

For $i < \kappa^+$ let $\zeta^i = \mathrm{ssup}(\mathrm{rge}(s^i))$. By thinning again (if necessary), again using $2^{<\kappa} = \kappa$, we may assume that there is some $\alpha < \kappa$, some

$s \in (\theta_\alpha)^{\delta+1}$ and some maps $f^i \in \mathcal{F}_{\alpha\kappa}$ for $i < \kappa^+$, such that for all $i < \kappa^+$ we have $s^i(\gamma) = f^i(s(\gamma))$ for all $\gamma < \delta$ and $f^i(s(\delta)) = \zeta^i$.

Suppose that $i < j < \kappa^+$ and $s^i \not< s^j$. Let $\beta \in [\alpha, \kappa)$ be least such that there is $\gamma < \kappa$ with $(s^j(\gamma))_{\beta+1} < (s^i(\gamma))_{\beta+1}$. Then we have that

$$\sigma_\beta \leq (s^i(\gamma))_\beta \leq (s^j(\gamma))_\beta = (s^j(\gamma))_{\beta+1} < \zeta^j_{\beta+1} < \theta_\beta \leq (s^i(\gamma))_{\beta+1} < \zeta^i_{\beta+1}.$$

Hence we have shown that if X is an antichain of size κ^+ in the product ordering derived from $\prec$ then $\{\zeta^j \mid j < \kappa^+\}$ is an antichain in $\prec$. $\star_{2.12}$

3. Destroying the antichain property

It is worthwhile observing that there are no long chains in $\prec$.

Proposition 3.1. *There are no $\prec$-chains of length κ^+.*

Proof. For s, $t \in (\kappa)^\kappa$ set $s < t$ if $s(\alpha) \leq t(\alpha)$ for all $\alpha < \kappa$ and there is some $\alpha < \kappa$ such that $s(\alpha) < t(\alpha)$. It is well known that there are no (strictly increasing) chains of length κ^+ in $((\kappa)^\kappa, <)$.

(To see the latter, if $\langle s^i \mid i < \kappa^+\rangle$ enumerates a strictly increasing chain in $<$, in increasing order then set $I_0 = \kappa^+$, and, by induction on $\beta \leq \kappa$ set $I_\beta = \bigcap_{\alpha<\beta}\{i \in I_\alpha \mid s^i(\alpha) = \tau_\alpha\}$ and $\tau_\beta = \max(\{s^i(\beta) \mid i \in I_\beta\})$, Then I_κ has size κ^+ and for all $i \in I_\kappa$ and $\alpha < \kappa$ we have $s^i(\alpha) = \tau_\alpha$. This is a contradiction to the chain being increasing.)

In order to prove the proposition, now suppose, towards a contradiction, that $\langle \xi^i \mid i < \kappa^+\rangle$ is a $\prec$-chain of length κ^+. By thinning, if necessary, suppose that there is some $\alpha^* < \kappa$ such that for all $i < \kappa^+$ we have that α^* is the least α such that ξ^i_α is defined. Apply the result that there are no κ^+-chains of length κ^+ in $((\kappa)^\kappa, <)$ to $\langle s^i \mid i < \kappa^+\rangle$ where for $i < \kappa^+$ and $\alpha \in [\alpha^*, \kappa)$ we take $s^i(\alpha) = \xi^i_\alpha$.[1] $\star_{3.1}$

[1] *Morass-y version of the same proof.* Suppose, towards a contradiction, that $X \in [\kappa^+]^{\kappa^+}$ is a $\prec$-chain. For $\alpha < \kappa$ let $X_\alpha = \{\xi \in X \mid \xi_{\alpha+1} < \theta_\alpha\}$. For each $\alpha < \kappa$ we have thatX_α is an initial segment of X, and hence either $\overline{\overline{X_\alpha}} \leq \kappa$ or $X_\alpha = X$. Set $Y = X \setminus \bigcup\{X_\alpha \mid \alpha < \kappa \ \&\ \overline{\overline{X_\alpha}} \leq \kappa\}$. Thus $\overline{\overline{Y}} = \kappa^+$.

Now let $\xi, \zeta \in Y$ and set $\alpha = b(\xi, \zeta)$, so that $\xi_\alpha = \zeta_\alpha$ and $\xi_{\alpha+1} \neq \zeta_{\alpha+1}$. We derive a contradiction. If $\overline{\overline{X_\alpha}} \leq \kappa$ then we would have that $\xi, \zeta \neq X_\alpha$ and hence $\theta_\alpha \leq \xi_{\alpha+1}$, $\zeta_{\alpha+1}$, and so $\xi_{\alpha+1} = h_\alpha(\xi_\alpha) = h_\alpha(\zeta_\alpha) = \zeta_{\alpha+1}$, and thus $b(\xi, \zeta) \neq \alpha$. On the other hand, if $X_\alpha = X$ then $\xi_{\alpha+1}$, $\zeta_{\alpha+1} < \theta_\alpha$ and hence, again $\xi_{\alpha+1} = \xi_\alpha = \zeta_\alpha = \zeta_{\alpha+1}$, and so $b(\xi, \zeta) \neq \alpha$. $\star_{3.1}$

Corollary 3.2. *For each $\delta < \kappa$ there are no $\prec$-chains of length κ^+ in the product order on $(\kappa^+)^\delta$.*

However we can actually prove a stronger result. In order to state it we need to make a definition.

Definition 3.3. Let $\mathbb{P}_\mathcal{M}$ be the forcing consisting of conditions which are small antichains in $\prec$: so $p \in \mathbb{P}_\mathcal{M}$ if $p \in [\kappa^+]^{<\kappa}$ and for all ξ, $\zeta \in p$ we have $\xi \not\prec \zeta$, ordered by $q \leq p$ if $p \subseteq q$.

Observe that with this definition we can restate Proposition (3.1) as that if $\mathfrak{X} \in [\mathbb{P}_\mathcal{M}]^{\kappa^+}$ and for all $p \in \mathfrak{X}$ we have $\overline{\overline{p}} = 1$ then $\mathfrak{X}$ is not an antichain in $\mathbb{P}_\mathcal{M}$.

Proposition 3.4. *Suppose $2^{<\kappa} = \kappa$. If $\mathcal{M}$ has the κ^+-antichain property then $\mathbb{P}_\mathcal{M}$ has the κ^+-chain condition.*

Proof. Let $\mathfrak{A} \in [\mathbb{P}_\mathcal{M}]^{\kappa^+}$ be an antichain and assume, after thinning if necessary, that the set forms a Δ-system with root a, with $\mathrm{ssup}(p) < \min(q \setminus a)$ or vice versa for each pair p, $q \in \mathfrak{A}$.

Let $N \prec H_{\kappa^{++}}$ be an elementary submodel of size κ with $N \cap \kappa^+ = \delta \in \kappa^+$, P, $\mathfrak{A}$, $\mathcal{M} \in N$ and a, $N^{<\kappa} \subseteq N$. Then $\mathcal{M} \cap N = \langle\langle\theta_\alpha \mid \alpha < \kappa\rangle, \langle\mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta < \kappa\rangle\rangle^\frown\langle\delta, \langle\{f \in \mathcal{F}_{\alpha\kappa} \mid \mathrm{rge}(f) \subseteq \delta\} \mid \alpha < \kappa\rangle\rangle$. Thus if $\xi < \delta$, $\alpha < \kappa$ and $\tau < \theta_\alpha$ then $N \models$ "$\xi_\alpha = \tau$" if and only if $\xi_\alpha = \tau$.

Choose p, $q \in \mathfrak{A} \setminus N$ with $\delta \leq \min(p \setminus a)$ and $\mathrm{ssup}(p) < \min(q \setminus a)$. Let $\beta^* =$ the least $\alpha < \kappa$ such that there is some map $f \in \mathcal{F}_{\alpha\kappa}$ with $p \cup q \subseteq \mathrm{rge}(f)$. Note that if $\xi \in p$ and $\zeta \in q \setminus a$ there is some $\alpha \leq \beta^*$ such that $\xi_\alpha < \zeta_\alpha$.

Let $\{q(\gamma) \mid \gamma < \varepsilon\}$ enumerate q in increasing order. For $\gamma < \varepsilon$ and $\alpha \leq \beta^*$ set $y(\gamma, \alpha) = q(\gamma)_\alpha$.

By elementarity and the closure of N there is some $r \in \mathfrak{A} \cap N$ such that, letting $\{r(\gamma) \mid \gamma < \varepsilon\}$ enumerate r in increasing order, for all $\gamma < \varepsilon$ and $\alpha \leq \beta^*$ we have $r(\gamma)_\alpha = y(\gamma, \alpha)$.

But then we have that if $\xi = r(\gamma) \in r$ and $\zeta \in p \setminus a$ there is some $\alpha \leq \beta^*$ such that $q(\gamma)_\alpha < \zeta_\alpha$, and hence $\xi_\alpha = r(\gamma)_\alpha = y(\gamma, \alpha) = q(\gamma)_\alpha < \zeta_\alpha$; and as $r \subseteq \delta$ and $\delta \leq \min(p \setminus a)$ there is some $\alpha < \kappa$ such that $\xi_\alpha < \zeta_\alpha$.

Thus $p \cup r$ is a $\prec$-antichain and hence is a condition in $\mathbb{P}_{\mathcal{M}}$. However we then have that $p \cup r \leq p$, r, thus contradicting $\mathfrak{A}$ being an antichain in $\mathbb{P}_{\mathcal{M}}$. $\star_{3.4}$

Observe that there is some freedom in the argument above: if we also 'reflect' p to a condition s in N, so that – writing informally – we have $s << r << p << q$, rather than amalgamating r and p we could instead amalgamate s and q.

Proposition 3.5. *Suppose $2^{<\kappa} = \kappa$. If $V \models$ "$\mathcal{M}$ has the κ^+-antichain property" and G is $\mathbb{P}_{\mathcal{M}}$-generic over V then $\mathrm{Card}^{V[G]} = \mathrm{Card}^V$ and*

$$V[G] \Vdash \text{“ } \mathcal{M} \text{ does not have the } \kappa^+\text{-antichain property. ”}$$

Proof. The antichain property is destroyed as $\mathbb{P}_{\mathcal{M}}$ generically adds an antichain of length κ^+. Cardinals are preserved since $\mathbb{P}_{\mathcal{M}}$ has the κ^+-chain condition and is κ-closed. $\star_{3.5}$

Let us focus briefly on the case $\kappa = \omega$. Recall Velleman's theorem ([8]) that ZFC implies there are always $(\omega, 1)$-simplified morasses. In contrast we have the following regarding simplified morasses with the ω_1-antichain property.

Corollary 3.6. *MA_{ω_1} implies no $(\omega, 1)$-simplified morass has the ω_1-antichain property.*

Proof. Suppose $\mathcal{M}$ is an $(\omega, 1)$-simplified morass with the ω_1-antichain property. By Proposition (3.4) in the case $\kappa = \omega$ there is there is a ccc forcing to destroy the property and so, applying MA_{ω_1}, $\mathcal{M}$ does not have the ω_1-antichain property – a contradiction. $\star_{3.6}$

Unfortunately one cannot directly generalize Corollary (3.6) to higher cardinals and obtaining a similar independence result. The Appendix of [4] gives examples showing that no generalization of Martin's Axiom for forcings with the κ^+-cc or strengthenings of it can hold for collections of forcings which would include $\mathbb{P}_{\mathcal{M}}$. Those results do not preclude that one could, in principle, iterate this specific forcing in order to reach a model in which no $(\kappa, 1)$-simplified morass has the κ^+-antichain property, however we are not aware of any applicable iteration theorems.

One might wonder whether there is a simpler forcing notion which destroys the antichain property, which one could use instead of $\mathbb{P}_{\mathcal{M}}$, in the hope of side-stepping these difficulties. However there are severe inherent difficulties with such a plan.

Definition 3.7. A forcing notion P has the κ^+-*Knaster property* if given any $X \in [P]^{\kappa^+}$ there is some $Z \in [X]^{\kappa^+}$ such that any two elements of Z are compatible.

Proposition 3.8. *If $\mathcal{M}$ has the κ^+-antichain property and $\mathbb{P}$ has the κ^+-Knaster property then $\Vdash_{\mathbb{P}}$ " $\mathcal{M}$ has the κ^+-antichain property "*

Proof. Suppose that $\Vdash_{\mathbb{P}}$ " $\dot{\mathcal{A}}$ is an antichain of size κ^+ ". Let $p_i \Vdash_{\mathbb{P}}$ " $\xi^i \in \dot{\mathcal{A}}$ " and $\xi^i \in p_i$ for $i < \kappa^+$ and a strictly increasing sequence $\langle \xi^i \mid i < \kappa^+ \rangle$. By the Knaster property let $I \in [\kappa^+]^{\kappa^+}$ be such that for $i, j \in I$ we have that p_i and p_j are compatible; for such i, j let $p_{ij} \leq p_i$, p_j. Then, for $i, j \in I$ we have $p_{ij} \Vdash_{\mathbb{P}}$ " $\xi^i \not\leq \xi^j$ ", and hence $\xi^i \not\leq \xi^j$. So $\langle \xi^i \mid i < \kappa^+ \rangle$ is an antichain of size κ^+ in the ground model. $\star_{3.8}$

Clearly the forcing $\mathbb{P}_{\mathcal{M}}$ used in Proposition (3.4) does not have the κ^+-Knaster property, but we are not aware of any iteration technology which works sucessfully for iterands of this type.

4. Complete amalgamation systems and the antichain property

We make a couple of remarks about a strengthening of the notion of a complete amalgamation system ([9]) and the κ^+-antichain property.

Definition 4.1. ([9]) Let $\langle \langle \rho_\alpha, X_\alpha, Y_\alpha \rangle \mid \alpha < \kappa \rangle$ be a sequence of triples where $\rho_\alpha < \kappa$ and X_α, $Y_\alpha \subseteq \theta_\alpha$ for each $\alpha < \kappa$. Define, by induction on $\alpha \leq \kappa$,

$$\begin{aligned} A_0 =& \emptyset, \\ A_{\alpha+1} =& \{ \langle \rho, f``X, f``Y \rangle \mid f \in \mathcal{F}_{\alpha\alpha+1} \;\&\; \langle \rho, X, Y \rangle \in A_\alpha \} \cup \\ & \qquad \{ \langle \rho_\alpha, X_\alpha, h``Y_\alpha \rangle \}, \text{ and} \\ A_\lambda =& \{ \langle \rho, f``X, f``Y \rangle \mid \exists \alpha < \lambda \, f \in \mathcal{F}_{\alpha\lambda} \;\&\; \langle \rho, X, Y \rangle \in A_\alpha \} \\ & \qquad \text{for limit } \lambda \leq \kappa. \end{aligned}$$

The sequence $\langle \langle \rho_\alpha, X_\alpha, Y_\alpha \rangle \mid \alpha < \kappa \rangle$ is an *amalgamation system* if for all $\alpha < \kappa$ either $X_\alpha = Y_\alpha$ or $\langle \rho_\alpha, X_\alpha, Y_\alpha \rangle \in A_\alpha$ or $\langle \rho_\alpha, Y_\alpha, X_\alpha \rangle \in A_\alpha$.

(New in this paper and *not* taken from [9].) It is a *strong amalgamation system* if for all $\alpha < \kappa$ either $X_\alpha = Y_\alpha$ or $\langle \rho_\alpha, X_\alpha, Y_\alpha \rangle \in A_\alpha$.

It is *complete* if in addition whenever $\rho < \kappa$ and $\mathfrak{X} \in [[\kappa^+]^{<\kappa}]^{\kappa^+}$ there are distinct X, $Y \in \mathfrak{X}$ such that $\langle \rho, X, Y \rangle \in A_\kappa$.

Lemma 4.2. *If $\langle \langle \rho_\alpha, X_\alpha, Y_\alpha \rangle \mid \alpha < \kappa \rangle$ is a complete amalgamation system then $2^{<\kappa} = \kappa$.*

Proof. Let $\lambda < \kappa$. For each $X \subseteq \lambda$ let $\mathfrak{X}_X = \{ X \cup \{ \lambda, \tau \} \mid \tau \in (\lambda, \kappa^+) \}$. By the completeness of the amagamation system there is some $\alpha < \kappa$ and $f \in \mathcal{F}_{\alpha\kappa}$ such that $X_\alpha = \overline{X} \cup \{ \overline{\lambda}, \overline{\tau} \}$, $\overline{X} \subseteq \overline{\lambda} < \overline{\tau} < \theta_\alpha$ $X = f``\overline{X}$, $f(\overline{\lambda}) = \lambda$ and $f(\overline{\tau}) = \tau$. Thus $\mathcal{P}(\lambda) \subseteq \{ \psi^\alpha_{\overline{\lambda}}``(X_\alpha \cap \overline{\lambda}) \mid \alpha < \kappa \}$. $\star_{4.2}$

Theorem 4.3. ([9]). *If $\kappa = \mu^+$ and $2^\mu = \kappa$ there is a complete amalgamation system for every $(\kappa, 1)$-simplified morass.*

Corollary 4.4. *If $\kappa = \mu^+$ and $\mathcal{M}$ is a $(\kappa, 1)$-simplified morass there is a complete amalgamation system for $\mathcal{M}$ if and only if $2^\mu = \kappa$.*

Theorem 4.5. ([9]) *If $V \models 2^{<\kappa} = \kappa$ and c is Add$(\kappa, 1)$-generic over V there is a complete amalgamation system for every $(\kappa, 1)$-simplified morass in $V[c]$. (Of course, as $(\kappa^+)^V = (\kappa^+)^{V[c]}$, every $(\kappa, 1)$-simplified morass in V remains such in $V[c]$.)*

Proposition 4.6. *If there is a complete strong amalgamation system for $\mathcal{M}$ then $\mathcal{M}$ satisfies the κ^+-antichain property.*

Proof. Immediate from the definitions. $\star_{4.6}$

Proposition 4.7. *If $\kappa = \mu^+$ and $2^\mu = \kappa$ there is a complete strong amalgamation system for every $(\kappa, 1)$-simplified morass which satisfies the κ^+-antichain property. If $V \models 2^{<\kappa} = \kappa$ and c is Add$(\kappa, 1)$-generic over V there is a complete strong amalgamation system for every $(\kappa, 1)$-simplified morass in $V[c]$ with the κ^+-antichain property.*

Proof. Exactly as Velleman's proofs, but using the κ^+-antichain property to ensure that one can choose strong amalgamation systems. $\star_{4.7}$

Note that if $V \models 2^{<\kappa} = \kappa$ and c is Add$(\kappa, 1)$-generic over V then every $(\kappa, 1)$-simplified morass in V with the κ^+-antichain property continues to have the κ^+-antichain property in $V[c]$ because Add$(\kappa, 1)$ trivially has the κ^+-Knaster condition.

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