$\Sigma_1(\kappa)$-DEFINABLE SUBSETS OF $\mathrm{H}(\kappa^+)$

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Abstract. We study $\Sigma_1(\omega_1)$-definable sets (i.e. sets that are equal to the collection of all sets satisfying a certain Σ_1-formula with parameter ω_1) in the presence of large cardinals. Our results show that the existence of a Woodin cardinal and a measurable cardinal above it imply that no well-ordering of the reals is $\Sigma_1(\omega_1)$-definable, the set of all stationary subsets of ω_1 is not $\Sigma_1(\omega_1)$-definable and the complement of every $\Sigma_1(\omega_1)$-definable Bernstein subset of ${}^{\omega_1}\omega_1$ is not $\Sigma_1(\omega_1)$-definable. In contrast, we show that the existence of a Woodin cardinal is compatible with the existence of a $\Sigma_1(\omega_1)$-definable well-ordering of $\mathrm{H}(\omega_2)$ and the existence of a $\Delta_1(\omega_1)$-definable Bernstein subset of ${}^{\omega_1}\omega_1$. We also that, if there are infinitely many Woodin cardinals and a measurable cardinal above them, then there is no $\Sigma_1(\omega_1)$-definable uniformization of the club filter on ω_1. Moreover, we prove a perfect set theorem for $\Sigma_1(\omega_1)$-definable subsets of ${}^{\omega_1}\omega_1$, assuming that there is a measurable cardinal and the non-stationary ideal on ω_1 is saturated. The proofs of these results use iterated generic ultrapowers and Woodin's $\mathbb{P}_{\max}$-forcing. Finally, we also prove variants of some of these results for $\Sigma_1(\kappa)$-definable subsets of ${}^{\kappa}\kappa$, in the case where κ itself has certain large cardinal properties.

1. Introduction

Given an uncountable regular cardinal κ, we study subsets of the collection $\mathrm{H}(\kappa^+)$ of all sets of hereditary cardinality at most κ that are definable over $\mathrm{H}(\kappa^+)$ by simple formulas.

Definition 1.1. Let M be a non-empty class, let $R_0,\ldots,R_{n-1}$ be relations on M and let $a_0,\ldots,a_{m-1}$ be elements of M. Set $\mathbb{M}=\langle M,\in,R_0,\ldots,R_{n-1}\rangle$.

(i) A subset X of M is $\Sigma_1(a_0,\ldots,a_{m-1})$-*definable over* $\mathbb{M}$ if there is a Σ_1-formula $\varphi(v_0,\ldots,v_m)$ in the language of set theory extended by predicate symbols $\dot{P}_0,\ldots,\dot{P}_{n-1}$ such that $X=\{x\in M\mid \mathbb{M}\models\varphi(a_0,\ldots,a_{m-1},x)\}$.

(ii) A subset Y of M is $\Pi_1(a_0,\ldots,a_{m-1})$-*definable over* $\mathbb{M}$ if $M\setminus Y$ in M is $\Sigma_1(a_0,\ldots,a_{m-1})$-definable over $\mathbb{M}$.

(iii) A subset of M is $\Delta_1(a_0,\ldots,a_{m-1})$-*definable over* $\mathbb{M}$ if the subset is both $\Sigma_1(a_0,\ldots,a_{m-1})$- and $\Pi_1(a_0,\ldots,a_{m-1})$-definable over $\mathbb{M}$.

Since Σ_1-formulas are absolute between V and $\mathrm{H}(\kappa^+)$, we will not mention the models $\langle \mathrm{V},\in\rangle$ and $\langle \mathrm{H}(\kappa^+),\in\rangle$ in our statements about Σ_1-definability.

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In this paper, we will focus on the following subjects: $\Sigma_1(\kappa)$-definable well-orderings of $\mathrm{H}(\kappa^+)$, $\Delta_1(\kappa)$-definitions of the club filter on κ and $\Delta_1(\kappa)$-definable Bernstein subsets of ${}^\kappa\kappa$ (see Definition 1.3 below). In the case of formulas containing arbitrary parameters from $\mathrm{H}(\kappa^+)$, it was shown that the existence of such objects is independent from ZFC together with large cardinal axioms (see [9], [16] and [18]). Moreover, it is known that such $\Sigma_1(\kappa)$-definitions exists in certain models of set theory that do not contain larger large cardinals (see [4] and [8]). This leaves open the question whether such $\Sigma_1(\kappa)$-definitions are compatible with larger large cardinals. The main results of this paper show that large cardinal axioms imply the non-existence of such definitions for $\kappa = \omega_1$.

Using results of Woodin on the Π_2-maximality of the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$ (see [14] and [31]), it is easy to show that the assumptions that there are infinitely many Woodin cardinals with a measurable cardinal above them all implies that no well-ordering of the reals is $\Sigma_1(\omega_1)$-definable. We will derive this conclusion from a much weaker assumption that is in some sense optimal (see remarks below).

Theorem 1.2. *Assume that there is a Woodin cardinal and a measurable cardinal above it. Then no well-ordering of the reals is $\Sigma_1(\omega_1)$-definable.*

In contrast, we will show that the existence of a $\Sigma_1(\omega_1)$-definable well-ordering of $\mathrm{H}(\omega_2)$ is compatible with the existence of a Woodin cardinal (see Theorem 5.2). Together with the above theorem, this answers [8, Question 1.9].

Given a regular cardinal κ, the *generalized Baire space* for κ consists of the the set ${}^\kappa\kappa$ of all functions from κ to κ equipped with the topology whose basic open sets are of the form $N_s = \{x \in {}^\kappa\kappa \mid s \subseteq x\}$ for some $s : \alpha \longrightarrow \kappa$ with $\alpha < \kappa$.

Definition 1.3. Let κ be a regular cardinal.

(i) A *perfect subset* of ${}^\kappa\kappa$ is the set of branches $[T]$ of a *perfect subtree* of ${}^{<\kappa}\kappa$, i.e. a $<\kappa$-closed tree with branching nodes above all nodes.
(ii) A subset A of ${}^\kappa\kappa$ has the *perfect set property* if either A has cardinality at most κ or A contains a perfect subset.
(iii) A *Bernstein set* is a subset of ${}^\kappa\kappa$ with the property that neither A nor its complement contains a perfect subset.

Theorem 1.4. *Assume that there is a Woodin cardinal and a measurable cardinal above it. Then no Bernstein subset of ${}^{\omega_1}\omega_1$ is $\Delta_1(\omega_1)$-definable over $\langle \mathrm{H}(\omega_2), \in\rangle$.*

We will also show that the large cardinal assumption of the above result is close to optimal by showing that the existence of such a Bernstein subset is compatible with the existence of a Woodin cardinal (see Lemma 5.6).

Next, we consider $\Delta_1(\omega_1)$-definitions of the club filter C_{ω_1} and the nonstationary ideal NS_{ω_1} on ω_1. In [3], Friedman and Wu showed that the existence of a proper class of Woodin cardinals implies that NS_{ω_1} is not $\Delta_1(\omega_1)$-definable. We will derive a stronger conclusion from a weaker hypothesis. In the following, we say that a subset X of $\mathcal{P}(\kappa)$ *separates the club filter from the nonstationary ideal* if X contains C_{ω_1} as a subset and is disjoint from NS_{ω_1}.

Theorem 1.5. *Assume that there is a Woodin cardinal and a measurable cardinal above it. Then no subset of $\mathcal{P}(\omega_1)$ that separates the club filter from the nonstationary ideal is $\Delta_1(\omega_1)$-definable over $\langle \mathrm{H}(\omega_2), \in\rangle$.*

We will in fact prove more general versions of the above theorems. First, we will derive the above conclusions from the assumption that $M_1^\#(A)$ exists for every subset A of ω_1 (see [22, p. 1738] and [28, p. 1660]). This assumption follows from the existence of a Woodin cardinal and a measurable cardinal above it (see [19] and [27]). In Section 2, we will show that it also follows from BMM (Bounded Martin's Maximum) together with the assumption that the nonstationary ideal NS_{ω_1} on ω_1 is precipitous. Second, we will allow as parameters subsets of ω_1 that are $\mathbf{\Sigma}^1_2$-definable in the codes. We will also prove this for all subsets of ω_1 which are universally Baire in the codes, assuming that there is a proper class of Woodin cardinals. Finally, we will prove results on perfect subsets of $\Sigma_1(\omega_1)$-subsets of ${}^{\omega_1}\omega_1$ (see Section 4.3), the nonexistence of $\Sigma_1(\omega_1)$-definable uniformizations of the club filter (see Section 4.5) and the absoluteness of $\Sigma_1(\omega_1)$-statements (see Section 4.6).

The above results raise the question whether large cardinals have a similar influence on $\Sigma_1(\kappa)$-definability for regular cardinals $\kappa > \omega_1$. Variations of the techniques used in the proofs of the above results will allow us to prove analogous statements hold for $\Sigma_1(\kappa)$-definable subsets of $\mathrm{H}(\kappa^+)$ in the case where κ itself has certain large cardinal properties.

Theorem 1.6. *If κ is either a measurable cardinal above a Woodin cardinal or a Woodin cardinal below a measurable cardinal, then there is no $\Sigma_1(\kappa)$-definable well-ordering of the reals.*

Theorem 1.7. *If κ is a measurable cardinal with the property that there are two distinct normal ultrafilters on κ, then no Bernstein subset of ${}^\kappa\kappa$ is $\Delta_1(\kappa)$-definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$.*

In contrast, we will show that consistently there can be a measurable cardinal κ and a Bernstein subset of ${}^\kappa\kappa$ that is $\Delta_1(\kappa)$-definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$.

Next, we consider the $\Pi_1(\kappa)$-definability of sets separating the club filter from the non-stationary ideal at ω_1-*iterable cardinals* (see Definition 6.1).

Theorem 1.8. *If κ is an ω_1-iterable cardinal and X is a subset of $\mathcal{P}(\kappa)$ that separates the club filter from the nonstationary ideal, then X is not $\Delta_1(\kappa)$-definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$.*

Friedman and Wu showed that the club filter on κ is not $\Pi_1(\kappa)$-definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$ if κ is a weakly compact cardinal (see [3, Proposition 2.1]). We will show that this conclusion also holds for stationary limits of ω_1-iterable cardinals. Note that these cardinal need not be weakly compact and Woodin cardinals are stationary limits of ω_1-iterable cardinals.

Theorem 1.9. *If κ is a regular cardinal that is a stationary limit of ω_1-iterable cardinals, then the club filter on κ is not $\Pi_1(\kappa)$-definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$.*

We outline the content of this paper. In Section 2 we will show that the condition that $M_1^\#(A)$ exists for all subsets A of ω_1 follows from BMM and the assumption that the non-stationary ideal NS_{ω_1} on ω_1 is saturated. In Section 3 we characterize $\Sigma_1(\omega_1)$-definable sets of reals and extend this characterization to formulas with universally Baire parameters, assuming that there is a proper class of Woodin cardinals. In Section 4 we prove the main results about $\Sigma_1(\omega_1)$-definable subsets of $H(\kappa^+)$. In Section 5 we show that the assumptions of some of the previous results are optimal by showing that some of the results fail in M_1. In Section 6 we prove version of some of the previous results for $\Sigma_1(\kappa)$-definable subsets of $H(\kappa^+)$, where κ is a large cardinal, for instance a measurable cardinal or an ω_1-iterable cardinal.

2. Forcing axioms and $M_1^{\#}(A)$

We will frequently make use of the hypothesis that $M_1^{\#}(A)$ exists for every subset A of ω_1. We show that this follows from BMM together with the assumption that the nonstationary ideal NS_{ω_1} on ω_1 is precipitous, by varying arguments from [2].

Theorem 2.1. *Assume* BMM *and that* NS_{ω_1} *is precipitous. Then* $M_1^{\#}(A)$ *exists for every* $A \subseteq \omega_1$.

Proof. Let us first assume that there is no inner model with a Woodin cardinal, and let K denote the core model (see for example [11]). By [2, Theorem 0.3], the fact that NS_{ω_1} is precipitous (or just the fact that there is a normal precipitous ideal on ω_1) yields $(\omega_1^{\mathrm{V}})^{+\mathrm{K}} = \omega_2^{\mathrm{V}}$, whereas by [2, Lemma 7.1], BMM (or just BPFA) gives that $(\omega_1^{\mathrm{V}})^{+\mathrm{K}} < \omega_2^{\mathrm{V}}$. This is a plain contradiction, so that there must be an inner model with a Woodin cardinal.

By [23, Theorem 1.3], BMM yields that V is closed under $X \mapsto X^{\#}$. By a theorem of Woodin, the facts that there is an inner model with a Woodin cardinal and V is closed under the sharp operation imply that $M_1^{\#}$ exists and is fully iterable.[1] This argument relativizes to show that for any real x, $M_1^{\#}(x)$ exists and is fully iterable.

Let us now fix $A \subseteq \omega_1$ and prove that $M_1^{\#}(A)$ exists and is countably iterable. Let $j : \mathrm{V} \longrightarrow M \subseteq \mathrm{V}[G]$, where G is NS_{ω_1}–generic over V and j is the induced generic elementary embedding such that M is transitive. By elementarity, $M_1^{\#}(A)$ exists in M and is fully iterable in M. We aim to see that $(M_1^{\#}(A))^M \in \mathrm{V}$ and it is fully iterable in V.

As V is closed under the sharp operation, $F = \{\langle x, x^{\#}\rangle \mid x \in \mathbb{R}\}$ is universally Baire. Suppose that T and U are (class sized) trees such that $F = p[T]$ in V and $p[U] = \mathbb{R}^2 \setminus p[T]$ in every generic extension of V. By well–known arguments, we must have $p[j(T)] = p[T]$ in $\mathrm{V}[G]$ and in fact in every generic extension of $\mathrm{V}[G]$.

We first claim that $(M_1^{\#}(A))^M$ is ω_1-iterable in $\mathrm{V}[G]$ and in fact in every generic extension $\mathrm{V}[G][H]$ of $\mathrm{V}[G]$ via its unique iteration strategy. In order to see this, let $W \in M$ be a canonical tree of attempts to find

- (a) $\sigma\colon N \to (M_1^{\#}(A))^M$, where N is countable,
- (b) $\mathcal{T}$ is a countable iteration tree on N
- (c) $(\mathcal{Q}_\lambda\colon \lambda \in \mathrm{Lim} \cap \mathrm{lh}(\mathcal{T})+1)$ is such that for every $\lambda \in \mathrm{Lim} \cap \mathrm{lh}(\mathcal{T})+1$, $\mathcal{Q}_\lambda \trianglelefteq (\mathcal{M}(\mathcal{T} \restriction \lambda))^{\#}$ is a $\mathcal{Q}$–structure for $\mathcal{M}(\mathcal{T})$, and for every $\lambda \in \mathrm{Lim} \cap \mathrm{lh}(\mathcal{T})$, $\mathcal{Q}_\lambda \trianglelefteq \mathcal{M}_\lambda^{\mathcal{T}}$, and either
- (d1) $\mathcal{T}$ has a last ill–founded model, or else
- (d2) $\mathcal{T}$ has limit length but no cofinal branch b such that $\mathcal{Q}_{\mathrm{lh}(\mathcal{T})} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$.

Notice that we may use $j(T)$ to certify the first part of (c). If $(M_1^{\#}(A))^M$ were not ω_1-iterable in $\mathrm{V}[G][H]$, then W would be ill–founded in $\mathrm{V}[G][H]$, hence in M, and then $(M_1^{\#})^M$ would not be iterable in M. Contradiction!

Let $j' : \mathrm{V} \longrightarrow M' \subseteq \mathrm{V}[H] \subseteq \mathrm{V}[G][H]$, where H is $(\mathrm{NS}_{\omega_1})^{\mathrm{V}}$–generic over $\mathrm{V}[G]$ and j' is the induced generic elementary embedding such that M' is transitive. By the above argument, $(M_1^{\#}(A))^M$ and $(M_1^{\#}(A))^{M'}$ may be successfully coiterated inside $\mathrm{V}[G][H]$, so that in fact $(M_1^{\#}(A))^M = (M_1^{\#}(A))^{M'}$, and hence $(M_1^{\#}(A))^M \in \mathrm{V}$.

Assume $(M_1^{\#}(A))^M$ were not ω_1-iterable in some generic extension $\mathrm{V}[H]$ of V. We may without loss of generality assume that H is generic over $\mathrm{V}[G]$. Let $W' \in \mathrm{V}$

[1] This result is unpublished, but the methods used in the (known) proof can be found in [29].

be defined exactly as the tree W above, except for that we use T instead of $j(T)$ to certify the first part of (c). By $p[j(T)] = p[T]$ in $\mathrm{V}[G][H]$, we must have $p[W'] = p[W]$ in $\mathrm{V}[G][H]$. As we assume $(M_1^\#(A))^M$ to be not ω_1-iterable in $\mathrm{V}[G][H]$, W' would be ill–founded in $\mathrm{V}[G][H]$, so that W would be ill-fouded in $\mathrm{V}[G][H]$ and hence in M. Contradiction!

The argument given shows that $(M_1^\#(A))^M \in \mathrm{V}$ is fully iterable in V. □

3. $\Sigma_1(\omega_1)$-definable sets and Σ^1_3 sets

We give a characterization of $\Sigma_1(\omega_1)$-definable sets of reals which we will use in the proof of Theorem 1.2. Let WO denote the Π^1_1-set of all reals that code a well-ordering of ω (in some fixed canonical way) and, given $z \in \mathrm{WO}$, let $\|z\|$ denote the order-type of the well-ordering coded by z. Remember that, given a class Γ of subsets of $\mathbb{R}$, a subset A of ω_1 is Γ *in the codes* if there is $W \in \Gamma$ such that $A = \{\|z\| \mid z \in W \cap \mathrm{WO}\}$. Note that ω_1 is Σ^1_2 in the codes.

Lemma 3.1. *If $a \in \mathbb{R}$, X is a $\Sigma^1_3(a)$-subset of $\mathbb{R}$ and κ is an uncountable cardinal, then X is $\Sigma_1(\kappa, a)$-definable.*

Proof. Pick a Σ^1_3-formula $\psi(v_0, v_1)$ that defines X using the parameter a. In this situation, Shoenfield absoluteness implies that the set X is equal to the set of all $x \in \mathbb{R}$ with the property that there is a transitive model M of ZFC^- in $\mathrm{H}(\kappa^+)$ such that $a, x \in M$, $\kappa \subseteq M$ and $\psi(a, x)^M$. This yields a $\Sigma_1(\kappa, a)$-definition of X. □

In the following, we will show that the converse of the above implication for ω_1 holds in the presence of large cardinals. This argument makes use of the *countable stationary tower* $\mathbb{Q}_{<\delta}$ introduced by Woodin (see [13, Section 2.7]) and results of Woodin on generic iteration (see [31, Lemma 3.10 & Remark 3.11]).

Lemma 3.2. *Let M be a transitive model of ZFC^- with a largest cardinal κ and let $\mathbb{P}$ be a partial order in M of cardinality less than κ such that the following conditions hold:*

(i) Forcing with $\mathbb{P}$ adds a (μ, ν)-extender over M for some $\mu, \nu < \kappa$.
(ii) There is an ω_1-iterable M-ultrafilter U on κ.

Then M is ω_1-iterable with respect to $\mathbb{P}$ and its images.

Proof. We first suppose that M is countable. Let $\langle M^\alpha, \kappa_\alpha, j_{\alpha,\beta} \mid \alpha \leq \beta < \omega_1\rangle$ denote the iteration of M with U of length ω_1. Then $M_\alpha = \mathrm{H}(\kappa_\alpha)^{M_{\alpha+1}}$. Then M^α is α-iterable by [31, Lemma 3.10 & Remark 3.11].

We show that M is α-iterable. Suppose that $\langle M^0_\beta \mid \beta < \alpha\rangle$ is a generic iteration of $M = M^0$ with a sequence $\langle G_\beta \mid \beta < \alpha\rangle$ of filter. This induces generic iterations $\langle M^\gamma_\beta \mid \beta < \alpha\rangle$ of M^γ for all $\gamma \leq \alpha$. These iterations and the induced elementary embeddings commute with the iterated ultrapowers with U and its images, since for all $\gamma, \delta < \alpha$, $M^\delta_\gamma = \mathrm{H}(\lambda^+)^{M^\delta_{\gamma+1}}$, where λ is the image of κ. Since M^α is α-iterable, the iterates of M^α are well-founded. Since and the corresponding diagrams commute, the iterates of M are well-founded.

For arbitrary M, the claim follows by forming a countable elementary substructure of some $\mathrm{H}(\theta)$. □

Lemma 3.3. *Assume that $M_1^\#(A)$ exists for every $A \subseteq \omega_1$. Given $a \in \mathbb{R}$, the following conditions are equivalent for any subset X of $\mathbb{R}$.*

(i) *X is $\Sigma_1(A)$-definable for some $A \subseteq \omega_1$ that is $\Sigma^1_2(a)$ in the codes.*
(ii) *X is a $\Sigma^1_3(a)$-subset of $\mathbb{R}$.*

Proof. Assume that (i) holds. Fix a Σ_1-formula $\varphi(v_0, v_1)$ and a Σ^1_2-formula $\psi(v_0, v_1)$ with the property that $X = \{x \in \mathbb{R} \mid \varphi(A, x)\}$, where $A = \{\|z\| \mid z \in W \cap \mathrm{WO}\}$ and $W = \{z \in \mathbb{R} \mid \psi(a, z)\}$. Define Y to be the set of all $y \in \mathbb{R}$ with the property that there is a countable transitive model M of ZFC^- and $\delta, A_0, W_0 \in M$ such that $a, y \in M$ and the following statements hold:

(i) δ is a Woodin cardinal in M and M is ω_1-iterable with respect to $\mathbb{Q}^M_{<\delta}$ and its images.
(ii) In M, we have $W_0 = \{z \in \mathbb{R} \mid \psi(a, z)\}$, $A_0 = \{\|z\| \mid z \in W_0 \cap \mathrm{WO}\}$ and $\varphi(A_0, y)$ holds.

Claim. *The set Y is a $\Sigma^1_3(a)$-subset of $\mathbb{R}$.*

Proof. The only condition on M which is not first-order is ω_1-iterability. This condition states that all countable generic iterates are well-founded and hence it is a Π^1_2-statement. □

Claim. *$Y \subseteq X$.*

Proof. Fix $y \in Y$ and pick a countable transitive model M_0 and $\delta, A_0, W_0 \in M_0$ witnessing this. Let $\langle M_\alpha \mid \alpha \leq \omega_1\rangle$ be a generic iteration of M_0 using $\mathbb{Q}^{M_0}_{<\delta}$ and its images. Set $N = M_{\omega_1}$ and let $j : M_0 \longrightarrow N$ denote the corresponding elementary embedding. Then N is a transitive model of ZFC^- and $j(\omega_1^{M_0}) = \omega_1^N = \omega_1$. Pick $\alpha \in A$. Then there is $u \in \mathrm{WO}^N$ such that $\alpha = \alpha_u$ and $\exists z \in \mathrm{WO}\ [\|u\| = \|z\| \wedge \psi(a, z)]$ holds. Since $\omega_1 \subseteq N$, Shoenfield absoluteness implies that there is $z \in \mathrm{WO}^N$ with $\alpha = \|z\|$ and $\psi(a, z)^N$. By elementarity, this shows that $z \in j(W_0)$ and $\alpha \in j(A_0)$. In the other direction, fix $z \in j(W_0)$. Then $\psi(a, z)^N$ holds and Shoenfield absoluteness implies that $z \in W$ and $\|z\| \in A$. We can conclude that $A = j(A_0)$ and $\varphi(A, y)^N$ holds. By Σ_1-upwards absoluteness, this shows that $\varphi(a, A)$ holds and hence $y \in X$. □

Claim. *$X \subseteq Y$.*

Proof. Pick $x \in X$. Then $\varphi(A, x)$ holds and we can find a subset C of ω_1 such that $a, x, A \in M_1^\#[C]$, $\omega_1 = \omega_1^{M_1^\#[C]}$ and $\varphi(A, x)^{M_1^\#[C]}$. Shoenfield absoluteness implies that

$$\bar{W} \ = \ W \cap M_1^\#(C) \ = \ \{z \in \mathbb{R}^{M_1^\#(C)} \mid \psi(a, z)^{M_1^\#(C)}\} \ \in \ M_1^\#(C).$$

As in the proof of the above claim, we can now use Shoenfield absoluteness to see that $A = \{\|z\| \mid z \in \bar{W} \cap \mathrm{WO}^{M_1^\#(C)}\}$.

Let N be a countable elementary submodel of $M_1^\#(C)$ and let $\pi : N \longrightarrow M$ denote the corresponding transitive collapse. Then M is a countable transitive model of ZFC^- with $a, x \in M$ and there is $\delta \in M$ such that δ is a Woodin cardinal in M and M is iterable with respect $\mathbb{Q}^M_{<\delta}$ and its images by Lemma 3.2. In M, we have $\pi(\bar{W}) = \{z \in \mathbb{R} \mid \psi(a, z)\}$, $\pi(A) = \{\|z\| \mid z \in \pi(\bar{W}) \cap \mathrm{WO}\}$ and $\varphi(\pi(A), y)$ holds. Together, this shows that M and $\delta, \pi(A), \pi(\bar{W}) \in M$ witness that x is an element of Y. □

This completes the proof of the implication from (i) to (ii). The converse implication is a direct consequence of Lemma 3.1. □

Note that the assumptions of Lemma 3.3 hold for instance in M_2.

Remark 3.4. *The assumptions for the implication from (i) to (ii) in Lemma 3.3 are optimal in the following sense:*

(i) The implication is not a theorem of ZFC*. If* CH *holds and the set* $\{\mathbb{R}\}$ *is* $\Sigma_1(\omega_1)$*-definable, then the projective truth predicate is a* $\Sigma_1(\omega_1)$*-definable subset of* $\mathbb{R}$ *that is not projective. Note that the above assumptions holds for instance in* L*. Moreover, we will later prove results that show that the assumption also holds in* M_1 *(see Lemma 5.2). This shows that the implication does not follow from the existence of a single Woodin cardinal.*

(ii) The implication does not follow from $\neg$CH*. Suppose that* $\mathrm{L}[G]$ *is an* $\mathrm{Add}(\omega,\omega_2)$*-generic extension of* L*. Since* $\{\mathbb{R}^{\mathrm{L}}\}$ *is* $\Sigma_1(\omega_1)$*-definable in* $\mathrm{L}[G]$*, the projective truth predicate of* L *is* $\Sigma_1(\omega_1)$*-definable in* $\mathrm{L}[G]$*. Assume that this set is projective in* $\mathrm{L}[G]$*. By a result of Woodin (see* [16, Lemma 9.1]*), there is an* $\mathrm{Add}(\omega,\omega_1)$*-generic filter* H *over* L *and an elementary embedding of* $\mathrm{L}(\mathbb{R})^{\mathrm{L}[H]}$ *into* $\mathrm{L}(\mathbb{R})^{\mathrm{L}[G]}$*. Then the projective truth predicate of* L *is also projective in* $\mathrm{L}[H]$*. Since* $\mathrm{Add}(\omega,\omega_1)$ *is definable over* $\mathrm{H}(\omega_1)^{\mathrm{L}}$ *and satisfies the countable chain condition, the forcing relation for* $\mathrm{Add}(\omega,\omega_1)$ *for projective statements with parameters in* $\mathbb{R}^{\mathrm{L}}$ *is projective in* L*. Using the homogeneity of* $\mathrm{Add}(\omega,\omega_1)$*, this shows that the projective truth predicate is projective in* L*, a contradiction.*

A simpler version of the proof of Lemma 3.3, using Lemma 3.2 and generic iterations of countable substructures of $\mathrm{H}(\theta)$, where θ is above a measurable cardinal, yields the following result.

Lemma 3.5. *The equivalence in Lemma 3.3 holds if there is a precipitous ideal on* ω_1 *and a measurable cardinal.* □

In the following, we will add a predicate A for sets of reals to the language to obtain a stronger version of Lemma 3.3. Note that quantifiers over A are unbounded in this language. We consider universally Baire (uB) subsets of $\mathbb{R}$.

Definition 3.6. Suppose that $\langle M,\in,I\rangle$ is a countable transitive model of ZF^- and $B\subseteq\mathbb{R}$. The structure $\langle M,\in,I\rangle$ is *B-iterable* if the following conditions hold.

(i) $\langle M,\in,I\rangle$ is ω_1-iterable, i.e. all countable iterates are well-founded.
(ii) $B\cap M\in M$.
(iii) If $i\colon M\to N$ is a countable iteration, then $i(B\cap M)=B\cap N$.

Suppose that B is a subset of $\mathbb{R}$. A set of reals is $\Sigma^1_n(B)$ if it is defined by a Σ^1_n-formula, where $x\in B$ and $x\notin B$ are allowed as atomic formulas.

Lemma 3.7. *Assume that there is a proper class of Woodin cardinals. If* B *is a uB set of reals and* X *is a subset of* $\mathbb{R}$ *that is* $\Sigma_1(\omega_1)$*-definable over* $\langle\mathrm{H}(\omega_2),\in,B,\mathrm{NS}_{\omega_1}\rangle$*, then* X *is a* $\Sigma^1_3(B)$*-subset of* $\mathbb{R}$*.*

Proof. Suppose that X is defined by a Σ_1-formula $\varphi(x,\dot{B},\dot{\mathrm{NS}})$ over the structure $\langle\mathrm{H}(\omega_2),\in,B,\mathrm{NS}_{\omega_1}\rangle$. We define Y as the set of all reals x such that there is a B-iterable structure $\langle M,\in,I\rangle$ with $x\in M$ and $M\vDash\varphi(x,B\cap M)$.

Claim. $Y\subseteq X$.

Proof. Suppose that $x \in Y$ and that this is witnessed by a B-iterable structure $\langle M, \in, I\rangle$. Let $j\colon \langle M, \in, I\rangle \to \langle M', \in, I'\rangle$ be an iteration of length ω_1. Since M is B-iterable, $j(B \cap M) = B \cap M'$. It follows from the normality of I that $I' = \mathrm{NS}_{\omega_1} \cap M'$. Hence $\varphi(x, B \cap M', \mathrm{NS}_{\omega_1} \cap M')$ holds in M' and therefore in V. □

Claim. $X \subseteq Y$.

Proof. Suppose that $x \in X$. We first argue that the required B-iterable structure exists in a generic extension. Let κ be measurable and let G be $\mathrm{Col}(\omega, {<}\kappa)$-generic over V. Then NS_{ω_1} is precipitous in $\mathrm{V}[G]$ by [10, Theorem 22.33]. Suppose that μ is the least measurable cardinal and ν is the least inaccessible cardinal above μ in $\mathrm{V}[G]$. Suppose that H is $\mathrm{Col}(\omega, \nu)$-generic over $\mathrm{V}[G]$. Let $I = \mathrm{NS}_{\omega_1}^{\mathrm{V}[G]}$. Suppose that T, U are trees in V with $p[T] = B$ and $p[U] = \mathbb{R} \setminus B$ witnessing that B is uB.

Subclaim. *Then $\langle \mathrm{V}[G]_\nu, \in, I\rangle$ is $p[T]$-iterable in $\mathrm{V}[G * H]$.*

Proof. We work in $\mathrm{V}[G * H]$. Since there is a measurable cardinal in $\mathrm{V}[G]_\nu$, the structure $\langle \mathrm{V}[G]_\nu, \in, I\rangle$ is ω_1-iterable by Lemma 3.2. Let $M = \mathrm{V}[G]_\nu$ and $B_{G*H} = p[T]^{\mathrm{V}[G*H]}$. Suppose that $j\colon M \to M'$ is a countable iteration. We argue that $p[j(T)] \cap M' = B_{G*H} \cap M'$. Since the statement $p[j(T)] \cap p[j(U)] \neq \emptyset$ is absolute between M' and $\mathrm{V}[G * H]$, this holds in $\mathrm{V}[G * H]$. Since $p[T] \subseteq p[j(T)]$ and $p[U] \subseteq p[j(U)]$ and $p[T] \cup p[U] = \mathbb{R}$ in $\mathrm{V}[G * H]$. This implies $B_{G*H} \cap M' = p[T]^{\mathrm{V}[G*H]} = p[j(T)]^{\mathrm{V}[G*H]}$. □

The existence of the required B-iterable structure is projective in B. Since there is a proper class of Woodin cardinals, the universally Baire sets are closed under projection [13, Theorem 3.3.3 & Theorem 3.3.14]. Hence this statement is absolute to generic extensions. □

Since Y is a $\Sigma^1_3(B)$-subset of $\mathbb{R}$, this completes the proof. □

4. $\Sigma_1(\omega_1)$-definable subsets of $\omega_1^{\omega_1}$

In this section, we present the proofs of the main results about $\Sigma_1(\omega_1)$-definable subset of $H(\kappa^+)$ stated in the introduction.

4.1. **Well-orderings of the reals.**

The above lemma directly yields the following strengthening of Theorem 1.2.

Theorem 4.1. *Assume that either $M_1^\#(A)$ exists for every $A \subseteq \omega_1$ or that there is a precipitous ideal on ω_1 and a measurable cardinal. If $A \subseteq \omega_1$ is $\mathbf{\Sigma}^1_2$ in the codes, then no well-ordering of the reals is $\Sigma_1(A)$-definable.*

Proof. Assume that there is a well-ordering of the reals that is $\Sigma_1(A)$-definable. By Lemma 3.3 and Lemma 3.5, this assumption implies that there is a $\mathbf{\Sigma}^1_3$-well-ordering of the reals. This contradicts our assumptions, because these assumption imply that $\mathbf{\Sigma}^1_2$-determinacy holds (see [21]), every $\mathbf{\Sigma}^1_3$-set of reals has the Baire property (see [20, 6G.11]) and hence there are no $\mathbf{\Sigma}^1_3$-well-orderings of the reals. □

We will consider Σ_1-well-orderings of the reals that allow more complicated parameters. As mentioned above, results of Woodin on the Π_2-maximality of the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$ imply that no well-ordering of the reals is $\Sigma_1(A)$-definable over $\langle \mathrm{H}(\omega_2), \in, B\rangle$ for some $A \in \mathcal{P}(\omega_1)^{\mathrm{L}(\mathbb{R})}$ and $B \in \mathcal{P}(\mathbb{R})^{\mathrm{L}(\mathbb{R})}$. In the following, we will use $\mathbb{P}_{max}$-forcing to derive a stronger conclusion from a stronger assumption.

Theorem 4.2. *Suppose that there is a proper class of Woodin cardinals. If B is uB, then there is no well-ordering of the reals which is $\Sigma_1(\omega_1)$-definable over the structure $\langle \mathrm{H}(\omega_2), \in, B, \mathrm{NS}_{\omega_1}\rangle$.*

Proof. If there is a proper class of Woodin cardinals, then every uB set of reals is determined by [13, Theorem 3.3.4 & Theorem 3.3.14]. Hence the claim follows from Lemma 3.7. □

4.2. **Bernstein subsets.** The next lemma shows how to construct perfect subsets of $\Sigma_1(\omega_1)$-definable subsets of ${}^{\omega_1}\omega_1$. It will allow us to prove that the existence of large cardinals implies the non-existence of $\Delta_1(\omega_1)$-definable Bernstein subsets of ${}^{\omega_1}\omega_1$. The lemma will also be used for a result about the non-stationary ideal (see Section 4.4). We interpret a function $x \in {}^{\omega_1}\omega_1$ as a code for $\{\alpha < \omega_1 \mid x(\alpha) > 0\}$.

Lemma 4.3. *Assume that $M_1^\#(A)$ exists for every $A \subseteq \omega_1$. Let $A \subseteq \omega_1$ be $\boldsymbol{\Sigma}^1_2$ in the codes and let X be a $\Sigma_1(A)$-definable subset of ${}^{\omega_1}\omega_1$. If some $x \in X$ codes a bistationary subset of ω_1, then for every $\xi < \omega_1$ there is*

(i) *a continuous injection $\iota : {}^{\omega_1}2 \longrightarrow X$*
(ii) *a club D in ω_1*

such that for the monotone enumeration $\langle \delta_\alpha \mid \alpha < \omega_1\rangle$ of D

(i) *$\mathrm{ran}(\iota) \subseteq N_{x\restriction\xi} \cap X$*
(ii) *for all $z \in {}^{\omega_1}2$ and $\alpha < \omega_1$, then $z(\alpha) = 1$ if and only if $\iota(z)(\delta_\alpha) > 0$.*

Proof. Fix $\xi < \omega_1$ and a Σ_1-formula $\varphi(v_0, v_1)$ with $X = \{z \in {}^{\omega_1}\omega_1 \mid \varphi(A, z)\}$. Pick $a \in \mathbb{R}$ and a Σ^1_2-formula $\psi(v_0, v_1)$ with $A = \{\|w\| \mid w \in \mathrm{WO},\ \psi(a, w)\}$. We can find $C \subseteq \omega_1$ such that $a, x, A \in M_1^\#(C)$, $\omega_1 = \omega_1^{M_1^\#(C)}$ and $\varphi(A, x)^{M_1^\#(C)}$. Then y is a bistationary subset of ω_1 in $M_1^\#(C)$. Note that every stationary subset of ω_1 is a condition in $\mathbb{Q}_{<\delta}$. Let N be a countable elementary submodel of $M_1^\#(C)$ with $a, x \in N$ and $\xi + 1 \subseteq N$, let $\pi : N \longrightarrow M$ be the corresponding transitive collapse and let δ denote the unique Woodin cardinal in M. Since Lemma 3.2 shows that M is ω_1-iterable with respect to $\mathbb{Q}^M_{<\delta}$ and its images, there is a directed system

$$\langle\langle M_s \mid s \in {}^{\leq\omega_1}2\rangle,\ \langle j_{s,t} : M_s \longrightarrow M_t \mid s, t \in {}^{\leq\omega_1}2,\ s \subseteq t\rangle\rangle$$

of transitive models of ZFC^- and elementary embeddings such that the following statements hold.

(i) $M = M_\emptyset$.
(ii) If $s \in {}^{<\omega_1}2$, then there are M_s-generic filters G^s_0 and G^s_1 over $j_{\emptyset,s}(\mathbb{Q}^M_{<\delta})$ such that $(j_{\emptyset,s} \circ \pi)(y) \in G^s_0$, $(j_{\emptyset,s} \circ \pi)(\omega_1 \setminus y) \in G^s_1$, $M_{s^\frown\langle i\rangle} = \mathrm{Ult}(M_s, G^s_i)$ and $j_{s,s^\frown\langle i\rangle}$ is the ultrapower map induced by G^s_i for all $i < 2$.
(iii) If $s \in {}^{\leq\omega_1}2$ with $\mathrm{lh}(s) \in \mathrm{Lim}$, then

$$\langle M_s,\ \langle j_{s\restriction\alpha,s} : M_{s\restriction\alpha} \longrightarrow M_s \mid \alpha < \mathrm{lh}(s)\rangle\rangle$$

is the direct limit of the directed system

$$\langle\langle M_{s\restriction\alpha} \mid \alpha < \mathrm{lh}(s)\rangle,\ \langle j_{s\restriction\bar\alpha,s\restriction\alpha} : M_{s\restriction\bar\alpha} \longrightarrow M_{s\restriction\alpha} \mid \bar\alpha \leq \alpha < \mathrm{lh}(s)\rangle\rangle.$$

Let $j_s = j_{\emptyset,s}$ for all $s \in {}^{\leq\omega_1}2$. Since $\omega_1 = \omega_1^{M_z}$ for all $z \in {}^{\omega_1}2$, we can define

$$i : {}^{\omega_1}2 \longrightarrow {}^{\omega_1}\omega_1;\ z \longmapsto (j_z \circ \pi)(x).$$

In this situation, elementarity and Σ_1-upwards absoluteness imply that $A \in M_z$, $x \restriction \xi = i(z) \restriction \xi$ and $\varphi(A, i(z))$ for all $z \in {}^{\omega_1}2$. This shows that $\mathrm{ran}(i) \subseteq N_{x\restriction\xi} \cap X$.

Given $z \in {}^{\omega_1}2$, we define

$$c_z : \omega_1 \longrightarrow \omega_1;\ \alpha \longmapsto \omega_1^{M_{z\restriction\alpha}}.$$

By definition, c_z is strictly increasing and continuous for every $z \in {}^{\omega_1}2$. Moreover, we have $c_{z_0} \restriction \alpha = c_{z_1} \restriction \alpha$ for all $z_0, z_1 \in {}^{\omega_1}2$ and $\alpha < \omega_1$ with $z_0 \restriction \alpha = z_1 \restriction \alpha$.

Claim. *Given $z \in {}^{\omega_1}2$ and $\alpha < \omega_1$, then $z(\alpha) = 1$ if and only if $c_z(\alpha) > 0$.*

Proof. Given $z \in {}^{\omega_1}2$ and $\alpha < \omega_1$, we know that $c_z(\alpha)$ is smaller than the ciritical point of $j_{z\restriction(\alpha+1),z}$ and this allows us to use [13, Fact 2.7.3.] to conclude that

$$\begin{aligned} z(\alpha) = 1 \iff & (j_{z\restriction\alpha} \circ \pi)(y) \in G^{z\restriction\alpha}_{z(\alpha)} \\ \iff & \omega_1^{M_{z\restriction\alpha}} \in (j_{z\restriction(\alpha+1)} \circ \pi)(y) \\ \iff & c_z(\alpha) \in (j_{z\restriction(\alpha+1)} \circ \pi)(y) \\ \iff & (((j_{z\restriction(\alpha+1)} \circ \pi)(x))(c_z(\alpha)) > 0 \\ \iff & (((j_{z\restriction(\alpha+1),z} \circ j_{z\restriction(\alpha+1)} \circ \pi)(x))(c_z(\alpha)) > 0 \\ \iff & i(z)(c_z(\alpha)) > 0. \end{aligned}$$ □

In particular, this shows that the function i is injective.

Claim. *The function i is continuous.*

Proof. Let $z \in {}^{\kappa}2$ and $\beta < \kappa$. Then there is $\alpha < \kappa$ with $\beta < c_z(\alpha) < \mathrm{crit}\,(j_z)$. Given $\bar{z} \in {}^{\omega_1}2$, we know that $c_{\bar{z}}(\alpha)$ is the critical point of $j_{\bar{z}\restriction\alpha,z}$ and hence

$$i(\bar{z}) \restriction \beta \ = \ (j_{\bar{z}} \circ \pi)(x) \restriction \beta \ = \ (j_{\bar{z}\restriction\alpha} \circ \pi)(x) \restriction \beta.$$

If $\bar{z} \in N_{z\restriction\alpha} \cap {}^{\omega_1}2$, then $j_{z\restriction\alpha} = j_{\bar{z}\restriction\alpha}$ and therefore $i(z) \restriction \beta = i(\bar{z}) \restriction \beta$. □

Claim. *There is a club D in ω_1 such that $c_z \restriction D = \mathrm{id}_D$ for all $z \in {}^{\omega_1}2$.*

Proof. Suppose that z_M is a real coding M. We define $D = \mathrm{Card}^{\mathrm{L}[z_M]} \cap \omega_1$. A statement and proof analogous to [2, Lemma 19] for forcing with $\mathbb{Q}_{<\delta}$ instead of a precipitous ideal shows that the cardinals in $\mathrm{L}[z_M]$ are closure points of the images of c_z for all $z \in {}^{\omega_1}2$. We can conclude that $c_z \restriction D = \mathrm{id}_D$ for all $z \in {}^{\omega_1}2$. □

Let $\langle \delta_\alpha \mid \alpha < \omega_1 \rangle$ denote the monotone enumeration of D and let $e : {}^{\omega_1}2 \longrightarrow {}^{\omega_1}2$ denote the unique continuous injection with $e(z)^{-1}\{1\} = \{\delta_\alpha \mid \alpha < \omega_1,\ z(\alpha) = 1\}$ for all $z \in {}^{\omega_1}2$. Set $\iota = i \circ e$. Given $z \in {}^{\omega_1}2$ and $\alpha < \omega_1$, we then have

$$z(\alpha) > 0 \iff e(z)(\delta_\alpha) > 0 \iff i(e(z))(c_{e(z)}(\delta_\alpha)) > 0 \iff \iota(z)(\delta_\alpha) > 0.$$ □

A simpler version of the proof of Lemma 4.3 shows the following.

Lemma 4.4. *The conclusion of Lemma 4.3 follows from the existence of a precipitous ideal on ω_1 and a measurable cardinal.* □

The above lemmas allow us to prove the following strengthening of Theorem 1.4.

Theorem 4.5. *Assume that either $M_1^\#(A)$ exists for every $A \subseteq \omega_1$ or that there is a precipitous ideal on ω_1 and a measurable cardinal. Let Γ denote the collection of subsets of ${}^{\omega_1}\omega_1$ that are $\Sigma_1(A)$-definable for some $A \subseteq \omega_1$ that is $\mathbf{\Sigma}^1_2$ in the codes. If $\Delta \subseteq \Gamma$ with $\bigcup \Delta = {}^{\omega_1}\omega_1$, then some element of Δ contains a perfect subset.*

Proof. Pick some $x \in {}^{\omega_1}\omega_1$ which codes a bistationary subset of ω_1. Then there is $X \in \Delta$ with $x \in X$. In this situation, Lemma 4.3 and Lemma 4.4 imply that X contains a perfect subset. □

Theorem 4.6. *Assume that either $M_1^\#(A)$ exists for every $A \subseteq \omega_1$ or that there is a precipitous ideal on ω_1 and a measurable cardinal. If $A \subseteq \omega_1$ is $\mathbf{\Sigma}^1_2$ in the codes, then no Bernstein subset of ${}^{\omega_1}\omega_1$ is $\Delta_1(A)$-definable over $\langle \mathrm{H}(\omega_2), \in\rangle$.*

Proof. Apply Theorem 4.5 with $\Delta = \{A, \omega_1 \setminus A\} \subseteq \Gamma$. □

We will see in Lemma 5.6 below that the existence of a $\Sigma_1(\omega_1)$-definable Bernstein subset of ${}^{\omega_1}\omega_1$ is consistent with the existence of a Woodin cardinal.

4.3. **A perfect set theorem.**

We aim to prove a perfect set theorem for $\Sigma_1(\omega_1)$-definable subsets of ${}^{\omega_1}\omega_1$. This is motivated by the following result.

Theorem 4.7 (Woodin, [14, Corollary 7.11])**.** *Assume $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ and suppose that G is $\mathbb{P}_{max}$-generic over $\mathrm{L}(\mathbb{R})$. Work in $\mathrm{L}(\mathbb{R})[G]$. Suppose that A is a subset of $\omega_1^{\omega_1}$ which is defined from a parameter in $\mathrm{L}(\mathbb{R})$. Then at least one of the following conditions hold.*

(i) A contains a perfect subset.
(ii) $A \subseteq \mathrm{L}(\mathbb{R})$.

We will prove a similar result for $\Sigma_1(\omega_1)$-definable sets in V from the assumption that NS_{ω_1} is saturated and there is a measurable cardinal. We do not know if our result is a true dichotomy, i.e. whether the two cases are mutually exclusive.

Assuming that NS_{ω_1} is saturated, the following result of Woodin shows that there is a canonical iteration of length ω_1 of any countable substructure of $\mathrm{H}(\omega_2)$.

Lemma 4.8 (Woodin)**.** *Suppose that the non-stationary ideal NS_{ω_1} on ω_1 is saturated. If $A \subseteq \omega_1$ and $i : \langle M, \in, I, \bar{A}\rangle \longrightarrow \langle \mathrm{H}(\theta), \in, \mathrm{NS}_{\omega_1}, A\rangle$ is an elementary embedding with $\theta \geq \omega_2$ and M is countable, then there is a generic iteration $j : M \longrightarrow N$ of length ω_1 with N well-founded and $j(\bar{A}) = A$.*

Proof. We inductively construct a generic iteration

$$\langle\langle\langle M_\alpha, \in, I_\alpha, \bar{A}_\alpha\rangle \mid \alpha < \omega_1\rangle, \langle i_{\alpha,\beta} : M_\alpha \longrightarrow M_\beta \mid \alpha \leq \beta < \omega_1\rangle\rangle$$

with $M = M_0$ and elementary embeddings $\langle j_\alpha : M_\alpha \longrightarrow M \mid \alpha < \omega_1\rangle$ such that $j_\alpha = j_\beta \circ i_{\alpha,\beta}$ for all $\alpha \leq \beta < \omega_1$. Suppose that $\langle M_\alpha, \in, I_\alpha, \bar{A}_\alpha\rangle$, $i_{\alpha,\beta}$ and j_α are defined for $\alpha \leq \beta \leq \gamma$. Set $\kappa = i_{0,\gamma}(\omega_1^{M_\gamma})$ and $U_\gamma = \{X \in \mathcal{P}(\kappa)^{M_\gamma} \mid \omega_1 \in j_\gamma(X)\}$.

Claim. *U_γ is $\mathcal{P}(\kappa)/I_\gamma$-generic over M_γ.*

Proof. Suppose that $A \in M_\gamma$ is a maximal antichain in $\mathcal{P}(\kappa)/I_\gamma$. Since NS_{ω_1} is saturated, $\mathcal{P}(\kappa)/I_\gamma$ satisfies the $\omega_2^{M_\gamma}$-chain condition in M_γ. Let $\langle X_\alpha \mid \alpha < \kappa\rangle$ enumerate A in M_γ and assume that $X_\alpha \notin U_\gamma$ for all $\alpha < \kappa$. By the definition of U_γ, we have $X = \triangle_{\alpha<\kappa}(\kappa \setminus X_\alpha) \in U_\gamma$. Since U_γ is normal, the set X is stationary. This contradicts the assumption that A is maximal. □

We define $M_{\gamma+1} = \mathrm{Ult}(M_\gamma, U_\gamma)$, $i_{\gamma,\gamma+1} : M_\gamma \longrightarrow M_{\gamma+1}$ the ultrapower map, and $j_{\gamma+1} : M_{\gamma+1} \longrightarrow \mathrm{H}(\theta)$ by $j_{\gamma+1}([f]) = j_\gamma(f)(\omega_1)$. It is straightforward to check that $j_{\gamma+1}$ is well-defined and elementary.

Claim. $j_\gamma = j_{\gamma+1} \circ i_{\gamma,\gamma+1}$.

Proof. If $x \in M_\gamma$, then

$$j_{\gamma+1}(i_{\gamma,\gamma+1}(x)) \;=\; j_{\gamma+1}([c_x]) \;=\; j_{\gamma+1}(c_x)(\omega_1) \;=\; c_{j_\gamma(x)}(\omega_1) \;=\; j_\gamma(x). \qquad \square$$

This completes the proof of the lemma. □

Theorem 4.9. *Suppose that* NS_{ω_1} *is saturated and there is a measurable cardinal. Suppose that* X *is a* $\Sigma_1(\omega_1)$*-definable subset of* $\omega_1^{\omega_1}$*. Then at least one of the following conditions holds.*

(i) *X contains a perfect subset.*
(ii) *$X \subseteq \mathrm{L}(\mathbb{R})$.*

Proof. Suppose that μ is measurable and $\theta = \mu^+$. Suppose that $X \not\subseteq \mathrm{L}(\mathbb{R})$. Then there is some $A \in X \setminus \mathrm{L}(\mathbb{R})$. Suppose that $i : \langle M, \in, I, \bar{A}\rangle \longrightarrow \langle \mathrm{H}(\theta), \in, \mathrm{NS}_{\omega_1}, A\rangle$ is elementary and M is countable. Let $\bar{\mu} = i^{-1}(\mu)$. Since NS_{ω_1} is saturated and $\mathcal{P}(\omega_1)^\#$ exists, $\langle M, \in, I, \bar{A}\rangle$ is ω_1-iterable by [31, Theorem 3.10 & Theorem 4.29].

Claim. *Suppose that for all countable iterations* $i_0 \colon M \to N_0$, $i_1 \colon M \to N_1$ *and* $\alpha = \min(\{i_0(\omega_1^M), i_1(\omega_1^M)\})$*, we have* $i_0(\bar{A}) \cap \alpha = i_1(\bar{A}) \cap \alpha$*. Then* $X \subseteq \mathrm{L}(\mathbb{R})$*.*

Proof. It follows from Lemma 4.8 that $i_0(\bar{A}) \cap \alpha = A \cap \alpha$. Hence A can be reconstructed from $(M, I, \bar{A})$ in $\mathrm{L}(\mathbb{R})$ by considering generic iterations of arbitrarily large countable length in $\mathrm{L}(\mathbb{R})$. □

Claim. *Suppose that there are countable iterations* $i_0 \colon M \to N_0$, $i_1 \colon M \to N_1$ *such that* $i_0(\bar{A}) \cap \alpha \neq i_1(\bar{A}) \cap \alpha$ *for* $\alpha = \min(\{i_0(\omega_1^M), i_1(\omega_1^M)\})$*. Then this remains true in every countable iterate of* M*.*

Proof. Let $\gamma = \max(\{i_0(\omega_1^M), i_1(\omega_1^M)\})$. Suppose that $\bar{U}$ is a normal measure on $\bar{\mu}$ in M. Suppose that $j \colon M \to M^\gamma$ is the iterate of M of length γ with $\bar{U}$. Then $j(\bar{\mu}) > \gamma$. As in the proof of Lemma 3.2, the iterated ultrapowers of M with $\bar{U}$ commute with the generic ultrapower since $\bar{\mu} > (2^{\omega_1})^M$. The same argument works for all further steps in the generic iteration of M and hence we obtain a commutative diagram. This shows that the generic iteration of M^γ commutes with the generic iteration of M. In any $\mathrm{Col}(\omega, j(\gamma))$-generic extension of M^γ, there are sequences of ultrafilters which induce i_0, i_1 as in the statement of the claim by Σ^1_2-absoluteness. Hence such iterations exist in any $\mathrm{Col}(\omega, \gamma)$-generic extension of M by elementarity. This statement is preserved in generic iterations of M by elementarity and guarantees the existence of i_0, i_1. □

The last claim allows us to build a perfect tree T of height ω_1 of generic iterates of M with the property that the set of images of $\bar{A}$ along the branches of T form a perfect subset of X. □

Remark 4.10. *If* CH *fails, then the set* $X = \{x \in {}^{\omega_1} \mid \forall \alpha \geq \omega\ x(\alpha) = 0\}$ *is a* $\Delta_1(\omega_1)$*-definable subset of* ${}^{\omega_1}\omega_1$ *without the perfect set property.*

4.4. **The club filter and the non-stationary ideal.** In this section, we will use the above lemma to prove a strengthening of Theorem 1.5.

Lemma 4.11. *Assume that either* $M_1^\#(A)$ *exists for every* $A \subseteq \omega_1$ *or that there is a precipitous ideal on* ω_1 *and a measurable cardinal. Let* A *be an unbounded subset of* ω_1 *that is* $\mathbf{\Sigma}^1_2$ *in the codes and let* Y *be a* $\Sigma_1(A)$*-definable subset of* $\mathcal{P}(\omega_1)$*. Then the following statements hold for all* $y \in Y$ *and* $\xi < \omega_1$*.*

(i) *If y is a stationary subset of ω_1, then there is $z \in Y$ such that z is an element of the club filter on ω_1 and $y \cap \xi = z \cap \xi$.*
(ii) *If y is a costationary subset of ω_1, then there is $z \in Y$ such that z is an element of the nonstationary ideal on ω_1 and $y \cap \xi = z \cap \xi$.*

Proof. Let $X \subseteq {}^{\omega_1}2$ denote the set of characteristic functions of elements of the set Y. Since A is unbounded in ω_1, the set X is $\Sigma_1(A)$-definable. Fix $y \in Y$ and $\xi < \omega_1$. In the following, we may assume that y is a bistationary subset of ω_1, because otherwise the above statements hold trivially. Let $x \in X$ denote the characteristic function of y. We can apply Lemma 4.3 and Lemma 4.4 to find $x_0, x_1 \in N_{x \restriction \xi} \cap X$ and a strictly increasing continuous sequence $\langle c_\alpha \mid \alpha < \omega_1 \rangle$ such that $x_i(c_\alpha) = i$ for all $\alpha < \omega_1$ and $i < 2$. Let $C = \mathrm{ran}(c_i)$ and $z_i = \{\alpha < \omega_1 \mid x_i(\alpha) > 0\} \in Y$ for $i < 2$. Then C is a club in ω_1 witnessing that z_0 is an element of the club filter on ω_1 and that z_1 is an element of the nonstationary ideal on ω_1. □

Theorem 4.12. *Assume that either $M_1^\#(A)$ exists for every $A \subseteq \omega_1$ or that there is a precipitous ideal on ω_1 and a measurable cardinal. If $A \subseteq \omega_1$ is $\mathbf{\Sigma}^1_2$ in the codes and X is a subset of $\mathcal{P}(\omega_1)$ that separates the club filter from the non-stationary ideal, then X is not $\Delta_1(A)$-definable.*

Proof. Assume that the set X is $\Delta_1(A)$-definable over $\langle \mathrm{H}(\omega_2), \in \rangle$. Since X is disjoint from the nonstationary ideal on ω_1 and therefore contains no countable subsets of ω_1, Σ_1-reflection implies that A is unbounded in ω_1 and the second part of Lemma 4.11 shows that X contains no costationary subsets of ω_1. But this implies that X is equal to the club filter on ω_1 and therefore $\mathcal{P}(\omega_1) \setminus X$ contains a stationary subset of ω_1. In this situation, the first part of Lemma 4.11 implies that $\mathcal{P}(\omega_1) \setminus X$ contains an element of the club filter on ω_1, a contradiction. □

Corollary 4.13. *Assume that either $M_1^\#(A)$ exists for every $A \subseteq \omega_1$ or that there is a precipitous ideal on ω_1 and a measurable cardinal. If $A \subseteq \omega_1$ is $\mathbf{\Sigma}^1_2$ in the codes, then the club filter on ω_1 is not $\Pi_1(A)$-definable over $\langle \mathrm{H}(\omega_2), \in \rangle$.*

Proof. This is immediate from Theorem 4.12. □

We can also use Lemma 4.11 to study $\Sigma_1(\omega_1)$-definable singletons.

Lemma 4.14. *Assume that either $M_1^\#(A)$ exists for every $A \subseteq \omega_1$ or that there is a precipitous ideal on ω_1 and a measurable cardinal. If $A \subseteq \omega_1$ is $\mathbf{\Sigma}^1_2$ in the codes and x is a subset of ω_1 with the property that $\{x\}$ is $\Sigma_1(A)$-definable, then x is either contained in the club filter on ω_1 or in the nonstationary ideal on ω_1.*

Proof. If A is bounded in ω_1, then Σ_1-reflection implies that $x \in \mathrm{H}(\omega_1)$ and hence x is contained in the nonstationary ideal on ω_1. Otherwise A is unbounded in ω_1 and the claim follows directly from Lemma 4.11. □

Remark 4.15. *If $\mathrm{V} = \mathrm{L}$ and κ is an uncountable regular cardinal, then there is a bistationary subset x of κ such that $\{x\}$ is $\Sigma_1(\kappa)$-definable. Such subsets can be constructed from the canonical $\diamondsuit_\kappa$-sequence in L, using the facts that this sequence is definable over $\langle \mathrm{L}_\kappa, \in \rangle$ by a formula without parameters and the set $\{\mathrm{L}_\kappa\}$ is $\Sigma_1(\kappa)$-definable. Another way to construct such subsets is described in* [8, Section 7].

4.5. **Uniformization of the club filter.** We show that the existence of large cardinals implies that the club filter on ω_1 has no $\Sigma_1(\omega_1)$-definable uniformization.

Definition 4.16. Let κ be an uncountable regular cardinal. A *uniformization* of the club filter on κ is a function $f : \mathrm{C}_\kappa \longrightarrow \mathrm{C}_\kappa$ such that $f(X) \subseteq X$ is a club for all $X \in \mathrm{C}_\kappa$.

Lemma 4.17. *If in a model of* ZF*, the club filter* C_{ω_1} *on* ω_1 *is an ultrafilter, then there is no uniformization of* C_{ω_1} *which is definable from a set of ordinals.*

Proof. Suppose that the club filter C_{ω_1} is an ultrafilter and there is a uniformization of C_{ω_1} which is definable from a set of ordinals z. Then there is a function $f : \mathcal{P}(\omega_1) \longrightarrow \mathrm{C}_{\omega_1}$ definable from z such that for all $A \in \mathcal{P}(\omega_1)$, $f(A)$ is a club subset of A or of its complement. Let HOD_z denote the class of sets which are hereditarily ordinal definable from z. Since ω_1 is regular in HOD_z, there is a subset of ω_1 which is bistationary in HOD_z. The least such set S in a definable enumeration of HOD_z is definable from z and ω_1. Then $f(S) \in \mathrm{HOD}$ and hence S is not bistationary in HOD. □

Remark 4.18. *Suppose that in a model of* ZF*,* $x^{\#}$ *exists for every real* x *(and hence for every* $x \in [\omega_1]^{<\omega_1}$*), and there is no uniformization of* C_{ω_1}*. Then there is no function* $f\colon P(\omega_1) \to [\omega_1]^{<\omega_1}$ *such that* $A \in L[f(A)]$ *for all* $A \subseteq \omega_1$*. Suppose that* f *is such a function. For* $A \subseteq \omega_1$ *let* x_A *denote the inclusion-least finite set of* $f(A)$*-indiscernibles such that* A *is definable from* $f(A)$ *and* x_A *in* $L[f(A)]$*. Then the club* C_A *of* $f(A)$*-indiscernibles (i.e. Silver indiscernibles) between* $\sup(x_A \cap \omega_1)$ *and* ω_1 *is either contained in* A *or disjoint from* A*. Since* C_A *is definable from* $f(A)^{\#}$*, this defines a uniformization of* C_{ω_1}*, contradicting the assumption.*

Theorem 4.19. *Suppose that there are infinitely many Woodin cardinals and a measurable cardinal above them.*

(i) In $\mathrm{L}(\mathbb{R})$*, there is no uniformization of the club filter on* ω_1*.*
(ii) There is no $\Sigma_1(\omega_1)$*-definable uniformization of the club filter on* ω_1*.*

Proof. (i) In $\mathrm{L}(\mathbb{R})$, every element is ordinal definable from a real and our assumptions imply that the club filter on ω_1 is an ultrafilter. By Lemma 4.17, there is no uniformization of the club filter on ω_1.

(ii) Assume that there is a $\Sigma_1(\omega_1)$-definable uniformization of C_{ω_1}. By the Π_2-maximality of the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$ (see [14, Theorem 7.3]), the same Σ_1-formula defines a uniformization of C_{ω_1} in the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$. Since $\mathbb{P}_{max}$ is weakly homogeneous in $\mathrm{L}(\mathbb{R})$ (see [14, Lemma 2.10]), this shows that there is a uniformization of C_{ω_1} in $\mathrm{L}(\mathbb{R})$, contradicting the first part of the theorem. □

Remark 4.20. *Unpublished results of Woodin (see* [13, Remark 3.3.12] *and* [15, End of Section 6.3]*) show that the existence of a proper class of Woodin limits of Woodin cardinals implies that the axiom of determinacy holds in the Chang model* $\mathrm{L}(\mathrm{On}^\omega)$*. Hence* C_{ω_1} *is an ultrafilter in* $\mathrm{L}(\mathrm{On}^\omega)$*. It follows from Lemma 4.17 that there is no uniformization of* C_{ω_1} *in* $\mathrm{L}(\mathrm{On}^\omega)$*.*

Remark 4.21. *Let* κ *be inaccessible in* L *and let* G *be* $\mathrm{Col}(\omega, {<}\kappa)$*-generic over* L*. Since* $\mathrm{Col}(\omega, {<}\kappa)$ *satisfies the* κ*-chain condition in* L*, every element of* $\mathrm{C}_{\omega_1}^{\mathrm{L}[G]}$ *contains a constructible club and there is a uniformization of* C_{ω_1} *in* $\mathrm{L}(\mathbb{R})^{\mathrm{L}[G]}$*.*

4.6. $\Sigma_1(\omega_1)$**-absoluteness.** In this section, we observe that for $\Sigma_1(\omega_1)$-formulas, absoluteness to ω_1-preserving forcings holds for formulas without parameters, but not for formulas with subsets of ω_1 as parameters.

Lemma 4.22. *Let δ be a Woodin cardinal below a measurable cardinal.*

(i) $\Sigma_1(\omega_1)$ statements (without parameters) are absolute to generic extensions for forcings of size less than δ.[2]

(ii) The set of $\Sigma_1(\omega_1)$-formulas defining sets $\{x\}$ with $x \subseteq \omega$ is absolute for forcings of size less than δ. Moreover, the set of $\Sigma_1(\omega_1)$-definable singletons $\{x\}$ with $x \subseteq \omega_1$ is absolute for ω_1-preserving forcings of size less than δ.

(iii) The canonical code for $M_1^\#$ is a subset of ω which is not $\Sigma_1(\omega_1)$-definable in any generic extension by forcings of size less than δ.

Proof. The first statement follows directly from Lemma 3.3, since it is equivalent to a Σ^1_3-statement. The second statement follows from the first statement. For the third statement, suppose that the canonical code for $M_1^\#$ is $\Sigma_1(\omega_1)$-definable. Then it is Σ^1_3-definable by Lemma 3.3. It is well known that forcing of size less than δ preserves $M_1^\#$ (see [25, Lemma 3.7]). Since Σ^1_3-truth can be computed in $M_1^\#$ (see [28, p. 1660]), the canonical code for $M_1^\#$ is an element of $M_1^\#$, a contradiction. □

Remark 4.23. *The existence of large cardinals does not imply that $\Sigma_1(\omega_1)$-formulas with parameters in $H(\omega_2)$ are absolute to generic extensions which preserve ω_1. For instance, we can add a Suslin tree T by adding a Cohen real (see* [10, Theorem 28.12]*). When we add a branch through T by forcing with T, ω_1 is not collapsed. Note that the existence of a branch through T is $\Sigma_1(T)$.*

5. $\Sigma_1(\omega_1)$-DEFINABLE SETS IN M_1

We show that for some of the results above, large cardinal assumptions are necessary, because these results fail in M_1. We start by showing that the assumption of Theorem 4.1 is optimal. For other applications, we will construct well-orderings of $\mathrm{H}(\kappa^+)$ with the property that the initial segments are uniformly $\Sigma_1(\kappa)$-definable.

Definition 5.1. Given an infinite cardinal κ, a well-ordering $\lhd$ of a subset of $\mathrm{H}(\kappa^+)$ is a *good* $\Sigma_1(\kappa)$*-well-ordering* if the set $I(\lhd) = \{\{x \mid x \lhd y\} \mid y \in \mathrm{ran}(\lhd)\}$ of all proper initial segments of $\lhd$ is $\Sigma_1(\kappa)$-definable.

Theorem 5.2. *Suppose that M_1 exists. In M_1, the canonical well-ordering of M_1 restricted to* $\mathrm{H}(\omega_2)$ *is a good $\Sigma_1(\omega_1)$-definable well-order.*

Proof. Let δ be the unique Woodin cardinal in M_1. Work in $M_1|\delta$. Then there is no inner model with a Woodin cardinal, because $M_1|\delta$ is closed under sharps and, by a theorem of Woodin, the existence of such an inner model would imply that $M_1^\#$ is an element of $M_1|\delta$.[3]

By a *mouse* we mean a premouse in the sense of Mitchell-Steel [19] such that all countable elementary substructures are ω_1-iterable. The previous argument allows us to use [2, Lemma 2.1] to conclude that a premouse $M \in \mathrm{H}(\omega_2)$ with no definable Woodin cardinals is a mouse if and only if there is a transitive model $U \in \mathrm{H}(\omega_2)$

[2]Given a Σ_1-formula $\varphi(v)$, a partial order $\mathbb{P}$ of cardinality less than δ and G $\mathbb{P}$-generic over V, then this statement says that $\varphi(\omega_1^{\mathrm{V}})^{\mathrm{V}}$ holds if and only if $\varphi(\omega_1^{\mathrm{V}[G]})^{\mathrm{V}[G]}$ holds.

[3]This result is unpublished, but the methods used in the (known) proof can be found in [29].

of ZFC^- plus "there is no inner model with a Woodin cardinal" with $\omega_1 \subseteq U$ and $\langle U, \in\rangle \models$ "*M is a mouse*". This shows that the set

$$A \;=\; \{M \in \mathrm{H}(\omega_2) \mid M \text{ is a mouse}, \, \omega_1^M = \omega_1, \, \rho_\omega(M) = \omega_1\}$$

is $\Sigma_1(\omega_1)$-definable. Since $N \in \mathrm{H}(\omega_2)$ is an initial segment of $M_1|\omega_2$ if and only N is an initial segment of some M in A, the above computations show that the collection of all initial segments of $M_1|\omega_2$ is also $\Sigma_1(\omega_1)$-definable.

Let $\lhd$ denote the canonical well-ordering of $\mathrm{H}(\omega_2)$ in M_1. Given $x, y \in \mathrm{H}(\omega_2)$, we have $x \lhd y$ if and only if there is an initial segment N of $M_1|\omega_2$ such that $x, y \in N$ and $x <_N y$, where $<_N$ is the canonical well-ordering of N. By the above computations, this shows that $\lhd$ is a good Σ_1-definable well-order of $H(\omega_2)^{M_1}$. □

Theorem 5.3. *Suppose that M_1 exists. There is a generic extension of M_1 in which $\neg\mathsf{CH}$ holds and there is a good $\Sigma_1(\omega_1)$-definable well-order of $H(\omega_2)$.*

Proof. Let δ denote the unique Woodin cardinal in M_1. Work in M_1 and let $\lhd$ denote the canonical well-ordering of M_1. Given $\alpha \in \omega_1 \cap \mathrm{Lim}$, let C_α denote the $\lhd$-least cofinal subset of α of order-type ω. Then $\vec{C} = \langle C_\alpha \mid \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ is a C-sequence. Let $\nu < \delta$ be a Mahlo cardinal and let $\kappa < \nu$ be Σ_1-reflecting in $M_1|\nu$. In this situation, let $\mathbb{P}$ denote the partial order constructed in [7] that forces BPFA to hold in a generic extension of $M_1|\nu$ using the reflecting cardinal κ and let G be $\mathbb{P}$-generic over M_1. Then $\omega_1^{M_1} = \omega_1^{M_1[G]}$, $\mathrm{H}(\omega_2)^{(M_1|\nu)[G]} = \mathrm{H}(\omega_2)^{M_1[G]}$, $\vec{C}$ is still a C-sequence in $(M_1|\nu)[G]$ and, by [1, Theorem 2], there is a good $\Sigma_1(\vec{C})$-definable well-ordering of $\mathrm{H}(\omega_2)$ in $M_1[G]$. The forcing does not add an inner model with a Woodin cardinal, since (as in the proof of Lemma 5.2) this would imply that $M_1^\#$ is an element of $(M_1|\delta)[G]$ and hence of $M_1|\delta$, by using two mutual generics and the fact that M_1 is Σ^1_3-correct in V. Hence we can use the same $\Sigma_1(\omega_1)$-definition of the initial segments of M_1 as in the proof of Lemma 5.2. Therefore the set $\{\vec{C}\}$ is $\Sigma_1(\omega_1)$-definable in $M_1[G]$. This yields the statement of the theorem. □

Theorem 5.4. *Suppose that M_1 exists. Then the following statements hold in a forcing extension $M_1[G]$ of M_1.*

(i) *There is a Woodin cardinal.*
(ii) *The* GCH *fails at ω_1.*
(iii) *There is a $\Sigma_1(\omega_1)$-definable well-ordering of* $\mathrm{H}(\omega_2)$.

Proof. If δ is the unique Woodin cardinal in M_1 and $\lhd$ is the canonical well-ordering of M_1 restricted to $\mathrm{H}(\omega_2)^{M_1}$, then the following statements hold in M_1:

(i) $\lhd$ is a good $\Sigma_1(\omega_1)$-definable well-ordering.
(ii) If $\mathbb{P}$ is a partial order of cardinality less than δ with the property that forcing with $\mathbb{P}$ preserves cofinalities less than or equal to ω_2 and G is $\mathbb{P}$-generic over V, then $\mathrm{H}(\omega_2)^{\mathrm{V}}$ is $\Sigma_1(\omega_1)$-definable in $\mathrm{V}[G]$.
(iii) There is a closed unbounded subset of $[\mathrm{H}(\omega_2)]^\omega$ consisting of elementary submodels M of $\mathrm{H}(\omega_2)$ with $\pi[I(\lhd) \cap M] \subseteq I(\lhd)$, where $\pi : M \longrightarrow N$ denotes the corresponding transitive collapse.

The proof of (i) and (ii) work as in the proofs of Theorem 5.2 and Theorem 5.3. The statement (iii) can be derived from the version of the *condensation lemma* (see [32, Theorem 9.3.2]) for M_1, where the cases (a), (b) and (d) can be ruled out.

This shows that the tuple $\langle \delta, \omega_2, \omega_1, \lhd\rangle$ is *suitable for* ω_1 as in [8, Definition 7.1]. Suppose that G is $\mathrm{Add}(\omega_1, \mu)$-generic for some cardinal $\mu < \delta$ with $\mathrm{cof}(\mu) > \omega_1$.

Then [8, Corollar 7.9] shows that there is a cofinality preserving forcing extension of $\mathrm{V}[G]$ that contains a $\Sigma_1(\omega_1)$-definable well-order of $\mathrm{H}(\omega_2)$. □

The following result shows that the assumption in Theorem 4.6 is optimal.

Lemma 5.5. *Let κ be an uncountable regular cardinal. If there is a good $\Sigma_1(\kappa)$-definable well-ordering of $\mathrm{H}(\kappa^+)$, then there is a Bernstein subset of ${}^\kappa\kappa$ that is $\Delta_1(\kappa)$-definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$.*

Proof. A $\Sigma_1(\kappa)$-definable Bernstein set can be constructed by a Σ-recursion along the good $\Sigma_1(\kappa)$-definable well-ordering $\lhd$ of $\mathrm{H}(\kappa^+)$. We fix a $\Sigma_1(\kappa)$-definable enumeration of perfect subtrees of ${}^\kappa\kappa$ of length κ^+. In each step, we choose two distinct elements of the next perfect subset of ${}^\kappa\kappa$. We add one of these to the Bernstein set and the other one to its complement. Moreover we add the next element in $\lhd$ either to the Bernstein set or to its complement. □

Lemma 5.6. *The existence of a $\Delta_1(\omega_1)$-definable Bernstein subset of $\omega_1^{\omega_1}$ is consistent with the existence of a Woodin cardinal.*

Proof. This follows from Theorem 5.2 and Lemma 5.5. □

6. $\Sigma_1(\kappa)$-definable sets at large cardinals

In this section, we generalize some of the previous results to large cardinals.

Definition 6.1 (Gitman-Welch, [6])**.** Let κ be an uncountable cardinal.

(i) A *weak κ-model* is a transitive model M of ZFC^- of size κ with $\kappa \in M$.
(ii) The cardinal κ is *ω_1-iterable* if for every subset A of κ there is a weak κ-model M and a weakly amenable M-ultrafilter U on κ such that $A \in M$ and $\langle M, \in, U\rangle$ is ω_1-iterable.

We start by proving the following analog of Lemma 3.3.

Lemma 6.2. *Assume that κ is either an ω_1-iterable cardinal or a regular cardinal that is a stationary limit of ω_1-iterable cardinals. Then the following statements are equivalent for every subset X of $\mathbb{R}$.*

(i) The set X is $\Sigma_1(\kappa)$-definable.
(ii) The set X is Σ^1_3-definable.

Proof. By Lemma 3.1, it suffices to show that (i) implies (ii). Assume that $\varphi(v_0, v_1)$ is a Σ_0-formula with $X = \{x \in \mathbb{R} \mid \varphi(x, \kappa)\}$. Define Y to be the set of all $y \in \mathbb{R}$ with the property that there is a countable transitive model M of ZFC^-, a cardinal δ of M with $\varphi(y, \delta)^M$ and a weakly amenable M-ultrafilter F on δ such that the structure $\langle M, \in, F\rangle$ is ω_1-iterable.

Claim. *The set Y is a Σ^1_3-subset of $\mathbb{R}$.*

Proof. Since ω_1-iterability is a Π^1_2-statement and all other conditions are first order statements about $\langle M, \in, F\rangle$, the existence of such a structure is a Σ^1_3-statement. □

Claim. $X \subseteq Y$.

Proof. First, assume that κ is ω_1-iterable and pick $x \in X$. Then we can find $A \subseteq \kappa$ with $x \in \mathrm{L}[A]$ and $\varphi(x, \kappa)^{\mathrm{L}[A]}$. By our assumption, there is a transitive model N of ZFC^- of cardinality κ with $\kappa, A \in N$ and an N-ultrafilter U on κ such that the structure $\langle N, \in, U\rangle$ is iterable. Then $x \in N$ and $\varphi(x, \kappa)^N$. Let $\langle N_0, \in, U_0\rangle$ be a countable elementary submodel of $\langle N, \in, U\rangle$ with $x, A \in N_0$ and let $\pi : N_0 \longrightarrow M$ denote the corresponding transitive collapse. Set $\delta = \pi(\kappa)$ and $F = \pi[U_0]$. In this situation, [12, Theorem 19.15] shows that the structure $\langle M, \in, F\rangle$ is iterable. Since $\varphi(x, \delta)^M$ holds by elementarity, we can conclude that x is an element of Y.

Now, assume that κ is a stationary limit of ω_1-iterable cardinals. Pick $x \in X$ and a strictly increasing continuous chain $\langle N_\alpha \mid \alpha < \kappa\rangle$ of elementary submodels of $\mathrm{H}(\kappa^+)$ of cardinality less than κ such that $x \in N_0$ and $\kappa_\alpha = \kappa \cap N_\alpha \in \kappa$ for all $\alpha < \kappa$. Then $C = \{\kappa_\alpha \mid \alpha \in \kappa \cap \mathrm{Lim}\}$ is a club in κ and there is an $\bar{\kappa} < \kappa$ such that $\kappa_{\bar{\kappa}}$ is ω_1-iterable. Since ω_1-iterability implies inaccessibility, we have $\bar{\kappa} = \kappa_{\bar{\kappa}}$. By elementarity and Σ_1-upwards absoluteness, we know that $\varphi(x, \bar{\kappa})$ holds. In this situation, we can repeat the construction of the first case to obtain a countable iterable structure $\langle M, \in, F\rangle$ that witnessing that x is an element of Y. □

Claim. $Y \subseteq X$.

Proof. Pick $y \in Y$ and let $\langle M_0, \in, F_0\rangle$ and $\delta \in M_0$ witness this. Then $\langle M_0, \in, F_0\rangle$ is iterable and $\varphi(y, \delta)^{M_0}$ holds. Let

$$\langle\langle\langle M_\alpha, \in, F_\alpha\rangle \mid \alpha \in \mathrm{On}\rangle, \langle j_{\bar{\alpha},\alpha} : M_{\bar{\alpha}} \longrightarrow M_\alpha \mid \bar{\alpha} \leq \alpha \in \mathrm{On}\rangle\rangle$$

denote the corresponding system of models and elementary embeddings. Then $j_{0,\kappa}(\delta) = \kappa$ and $\varphi(x, \kappa)$ holds by elementarity and Σ_1-upwards absoluteness. This shows that y is an element of X. □

This completes the proof of the lemma. □

Lemma 6.3. *Assume that κ is either an ω_1-iterable cardinal or a regular cardinal that is a stationary limit of ω_1-iterable cardinals. If there is a $\Sigma_1(\kappa)$-definable well-ordering of the reals, then there is a Σ^1_3-well-ordering of the reals.* □

If κ is either a Woodin cardinal below a measurable cardinal or a measurable cardinal above a Woodin cardinal, then the above results allow us to show that there is no $\Sigma_1(\kappa)$-definable well-ordering of the reals.

Proof of Theorem 1.6. Let κ either be a measurable cardinal above a Woodin cardinal or a Woodin cardinal below a measurable cardinal. Then $\mathbf{\Sigma}^1_2$-determinacy holds and no well-ordering of the reals is Σ^1_3-definable. If κ is a measurable cardinal, then κ is ω_1-iterable (see [5]) and Corollary 6.3 implies that no well-ordering of the reals is $\Sigma_1(\kappa)$-definable. In the other case, if κ is a Woodin cardinal, then κ is a stationary limit of measurable cardinals (and hence a stationary limit of ω_1-iterable cardinals) and Corollary 6.3 implies that no well-ordering of the reals is $\Sigma_1(\kappa)$-definable. □

In the following, we prove a large cardinal version of Lemma 4.3. This result will allow us to prove Theorem 1.7.

Lemma 6.4. *Let κ be a measurable cardinal and let X be a $\Sigma_1(\kappa)$-definable subset of ${}^\kappa\kappa$. If there is an $x \in X$ such that are normal ultrafilters U_0 and U_1 on κ with $y = \{\alpha < \kappa \mid x(\alpha) = 0\} \in U_1 \setminus U_0$, then for every $\xi < \kappa$ there is*

(i) a continuous injection $\iota : {}^{\omega_1}2 \longrightarrow X$
(ii) a club D in κ

such that for the increasing enumeration $\langle \delta_\alpha \mid \alpha < \kappa\rangle$ of D

(i) $\mathrm{ran}(\iota) \subseteq N_{x \restriction \xi} \cap X$

(ii) *for all* $z \in {}^\kappa 2$ *and* $\alpha < \kappa$, *then* $z(\alpha) = 1$ *if and only if* $\iota(z)(\delta_\alpha) > 0$.

Proof. Fix $\xi < \kappa$ and and a regular cardinal $\theta > \kappa$ with $\mathcal{P}(\mathcal{P}(\kappa)) \in \mathrm{H}(\theta)$. Pick a Σ_1-formula $\varphi(v_0, v_1)$ with $X = \{z \in {}^\kappa\kappa \mid \varphi(\kappa, z)\}$ and an elementary submodel N of $\mathrm{H}(\theta)$ of cardinality less than κ with $\kappa, x, U_0, U_1 \in N$ and $\xi + 1 \subseteq N$. Let $\pi : N \longrightarrow M$ denote the corresponding transitive collapse.

In this situation [26, Theorem 2.3] shows that there is a directed system

$$\langle\langle M_s \mid s \in {}^{\leq\kappa}2\rangle,\ \langle j_{s,t} : M_s \longrightarrow M_t \mid s,t \in {}^{\leq\kappa}2,\ s \subseteq t\rangle\rangle$$

of transitive models of ZFC^- and elementary embeddings such that the following statements hold:

(i) $M = M_\emptyset$.

(ii) If $s \in {}^{<\omega_1}2$ and $i < 2$, then $M_{s^\frown\langle i\rangle} = \mathrm{Ult}(M_s, (j_{\emptyset,s} \circ \pi)(U_i))$ and $j_{s,s^\frown\langle i\rangle}$ is the corresponding ultrapower map induced by $(j_{\emptyset,s} \circ \pi)(U_i)$.

(iii) If $s \in {}^{\leq\kappa}2$ with $\mathrm{lh}(s) \in \mathrm{Lim}$, then

$$\langle M_s,\ \langle j_{s\restriction\alpha,s} : M_{s\restriction\alpha} \longrightarrow M_s \mid \alpha < \mathrm{lh}(s)\rangle\rangle$$

is the direct limit of the directed system

$$\langle\langle M_{s\restriction\alpha} \mid \alpha < \mathrm{lh}(s)\rangle,\ \langle j_{s\restriction\bar\alpha,s\restriction\alpha} : M_{s\restriction\bar\alpha} \longrightarrow M_{s\restriction\alpha} \mid \bar\alpha \leq \alpha < \mathrm{lh}(s)\rangle\rangle.$$

Set $j_s = j_{\emptyset,s}$ for all $s \in {}^{\leq\kappa}2$. Since $\kappa = (j_z \circ \pi)(\kappa)$ for all $z \in {}^\kappa 2$, we can define

$$i : {}^\kappa 2 \longrightarrow {}^\kappa\kappa;\ z \longmapsto (j_z \circ \pi)(x).$$

In this situation, elementarity and Σ_1-upwards absoluteness imply that $\varphi(\kappa, i(z))$ and $x \restriction \xi = i(z) \restriction \xi$ holds for all $z \in {}^\kappa 2$. In particular, we have $\mathrm{ran}(i) \subseteq N_{x\restriction\xi} \cap X$.

Given $z \in {}^\kappa 2$, we define

$$c_z : \kappa \longrightarrow \kappa;\ \alpha \longmapsto (j_{z\restriction\alpha} \circ \pi)(\kappa).$$

Then $\mathrm{ran}(c_z)$ is strictly increasing and continuous for every $z \in {}^\kappa 2$. By definition, we have $c_{z_0} \restriction \alpha = c_{z_1} \restriction \alpha$ for all $z_0, z_1 \in {}^\kappa 2$ and $\alpha < \kappa$ with $z_0 \restriction \alpha = z_1 \restriction \alpha$. Given $z \in {}^\kappa 2$ and $\alpha < \kappa$, we have

$$\mathrm{crit}\left(j_{z\restriction\alpha,z\restriction(\alpha+1)}\right) \ = \ c_z(\alpha) \ < \ c_z(\alpha+1) \ = \ \mathrm{crit}\left(j_{z\restriction(\alpha+1),z}\right)$$

and

$$(j_{z\restriction\alpha} \circ \pi)(y) \ \in \ (j_{z\restriction\alpha} \circ \pi)(U_1) \setminus (j_{z\restriction\alpha} \circ \pi)(U_0).$$

This allows us to conclude that

$$\begin{aligned} z(\alpha) = 1 \iff & c_z(\alpha) \in (j_{z\restriction(\alpha+1)} \circ \pi)(y) \\ \iff & (((j_{z\restriction(\alpha+1)} \circ \pi)(x))(c_z(\alpha)) > 0 \\ \iff & (((j_{z\restriction(\alpha+1),z} \circ j_{z\restriction(\alpha+1)} \circ \pi)(x))(c_z(\alpha)) > 0 \\ \iff & (i(z)(c_z(\alpha)) > 0 \end{aligned}$$

holds for all $z \in {}^\kappa 2$ and $\alpha < \kappa$. In particular, this shows that i is injective.

Now, fix $z \in {}^{\omega_1}2$ and $\beta < \omega_1$. Pick $\alpha < \omega_1$ with $c_z(\alpha) > \beta$. Since we have $c_{\bar z}(\alpha) = \mathrm{crit}\left(j_{\bar z\restriction\alpha,z}\right)$ and $i(\bar z) \restriction \beta = (j_{\bar z\restriction\alpha} \circ \pi)(x) \restriction \beta$ for all $\bar z \in {}^{\omega_1}2$, we can conclude that $i(z) \restriction \beta = i(\bar z) \restriction \beta$ holds for all $\bar z \in N_{z\restriction\alpha} \cap {}^\kappa 2$. This shows that i is continuous.

Let $\langle\delta_\alpha \mid \alpha < \kappa\rangle$ denote the monotone enumeration of the club D of all uncountable cardinals less than κ and let $e : {}^\kappa 2 \longrightarrow {}^\kappa 2$ denote the unique continuous injection with $e(z)^{-1}\{1\} = \{\delta_\alpha \mid \alpha < \kappa,\ z(\alpha) = 1\}$ for all $z \in {}^\kappa 2$. Then $c_z \restriction D = \mathrm{id}_D$ for all $z \in {}^\kappa 2$. Set $\iota = i \circ e$. Given $z \in {}^\kappa 2$ and $\alpha < \kappa$, we then have

$$z(\alpha) > 0 \iff e(z)(\delta_\alpha) > 0 \iff i(e(z))(c_{e(z)}(\delta_\alpha)) > 0 \iff \iota(z)(\delta_\alpha) > 0. \quad \square$$

The above lemma allows us to prove the following strengthening of Theorem 1.7.

Theorem 6.5. *Let κ be a measurable cardinal with the property that there are two distinct normal ultrafilters on κ and let Γ be a set of $\Sigma_1(\kappa)$-definable subsets of ${}^\kappa\kappa$. If $\bigcup\Gamma = {}^\kappa\kappa$, then some element of Γ contains a perfect subset.*

Proof. Pick normal ultrafilters U_0 and U_1 on κ with $U_0 \neq U_1$. Then there is $x \in {}^\kappa\kappa$ with $\{\alpha < \kappa \mid x(\alpha) > 0\} \in U_1 \setminus U_0$ and $X \in \Gamma$ with $x \in X$. In this situation, Lemma 6.4 implies that X contains a perfect subset. □

The following result shows that the conclusion of Theorem 1.7 does not hold for all measurable cardinals.

Theorem 6.6. *Assume that κ is a measurable cardinal and U is a normal ultrafilter on κ with $\mathrm{V} = \mathrm{L}[U]$. Then there is a Bernstein subset of ${}^\kappa\kappa$ that is $\Delta_1(\kappa)$-definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$.*

Proof. Following [12, p. 264], we define a ZFC^--*mouse at* λ to be a structure $\langle M, \in, F\rangle$ such that M is a transitive model of ZFC^- with $M = \mathrm{L}_\alpha[F]$ for some ordinal α and F is a weakly amenable M-ultrafilter on λ such that $\langle M, \in, F\rangle$ is ω_1-iterable. Note that ω_1-iterability implies full iterability and our assumptions imply that every element of $\mathrm{H}(\kappa^+)$ is contained in a ZFC^--mouse at some $\lambda > \kappa$. We define a well-order $\lhd$ on $\mathrm{H}(\kappa^+)$ by setting $x \lhd y$ if there is a ZFC^--mouse $\langle M, \in, F\rangle$ at some $\lambda > \kappa$ with $x, y \in M$ and $x <_{\mathrm{L}[F]} y$.

Claim. *$\lhd$ is a good $\Sigma_1(\kappa)$-definable well-order of $\mathcal{P}(\kappa)^{\mathrm{L}[U]}$.*

Proof. Let M be a ZFC^--mouse. By [12, Lemma 20.8], there are elementary embeddings $i : M \longrightarrow \mathrm{L}_\gamma[F]$ and $j : \mathrm{Ult}(\mathrm{V}, U) \longrightarrow \mathrm{L}[F]$ with critical points greater than κ and $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^{\mathrm{L}_\gamma[F]} \subseteq \mathcal{P}(\kappa)^{\mathrm{L}[F]} = \mathcal{P}(\kappa)^{\mathrm{V}}$. Hence $\lhd$ is equal to the restriction of the canonical well-order of $\mathrm{Ult}(\mathrm{V}, U)$ to $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ and every ZFC^--mouse is downwards-closed with respect to $\lhd$. Since ω_1-iterability can be checked by transitive models of some fragments of ZFC containing ω_1 as a subset and is therefore a $\Sigma_1(\kappa)$ condition, the above computations yield the statement of the claim. □

By Lemma 5.5, the above claim implies the statement of the theorem. □

In the remainder of this section, we study the Π_1-definability of the club filter at large cardinals. We start by proving Theorem 1.9, which shows that the club filter on κ is not $\Pi_1(\kappa)$-definable if κ is a stationary limit of ω_1-iterable cardinals.

Proof of Theorem 1.9. Let κ be a regular cardinal that is a stationary limit of ω_1-iterable cardinals. Fix a Σ_1-formula $\varphi(v_0, v_1)$ and assume, towards a contradiction, that the complement of the club filter on κ is equal to the set $\{x \subseteq \kappa \mid \varphi(\kappa, x)\}$. Let y denote the set of ω_1-iterable cardinals less than κ and set $z = \kappa \setminus y$. Then z is a bistationary subset of κ and $\varphi(\kappa, z)$ holds.

Pick a strictly increasing continuous chain $\langle N_\alpha \mid \alpha < \kappa\rangle$ of elementary submodels of $\mathrm{H}(\kappa^+)$ of cardinality less than κ such that $z \in N_0$ and $\kappa_\alpha = \kappa \cap N_\alpha \in \kappa$ for all $\alpha < \kappa$. Then $C = \{\kappa_\alpha \mid \alpha \in \kappa \cap \mathrm{Lim}\}$ is a club in κ. Let δ denote the minimal element of $\kappa \cap \mathrm{Lim}$ with $\kappa_\delta \in y$. Since κ_δ is an ω_1-iterable cardinal and therefore regular, we know that $\delta = \kappa_\delta$. Let $\pi : N_\delta \longrightarrow N$ denote the transitive collapse of N_δ. Then $\pi(\kappa) = \delta$, $\pi(z) = z \cap \delta$. In this situation, Σ_1-upwards absoluteness implies that $\varphi(\delta, z \cap \delta)$ holds in V. Moreover, $C \cap \delta$ is a club in δ and the minimality of δ implies that $C \cap \delta$ is a subset of $z \cap \delta$.

Since δ is ω_1-iterable, we can find a weak κ-model M_0 and an M_0-ultrafilter F_0 on δ such that $z \cap \delta, C \cap \delta \in M_0$, $\varphi(\delta, z \cap \delta)^{M_0}$ holds and $\langle M_0, \in, F_0 \rangle$ is iterable. Let

$$\langle \langle \langle M_\alpha, \in, F_\alpha \rangle \mid \alpha \in \mathrm{On} \rangle, \langle j_{\bar{\alpha},\alpha} : M_{\bar{\alpha}} \longrightarrow M_\alpha \mid \bar{\alpha} \leq \alpha \in \mathrm{On} \rangle \rangle$$

denote the corresponding system of models and elementary embeddings. Then $j_{0,\kappa}(\delta) = \kappa$ and $j_{0,\kappa}(C \cap \delta)$ is a club in κ that witnesses that the set $j_{0,\kappa}(z \cap \delta)$ is contained in the club filter on κ. But Σ_1-upwards absoluteness and elementarity imply that $\varphi(\kappa, j_{0,\kappa}(z \cap \delta))$ holds, a contradiction. □

Next, we prove an analog of Lemma 4.11 for certain large cardinals.

Lemma 6.7. *Let κ be an uncountable regular cardinal, let M be a weak κ-model and let U be an M-ultrafilter such that $\langle M, \in, U \rangle$ is ω_1-iterable. If $\varphi(v_0, v_1)$ is a Σ_1-formula, then the following statements hold for all $\xi < \kappa$ and $x \in M \cap \mathcal{P}(\kappa)$ with the property that $\varphi(\kappa, x)^M$ holds:*

- *(i) If $x \in U$, then there is an element y of the club filter on κ such that $x \restriction \xi = y \restriction \xi$ and $\varphi(\kappa, y)$ holds.*
- *(ii) If $x \notin U$, then there is an element y of the nonstationary ideal on κ such that $x \restriction \xi = y \restriction \xi$ and $\varphi(\kappa, y)$ holds.*

Proof. Pick an elementary submodel $\langle N, \in, F \rangle$ of $\langle M, \in, U \rangle$ of cardinality less than κ with $\kappa, x \in N$ and $\xi + 1 \subseteq N$. Let $\pi : N \longrightarrow M_0$ denote the corresponding transitive collapse. Set $F_0 = \pi[F]$. Then F_0 is an M_0-ultrafilter and [12, Theorem 19.15] implies that the structure $\langle M_0, \in, F_0 \rangle$ is iterable. Let

$$\langle \langle \langle M_\alpha, \in, F_\alpha \rangle \mid \alpha \in \mathrm{On} \rangle, \langle j_{\bar{\alpha},\alpha} : M_{\bar{\alpha}} \longrightarrow M_\alpha \mid \bar{\alpha} \leq \alpha \in \mathrm{On} \rangle \rangle$$

denote the corresponding system of models and elementary embeddings. Define $y = (j_{0,\kappa} \circ \pi)(x)$. Since $\kappa = (j_{0,\kappa} \circ \pi)(\kappa)$, Σ_1-upwards absoluteness and elementarity imply that $\varphi(\kappa, y)$ holds and $x \restriction \xi = y \restriction \xi$. Moreover, the set $C = \{(j_{0,\alpha} \circ \pi)(\kappa) \mid \alpha < \kappa\}$ is a club in κ.

Now, assume $x \in U$. Then $(j_{0,\alpha} \circ \pi)(x) \in F_\alpha$ and $(j_{0,\alpha} \circ \pi)(\kappa) \in (j_{0,\alpha+1} \circ \pi)(x)$ for all $\alpha < \kappa$. Since we have $(j_{0,\alpha} \circ \pi)(x) < (j_{0,\alpha+1} \circ \pi)(x) = \mathrm{crit}\,(j_{\alpha+1,\kappa})$ for all $\alpha < \kappa$, we can conclude that C is a subset of y in this case and therefore y is contained in the club filter on κ.

Finally, assume $x \notin U$. Then $(j_{0,\alpha} \circ \pi)(x) \notin F_\alpha$ and $(j_{0,\alpha} \circ \pi)(\kappa) \notin (j_{0,\alpha+1} \circ \pi)(x)$ for all $\alpha < \kappa$. As above, we can conclude that C is disjoint from y in this case and therefore y is an element of the nonstationary ideal. □

The previous lemma allows us to show that the club filter and the non-stationary ideal cannot be separated by a $\Delta_1(\kappa)$-set for certain large cardinals κ.

Proof of Theorem 1.8. Let κ be an ω_1-iterable cardinal and assume that there are Σ_1-formulas $\varphi(v_0, v_1)$ and $\psi(v_0, v_1)$ with the property that the subset $X = \{x \subseteq \kappa \mid \varphi(\kappa, x)\}$ of $\mathcal{P}(\kappa)$ separates the club filter from the nonstationary ideal and $\mathcal{P}(\kappa) \setminus X = \{x \subseteq \kappa \mid \psi(\kappa, x)\}$. Pick an elementary submodel M of $\mathrm{H}(\kappa^+)$ of cardinality κ with $\kappa + 1 \subseteq M$. By our assumptions, there is a κ-model N and an N-ultrafilter U on κ such that $M \in N$ and $\langle N, \in, U \rangle$ is iterable. Set $F = M \cap U$.

Claim. *$F = M \cap X$.*

Proof. Assume that there is $x \in F$ with $x \notin X$. Then elementarity implies that $\psi(\kappa, x)^M$ holds and Σ_1-upwards absoluteness implies that $\psi(\kappa, x)^N$ holds. By the first part of Lemma 6.7, this shows that there is an element y of the club filter on κ such that $\psi(\kappa, y)$ holds, a contradiction. This shows that $F \subseteq M \cap X$.

Now, assume that $x \in M \cap X$ with $x \notin U$. Then elementarity implies that $\varphi(\kappa, x)^M$ holds and Σ_1-upwards absoluteness implies that $\varphi(\kappa, x)^N$ holds. By the second part of Lemma 6.7, there is an element y of the nonstationary ideal on κ such that $\varphi(\kappa, y)$ holds, a contradiction. Together with the above computations, this shows that $F = M \cap X$. □

Since $\langle M, \in, F\rangle \models$ "*F is a normal ultrafilter on κ*" and F is $\Delta_1(\kappa)$-definable over $\langle M, \in\rangle$, elementarity implies that X is a normal ultrafilter over κ in V. Let $\mathrm{Ult}(\mathrm{V}, X)$ denote the corresponding ultrapower of V. Then $\mathrm{H}(\kappa^+) = \mathrm{H}(\kappa^+)^{\mathrm{Ult}(\mathrm{V},X)}$. Since X is definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$, we can conclude that X is an element of $\mathrm{Ult}(\mathrm{V}, X)$, a contradiction. □

For measurable cardinals κ, we obtain a result similar to Lemma 4.22.

Lemma 6.8. *Let κ be an ω_1-iterable cardinal and let λ be a measurable cardinal.*

- *(i) $\Sigma_1(\kappa)$-statements (without parameters) are absolute to generic extensions for forcings of size less than λ which preserve the ω_1-iterability of κ.*
- *(ii) The set of $\Sigma_1(\kappa)$-definable singletons $\{x\}$ with $x \subseteq \kappa$ is absolute for forcings of size less than λ which preserve the ω_1-iterability of κ.*

Proof. The first claim follows from Lemma 6.2, since the statement is equivalent to a Σ^1_3-statement and Σ^1_3-absoluteness holds for forcings of size less than λ (see [25, Lemma 3.7]). The second claim follows from the first claim. □

7. Open questions

We close this paper with a collection of questions raised by the above results. Lemma 3.3 and Lemma 3.7 suggest the following question.

Question 7.1. *Assume that there is a proper class of Woodin cardinals. If B is a uB set of reals, is every $\Sigma^1_3(B)$-set $\Sigma_1(\omega_1)$-definable over $\langle \mathrm{H}(\omega_2), \in, B, \mathrm{NS}_{\omega_1}\rangle$?*

Theorem 4.19 leaves open the following question.

Question 7.2. *Suppose that there is a Woodin cardinal and a measurable cardinal above it. Is there no $\Sigma_1(\omega_1)$-definable uniformization of the club filter on ω_1?*

Note that the existence of a good $\Sigma_1(\omega_1)$-definable well-order of $\mathcal{P}(\omega_1)$ yields a $\Sigma_1(\omega_1)$-definable uniformization of the club filter on ω_1 and Theorem 5.2 shows that such a uniformization is compatible with the existence of a Woodin cardinal.

Next, we ask if the assumption in Theorem 4.9 is optimal. The conclusion does not follow from the existence of a Woodin cardinal by the proof of Lemma 5.6. Moreover the perfect set property for all definable subsets of ${}^{\omega_1}\omega_1$ can be forced by Levy-collapsing an inaccessible cardinal (see [24]).

Question 7.3. *Suppose that NS_{ω_1} is saturated or that there is a Woodin cardinal and a measurable cardinal above it. Does the perfect set dichotomy over $\mathrm{L}(\mathbb{R})$ in Theorem 4.9 hold?*

Moreover, we do not know if the two cases in the perfect set dichotomy in Theorem 4.9 are mutually exclusive unless $2^\omega < 2^{\omega_1}$. This is related to the question over which models it is possible to add perfect subsets of the ground model (see [30] and [17, Lemma 6.2]).

Question 7.4. *Is it consistent with the existence of a Woodin cardinal and a measurable cardinal above it that there is a perfect subset of* ${}^{\omega_1}\omega_1 \cap \mathrm{L}(\mathbb{R})$*? In particular, does this statement fail in the* $\mathbb{P}_{max}$*-extension of* $\mathrm{L}(\mathbb{R})$ *if there are infinitely many Woodin cardinals?*

We ask about generalizations of the results of this paper to ω_2 and larger cardinals. In this situation, the method of iterations of generic ultrapowers fails, since generics need not exist over uncountable models.

Question 7.5. *Is the existence of a* $\Sigma_1(\omega_2)$*-definable well-ordering of the reals compatible with the existence of a supercompact cardinal?*

We also ask about a perfect set dichotomy for large cardinals.

Question 7.6. *Let* κ *be a supercompact cardinal and let* X *be a subset of* ${}^\kappa\kappa$ *that is* Σ_1*-definable over* $\langle \mathrm{H}(\kappa^+), \in\rangle$ *and has cardinality greater than* κ*. Does* X *contain a perfect subset?*

The motivation for this question is that for supercompact cardinals, there are many different normal ultrafilters on κ. Let κ be a measurable cardinal and let D denote the collection of all subsets y of κ with the property that there are ultrapowers I_0 and I_1 of V with normal ultrafilters on κ such that $j_{I_0}(\kappa) = j_{I_1}(\kappa)$ and $j_{I_0}(y) \neq j_{I_1}(y)$. Then the above proofs show: If X is a $\Sigma_1(\kappa)$-definable subset of ${}^\kappa\kappa$ and there is an element x of X with $\{\alpha < \kappa \mid x(\alpha) > 0\} \in D$, then X contains a perfect subset.

Finally, Lemma 6.8 leaves open the following question.

Question 7.7. *Suppose that* $\Phi(\kappa)$ *holds, where* $\Phi(\kappa)$ *is a large cardinal property that implies that* κ *is weakly compact. Are* $\Sigma_1(\kappa)$*-formulas with parameters in* $\mathrm{H}(\kappa^+)$ *absolute to generic extensions for* $<\kappa$*-distributive forcings which preserve* $\Phi(\kappa)$*?*

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