The higher sharp I

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Abstract

We establish the descriptive set theoretic representation of the mouse $M_n^{\#}$, which is called $0^{(n+1)\#}$. This part deals with the case $n \leq 3$.

1 Introduction

The collection of projective subsets of $\mathbb{R}$ is the minimum one which contains all the Borel sets and is closed under both complements and continuous images. Despite its natural-looking definition, many fundamental problems about projective sets are undecidable in ZFC, for instance, if all projective sets are Lebesgue measurable. The axiom of Projective Determinacy (PD) is the most satisfactory axiom that settles these problems by producing a rich structural theory of the projective sets. PD implies certain regularity properties of projective sets: all projects of reals are Lebesgue measurable (Mycielski, Swierczkowski), have the Baire property (Banach, Mazur) and are either countable or have a perfect subset (Davis) (cf. [36]). The structural theory of the projective sets are centered at good Suslin representations of projective sets. Moschovakis [36] shows that PD implies the scale property of the pointclasses $\boldsymbol{\Pi}^1_{2n+1}$ and $\boldsymbol{\Sigma}^1_{2n+2}$. It follows that there is a nicely behaved tree T_{2n+1} that projects to the good universal Σ^1_{2n+2} set. So the analysis of $\boldsymbol{\Sigma}^1_{2n+2}$ sets is reduced to that of the tree T_{2n+1}, the canonical model $L[T_{2n+1}]$

and its relativizations. The canonicity of $L[T_{2n+1}]$ is justified by Becker-Kechris [3] in the sense that $L[T_{2n+1}]$ does not depend on the choice of T_{2n+1}. The model $L[T_{2n+1}]$ turns out to have many analogies with $L = L[T_1]$. These analogies support the generalizations of classical results on $\boldsymbol{\Sigma}^1_2$ sets to $\boldsymbol{\Sigma}^1_{2n+2}$ sets.

The validity of PD is further justified by Martin-Steel [34]. They show that PD is a consequence of large cardinals: if there are n Woodin cardinals below a measurable cardinal, then $\boldsymbol{\Pi}^1_{n+1}$ sets are determined. Inner model quickly developed into the region of Woodin cardinals. $M_n^{\#}$, the least active mouse with n Woodin cardinals, turns out to have its particular meaning in descriptive set theory. Martin [33] (for $n = 0$) and Neeman [37,38] (for $n \geq 1$) show that $M_n^{\#}$ is many-one equivalent to the good universal $\Game^{n+1}(<\omega^2\text{-}\Pi^1_1)$ real. Steel [49] shows that $L[T_{2n+1}] = L[M^{\#}_{2n,\infty}|\boldsymbol{\delta}^1_{2n+1}]$ where $M^{\#}_{2n,\infty}$ is the direct limit of all the countable iterates of $M^{\#}_{2n}$, and that $\boldsymbol{\delta}^1_{2n+1}$ is the least cardinal that is strong up to the least Woodin of $M^{\#}_{2n,\infty}$. This precisely explains the analogy between $L[T_{2n+1}]$ and L. The mechanism of inner model theory is therefore applicable towards understanding the structure $L[T_{2n+1}]$.

In this paper and its sequel, we generalize the Silver indiscernibles for L to the level-$(2n+1)$ indiscernibles of $L[T_{2n+1}]$. The theory of $L[T_{2n+1}]$ with the level-$(2n+1)$ indiscernibles will be called $0^{(2n+1)\#}$, which is many-one equivalent to $M^{\#}_{2n}$. At the level of mice with an odd number of Woodins, $M^{\#}_{2n-1}$ is the optimal real with the basis result for Σ^1_{2n+1} sets (cf. [51, Section 7.2]): Every nonempty Σ^1_{2n+1} set has a member recursive in $M^{\#}_{2n-1}$. The basis result for Σ^1_{2n+1} was originally investigated in [27], with the intention of generalizing Kleene's basis theorem: Every nonempty Σ^1_1 set of real has a member recursive in Kleene's $\mathcal{O}$. The real y_{2n+1}, defined [27], turns out Δ^1_{2n+1} equivalent to $M^{\#}_{2n-1}$. In this paper and its sequel, we define the canonical tree T_{2n} that projects to a good universal Π^1_{2n} set. It is the natural generalization of the Martin-Solovay tree T_2 that projects a good universal Π^1_2 set. We show that $L_{\kappa_{2n+1}}[T_{2n}]$, the minimum admissible set over T_{2n}, shares most of the standard properties of $L_{\omega_1^{CK}}$, in particular, the higher level analog of the Kechris-Martin theorem [21, 23]. We define $0^{(2n)\#}$ as the set of truth values in $L_{\kappa_{2n+1}}[T_{2n}]$ for formulas of complexity slightly higher than Σ_1. $0^{(2n)\#}$ is many-one equivalent to both $M^{\#}_{2n+1}$ and y_{2n+1}. Summing up, we have

$$0^{(n+1)\#} \equiv_m M_n^{\#}.$$

We start to give a detailed explanation of the influence of the higher sharp in the structural theory of projective sets and in inner model theory. The set theoretic structures tied to Π^1_1 sets are $L_{\omega_1^{CK}}$ and its relativizations. The classical results on Π^1_1 sets and $L_{\omega_1^{CK}}$ include:

1. (Model theoretic representation of Π^1_1) $A \subseteq \mathbb{R}$ is Π^1_1 iff there is a Σ_1 formula φ such that $x \in A \leftrightarrow L_{\omega_1^x}[x] \models \varphi(x)$.

2. (Mouse set) $x \in \mathbb{R} \cap L_{\omega_1^{CK}}$ iff x is Δ^1_1 iff x is Δ^1_1 in a countable ordinal.

3. (The transcendental real over $L_{\omega_1^{CK}}$) $\mathcal{O}$ is the Σ_1-theory of $L_{\omega_1^{CK}}$.

4. (Π^1_1-coding of ordinals below ω_1) $x \in \mathrm{WO}$ iff x codes a wellordering of a subset of ω. Every ordinal below ω_1 is coded by a member of WO. WO is Π^1_1.

Σ^1_2 sets are ω_1-Suslin via the Shoenfield tree T_1. The complexity of T_1 is essentially that of WO, or Π^1_1. The set-theoretic structures in our attention are $L = L[T_1]$ and its relativizations. Assuming every real has a sharp, the classical results related to L include:

1. (Model theoretic representation of Σ^1_2) $A \subseteq \mathbb{R}$ is Σ^1_2 iff there is a Σ_1 formula φ such that $x \in A \leftrightarrow L[x] \models \varphi(x)$.

2. (Mouse set) $x \in \mathbb{R} \cap L$ iff x is Δ^1_2 in a countable ordinal.

3. (The transcendental real over L) $0^{\#}$ is the theory of L with Silver indiscernibles, or equivalently, the least active sound mouse projecting to ω.

4. (Δ^1_3-coding of ordinals below u_ω) WO_ω is the set of sharp codes. Every ordinal $\alpha < u_\omega$ has a sharp code $\langle \ulcorner \tau \urcorner, x^{\#} \rangle$ so that $\alpha = \tau^{L[x]}(x, u_1, \ldots, u_k)$. The comparison of sharp codes is Δ^1_3.

Inner model theory start to participate at this level. Based on the theory of sharps for reals, the Martin-Solovay tree T_2 is defined. T_2 is essentially a tree on u_ω. The complexity of T_2 is Δ^1_3 via the sharp coding of ordinals.

Σ^1_3 sets are u_ω-Suslin via the Martin-Solovay tree T_2. The structures tied to Π^1_3 sets are $L_{\kappa_3}[T_2]$ and its relativizations. The theory at this level is in parallel to Π^1_1 sets and $L_{\omega_1^{CK}}$:

1. (Model theoretic representation of Π^1_3, [21, 23]) $A \subset \mathbb{R}$ is Π^1_3 iff there is a Σ_1 formula φ such that $x \in A \leftrightarrow L_{\kappa_3^x}[T_2, x] \models \varphi(T_2, x)$.

2. (Mouse set, [21, 23, 27, 47]) $x \in \mathbb{R} \cap L_{\kappa_3}[T_2]$ iff x is Δ^1_3 in a countable ordinal iff $x \in \mathbb{R} \cap M_1^{\#}$.

3. (The transcendental real over $L_{\kappa_3}[T_2]$, Theorem 4.8) $M_1^{\#} \equiv_m 0^{2\#}$.

4. (Π^1_3-coding of ordinals below $\boldsymbol{\delta}^1_3$, essentially by Kunen in [46]) $\mathrm{WO}^{(3)}$ is the set of reals that naturally code a wellordering of u_ω. $\mathrm{WO}^{(3)}$ is Π^1_3.

In general, if Γ is a pointclass, α is an ordinal, and $f : \mathbb{R} \twoheadrightarrow \alpha$ is a surjection, then $\mathrm{Code}(f) = \{(x,y) : f(x) \leq f(y)\}$ and f is in Γ iff $\mathrm{Code}(f)$ is in Γ; α is Γ-wellordered cardinal iff there is a surjection $f : \mathbb{R} \twoheadrightarrow \alpha$ such that f is in Γ but there is no $\beta < \alpha$ and surjections $g : \mathbb{R} \twoheadrightarrow \beta$, $h : \beta \twoheadrightarrow \alpha$ such that both g and $\{(x,y) : f(x) = h \circ g(y)\}$ is in Γ. The above list can be continued:

5. The uncountable $\boldsymbol{\Delta}^1_3$ wellordered cardinals are $(u_k : 1 \leq k \leq \omega)$.

The heart of the new knowledge is the equality of pointclass in Theorem 4.5: $\Game^2({<}\omega^2\text{-}\Pi^1_1) = {<}u_\omega\text{-}\Pi^1_3$. Philosophically speaking, as $\Game^2\Pi^1_1 = \Pi^1_3$, this equality reduces the "non-linear" part $\Game^2$ to the "linear" part ${<}u_\omega$. Based on this equality, $0^{2\#}$ is defined to be the set of truth of $L_{\kappa_3}[T_2]$ for formulas of complexity slightly larger than Σ_1, cf. Definitions 4.6-4.7. $0^{2\#}$ is essentially y_3, defined in [27]. It is a good universal ${<}u_\omega\text{-}\Pi^1_3$ subset of ω. The many-one equivalence $M_1^\# \equiv_m 0^{2\#}$ is thus obtained using Neeman [37, 38]. Under AD, we have $u_k = \aleph_k$, and [25] summarizes the further structural theory at this level. The expression of $0^{2\#}$ opens the possibility of running recursion-theoretic arguments in $L_{\kappa_3}[T_2]$ that generalize those in $L_{\omega_1^{CK}}$.

The Moschovakis tree T_{2n+1} projects to the good universal Σ^1_{2n+2} set. The structures tied to Σ^1_{2n+2} sets are $L[T_{2n+1}]$ and its relativizations. $L[T_{2n+1}]$ is the higher level analog of L:

1. (Model theoretic representation of Σ^1_{2n+2}) $A \subseteq \mathbb{R}$ is Σ^1_{2n+2} iff there is a Σ_1 formula φ such that $x \in A \leftrightarrow L[T_{2n+1}, x] \models \varphi(T_{2n+1}, x)$.

2. (Mouse set, [47]) $x \in \mathbb{R} \cap L[T_{2n+2}]$ iff x is Δ^1_{2n+2} in a countable ordinal iff $x \in \mathbb{R} \cap M_{2n}^\#$.

3. (The transcendental real over $L[T_{2n+2}]$, Theorem 5.13 for $n = 1$) $M_{2n}^\# \equiv_m 0^{(2n+1)\#}$.

4. (Δ^1_{2n+3}-coding of ordinals below $u^{(2n+1)}_{E(2n+1)}$) $\mathrm{WO}^{(2n+1)}_{E(2n+1)}$ is the set of level-$(2n+1)$ sharp codes for ordinals in $u^{(2n+1)}_{E(2n+1)}$. The comparison of level-$(2n+1)$ sharp codes is Δ^1_{2n+3}.

$0^{(2n+1)\#}$ is the theory of $L[T_{2n+1}]$ with level-$(2n+1)$ indiscernibles. The structure of the level-$(2n+1)$ indiscernibles is more complicated than their order, as opposed to the order indiscernibles for L. The level-$(2n+1)$ indiscernibles form a tree structure, and the type realized in $L[T_{2n+1}]$ by finitely many of them depends only on the finite tree structure that relates them. This tree structure resembles the structure of measures (under AD) witnessing the homogeneity of S_{2n+1}, a tree on $\omega \times \boldsymbol{\delta}^1_3$ that projects to the good universal Π^1_{2n+1} set. We give a purely syntactical definition of $0^{(2n+1)\#}$ as the unique iterable, remarkable, level $\leq 2n$ correct level-$(2n+1)$ EM blueprint. This is the higher level analog of $0^\#$ as the unique wellfounded remarkable EM blueprint. The "iterability" part takes the form $\forall^{\mathbb{R}}(\Pi^1_{2n+1} \to \Pi^1_{2n+1})$, making the complexity of the whole definition Π^1_{2n+2}. The ordinal $u^{(2n+1)}_{E(2n+1)}$ is a level-$(2n+1)$ uniform indiscernible. It will be discussed in the next paragraph. When $n = 0$, $u^{(1)}_{E(1)} = u_\omega$.

The structure tied to arbitrary Π^1_{2n+1} sets are defined. By induction, we have level-$(2n-1)$ indiscernibles for $L_{\boldsymbol{\delta}^1_{2n-1}}[T_{2n-1}]$ and the real $0^{(2n-1)\#}$. Based on the EM blueprint formulation of $0^{(2n-1)\#}$, we define the level-$2n$ Martin-Solovay tree T_{2n}. It is the higher level analog of T_2. This is the most canonical tree that enables the correct generalization of the structural theory related to Π^1_{2n+1} sets. The structures in our attention are $L_{\kappa_{2n+1}}[T_{2n}]$, the least admissible set over T_{2n}, and its relativizations:

1. (Model theoretic representation of Π^1_{2n+1}, Theorem 7.4 for $n = 2$) $A \subseteq \mathbb{R}$ is Π^1_{2n+1} iff there is a Σ_1 formula φ such that $x \in A \leftrightarrow L_{\kappa^x_{2n+1}}[T_{2n}, x] \models \varphi(T_{2n}, x)$.

2. (Mouse set, [47]) $x \in \mathbb{R} \cap L_{\kappa_{2n+3}}[T_{2n+1}]$ iff x is Δ^1_{2n+1} in a countable ordinal iff $x \in M^\#_{2n-1}$.

3. (The transcendental real over $L_{\kappa^x_{2n+1}}[T_{2n}, x]$, Theorem 7.10 for $n = 2$) $M^\#_{2n-1} \equiv_m 0^{(2n)\#}$.

4. (Π^1_{2n+1}-coding of ordinals below $\boldsymbol{\delta}^1_{2n+1}$) $\mathrm{WO}^{(2n+1)}$ is the set of reals that naturally code a wellordering of $u^{(2n-1)}_{E(2n-1)}$. $\mathrm{WO}^{(2n+1)}$ is Π^1_{2n+1}.

5. The uncountable $\boldsymbol{\Delta}^1_{2n+1}$ wellordered cardinals are $(u_k : 1 \leq k \leq \omega)$, $(u^{(3)}_\xi : 1 \leq \xi \leq E(3))$, …, $(u^{(2n-1)}_\xi : 1 \leq \xi \leq E(2n-1))$, where $E(0) = 1$, $E(i+1) = \omega^{E(i)}$ via ordinal exponentiation.

The equivalence $M^\#_{2n-1} \equiv_m 0^{(2n)\#}$ is based on the equality of pointclasses (Theorem 7.7 for $n = 2$): $\Game^{2n}(<\omega^2\text{-}\Pi^1_1) = <u^{(2n-1)}_{E(2n-1)}\text{-}\Pi^1_{2n+1}$. $\{u^{(2n-1)}_\xi : 1 \leq \xi \leq E(2n-1)\}$ is the set of level-$(2n-1)$ uniform indiscernibles. It is the higher level analog of the first $\omega+1$ uniform indiscernibles $\{u_n : 1 \leq n \leq \omega\}$. Under full AD, the uncountable $\boldsymbol{\Delta}^1_{2n+1}$ wellordered cardinals enumerate all the uncountable cardinals below $\boldsymbol{\delta}^1_{2n+1}$: $u_k = \aleph_k$ for $1 \leq k < \omega$, $u^{(2i+1)}_\xi = \aleph_{E(2i-1)+\xi}$ for $1 \leq \xi \leq E(2i+1)$. Assume AD for the moment. The equation

$\boldsymbol{\delta}^1_{2n+1} = \aleph_{E(2n-1)+1}$ is originally proved by Jackson in [12, 15]. Jackson shows that every successor cardinal in the interval $[\boldsymbol{\delta}^1_{2n-1}, \aleph_{E(2n-1)})$ is the image of $\boldsymbol{\delta}^1_{2n-1}$ via an ultrapower map induced by a measure on $\boldsymbol{\delta}^1_{2n-1}$. [16] goes on to show that for a certain collection of measures μ on $\boldsymbol{\delta}^1_3$, every description leads to a canonical function representing a cardinal modulo μ. [16, 17] compute the cofinality of the cardinals below $\boldsymbol{\delta}^1_\omega$. In this paper and its sequel, we demonstrate the greater importance of the set theoretic *structures* tied to these cardinals over their *order type*. It is the inner model $L[T_{2n-1}]$ and its images via different ultrapower maps that give birth to the uncanny order type $E(2n-1)+1$. The level-$(2n-1)$ uniform indiscernibles $(u^{(2n-1)}_\xi : 1 \leq \xi \leq E(2n-1))$ are defined under this circumstance. Recall that the first ω uniform indiscernibles can be generated by $j^{\mu^n}(L_{\omega_1}) = L_{u_{n+1}}$, where μ^n is the n-fold product of the club measure on ω_1; if $1 \leq i \leq n+1$, then u_i is represented modulo μ^n by a projection map; every ordinal below u_{n+1} is in the Skolem hull of $\{x, u_1, \dots, u_n\}$ over $L[x]$ for some $x \in \mathbb{R}$. This scenario is generalized by the level-$(2n-1)$ uniform indiscernibles. As a by-product, we simplify the arguments in [12, 15–17], show in full generality that any description represents a cardinal modulo any measure on $\boldsymbol{\delta}^1_{2n-1}$, and establish the effective version of the cofinality computations.

The whole argument is inductive. Assume AD for simplicity. In the computation of $\boldsymbol{\delta}^1_{2n+1}$ in [12, 15], the strong partition property of $\boldsymbol{\delta}^1_{2n+1}$ is proved and used inductively in the process. Our argument reproves the strong partition property of $\boldsymbol{\delta}^1_{2n+1}$ using the EM blueprint formulation of $0^{(2n+1)\#}$. The definition of $0^{(2n+1)\#}$ is based on the analysis of level-$(2n+1)$ indiscernibles, whose existence depend on the homogeneous Suslin representations of $\boldsymbol{\Pi}^1_{2n}$ sets, which in turn follow from the strong partition property of $\boldsymbol{\delta}^1_{2n-1}$. Just as the main ideas of the computation of $\boldsymbol{\delta}^1_{2n+1}$ boil down to that of $\boldsymbol{\delta}^1_5$, this paper defines $0^{2\#}, 0^{3\#}, 0^{4\#}$, which contains all the key ideas in a general inductive step. The sequel to this paper will deal with the general inductive step. It will be merely a technical manifestation.

A deeper insight into the interaction between inner model theory and Jackson's computation of projective ordinals in [12, 15] is the concrete information on the direct system of countable iterates of $M^\#_{2n}$. Put $n = 1$ and assume AD for simplicity sake. Put $M^-_{2,\infty} = L_{\boldsymbol{\delta}^1_3}[T_3]$. We define $(c^{(3)}_\xi : \xi < \boldsymbol{\delta}^1_3)$, a continuous sequence in $\boldsymbol{\delta}^1_3$ that generates the set of level-3 indiscernibles for $M^-_{2,\infty}$. Each $M^-_{2,\infty}|c^{(3)}_\xi$ is the direct limit of Π^1_3-iterable mice whose Dodd-Jensen order is $c^{(3)}_\xi$. We define an alternative direct limit system indexed by ordinals in u_ω which is dense in the system leading to $M^-_{2,\infty}|c^{(3)}_\xi$. The advantage of this dense subsystem is that it leads to a good coding of $M^-_{2,\infty}|c^{(3)}_\xi$ by a subset of u_ω. The indexing ordinals are represented by wellorderings on ω_1 of order type $\omega_1 + 1$ modulo measures on ω_1 arising from the strong partition property on ω_1. Any order-preserving injection between two such wellorderings corresponds to an elementary embedding between models of this new direct limit. This injection is an isomorphism just in case its corresponding elementary embedding is essentially an iteration map, i.e., commutes with the comparison maps. The new direct system is then guided by isomorphisms between wellorderings on ω_1 of order type $\omega_1 + 1$. In this regard, the Dodd-Jensen property of mice corresponds to the simple fact that if f is an order preserving map between ordinals, then $\alpha \leq f(\alpha)$ pointwise. This observation is not surprising at all, as the Dodd-Jensen property on iterates of $0^{\#}$ *is* originated from this simple fact. This viewpoint might be a prelude to understanding the combinatorial nature of iteration trees on mice with finitely many Woodin cardinals.

A key step in computing the upper bound of $\boldsymbol{\delta}^1_5$ in [12] is the (level-3) Martin tree. For the reader familiar with the Martin tree and the purely descriptive set theoretical proof of the Kechris-Martin theorem in [13, Section 4.4], the level-1 version of the Martin tree is essentially an analysis of partially iterable sharps. The level-3 Martin tree is therefore replaced by an analysis of partially iterable level-3 sharps in this paper. The aforementioned new direct limit system indexed by ordinals in u_ω applies to any partially iterable mouse, so that its possibly illfounded direct limit is naturally coded by a subset of u_ω. This is yet another incidence that descriptive set theory and inner model theory are two sides of the same coin.

Apart from inner model theory, the pure computational component in [12, 15] has a major simplification. Under AD, a successor cardinal in the interval $[\boldsymbol{\delta}^1_3, \aleph_{\omega^{\omega^\omega}})$ is represented by a measure μ on $\boldsymbol{\delta}^1_3$ and a description. The original definition of description involves a finite iteration of ultrapowers on u_ω. The "finite iteration of ultrapowers" part is now simplified to a single ultrapower, due to Lemma 4.53.

As $L_{\kappa_{2n+1}}[T_{2n}]$ is the correct structure tied to Π^1_{2n+1} sets, it is natural to investigate its intrinsic structure. However, little is known at this very step. The closest result is on the full model $L[T_{2n}]$. The uniqueness of $L[T_{2n}]$ is proved by Hjorth [9] for $n = 1$ and Atmai [2] for general n. Here, uniqueness means that if T' is the tree of another Δ^1_{2n+1}-scale on a good universal Π^1_{2n} set, then $L[T_{2n}] = L[T']$. Atmai-Sargsyan [2] goes on to show that the full model $L[T_{2n}]$ is just $L[M^{\#}_{2n-1,\infty}]$, where $M^{\#}_{2n-1,\infty}$ is the direct limit of all the countable iterates of $M^{\#}_{2n-1}$. A test question that separates $L_{\kappa_3}[T_2]$ from $L[T_2]$ is the inner model theoretic characterization of C_3, the largest countable Π^1_3 set: if $x \in C_3$, must x be Δ^1_3-equivalent to a master code in M_2? (cf. [50, p.13]) Section 4.5 sets up a good preparation for tackling this problem.

Looking higher up, the technique in this paper and its sequel should generalize to arbitrary projective-like pointclasses in $L(\mathbb{R})$ and beyond. The descriptive set theory counterpart of larger mice should enhance our understanding of large cardinals. Typical open questions in the higher level include:

1. (cf. [1, Problem 19]) Assume AD. Let $\boldsymbol{\Gamma}$ be a $\boldsymbol{\Pi}^1_1$-like scaled pointclass (i.e., closed under $\forall^{\mathbb{R}}$, continuous preimages and non-self-dual) and Let $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \boldsymbol{\Gamma}^{\smile}$, $\boldsymbol{\delta} = \sup\{|<| : <$ is a prewellordering in $\boldsymbol{\Delta}\}$. Is $\boldsymbol{\Gamma}$ closed under unions of length $< \boldsymbol{\delta}$?

2. Assume AD. Let $\boldsymbol{\Gamma}$, $\boldsymbol{\delta}$ be as in 1. Must $\boldsymbol{\delta}$ have the strong partition property?

3. Assume AD. If $\kappa \leq \lambda$ are cardinals, must $\text{cf}(\kappa^{++}) \leq \text{cf}(\lambda^{++})$?

We now switch to some immediate applications on the theory of higher level indiscernibles. Our belief is that any result in set theory that involves sharp and Silver indiscernibles should generalize to arbitrary projective levels.

Woodin [43] proves that boldface $\boldsymbol{\Pi}^1_{2n+1}$-determinacy is equivalent to "for any real x, there is an (ω, ω_1)-iterable $M^{\#}_{2n}(x)$". The lightface scenario is tricky however. Neeman [37, 38] proves that the existence of an ω_1-iterable $M^{\#}_n$ implies boldface $\boldsymbol{\Pi}^1_n$-determinacy and lightface Π^1_{n+1}-determinacy.

Question 1.1 (cf. [4, #9])**.** Assume $\boldsymbol{\Pi}^1_n$-determinacy and Π^1_{n+1}-determinacy. Must there exist an ω_1-iterable $M^{\#}_n$?

Note that the assumption of boldface $\boldsymbol{\Pi}^1_n$-determinacy in Question 1.1 is necessary, as Δ^1_2-determinacy alone is enough to imply that there is a model of OD-determinacy (Kechris-Solovay [28]). The cases $n \in \{0, 1\}$ in Question 1.1 are solved positively by Harrington in [8] and by Woodin in [48]. The proof of the $n = 1$ case heavily relies on the theory of Silver indiscernibles for L. The theory of level-3 indiscernibles for $L_{\boldsymbol{\delta}^1_3}[T_3]$ is thus involved in proving the general case when n is odd.

Theorem 1.2. *Assume $\boldsymbol{\Pi}^1_{2n+1}$-determinacy and Π^1_{2n+2}-determinacy. Then there exists an (ω, ω_1)-iterable $M^{\#}_{2n+1}$.*

The proof of Theorem 1.2 will appear in further publications. The case $n \geq 2$ even in Question 1.1 remains open.

Another application is the δ-ordinal of intermediate pointclasses between $\boldsymbol{\Pi}^1_m$ and $\boldsymbol{\Delta}^1_{m+1}$. If Γ is a pointclass, $\delta(\Gamma)$ is the supremum of the lengths of

Γ-prewellorderings on $\mathbb{R}$. $A \subseteq \mathbb{R}$ is $\Gamma_{m,n}(z)$ iff for some formula ψ we have $x \in A \leftrightarrow M_{m-1}[x, z] \models \psi(x, z, \aleph_1, \dots, \aleph_n)$. A is $\mathbf{\Gamma}_{m,n}$ iff A is $\Gamma_{m,n}(z)$ for some real z. Hjorth [11] proves that $\delta(\mathbf{\Gamma}_{1,n}) = u_{n+2}$ under $\mathbf{\Delta}^1_2$-determinacy. Sargsyan [41] proves that under AD, $\sup_{n<\omega} \delta(\mathbf{\Gamma}_{2k+1,n})$ is the cardinal predecessor of $\boldsymbol{\delta}^1_{2k+3}$. The exact value of $\delta(\mathbf{\Gamma}_{2k+1,n})$ remains unknown. Based on the theory of higher level indiscernibles, we can define the pointclasses $\mathbf{\Lambda}_{2k+1,\xi}$ for $0 < \xi \leq E(2k+1)$. For the moment we need the notations in this paper. $A \subseteq \mathbb{R}$ is $\Lambda_{3,\xi+1}(z)$ iff for some level-3 tree R such that $[\![\emptyset]\!]_R = \widehat{\xi}$, for some $\mathcal{L}^{\underline{x},R}$-formula ψ we have $x \in A \leftrightarrow \ulcorner\psi\urcorner \in (x, z)^{3\#}(R)$. When ξ is a limit, $\Lambda_{3,\xi}(z) = \bigcup_{\eta<\xi} \Lambda_{3,\eta}(z)$. A is $\mathbf{\Lambda}_{3,\xi}$ iff A is $\Lambda_{3,\xi}(z)$ for some real z.

Theorem 1.3. *Assume $\mathbf{\Delta}^1_4$-determinacy and $0 < \xi < \omega^{\omega^\omega}$. If ξ is a successor ordinal, then $\delta(\mathbf{\Lambda}_{3,\xi}) = u^{(3)}_{\xi+1}$. If ξ is a limit ordinal, then $\delta(\mathbf{\Lambda}_{3,\xi}) = u^{(3)}_{\xi}$.*

The proof of Theorem 1.3 and its higher level analog will appear in further publications. The question on the value of $\delta(\mathbf{\Gamma}_{3,n})$ is then reduced to the relative position of $\mathbf{\Gamma}_{3,n}$ in the hierarchy $(\mathbf{\Lambda}_{3,\xi} : 0 < \xi < \omega^{\omega^\omega})$. The results of this paper combined with Neeman [37, 38] yields the following estimate:

$$\mathbf{\Lambda}_{3,\omega^{\omega^n}} \subseteq \mathbf{\Gamma}_{3,n} \subseteq \mathbf{\Lambda}_{3,\omega^{\omega^{n+1}}+1}.$$

We conjecture that $\mathbf{\Lambda}_{3,\omega^{\omega^{n+1}}} \subsetneq \mathbf{\Gamma}_{3,n} \subsetneq \mathbf{\Lambda}_{3,\omega^{\omega^{n+1}}+1}$ and $\delta(\mathbf{\Gamma}_{3,n}) = u^{(3)}_{\omega^{\omega^{n+1}}+1}$.

We try to make this paper as self-contained as possible. The reader is assumed to have some minimum background knowledge in descriptive set theory and inner model theory. On the descriptive set theory side, we assume basic knowledge of determinacy, scale and its tree representation, homogeneous tree and its ultrapower representation, and at least the results of Moschovakis periodicity theorems. We will briefly recall them in Section 2. Theorem 2.1 by Becker-Kechris [3] and Kechris-Martin [21, 23] will basically be treated as a black box. Knowing its proof would help, though not necessary. On the inner model theory side, we assume basic knowledge of mice and iteration trees in the region of finitely many Woodin cardinals, especially Theorem 6.10 in [51]. The level-wise projective complexity associated to mice will be recalled in Section 2.5. Theorem 2.18 by Steel [49] will be treated as a black box. In particular, we require absolutely no knowledge of Jackson's analysis in [12, 15].

This paper is structured as follows. Section 2 fixes notations and briefly reviews the background knowledge. Section 3 is basically a review of sharps and the Martin-Solovay tree, expressed in a form that is easy to generalize. Section 4 proves the many-one equivalence of $0^{2\#}$ and $M_1^{\#}$, generalizes Jackson's level-2 and level-3 analysis, and establishes useful properties of the coding system for ordinals in $\boldsymbol{\delta}^1_3$. Built on these results, Section 5 defines the level-3 indiscernibles for $L_{\boldsymbol{\delta}^1_3}[T_3]$, proves the many-one equivalence of $0^{3\#}$ and $M_2^{\#}$, and gives a Π^1_4-axiomatization of the real $0^{3\#}$. Section 6 defines the uniform level-3 indiscernibles and the level-4 Martin-Solovay tree. Section 7 proves the level-4 Kechris-Martin theorem and the many-one equivalence of $0^{4\#}$ and $M_3^{\#}(x)$, which prepares for the induction into the next level.

2 Backgrounds and preliminaries

Following the usual treatment in descriptive set theory, $\mathbb{R} = \omega^\omega$ is the Baire space, which is homeomorphic to the irrationals of the real line. If $A \subseteq \mathbb{R}\times X$, then $y \in \exists^{\mathbb{R}} A$ iff $\exists x \in \mathbb{R}\ (x, y) \in A$, $y \in \forall^{\mathbb{R}} A$ iff $\forall x \in \mathbb{R}\ (x, y) \in A$, $y \in \Game A$ iff Player I has a winning strategy in the game with output $A_y =_{\text{DEF}} \{x : (x, y) \in A\}$. $\Game^{n+1} A = \Game(\Game^n(A))$ when A is a subset of an appropriate product space. A pointclass is a collection of subsets of Polish spaces (typically finite products of ω and $\mathbb{R}$). If Γ is a pointclass, then $\exists^{\mathbb{R}}\Gamma = \{\exists^{\mathbb{R}} A : A \in \Gamma\}$, and similarly for $\forall^{\mathbb{R}}\Gamma$, $\Game\Gamma$, $\Game^n\Gamma$. $\boldsymbol{\Sigma}^0_1 = \boldsymbol{\Sigma}^1_0$ is the pointclass of open sets. $\Sigma^0_1 = \Sigma^1_0$ is the pointclass of effectively open sets. $\boldsymbol{\Pi}^1_{n+1} = \forall^{\mathbb{R}}\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Sigma}^1_{n+1} = \exists^{\mathbb{R}}\boldsymbol{\Pi}^1_n$, $\Pi^1_{n+1} = \forall^{\mathbb{R}}\Sigma^1_n$, $\Sigma^1_{n+1} = \exists^{\mathbb{R}}\Pi^1_n$.

If α is an ordinal and $A \subseteq \alpha \times X$, then

$$x \in \text{Diff } A \leftrightarrow \exists i < \alpha\ (\alpha \text{ is odd } \wedge \forall j < i((j, x) \in A) \wedge (i, x) \notin A).$$

If $\alpha < \omega_1^{CK}$ then $A \subseteq X$ is α-Π^1_1 iff $A = \text{Diff } B$ for some Π^1_1 $B \subseteq \alpha \times X$. A is $<\alpha$-Π^1_1 iff A is β-Π^1_1 for some $\beta < \alpha$. Martin [33] proves that Π^1_1-determinacy implies $<\omega^2$-Π^1_1-determinacy.

A tree on X is a subset of $X^{<\omega}$ closed under initial segments. If T is a tree on X, $[T]$ is the set of infinite branches of T, i.e., $x \in T$ iff $\forall n\ (x \restriction n) \in T$. If T is a tree on λ, λ is an ordinal, $[T] \neq \emptyset$, the leftmost branch is $x \in [T]$ such that for any $y \in [T]$, $(x(0), x(1), \dots)$ is lexicographically smaller than or equal to $(y(0), y(1), \dots)$. In addition, if $x \in [T]$ and for any $y \in [T]$ we have $\forall n\ x(n) \leq y(n)$, then x is the honest leftmost branch of T. A tree T on $\omega\times X$ is identified with a subset of $\omega^{<\omega}\times X^{<\omega}$ consisting of (s, t) so that $\text{lh}(s) = \text{lh}(t)$ and $((s(i), t(i)))_{i<\text{lh}(s)} \in T$. If T is a tree on $\omega \times X$, $[T] \subseteq \omega^{<\omega} \times X^{<\omega}$ is the set of infinite branches of T. $p[T] = \{x : \exists y\ (x, y) \in [T]\}$ is the projection of T. If T is a tree on $\omega \times \lambda$ and $p[T] \neq \emptyset$, then x is the leftmost real of T iff $\exists \vec{\alpha}\ (x, \vec{\alpha})$ is the leftmost branch of T.

Suppose $A \subseteq \mathbb{R}$. A norm on A is a function $\varphi : A \to \text{Ord}$. φ is regular iff $\text{ran}(\varphi)$ is an ordinal. A scale on A is a sequence of norms $\vec{\varphi} = (\varphi_n)_{n<\omega}$ on A such that if $(x_i)_{i<\omega} \subseteq A$, $x_i \to x(i \to \infty)$ in the Baire topology, and for all n, $\varphi_n(x_i) \to \lambda_n(i \to \infty)$ in the discrete topology, then $x \in A$ and $\forall n\ \varphi_n(x) \leq \lambda_n$.

$\vec{\varphi}$ is regular iff each φ_n is regular. If $A = p[T]$, T is a tree on $\omega \times \lambda$, the λ-scale associated to T is $(\varphi_n)_{n<\omega}$ where $\varphi_n(x) = \langle \alpha_x^0, \ldots, \alpha_x^n \rangle$, $(\alpha_x^n)_{n<\omega}$ is the leftmost branch of $T_x =_{\mathrm{DEF}} \{\vec{\beta} : (x, \vec{\beta}) \in [T]\}$, $\langle \ldots \rangle : \lambda^{n+1} \to \mathrm{Ord}$ is order preserving with respect to the lexicographic order and is onto an ordinal. Suppose Γ is a pointclass. If φ is a norm on A, then φ is a Γ-norm iff the relations

$$x \leq_\varphi y \leftrightarrow x \in A \wedge (y \in A \to \varphi(x) \leq \varphi(y)),$$
$$x <_\varphi y \leftrightarrow x \in A \wedge (y \in A \to \varphi(x) < \varphi(y)).$$

are both in Γ. $\vec{\varphi} = (\varphi_n)_{n<\omega}$ is a Γ-scale iff the relations $x \leq_{\varphi_n} y$ and $x <_{\varphi_n} y$ in (x, y, n) are both in Γ. Γ has the prewellordering property iff every set in Γ has a Γ-norm. Γ has the scale property iff every set in Γ has a Γ-scale. Assuming PD, Moschovakis [36] shows that the pointclasses Π^1_{2n+1}, $\boldsymbol{\Pi}^1_{2n+1}$, Σ^1_{2n+2}, $\boldsymbol{\Sigma}^1_{2n+2}$ have the scale property.

For a nonempty finite tuple $t = (a_0, \ldots, a_k)$, put $t^- = (a_0, \ldots, a_{k-1})$. This notation will be followed throughout this paper. If $<_i$ is a linear ordering on A_i for $i < \omega$, then $<_{BK}^{(<_i)_i}$ is the Brouwer-Kleene order on $\bigcup_{n<\omega}(\Pi_{i<n} A_i)$ where $(a_0, \ldots, a_n) <_{BK}^{(<_i)_i} (b_0, \ldots, b_m)$ iff either $(a_0, \ldots, a_n)$ is a proper lengthening of $(b_0, \ldots, b_m)$ or there exists $k \leq \min(m, n)$ such that $\forall i < k\ a_i = b_i \wedge a_k <_k b_k$. In our applications, these orderings $<_i$ will be apparent enough so that $(<_i)_i$ can be omitted from the superscript without confusion.

Put $\mathbb{L} = \bigcup_{x \in \mathbb{R}} L[x]$, $\mathbb{L}_\alpha = \bigcup_{x \in \mathbb{R}} L_\alpha[x]$. If A is a set, put $\mathbb{L}[A] = \bigcup_{x \in \mathbb{R}} L[A, x]$, $\mathbb{L}_\alpha[A] = \bigcup_{x \in \mathbb{R}} L_\alpha[A, x]$. $\mathbb{L}$ and $\mathbb{L}[A]$ are in general not models of ZF. Nonetheless, cardinality and cofinality in $\mathbb{L}[A]$ are well defined. So for example, $\mathrm{cf}^{\mathbb{L}[A]}(\alpha) = \min\{\mathrm{cf}^{L[A,x]}(\alpha) : x \in \mathbb{R}\}$.

If R is a wellfounded relation, $\|x\|_R$ denotes the R-rank of x, i.e., $\|x\|_R = \sup\{\|y\|_R + 1 : yRx\}$. If $<$ is a linear order, then $\mathrm{pred}_<(a)$, $\mathrm{succ}_<(a)$ denote the $<$-predecessor and $<$-successor of a respectively, if exists.

We recall the basic theory of the first $\omega + 1$ uniform indiscernibles. γ is a uniform indiscernible iff for every $x \in \mathbb{R}$, γ is an x-indiscernible. The uniform indiscernibles form a club in Ord, which are listed $u_1, u_2, \ldots$ in the increasing order. In particular, $u_1 = \omega_1$ and $u_\omega = \sup_{n<\omega} u_n$.

2.1 The Martin-Solovay tree and Q-theory

In Sections 2.1-2.3, we assume $\boldsymbol{\Delta}^1_2$-determinacy.

The set $\{x^\# : x \in \mathbb{R}\}$ is Π^1_2. $\mathrm{WO} = \mathrm{WO}_1$ is the set of codes for countable ordinals. For $1 \leq m < \omega$, WO_{m+1} is the set of $\langle \ulcorner \tau \urcorner, x^\# \rangle$ where τ is an $(m+1)$-ary Skolem term for an ordinal in the language of set theory and $x \in \mathbb{R}$. The ordinal coded by $w = \langle \ulcorner\tau\urcorner, x^{\#}\rangle \in \mathrm{WO}_{m+1}$ is

$$|w| = \tau^{L[x]}(x, u_1, \ldots, u_m).$$

Every ordinal in u_{m+1} is of the form $|w|$ for some $w \in \mathrm{WO}_{m+1}$. For each $1 \leq m < \omega$,

$$\{\tau^{L[x]}(x, u_m) : \langle \ulcorner\tau\urcorner, x^{\#}\rangle \in \mathrm{WO}_2\}$$

is a cofinal subset of u_{m+1}. $\mathrm{WO}_\omega = \bigcup_{1<m<\omega} \mathrm{WO}_m$. WO is Π^1_1, and WO_{m+1} is Π^1_2 for $1 \leq m < \omega$.

If $\mathcal{X}$ is a Polish space, $A \subseteq \mathcal{X} \times u_\omega$ and Γ is a pointclass, say that A is in Γ iff

$$A^* = \{(x, w) : x \in \mathrm{WO}_\omega \wedge (x, |w|) \in A\}$$

is in Γ. Γ acting on product spaces are similarly defined.

T_2 refers to the Martin-Solovay tree on $\omega \times u_\omega$ that projects to $\{x^{\#} : x \in \mathbb{R}\}$, giving the scale

$$\varphi_{\ulcorner\tau\urcorner}(x^{\#}) = \tau^{L[x]}(x, u_1, \ldots, u_{k_\tau}),$$

where $\ulcorner\tau\urcorner$ is the Gödel number of τ, τ is $k_\tau + 1$-ary. Details can be found in [31] or [3,23], or in Section 3.2 of this paper. T_2 is a Δ^1_3 subset of $(\omega \times u_\omega)^{<\omega}$. From T_2 one can compute a tree $\widehat{T}_2$ on $\omega \times u_\omega$ that projects to a good universal $\boldsymbol{\Pi}^1_2$ set. The definition of $\widehat{T}_2$ will be recalled in Section 3.2.

For $x \in \mathbb{R}$, $L_{\kappa^x_3}[T_2, x]$ is the minimum admissible set containing (T_2, x). The fact that the $\widehat{T}_2$ projects to a good universal Π^1_2 set implies for every Π^1_3 set of reals A, there is a Σ_1-formula φ such that $x \in A$ iff $L_{\kappa^x_3}[T_2, x] \models \varphi(T_2, x)$; φ can be effectively computed from the definition of A. Becker-Kechris in [3] strengthens this fact by allowing a parameter in u_ω. The converse direction is shown by Kechris-Martin in [21, 23]. The back-and-forth conversion is concluded in [3].

Theorem 2.1 (Becker-Kechris, Kechris-Martin)**.** *Assume $\boldsymbol{\Delta}^1_2$-determinacy. Then for each $A \subseteq u_\omega \times \mathbb{R}$, the following are equivalent.*

1. *A is Π^1_3.*
2. *There is a Σ_1 formula φ such that $(\alpha, x) \in A$ iff $L_{\kappa^x_3}[T_2, x] \models \varphi(T_2, \alpha, x)$.*

The conversion between the Π^1_3 definition of A and the Σ_1-formula φ are effective. The original proof of $2 \Rightarrow 1$ in Theorem 2.1 is based on Theorem 2.2 and Corollary 2.3.

Theorem 2.2 (Kechris-Martin, [21,23])**.** *Assume $\boldsymbol{\Delta}^1_2$-determinacy. Let $x \in \mathbb{R}$. If A is a nonempty $\Pi^1_3(x)$ subset of u_ω, then $\exists w \in \Delta^1_3(x) \cap WO_\omega(|w| \in A)$.*

Corollary 2.3 (Kechris-Martin, [21, 23]). *Assume* $\boldsymbol{\Delta}^1_2$*-determinacy. Then* Π^1_3 *is closed under quantifications over* u_ω*, i.e., if* $A \subseteq (u_\omega)^2 \times \mathbb{R}$ *is* Π^1_3*, then so are*

$$B = \{(\alpha, x) : \exists \beta < u_\omega \ (\beta, \alpha, x) \in A\},$$
$$C = \{(\alpha, x) : \forall \beta < u_\omega \ (\beta, \alpha, x) \in A\}.$$

Suppose $\mathcal{X}$ is a Polish space. For $x \in \mathbb{R}$ and $\alpha < u_\omega$, $A \subseteq \mathcal{X}$ is $\Sigma^1_3(x, \alpha)$ iff there is a $\Sigma^1_3(x)$ set $B \subseteq u_\omega \times \mathcal{X}$ such that $y \in A$ iff $(\alpha, y) \in B$. Or equivalently, A is $\Sigma^1_3(x, \alpha)$ iff there is a $\Sigma^1_3(x)$ set $B \subseteq \mathbb{R} \times \mathcal{X}$ such that $y \in A$ iff $\exists w \in \mathrm{WO}_\omega$ $(|w| = \alpha \wedge (w, \alpha) \in B)$. A is $\Pi^1_3(x, \alpha)$ iff $\mathcal{X} \setminus A$ is $\Sigma^1_3(x, \alpha)$. A is $\Delta^1_3(x, \alpha)$ iff A is both $\Sigma^1_3(x, \alpha)$ and $\Pi^1_3(x, \alpha)$. $\Sigma^1_3(x, {<}\beta)$ means $\Sigma^1_3(x, \alpha)$ for some $\alpha < \beta$. Similarly define $\Pi^1_3(x, {<}\beta)$ and $\Delta^1_3(x, {<}\beta)$.

In the proof of Theorem 2.1, the prewellordering property for Π^1_3 subsets of $\omega \times u_\omega$, originally proved by Solovay, is used.

Theorem 2.4 (Solovay, [24, Theorem 3.1]). *Assume* $\boldsymbol{\Delta}^1_2$*-determinacy. Suppose* $A \subseteq u_\omega \times \mathbb{R}$ *is* $\Pi^1_3(x, \alpha)$*, where* $x \in \mathbb{R}$*,* $\alpha < u_\omega$*. Then there is a* $\Pi^1_3(x, \alpha)$ *norm* $\varphi : A \to \mathrm{Ord}$*, i.e., the relations*

$$(\beta, y) \leq^*_\varphi (\gamma, z) \leftrightarrow (\beta, y) \in A \wedge ((\gamma, z) \in A \to \varphi(\beta, y) \leq \varphi(\gamma, z))$$
$$(\beta, y) <^*_\varphi (\gamma, z) \leftrightarrow (\beta, y) \in A \wedge ((\gamma, z) \in A \to \varphi(\beta, y) < \varphi(\gamma, z))$$

are $\Pi^1_3(x, \alpha)$*.*

Corollary 2.5 (Reduction). *Assume* $\boldsymbol{\Delta}^1_2$*-determinacy. Suppose* $A, B \subseteq u_\omega \times \mathbb{R}$ *are both* $\Pi^1_3(x, \alpha)$*, where* $x \in \mathbb{R}$*,* $\alpha < u_\omega$*. Then there exist* $\Pi^1_3(x, \alpha)$ *sets* $A', B' \subseteq u_\omega \times \mathbb{R}$ *such that* $A' \subseteq A$*,* $B' \subseteq B$*,* $A \cup B = A' \cup B'$ *and* $A' \cap B' = \emptyset$*.*

Corollary 2.6 (Easy uniformization). *Assume* $\boldsymbol{\Delta}^1_2$*-determinacy. Suppose* $A \subseteq (u_\omega \times \mathbb{R}) \times u_\omega$ *is* $\Pi^1_3(x, \alpha)$*, where* $x \in \mathbb{R}$*,* $\alpha < u_\omega$*. Then* A *can be uniformized by a* $\Pi^1_3(x, \alpha)$ *function, i.e., there is a* $\Pi^1_3(x, \alpha)$ *function* f *such that* $\mathrm{dom}(f) = \{(\beta, y) : \exists \gamma \ ((\beta, y), \gamma) \in A\}$ *and that* $((\beta, y), f(\beta, y)) \in A$ *for all* $(\beta, y) \in \mathrm{dom}(f)$*.*

The Π^1_3 coding system for Δ^1_3 sets (e.g., [7, Theorem 3.3.1]) applies to the larger pointclass $\Delta^1_3({<}u_\omega)$. The proof is similar.

Corollary 2.7 (Π^1_3-codes for $\Delta^1_3({<}u_\omega)$). *Assume* $\boldsymbol{\Delta}^1_2$*-determinacy. Then there is a* Π^1_3 *set* $C \subseteq u_\omega$ *and sets* $P, S \subseteq u_\omega \times \mathbb{R}$ *in* Π^1_3, Σ^1_3 *respectively such that for any* $\alpha \in C$*,*

$$P_\alpha = S_\alpha =_{DEF} D_\alpha$$

and

$$\{D_\alpha : \alpha \in C\} = \{A \subseteq \mathbb{R} : A \text{ is } \Delta^1_3({<}u_\omega)\}.$$

Proof. Let $U \subseteq \omega \times \mathbb{R}^2$ be a good universal Π^1_3 set. Define

$$((n,\alpha),(m,\beta),x) \in A \leftrightarrow \forall w \in \mathrm{WO}_\omega\ (|w| = \alpha \to (n,w,x) \in U)$$
$$((n,\alpha),(m,\beta),x) \in B \leftrightarrow \forall w \in \mathrm{WO}_\omega\ (|w| = \beta \to (m,w,x) \in U)$$

Then A, B are Π^1_3 subsets of $(\omega \times u_\omega)^2$. Reduce them to A', B' according to Corollary 2.5. Define

$$((n,\alpha),(m,\beta)) \in C \leftrightarrow (A')_{(n,\alpha),(m,\beta)} \cup (B')_{(n,\alpha),(m,\beta)} = \mathbb{R}$$

C is a Π^1_3 subset of $(\omega \times u_\omega)^2$. Let $P = A'$, $S = (\omega \times u_\omega)^2 \times \mathbb{R} \setminus B'$. Identifying $(\omega \times u_\omega)^2$ with u_ω with the Gödel pairing function, C, P, S are as desired. $\square$

Theorem 2.1 provides a model-theoretic view of Q-theory [27] at the level of Q_3-degrees. We give an exposition of these results, probably with simple strengthenings thereof.

The higher level analog of the hyperarithmetic reducibility on reals is Q_3 reducibility. Q_3-degrees are coarser than Δ^1_3-degrees. $y \in Q_3(x)$ iff y is $\Delta^1_3(x)$ in a countable ordinal, i.e., there is $\alpha < \omega_1$ such that $\forall w \in \mathrm{WO}(|w| = \alpha \to y \in \Delta^1_3(x)$. y is $\Delta^1_3(x)$ in an ordinal $< u_\omega$ iff there is $\alpha < u_\omega$ such that $\forall w \in \mathrm{WO}_\omega(|w| = \alpha \to y \in \Delta^1_3(x))$. $y \leq_{\Delta^1_3} x$ iff $y \in \Delta^1_3(x)$. $y \equiv_{\Delta^1_3} x$ iff $y \leq_{\Delta^1_3} x \leq_{\Delta^1_3} y$. $y \leq_{Q_3} x$ iff $y \in Q_3(x)$. $y \equiv_{Q_3} x$ iff $y \leq_{Q_3} x \leq_{Q_3} y$.

Proposition 2.8 ([20, 21, 23, 27, 47])**.** *1. Let $x, y \in \mathbb{R}$. Then $y \in L_{\kappa^x_3}[T_2, x]$ iff $y \in M^\#_1(x)$ iff y is $\Delta^1_3(x)$ in a countable ordinal iff y is $\Delta^1_3(x)$ in an ordinal $< u_\omega$.*

2. The relation $y \in L_{\kappa^x_3}[T_2, x]$ is Π^1_3, where x, y ranges over $\mathbb{R}$.

3. The relation $y \in \Delta^1_3(x)$ is Π^1_3, where x, y ranges over $\mathbb{R}$.

κ^x_3 is the higher level analog of ω^x_1, the least x-admissible. It is defined in a different way in [27, Section 14]. As in [23, 27], we define

$$\begin{aligned}\lambda^x_3 &= \sup\{|W| : W \text{ is a } \Delta^1_3(x) \text{ prewellordering on } \mathbb{R}\}\\ &= \sup\{\xi < \kappa^x_3 : \xi \text{ is } \Delta_1\text{-definable over } L_{\kappa^x_3}[T_2, x] \text{ from } \{T_2, x\}\}.\end{aligned}$$

The equivalence of these two definitions of κ^x_3 is proved in [23]:

$$\begin{aligned}\kappa^x_3 &= \sup\{\mathrm{o.t.}(W) : W \text{ is a } \Delta^1_3(x, {<}u_\omega) \text{ wellordering on } \mathbb{R}\}\\ &= \sup\{\lambda^{x,y}_3 : M^\#_1(x) \not\leq_{\Delta^1_3} (x, y)\}.\end{aligned}$$

Moreover,

$$\forall \alpha < u_\omega\ \exists w \in \mathrm{WO}_\omega\ (|w| = \alpha \wedge \lambda^{x,w}_3 < \kappa^x_3).$$

Note that $\kappa_3^x < \lambda_3^{M_1^\#(x)} < \boldsymbol{\delta}^1_3$, as proved in [27, Lemma 14.2].

The Kunen-Martin theorem implies that κ_3^x is a bound on the rank of any $\Sigma^1_3(x, {<}u_\omega)$ wellfounded relation.

Theorem 2.9 (Kunen-Martin, [36, 2G.2])**.** *Suppose W is a wellfounded relation on $\mathbb{R}$. Suppose γ is an ordinal and T is a tree on $(\omega\times\omega)\times\gamma$ such that $W = p[T]$. Let $L_\kappa[T]$ be the least admissible set containing T as an element. Then the rank of W is smaller than*

$$\sup\{\xi < \kappa : \xi \text{ is } \Delta_1\text{-definable over } L_\kappa[T] \text{ from } \{T\}\}.$$

Corollary 2.10. *Suppose W is a $\Sigma^1_3(x, {<}u_\omega)$ wellfounded relation on $\mathbb{R}$. Then the rank of W is smaller than κ_3^x.*

2.2 A Δ^1_3 coding of subsets of u_ω in $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$

As a corollary to Theorem 2.1, every subset of u_ω in $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$ is $\boldsymbol{\Delta}^1_3$. The proof of Theorem 2.1 gives a better definability estimate of $\mathcal{P}(u_\omega)\cap\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$.

For $x \in \mathbb{R}$, A putative x-sharp is a remarkable EM blueprint over x. Suppose x^* is a putative x-sharp. For any ordinal α, $\mathcal{M}_{x^*,\alpha}$ is the EM model built from x^* and indiscernibles of order type α. The wellfounded part of $\mathcal{M}_{x^*,\alpha}$ is transitive. For any limit ordinal $\alpha < \beta$, $\mathcal{M}_{x^*,\alpha}$ is a rank initial segment of $\mathcal{M}_{x^*,\beta}$. Say that x^* is α-wellfounded iff $\alpha \in \mathrm{wfp}(\mathcal{M}_{x^*,\alpha})$. A *putative sharp code for an increasing function* is $w = \langle \ulcorner\tau\urcorner, x^*\rangle$ such that x^* is a putative x-sharp, τ is a $\{\underline{\in}, \underline{x}\}$-unary Skolem term for an ordinal and

$$\text{“}\forall v, v'((v, v' \in \mathrm{Ord} \wedge v < v') \to (\tau(v) \in \mathrm{Ord} \wedge \tau(v) < \tau(v')))\text{”}$$

is a true formula in x^*. The statement "$\langle\ulcorner\tau\urcorner, x^*\rangle$ is a putative sharp code for an increasing function, x^* is α-wellfounded, r codes the order type of $\tau^{\mathcal{M}_{x^*,\alpha}}(\alpha)$" about $(\langle\ulcorner\tau\urcorner, x^*\rangle, r)$ is $\boldsymbol{\Sigma}^1_1$ in the code of α. In addition, when $x^* = x^\#$, $\langle\ulcorner\tau\urcorner, x^*\rangle$ is called a *(true) sharp code for an increasing function.*

A subset $A \subseteq u_n$ is coded by $\mathrm{Code}_n(A) = \{w \in \mathrm{WO}_n : |w| \in A\}$.

Lemma 2.11. *Assume $\boldsymbol{\Delta}^1_2$-determinacy. Suppose $n < \omega$, $A \subseteq u_n$ and $A \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. Then* $\mathrm{Code}_n(A)$ *is in* $\Game(\omega(n+1)\text{-}\boldsymbol{\Pi}^1_1)$.

Proof. By Kechris-Woodin [29], $\Game({<}\omega^2\text{-}\Pi^1_1)$ sets are determined. We prove by induction on n the following claim:

> Suppose $A \subseteq u_n$. Suppose $B, C \subseteq \mathbb{R}$ are $\boldsymbol{\Pi}^1_2$ subsets of $\mathbb{R}^2$ such that $(w \in \mathrm{WO}_n \wedge |w| \in A)$ iff $\exists z((w, z) \in B)$ iff $\neg\exists z((w, z) \in C)$. Then there is $x \in \mathbb{R}$ such that
>
> $$\forall v \in \mathrm{WO}_n\, \exists w \in \mathrm{WO}_n \cap L[v, x]\, (|v| = |w| \wedge L[v, x] \models \exists z((w, z) \in B \cup C)).$$

By Shoenfield absoluteness, this claim gives a uniform definition of the relation $|v| \in A$ over $L[v,x]$ from parameters in $\{u_1, \ldots, u_n\}$. In the definition, the parameters $u_1, \ldots, u_n$ are used to decide whether or not $|v| = |w|$ for $v, w \in \mathrm{WO}_n$. Combined with the fact from Theorem 2.1 that every subset of u_ω in $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$ is $\boldsymbol{\Delta}^1_3$, the lemma will follow from our claim.

We start the induction with $n = 1$. Consider the game $G(B, C, 0)$, where I produces v, II produces (w, y). II wins iff either $v \notin \mathrm{WO}$ or

$$\begin{aligned} &v, w \in \mathrm{WO} \wedge |v| < |w| \wedge \\ &\qquad \forall \alpha < |v| \ \exists (\bar{w}, z) \leq_T y \ (\bar{w} \in \mathrm{WO} \wedge |\bar{w}| = \alpha \wedge (\bar{w}, z) \in B \cup C). \end{aligned}$$

This game is $\boldsymbol{\Pi}^1_2$ for Player II, hence determined. I does not have a winning strategy by $\boldsymbol{\Sigma}^1_1$-boundedness. So II has a winning strategy g. g plays the role of x in the claim, verifying the $n = 1$ case.

Suppose the claim holds for n and we want to prove for $n+1$. Consider the game $G(B, C, n+1)$, where I produces $\langle \ulcorner \tau \urcorner, a^* \rangle$, II produces $(\langle \ulcorner \sigma \urcorner, b^* \rangle, y)$. II wins iff

1. If $\langle \ulcorner \tau \urcorner, a^* \rangle$ is a putative sharp code for an increasing function, then so is $\langle \ulcorner \sigma \urcorner, b^* \rangle$. Moreover, for any $\eta < \omega_1$, if
$$a^* \text{ is } \eta\text{-wellfounded } \wedge \tau^{\mathcal{M}_{a^*,\eta}}(\eta) \in \mathrm{wfp}(\mathcal{M}_{a^*,\eta})$$
then
$$\begin{aligned} &b^* \text{ is } \eta\text{-wellfounded } \wedge \sigma^{\mathcal{M}_{b^*,\eta}}(\eta) \in \mathrm{wfp}(\mathcal{M}_{b^*,\eta}) \\ &\qquad \wedge \tau^{\mathcal{M}_{a^*,\eta}}(\eta) < \sigma^{\mathcal{M}_{b^*,\eta}}(\eta). \end{aligned}$$
2. If $\langle \ulcorner \tau \urcorner, a^* \rangle$ is a true sharp code for an increasing function, $a^* = a^{\#}$, then
$$\begin{aligned} &\forall v \in \mathrm{WO}_{n+1}(|v| < \tau^{L[a]}(u_n) \to \exists (\bar{w}, z) \in L[v, y] \\ &\qquad (\bar{w} \in \mathrm{WO}_{n+1} \wedge |\bar{w}| = |v| \wedge (\bar{w}, z) \in B \cup C)). \end{aligned}$$

This game is $\Game(\omega(n+1)\text{-}\boldsymbol{\Pi}^1_1)$, hence determined. If Player I has a winning strategy f, then for each η, let X_η be the set of $r \in \mathbb{R}$ such that there are putative sharp codes for increasing functions on ordinals $\langle \ulcorner \tau \urcorner, a^* \rangle, \langle \ulcorner \sigma \urcorner, b^* \rangle$ and an ordinal $\beta \leq \eta$ such that

1. $\langle \ulcorner \tau \urcorner, a^* \rangle = f * \langle \ulcorner \sigma \urcorner, b^* \rangle$;

2. for any $\bar\beta < \beta$, b^* is $\bar\beta$-wellfounded, $\sigma^{\mathcal{M}_{b^*,\eta}}(\bar\beta) \in \text{wfp}(\mathcal{M}_{b^*,\eta})$, $\sigma^{\mathcal{M}_{b^*,\eta}}(\bar\beta) \leq \eta$;

3. a^* is β-wellfounded, $\tau^{\mathcal{M}_{a^*,\eta}}(\beta)$ has order type coded in r.

X_η is a $\boldsymbol{\Sigma}^1_1$ set in the code of η. Since f is a winning strategy for I, $X_\eta \subseteq$ WO. Let C be the set of countable f-admissibles and their limits. By $\boldsymbol{\Sigma}^1_1$-boundedness, if $\langle \ulcorner\sigma\urcorner, b^\#\rangle$ is a true sharp code for an increasing function, such that $\forall\beta < \omega_1\ \sigma^{L[b]}(\beta) \in C$, then $\langle \ulcorner\tau\urcorner, a^\#\rangle =_{\text{DEF}} f * \langle \ulcorner\sigma\urcorner, b^\#\rangle$ is a true sharp code for an increasing function, and for any $\eta \in C$ such that $\forall\beta < \eta\ \sigma^{L[b]}(\beta) < \eta$, $\tau^{L[a]}(\eta) < \min(C \setminus \eta + 1)$, and in particular, $\tau^{L[a]}(u_n) <$the least f-admissible above u_n. Let ξ be the least f-admissible above u_n. In $L[f]$, there is a bijection $\pi : u_n \to \xi$, definable from $\{u_n\}$. $A \cap \beta$ is thus identified with $(\pi^{-1})''(A \cap \beta)$, as a subset of u_n. π induces a $\Delta^1_3(f)$ map π_* such that for any $v \in \text{WO}_n$, $\pi_*(v) \in \text{WO}_{n+1}$ and $|\pi_*(v)| = \pi(|v|)$. Let $\pi(v) = w$ iff $\exists z(v, w, z) \in D$, where D is $\Pi^1_2(f)$.

Let $(v, z) \in B'$ iff $v \in \text{WO}_n$, $(v, (z)_0, (z)_1) \in D$, and $((z)_0, (z)_2) \in B$. Similarly define C'. Then $|v| \in (\pi^{-1})''(A \cap \beta)$ iff $\exists z\ (v, z) \in B'$ iff $\neg\exists z\ (v, z) \in C'$. B', C' are $\boldsymbol{\Pi}^1_2$. By induction hypothesis, there is a real x^* such that

$$\forall v \in \text{WO}_n \exists w \in \text{WO}_n \cap L[v, x^*](|v| = |w| \wedge L[v, x^*] \models \exists z((w, z) \in B' \cup C'))$$

In $G(B, C, n+1)$, II defeats f by playing $((\ulcorner\sigma^*\urcorner, f^\#), f \oplus x^*)$, where $(\sigma^*)^{L[f]}(\beta)$ is the β-th f-admissible. This is a contradiction.

Thus, II has a winning strategy g in $G(B, C, n+1)$. g plays the role of x in the claim, verifying the inductive case. □

As a corollary to Lemma 2.11, we obtain a Δ^1_3 coding of subsets of u_ω that lie in $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. The Δ^1_3 coding was first established by Kunen under AD in a less effective way in [46].

Corollary 2.12. *Assume $\boldsymbol{\Delta}^1_2$-determinacy. There is Δ^1_3 set $X \subseteq \mathbb{R} \times u_\omega$ such that $\{X_v : v \in \mathbb{R}\} = \mathcal{P}(u_\omega) \cap \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. Here $X_v = \{\alpha < u_\omega : (x, \alpha) \in X\}$.*

Proof. If $v = \langle k, \ulcorner\varphi\urcorner, z\rangle$, $k < \omega$, $z \in \mathbb{R}$, (φ, z) defines a $\Pi^1_1(z)$ subset A_φ of $\omega k \times \mathbb{R}^2$, put $X_{v;k} = \Game(\text{Diff}\ A_\varphi)$. Put $X_v = \bigcup_{k<\omega} X_{(v)_k;k}$. $X = \{(v, \beta) : \beta \in X_v\}$. X is clearly Δ^1_3. The map $v \mapsto X_v$ is onto $\mathcal{P}(u_\omega) \cap \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$ by Lemma 2.11. □

As a corollary, assuming $\boldsymbol{\Delta}^1_2$-determinacy, if $A \subseteq \omega_1$, then $A \in \mathbb{L}$ iff $A \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$.

2.3 Silver's dichotomy on $\mathbf{\Pi}^1_3$ equivalence relations

Harrington's proof [22], [18, Chapter 32] of Silver's dichotomy [44] on $\mathbf{\Pi}^1_1$ equivalence relations generalizes to $\mathbf{\Pi}^1_3$ in a straightforward fashion. This folklore generalization is stated in [9, 10] in a slightly weaker form.

An equivalence relation E on $\mathbb{R}$ is thin iff there is no perfect set P such that $\forall x, y \in P\ (xEy \to x = y)$. If Γ is a pointclass, for equivalence relations E, F (possibly on different spaces of the form $\mathbb{R}^m \times (u_\omega)^n$), E is Γ-reducible to F iff there is a function π in Γ such that $xEy \leftrightarrow \pi(x)F\pi(y)$.

Theorem 2.13 (Folklore)**.** *Assume $\mathbf{\Delta}^1_2$-determinacy. Let $x \in \mathbb{R}$. If E is a thin $\Pi^1_3(x)$ equivalence relation on $\mathbb{R}$, then E is $\Delta^1_3(x)$ reducible to a $\Pi^1_3(x)$ equivalence relation on a $\Pi^1_3(x)$ subset of u_ω.*

Proof. For simplicity, let $x = 0$. The generalization of Harrington's proof of Silver's dichotomy shows that for every $y \in \mathbb{R}$, there is a $\Delta^1_3(<u_\omega)$ set A such that $y \in A \subseteq [y]_E$.

Let $C, P, S, (D_\alpha)_{\alpha \in C}$ be the Π^1_3 coding system for $\Delta^1_3(<u_\omega)$ subsets of $\mathbb{R}$, given by Corollary 2.7. Let $\alpha \in C'$ iff $\alpha \in C$ and $\forall y \in D_\alpha \forall z \in D_\alpha (yEz)$. C' is Π^1_3. The set

$$A = \{(y, \alpha) : \alpha \in C' \wedge y \in D_\alpha\}$$

is Π^1_3. By Corollary 2.6, A can be uniformized by a Π^1_3 function π. Let $\alpha F \beta$ iff $\alpha \in C'$, $\beta \in C'$, and $\forall y \in D_\alpha \forall z \in D_\beta (yEz)$. F is a Π^1_3 equivalence relation on C'. π is a reduction from E to F. To see that π is also Σ^1_3, apply Corollary 2.3 and use the fact that π is a total function taking values in u_ω. □

The reduction π and the target equivalence relation F in Theorem 2.13 are uniformly definable from the $\Pi^1_3(x)$ definition of E, independent of x. A similar uniformity applies to the following corollary.

Corollary 2.14. *Assume $\mathbf{\Delta}^1_2$-determinacy. Let $x \in \mathbb{R}$. If E is a thin $\Delta^1_3(x)$ equivalence relation on $\mathbb{R}$, then E is $\Delta^1_3(x)$ reducible to $=_{u_\omega}$. Here $\alpha =_{u_\omega} \beta$ iff $\alpha = \beta < u_\omega$.*

Proof. Assume $x = 0$. Proceed as in the proof of Theorem 2.13 until we reach the set A. We now show that A can be uniformized by a Π^1_3 function π such that yEz iff $\pi(y) = \pi(z)$. Indeed, let φ be a Π^1_3-norm on A, given by Theorem 2.4, and let $\pi(y) = \alpha$ iff $(y, \alpha) \in A$ and $(\varphi(y, \alpha), \alpha)$ is lexicographically minimal among the set $\{(\varphi(z, \beta), \beta) : zEy \wedge (z, \beta) \in A\}$. Similarly to the proof of Corollary 2.6, π is Π^1_3 (we use $E \in \Delta^1_3$ here). Again, π is Σ^1_3. π is the desired reduction from E to $=_{u_\omega}$. □

It should be possible to give an alternative proof of Corollary 2.14 using the forceless proof of the dichotomy of chromatic numbers of graphs in [35], but the author has not checked the details.

Corollary 2.15. *Assume $\mathbf{\Delta}^1_2$-determinacy. Let $x \in \mathbb{R}$. If $\leq^*$ is a $\Delta^1_3(x)$ prewellordering on $\mathbb{R}$ and A is a $\Sigma^1_3(x)$ subset of $\mathbb{R}$, then $|\leq^*|$ and $\{\|x\|_{\leq^*} : x \in A\}$ are both in $L_{\kappa_3^{M_1^\#(x)}}[T_2, M_1^\#(x)]$ and Δ_1-definable over $L_{\kappa_3^{M_1^\#(x)}}[T_2, M_1^\#(x)]$ from parameters in $\{T_2, M_1^\#(x)\}$.*

Proof. The equivalence relation $a \equiv^* b \leftrightarrow a \leq^* b \leq^* a$ is thin. By Corollary 2.14, we get a $\Delta^1_3(x)$-function $\pi : \mathbb{R} \to u_\omega$ such that $a \equiv^* b$ iff $\pi(a) = \pi(b)$. π induces a wellordering $<^{**}$ on $\operatorname{ran}(\pi)$ where $\pi(a) <^{**} \pi(b)$ iff $a <^* b$. $|\leq^*|$ is then the order type of $<^{**}$. $\operatorname{ran}(\pi)$ and $<^{**}$ are Σ^1_3, hence Π_1-definable over $L_{\kappa_3^x}[T_2, x]$ from $\{T_2, x\}$ by Theorem 2.1. Put $w = M_1^\#(x)$. By [27, Lemma 14.2], $\kappa_3^x < \kappa_3^w$. So $\operatorname{ran}(\pi)$ and $<^{**}$ are Δ_1-definable over $L_{\kappa_3^w}[T_2, w]$ from $\{T_2, w\}$. By admissibility, $|\leq^*|$ is Δ_1-definable in $L_{\kappa_3^w}[T_2, w]$ from $\{T_2, w\}$. The part concerning $\{\|x\|_{\leq^*} : x \in A\}$ is similar. □

Remark 2.16. We do not know if $M_1^\#(x)$ can be replaced by x in the conclusion of Corollary 2.15.

2.4 N-homogeneous trees

As this paper deals with restricted ultrapowers and "restricted homogeneous trees" over and over again, it is convenient to abstract the relevant properties.

A transitive set or class N is *admissibly closed* iff

$$\forall M \in N \exists M' \in N (M' \text{ is admissible } \wedge M \in M')$$

Suppose N is admissibly closed and $X \in N$. ν is an *N-filter* on X iff there is a filter ν^* on X such that $\nu = \nu^* \cap N$. An N-filter ν is an *N-measure* on X iff ν is countably complete and for any $A \in \mathcal{P}(X) \cap N$, either $A \in \nu$ or $X \setminus A \in \nu$. If ν is an N-measure on X, then $\operatorname{Ult}(N, \nu)$ is the ultrapower consisting of equivalence classes of functions $f : X \to N$ that lie in N. Denote by $j^\nu_N : N \to \operatorname{Ult}(N, \nu)$ the ultrapower map and $[f]^\nu_N$ the ν-equivalence class of f in $\operatorname{Ult}(N, \nu)$. The ultrapower is well-defined by admissible closedness of N, and is wellfounded by countable completeness of ν. The usual Łoś proof shows for any transitive $M \in N$ containing $\{X\}$, for any first order formula φ, for any $f_i : X \to M$ that belongs to N, $1 \leq i \leq n$,

$$j^\nu_N(M) \models \varphi([f_1]^\nu_N, \ldots, [f_n]^\nu_N)$$

iff

$$\text{for } \nu\text{-a.e. } a \in X,\ M \models \varphi(f_1(a), \dots, f_n(a)).$$

Suppose ν is an N-measure on X^n and μ is an N-measure on X^m, $m \leq n$. ν *projects to* μ iff for all $A \subseteq X^m$, $A \in \mu$ iff $\{\vec{\alpha} : \vec{\alpha} \restriction m \in A\} \in \nu$. $\vec{\nu} = (\nu_n)_{n<\omega}$ is a *tower of N-measures* on X iff for each n, ν_n is an N-measure on X^n and ν_n projects to ν_m for all $m < n$.

Suppose N is admissibly closed, $X \in N$, and $\vec{\nu} = (\nu_n)_{n<\omega}$ is a tower of N-measures on X. This naturally induces factor maps $j_N^{\nu_m,\nu_n}$ from $\mathrm{Ult}(N, \nu_m)$ to $\mathrm{Ult}(N, \nu_n)$. We say $\vec{\nu}$ is *close to N* iff whenever $(A_n)_{n<\omega}$ is a sequence such that $A_n \in \nu_n \cap N$ for all n, there exists $(B_n)_{n<\omega} \in N$ such that $B_n \subseteq A_n$ and $B_n \in \nu_n$ for all n. If $\vec{\nu}$ is close to N, we say $\vec{\nu}$ is *N-countably complete* iff whenever $(A_n)_{n<\omega}$ is a sequence such that $A_n \in \nu_n \cap N$ for all n, there exists $(a_n)_{n<\omega}$ such that $(a_1, \dots, a_n) \in A_n$ for all n. The usual homogeneous tree argument shows:

Proposition 2.17. *Suppose $\vec{\nu} = (\nu_n)_{n<\omega}$ is close to N. Then $\vec{\nu}$ is N-countably complete iff the direct limit of $(j_N^{\nu_m,\nu_n})_{m<n<\omega}$ is wellfounded.*

Proof. The new part is to show N-countable completeness of $\vec{\nu}$ from wellfoundedness of the direct limit of $(j_N^{\nu_m,\nu_n})_{m<n<\omega}$. Given $(A_n)_{n<\omega}$ such that $A_n \in \nu_n \cap N$ for all n, suppose towards contradiction that there does not exist $(a_n)_{n<\omega}$ such that $(a_1, \dots, a_n) \in A_n$ for all n. By closedness of $\vec{\nu}$ to N, let $(B_n)_{n<\omega} \in N$ such that $B_n \subseteq A_n$ and $B_n \in \nu_n$ for all n. The tree T consisting of $(a_1, \dots, a_n)$ such that $a_i \in B_i$ for all i is wellfounded. The ranking function f of T belongs to N by admissible closedness. From f we can construct $f_n : X^n \to N$ so that $f_n \in N$ and $[f_n]_{\nu_n} > [f_{n+1}]_{\nu_{n+1}}$ as usual, contradicting to wellfoundedness of $(j_N^{\nu_m,\nu_n})_{m<n<\omega}$. $\square$

An N-homogeneous system is a sequence $(\nu_s)_{s\in\omega^{<\omega}}$ such that for any $x \in \mathbb{R}$, $\nu_x =_{DEF} (\nu_{x\restriction n})_{n<\omega}$ is a tower of N-measures which is close to N. For $X \in N$, a tree T on $\omega \times X$ is N-homogeneous iff there is an N-homogeneous system $(\nu_s)_{s\in\omega^{<\omega}}$ such that $T_s \in \nu_s$ for all $s \in \omega^{<\omega}$ and for all $x \in p[T]$, ν_x is N-countably complete. If T is N-homogeneous, by Proposition 2.17 and standard arguments, $x \in p[T]$ iff the direct limit of $(j_N^{\nu_{x\restriction m},\nu_{x\restriction n}})_{m<n<\omega}$ is wellfounded.

2.5 $L[T_3]$ as a mouse

The notations concerning inner model theory follow [51]. If $\mathcal{M}$ is a premouse, $o(\mathcal{M})$ denotes $\mathrm{Ord} \cap \mathcal{M}$. In Steel [47], the level-wise projective complexity associated to mice is discussed in detail. In this paper, we find it more convenient to work with Π^1_{n+1}-iterability rather than Π^{HC}_n-iterability in [47].

A countable normal iteration tree $\mathcal{T}$ on a countable premouse is Π^1_1-guided iff for any limit $\lambda \leq \mathrm{lh}(\mathcal{T})$, there is $\xi \leq o(\mathcal{M}^{\mathcal{T}}_\lambda)$ such that $\mathcal{M}^{\mathcal{T}}_\lambda|\xi = J_\xi[\mathcal{M}(\mathcal{T}\restriction\alpha)]$ and $J_{\xi+1}[\mathcal{M}(\mathcal{T}\restriction\alpha)] \models$"$\delta(\mathcal{T}\restriction\alpha)$ is not Woodin". A countable stack of countable normal iteration trees $\vec{\mathcal{T}}$ is Π^1_1-guided iff every normal component of $\vec{\mathcal{T}}$ is Π^1_1-guided.

$x \in \mathbb{R}$ codes a Π^1_2-iterable mouse iff x codes a 1-small premouse $\mathcal{P}_x$ such that for any Π^1_1-guided normal iteration trees $\vec{\mathcal{T}} \in HC$ on $\mathcal{P}_x$, either $\mathcal{T}$ has a last wellfounded model or $\mathrm{lh}(\mathcal{T})$ is a limit ordinal and for any $\xi \geq o(\mathcal{M}(\mathcal{T}))$, if $J_\xi[\mathcal{M}(\mathcal{T})] \models$"$\delta(\mathcal{T}\restriction\alpha)$ is Woodin", then there is a cofinal branch b through $\mathcal{T}$ such that either $J_\xi[\mathcal{M}(\mathcal{T})] \trianglelefteq \mathcal{M}^{\mathcal{T}}_b$ or $\mathcal{M}^{\mathcal{T}}_b \trianglelefteq J_\xi[\mathcal{M}(\mathcal{T})]$.

Π^1_2-iterability is enough to compare countable 1-small premice that project to ω. A countable normal iteration tree $\mathcal{T}$ on a countable premouse is Π^1_2-guided iff for any limit $\lambda \leq \mathrm{lh}(\mathcal{T})$, there is $\xi \leq o(\mathcal{M}^{\mathcal{T}}_\lambda)$ such that $\mathcal{M}^{\mathcal{T}}_\lambda|\xi$ is Π^1_2-iterable above $\delta(\mathcal{T}|\lambda)$ and $\mathrm{rud}(\mathcal{M}(\mathcal{T}\restriction\alpha)) \models$"$\delta(\mathcal{T}\restriction\alpha)$ is not Woodin". A countable stack of countable normal iteration trees $\vec{\mathcal{T}}$ is Π^1_2-guided iff every normal component of $\vec{\mathcal{T}}$ is Π^1_2-guided.

Assume $\mathbf{\Delta}^1_2$-determinacy. $x \in \mathbb{R}$ codes a Π^1_3-iterable mouse iff x codes a countable 2-small premouse $\mathcal{P}_x$ such that for any $v \in \mathbb{R}$ coding Π^1_2-guided stack of normal iteration trees $\vec{\mathcal{T}} = (\mathcal{T}_i)_{i<\alpha}$ on $\mathcal{P}_x$, either

1. $\mathcal{M}^{\vec{\mathcal{T}}}_\infty$ exists, either as the last model of $\mathcal{T}_{\alpha-1}$ when α is a successor or as the direct limit of $(\mathcal{M}^{\mathcal{T}}_i : i < \alpha)$ when α is a limit, and there is $\mathcal{Q} \triangleright \mathcal{M}^{\vec{\mathcal{T}}}_\infty$ such that $\mathcal{Q} \in M^\#_1(x,v)$, $\mathcal{Q}$ is Π^1_2-iterable above $o(\mathcal{M}^{\vec{\mathcal{T}}}_\infty)$, $\mathrm{rud}(\mathcal{Q}) \models$"there is no Woodin cardinal $\leq o(\mathcal{M}^{\vec{\mathcal{T}}}_\infty)$", or

2. α is a successor cardinal and there is $b \in M^\#_1(x,v)$ such that b is a maximal branch through $\mathcal{T}_{\alpha-1}$, and there is $\mathcal{Q} \triangleright \mathcal{M}^{\mathcal{T}_{\alpha-1}}_b$ such that $\mathcal{Q} \in M^\#_1(x,v)$, $\mathcal{Q}$ is Π^1_2-iterable above $o(\mathcal{M}^{\mathcal{T}_{\alpha-1}}_b)$, $\mathrm{rud}(\mathcal{Q}) \models$"there is no Woodin cardinal $\leq o(\mathcal{M}^{\mathcal{T}_{\alpha-1}}_b)$".

Π^1_3-iterability is a Π^1_3 property by Spector-Gandy. "countable" and "2-small" are usually omitted from prefixing "Π^1_3-iterable mouse". $\mathcal{P}$ is a Π^1_3-iterable mouse iff there is $x \in \mathbb{R}$ that codes a Π^1_3-iterable mouse $\mathcal{P} = \mathcal{P}_x$. Note that Π^1_3-iterable mice are genuinely (ω_1,ω_1)-iterable. $\leq_{DJ}$ is the Dodd-Jensen prewellordering on Π^1_3-iterable mice. $\mathcal{M} \leq_{DJ} \mathcal{N}$ iff $\mathcal{M},\mathcal{N}$ are Π^1_3-iterable mice and in the comparison between $\mathcal{M}$ and $\mathcal{N}$, the main branch on the $\mathcal{M}$-side does not drop. $\mathcal{M} \sim_{DJ} \mathcal{N}$ iff $\mathcal{M} \leq_{DJ} \mathcal{N} \leq_{DJ} \mathcal{M}$. $\mathcal{M} <_{DJ} \mathcal{N}$ iff $\mathcal{M} \leq_{DJ} \mathcal{N} \not\leq_{DJ} \mathcal{M}$. The norm $x \mapsto \|\mathcal{P}_x\|_{<_{DJ}}$ for x coding a Π^1_3-iterable mouse $\mathcal{P}_x$ is Π^1_3. For instance, ($\mathcal{P}_x$ is a Π^1_3-iterable mouse $\wedge$($\mathcal{P}_y$ is a Π^1_3-iterable mouse $\to \mathcal{P}_x \leq_{DJ} \mathcal{P}_y$)) iff $\mathcal{P}_x$ is a Π^1_3-iterable mouse and for any Π^1_2-guided normal iteration trees $\mathcal{T},\mathcal{U}$ on $\mathcal{P}_x, \mathcal{P}_y$ respectively, if $\mathcal{T},\mathcal{U}$ have the common last model $\mathcal{Q}$ and the main branch of $\mathcal{T}$ drops, then the main branch of $\mathcal{U}$ also drops.

If $\mathcal{N}$ is a Π^1_3-iterable mouse, then $\mathcal{I}_{\mathcal{N}}$ is the direct system consisting of countable nondropping iterates of $\mathcal{N}$, and $\mathcal{N}_\infty$ is the direct limit of $\mathcal{I}_{\mathcal{N}}$, $\pi_{\mathcal{N},\infty} : \mathcal{N} \to \mathcal{N}_\infty$ is the direct limit map. $o(\mathcal{N}_\infty) < \boldsymbol{\delta}^1_3$ as it is the length of a $\boldsymbol{\Delta}^1_3$-prewellordering.

For a real z, all the iterability notions relativize to z-mice. $<_{DJ(z)}$ is the Dodd-Jensen prewellordering on Π^1_3-iterable z-mice.

Assume $\boldsymbol{\Pi}^1_3$-determinacy. $\mathcal{F}_{2,z}$ is the direct system consisting of countable iterates of $M_2^{\#}(z)$. $M_{2,\infty}^{\#}(z)$ is the direct limit of $\mathcal{F}_{2,z}$. $M^-_{2,\infty}(z) = M^{\#}_{2,\infty}(z)|\boldsymbol{\delta}^1_3$. $(\mathcal{F}_2, \mathcal{M}^{\#}_{2,\infty}, \mathcal{M}^-_{2,\infty}) = (\mathcal{F}_{2,0}, \mathcal{M}^{\#}_{2,\infty}(0), \mathcal{M}^-_{2,\infty}(0))$.

Theorem 2.18 (Steel [49])**.** *Assume $\boldsymbol{\Pi}^1_3$-determinacy. Then for any real z,*

1. *$\boldsymbol{\delta}^1_3$ is the least $< \delta^z_{2,\infty}$-strong cardinal of $M^{\#}_{2,\infty}(z)$, where $\delta^z_{2,\infty}$ is the least Woodin cardinal of $M^{\#}_{2,\infty}(z)$.*

2. *$M^-_{2,\infty}(z) = L[T_3, z]$.*

3 The level-1 sharp

The level-1 sharp is the usual sharp, originally published in [45]. We present the usual arguments of Martin's proof of $\boldsymbol{\Pi}^1_1$-determinacy and the Martin-Solovay tree on a $\boldsymbol{\Pi}^1_2$-complete set in a form that conveniently generalizes to higher levels.

3.1 The tree S_1, level-1 description analysis

We are working under ZF + DC.

The technical definition of *tree of uniform cofinalities* is extracted from [26], defined in [14], and redefined in our paper in a more convenient way. A tree of uniform cofinality pinpoints a particular measure that appears in a homogeneity system for a projective set. A *level-1 tree of uniform cofinalities*, or a *level-1 tree*, is a set $P \subseteq \omega^{<\omega}$ such that:

1. $\emptyset \notin A$.

2. If $(i_1, \ldots, i_{k+1}) \in T$, $k \geq 1$, then $(i_1, \ldots, i_k) \in T$ and for every $j < i_{k+1}$, $(i_1, \ldots, i_k, j) \in T$.

Any countable linear ordering is isomorphic to $<_{BK}\upharpoonright P$ for some level-1 tree P. If P, P' are finite level-1 trees, $s \notin P$, $P' = P \cup \{s\}$, then the $<_{BK}\upharpoonright P'$-predecessor of s^- is s. Level-1 trees are just convenient representations of countable linear orderings and their extensions.

A level-1 tree P is said to be *regular* iff $(1) \notin P$. In other words, when P is regular and $P \neq \emptyset$, (0) must be the $<_{BK}$-maximal node of P.

The *ordinal representation* of P is

$$\mathrm{rep}(P) = \{(p) : p \in P\} \cup \{(p, n) : p \in P, n < \omega\}.$$

$\mathrm{rep}(P)$ is endowed with the ordering

$$<^P = <_{BK}\upharpoonright \mathrm{rep}(P).$$

Thus, for $p \in P$, (p) is the $<^P$-supremum of (p, n) for $n < \omega$. If $B \subseteq \omega_1$ is in $\mathbb{L}$, let $B^{P\uparrow}$ the set of functions $f : \mathrm{rep}(P) \to B$ which are continuous, order preserving (with respect to $<^P$ and $<$) and belong to $\mathbb{L}$. If $f \in \omega_1^{P\uparrow}$, let

$$[f]^P = ([f]^P_p)_{p \in P},$$

where $[f]^P_p = f((p))$ for $p \in P$. Let $[B]^{P\uparrow} = \{[f]^P : f \in B^{P\uparrow}\}$. P is said to be Π^1_1-*wellfounded* iff $P \cup \{\emptyset\}$ is a wellfounded tree, or equivalently, $<^P$ is a wellordering. Π^1_1-wellfoundedness of a level-1 tree is a $\boldsymbol{\Pi}^1_1$ property in the real coding the tree. A tuple $\vec{\alpha} = (\alpha_p)_{p \in P}$ is said to *respect* P iff $\vec{\alpha} \in [\omega_1]^{P\uparrow}$. In other words, each α_p is a countable limit ordinal, and the map $p \mapsto \alpha_p$ is an isomorphism between $(P; <_{BK}\upharpoonright P)$ and $(\{\alpha_p : p \in P\}; <)$. In particular, when P is regular, $P \neq \emptyset$ and $\vec{\alpha}$ respects P, then $\alpha_{(0)} > \alpha_p$ whenever $p \in P \setminus \{(0)\}$.

A *finite level-1 tower* is a tuple $(P_i)_{i \leq n}$ such that $n < \omega$, P_i is a level-1 tree of cardinality i for any i, and $i < j \to P_i \subseteq P_j$. An *infinite level-1 tower* is $(P_i)_{i<\omega}$ such that $(P_i)_{i \leq n}$ is a finite level-1 tower for any $n < \omega$. A *level-1 system* is a sequence $\vec{P} = (P_s)_{s \in \omega^{<\omega}}$ such that for each $s \in \omega^{<\omega}$, $(P_{s\upharpoonright i})_{i < \mathrm{lh}(s)}$ is a finite level-1 tower. $\vec{P}$ is *regular* iff each P_s is regular. Associated to a $\boldsymbol{\Pi}^1_1$ set A we can assign a regular level-1 system $(P_s)_{s \in \omega^{<\omega}}$ so that $x \in A$ iff the infinite regular level-1 tree $P_x =_{\mathrm{DEF}} \cup_{n<\omega} P_{x\upharpoonright n}$ is Π^1_1-wellfounded. If A is lightface Π^1_1, then $(P_s)_{s \in \omega^{<\omega}}$ can be picked effective.

Definition 3.1. S_1 is the tree on $V_\omega \times \omega_1$ such that $(\emptyset, \emptyset) \in S_1$ and a nonempty node

$$(\vec{P}, \vec{\alpha}) = ((P_i)_{i \leq n}, (\alpha_i)_{i \leq n}) \in S_1$$

iff $(P_i)_{i \leq n}$ is a finite regular level-1 tower and putting $p_i \in P_{i+1} \setminus P_i$, $\beta_{p_i} = \alpha_i$, then $(\beta_p)_{p \in P_n}$ respects P_n.

Since every tree occurring in S_1 is regular, for a nonempty node $(\vec{P}, \vec{\alpha}) \in S_1$, we must have $\alpha_0 > \max(\alpha_1, \ldots, \alpha_n)$.

S_1 projects to the universal Π^1_1 set:

$$p[S_1] = \{\vec{P} : \vec{P} \text{ is a } \Pi^1_1\text{-wellfounded regular level-1 tower}\}.$$

The (non-regular) ω_1-scale associated to S_1 is Π^1_1.

Definition 3.2.

1. Suppose P is a level-1 tree. The set of *P-descriptions* is $\mathrm{desc}(P) =_{\mathrm{DEF}} P \cup \{\emptyset\}$. The *constant P-description* is $\emptyset$.
2. $p \prec p'$ iff $p, p' \in \mathrm{desc}(P)$ and $p <_{BK} p'$.
3. Suppose P, W are level-1 trees. A function $\sigma : P \cup \{\emptyset\} \to W \cup \{\emptyset\}$ is said to *factor* (P, W) iff $\sigma(\emptyset) = \emptyset$ and σ preserves the $<_{BK}$-order. (σ does not necessarily preserve the tree order.)
4. Suppose P is a level-1 tree. σ factors $(P, *)$ iff σ factors (P, W) for some level-1 tree W.

Suppose P, W are Π^1_1-wellfounded. Then $\mathrm{o.t.}(<^P) \leq \mathrm{o.t.}(<^W)$ is equivalent to "$\exists \sigma$ (σ factors (P, W))". $\mathrm{o.t.}(<^P) < \mathrm{o.t.}(<^W)$ is equivalent to "$\exists \sigma \exists w \in W$ (σ factors $(P, W) \wedge \forall p \in P\ \sigma(p) \prec^W w$)". The higher level analog of this simple fact will be established in Section 4.9, which will be an ingredient in the axiomatization of $0^{3\#}$ in Section 5.

3.2 Homogeneity properties of S_1

From now on, we assume $\mathbf{\Pi}^1_1$-determinacy. This is equivalent to $\forall x \in \mathbb{R}(x^\#$ exists) by Martin [32] and Harrington [8].

The first ω uniform indiscernibles $(u_n)_{n<\omega}$ can be generated by restricted ultrapowers of $\mathbb{L}$. Recall that $\mathbb{L} = \bigcup_{x \in \mathbb{R}} L[x]$, which is admissibly closed. Then for every subset $A \subseteq \omega_1$ in $\mathbb{L}$, there is a real x such that A is Σ_1-definable over $(L_{\omega_1}[x]; \in, x)$. Let

$$\mu_{\mathbb{L}}$$

be the $\mathbb{L}$-club measure on ω_1, i.e., $A \in \mu_{\mathbb{L}}$ iff $A \in \mathbb{L}$ and $\exists C \in \mathbb{L}$ ($C \subseteq A \wedge C$ is a club in ω_1). When P is a finite level-1 tree, μ^P is the $\mathbb{L}$-measure on $\mathrm{card}(P)$-tuples in ω_1 given by: $A \in \mu^P$ iff there is $C \in \mu_{\mathbb{L}}$ such that $[C]^{P\uparrow} \subseteq A$. So μ^P is essentially a variant of the $\mathrm{card}(P)$-fold product of $\mu_{\mathbb{L}}$, concentrating on tuples whose ordinals are ordered according to the $<_{BK}$-order of P. In particular, $\mu^\emptyset$ is the principal ultrafilter concentrating on

$\{\emptyset\}$. Put $j^P = j^{\mu^P}_{\mathbb{L}}$, $[f]_{\mu^P} = [f]^{\mu^P}_{\mathbb{L}}$ for $f \in \mathbb{L}$. Standard arguments show that $\text{Ult}(\mathbb{L}, \mu^P) = \mathbb{L}$, and $j^P(\omega_1) = u_{\text{card}(P)+1}$. For any real x, $j^P \restriction L[x]$ is elementary from $L[x]$ to $L[x]$.

The set of uncountable $\mathbb{L}$-regular cardinals below u_ω is $\{u_n : 1 \leq n < \omega\}$. The relation "$\beta = \text{cf}^{\mathbb{L}}(\alpha)$" is Δ^1_3 (in the sharp codes). Suppose P is a finite level-1 tree, $p \in \text{desc}(P)$. Then

$$\text{seed}^P_p \in \mathbb{L}$$

is the element represented modulo μ^P by the projection map sending $\vec{\alpha} = (\alpha_{p'})_{p' \in P}$ to α_p if $p \in P$, by the constant function with value ω_1 if $p = \emptyset$. We have $\text{seed}^P_p = u_{\|p\|_{\prec^P}+1}$, where $\|p\|_{\prec^P}$ is the $\prec^P$-rank of p. In particular, $\text{seed}^P_\emptyset = u_{\text{card}(P)+1} = j^P(\omega_1)$. For each $p \in P$, μ^P projects to $\mu_{\mathbb{L}}$ via the map $\vec{\alpha} \mapsto \alpha_p$.

$$p^P : \mathbb{L} \to \mathbb{L}$$

is the induced factoring map that sends $j_{\mu_{\mathbb{L}}}(h)(\omega_1)$ to $j^P(h)(\text{seed}^P_p)$. Thus, p^P is the unique map such that for any $z \in \mathbb{R}$, p^P is elementary from $L[z]$ to $L[z]$ and $p^P \circ j_{\mu_{\mathbb{L}}} = j^P$, $p^P(\omega_1) = \text{seed}^P_p$. If p is the $\prec^P$-predecessor of p', then $(p^P)''u_2$ is a cofinal subset of $\text{seed}^P_{p'}$. Put

$$\text{seed}^P = (\text{seed}^P_p)_{p \in \text{desc}(P)},$$

So $p \prec^P p'$ iff $\text{seed}^P_p < \text{seed}^P_{p'}$. Every element in $\mathbb{L}$ is expressible in the form $j^P(h)(\text{seed}^P)$ for some $h \in \mathbb{L}$.

If P, P' are finite level-1 trees, P is a subtree of P', then $\mu^{P'}$ projects to μ^P in the language of Section 2.4, i.e., the identity map factors (P, P'). Let

$$j^{P,P'} = j^{\mu^P, \mu^{P'}}_{\mathbb{L}} : \mathbb{L} \to \mathbb{L}$$

be the factor map given by Section 2.4. Thus, for any real x,

$$j^{P,P'} \restriction L[x] : L[x] \to L[x]$$

is elementary and

$$j^{P,P'}(\tau^{L[x]}(\text{seed}^P_{p_1}, \ldots, \text{seed}^P_{p_n})) = \tau^{L[x]}(\text{seed}^{P'}_{p_1}, \ldots, \text{seed}^{P'}_{p_n})$$

for $p_1, \ldots, p_n \in P$. If $(P_n)_{n<\omega}$ is an infinite level-1 tower, the associated measure tower $(\mu^{P_n})_{n<\omega}$ is easily seen close to $\mathbb{L}$.

The proof of $\mathbf{\Pi}^1_1$-determinacy [32] shows:

Theorem 3.3 (Martin). *Assume $\mathbf{\Pi}^1_1$-determinacy. Let $(P_n)_{n<\omega}$ be an infinite level-1 tower. The following are equivalent.*

1. $(P_n)_{n<\omega}$ *is* Π^1_1*-wellfounded.*

2. $[\omega_1]^{\bigcup\{P_n : n<\omega\}\uparrow} \neq \emptyset$.

3. $(\mu^{P_n})_{n<\omega}$ *is* $\mathbb{L}$*-countably complete.*

4. *The direct limit of* $(j^{P_m,P_n})_{m<n<\omega}$ *is wellfounded.*

If $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n'\}$ is order preserving, let

$$j^\sigma : \mathbb{L} \to \mathbb{L}$$

where $j^\sigma(\tau^{L[x]}(u_1, \ldots, u_n)) = \tau^{L[x]}(u_{\sigma(1)}, \ldots, u_{\sigma(n)})$. Let

$$j^\sigma_{\text{sup}} : u_{n+1} \to u_{n'+1}$$

where $j^\sigma_{\text{sup}}(\beta) = \sup(j^\sigma)''\beta$. So j^σ is continuous at β iff $j^\sigma(\beta) = j^\sigma_{\text{sup}}(\beta)$. The continuity points of j^σ are characterized by their $\mathbb{L}$-cofinalities:

Lemma 3.4. *Suppose* $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n'\}$ *is order preserving,* $\beta < u_{n+1}$. *Put* $\sigma(0) = 0$. *Then* $j^\sigma(\beta) \neq j^\sigma_{\text{sup}}(\beta)$ *iff for some* k, $\text{cf}^{\mathbb{L}}(\beta) = u_k$ *and* $\sigma(k) > \sigma(k-1) + 1$. *If* $\text{cf}^{\mathbb{L}}(\beta) = u_k$ *and* $\sigma(k) > \sigma(k-1) + 1$, *then* $j^\sigma_{\text{sup}}(\beta) = j^{\sigma_k} \circ j^{\tau_k}_{\text{sup}}(\beta)$, *where* $\sigma = \sigma_k \circ \tau_k$, $\sigma_k(i) = \sigma(i)$ *for* $1 \leq i < k$, $\sigma_k(k) = \sigma(k-1) + 1$, $\sigma_k(i) = \sigma(i-1)$ *for* $k < i \leq n+1$.

The second half of this lemma states that j^σ_{sup} acting on points of $\mathbb{L}$-cofinality u_k is factored into the "continuous part" j^{σ_k} and the "discontinuous part" $j^{\tau_k}_{\text{sup}}$. This simple fact about factoring j^σ_{sup} is essentially part of effectivized Kunen's analysis on u_ω in [46].

A *partial level* ≤ 1 *tree* is a pair (P, t) such that P is a finite regular level-1 tree, and either

1. $t \notin P \wedge P \cup \{t\}$ is a regular level-1 tree, or
2. $P \neq \emptyset$, $t = -1$.

-1 is regarded as the "level-0" component, hence the name "level ≤ 1". (P, t) is of degree 0 if $t = -1$, of degree 1 otherwise. We put $\text{dom}(P, t) = P \cup \{t\}$. The *uniform cofinality* of (P, t) is

$$\text{ucf}(P, t) = \begin{cases} -1 & \text{if } t = -1, \\ t^- & \text{if } t \neq -1. \end{cases}$$

$\vec{\alpha} = (\alpha_s)_{s \in P \cup \{t\}}$ *respects* (P, t) iff $\vec{\alpha} \restriction P$ respects P and $t = -1 \to \alpha_t < \omega$, $t \neq -1 \to \alpha_t < \alpha_{t^-}$. The *cardinality* of (P, t) is $\text{card}(P, t) = \text{card}(P) + 1$.

The unique partial level ≤ 1 tree of cardinality 1 is $(\emptyset, (0))$. If (P,t) is of degree 1, its *completion* is $P \cup \{t\}$. $(P,-1)$ has no completion. (P,t) is a *partial subtree* of P' iff the completion of (P,t) exists and is a subtree of P'.

A *partial level* ≤ 1 *tower of discontinuous type* is a nonempty finite sequence $(\vec{P},\vec{p}) = (P_i,p_i)_{i\leq k}$ such that $\mathrm{card}(P_0,p_0)=1$, each (P_i,p_i) is a partial level ≤ 1 tree, and P_{i+1} is the completion of (P_i,p_i). Its *signature* is $(p_i)_{i<k}$. Its *uniform cofinality* is $\mathrm{ucf}(P_k,p_k)$. A *partial level* ≤ 1 *tower of continuous type* is $(P_i,p_i)_{i<k}{}^\frown(P_*)$ such that either $k=0 \wedge P_*=\emptyset$ or $(P_i,p_i)_{i<k}$ is a partial level ≤ 1 tower of discontinuous type $\wedge P_*$ is the completion of (P_{k-1},p_{k-1}). Its *signature* is $(p_i)_{i<k}$. When $k>0$, its *uniform cofinality* is p_{k-1}. For notational convenience, the information of a partial level ≤ 1 tower is compressed into a potential partial level ≤ 1 tower. We say a *potential partial level* ≤ 1 *tower* is $(P_*,\vec{p}) = (P_*,(p_i)_{i<\mathrm{lh}(\vec{p})})$ such that for some level-1 tower $\vec{P}=(P_i)_{i\leq k}$, either $P_*=P_k \wedge (\vec{P},\vec{p})$ is a partial level ≤ 1 tower of discontinuous type or $(\vec{P},\vec{p}){}^\frown(P_*)$ is a partial level ≤ 1 tower of continuous type. The signature, (dis-)continuity type, uniform cofinality of $(P_*,\vec{p})$ are defined according to the partial level ≤ 1 tree generating $(P_*,\vec{p})$.

$$\mathrm{ucf}(P_*,\vec{p})$$

denotes the uniform cofinality of $(P_*,\vec{p})$. If $(P_*,(p_i)_{i\leq k})$ is a potential partial level ≤ 1 tower of discontinuous type, its *completion* is the completion of (P,p_k).

Clearly, a potential partial level ≤ 1 tower $(P_*,\vec{p})$ is of continuous type iff $\mathrm{card}(P_*)=\mathrm{lh}(\vec{p})$, of discontinuous type iff $\mathrm{card}(P_*)=\mathrm{lh}(\vec{p})-1$.

Suppose P,W are level-1 trees, σ factors (P,W). Given a tuple $\vec{\alpha} = (\alpha_w)_{w\in W} \in [\omega_1]^{W\uparrow}$, define

$$\vec{\alpha}_\sigma = (\alpha_{\sigma,p})_{p\in P} \in [\omega_1]^{P\uparrow}$$

where $\alpha_{\sigma,p}=\alpha_{\sigma(p)}$. If W is finite, put $\mathrm{seed}^W_\sigma = (\mathrm{seed}^W_{\sigma(p)})_{p\in P}$, i.e., seed^W_σ is represented modulo μ^W by the function $\vec{\alpha}\mapsto\vec{\alpha}_\sigma$.

If P,W are finite, σ factors (P,W), then for any $A\in\mu^P$, for μ^W-a.e. $\vec{\alpha}$, $\vec{\alpha}_\sigma\in A$. Thus, for any $A\in\mu^P$, $\mathrm{seed}^W_\sigma \in j^W(A)$. Thus, we can unambiguously define

$$\sigma^W : \mathbb{L}\to\mathbb{L}$$

by $\sigma^W(j^P(h)(\mathrm{seed}^P)) = j^W(h)(\mathrm{seed}^W_\sigma)$. σ^W is the unique map such that for any $z\in\mathbb{R}$, σ^W is elementary from $L[z]$ to $L[z]$, $\sigma^W\circ j^P = j^W$, and for any $p\in P$, $\sigma^W\circ p^P = \sigma(p)^W$. Define $\sigma^W_{\sup}(\beta) = \sup(\sigma^W)''\beta$.

The next two lemmas compute the "effective uniform cofinality" of the image of certain ordinals under level-1 tree factoring maps. They will be useful in the level-2 description analysis in Section 4.4.

Lemma 3.5. *Suppose (P^-, p) is a partial level ≤ 1 tree whose completion is P. σ, σ' both factor (P, W). σ and σ' agree on P^-, $\sigma'(p)$ is the $\prec^W$-predecessor of $\sigma(p)$. Then for any $\beta < j^{P^-}(\omega_1)$ such that $\mathrm{cf}^{\mathbb{L}}(\beta) = \mathrm{seed}^{P^-}_{p^-}$,*

$$\sigma^W \circ j^{P^-,P}_{\sup}(\beta) = (\sigma')^W_{\sup} \circ j^{P^-,P}(\beta).$$

Proof. Note that $\mathrm{cf}^{\mathbb{L}}(j^{P^-,P}(\beta)) = \mathrm{seed}^P_{p^-}$. As in Lemma 3.4, $(\sigma')^W_{\sup}$ acting on points of $\mathbb{L}$-cofinality $\mathrm{seed}^P_{p^-}$ is decomposed into the discontinuous part $j^{P,P^+}_{\sup}$ and the continuous part $(\sigma^+)^W$, where P^+ is the completion of the partial level ≤ 1 tree (P, p^+), $(p^+)^- = p^-$, σ^+ factors (P^+, W), σ' and σ^+ agree on P, $\sigma^+(p^+) = \sigma(p)$. Let ι factor (P, P^+) where $\iota \restriction P^- = \mathrm{id}$, $\iota(p) = p^+$. So $\sigma^+ \circ \iota = \sigma$. By considering the $\mathrm{seed}^{P^-}_{p^-}$-cofinal sequence in β, it is not hard to show that $j^{P,P^+}_{\sup} \circ j^{P^-,P}(\beta) = \iota^{P^+} \circ j^{P^-,P}_{\sup}(\beta)$. Hence,

$$\begin{aligned}(\sigma')^W_{\sup} \circ j^{P^-,P}(\beta) &= (\sigma^+)^W \circ j^{P,P^+}_{\sup} \circ j^{P^-,P}(\beta) \\ &= (\sigma^+)^W \circ \iota^{P^+} \circ j^{P^-,P}_{\sup}(\beta) \\ &= \sigma^W \circ j^{P^-,P}_{\sup}(\beta).\end{aligned}$$

□

Lemma 3.6. *Suppose (P, p) is a partial level ≤ 1 tree, σ factors (P, W). Suppose $\beta < j^{P^-}(\omega_1)$ and either*

1. *$p = -1$, $P^+ = P$, $\sigma' = \sigma$, $\mathrm{cf}^{\mathbb{L}}(\beta) = \omega$, or*
2. *$p \neq -1$, P^+ is the completion of (P, p), σ' factors (P^+, W), $\sigma = \sigma' \restriction P$, $\sigma'(p)$ is the $\prec^W$-predecessor of $\sigma(p^-)$, $\mathrm{cf}^{\mathbb{L}}(\beta) = \mathrm{seed}^P_{p^-}$.*

Then

$$\sigma^W(\beta) = (\sigma')^W_{\sup} \circ j^{P,P^+}(\beta).$$

Proof. By commutativity of factoring maps, $\sigma^W(\beta) = (\sigma')^W \circ j^{P,P^+}(\beta)$. Note that $\mathrm{cf}^{\mathbb{L}}(j^{P,P^+}(\beta)) = \mathrm{seed}^{P^+}_{p^-}$ when $p \neq -1$, $\mathrm{cf}^{\mathbb{L}}(j^{P,P^+}(\beta)) = \omega$ when $p = -1$. In either case, by Lemma 3.4, $(\sigma')^W$ is continuous at $j^{P,P^+}(\beta)$. □

To conclude this section, we define the Martin-Solovay tree T_2 projecting to $\{x^\# : x \in \mathbb{R}\}$ and its variant $\widehat{T}_2$ projecting to a good universal Π^1_2 set. This formulation of T_2 and $\widehat{T}_2$ will generalize to the higher level in Section 6.3. Let $T \subseteq 2^{<\omega}$ be a recursive tree such that $[T]$ is the set of remarkable EM blueprints over some real. Here we have fixed in advance an effective Gödel coding of first order formulas in the language $\{\underline{\in}, \underline{x}, \underline{c_n} : n < \omega\}$, so that an infinite string $x \in 2^\omega$ represents the theory $\{\varphi : x_{\ulcorner \varphi \urcorner} = 0\}$. Fix an effective list of Skolem terms $(\tau_k)_{k<\omega}$ in the language of set theory, where τ_k is $f(k)+1$-ary, f is effective. T_2 is defined as a tree on $2 \times u_\omega$ where

$$(s, (\alpha_0, \dots, \alpha_{n-1})) \in T_2$$

iff $s \in T$, $\text{lh}(s) = n$, and for any $k, l < n$, for any order preserving $\sigma : \{1, \dots, f(k)\} \to \{1, \dots, f(l)\}$,

1. if "$\tau_k(\underline{x}, \underline{c_{\sigma(1)}}, \dots, \underline{c_{\sigma(f(k))}}) = \tau_l(\underline{x}, \underline{c_1}, \dots, \underline{c_{f(l)}})$" is true in s, then $j^\sigma(\alpha_k) = \alpha_l$;

2. if "$\tau_k(\underline{x}, \underline{c_{\sigma(1)}}, \dots, \underline{c_{\sigma(f(k))}}) < \tau_l(\underline{x}, \underline{c_1}, \dots, \underline{c_{f(l)}})$" is true in s, then $j^\sigma(\alpha_k) < \alpha_l$;

3. if "$\tau_k(\underline{x}, \underline{c_{\sigma(1)}}, \dots, \underline{c_{\sigma(f(k))}}) > \tau_l(\underline{x}, \underline{c_1}, \dots, \underline{c_{f(l)}})$" is true in s, then $j^\sigma(\alpha_k) > \alpha_l$.

In essence, the second coordinate of T_2 attempts to verify the wellfoundedness of the EM blueprint coded in the first coordinate. From T_2 we compute $\widehat{T}_2$, a tree on $\omega \times (\omega \times u_\omega)$ that projects to a good universal Π^1_2 set. By Shoenfield absoluteness, if $\varphi(v)$ is a Π^1_2 formula, effectively from $\ulcorner\varphi\urcorner$ we can compute a unary Skolem term $\tau_{\ulcorner\varphi\urcorner}$ such that $\tau^{L[x]}_{\ulcorner\varphi\urcorner}(x) = 0$ iff $\varphi(x)$ holds. Define $(\ulcorner\varphi\urcorner^\frown(v), (s, \vec{\alpha})) \in \widehat{T}_2$ iff $(s, \vec{\alpha}) \in T_2$ and

1. if "$\underline{x}(m) = n$" is true in s, then $v(m) = n$;

2. "$\tau_{\ulcorner\varphi\urcorner}(\underline{x}) \neq 0$" is not true in s.

So $p[\widehat{T}_2] = \{\ulcorner\varphi\urcorner^\frown(x) : \varphi(x)\}$.

3.3 The tree S_2

In this section, we redefine the tree S_2 introduced in [26, Section 2] in the language of trees of uniform cofinalities in [14].

A *tree of level-1 trees* is a tree T on $\omega^{<\omega}$ (i.e., $T \subseteq (\omega^{<\omega})^{<\omega}$ and closed under $\subseteq$) and such that for any $s \in T$, $\{a \in \omega^{<\omega} : s^\frown(a) \in T\}$ is a level-1 tree.

A *level-2 tree of uniform cofinalities*, or *level-2 tree*, is a function Q such that $\text{dom}(Q)$ is a tree of level-1 trees, $\emptyset \in \text{dom}(Q)$ and for any $q \in \text{dom}(Q)$, $(Q(q \restriction l))_{l \leq \text{lh}(q)}$ is a partial level ≤ 1 tower of discontinuous type. In particular, $Q(\emptyset) = (\emptyset, (0))$.

We denote $Q(q) = (Q_{\text{tree}}(q), Q_{\text{node}}(q))$ and $Q[q] = (Q_{\text{tree}}(q), (Q_{\text{node}}(q \restriction l))_{l \leq \text{lh}(q)})$. So $Q[q]$ is a potential partial level ≤ 1 tower of discontinuous type. Denote $Q\{q\} = \{a \in \omega^{<\omega} : q^\frown(a) \in \mathrm{dom}(Q)\}$, which is a level-1 tree. The *cardinality* of Q is $\mathrm{card}(Q) = \mathrm{card}(\mathrm{dom}(Q))$. $\mathrm{card}(Q)$ could be finite or $\aleph_0$.

For Q a level-2 tree, Let

$$\mathrm{dom}^*(Q) = \mathrm{dom}(Q) \cup \{q^\frown(-1) : q \in \mathrm{dom}(Q)\}.$$

Here -1 is a distinguished element which is $<_{BK}$-smaller than any node in $\omega^{<\omega}$. So $<_{BK}\restriction \mathrm{dom}^*(Q)$ extends $<_{BK}\restriction \mathrm{dom}(Q)$ where $q^\frown(-1)$ comes before any $q^\frown(s) \in \mathrm{dom}(Q)$. If $q \neq \emptyset$, denote $Q\{q,-\} = \{q^{-\frown}(-1)\} \cup \{q^{-\frown}(a) : Q_{\mathrm{tree}}(q^{-\frown}(a)) = Q_{\mathrm{tree}}(q) \wedge a <_{BK} q(\mathrm{lh}(q)-1)\}$, $Q\{q,+\} = \{q^-\} \cup \{q^{-\frown}(a) : Q_{\mathrm{tree}}(q^{-\frown}(a)) = Q_{\mathrm{tree}}(q) \wedge a >_{BK} q(\mathrm{lh}(q)-1)\}$, For $q \in \mathrm{dom}^*(Q)$, q is *of discontinuous type* if $q \in \mathrm{dom}(Q)$; q is *of continuous type* if $q \in \mathrm{dom}^*(Q) \setminus \mathrm{dom}(Q)$. In particular, $\{\emptyset, (-1)\} \subseteq \mathrm{dom}^*(Q)$. Put $Q[q^\frown(-1)] = (P, (Q_{\mathrm{node}}(q\restriction l))_{l \leq \mathrm{lh}(q)})$, where P is the completion of $Q(q)$. So $Q[q^\frown(-1)]$ is a potential partial level ≤ 1 tower of continuous type.

Definition 3.7. Suppose Q is a level-2 tree. A *Q-description* is a triple

$$\mathbf{q} = (q, P, \vec{p})$$

such that $q \in \mathrm{dom}^*(Q)$ and $(P, \vec{p}) = Q[q]$. $\mathrm{desc}(Q)$ is the set of Q-descriptions. A Q-description $(q, P, \vec{p})$ is of *(dis-)continuous type* iff q is of (dis-)continuous type. The *constant Q-description* is $(\emptyset, \emptyset, \emptyset)$.

If $\mathbf{q} = (q, P, \vec{p}) \in \mathrm{desc}(Q)$ is of discontinuous type, put $\mathbf{q}^\frown(-1) = (q^\frown(-1), P^+, \vec{p})$ where P^+ is the completion of $(P, \vec{p})$. If $\vec{\alpha} = (\alpha_p)_{p \in N}$ is a tuple indexed by N, $q \in \mathrm{dom}^*(Q)$, $\mathrm{dom}(Q(q^-)) \subseteq N$ if $q \neq \emptyset$, we put

$$\vec{\alpha} \oplus_Q q = (\alpha_{p_0}, q(0), \ldots, \alpha_{p_{\mathrm{lh}(q)-1}}, q(\mathrm{lh}(q)-1)),$$

where $p_i = Q_{\mathrm{node}}(q\restriction i)$.

The ordinal representation of Q is the set

$$\begin{aligned}\mathrm{rep}(Q) =& \{\vec{\alpha} \oplus_Q q : q \in \mathrm{dom}(Q), \vec{\alpha} \text{ respects } Q_{\mathrm{tree}}(q)\} \\ & \cup \{\vec{\alpha} \oplus_Q q^\frown(-1) : q \in \mathrm{dom}(Q), \vec{\alpha} \text{ respects } Q(q)\}.\end{aligned}$$

$\mathrm{rep}(Q)$ is endowed with the $<_{BK}$ ordering:

$$<^Q = <_{BK}\restriction \mathrm{rep}(Q).$$

Thus, the $<^Q$-greatest element is $\emptyset = \emptyset \oplus_Q \emptyset$, and the set $\{(\beta, -1) : \beta < \omega_1\}$ is $<^Q$-cofinal below $\emptyset$. In general, if $q \in \mathrm{dom}(Q)$ and $\vec{\alpha}$ respects $Q_{\mathrm{tree}}(q)$, then $\vec{\alpha} \oplus_Q q$ is the $<^Q$-sup of $\vec{\alpha}^\frown(\beta) \oplus q^\frown(-1) \in \mathrm{rep}(Q)$. The fact that (0) is the $<_{BK}$-maximum node of a nonempty regular level-1 tree implies that if $(q,P) \in \mathrm{desc}(Q)$, $q \neq \emptyset$, $(\alpha_p)_{p\in P}$ respects P, then $\alpha_{(0)}$ is bigger than α_p for any $p \in P \setminus \{(0)\}$. Hence, when Q is finite, $<^Q$ has order type $\omega_1 + 1$. If $B \in \mathbb{L}$ is a subset of ω_1, we put

$$f \in B^{Q\uparrow}$$

iff $f \in \mathbb{L}$ is an order preserving, continuous function from $\mathrm{rep}(Q)$ to $B \cup \{\omega_1\}$. If $f \in B^{Q\uparrow}$, for each $q \in \mathrm{dom}(Q)$, letting $P_q = Q_{\mathrm{tree}}(q)$, f_q is the function on $[\omega_1]^{P_q\uparrow}$ that sends $\vec{\alpha}$ to $f(\vec{\alpha} \oplus_Q q)$, and

$$[f]^Q = ([f]^Q_q)_{q\in\mathrm{dom}(Q)}$$

where $[f]^Q_q = [f_q]_{\mu^{P_q}}$. A tuple $\vec{\beta}$ *respects* Q iff $\vec{\beta} = [f]^Q$ for some $f \in \omega_1^{Q\uparrow}$; $\vec{\beta}$ *weakly respects* Q iff $\beta_\emptyset = \omega_1$ and for any $q \in \mathrm{dom}(Q) \setminus \{\emptyset\}$, $\beta_q < j^{Q_{\mathrm{tree}}(q^-),Q_{\mathrm{tree}}(q)}(\beta_{q^-})$.

If $y \in [\mathrm{dom}(Q)]$, let $Q(y) =_{\mathrm{DEF}} \cup_{n<\omega} Q_{\mathrm{tree}}(y\restriction n)$ be an infinite level-1 tree. Q is Π^1_2-*wellfounded* iff

1. $\forall q \in \mathrm{dom}(Q)$ $Q\{q\}$ is Π^1_1-wellfounded,
2. $\forall y \in [\mathrm{dom}(Q)]$ $Q(y)$ is not Π^1_1-wellfounded.

In particular, finite level-2 trees are Π^1_2-wellfounded. Π^1_2-wellfoundedness of a level-2 tree is a $\boldsymbol{\Pi}^1_2$ property in the real coding the tree.

A level-2 tree Q is a called a *subtree* of Q' iff Q is a subfunction of Q'. A *finite level-2 tower* is a (possibly empty) sequence $(Q_i)_{1\leq i\leq n}$ such that Q_i is a level-2 tree for $1 \leq i \leq n$, $\mathrm{card}(Q_i) = i$ and $i < j \to Q_i$ is a subtree of Q_j. An *infinite level-2 tower* is a sequence $\vec{Q} = (Q_n)_{1\leq n<\omega}$ such that for each n, $(Q_i)_{1\leq i\leq n}$ is a finite level-2 tower. *A level-2 system* is $(Q_s)_{s\in\omega^{<\omega}}$ such that for each s, $(Q_{s\restriction i})_{1\leq i<\mathrm{lh}(s)}$ is a finite level-2 tower. Associated to a $\boldsymbol{\Pi}^1_2$ set A we can assign a level-2 system $(Q_s)_{s\in\omega^{<\omega}}$ so that $x \in A$ iff the level-2 tower $Q_x =_{\mathrm{DEF}} (Q_{x\restriction n})_{n<\omega}$ is Π^1_2-wellfounded. If A is lightface Π^1_2, then $(Q_s)_{s\in\omega^{<\omega}}$ can be picked effective.

In our language, the level-2 tree S_2, originally defined in [26, Section 2], takes the following form.

Definition 3.8. Assume $\boldsymbol{\Pi}^1_1$-determinacy.

1. S_2^- is the tree on $V_\omega \times u_\omega$ such that $(\emptyset,\emptyset) \in S_2^-$ and a nonempty node

$$(\emptyset,\emptyset) \neq (\vec{Q},\vec{\alpha}) = ((Q_i)_{1\leq i\leq n}, (\alpha_i)_{1\leq i\leq n}) \in S_2^-$$

iff $\vec{Q}$ is a finite level-2 tower, and putting $Q_0 = \emptyset$, $q_i \in \mathrm{dom}(Q_{i+1}) \setminus \mathrm{dom}(Q_i)$, $\beta_{q_i} = \alpha_i$, then $(\beta_q)_{q\in\mathrm{dom}(Q_n)}$ respects Q_n.

2. S_2^- is the tree on $V_\omega \times u_\omega$ such that $(\emptyset, \emptyset) \in S_2^-$ and a nonempty node

$$(\emptyset, \emptyset) \neq (\vec{Q}, \vec{\alpha}) = ((Q_i)_{1 \leq i \leq n}, (\alpha_i)_{1 \leq i \leq n}) \in S_2^-$$

iff $\vec{Q}$ is a finite level-2 tower, and putting $Q_0 = \emptyset$, $q_i \in \mathrm{dom}(Q_{i+1}) \setminus \mathrm{dom}(Q_i)$, $\beta_{q_i} = \alpha_i$, then $(\beta_q)_{q \in \mathrm{dom}(Q_n)}$ weakly respects Q_n.

By Theorem 3.3,

$$p[S_2^-] = p[S_2] = \{\vec{Q} : \bigcup \vec{Q} \text{ is } \Pi^1_2\text{-wellfounded}\}.$$

The (non-regular) u_ω-scale associated to S_2 is Δ^1_3 (cf. [26]).

A *level* ≤ 2 *tree* is a pair $Q = ({}^1Q, {}^2Q)$ such that ${}^dQ$ is a level-d tree for $d \in \{1, 2\}$. Its *cardinality* is $\mathrm{card}(Q) = \sum_d \mathrm{card}({}^dQ)$. We follow the convention that ${}^dQ$ always stands for the level-d component of a level ≤ 2 tree Q. Q is a *level* ≤ 2 *subtree* of Q' iff ${}^dQ$ is a level-d subtree of ${}^dQ'$ for $d \in \{1, 2\}$. $\mathrm{rep}(Q) = \bigcup_d(\{d\} \times \mathrm{rep}({}^dQ))$. $<^Q = <_{BK} \restriction \mathrm{rep}(Q)$. So $<^Q$ is essentially the concatenation of $<^{{}^1Q}$ and $<^{{}^2Q}$. $\mathrm{dom}(Q) = \bigcup_d(\{d\} \times \mathrm{dom}({}^dQ))$, $\mathrm{dom}^*(Q) = \bigcup_d(\{d\} \times \mathrm{dom}^*({}^dQ))$, where $\mathrm{dom}^*({}^1Q) = \mathrm{dom}({}^1Q) = {}^1Q$. $\mathrm{desc}(Q) = \bigcup_d(\{d\} \times \mathrm{desc}({}^dQ))$ is the set of Q-descriptions. $(d, \mathbf{q}) \in \mathrm{desc}(Q)$ is *of continuous type* iff $d = 2$ and $\mathbf{q}$ is of continuous type; otherwise, $(d, \mathbf{q})$ is *of discontinuous type*. Q is Π^1_2-*wellfounded* iff ${}^1Q$ is Π^1_1-wellfounded and ${}^2Q$ is Π^1_2-wellfounded. By virtue of the Brower-Kleene ordering, the next proposition is a corollary of Theorem 3.3.

Proposition 3.9. *Let Q be a level ≤ 2 tree. Then Q is Π^1_2-wellfounded iff $<^Q$ is a wellordering on* $\mathrm{rep}(Q)$.

As a corollary, if Q is Π^1_2-wellfounded, then $\mathrm{o.t.}(<^Q) = \omega_1 + 1$.

If f is a function on $\mathrm{rep}(Q)$, let ${}^df$ be the function on $\mathrm{rep}({}^dQ)$ that sends v to $f(d, v)$. If $B \in \mathbb{L}$ is a subset of ω_1, we put

$$f \in B^{Q\uparrow}$$

iff $f \in \mathbb{L}$ is an order preserving, continuous function on $\mathrm{rep}(Q)$, and ${}^df \in B^{{}^dQ\uparrow}$ for $d \in \{1, 2\}$. f represents a $\mathrm{card}(Q)$-tuple of ordinals

$$[f]^Q = ({}^d[f]^Q_q)_{(d,q) \in \mathrm{dom}(Q)}$$

where ${}^d[f]^Q_q = [{}^df]^{{}^dQ}_q$. In particular, we must have ${}^2[f]^Q_\emptyset = \omega_1$. Let

$$[B]^{Q\uparrow} = \{[f]^Q : f \in B^{Q\uparrow}\}.$$

The properties of a tuple $[f]^Q$ for $f \in \omega_1^{Q\uparrow}$ are analyzed in [26, 46]. We restate the key results in the effective context.

Definition 3.10. Suppose Q is a level ≤ 2 tree. An *extended Q-description* is either a Q-description or of the form $(2,(q,P,\vec{p}))$ such that $(2,(q^\frown(-1),P,\vec{p}))$ is a Q-description of continuous type. $\operatorname{desc}^*(Q)$ is the set of extended Q-descriptions. $(d,\mathbf{q}) \in \operatorname{desc}^*(Q)$ is *regular* iff either $(d,\mathbf{q}) \in \operatorname{desc}(Q)$ of discontinuous type or $(d,\mathbf{q}) \notin \operatorname{desc}(Q)$.

Suppose $(2,\mathbf{q}) = (2,(q,P,\vec{p})) \in \operatorname{desc}^*(Q)$. If $f \in \omega_1^{Q\uparrow}$, ${}^2f_{\mathbf{q}}$ is the function on $[\omega_1]^{P\uparrow}$ defined as follows: ${}^2f_{\mathbf{q}} = {}^2f_q$ if $(2,\mathbf{q}) \in \operatorname{desc}(Q)$; ${}^2f_{\mathbf{q}}(\vec{\alpha}) = {}^2f_q(\vec{\alpha} \restriction {}^2Q_{\text{tree}}(q))$ if $(2,\mathbf{q}) \notin \operatorname{desc}(Q)$. If $\vec{\beta} = ({}^d\beta_q)_{(d,q)\in\operatorname{dom}(Q)} \in [\omega_1]^{Q\uparrow}$, we define ${}^d\beta_{\mathbf{q}}$ for $(d,\mathbf{q}) \in \operatorname{desc}^*(Q)$: if $d = 2$, $\mathbf{q} = (q,P,\vec{p})$, put ${}^d\beta_{\mathbf{q}} = [{}^df_{\mathbf{q}}]_{\mu^P}$ where $\vec{\beta} = [f]^Q$. Clearly, ${}^2\beta_{\mathbf{q}} = {}^2\beta_q$ if $(2,\mathbf{q}) \in \operatorname{desc}(Q)$ of discontinuous type, ${}^2\beta_{\mathbf{q}} = j^{{}^2Q_{\text{tree}}(q),P}({}^2\beta_q)$ if $(2,\mathbf{q}) \notin \operatorname{desc}(Q)$. The next lemma computes the remaining case when $\mathbf{q} \in \operatorname{desc}(Q)$ is of continuous type, justifying that ${}^d\beta_{\mathbf{q}}$ does not depend on the choice of f.

Lemma 3.11. *Suppose Q is a level ≤ 2 tree. Suppose $\vec{\beta} = ({}^d\beta_q)_{(d,q)\in\operatorname{dom}(Q)} \in [\omega_1]^{Q\uparrow}$, $(2,\mathbf{q}) = (2,(q,P,\vec{p})) \in \operatorname{desc}(Q)$ is of continuous type, $P^- = Q_{\text{tree}}(q^-)$, then ${}^2\beta_{\mathbf{q}} = j_{\sup}^{P^-,P}({}^2\beta_{q^-})$.*

Proof. Let $\vec{\beta} = [f]^Q$, $f \in [\omega_1]^{Q\uparrow}$. Let $v = p_{\operatorname{lh}(q)-1}$. So P is the completion of $Q(q^-) = (P^-, v)$.

Suppose $\gamma = [g]_{\mu^{P^-}} < {}^2\beta_{q^-}$, $g \in \mathbb{L}$. So for μ^{P^-}-a.e. $\vec{\alpha}$, $g(\vec{\alpha}) < {}^2f_{q^-}(\vec{\alpha}) = \sup_{\xi<\alpha_{v^-}} {}^2f_q(\vec{\alpha}^\frown(\xi))$, where $\vec{\alpha}^\frown(\xi)$ is the extension of $\vec{\alpha}$ whose entry indexed by v is ξ. Let $h(\vec{\alpha})$ be the least $\xi < \alpha_{v^-}$ such that $g(\vec{\alpha}) < {}^2f_q(\vec{\alpha}^\frown(\xi))$. Then $h \in \mathbb{L}$. By remarkability of level-1 sharps, we get $C \in \mu_{\mathbb{L}}$ such that for any $\vec{\alpha} \in [C]^{P\uparrow}$, $h(\vec{\alpha} \restriction \operatorname{dom}(P^-)) < \alpha_v$. Hence for any $\vec{\alpha} \in [C]^{P\uparrow}$, $g(\vec{\alpha} \restriction \operatorname{dom}(P^-)) < {}^2f_q(\vec{\alpha})$. Hence $j^{P^-,P}(\gamma) < {}^2\beta_q$.

Suppose on the other hand $\gamma = [g]_{\mu^P} < {}^2\beta_{\mathbf{q}}$. Then for μ^P-a.e. $\vec{\alpha}$, $g(\vec{\alpha}) < {}^2f_q(\vec{\alpha}) = \sup_{\xi<\alpha_v} {}^2f_q(\vec{\alpha} \restriction \operatorname{dom}(P^-)^\frown(\xi))$. Let $h(\vec{\alpha})$ be the least $\xi < \alpha_v$ such that $g(\vec{\alpha}) < {}^2f_q(\vec{\alpha} \restriction \operatorname{dom}(P^-)^\frown(\xi))$. By remarkability, we get $C \in \mu_{\mathbb{L}}$ and $h' \in \mathbb{L}$ such that for any $\vec{\alpha} \in [C]^{P\uparrow}$, $h(\vec{\alpha}) = h'(\vec{\alpha} \restriction \{p : p \prec^P v\})$. Hence, $g(\vec{\alpha}) < {}^2f_q(\alpha \restriction \operatorname{dom}(P^-)^\frown h'(\vec{\alpha} \restriction \{p : p \prec^P v\})) = j^{P^-,P}(\eta)$, where $\eta = [\vec{\alpha} \mapsto {}^2f_q(\vec{\alpha}^\frown h'(\vec{\alpha} \restriction \{p : p \prec^{P^-} v^-\}))]_{\mu^{P^-}}$. Clearly, $\eta < {}^2\beta_{q^-}$. So $\gamma < j_{\sup}^{P^-,P}({}^2\beta_{q^-})$. $\square$

A tuple $\vec{\beta} = ({}^d\beta_q)_{(d,q)\in\operatorname{dom}(Q)}$ *respects* Q iff $\vec{\beta} \in [\omega_1]^{Q\uparrow}$. In particular, if $\vec{\beta}$ respects Q, then ${}^2\beta_\emptyset = \omega_1$. $\vec{\beta}$ *weakly respects* Q iff $({}^1\beta_q)_{q\in{}^1Q}$ respects ${}^1Q$ and $({}^2\beta_q)_{q\in{}^2Q}$ weakly respects ${}^2Q$.

The relation of weak respectability is clearly Δ^1_3. It is essentially shown in [46] that respectability is also Δ^1_3. We restate the relevant definitions in a more applicable fashion.

Definition 3.12. Suppose W is a finite level-1 tree, $\vec{w} = (w_i)_{i<m}$ is a distinct enumeration of a subset of W. Suppose $f : [\omega_1]^{W\uparrow} \to \omega_1$ is a function which lies in $\mathbb{L}$. The *signature* of f is $\vec{w}$ iff there is $C \in \mu_{\mathbb{L}}$ such that

1. for any $\vec{\alpha}, \vec{\beta} \in [C]^{W\uparrow}$, if $(\alpha_{w_0}, \ldots, \alpha_{w_{m-1}}) <_{BK} (\beta_{w_0}, \ldots, \beta_{w_{m-1}})$ then $f(\vec{\alpha}) < f(\vec{\beta})$;
2. for any $\vec{\alpha}, \vec{\beta} \in [C]^{W\uparrow}$, if $(\alpha_{w_0}, \ldots, \alpha_{w_{m-1}}) = (\beta_{w_0}, \ldots, \beta_{w_{m-1}})$ then $f(\vec{\alpha}) = f(\vec{\beta})$.

In particular, f is constant on a μ^W-measure one set iff the signature of f is $\emptyset$.

Suppose the signature of f is $\vec{w} = (w_i)_{i<m}$. f is *essentially continuous* iff $m > 0$ and for μ^W-a.e. $\vec{\alpha}$, $f(\vec{\alpha}) = \sup\{f(\vec{\beta}) : (\beta_{w_0}, \ldots, \beta_{w_{m-1}}) < (\alpha_{w_0}, \ldots, \alpha_{w_{m-1}})\}$. Otherwise, f is *essentially discontinuous.* Put $[B]^{W\uparrow -1} = [B]^{W\uparrow} \times \omega$. For $w \in \mathrm{dom}(W)$, put $[B]^{W\uparrow w} = \{(\vec{\beta}, \gamma) : \vec{\beta} \in [B]^{W\uparrow}, \gamma < \beta_w\}$. For $v \in \{-1\} \cup W$, say that the *uniform cofinality* of f is v iff there is $g : [\omega_1]^{W\uparrow v} \to \omega_1$ such that $g \in \mathbb{L}$ and for μ^W-a.e. $\vec{\alpha}$, $F(\vec{\alpha}) = \sup\{G(\vec{\alpha}, \beta) : (\vec{\alpha}, \beta) \in [\omega_1]^{W\uparrow v}\}$ and the function $\beta \mapsto G(\vec{\alpha}, \beta)$ is order preserving. It is essentially shown in [46] that every $f : [\omega_1]^{W\uparrow} \to \omega_1$ in $\mathbb{L}$ has a unique signature and uniform cofinality. Let $(P_i, p_i)_{i<m}{}^\frown(P_m)$ be the partial level ≤ 1 tower of continuous type and let σ factor (P_m, W) such that $\sigma(p_i) = w_i$ for each $i < m$. Note that $w_i \prec^W w_0$ for $0 < i < m$, so each P_i is indeed a regular level-1 tree. $\vec{P} = (P_i)_{i\leq m}$ is called the *level-1 tower induced by* f, and σ is called the *factoring map induced by* f. Note that $\sigma \restriction P_i$ factors (P_i, W) for each i. The *potential partial level* ≤ 1 *tower* induced by f is

1. $(P_m, (p_i)_{i<m})$, if f is essentially continuous;
2. $(P_m, (p_i)_{i<m}{}^\frown(-1))$, if f is essentially discontinuous and has uniform cofinality -1;
3. $(P_m, (p_i)_{i<m}{}^\frown(p^+))$, if f is essentially discontinuous and has uniform cofinality $w_* \in W$, (P_m, p^+) is a partial level ≤ 1 tree, $\sigma((p^+)^-) = w_*$.

In particular, if $w_* \in W$, $f(\vec{\alpha}) = \alpha_{w_*}$ is the projection map, then the potential partial level ≤ 1 tower induced by f is $(\emptyset, (0))$. The *approximation sequence* of f is $(f_i)_{i\leq m}$ where $\mathrm{dom}(f_i) = [\omega_1]^{P_i\uparrow}$, f_0 is the constant function with value ω_1, $f_i(\vec{\alpha}) = \sup\{f(\vec{\beta}) : \vec{\beta} \in [\omega_1]^{W\uparrow}, (\beta_{w_0}, \ldots, \beta_{w_{i-1}}) = (\alpha_{p_0}, \ldots, \alpha_{p_{i-1}})\}$ for $1 \leq i \leq m$. In particular, $f_m(\vec{\beta}_\sigma) = f(\vec{\beta})$ for μ^W-a.e. $\vec{\beta}$.

Note that all the relevant properties of f depend only on the value of f on a μ^W-measure one set. We will thus be free to say the signature, etc. of f when f is defined on a μ^W-measure one set.

Definition 3.13. Suppose $\omega_1 \leq \beta < u_\omega$ is a limit ordinal. Suppose W is a finite level-1 tree, $\beta = [f]_{\mu^W} < u_{\mathrm{card}(W)+1}$, the signature of f is $(w_i)_{i<m}$, the approximation sequence of f is $(f_i)_{i\leq m}$, the level-1 tower induced by f is $(P_i)_{i\leq m}$, the factoring map induced by f is σ. Then the *signature* of β is $(\mathrm{seed}^W_{w_i})_{i<m}$, the *approximation sequence* of β is $([f_i]_{\mu^{P_i}})_{i\leq m}$, β is *essentially continuous* iff f is essentially continuous. The *uniform cofinality* of β is ω if f has uniform cofinality -1, $\mathrm{seed}^W_{w_*}$ if f has uniform cofinality $w_* \in W \cup \{\emptyset\}$. The *potential partial level* ≤ 1 *tower induced by* β is the potential partial level ≤ 1 tower induced by f.

The uniform cofinality of β is exactly $\mathrm{cf}^{\mathbb{L}}(\beta)$. The signature, approximation sequence and essential continuity of β are independent of the choice of (W, f) in Definition 3.13, and moreover Δ^1_3 in β uniformly.

Suppose the signature of β is $(u_{l_i})_{i<m}$, the approximation sequence of β is $(\gamma_i)_{i\leq m}$. For $i \leq m$, let $\tau_{i,m} : \{1, \ldots, i+1\} \to \{l_0, \ldots, l_i\}$ be order preserving. For $i < k < m$, let $\tau_{i,k} = \tau^{-1}_{k,m} \circ \tau_{i,m}$. A straightforward analysis on the representative function of β yields the following:

1. For $i < k < m$, $j^{\tau_{i,k}}_{\sup}(\gamma_i) < \gamma_k < j^{\tau_{i,k}}(\gamma_i)$.
2. For $i < m$, $j^{\tau_{i,m}}_{\sup}(\gamma_i) \leq \gamma_m < j^{\tau_{i,m}}(\gamma_i)$.
3. For $i < m$, $j^{\tau_{i,m}}_{\sup}(\gamma_i) = \gamma_m$ iff $i = m-1$ and β is essentially continuous.
4. $\beta = j^{\tau_{m,m}}(\gamma_m)$.

The next lemma is a version of the "converse direction". In its statement, the inequality $j^\pi_{\sup}(\gamma) < \gamma' < j^\pi(\gamma)$ forces π to move the signature of γ to a proper initial segment of that of γ', and forces the approximation sequence of γ to be a proper initial segment of that of γ'. It will be useful in the analysis of descriptions and tree factoring maps in Sections 4.4-4.8, which eventually justifies the axiomatization of $0^{3\#}$ in Section 5.4. The proof is again based on an analysis of the representative function of γ and γ'.

Lemma 3.14. *Suppose A is a finite subset of ω. Let $\pi : \{1, \ldots, \mathrm{card}(A)\} \to A$ be order preserving. Suppose that $\gamma < u_{\mathrm{card}(A)+1}$ and $j^\pi_{\sup}(\gamma) < \gamma' < j^\pi(\gamma)$. Let $(u_{l_k})_{k<v}$, $(\gamma_k)_{k\leq v}$, $(P, \vec{p})$ be the signature, approximation sequence and potential partial level ≤ 1 tower induced by γ respectively. Let $(u_{l'_k})_{k<v}$, $(\gamma'_k)_{k\leq v'}$, $(P', \vec{p}')$ be the signature, approximation sequence and potential partial level ≤ 1 tower induced by γ' respectively. Let $\mathrm{cf}^{\mathbb{L}}(\gamma) = u_{l_*}$. Then*

1. *$v < v'$, $\pi(l_k, \gamma_k) = (l'_k, \gamma'_k)$. γ is essentially discontinuous $\to \gamma_v = \gamma_{v'}$. γ is essentially continuous$\to \gamma_v < \gamma_{v'}$.*

2. $l'_k \notin A$ *for* $v \le k < v'$.

3. For any $k < v$, $l'_v < \pi(l_k) \leftrightarrow l_* \le l_k$.

4. P *is a proper subtree of* P' *and* $\vec{p}$ *is an initial segment of* $\vec{p}'$.

Moreover, if $\gamma' < \gamma'' < j^\pi(\gamma)$ *and* $(\gamma''_k)_{k \le v''}$ *is the approximation sequence of* γ''*, then* $\gamma'_v < \gamma''_v$.

The next few lemmas are essentially part of effectivized Kunen's analysis [46] of tuples of ordinals in u_ω. The proofs are rather routine.

Suppose E is a club in ω_1. For a partial level ≤ 1 tree (P,t), put $\vec{\alpha} = (\alpha_p)_{p \in P \cup \{t\}} \in [E]^{(P,t)\uparrow}$ iff $\vec{\alpha}$ respects (P,t), $(\alpha_p)_{p \in P} \in [E]^{P\uparrow}$, and $t \ne -1 \to \alpha_t \in E$. For a level ≤ 2 tree Q, put

$$\begin{aligned}\operatorname{rep}({}^2Q) \restriction E = &\{\vec{\alpha} \oplus_{{}^2Q} q : q \in \operatorname{dom}({}^2Q), \vec{\alpha} \in [E]^{{}^2Q_{\operatorname{tree}(q)}\uparrow}\} \\ &\cup \{\vec{\alpha} \oplus_{{}^2Q} q^\frown(-1) : q \in \operatorname{dom}({}^2Q), \vec{\alpha} \in [E]^{{}^2Q(q)\uparrow}\}.\end{aligned}$$

Put $\operatorname{rep}(Q) \restriction E = (\{1\} \times \operatorname{rep}({}^1Q)) \cup (\{2\} \times \operatorname{rep}({}^2Q) \restriction E)$. Then $\operatorname{rep}(Q) \restriction E$ is a closed subset of $\operatorname{rep}(Q)$ (in the order topology of $<^Q$).

Lemma 3.15. *Suppose* Q *is a finite level* ≤ 2 *tree,* $C \in \mu_{\mathbb{L}}$ *is a club. Then* $\vec{\beta} \in [C]^{Q\uparrow}$ *iff there exist* $f \in \omega_1^{Q\uparrow}$ *and* $E \in \mu_{\mathbb{L}}$ *such that* $\vec{\beta} = [f]^Q$ *and for any* $q \in {}^1Q$, ${}^1f(q)$ *is a limit point of* C*; for any* $q \in \operatorname{dom}({}^2Q)$*, for any* $\vec{\alpha} \in [E]^{{}^2Q_{\operatorname{tree}(q)}\uparrow}$, ${}^2f_q(\vec{\alpha})$ *is a limit point of* C.

Proof. The nontrivial direction is $\Leftarrow$. Suppose $f \in \omega_1^{Q\uparrow}$ and $E \in \mu_{\mathbb{L}}$ are as given. For $q \in \operatorname{dom}({}^2Q) \setminus \{\emptyset\}$, let ${}^2Q(q) = (P_q, p_q)$, and let q^* be the $<_{BK}$-maximum of ${}^2Q\{q,-\}$.

Claim 3.16. *There is* $E' \in \mu_{\mathbb{L}}$ *such that* $E' \subseteq E$ *and for any* $q \in \operatorname{dom}({}^2Q) \setminus \{\emptyset\}$*, for any* $\vec{\alpha} \in [E']^{P_q\uparrow}$*, if* $p_q \ne -1$ *then* $C \cap ({}^2f_{q^*}(\vec{\alpha}), {}^2f_q(\vec{\alpha}))$ *has order type* $\alpha_{p_q^-}$.

Proof of Claim 3.16. Otherwise, there is $q \in \operatorname{dom}({}^2Q) \setminus \{\emptyset\}$ such that $p_q \ne -1$ and for μ^{P_q}-a.e. $\vec{\alpha}$, $C \cap ({}^2f_{q^*}(\vec{\alpha}), {}^2f_q(\vec{\alpha}))$ has order type smaller than $\alpha_{p_q^-}$. However, by assumption, $C \cap ({}^2f_{q^*}(\vec{\alpha}), {}^2f_q(\vec{\alpha}))$ is cofinal in ${}^2f_q(\vec{\alpha})$, and ${}^2f_{q^\frown(-1)}$ witnesses that ${}^2f_q$ has uniform cofinality p_q^-. This leads to a function $h \in \mathbb{L}$ where for μ^{P_q}-a.e. $\vec{\alpha}$, $h(\vec{\alpha})$ is a cofinal sequence in $\alpha_{p_q^-}$ of order type $< \alpha_{p_q^-}$. Hence, $\operatorname{cf}^{\mathbb{L}}(\operatorname{seed}^{P_q}_{p_q^-}) < \operatorname{seed}^{P_q}_{p_q^-}$ by Łoś, which is absurd. $\square$

Fix E' as in Claim 3.16. We are able to define $f' : \operatorname{rep}(Q) \restriction E' \to C$ such that $f(1,q) = f'(1,q)$ for $q \in {}^1Q$, $f(2, \vec{\alpha} \oplus_{{}^2Q} q) = f'(2, \vec{\alpha} \oplus_{{}^2Q} q)$ for

$q \in \mathrm{dom}({}^2Q) \setminus \{\emptyset\}$, $\vec{\alpha} \in [E'']^{P_q\uparrow}$. Let $\theta : \mathrm{rep}(Q) \to \mathrm{rep}(Q) \restriction E'$ be an order preserving bijection. Let $E'' \in \mu_{\mathbb{L}}$ where $\eta \in E''$ iff $E' \cap \eta$ has order type η. It is easy to see that $\theta \restriction (\mathrm{rep}(Q) \restriction E'')$ is the identity map. Define $g = f' \circ \theta$. Then $g \in C^{Q\uparrow}$ and $[g]^Q = [f]^Q$. □

Lemma 3.17. *Suppose Q is a finite level ≤ 2 tree, ${}^2Q(q) = (P_q, p_q)$ for $q \in \mathrm{dom}(Q)$, $E \in \mu_{\mathbb{L}}$ is a club. Suppose $f : \mathrm{rep}(Q) \restriction E \to \omega_1 + 1$ satisfies*

1. *$f \restriction (\{1\} \times \mathrm{rep}({}^1Q))$ is continuous, order preserving;*
2. *if $q \in \mathrm{dom}({}^2Q)$, then the potential partial level ≤ 1 tower induced by ${}^2f_q$ is ${}^2Q[q]$, the approximation sequence of ${}^2f_q$ is $({}^2f_{q \restriction i})_{i \leq \mathrm{lh}(q)}$, and the uniform cofinality of ${}^2f_q$ on $[E]^{P_q\uparrow}$ is witnessed by ${}^2f_{q^\frown(-1)}$, i.e., if $\vec{\alpha} \in [E]^{P_q\uparrow}$, then ${}^2f_q(\vec{\alpha}) = \sup\{{}^2f_{q^\frown(-1)}(\vec{\alpha}^\frown(\beta)) : \vec{\alpha}^\frown(\beta) \in \mathrm{rep}({}^2Q) \restriction E\}$, and the map $\vec{\beta} \mapsto {}^2f_{q^\frown(-1)}(\vec{\alpha}^\frown(\beta))$ is continuous, order preserving;*
3. *if $a, b \in {}^2Q\{q\}$ and $a <_{BK} b$, then $[f_{q^\frown(a)}]_{\mu^{P_{q^\frown(a)}}} < [f_{q^\frown(b)}]_{\mu^{P_{q^\frown(b)}}}$.*

Then there is $E' \in \mu_{\mathbb{L}}$ such that $E' \subseteq E$ and $f \restriction (\mathrm{rep}(Q) \restriction E')$ is order preserving.

Proof. We know by assumption that for μ^{P_q}-a.e. $\vec{\alpha}$, $f_q(\vec{\alpha}) = \sup\{f_{q^\frown(a)}(\vec{\alpha}^\frown(\beta)) : \beta < \alpha_{p_q^-}\}$. Fix for the moment q such that $p_q \neq -1$. For $\vec{\alpha} = (\alpha_p)_{p \in P_q}$, put $\vec{\alpha}^- = (\alpha_p)_{p <_{BK} p_k^-}$. By remarkability of (level-1) sharps, there is a function $h \in \mathbb{L}$ and $E'_q \in \mu_{\mathbb{L}}$ such that for any $\vec{\alpha} \in [E'_q]^{P_q\uparrow}$, $h(\vec{\alpha}^-) < \alpha_{p_q^-}$ and for any $\beta \in \alpha_{p_q^-} \cap E'_q$, for any $a, b \in {}^2Q\{q\}$, $f_{q^\frown(a)}(\vec{\alpha}^\frown(\beta)) < f_{q^\frown(b)}(\vec{\alpha}^\frown(h(\vec{\alpha}^-)))$. Let $\eta \in E''_q$ iff for any $\vec{\alpha} \in [E'_q]^{P_q\uparrow}$, if $\forall p <_{BK} p_k^-\ \alpha_p < \eta$ then $h(\vec{\alpha}^-) < \eta$. Finally, let $E'' = \bigcap\{E''_q : p_q \neq -1\}$. E'' works for the lemma. □

Lemma 3.18. *Suppose that Q is a finite level ≤ 2 tree and $\vec{\beta} = ({}^d\beta_q)_{(d,q) \in \mathrm{dom}(Q)}$ is a tuple of ordinals in u_ω. Then $\vec{\beta}$ respects Q iff all of the following holds:*

1. *$({}^1\beta_q)_{q \in {}^1Q}$ respects ${}^1Q$.*
2. *For any $q \in \mathrm{dom}({}^2Q)$, the potential partial level ≤ 1 tower induced by β_q is $Q[q]$, and the approximation sequence of β_q is $(\beta_{q \restriction l})_{l \leq \mathrm{lh}(q)}$.*
3. *If $a, b \in {}^2Q\{q\}$ and $a <_{BK} b$ then ${}^2\beta_{q^\frown(a)} < {}^2\beta_{q^\frown(b)}$.*

Moreover, if $C \in \mu_{\mathbb{L}}$ is a club, then $\vec{\beta} \in [C]^{Q\uparrow}$ iff $\vec{\beta}$ respects Q and letting C' be the set of limit points of C, then ${}^1\beta_q \in C'$ for $q \in {}^1Q$, ${}^2\beta_q \in j^{{}^2Q_{\mathrm{tree}(q)}}(C')$ for $q \in \mathrm{dom}({}^2Q)$.

Lemma 3.19. *The relation "Q is a finite level ≤ 2 tree $\wedge$ $\vec{\beta}$ respects Q" is* Δ^1_3.

Lemma 3.20. *Suppose Q and Q' are level ≤ 2 trees with the same domain. Suppose $\vec{\beta}$ respects both Q and Q'. Then $Q = Q'$.*

Suppose Q is a finite level ≤ 2 tree. Suppose $(d, \mathbf{q}) \in \operatorname{desc}^*(Q)$, and if $d = 2$ then $\mathbf{q} = (q, P, \vec{p})$. Put

$$[\![(d,\mathbf{q})]\!]_Q = \begin{cases} \|(1,(\mathbf{q}))\|_{<Q} & \text{if } d = 1, \\ [\vec{\alpha} \mapsto \|(2, \vec{\alpha} \oplus_{^2Q} q)\|_{<Q}]_{\mu^P} & \text{if } d = 2. \end{cases}$$

To save ink, put $[\![d,\mathbf{q}]\!]_Q = [\![(d,\mathbf{q})]\!]_Q$. If in addition, $d = 2$ and $\mathbf{q} \in \operatorname{desc}(Q)$ of discontinuous type, put $[\![2,q]\!]_Q = [\![2,\mathbf{q}]\!]_Q$. It is easy to compute $[\![d,\mathbf{q}]\!]_Q$ only from the syntactics.

Definition 3.21. To every ordinal $\xi < \omega^{\omega^\omega}$ (ordinal arithmetic), we assign $\widehat{\xi}$ as follows:

1. $\widehat{0} = 0$.
2. $\widehat{1} = \omega$.
3. If $0 < \eta = \omega^{n_1} + \cdots + \omega^{n_k} < \omega^\omega$, $\omega > n_1 \geq \cdots \geq n_k$ in the Cantor normal form, then $\widehat{\omega^\eta} = u_{n_1+1} \cdot \cdots \cdot u_{n_k+1}$.
4. If $0 < \xi = \omega^{\eta_1} + \cdots + \omega^{\eta_k}$, $\omega^\omega > \eta_1 \geq \cdots \geq \eta_k$ in the Cantor normal form, then $\widehat{\xi} = \widehat{\omega^{\eta_1}} + \cdots + \widehat{\omega^{\eta_k}}$.

Then

$$\{\widehat{\xi} : 0 < \xi < \omega^{\omega^\omega}\} = \{[\![d,\mathbf{q}]\!]_Q : \ Q \text{ finite level} \leq 2 \text{ tree}, (d,\mathbf{q}) \in \operatorname{desc}^*(Q)\}$$

and the relation $\widehat{\xi} = [\![d,\mathbf{q}]\!]_Q$ is effective. The ordering among different $[\![d,\mathbf{q}]\!]_Q$ can be computed in the following concrete way. Put $\langle 1,\mathbf{q}\rangle = (1,\mathbf{q})$. For $\mathbf{q} = (q, P, \vec{p})$, $k = \operatorname{lh}(q)$, $\vec{p} = (p_i)_{i<\operatorname{lh}(\vec{p})}$, put

$$\langle 2,\mathbf{q}\rangle = \begin{cases} (2, \|p_0\|_{\prec^P}, q(0), \ldots, \|p_{k-2}\|_{\prec^P}, q(k-2), -1) \\ \qquad \text{if } \mathbf{q} \in \operatorname{desc}({}^2Q) \text{ of continuous type}, \\ (2, \|p_0\|_{\prec^P}, q(0), \ldots, \|p_{k-1}\|_{\prec^P}, q(k-1), -1) \\ \qquad \text{if } \mathbf{q} \in \operatorname{desc}({}^2Q) \text{ of discontinuous type}, \\ (2, \|p_0\|_{\prec^P}, q(0), \ldots, \|p_{k-1}\|_{\prec^P}, q(k-1), \|p_k\|_{\prec^P}) \\ \qquad \text{if } \mathbf{q} \notin \operatorname{desc}({}^2Q). \end{cases}$$

Define

$$(d, \mathbf{q}) \prec (d', \mathbf{q}')$$

iff $\langle d, \mathbf{q} \rangle <_{BK} \langle d', \mathbf{q}' \rangle$. Define

$$(d, \mathbf{q}) \sim (d', \mathbf{q}')$$

iff $\langle d, \mathbf{q} \rangle = \langle d', \mathbf{q}' \rangle$. Then for any finite level ≤ 2 tree Q, $[\![d, \mathbf{q}]\!]_Q < [\![d', \mathbf{q}']\!]_Q$ iff $(d, \mathbf{q}) \prec (d', \mathbf{q}')$; $[\![d, \mathbf{q}]\!]_Q = [\![d', \mathbf{q}']\!]_Q$ iff $(d, \mathbf{q}) \sim (d', \mathbf{q}')$. In fact, $(d, \mathbf{q}) \sim (d', \mathbf{q}')$ iff either $(d, \mathbf{q}) = (d', \mathbf{q}')$ or $\{(d, \mathbf{q}), (d', \mathbf{q}')\} = \{(2, (\emptyset, \emptyset, \emptyset)), (2, ((-1), \{(0)\}, ((0))))\}$.

Define $\prec^Q = \prec \restriction \mathrm{desc}^*(Q)$, $\sim^Q = \sim \restriction \mathrm{desc}^*(Q)$. It is also easy to verify the next lemma on the order of the entries of $\vec{\beta}$ which respects Q.

Lemma 3.22. *Suppose Q is a level ≤ 2 tree and $\vec{\beta}$ respects Q. Suppose $(d, \mathbf{q}), (d', \mathbf{q}') \in \mathrm{desc}^*(Q)$. Then ${}^d\beta_{\mathbf{q}} < {}^{d'}\beta_{\mathbf{q}'}$ iff $(d, \mathbf{q}) \prec^Q (d, \mathbf{q}')$; ${}^d\beta_{\mathbf{q}} = {}^{d'}\beta_{\mathbf{q}'}$ iff $(d, \mathbf{q}) \sim^Q (d, \mathbf{q}')$.*

4 The level-2 sharp

4.1 The equivalence of $x^{2\#}$ and $M_1^\#(x)$

From now on, we assume $\mathbf{\Delta}^1_2$-determinacy. By Kechris-Woodin [29], $\Game(<\omega^2\text{-}\mathbf{\Pi}^1_1)$-determinacy follows. By Neeman [37, 38] and Woodin [30, 48], this is also equivalent to "for every $x \in \mathbb{R}$, there is an (ω, ω_1)-iterable $M_1^\#(x)$".

Definition 4.1. Suppose $\mathcal{X} = \omega^k \times \mathbb{R}^l$ is a product space. Suppose x is a real and $\beta \leq u_\omega$. A subset $A \subseteq \mathcal{X}$ is β-$\Pi^1_3(x)$ iff there is a $\Pi^1_3(x)$ set $B \subseteq u_\omega \times \mathcal{X}$ such that $A = \mathrm{Diff}\, B$. A is β-Π^1_3 iff A is β-$\Pi^1_3(0)$. A is β-$\mathbf{\Pi}^1_3$ iff A is β-$\Pi^1_3(x)$ for some real x.

By Theorem 2.1, when β is a limit ordinal, $A \subseteq \mathcal{X}$ is β-$\Pi^1_3(x)$ iff there is a pair of Σ_1-formulas (φ, ψ) such that

$$(\vec{n}, \vec{y}) = (n_1, \ldots, n_k, y_1, \ldots, y_l) \in A$$

iff

$$L_{\kappa_3^{x,\vec{y}}}[T_2, x, \vec{y}] \models \exists \alpha < \beta (\forall \eta < \alpha\ \varphi(\eta, \vec{n}, \vec{y}, T_2, x) \wedge \neg\psi(\alpha, \vec{n}, \vec{y}, T_2, x)).$$

Lemma 4.2. *Assume $\mathbf{\Delta}^1_2$-determinacy. Suppose n, m are positive integers. If A is $(u_n)^m$-$\Pi^1_3(x)$, then A is $\Game^2(\omega n\text{-}\Pi^1_1(x))$.*

Proof. Without loss of generality, we assume $x = 0$, $m = 1$, and $A \subseteq \mathbb{R}$. Let B be a Π^1_3 subset of $u_\omega \times \mathbb{R}$ such that $A = \operatorname{Diff} B$. Let $B^* = \{(w, y) : (|w|, y) \in B\}$ be the Π^1_3 code set of B. Let $C \subseteq \mathbb{R}^3$ be a Σ^1_2 set such that

$$(w, y) \in B^* \leftrightarrow \forall r(w, y, r) \in C.$$

Consider the game $H(y)$, where I produces $w, r \in \mathbb{R}$, II produces $w', r' \in \mathbb{R}$. The game is won by I iff both of the following hold:

1. $w \in \mathrm{WO}_n$, $|w|$ is odd, and $(w, y, r) \notin C$.
2. If $w' \in \mathrm{WO}_n$, $|w'|$ is even, and $(w', y, r') \notin C$, then $|w| < |w'|$.

Therefore, $y \in A$ iff I has a winning strategy in $H(y)$.

Since $L[y, w, r, w', r']$ is Σ^1_2-absolute, and since the relation $|w| \leq |w'|$ for $w, w' \in \mathrm{WO}_n$ is definable over $L[y, w, r, w', r']$ from parameters $u_1, \ldots, u_{n-1}$, the payoff set of the game $H(y)$ can be expressed as a first order statement over $L[y, \cdot]$ from parameters $u_1, \ldots, u_{n-1}$. That is, there is a formula θ such that an infinite run

$$(w, r, w', r')$$

is won by I iff

$$L[y, w, r, w', r'] \models \theta(y, w, r, w', r', u_1, \ldots, u_{n-1}).$$

It follows by Martin [33] that the payoff set of $H(y)$ is $\Game(\omega n\text{-}\Pi^1_1(y))$, uniformly in y, hence determined. Hence A is in $\Game^2(\omega n\text{-}\Pi^1_1)$. $\square$

Lemma 4.3. *Assume $\mathbf{\Delta}^1_2$-determinacy. Let $n < \omega$. If A is $\Game^2(\omega n\text{-}\Pi^1_1(x))$, then A is $u_{n+2}\text{-}\Pi^1_3(x)$.*

Proof. Without loss of generality, assume $x = 0$ and $A \subseteq \mathbb{R}$. We produce an effective transformation from a $\Game^2(\omega n\text{-}\Pi^1_1)$ definition to the desired $u_{n+2}\text{-}\Pi^1_3$ definition. By Martin [33], if $(y, r) \in \mathbb{R}^2$, $C \subseteq \mathbb{R}$ is $\omega n\text{-}\Pi^1_1(y, r)$, then there is a formula φ such that Player I has a winning strategy in $G(C)$ iff

$$L[y, r] \models \varphi(y, r, u_1, \ldots, u_n).$$

The transform from the $\omega n\text{-}\Pi^1_1(y, r)$ definition of C to φ is uniform, independent of (y, r). Suppose $A = \Game B$, where $B \subseteq \mathbb{R}^2$ is $\Game(\omega n\text{-}\Pi^1_1)$. Suppose φ is a formula such that

$$(y, r) \in B \leftrightarrow \; L[y, r] \models \varphi(y, r, u_1, \ldots, u_n).$$

To establish a $u_{n+2}\text{-}\Pi^1_3$ definition of A, we have to decide which player has a winning strategy in $G(B_y)$, for $y \in \mathbb{R}$. For ordinals $\xi_1 < \cdots < \xi_n < \eta < \omega_1$, we say that M is a Kechris-Woodin non-determined set with respect to $(y, \xi_1, \ldots, \xi_n, \eta)$ iff

1. M is a countable subset of $\mathbb{R}$;

2. M is closed under join and Turing reducibility;

3. $\forall \sigma \in M\ \exists v \in M\ L_\eta[y, \sigma \otimes v] \models \neg\varphi(y, \sigma \otimes v, \xi_1, \ldots, \xi_n)$;

4. $\forall \sigma \in M\ \exists v \in M\ L_\eta[y, v \otimes \sigma] \models \varphi(y, v \otimes \sigma, \xi_1, \ldots, \xi_n)$.

In clause 3, "$\forall \sigma \in M$" is quantifying over all strategies σ for Player I that is coded in some member of M; $\sigma * v$ is Player I's response to v according to σ, and $\sigma \otimes v = (\sigma * v) \oplus w$ is the combined infinite run. Similarly for clause 4, roles between two players being exchanged. Say that z is $(y, \xi_1, \ldots, x_n, \eta)$-stable iff z is not contained in any Kechris-Woodin non-determined set with respect to $(y, \xi_1, \ldots, \xi_n, \eta)$. z is y-stable iff z is $(y, \xi_1, \ldots, \xi_n, \eta)$-stable for all $\xi_1 < \ldots < \xi_n < \eta < \omega_1$. The set of (y, z) such that z is y-stable is Π^1_2. By the proof of Kechris-Woodin [29], for all $y \in \mathbb{R}$, there is $z \in \mathbb{R}$ which is y-stable.

Note that if z is $(y, \xi_1, \ldots, \xi_n, \eta)$-stable and $z \leq_T z'$, then z' is $(y, \xi_1, \ldots, \xi_n, \eta)$-stable. Let $<_y^{\xi_1,\ldots,\xi_n,\eta}$ be the following wellfounded relation on the set of z which is $(y, \xi_1, \ldots, \xi_n, \eta)$-stable:

$$\begin{aligned} z' <_y^{\xi_1,\ldots,\xi_n,\eta} z \leftrightarrow\ & z \text{ is } (y, \xi_1, \ldots, \xi_n, \eta)\text{-stable} \wedge z \leq_T z' \wedge \\ & \forall \sigma \leq_T z\ \exists v \leq_T z'\ L_\eta[y, \sigma \otimes v] \models \neg\varphi(y, \sigma \otimes v, \xi_1, \ldots, \xi_n) \\ & \forall \sigma \leq_T z\ \exists v \leq_T z'\ L_\eta[y, v \otimes \sigma] \models \varphi(y, v \otimes \sigma, \xi_1, \ldots, \xi_n). \end{aligned}$$

Wellfoundedness of $<_y^{\xi_1,\ldots,\xi_n,\eta}$ follows from the definition of $(y, \xi_1, \ldots, \xi_n, \eta)$-stableness. If z is $(y, \xi_1, \ldots, \xi_n, \eta)$-stable, then $<_y^{\xi_1,\ldots,\xi_n,\eta} \restriction \{z' : z' <_y^{\xi_1,\ldots,\xi_n,\eta} z\}$ is a $\boldsymbol{\Sigma}^1_1$ wellfounded relation in parameters (y, z) and the code of $(\xi_1, \ldots, \xi_n, \eta)$, hence has rank $< \omega_1$ by Kunen-Martin. If z is y-stable, let f^z_y be the function that sends $(\xi_1, \ldots, \xi_n, \eta)$ to the rank of z in $<_y^{\xi_1,\ldots,\xi_n,\eta}$. Then f^z_y is a function into ω_1. By Σ^1_2-absoluteness between V and $L[y, z]^{\mathrm{Coll}(\omega,\eta)}$, we can see $f^z_y \in L[y, z]$. Furthermore, f^z_y is definable over $L[y, z]$ in a uniform way, so there is a $\{\underline{\in}\}$-Skolem term τ such that for all $(y, z) \in \mathbb{R}^2$, if z is y-stable, then

$$f^z_y(\xi_1, \ldots, \xi_n, \eta) = \tau^{L[y,z]}(y, z, \xi_1, \ldots, \xi_n, \eta).$$

Let

$$\beta^z_y = \tau^{L[y,z]}(y, z, u_1, \ldots, u_{n+1}).$$

The function

$$(y, z) \mapsto \beta^z_y$$

is Δ^1_3 in the sharp codes. We say that z is y-ultrastable iff z is y-stable and $\beta^z_y = \min\{\beta^w_y : w \text{ is } y\text{-stable}\}$.

Claim 4.4. *If z is y-ultrastable, then there is $\sigma \leq_T z$ such that σ is a winning strategy for either of the players in $G(B_y)$.*

Proof of Claim 4.4. Suppose otherwise. For any $\sigma \leq_T z$ which is a strategy for either player, pick w_σ which defeats σ in $G(B_y)$. Let w be a real coding $\{(\sigma, w_\sigma) : \sigma \leq_T z\}$. By an indiscernability argument, for any (y, w)-indiscernibles $\xi_1 < \cdots < \xi_n < \eta$, for any $\sigma \leq_T z$, if σ is a strategy for Player I, then

$$L_\eta[y, \sigma \otimes w_\sigma] \models \neg\varphi(y, \sigma \otimes w_\sigma, \xi_1, \ldots, \xi_n);$$

if σ is a strategy for Player II, then

$$L_\eta[y, w_\sigma \otimes \sigma] \models \varphi(y, w_\sigma \otimes \sigma, \xi_1, \ldots, \xi_n).$$

This exactly means

$$w <_y^{\xi_1, \ldots, \xi_n, \eta} z,$$

and hence

$$f_y^w(\xi_1, \ldots, \xi_n, \eta) < f_y^z(\xi_1, \ldots, \xi_n, \eta).$$

Since z is y-stable and $z \leq_T w$, w is y-stable. Therefore, β_y^w is defined and $\beta_y^w < \beta_y^z$, contradicting to y-ultrastableness of z. □

From Claim 4.4, if Player I (or II) has a winning strategy in $G(B_y)$, then for any y-ultrastable z, there is a winning strategy for Player I (or II) in $G(B_y)$ which is Turing reducible to z. Therefore, Player I has a winning strategy in $G(B_y)$ iff there is $\delta < u_{n+2}$ such that

$$\exists z \ (z \text{ is } y\text{-stable} \wedge \beta_y^z = \delta) \tag{1}$$

and

$$\forall \gamma \leq \delta \ \forall z \ ((z \text{ is } y\text{-stable} \wedge \beta_y^z = \gamma) \rightarrow \\ \exists \sigma \leq_T z \ (\sigma \text{ is a winning strategy for I in } G(B_y))). \tag{2}$$

Note that in (1),

$$\{(\delta, y) : \exists z \ (z \text{ is } y\text{-stable} \wedge \beta_y^z = \delta)\}$$

is a Σ^1_3 subset of $u_\omega \times \mathbb{R}$, and in (2),

$$\{(\gamma, y) : \forall z \ ((z \text{ is } y\text{-stable} \wedge \beta_y^z = \gamma) \rightarrow \\ \exists \sigma \leq_T z \ (\sigma \text{ is a winning strategy for I in } G(B_y)))\}$$

is a Π^1_3 subset of $u_\omega \times \mathbb{R}$. So $\exists \delta < u_{n+2}((1) \wedge (2))$ is a u_{n+2}-Π^1_3 definition of A. □

Lemma 4.2 and Lemma 4.3 are concluded in a simple equality between pointclasses.

Theorem 4.5. *Assume $\mathbf{\Delta}^1_2$-determinacy. Then for $x \in \mathbb{R}$,*

$$\Game^2(<\omega^2\text{-}\Pi^1_1(x)) = <u_\omega\text{-}\Pi^1_3(x).$$

Definition 4.6.

$$\mathcal{O}^{T_2,x} = \{(\ulcorner\varphi\urcorner, \alpha) : \varphi \text{ is a } \Sigma_1\text{-formula}, \alpha < u_\omega, L_{\kappa^x_3}[T_2, x] \models \varphi(T_2, x, \alpha)\}.$$

$\mathcal{O}^{T_2,x}$ is the u_ω-version of Kleene's $\mathcal{O}$ relative to (T_2, x). It is called $\mathcal{P}^x_3$ in [23].

Definition 4.7.

$$x^{2\#}_n = \{(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) : \exists\alpha < u_n((\ulcorner\varphi\urcorner, \alpha) \notin \mathcal{O}^{T_2,x} \wedge \forall\eta < \alpha(\ulcorner\psi\urcorner, \eta) \in \mathcal{O}^{T_2,x})\}.$$

$$x^{2\#} = \{(n, \ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) : n < \omega \wedge (\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \in x^{2\#}_n\}.$$

$\mathcal{O}^{T_2,x}$ splits into ω many parts $(\mathcal{O}^{T_2,x} \cap (\omega \times u_n))_{n<\omega}$. Each part is squeezed into a real $x^{2\#}_n$ by applying the difference operator on its second coordinate. The join of $(x^{2\#}_n)_{n<\omega}$ is $x^{2\#}$. In particular, $x^{2\#}_0$ is Turing equivalent to the good universal Π^1_3 real, which is called the Δ^1_3-jump of x. Each $x^{2\#}_n$ belongs to $L_{\kappa^x_3}[T_2, x]$, but $x^{2\#} \notin L_{\kappa^x_3}[T_2, x]$. The distinction between $x^{2\#}_0$ and $x^{2\#}$ does not have a lower level analog.

The expression of $0^{2\#}$ generalizes Kleene's $\mathcal{O}$ to the higher level. Note that the transformations between $\Game^2(<\omega^2\text{-}\Pi^1_1(x))$ and $<u_\omega\text{-}\Pi^1_3(x)$ definitions in Theorem 4.5 are uniform. Applying Theorem 4.5 to the space $\mathcal{X} = \omega$, in combination with Theorem 2.1, we get the equivalence between $x^{2\#}$ and $M^\#_1(x)$.

Theorem 4.8. *Assume $\mathbf{\Delta}^1_2$-determinacy. Then $x^{2\#}$ is many-one equivalent to $M^\#_1(x)$, the many-one reductions being independent of x.*

$0^{2\#}$ is essentially a fancy way of expressing y_3, the leftmost real of $\widehat{T_2}$ which is used in the standard uniformization argument. T_2 and y_3 are used in [31] to show that every nonempty Σ^1_3 set of reals contains a member which is recursive in y_3, or in our terminology, recursive in $0^{2\#}$. Basis theorems can also be proved with inner model theory. If $M^\#_{2n-1}$ exists, then every nonempty Σ^1_{2n+1} set of reals contains a member recursive in $M^\#_{2n-1}$ (cf. [47, 51]). At higher levels, the leftmost real basis arguments are investigated in [27]. It is shown by Harrington (modulo Neeman [37,38]) that under $\mathbf{\Delta}^1_{2n}$-determinacy, there is a Δ^1_{2n+1}-scale on a Δ^1_{2n+1} set whose leftmost real y_{2n+1} is Δ^1_{2n+1}-equivalent to $M^\#_{2n-1}$ and such that every nonempty Σ^1_{2n+1} set contains a real recursive in y_{2n+1}. It is asked in [27, Conjecture 11.2] whether y_{2n+1} is Turing equivalent to $M^\#_{2n-1}$. Theorem 4.8 solves this conjecture in the $n = 1$ case in an effective manner.

4.2 Homogeneity properties of S_2

By [27, Lemma 14.2], $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$ is admissibly closed. We shall define a system of $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$-measures on finite tuples in u_ω. This system of $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$-measures will witness S_2 being $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$-homogeneous. Under AD, these $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$-measures are total measures induced from the strong partition property on ω_1 (cf. [26]). These measures enable the Martin-Solovay tree construction of S_3 projecting to the universal Π^1_3 set, to be redefined in Section 4.3. In our situation, we must recast the effective version of the proof of the strong partition property on ω_1. Let $X^\uparrow$ be the set of strictly increasing functions $f : \omega_1 \to X$ that belong to $\mathbb{L}$. Only functions in $\mathbb{L}$ will be partitioned, and the partition must be guided by a $\boldsymbol{\Delta}^1_3$ surjection from $\omega_1^\uparrow$ onto u_ω and a subset $A \subseteq u_\omega$ which lies in $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$.

Every function $f \in \omega_1^\uparrow$ is of the form $\alpha \mapsto \tau^{L[x]}(x, \alpha)$ where $x \in \mathbb{R}$ and τ is a Skolem term. Thus, sharp codes for increasing functions is a good coding system for $\omega_1^\uparrow$.

Definition 4.9. ω_1 has the *level-2 strong partition property* iff for every function $\psi : \omega_1^\uparrow \to u_\omega$ such that the relation "$(\tau, x^\#)$ is a sharp code for an increasing function, $\alpha = \psi(\tau^{L[x]}(x, \cdot))$" is $\boldsymbol{\Delta}^1_3$, for every $B \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$, there is $X \subseteq \omega_1$ such that o.t.$(X) = \omega_1$, $X \in \mathbb{L}$ and either $\psi'' X^\uparrow \subseteq B$ or $\psi'' X^\uparrow \subseteq u_\omega \setminus B$.

In most applications, ψ will have the property that $\psi(f) = \psi(g)$ whenever $\forall \alpha < \omega_1 \ \sup f''\alpha = \sup g''\alpha$. The partition will be essentially on continuous functions only. In this case, when X is the homogeneous set produced by Definition 4.9, so is the set of limit points of X. We will henceforth demand that the homogeneous set is a club in ω_1.

Martin's proof of the strong partition property on ω_1 under AD carries over in a trivial way. For the reader's convenience, we include a proof.

Theorem 4.10 (Martin)**.** *Assume $\boldsymbol{\Delta}^1_2$-determinacy. Then ω_1 has the level-2 strong partition property.*

Proof. We imitate the proof in [19, Theorem 28.12], which builds on partially iterable sharps. We are given $\psi : \omega_1^\uparrow \to u_\omega$ whose complexity is $\boldsymbol{\Delta}^1_3$ (in the sharp codes for increasing functions) and the target of the partition $B \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. Define the game G in which I produces $\langle \ulcorner \tau \urcorner, a^* \rangle$, II produces $\langle \ulcorner \sigma \urcorner, b^* \rangle$. An infinite run is won by Player II iff

1. If $\langle \ulcorner \tau \urcorner, a^* \rangle$ is a putative sharp code for an increasing function, then so is $\langle \ulcorner \sigma \urcorner, b^* \rangle$. Moreover, for any $\eta < \omega_1$, if

$$a^* \text{ is } \eta\text{-wellfounded } \wedge \tau^{\mathcal{M}_{a^*,\eta}}(\eta) \in \text{wfp}(\mathcal{M}_{a^*,\eta})$$

then

$$b^* \text{ is } \eta\text{-wellfounded } \wedge \sigma^{\mathcal{M}_{b^*,\eta}}(\eta) \in \mathrm{wfp}(\mathcal{M}_{b^*,\eta}).$$

2. If $\langle \ulcorner \tau \urcorner, a^* \rangle, \langle \ulcorner \sigma \urcorner, b^* \rangle$ are true sharp codes for increasing functions, $a^* = a^\#$, $b^* = b^\#$, letting $h(\eta) = \sup\{\tau^{L[a]}(\omega\eta + n), \sigma^{L[b]}(\omega\eta + n) : n < \omega\}$ for $\eta < \omega_1$, then $\psi(h) \in B$.

The payoff set of G is in $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$ by Theorem 2.1, hence in $\Game({<}\omega^2\text{-}\boldsymbol{\Pi}^1_1)$ by Lemma 2.11. Hence G is determined.

Suppose that player I has a winning strategy φ in G. Let C be the set of φ-admissibles and limits of φ-admissibles. Similarly to the proof of Lemma 2.11, using boundedness, if $\langle \ulcorner \sigma \urcorner, v^\# \rangle$ is a true sharp code for an increasing function such that $\forall \beta < \omega_1\ \sigma^{L[v]}(\beta) \in C$, then $\langle \ulcorner \tau \urcorner, w^\# \rangle =_{\mathrm{DEF}} f * \langle \ulcorner \sigma \urcorner, v^\# \rangle$ is a true sharp code for an increasing function, and for any $\eta \in C$, for any $\beta \leq \eta$ such that $\forall \bar{\beta} < \beta\ \sigma^{L[v]}(\bar{\beta}) \leq \eta$, we have $\tau^{L[w]}(\beta) < \min(C \setminus \eta + 1)$.

Let $e : \omega_1 \to C$ enumerate C in the increasing order and let $X = \{\sup_{n<\omega} e(\omega\xi + n) : \xi < \omega_1\}$. We show that $\psi'' X^\uparrow \subseteq B$. Suppose that $f \in X^\uparrow$. By definition of X there is a function $g \in C^\uparrow$ such that $f(\alpha) = \sup_{n<\omega} g(\omega\alpha + n)$ for any $\alpha < \omega_1$. Let $b \in \mathbb{R}$ and σ be such that $g(\eta) = \tau^{L[b]}(\eta)$ for any $\eta < \omega_1$. Feed in $\langle \ulcorner \sigma \urcorner, b^\# \rangle$ for Player II in G. Then the response $\langle \ulcorner \tau \urcorner, a^\# \rangle =_{\mathrm{DEF}} f * \langle \ulcorner \sigma \urcorner, b^\# \rangle$ is a true sharp code for an increasing function, and for any $\alpha < \omega_1$, for any $n < \omega$,

$$\tau^{L[a]}(\omega\alpha + n) < \min(C \setminus (g(\omega\alpha + n) + 1)) \leq g(\omega\alpha + n + 1),$$

where the last inequality follows from the fact $g \in C^\uparrow$. Let h be given as in the definition of G. Then $h = f$. Since φ is a winning strategy for Player I, $\psi(h) \notin B$.

A symmetrical argument shows that if Player II has a winning strategy in G, then there is $X \in \mathbb{L}$ which is cofinal in ω_1 such that $\psi'' X^\uparrow \cap B = \emptyset$. $\square$

Definition 4.11. Let Q be a finite level ≤ 2 tree. We define

$$A \in \mu^Q$$

iff there is $C \in \mu_{\mathbb{L}}$ such that

$$[C]^{Q\uparrow} \subseteq A.$$

μ^Q is easily verified to be a countably complete filter concentrating on $[\omega_1]^{Q\uparrow}$. In particular, when $\mathrm{card}(Q) = 1$, μ^Q is the principal measure concentrating on $\{(\omega_1)_{(2,\emptyset)}\}$. Noticing the facts that $\mathrm{rep}(Q)$ has order type

$\omega_1 + 1$, and that $[f]^Q$ depends only on $\{f(v) : \|v\|_{<Q}$ is a limit ordinal$\}$. Theorem 4.10 implies that

$$\mu^Q \text{ is an } \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]\text{-measure.}$$

Let $j^Q = j^{\mu^Q}_{\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]}$ be the restricted ultrapower map of μ^Q on $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. Put $[f]_{\mu^Q} = [f]^{\mu^Q}_{\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]}$ for $f \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. Łoś' theorem reads: for any first order formula φ, for any $x \in \mathbb{R}$, for any $f_i \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$, with $\mathrm{ran}(f_i) \subseteq L_{\kappa_3^x}[T_2, x]$, $1 \le i \le n$,

$$j^Q(L_{\kappa_3^x}[T_2, x]) \models \varphi([f_1]_{\mu^Q}, \dots, [f_n]_{\mu^Q})$$

iff

$$\text{for } \mu^Q\text{-a.e. } \vec{\xi},\ L_{\kappa_3^x}[T_2, x] \models \varphi(f_1(\vec{\xi}), \dots, f_n(\vec{\xi})).$$

Lemma 4.12. *Assume $\boldsymbol{\Delta}^1_2$-determinacy. If P, W are finite level-1 trees, $A \subseteq [\omega_1]^{W\uparrow} \times \mathbb{R}$ is Π^1_3 (or Σ^1_3, Δ^1_3 resp.), then so are*

$$B = \{x : \textit{for } \mu^W\textit{-a.e. } \vec{\alpha}, (\vec{\alpha}, x) \in A\},$$
$$C = \{(\vec{\beta}, x) : \vec{\beta} \in j^P(A_x)\},$$

where $A_x = \{\vec{\alpha} : (\vec{\alpha}, x) \in A\}$.

Proof. $x \in B$ iff $\exists y\ \forall \vec{\alpha} \in [\omega_1]^{W\uparrow}(\vec{\alpha}$ are y-admissibles $\to (\vec{\alpha}, x) \in A)$. $x \notin B$ iff $\exists y\ \forall \vec{\alpha} \in [\omega_1]^{W\uparrow}(\vec{\alpha}$ are y-admissibles $\to (\vec{\alpha}, x) \notin A)$. The quantifier $\forall \vec{\alpha} \in [\omega_1]^{W\uparrow}$ does not increase the complexity due to Corollary 2.3. The complexity of C follows from that of B and Łoś. □

A purely descriptive set theoretic proof of Lemma 4.12 is given in [13, Lemma 4.40].

Lemma 4.13. *Assume $\boldsymbol{\Delta}^1_2$-determinacy. Suppose Q is a finite level ≤ 2 tree. If $A \subseteq \omega_1 \times \mathbb{R}$ is Π^1_3 (or Σ^1_3, Δ^1_3 resp.), then so is*

$$B = \{(\vec{\beta}, x) : \vec{\beta} \in [A_x]^{Q\uparrow}\},$$

where $A_x = \{\alpha : (\alpha, x) \in A\}$.

Proof. Put A'_x =the set of limit points of A_x. By Lemma 3.18, $\vec{\beta} \in [A_x]^{Q\uparrow}$ iff $\vec{\beta}$ respects Q and for any $q \in {}^1Q$, ${}^1\beta_q \in A'_x$, for any $q \in \mathrm{dom}({}^2Q)$, ${}^2\beta_q \in j^{Q_{\mathrm{tree}(q)}}(A'_x)$.

Now apply Lemma 4.12 and Lemma 3.19. □

Lemma 4.14. *Assume $\mathbf{\Delta}^1_2$-determinacy. Suppose Q is a finite level ≤ 2 tree. If $A \subseteq [\omega_1]^{Q\uparrow} \times \mathbb{R}$ is Π^1_3 (or Σ^1_3, Δ^1_3 resp.), then so is*

$$B = \{x : \textit{for } \mu^Q\textit{-a.e. } \vec{\beta}, (\vec{\beta}, x) \in A\},$$

Proof. Let $C = \{(y, \alpha) : \alpha < \omega_1 \wedge \alpha \text{ is } y\text{-admissible}\}$. C is Δ^1_3. Then $x \in B$ iff $\exists y\, \forall \vec{\beta}\, (\vec{\beta} \in [C_y]^{Q\uparrow} \to (\vec{\beta}, x) \in A)$. $x \notin B$ iff $\exists y\, \forall \vec{\beta}\, (\vec{\beta} \in [C_y]^{Q\uparrow} \to (\vec{\beta}, x) \notin A)$. Use Lemma 4.13. □

Corollary 4.15. *Assume $\mathbf{\Delta}^1_2$-determinacy. Suppose Q is a finite level ≤ 2 tree. Then $j^Q(\alpha) < \boldsymbol{\delta}^1_3$ for any $\alpha < \boldsymbol{\delta}^1_3$. $j^Q(T_2) \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$.*

Proof. By Lemma 4.14, for any $\alpha < \boldsymbol{\delta}^1_3$, $j^Q(\alpha)$ is the length of a $\mathbf{\Delta}^1_3$ prewellordering on $\mathbb{R}$. $j^Q(T_2) \in L_{\kappa_3^{M_1^\#}}[T_2, M_1^\#]$ by Corollary 2.15. □

Corollary 4.15 is the effective version of [46, Corollary 3.9]. Actually, $j^Q(T_2)$ is Δ^1_3 in the sharp codes, a fact to be shown in Section 4.5.

For Q a finite level ≤ 2 tree, by Corollary 4.15, $\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)] = \text{Ult}(\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2], \mu^Q)$.

If Q is a subtree of Q', both finite, then $\mu^{Q'}$ projects to μ^Q via the map that sends $(\beta_q)_{(d,q)\in\text{dom}(Q')}$to $({}^d\beta_q)_{(d,q)\in\text{dom}(Q)}$. Let

$$j^{Q,Q'} : \text{Ult}(\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2], \mu^Q) \to \text{Ult}(\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2], \mu^{Q'})$$

be the induced factor map. If $\vec{Q} = (Q_n)_{n<\omega}$ is a level ≤ 2 tower, the associated $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$-measure tower $(\mu^{Q_n})_{n<\omega}$ is easily seen close to $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$.

If $(P, \vec{p}) = (P, (p_i)_{i<m})$ is a potential partial level ≤ 1 tower, let $f \in B^{(P,\vec{p})\uparrow}$ iff $f : [\omega_1]^{P\uparrow} \to B$ is a function and

1. if $(P, \vec{p})$ is of continuous type, then the signature of f is $(p_i)_{i<m}$, f is essentially continuous;
2. if $(P, \vec{p})$ is of discontinuous type, then the signature of f is $(p_i)_{i<m-1}$, f is essentially discontinuous, f has uniform cofinality $\text{ucf}(P, \vec{p})$.

Let $\beta \in [B]^{(P,\vec{p})\uparrow}$ iff $\beta = [f]_{\mu^P}$ for some $f \in B^{(P,\vec{p})\uparrow}$.

$$\mu^{(P,\vec{p})}$$

is the $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$-measure where $A \in \mu^{(P,\vec{p})}$ iff there is $E \in \mu_{\mathbb{L}}$ such that $[E]^{(P,\vec{p})\uparrow} \subseteq A$. β *respects* $(P, \vec{p})$ iff $\beta \in [\omega_1]^{(P,\vec{p})\uparrow}$. Let $j^{(P,\vec{p})} = j^{\mu^{(P,\vec{p})}}_{\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]}$ be the induced $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$-ultrapower map. Let $\text{seed}^{(P,\vec{p})}$ be represented modulo $\mu^{(P,\vec{p})}$ by the identity map.

If $(d,\mathbf{q}) \in \operatorname{desc}(Q)$, let $\mu^{(d,\mathbf{q})} = \mu_{\mathbb{L}}$ if $d = 1$; $\mu^{(d,\mathbf{q})} = \mu^{(P,\vec{p})}$ if $d = 2$, $\mathbf{q} = (q, P, \vec{p})$. Thus, μ^Q projects to $\mu^{(d,\mathbf{q})}$ via the map $\vec{\beta} \mapsto {}^d\beta_{\mathbf{q}}$, i.e., $A \in \mu^{(d,\mathbf{q})}$ iff $\{\vec{\beta} : {}^d\beta_{\mathbf{q}} \in A\} \in \mu^Q$. (Recall the definition of ${}^d\beta_{\mathbf{q}}$ from $\vec{\beta}$ in Section 3.3.) Let

$$(d,\mathbf{q})^Q$$

be the induced factor map, so that $j^Q = (d,\mathbf{q})^Q \circ j_{\mu_{\mathbb{L}}}$ if $d = 1$, $j^Q = (d,\mathbf{q})^Q \circ j^{(P,\vec{p})}$ if $d = 2$. Let

$$\operatorname{seed}^Q_{(d,\mathbf{q})}$$

be represented modulo μ^Q by the map $\vec{\beta} \mapsto {}^d\beta_{\mathbf{q}}$.

The homogeneity property of the Martin-Solovay tree on a $\mathbf{\Pi}^1_2$ set (cf. [26]) translates to our context:

Theorem 4.16. *Assume $\mathbf{\Delta}^1_2$-determinacy. Let $\vec{Q} = (Q_n)_{n<\omega}$ be an infinite level-2 tower. Let $Q_\omega = \cup_{n<\omega} Q_n$. The following are equivalent.*

1. *Q_ω is Π^1_2-wellfounded.*
2. *$<^{Q_\omega}$ is a wellordering.*
3. *There is $\vec{\beta} = (\beta_t)_{t\in\operatorname{dom}(Q_\omega)}$ which respects Q_ω.*
4. *$(\mu^{Q_n})_{n<\omega}$ is $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$-countably complete.*
5. *The direct limit of $(j^{Q_m,Q_n})_{m<n<\omega}$ is wellfounded.*

Proof. 1 ⇔ 2: By Proposition 3.9.

2 ⇒ 4: Suppose $<^{Q_\omega}$ is a wellordering. Let $(A_n)_{n<\omega}$ be such that $A_n \in \mu^{Q_n} \cap \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. Let $x \in \mathbb{R}$ and $C \in L[x]$ be a club in ω_1 such that $[C]^{Q_n\uparrow} \subseteq A_n$ for all n. Let $f : \operatorname{dom}(<^{Q_\omega}) \to C$ be given by

$$f(\vec{\alpha} \oplus_{Q_\omega} t) = \text{ the } \|\vec{\alpha} \oplus_{Q_\omega} t\|_{<^{Q_\omega}}\text{-th element of } C.$$

Then $f \in L[x, Q_\omega]$ and is order preserving. Let $\beta_n = [f \restriction \operatorname{rep}(Q_n)]^{Q_n}$. Then for all n, $(\beta_1, \dots, \beta_n) \in A_n$.

4 ⇒ 3: This follows from the fact that μ^{Q_n} concentrates on tuples that respect Q_n.

3 ⇒ 1: If $x \in [\operatorname{dom}(Q_\omega)]$, then $j^{Q_\omega(x|k),Q_\omega(x|l)}(\beta_{x|k}) > \beta_{x|l}$ for all $k < l < \omega$. This means the direct limit of $j^{Q_\omega(x|k),Q_\omega(x|l)}$ is illfounded. Hence $Q_\omega(x)$ is not Π^1_1-wellfounded by Theorem 3.3.

4 ⇔ 5: By Proposition 2.17. □

Definition 4.17. Q^0, Q^1, Q^{20}, Q^{21} denote the following typical level ≤ 2 trees of cardinalities at most 2:

- ${}^1Q^0 = \emptyset$, ${}^1Q^1 = \{(0)\}$, $\mathrm{dom}({}^2Q^0) = \mathrm{dom}({}^2Q^1) = \{\emptyset\}$.
- For $d \in \{0,1\}$, ${}^1Q^{2d} = \emptyset$, $\mathrm{dom}({}^2Q^{2d}) = \{\emptyset, ((0))\}$, ${}^2Q^{2d}((0))$ is of degree d.

μ^{Q^0} is a principle measure. μ^{Q^1} is essentially $\mu_{\mathbb{L}}$. $\mu^{Q^{20}}$ and $\mu^{Q^{21}}$ are essentially refinements of the $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$-club filter on u_2, the former concentrates on ordinals of $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$-cofinality ω, the latter of $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$-cofinality ω_1.

4.3 The tree S_3

A *partial level* ≤ 2 *tree* is a pair $(Q,(d,q,P))$ such that Q is a finite level ≤ 2 tree, and one of the following holds:

1. $(d,q,P) = (0,-1,\emptyset)$, or
2. $d = 1$, $q \notin {}^1Q$, ${}^1Q \cup \{q\}$ is a level-1 tree, $P = \emptyset$, or
3. $d = 2$, $q \notin \mathrm{dom}({}^2Q)$, $\mathrm{dom}({}^2Q) \cup \{q\}$ is tree of level-1 trees, P is the completion of ${}^2Q(q^-)$. (In particular, ${}^2Q(q^-)$ must have degree 1.)

The *degree* of $(Q,(d,q,P))$ is d. We put $\mathrm{dom}(Q,(d,q,P)) = \mathrm{dom}(Q) \cup \{(d,q)\}$. The *cardinality* of $(Q,(d,q,P))$ is $\mathrm{card}(Q,(d,q,P)) = \mathrm{card}(Q)+1$. The *uniform cofinality* of a partial level ≤ 2 tree $(Q,(d,q,P))$ is

$$\mathrm{ucf}(Q,(d,q,P)),$$

defined as follows.

1. $\mathrm{ucf}(Q,(d,q,P)) = (0,-1)$ if $d = 0$;
2. $\mathrm{ucf}(Q,(d,q,P)) = (1,q^-)$ if $d = 1$, $\mathrm{lh}(q) > 1$;
3. $\mathrm{ucf}(Q,(d,q,P)) = (2,(\emptyset,\emptyset,\emptyset))$ if $d = 1$, $\mathrm{lh}(q) = 1$;
4. $\mathrm{ucf}(Q,(d,q,P)) = (2,(q',P,\vec{p}))$ if $d = 2$, ${}^2Q[q^-] = (P^-,\vec{p})$, and q' is the $<_{BK}$-least element of ${}^2Q\{q,+\}$.

So $\mathrm{ucf}(Q,(d,q,P))$ is either $(0,-1)$ or a regular extended Q-description. The *cofinality* of $(Q,\overrightarrow{(d,q,P)})$ is

$$\mathrm{cf}(Q,\overrightarrow{(d,q,P)}) = \begin{cases} 0 & \text{if } d = 0, \\ 1 & \text{if } d = 1 \text{ and } q = \min(\prec^{{}^1Q\cup\{q\}}), \\ 2 & \text{otherwise.} \end{cases}$$

A tuple $\vec{\beta} = ({}^{e}\beta_t)_{(e,t)\in\mathrm{dom}(Q,(d,q,P))}$ *respects* $(Q,(d,q,P))$ iff $\vec{\beta}\restriction\mathrm{dom}(Q)$ respects Q and ${}^{d}\beta_q < \omega$ if $d = 0$, ${}^{d}\beta_q < {}^{d}\beta_{\mathbf{t}}$ if $d > 0$ and $\mathrm{ucf}(Q,(d,q,P)) = (d,\mathbf{t})$. A partial level ≤ 2 tree of degree 0 has no completion. A *completion* of a partial level ≤ 2 tree $(Q,(d,q,P))$ of degree ≥ 1 is a level ≤ 2 tree Q^* such that $\mathrm{dom}(Q^*) = \mathrm{dom}(Q,(d,q,P))$, ${}^{2}Q^*\restriction\mathrm{dom}({}^{2}Q) = {}^{2}Q$, and either $d = 1$ or $d = 2 \wedge {}^{2}Q_{\mathrm{tree}}(t) = P$. For a level ≤ 2 tree Q', $(Q,(d,q,P))$ is a *partial subtree* of Q' iff a completion of $(Q,(d,q,P))$ is a subtree of Q'.

A *partial level* ≤ 2 *tower of discontinuous type* is a nonempty finite sequence $(Q_i,(d_i,q_i,P_i))_{1\leq i\leq k}$ such that $\mathrm{card}(Q_1) = 1$, each $(Q_i,(d_i,q_i,P_i))$ is a partial level ≤ 2 tree, and each Q_{i+1} is a completion of $(Q_i,(d_i,q_i,P_i))$. Its *signature* is $(d_i,q_i)_{1\leq i<k}$. Its *uniform cofinality* is $\mathrm{ucf}(Q_k,(d_k,q_k,P_k))$. A *partial level* ≤ 2 *tower of continuous type* is $(Q_i,(d_i,q_i,P_i))_{1\leq i<k}{}^\frown(Q_*)$ such that either $k = 0 \wedge Q_*$ is the level ≤ 2 tree of cardinality 1 or $(Q_i,(d_i,q_i,P_i))_{1\leq i<k}$ is a partial level ≤ 2 tower of discontinuous type $\wedge Q_*$ is a completion of $(Q_{k-1},(d_{k-1},q_{k-1},P_{k-1}))$. Its *signature* is $(d_i,q_i)_{1\leq i<k}$. If $k > 0$, its *uniform cofinality* is $(1,q_{k-1})$ if $d_{k-1} = 1$, $(2,(q_{k-1},P,\vec{p}))$ if $d_{k-1} = 2$ and ${}^{2}Q[q_{k-1}] = (P,\vec{p})$. For notational convenience, the information of a partial level ≤ 2 tower is compressed into a potential partial level ≤ 2 tower. A *potential partial level* ≤ 2 *tower* is $(Q_*,\overrightarrow{(d,q,P)}) = (Q_*,(d_i,q_i,P_i)_{1\leq i\leq\mathrm{lh}(\vec{q})})$ such that for some level ≤ 2 tower $\vec{Q} = (Q_i)_{1\leq i\leq k}$, either $Q_* = Q_k \wedge (\vec{Q},\overrightarrow{(d,q,P)})$ is a partial level ≤ 2 tower of discontinuous type or $(\vec{Q},\overrightarrow{(d,q,P)})^\frown(Q_*)$ is a partial level ≤ 2 tower of continuous type. The signature, (dis-)continuity type, uniform cofinality of $(Q_*,\overrightarrow{(d,q,P)})$ are defined according to the partial level ≤ 2 tree generating $(Q_*,\overrightarrow{(d,q,P)})$.

$$\mathrm{ucf}(Q_*,\overrightarrow{(d,q,P)})$$

denotes the uniform cofinality of $(Q_*,\overrightarrow{(d,q,P)})$. Clearly, a potential partial level ≤ 2 tower $(Q_*,\overrightarrow{(d,q,P)})$ is of continuous type iff $\mathrm{card}(Q_*) = \mathrm{lh}(\vec{q})$, of discontinuous type iff $\mathrm{card}(Q_*) = \mathrm{lh}(\vec{q}) - 1$.

Definition 4.18. A *level-3 tree of uniform cofinality*, or *level-3 tree*, is a function

$$R$$

such that $\emptyset \notin \mathrm{dom}(R)$, $\mathrm{dom}(R)\cup\{\emptyset\}$ is tree of level-1 trees and for any $r \in \mathrm{dom}(R)$, $(R(r\restriction l))_{1\leq l\leq\mathrm{lh}(r)}$ is a partial level ≤ 2 tower of discontinuous type. If $R(r) = (Q_r,(d_r,q_r,P_r))$, we denote $R_{\mathrm{tree}}(r) = Q_r$, $R_{\mathrm{node}}(r) = (d_r,q_r)$, $R[r] = (Q_r,(d_{r\restriction l},q_{r\restriction l},P_{r\restriction l})_{1\leq l\leq\mathrm{lh}(r)})$. $R[r]$ is a potential partial level ≤ 2 tower of discontinuous type. If Q is a completion of $R(r)$, put $R[r,Q] =$

$(Q, (d_{r\restriction l}, q_{r\restriction l}, P_{r\restriction l})_{1\le l\le \mathrm{lh}(r)})$, which is a potential partial level ≤ 2 tower of continuous type. For $r \in \mathrm{dom}(R) \cup \{\emptyset\}$, put $R\{r\} = \{a \in \omega^{<\omega} : r^\frown(a) \in \mathrm{dom}(R)\}$, which is a level-1 tree.

The *cardinality* of R is $\mathrm{card}(R) = \mathrm{card}(\mathrm{dom}(R))$. R is said to be *regular* iff $((1)) \notin \mathrm{dom}(R)$. In other words, when $R \neq \emptyset$, $((0))$ is the $<_{BK}$-maximum of $\mathrm{dom}(R)$.

Suppose R is a level-3 tree. Let $\mathrm{dom}^*(R) = \mathrm{dom}(R) \cup \{r^\frown(-1) : r \in \mathrm{dom}(R)\}$. For $r \in \mathrm{dom}(R)$, put $R\{r,-\} = \{r^{-\frown}(-1)\} \cup \{r^{-\frown}(a) : R_{\mathrm{tree}}(r^{-\frown}(a)) = R_{\mathrm{tree}}(r), a <_{BK} r(\mathrm{lh}(r)-1)\}$, $R\{r,-\} = \{r^-\} \cup \{r^{-\frown}(a) : R_{\mathrm{tree}}(r^{-\frown}(a)) = R_{\mathrm{tree}}(r), a >_{BK} r(\mathrm{lh}(r)-1)\}$,

If $\vec{\beta} = ({}^d\beta_q)_{(d,q)\in N}$ is a tuple indexed by N, $r \in \mathrm{dom}^*(R)$, $\mathrm{lh}(r) = k$, either $k = 1$ or $\mathrm{dom}(R(r^-)) \subseteq N$, we put

$$\vec{\beta} \oplus_R r = (r(0), {}^d\beta_{q_1}, r(1), \ldots, {}^d\beta_{q_{k-1}}, r(k-1)),$$

where $(d_i, q_i) = R_{\mathrm{node}}(r\restriction i)$. The *ordinal representation* of R is the set

$$\begin{aligned}\mathrm{rep}(R) = &\{\vec{\beta} \oplus_R r : r \in \mathrm{dom}(R), \vec{\beta} \text{ respects } R_{\mathrm{tree}}(r)\}\\ &\cup \{\vec{\beta} \oplus_R r^\frown(-1) : r \in \mathrm{dom}(R), \vec{\beta} \text{ respects } R(r)\}.\end{aligned}$$

$\mathrm{rep}(R)$ is endowed with the $<_{BK}$ ordering

$$<^R = <_{BK}\restriction \mathrm{rep}(R).$$

R is Π^1_3-*wellfounded* iff

1. $\forall r \in \mathrm{dom}(R) \cup \{\emptyset\}$ $R\{r\}$ is Π^1_1-wellfounded, and
2. $\forall z \in [\mathrm{dom}(R)]$ $R(z) =_{\mathrm{DEF}} \cup_{n<\omega}(R_{\mathrm{tree}}(z\restriction n))_{1\le n<\omega}$ is not Π^1_2-wellfounded.

For level-3 trees R and R', R is a *subtree* of R' iff R is a subfunction of R'. A *finite level-3 tower* is a sequence $(R_i)_{i\le n}$ such that $n < \omega$, each R_i is a regular level-2 tree, $\mathrm{card}(R_i) = i+1$ and $i < j \to R_i$ is a subtree of R_j. $\vec{R}$ is *regular* iff each R_i is regular. An *infinite level-3 tower* is a sequence $\vec{R} = (R_n)_{n<\omega}$ such that for each n, $(R_i)_{i\le n}$ is a finite level-3 tower. Π^1_3-wellfoundedness of a level-3 tower is a $\boldsymbol{\Pi}^1_3$ property in the real coding the tower. In particular, every finite level-3 tree is Π^1_3-wellfounded. Similarly to Proposition 3.9, we have

Proposition 4.19. *Assume $\boldsymbol{\Delta}^1_2$-determinacy. Suppose R is a level-3 tree. Then R is Π^1_3-wellfounded iff $<^R$ is a wellordering.*

Associated to a $\mathbf{\Pi}^1_3$ set A we can assign a level-3 system $(R_s)_{s\in\omega^{<\omega}}$ so that $x \in A$ iff the infinite level-3 tree $R_x =_{\text{DEF}} \cup_{n<\omega} R_{x\restriction n}$ is Π^1_3-wellfounded. If A is lightface Π^1_3, then $(R_s)_{s\in\omega^{<\omega}}$ can be picked effective.

Suppose $F \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$ is a function on $\text{rep}(R)$, $r \in \text{dom}(R)$. Then F_r is a function on $\omega_1^{R_{\text{tree}}(r)\uparrow}$ that sends $\vec{\beta}$ to $F(\vec{\beta}\oplus_R r)$. F represents a $\text{card}(R)$-tuple of ordinals

$$[F]^R = ([F]^R_r)_{r\in\text{dom}(R)}$$

where $[F]^R_r = [F_r]_{\mu^{R_{\text{tree}}(r)}}$ for $r \in \text{dom}(R)$. If $B \subseteq \boldsymbol{\delta}^1_3$, put

$$F \in B^{R\uparrow}$$

iff $F \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$ and F is an order-preserving continuous function from $\text{rep}(R)$ to B (with respect to $<^R$ and $<$). Let

$$[B]^{R\uparrow} = \{[F]^R : F \in B^{R\uparrow}\}.$$

A tuple of ordinals $\vec{\gamma} = (\gamma_r)_{r\in\text{dom}(R)}$ is said to *respect* R iff $\vec{\gamma} \in [\boldsymbol{\delta}^1_3]^{R\uparrow}$. $\vec{\gamma}$ is said to *weakly respect* R iff for any $t, t' \in \text{dom}(R)$, if t is a proper initial segment of t', then $j^{R_{\text{tree}}(t),R_{\text{tree}}(t')}(\gamma_t) > \gamma_{t'}$. By virtue of the order $<^R$, if $\vec{\gamma}$ respects R, then $\vec{\gamma}$ weakly respects R and whenever $R_{\text{tree}}(t^\frown(p)) = R_{\text{tree}}(t^\frown(q))$ and $p < q$, then $\gamma_{t^\frown(p)} < \gamma_{t^\frown(q)}$.

The trees S_3^- and S_3 are defined in [26]. They both project to the universal Π^1_3 set. In our language, they take the following form.

Definition 4.20. Assume $\mathbf{\Delta}^1_2$-determinacy.

1. S_3^- is the tree on $V_\omega \times \boldsymbol{\delta}^1_3$ such that $(\emptyset,\emptyset) \in S_3^-$ and

$$(\vec{R},\vec{\alpha}) = ((R_i)_{i\leq n}, (\alpha_i)_{i\leq n}) \in S_3^-$$

iff $\vec{R}$ is a finite regular level-3 tower and letting $r_i \in \text{dom}(R_{i+1}) \setminus \text{dom}(R_i)$, $\beta_{r_i} = \alpha_{i+1}$, then $(\beta_r)_{r\in\text{dom}(R_n)}$ respects R_n.

2. S_3 is the tree on $V_\omega \times \boldsymbol{\delta}^1_3$ such that $(\emptyset,\emptyset) \in S_3$ and

$$(\vec{R},\vec{\alpha}) = ((R_i)_{i\leq n}, (\alpha_i)_{i\leq n}) \in S_3$$

iff $\vec{R}$ is a finite regular level-3 tower and letting $r_i \in \text{dom}(R_{i+1}) \setminus \text{dom}(R_i)$, $\beta_{r_i} = \alpha_{i+1}$, then $(\beta_r)_{r\in\text{dom}(R_n)}$ weakly respects R_n.

By Theorem 4.16,

$$p[S_3^-] = p[S_3] = \{\vec{R} : \vec{R} \text{ is a } \Pi^1_3\text{-wellfounded level-3 tower}\}.$$

The (non-regular) scale associated to S_3 is Π^1_3. For $\xi < \boldsymbol{\delta}^1_3$, put $(\vec{R}, \vec{\alpha}) \in S_3 \restriction \xi$ iff $(\vec{R}, \vec{\alpha}) \in S_3$ and $(\vec{R}, \vec{\alpha}) \neq (\emptyset, \emptyset) \to \alpha_0 < \xi$.

The properties of a tuple respecting R is decided by the signature, approximation sequence and relative ordering of its entries, in a parallel way to the level-2 case. It is handled in [12]. We state the results in our language.

For level ≤ 2 trees Q, X, we say that $\pi : \mathrm{dom}(X) \to \mathrm{dom}(Q)$ *factors* (X, Q) iff putting $(d, {}^d\pi(x)) = \pi(d, x)$ for $(d, x) \in \mathrm{dom}(X)$,

1. ${}^1\pi$ factors $({}^1X, {}^1Q)$;
2. if $x \in \mathrm{dom}({}^2X)$ then ${}^2X(x) = {}^2Q({}^2\pi(x))$;
3. if $x, x' \in \mathrm{dom}({}^2X)$, then $x <_{BK} x' \to {}^2\pi(x) <_{BK} {}^2\pi(x')$, $x \subseteq x' \to {}^2\pi(x) \subseteq {}^2\pi(x')$.

For $d \in \{1, 2\}$, ${}^d\pi$ has this fixed meaning if π factors (Q, X). Extend the definition of ${}^2\pi$ on $\mathrm{dom}^*({}^2Q)$ and $\mathrm{desc}({}^2Q)$ is the following natural way: if $q^\frown(-1) \in \mathrm{dom}^*({}^2Q)$, define ${}^2\pi(q^\frown(-1)) = \pi(q)^\frown(-1)$; if $\mathbf{q} = (q, P, \vec{p}) \in \mathrm{desc}({}^2Q)$, define ${}^2\pi(\mathbf{q}) = ({}^2\pi(q), P, \vec{p})$. If $\vec{\beta} = ({}^d\beta_q)_{(d,q) \in \mathrm{dom}(Q)} \in [\omega_1]^{Q\uparrow}$, put $\vec{\beta}_\pi = ({}^d\beta_{\pi,x})_{(d,x) \in \mathrm{dom}(X)} \in [\omega_1]^{X\uparrow}$ where ${}^d\beta_{\pi,x} = {}^d\beta_{{}^d\pi(x)}$.

Definition 4.21. Suppose Q is a finite level ≤ 2 tree, $\overrightarrow{(d,q)} = ((d_i, q_i))_{1 \leq i < k}$ is a distinct enumeration of a subset of Q and such that $\{q_i : d_i = 2\} \cup \{\emptyset\}$ forms a tree on $\omega^{<\omega}$. Suppose $F : [\omega_1]^{Q\uparrow} \to \boldsymbol{\delta}^1_3$ is a function which lies is $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. The *signature* of F is $\overrightarrow{(d,q)}$ iff there is $E \in \mu_{\mathbb{L}}$ such that

1. for any $\vec{\beta}, \vec{\gamma} \in [E]^{Q\uparrow}$, if $({}^{d_0}\gamma_{q_0}, \ldots, {}^{d_{k-1}}\gamma_{q_{k-1}}) <_{BK} ({}^{d_0}\beta_{q_0}, \ldots, {}^{d_{k-1}}\beta_{q_{k-1}})$ then $f(\vec{\beta}) < f(\vec{\gamma})$;
2. for any $\vec{\beta}, \vec{\gamma} \in [E]^{Q\uparrow}$, if $({}^{d_0}\gamma_{q_0}, \ldots, {}^{d_{k-1}}\gamma_{q_{k-1}}) = ({}^{d_0}\beta_{q_0}, \ldots, {}^{d_{k-1}}\beta_{q_{k-1}})$ then $f(\vec{\beta}) = f(\vec{\gamma})$.

Clearly the signature of F exists and is unique. In particular, F is constant on a μ^Q-measure one set iff the signature of F is $\emptyset$.

Suppose the signature of F is $\overrightarrow{(d,q)} = ((d_i, q_i))_{1 \leq i < k}$. F is *essentially continuous* iff for μ^Q-a.e. $\vec{\beta}$, $F(\vec{\beta}) = \sup\{F(\vec{\gamma}) : \vec{\gamma} \in [\omega_1]^{Q\uparrow}, ({}^{d_0}\gamma_{q_0}, \ldots, {}^{d_{k-1}}\gamma_{q_{k-1}}) <_{BK} ({}^{d_0}\beta_{q_0}, \ldots, {}^{d_{k-1}}\beta_{q_{k-1}})\}$. Otherwise, F is *essentially discontinuous.* Put $[\omega_1]^{Q\uparrow(0,-1)} = [\omega_1]^{Q\uparrow} \times \omega$. For $(d, \mathbf{q}) \in \mathrm{desc}^*(Q)$ regular, put $[\omega_1]^{Q\uparrow(d,\mathbf{q})} = \{(\vec{\beta}, \gamma) : \vec{\beta} \in [\omega_1]^{Q\uparrow}, \gamma < {}^d\beta_{\mathbf{q}}\}$. For $(d, \mathbf{q})$ either $(0, -1)$ or in $\mathrm{desc}^*(Q)$ regular, say that the *uniform cofinality* of F is $\mathrm{ucf}(F) = (d, \mathbf{q})$ iff there is $G : [\omega_1]^{Q\uparrow(d,\mathbf{q})} \to \boldsymbol{\delta}^1_3$ such that $G \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$ and for any for μ^Q-a.e. $\vec{\beta}$, $F(\vec{\beta}) = \sup\{G(\vec{\beta}, \gamma) : (\vec{\beta}, \gamma) \in$

$[\omega_1]^{Q\uparrow(d,\mathbf{q})}\}$ and the function $\gamma \mapsto G(\vec{\beta}, \gamma)$ is order preserving. The *cofinality* of F is

$$\mathrm{cf}(F) = \begin{cases} 0 & \text{if } \mathrm{ucf}(F) = (0, -1), \\ 1 & \text{if } \mathrm{ucf}(F) = (1, q), q = \min(\prec^{{}^1 Q}), \\ 2 & \text{otherwise.} \end{cases}$$

Let $(X_i, (d_i, x_i, W_i))^\frown(X_k)$ be the partial level ≤ 2 tower of continuous type and let π factor (X_k, Q) such that $\pi(d_i, x_i) = (d_i, q_i)$ for each $1 \leq i < k$. The *potential partial level* ≤ 2 *tower* induced by F is

1. $(X_k, (d_i, x_i, W_i)_{1 \leq i < k})$, if F is essentially continuous;
2. $(X_k, (d_i, x_i, W_i)_{1 \leq i < k}{}^\frown(0, -1, \emptyset))$, if F is essentially discontinuous and has uniform cofinality $(0, -1)$;
3. $(X_k, (d_i, x_i, W_i)_{1 \leq i < k}{}^\frown(1, x^+, \emptyset))$, if F is essentially discontinuous and has uniform cofinality $(1, q_*)$, $(X_k, (1, x^+, \emptyset))$ is a partial level ≤ 2 tree, $\pi(1, (x^+)^-) = (1, q_*)$;
4. $(X_k, (d_i, x_i, W_i)_{1 \leq i < k}{}^\frown(2, x^+, P_*))$, if F is essentially discontinuous and has uniform cofinality $(2, \mathbf{q}_*)$, $\mathbf{q}_* = (q_*, P_*, \vec{p}_*)$, $(X_k, (2, x^+, P_*))$ is a partial level ≤ 2 tree, and either
    (a) $\mathbf{q}_* \in \mathrm{desc}({}^2 Q)$, $x^+ = (x^+)^{-\frown}(a)$, $\pi(2, (x^+)^{-\frown}(a^-)) = (2, q_*)$, or
    (b) $\mathbf{q}_* \notin \mathrm{desc}({}^2 Q)$, $\pi(2, (x^+)^-) = (2, q_*)$.

The *approximation sequence* of F is $(F_i)_{1 \leq i \leq k}$ where F_i is a function on $[\omega_1]^{X_i \uparrow}$, $F_i(\vec{\beta}) = \sup\{F(\vec{\gamma}) : \vec{\gamma} \in [\omega_1]^{Q\uparrow}, ({}^{d_1}\gamma_{q_1}, \dots, {}^{d_{i-1}}\gamma_{q_{i-1}}) = ({}^{d_1}\beta_{x_1}, \dots, {}^{d_{i-1}}\beta_{x_{i-1}})\}$ for $1 \leq i \leq k$.

The existence and uniqueness of the uniform cofinality of F will be proved in Section 4.5. In particular, if R is a level-3 tree, $H \in (\boldsymbol{\delta}^1_3)^{R\uparrow}$, then for any $r \in \mathrm{dom}(R)$, H_r has signature $(R_{\mathrm{node}}(r \restriction i))_{1 \leq i < \mathrm{lh}(r)}$, is essentially discontinuous, has uniform cofinality $\mathrm{ucf}(R(r))$ and cofinality $\mathrm{cf}(R(r))$, induces the potential partial level ≤ 2 tower $R[r]$, and $(H_{r \restriction i})_{1 \leq i \leq \mathrm{lh}(r)}$ is the approximation sequence of H_r. Again, all the relevant properties of F depends only on the value of F on a μ^Q-measure one set. We will thus be free to say the signature, etc. of F when F is defined on a μ^Q-measure one set.

Definition 4.22. Suppose $\omega_1 \leq \gamma < \boldsymbol{\delta}^1_3$ is a limit ordinal. Suppose Q is a finite level ≤ 2 tree, $\gamma = [F]_{\mu^Q}$, the signature of F is $((d_i, q_i))_{1 \leq i < k}$, the approximation sequence of F is $(F_i)_{1 \leq i \leq k}$. Then the *Q-signature* of β is $((d_i, q_i))_{1 \leq i < k}$, the *$Q$-approximation sequence* of γ is $([F_i]_{\mu^Q})_{1 \leq i \leq k}$, γ is

Q-essentially continuous iff F is essentially continuous. The *Q-uniform cofinality* of γ is ω if F has uniform cofinality $(0,-1)$, $\mathrm{seed}^Q_{(d,\mathbf{q})}$ if f has uniform cofinality $(d,\mathbf{q}) \in \mathrm{desc}^*(Q)$. The *Q-potential partial level* ≤ 2 *tower induced by* γ is the potential partial level ≤ 2 tower induced by F.

In Section 4.5, we will show that all the relevant properties in Definition 4.22 are independent of the choice of F (but depends on Q of course). We will also show that the Q-uniform cofinality of γ is exactly $\mathrm{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]}(\gamma)$, and $\mathrm{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]}(\gamma) = u_{\mathrm{cf}(F)}$, where we set $u_0 = \omega$.

Definition 4.23. We fix the notations for all the level-3 trees of cardinality 1. For $d \in \{0,1,2\}$, $\mathrm{dom}(R^d) = \{((0))\}$ and $R^d((0))$ is of degree d.

4.4 Level-2 description analysis

If Q is a level-2 tree, $\mathbf{q} = (q, P, \vec{p}) \in \mathrm{desc}(Q)$, $\mathrm{lh}(q) = k$, $\vec{p} = (p_i)_{i<\mathrm{lh}(\vec{p})}$, σ is a function whose domain contains P, we put

$$\sigma \oplus \mathbf{q} = \sigma \oplus_Q q = (\sigma(p_0), q(0), \ldots, \sigma(p_{k-1}), q(k-1)).$$

Definition 4.24. Suppose W is a finite level-1 tree and suppose Q is a level ≤ 2 tree. A *(Q,W)-description* is of the form

$$\mathbf{D} = (d, (\mathbf{q}, \sigma))$$

such that either

1. $d = 1$, $\mathbf{q} \in {}^1Q$, $\sigma = \emptyset$, or
2. $d = 2$, $\mathbf{q} = (q, P, \vec{p}) \in \mathrm{desc}({}^2Q)$, σ factors (P, W).

$\mathrm{desc}(Q,W)$ is the set of (Q,W)-descriptions. A $(Q,*)$-description is a (Q,W')-description for some finite level-1 tree W'. $\mathrm{desc}(Q,*)$ is the set of $(Q,*)$-descriptions. We sometimes abbreviate $(d,\mathbf{q},\sigma)$ for $(d,(\mathbf{q},\sigma)) \in \mathrm{desc}(Q,W)$ without confusion.

Suppose now $\mathbf{D} = (d, \mathbf{q}, \sigma)$ and if $d = 2$, then $\mathbf{q} = (q, P, \vec{p})$, $\vec{p} = (p_i)_{i<\mathrm{lh}(\vec{p})}$, $\mathrm{lh}(q) = k$. The *degree* of $\mathbf{D}$ is d. The *level-1 signature* of $\mathbf{D}$ is

$$\mathrm{sign}_1(\mathbf{D}) = \begin{cases} \emptyset & \text{if } d = 1, \\ (\sigma(p_i))_{i<k} & \text{if } d = 2. \end{cases}$$

$\mathbf{D}$ is *of level-1 continuous type* iff $d = 2$ and $\mathbf{q}$ is of continuous type; otherwise, $\mathbf{D}$ is *of level-1 discontinuous type*. The *level-1 uniform cofinality* of $\mathbf{D}$ is

$$\mathrm{ucf}_1(\mathbf{D}) = \begin{cases} -1 & \text{if } d = 1 \vee (d = 2 \wedge \mathrm{ucf}(P,\vec{p}) = -1), \\ \sigma(\mathrm{ucf}(P,\vec{p})) & \text{if } d = 2 \wedge \mathrm{ucf}(P,\vec{p}) \neq -1. \end{cases}$$

The *level-2 signature* of $\mathbf{D}$ is

$$\mathrm{sign}_2(\mathbf{D}) = \begin{cases} ((1, \mathbf{q})) & \text{if } d = 1, \\ ((2, q \restriction i))_{1 \le i \le k-1} & \text{if } d = 2, q \text{ of continuous type,} \\ ((2, q \restriction i))_{1 \le i \le k} & \text{if } d = 2, q \text{ of discontinuous type.} \end{cases}$$

$\mathbf{D}$ is *of level-2 W-continuous type* iff $d = 2$ and if $\mathrm{ucf}(P, \vec{p}) \neq -1 \wedge \sigma(\mathrm{ucf}(P, \vec{p})) \neq \min(\prec^W)$, then $\mathrm{pred}_{\prec^W}(\sigma(\mathrm{ucf}(P, \vec{p}))) \in \mathrm{ran}(\sigma)$. Otherwise, $\mathbf{D}$ is *of level-2 W-discontinuous type*. The *level-2 W-uniform cofinality* of $\mathbf{D}$ is

$$\mathrm{ucf}_2^W(\mathbf{D})$$

defined as follows. If $d = 1$, then $\mathrm{ucf}_2^W(\mathbf{D}) = (1, \mathbf{q})$. If $d = 2$, q is of continuous type,

1. if $\mathbf{D}$ is of level-2 W-continuous type, then $\mathrm{ucf}_2^W(\mathbf{D}) = (2, (q^-, P \setminus \{p_{k-1}\}, \vec{p}))$;
2. if $\mathbf{D}$ is of level-2 W-discontinuous type, then $\mathrm{ucf}_2^W(\mathbf{D}) = (2, (q^-, P, \vec{p}))$.

If $d = 2$, q is of discontinuous type,

1. if $\mathbf{D}$ is of level-2 W-continuous type, then $\mathrm{ucf}_2^W(\mathbf{D}) = (2, \mathbf{q})$;
2. if $\mathbf{D}$ is of level-2 W-discontinuous type, then $\mathrm{ucf}_2^W(\mathbf{D}) = (2, (q, P \cup \{p_k\}, \vec{p})$.

The *constant* $(Q, *)$*-description* is $(2, (\emptyset, \emptyset, \emptyset), \sigma_0)$ where σ_0 is the unique that factors $(\emptyset, *)$, i.e., $\sigma_0(\emptyset) = \emptyset$.

Note that if $\mathbf{D} \in \mathrm{desc}(Q, W)$ and W is a subtree of W', then $\mathbf{D} \in \mathrm{desc}(Q, W')$, but $\mathrm{ucf}_2^W(\mathbf{D})$ could be different from $\mathrm{ucf}_2^{W'}(\mathbf{D})$. If Q is finite, there are in total

$$\mathrm{card}({}^1Q) + \sum_{q \in \mathrm{dom}({}^2Q)} \binom{\mathrm{card}(W)}{\mathrm{lh}(q)} + \sum_{{}^2Q(q) \text{ of degree } 1} \binom{\mathrm{card}(W)}{\mathrm{lh}(q) + 1}$$

many (Q, W)-descriptions. We shall establish an exact correspondence between $\mathrm{desc}(Q, W)$ and uniform indiscernibles $\le j^Q \circ j^W(\omega_1)$.

Suppose $\mathbf{D} = (d, \mathbf{q}, \sigma) \in \mathrm{desc}(Q, W)$, and if $d = 2$, then $\mathbf{q} = (q, P, \vec{p})$, $\vec{p} = (p_i)_{i < \mathrm{lh}(\vec{p})}$, $\mathrm{lh}(q) = k$. For $g \in \omega_1^{Q\uparrow}$, let

$$g_{\mathbf{D}}^W : [\omega_1]^{W\uparrow} \to \omega_1 + 1$$

be the function as follows: if $d = 1$, then $g^W_{\mathbf{D}}(\vec{\alpha}) = {}^1[g]^Q_{\mathbf{q}}$ when $\min(\vec{\alpha}) > {}^1[g]^Q_{\mathbf{q}}$, $g^W_{\mathbf{D}}(\vec{\alpha}) = \|(1,(q))\|_{<Q}$ otherwise[1]; if $d = 2$, then $g^W_{\mathbf{D}}(\vec{\alpha}) = {}^2g_q(\vec{\alpha}_\sigma)$ (Recall the definition of $\vec{\alpha}_\sigma$ in Section 3.2). In particular, if $\mathbf{D}$ is the constant $(Q,*)$-description, then $g^W_{\mathbf{D}}$ is the constant function with value ω_1. Clearly, the signature of $g^W_{\mathbf{D}}$ is $\mathrm{sign}_1(\mathbf{D})$; $\mathbf{D}$ is of level-1 continuous type iff $g^W_{\mathbf{D}}$ is essentially continuous; the uniform cofinality of $g^W_{\mathbf{D}}$ is $\mathrm{ucf}_1(\mathbf{D})$. Suppose additionally that Q is finite. Let

$$\mathrm{id}^{Q,W}_{\mathbf{D}}$$

be the function $[g]^Q \mapsto [g^W_{\mathbf{D}}]_{\mu^W}$, or equivalently, $\vec{\beta} \mapsto \sigma^W({}^d\beta_{\mathbf{q}})$, where $\emptyset^W$ is interpreted as j^W. Clearly, the signature of $\mathrm{id}^{Q,W}_{\mathbf{D}}$ is $\mathrm{sign}^W_2(\mathbf{D})$; $\mathrm{id}^{Q,W}_{\mathbf{D}}$ is essentially continuous iff $\mathbf{D}$ is of level-2 W-continuous type; the uniform cofinality of $\mathrm{id}^{Q,W}_{\mathbf{D}}$ is $\mathrm{ucf}^W_2(\mathbf{D})$. Let

$$\mathrm{seed}^{Q,W}_{\mathbf{D}} \in \mathbb{L}_{\boldsymbol{\delta}^1_3}(j^Q \circ j^W[T_2])$$

be the element represented modulo μ^Q by $\mathrm{id}^{Q,W}_{\mathbf{D}}$. In particular, if $d = 1$ then $\mathrm{seed}^{Q,W}_{\mathbf{D}} = \mathrm{seed}^Q_{(1,\mathbf{q})}$; if $d = 2$, $P = W$ and $\sigma = \mathrm{id}_P$, then $\mathrm{seed}^{Q,W}_{\mathbf{D}} = \mathrm{seed}^Q_{(2,\mathbf{q})}$. By Łoś, if $\mathbf{D}$ is not the constant $(Q,*)$-description, for any $A \in \mu_{\mathbb{L}}$, $\mathrm{seed}^{Q,W}_{\mathbf{D}} \in j^Q \circ j^W(A)$. Thus, we can define

$$\mathbf{D}^{Q,W} : \mathbb{L}_{\boldsymbol{\delta}^1_3}[j_{\mu_{\mathbb{L}}}(T_2)] \to \mathbb{L}_{\boldsymbol{\delta}^1_3}(j^Q \circ j^W[T_2])$$

by sending $j_{\mu_{\mathbb{L}}}(h)(\omega_1)$ to $j^Q \circ j^W(h)(\mathrm{seed}^{Q,W}_{\mathbf{D}})$.

If Q is a level-2 tree, $\mathbf{q} = (q, P, \vec{p}) \in \mathrm{desc}(Q)$, $l \leq \mathrm{lh}(q)$, define

$$\mathbf{q} \restriction l = (q \restriction l, \{p_i : i < l\}, (p_i)_{i<l}).$$

which is a Q-description. If $\mathbf{D} = (2, \mathbf{q}, \sigma) \in \mathrm{desc}(Q,*)$, $\mathbf{q} = (q, P, \vec{p})$, $l \leq \mathrm{lh}(q)$, define

$$\mathbf{D} \restriction l = (2, \mathbf{q} \restriction l, \sigma \restriction \{p_i : i < l\})$$

which is a $(Q,*)$-description. Define

$$\mathbf{D} \lhd \mathbf{D}'$$

iff $\mathbf{D} = \mathbf{D}' \restriction l$ for some $l < \mathrm{lh}(\mathbf{D}')$. Define $\lhd^{Q,W} = \lhd \restriction \mathrm{desc}(Q,W)$.

The ordering of $\mathrm{seed}^{Q,W}_{\mathbf{D}}$ is definable in the following concrete way. Put

$$\langle \mathbf{D} \rangle = \begin{cases} (1,\mathbf{q}) & \text{if } d = 1, \\ (2, \sigma \oplus \mathbf{q}) & \text{if } d = 2. \end{cases}$$

[1]the split in definition is insignificant, only to ensure Lemma 4.25.

Define

$$\mathbf{D} \prec \mathbf{D}'$$

iff $\langle \mathbf{D} \rangle <_{BK} \langle \mathbf{D}' \rangle$, the ordering on subcoordinates in $\omega^{<\omega}$ again according to $<_{BK}$. For example, the constant $(Q,*)$-description $\mathbf{D}_0$ is the $\prec$-maximum, and we have $\langle \mathbf{D}_0 \rangle = (2, \emptyset)$. When $1 \leq \text{card}({}^1Q) < \aleph_0$, the $\prec$-least $(Q,*)$-description is $(1, q, \emptyset)$, where q is the $<_{BK}$-least node in ${}^1Q$. When $W \neq \emptyset$, the $\prec$-least (Q,W)-description of degree 2 is $\mathbf{D}_W = (2, ((-1), \{(0)\}, ((0))), \sigma_W)$, where $\sigma_W((0)) =$the $<_{BK}$-least node in W, and we have $\langle \mathbf{D}_W \rangle = (2, (\sigma_W(1), -1))$. Define $\prec^{Q,W} = \prec \restriction \text{desc}(Q,W)$. $\prec^{Q,W}$ exactly determines the order of the $\text{seed}^{Q,W}_{\mathbf{D}}$'s, as in the following lemma. It is parallel to Lemma 3.22.

Lemma 4.25. *Suppose* $\mathbf{D}, \mathbf{D}' \in \text{desc}(Q,W)$ *and* $\mathbf{D} \prec^{Q,W} \mathbf{D}'$. *Then*

1. *For any* $g \in \omega_1^{Q\uparrow}$, *for any* $\vec{\alpha} \in \omega_1^{W\uparrow}$, $g^W_{\mathbf{D}}(\vec{\alpha}) < g^W_{\mathbf{D}'}(\vec{\alpha})$.
2. *Suppose Q is finite. Then* $\text{seed}^{Q,W}_{\mathbf{D}} < \text{seed}^{Q,W}_{\mathbf{D}'}$. *Moreover, for any* $\beta < u_2$, $\mathbf{D}^{Q,W}(\beta) < \text{seed}^{Q,W}_{\mathbf{D}'}$.

Proof. 1. Simple computation.

2. Note that $\mathbf{D}^{Q,W}(\omega_1) = \text{seed}^{Q,W}_{\mathbf{D}}$. We directly prove the "moreover" part. We are given $\beta = j_{\mu_{\mathbb{L}}}(h)(\omega_1)$, where h is a function into ω_1. Let $E \in \mu_{\mathbb{L}}$ such that for any $\alpha \in E$, $h(\alpha) < \min(E \setminus \alpha + 1)$. We have $\mathbf{D}^{Q,W}(\beta) = j^Q \circ j^W(h)(\text{seed}^{Q,W}_{\mathbf{D}})$. By Łoś, it suffices to show that for any $g \in E^{Q\uparrow}$, $j^W(h)([g^W_{\mathbf{D}}]_{\mu^W}) < [g^W_{\mathbf{D}'}]_{\mu^W}$. By Łoś again, it suffices to show that for any $\vec{\alpha} \in [\omega_1]^{W\uparrow}$, $h(g^W_{\mathbf{D}}(\vec{\alpha})) < g^W_{\mathbf{D}'}(\vec{\alpha})$. We already know that $g^W_{\mathbf{D}}(\vec{\alpha}), g^W_{\mathbf{D}'}(\vec{\alpha}) \in E$. By our choice of E, it suffices to show that $g^W_{\mathbf{D}}(\vec{\alpha}) < g^W_{\mathbf{D}'}(\vec{\alpha})$. This is exactly part 1. □

Suppose W is a level-1 proper subtree of W', W' is finite, $w \in W \cup \{\emptyset\}$, $w' \in W' \setminus W$. Define

$$w \lhd^W_1 w'$$

iff $w' <_{BK} w$ and $\{w^* \in W : w' <_{BK} w^* <_{BK} w\} = \emptyset$.

$\lhd^W_1$ inherits the following trivial continuity property.

Lemma 4.26. *Suppose W is a level-1 proper subtree of W', W' is finite, $w \in W$, $w' \in W' \setminus W$, $w \lhd^W_1 w'$. Suppose $C \in \mu_{\mathbb{L}}$ is a club, C' is the set of limit points of C. Then for any $\vec{\alpha} \in [C']^{W\uparrow}$,*

$$\alpha_w = \sup\{\beta_{w'} : \vec{\beta} \in [C]^{W'\uparrow}, \vec{\beta} \textit{ extends } \vec{\alpha}\}.$$

Suppose W is a proper level-1 subtree of W'. For $\mathbf{D} \in \mathrm{desc}(Q, W)$ and $\mathbf{D}' \in \mathrm{desc}(Q, W') \setminus \mathrm{desc}(Q, W)$, define the level-1 end extension relation

$$\mathbf{D} \lhd_1^{Q,W} \mathbf{D}'$$

iff $\mathbf{D}' \prec \mathbf{D}$ and $\{\mathbf{D}^* \in \mathrm{desc}(Q, W) : \mathbf{D}' \prec \mathbf{D}^* \prec \mathbf{D}\} = \emptyset$. Thus, $\mathbf{D} \lhd_1^{Q,W} \mathbf{D}'$ iff both $\mathbf{D}, \mathbf{D}'$ are of degree 2 and letting $\mathbf{D} = (2, (q, P, \vec{p}), \sigma)$, $\mathbf{D}' = (2, (q', P', \vec{p}'), \sigma')$, $\mathrm{lh}(q) = k$, $\vec{p} = (p_i)_{i<\mathrm{lh}(\vec{p})}$, then either

1. q is of continuous type (hence $\mathrm{lh}(\vec{p}) = k$), $\mathbf{D}^- \lhd \mathbf{D}'$, $\sigma(p_{k-1}) \lhd_1^W \sigma'(p_{k-1})$, or
2. q is of discontinuous type (hence $\mathrm{lh}(\vec{p}) = k + 1$), $\mathbf{D} \lhd \mathbf{D}'$, $\sigma(p_k^-) \lhd_1^W \sigma'(p_k)$.

As a corollary to Lemma 4.26, $\lhd_1^{Q,W}$ inherits the following continuity property.

Lemma 4.27. *Suppose W is a proper subtree of W', $\mathbf{D} \in \mathrm{desc}(Q, W)$, $\mathbf{D}' \in \mathrm{desc}(Q, W')$, $\mathbf{D} \lhd_1^{Q,W} \mathbf{D}'$. Suppose $C \in \mu_{\mathbb{L}}$ is a club, C' is the set of limit points of C. Then for any $g \in \omega_1^{Q\uparrow}$, for any $\vec{\alpha} \in [C']^{W\uparrow}$,*

$$g_{\mathbf{D}}^W(\vec{\alpha}) = \sup\{g_{\mathbf{D}'}^{W'}(\vec{\beta}) : \vec{\beta} \in [C]^{W'\uparrow}, \vec{\beta} \text{ extends } \vec{\alpha}\}.$$

Suppose Q is a proper subtree of Q', both finite. For $(d, \mathbf{q}) \in \mathrm{desc}^*(Q)$, $(d', \mathbf{q}') \in \mathrm{desc}^*(Q')$, define the level-2 extension relation

$$(d, \mathbf{q}) \lhd_2^Q (d', \mathbf{q}')$$

iff $(d', \mathbf{q}') \prec (d, \mathbf{q})$ and $\{(d^*, \mathbf{q}^*) \in \mathrm{desc}^*(Q) : (d', \mathbf{q}') \prec (d^*, \mathbf{q}^*) \prec (d, \mathbf{q})\} = \emptyset$. Thus, $(d, \mathbf{q}) \lhd_2^Q (d', \mathbf{q}')$ iff either

1. $d = d' = 1$, $\mathbf{q} \lhd_1^{{}^1Q} \mathbf{q}'$, or
2. $d' = 1$, $\emptyset \lhd_1^{{}^1Q} \mathbf{q}'$, $d = 2$, $\mathbf{q} \in \{((-1), \{(0)\}, ((0))), (\emptyset, \emptyset, \emptyset)\}$, or
3. $d = d' = 2$, letting $\mathbf{q} = (q, P, \vec{p})$, $\vec{p} = (p_i)_{i<\mathrm{lh}(\vec{p})}$, $\mathbf{q} = (q', P', \vec{p}')$, $\vec{p}' = (p_i')_{i<\mathrm{lh}(\vec{p}')}$, $\mathrm{lh}(q) = k$, then either
   (a) $\mathbf{q} \in \mathrm{desc}(Q)$ is of continuous type, $k \geq 2$, $(P, \vec{p}) = (P', \vec{p}' \upharpoonright k)$, $(q^-)^- \subsetneq q'$, $q(k-2) \lhd_1^{{}^2Q\{(q^-)^-\}} q'(k-2)$, or
   (b) $\mathbf{q} \in \mathrm{desc}(Q)$ is of discontinuous type, $(P, \vec{p} \upharpoonright k) = (P', \vec{p}' \upharpoonright k)$, $q^- = (q')^-$, $q(k-1) \lhd_1^{{}^2Q\{q^-\}} q'(k-1)$, or

(c) $\mathbf{q} \notin \operatorname{desc}(Q)$, $q \subsetneq q'$, $\emptyset \lhd_1^{2Q\{q\}} q'(k)$.

As a corollary to Lemma 3.22 and Lemma 3.18, $\lhd_2^Q$ inherits the following continuity property.

Lemma 4.28. *Suppose $C \in \mu_{\mathbb{L}}$ is a club. Let $\eta \in C'$ iff $C \cap \eta$ has order type η. Suppose Q is a proper subtree of Q', Q, Q' are finite, $(d, \mathbf{q}) \in \operatorname{desc}^*(Q)$, $(d', \mathbf{q}') \in \operatorname{desc}^*(Q')$, $(d, \mathbf{q}) \lhd_2^Q (d', \mathbf{q}')$. Then for any $\vec{\beta} \in [C']^{Q\uparrow}$,*

$$^{d}\beta_{\mathbf{q}} = \sup\{^{d'}\gamma_{\mathbf{q}'} : \vec{\gamma} \in [C]^{Q'\uparrow}, \vec{\gamma} \textit{ extends } \vec{\beta}\}.$$

In the proof of Lemma 4.28, the construction of $\vec{\gamma}$ that witnesses the $\leq$ direction relies on the assumption that $\eta \in C'$ iff $C \cap \eta$ has order type η.

Suppose Q is a proper subtree of Q', both finite. For $\mathbf{D} \in \operatorname{desc}(Q, W)$, $\mathbf{D}' \in \operatorname{desc}(Q', W) \setminus \operatorname{desc}(Q, W)$. Define the level-2 end extension relation

$$\mathbf{D} \lhd_2^{Q,W} \mathbf{D}'$$

iff $\mathbf{D}' \prec \mathbf{D}$ and $\{\mathbf{D}^* \in \operatorname{desc}(Q, W) : \mathbf{D}' \prec \mathbf{D}^* \prec \mathbf{D}\} = \emptyset$. Thus, putting $\mathbf{D} = (d, \mathbf{q}, \sigma)$, $\mathbf{D} = (d', \mathbf{q}', \sigma')$, $\mathbf{D} \lhd_2^{Q,W} \mathbf{D}'$ iff either

1. $d = d' = 1$, $\mathbf{q} \lhd_1^{1Q} \mathbf{q}'$, or
2. $d' = 1$, $\emptyset \lhd_1^{1Q} \mathbf{q}'$, $d = 2$, $\mathbf{q} = ((-1), \{(0)\}, ((0)))$, $\sigma((0)) = \min(\prec^W)$, or
3. $d = d' = 2$, letting $\mathbf{q} = (q, P, \vec{p})$, $\vec{p} = (p_i)_{i<\operatorname{lh}(\vec{p})}$, $\mathbf{q} = (q', P', \vec{p}')$, $\vec{p}' = (p'_i)_{i<\operatorname{lh}(\vec{p}')}$, $\operatorname{lh}(q) = k$, then either
   (a) q is of continuous type (hence $\operatorname{lh}(\vec{p}) = k$), $\mathbf{D}^- \lhd \mathbf{D}'$, $\emptyset \lhd_1^{2Q\{q^-\}} q'(k-1)$, either $p_{k-1} = -1$ or $\sigma'(p_{k-1}) = \operatorname{pred}_{\prec^W}(\sigma(p_{k-1}))$, or
   (b) q is of discontinuous type (hence $\operatorname{lh}(\vec{p}) = k+1$), $\mathbf{D} \lhd \mathbf{D}'$, $\emptyset \lhd_1^{2Q\{q\}} q'(k)$, $\sigma'(p_k^-) = \operatorname{pred}_{\prec^W}(\sigma(p_k))$.

In particular, $\mathbf{D} \lhd_2^{Q,W} \mathbf{D}'$ implies that $\mathbf{D}$ is of level-2 W-discontinuous type.

$\lhd_2^{Q,W}$ inherits the following continuity property.

Lemma 4.29. *Suppose $C \in \mu_{\mathbb{L}}$ is a club. Let $\eta \in C'$ iff $\eta \in C$ and $C \cap \eta$ has order type η. Suppose Q is a proper subtree of Q', both finite, $\mathbf{D} = (d, \mathbf{q}, \sigma) \in \operatorname{desc}(Q, W)$, $\mathbf{D}' = (d', \mathbf{q}', \sigma') \in \operatorname{desc}(Q', W)$, $\mathbf{D} \lhd_2^{Q,W} \mathbf{D}'$. Then for any $\vec{\beta} \in [C']^{Q\uparrow}$,*

$$\sigma^W(^{d}\beta_{\mathbf{q}}) = \sup\{(\sigma')^W(^{d}\gamma_{\mathbf{q}'}) : \vec{\gamma} \in [C]^{Q'\uparrow}, \vec{\gamma} \textit{ extends } \vec{\beta}\}.$$

Proof. The $\geq$ direction follows from Lemma 4.25. We show the $\leq$ direction. When $d = d' = 1$, both sides are equal to ${}^{d}\beta_{\mathbf{q}}$ by Lemma 4.26. When $d = 2 \wedge d' = 1$, both sides are equal to ω_1 by Lemma 4.26 again. Suppose now $d = d' = 2$. Let $\mathbf{q} = (q, P, \vec{p})$, $\vec{p} = (p_i)_{i<\mathrm{lh}(\vec{p})}$, $\mathbf{q} = (q', P', \vec{p}')$, $\vec{p}' = (p'_i)_{i<\mathrm{lh}(\vec{p}')}$, $\mathrm{lh}(q) = k$.

Case 1: q is of continuous type.

Let $P^- = P \setminus \{p_{k-1}\}$. So ${}^{2}Q(q^-) = (P^-, p_{k-1})$. Let $q'' = q' \restriction k$, $\sigma'' = \sigma' \restriction P$, $p'' = {}^{2}Q_{\mathrm{node}}(q'')$. Then $(2, (q'', P, \vec{p}^\frown(p''))) \lhd^Q_2 (2, (q^-, P, \vec{p}))$. By Lemma 4.28,

$$j^{P^-,P}({}^{2}\beta_{q^-}) = \sup\{{}^{2}\gamma_{q''} : \vec{\gamma} \in [C]^{Q'\uparrow}, \vec{\gamma} \text{ extends } \vec{\beta}\}.$$

It suffices to show that

$$\sigma^W \circ j^{P^-,P}_{\sup}({}^{2}\beta_{q^-}) = (\sigma'')^W_{\sup} \circ j^{P^-,P}({}^{2}\beta_{q^-}).$$

This is exactly Lemma 3.5, using the fact $\mathrm{cf}^{\mathbb{L}}({}^{2}\beta_{q^-}) = \mathrm{seed}^{P^-}_{p^-_{k-1}}$.

Case 2. q is of discontinuous type.

Let P^+ be the completion of (P, p_k) if $p_k \neq -1$, $P^+ = P$ if $p_k = -1$. Let $q'' = q' \restriction k + 1$, $\sigma'' = \sigma' \restriction P^+$. Then $(2, (q'', P^+, \vec{p})) \lhd^Q_2 (2, (q, P^+, \vec{p}))$. By Lemma 4.28,

$$j^{P,P^+}({}^{2}\beta_q) = \sup\{{}^{2}\gamma_{\mathbf{q}'} : \vec{\gamma} \in [C]^{Q'\uparrow}, \vec{\gamma} \text{ extends } \vec{\beta}\}.$$

It remains to show

$$\sigma^W({}^{2}\beta_q) = (\sigma'')^W_{\sup} \circ j^{P,P^+}({}^{2}\beta_q).$$

This is exactly Lemma 3.6, using the fact $\mathrm{cf}^{\mathbb{L}}({}^{2}\beta_q) = \mathrm{seed}^P_{p^-_k}$ when $p_k \neq -1$, $\mathrm{cf}^{\mathbb{L}}({}^{2}\beta_q) = \omega$ when $p_k = -1$. $\square$

Definition 4.30. Suppose S is a finite regular level-1 tree and Q is a level ≤ 2 tree. Suppose $\tau : S \cup \{\emptyset\} \to \mathrm{desc}(Q, *)$ is a function. Then τ factors $(S, Q, *)$ iff

1. $\tau(\emptyset)$ is the constant $(Q, *)$-description.
2. If $s \prec^S s'$, then $\tau(s) \prec \tau(s')$.

For a level-1 tree W, τ factors (S, Q, W) iff τ factors $(S, Q, *)$ and $\mathrm{ran}(\tau) \subseteq \mathrm{desc}(Q, W)$. In particular, if every $\tau(s)$ is of degree 1, then τ factors $(S, Q, \emptyset)$.

If S is a level-1 tree, then

$$\mathrm{id}_{*,S}$$

factors (S, Q^0, S), where $\mathrm{id}_{*,S}(s) = (2, ((-1), \{(0)\}, ((0))), \sigma_s)$, $\sigma_s(0) = s$.

Suppose τ factors (S, Q, W). For $g \in \omega_1^{Q\uparrow}$, let

$$g_\tau^W : [\omega_1]^{W\uparrow} \to [\omega_1]^{S\uparrow}$$

be the function sending $\vec{\alpha}$ to $(g^W_{\tau(s)}(\vec{\alpha}))_{s\in \mathrm{dom}(S)}$. Lemma 4.25 ensures that g_τ^W is indeed a function into $[\omega_1]^{S\uparrow}$. In particular, $g^S_{\mathrm{id}_{*,S}}$ is the identity map on $[\omega_1]^{S\uparrow}$.

$$\mathrm{id}_\tau^{Q,W}$$

is the map sending $[g]^Q$ to $[g_\tau^W]_{\mu^W}$. So $\mathrm{id}_\tau^{Q,W}(\vec{\beta}) = (\mathrm{id}^{Q,W}_{\tau(s)}(\vec{\beta}))_{s\in S}$. Put

$$\mathrm{seed}_\tau^{Q,W} = [\mathrm{id}_\tau^{Q,W}]_{\mu^Q}$$

By Lemma 4.25 and Łoś, for any $A \in \mu^S$, $\mathrm{seed}_\tau^{Q,W} \in j^Q \circ j^W(A)$. Hence, we can unambiguously define

$$\tau^{Q,W} : \mathbb{L}_{\boldsymbol{\delta}_3^1}[j^S(T_2)] \to \mathbb{L}_{\boldsymbol{\delta}_3^1}[j^Q \circ j^W(T_2)]$$

by sending $j^S(h)(\mathrm{seed}^S)$ to $j^Q \circ j^W(h)(\mathrm{seed}_\tau^{Q,W})$. $\tau^{Q,W}$ is the unique map such that for any $z \in \mathbb{R}$, $\tau^{Q,W}$ is elementary from $L_{\kappa_3^z}[j^S(T_2), z]$ into $L_{\kappa_3^z}[j^Q \circ j^W(T_2), z]$ and for any $s \in S$, $\tau^{Q,W} \circ s^S = \tau(s)^{Q,W}$.

Lemma 4.31. *Suppose Q, W are finite.*

1. *If $\mathbf{D} = \min(\prec^{Q,W})$, then $\mathrm{seed}_{\mathbf{D}}^{Q,W} = \omega_1$. Hence $\mathbf{D}^{Q,W}$ is the identity on $\omega_1 + 1$.*
2. *If $\mathbf{E} = \mathrm{pred}_{\prec^{Q,W}}(\mathbf{D})$, then $(\mathbf{E}^{Q,W})''u_2$ is a cofinal subset of $\mathrm{seed}_{\mathbf{D}}^{Q,W}$.*

Proof. We only prove the case when ${}^1Q = \emptyset$. The general case takes an analogous additional argument.

Case 1: $W = \emptyset$.

The only (Q, W)-description is the constant $(Q, *)$-description $\mathbf{D}_0$. We only have to prove part 1. For any $x \in \mathbb{R}$, $\mathbf{D}_0^{Q,W} = j^Q \circ j^W$ is elementary from $L_{\kappa_3^x}[T_2, x]$ into $L_{\kappa_3^x}[j^Q \circ j^W(T_2), x]$. It follows that $\mathbf{D}_0^{Q,W} \restriction \omega_1$ is the identity map. It remains to show that $\mathrm{seed}_{\mathbf{D}_0}^{Q,W} = \omega_1$. We already know that $\mathrm{seed}_{\mathbf{D}_0}^{Q,W} = j^Q(\omega_1)$. Suppose $[g]_{\mu^Q} < j^Q(\omega_1)$ and we try to show that $[g]_{\mu^Q} \le \omega_1$. Let Q' be the completion of the partial level ≤ 2 tree $(Q, (1, (0), \emptyset))$. Let $\mathbf{D}' = (1, (0), \emptyset) \in \mathrm{desc}(Q', W)$. Then $\mathbf{D}_0 \lhd_2^{Q,W} \mathbf{D}'$. We partition functions $f \in \omega_1^{Q'\uparrow}$ according to whether ${}^1[f]_{(0)}^{Q'} \le g([f \restriction \mathrm{rep}(Q)]^Q)$. We obtain a club $C \in \mu_{\mathbb{L}}$ which is homogeneous for this property. Let $\eta \in C'$ iff $\eta \in C$ and $C \cap \eta$ has order type η. If the homogeneous side satisfies ${}^1[f]_{(0)}^{Q'} > g([f \restriction \mathrm{rep}(Q)]^Q)$, we let α_0 =the ω-th element of C, and so every $f \in [C']^{Q\uparrow}$ is extendable to $f' \in C^{Q'\uparrow}$ so that ${}^1[f']^{Q'}_{(0)} = \alpha_0$. Therefore, for every $\vec{\xi} \in [C']^{Q\uparrow}$, $g(\vec{\xi}) < \alpha_0$. Hence by Łoś, $[g]_{\mu^Q} < j^Q(\alpha_0) = \alpha_0$ and we are done. If the homogeneous side satisfies ${}^1[f]^{Q'}_{(0)} \leq g([f \restriction \text{rep}(Q)]^Q)$, then by Lemma 4.29, $\omega_1 = {}^2[f \restriction \text{rep}(Q)]^Q_\emptyset \leq g([f \restriction \text{rep}(Q)]^Q)$, contradicting to the assumption on g.

Case 2: $W \neq \emptyset$.

We firstly prove part 1. The $\prec^{Q,W}$-minimum is $\mathbf{D}_0 = (2, \mathbf{q}, \sigma)$, where $\mathbf{q} = ((-1), \{(0)\}, ((0)))$, $\sigma((0))$ is the $<_{BK}$-least node in W. $\text{seed}^{Q,W}_{\mathbf{D}_0}$ is represented modulo μ^Q by the function that sends $\vec{\beta}$ to $\sigma^W(\beta_{\mathbf{q}}) = \sigma^W(\omega_1) = \omega_1$. Hence, $\text{seed}^{Q,W}_{\mathbf{D}_0} = j^Q(\omega_1)$. Work with the same Q' as in Case 1 and argue with the same partition arguments.

Next, we prove part 2. Let $\mathbf{D} = (2, \mathbf{q}, \sigma)$, $\mathbf{q} = (q, P, \vec{p})$, $\mathbf{E} = (2, \mathbf{r}, \tau)$, $\mathbf{r} = (r, Z, \vec{z})$. Then $q \neq (-1)$. Put ${}^2Q^v = \{a \in \omega^{<\omega} : v^\frown(a) \in \text{dom}({}^2Q)\}$ for $v \in \text{dom}({}^2Q)$.

Subcase 2.1: r is of discontinuous type.

Let Q' be the level ≤ 2 tree extending Q such that $\text{dom}(Q') \setminus \text{dom}(Q) = \{(2, r')\}$, $r' = r^{-\frown}(a)$, $\emptyset \lhd_1^{{}^2Q\{r^-\}} a$, $Q'(r') = Q(r)$. Let $\mathbf{r}' = (r', Z, \vec{z})$, $\mathbf{E}' = (2, \mathbf{r}', \tau)$. Then $\mathbf{D} \lhd_2^{Q,W} \mathbf{E}'$. Our partition arguments will be based on Q'.

Suppose $[g]_{\mu^Q} < \text{seed}^{Q,W}_{\mathbf{D}}$ and we seek $\eta_0 < u_2$ such that $[g]_{\mu^Q} < \mathbf{E}^{Q,W}(\eta_0)$. We partition functions $f \in \omega_1^{Q'\uparrow}$ according to whether $\tau^W({}^2[f]^{Q'}_{r'}) \leq g([f \restriction \text{rep}(Q)]^Q)$. By Theorem 4.10, we obtain a club $C \in \mu_{\mathbb{L}}$ which is homogeneous for this property. Let $\eta \in C'$ iff $\eta \in C$ and $C \cap \eta$ has order type η. If the homogeneous side satisfies $\tau^W({}^2[f]^{Q'}_{r'}) > g([f \restriction \text{rep}(Q)]^Q)$, we let $\eta_0 = j_{\mu_{\mathbb{L}}}(h_0)(\omega_1)$, where $h_0(\alpha) = \min(C' \setminus (\alpha+1))$. This allows us to extend every $f \in (C')^{Q\uparrow}$ to $f' \in C^{Q'\uparrow}$ so that ${}^2[f']^{Q'}_{r'} = j^P(h_0)([f]^Q_r)$. Therefore, for every $\vec{\xi} \in [C']^{Q\uparrow}$, $g(\vec{\xi}) < \tau^W(j^Z(h_0)(\xi_r)) = j^W(h_0)(\tau^W(\xi_r))$. Hence $[g]_{\mu^Q} < j^Q \circ j^W(h_0)(\text{seed}^{Q,W}_{\mathbf{E}}) = \mathbf{E}^{Q,W}(j_{\mu_{\mathbb{L}}}(h_0)(\omega_1))$. Hence $[g]_{\mu^Q} < \mathbf{E}^{Q,W}(\eta_0)$. If the homogeneous side satisfies $\tau^W({}^2[f]^{Q'}_{r'}) \leq g([f \restriction \text{rep}(Q)]^Q)$, then by Lemma 4.29, $\sigma^W({}^2[f]^Q_q) \leq g([f \restriction \text{rep}(Q)]^Q)$. This contradicts our assumption on g.

Subcase 2.2: r is of continuous type.

Let Q' be the level ≤ 2 tree extending Q such that $\text{dom}(Q') \setminus \text{dom}(Q) = \{(2, r')\}$, $r' = (r^-)^{-\frown}(a)$, $\emptyset \lhd_1^{{}^2Q\{(r^-)^-\}} a$, $Q'(r') = Q(r^-)$. Let $\mathbf{r}' = (r'^\frown(-1), Z, \vec{z})$, $\mathbf{E}' = (2, \mathbf{r}', \tau)$. Then $\mathbf{D} \lhd_2^{Q,W} \mathbf{E}'$. The rest is similar to Subcase 2.1. □

At this point, it is convenient to label the nodes of a tree of uniform cofinalities using arbitrary sets instead of elements in $\omega^{<\omega}$ and $(\omega^{<\omega})^{<\omega}$. Suppose Q is a level ≤ 2 tree and W is a level-1 tree. *A representation of* $Q \otimes W$ is a pair (S, τ) such that S is a level-1 tree, τ factors (S, Q, W),

$\operatorname{ran}(\tau) = \operatorname{desc}(Q,W)$, and $s \prec^S s'$ iff $\tau(s) \prec^{Q,W} \tau(s')$. Representations of $Q \otimes W$ are clearly mutually isomorphic. We shall informally regard

$$Q \otimes W = \operatorname{desc}(Q,W) \setminus \{\text{the constant } (Q,W)\text{-description}\}$$

as a "level-1 tree" by identifying it with S via τ. We put $\operatorname{seed}^{Q\otimes W}_{\mathbf{D}} = \operatorname{seed}^S_{\tau^{-1}(\mathbf{D})}$ for $\mathbf{D} \in \operatorname{desc}(Q,W)$. If τ' factors (S',Q,W), then τ' also factors "level-1 trees" $(S', Q\otimes W)$, and $(\tau')^{Q\otimes W}$ makes sense. That is, $(\tau')^{Q\otimes W} = (\tau^{-1} \circ \tau')^S$, where $\tau^{-1}\circ\tau'$ factors (S',S). The identity map $\operatorname{id}_{Q\otimes W} : \mathbf{D} \mapsto \mathbf{D}$ factors $(Q\otimes W, Q, W)$. If Q is a subtree of Q' and W is a subtree of W', then $Q \otimes W$ is regarded as a subtree of $Q' \otimes W'$, and the map $j^{Q\otimes W, Q'\otimes W'}$ makes sense. In other words, let (S,τ) be a representation of $Q\otimes W$ and (S',τ') be a representation of $Q'\otimes W'$ such that S is a subtree of S' and $\tau \subseteq \tau'$, then $j^{Q\otimes W, Q'\otimes W'} = j^{S,S'}$. If π factors level ≤ 2 trees (Q,T), then

$$\pi \otimes W$$

factors level-1 trees $(Q\otimes W, T\otimes W)$, where $\pi(d,\mathbf{q},\sigma) = (d, {}^d\pi(\mathbf{q}), \sigma)$.

Lemma 4.32. *Suppose Q is a finite level ≤ 2 tree, W is a finite level-1 tree.*

1. *If $\mathbf{D} \in \operatorname{desc}(Q,W)$, then $\operatorname{seed}^{Q\otimes W}_{\mathbf{D}} = \operatorname{seed}^{Q,W}_{\mathbf{D}}$.*
2. *$(\operatorname{id}_{Q\otimes W})^{Q,W}$ is identity on $j^Q \circ j^W(\omega_1+1)$.*
3. *If S is a level-1 tree, τ factors (S,Q,W), then $\tau^{Q,W} = \tau^{Q\otimes W}$.*

Proof. Let $\mathbf{D}_0,\dots,\mathbf{D}_m$ enumerate $\operatorname{desc}(Q,W)$ in the $\prec^{Q,W}$-ascending order. We prove by induction on $l \leq m$ that $\operatorname{seed}^{Q\otimes W}_{\mathbf{D}_i} = \operatorname{seed}^{Q,W}_{\mathbf{D}_i}$ for any $i \leq l$.

Suppose $l = 0$. The fact $\operatorname{seed}^{Q,W}_{\mathbf{D}_0} = \omega_1$ follows from Lemma 4.31.

Suppose the induction hypothesis holds at $l < m$. That is, $(\mathbf{D}_l)^{Q,W}(\omega_1) = \operatorname{seed}^{Q,W}_{\mathbf{D}_l} = u_{l+1}$ for $l < m$. By Łoś, $(\operatorname{id}_{Q\otimes W})^{Q,W}$ is identity on u_{l+2}. But $((\mathbf{D}_l)^{Q,W})''u_2$ is a cofinal subset of $\operatorname{seed}^{Q,W}_{\mathbf{D}_{l+1}}$ by Lemma 4.31. Hence, $\operatorname{seed}^{Q,W}_{\mathbf{D}_{l+1}} = u_{l+2}$. This proves part 1. Parts 2-3 are immediate corollaries. □

$\mathbf{D} \in \operatorname{desc}(Q,W)$ is *direct* iff either $\mathbf{D}$ is of degree 1 or $\mathbf{D}$ is of the form $(2,(q,P,\vec{p}),\operatorname{id}_P)$. Lemma 4.32 has the following corollary on representations of uniform indiscernibles in the μ^Q-ultrapower.

Lemma 4.33. *Suppose Q is a finite level ≤ 2 tree. Then*

$$\{u_n : 1 \leq n < \omega\} = \{\operatorname{seed}^{Q,W}_{\mathbf{D}} : W \text{ finite}, \mathbf{D} \in \operatorname{desc}(Q,W) \text{ is direct}\}.$$

For $(d,q) \in \operatorname{dom}(Q)$, define

$$\operatorname{cf}^Q(d,q) = \begin{cases} 0 & \text{if } d = 1 \vee (d = 2 \wedge {}^2Q(q) \text{ of degree } 0), \\ 1 & \text{if } d = 2 \wedge \operatorname{ucf}({}^2Q(q)) = \min(\prec^{{}^2Q_{\text{tree}}(q)}), \\ 2 & \text{otherwise.} \end{cases}$$

By Lemma 4.31, if $\vec{\beta}$ respects Q, then

$$\operatorname{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]}({}^d\beta_q) = u_{\operatorname{cf}^Q(d,q)}$$

where $u_0 = \omega$.

4.5 Approximations of S_3 in $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$

Lemma 4.34. *Suppose Q is a level ≤ 2 tree, W is a level-1 subtree of W', all trees are finite. Then $j^Q(j^{W,W'} \restriction j^W(\omega_1+1)) = j^{Q\otimes W, Q\otimes W'} \restriction (j^{Q\otimes W}(\omega_1+1))$, and hence $j^Q(j^{W,W'}_{\sup} \restriction j^W(\omega_1+1)) = j^{Q\otimes W, Q\otimes W'}_{\sup}(j^{Q\otimes W}(\omega_1+1))$ by sufficient elementarity of j^Q.*

Proof. By Lemma 4.32, $\operatorname{seed}^{Q,W}_{\mathbf{D}} = \operatorname{seed}^{Q\otimes W}_{\mathbf{D}}$ for $\mathbf{D} \in \operatorname{desc}(Q,W)$, and similarly for W'. So $j^{Q\otimes W, Q\otimes W'}(\operatorname{seed}^{Q,W}_{\mathbf{D}}) = \operatorname{seed}^{Q,W'}_{\mathbf{D}}$ for $\mathbf{D} \in \operatorname{desc}(Q,W)$. since $j^Q(j^{W,W'})$ is elementary from $L[z]$ to $L[z]$ for any $z \in \mathbb{R}$, it suffices to show that $j^Q(j^{W,W'} \restriction j^W(\omega_1+1))(\operatorname{seed}^{Q,W}_{\mathbf{D}}) = j^{Q\otimes W, Q\otimes W'}(\operatorname{seed}^{Q,W}_{\mathbf{D}})$ for any $\mathbf{D} \in \operatorname{desc}(Q,W)$. Fix $\mathbf{D} \in \operatorname{desc}(Q,W)$. Suppose the typical case when $\mathbf{D} = (2, \mathbf{q}, \sigma)$ is of degree 2. Then by Łoś,

$$\begin{aligned} j^Q(j^{W,W'} \restriction j^W(\omega_1+1))(\operatorname{seed}^{Q,W}_{\mathbf{D}}) &= j^Q(j^{W,W'} \restriction j^W(\omega_1+1))([\vec{\xi} \mapsto \sigma^W({}^2\xi_{\mathbf{q}})]_{\mu^Q}) \\ &= [\vec{\xi} \mapsto j^{W,W'} \circ \sigma^W({}^2\xi_{\mathbf{q}})]_{\mu^Q} \\ &= [\vec{\xi} \mapsto \sigma^{W'}({}^2\xi_{\mathbf{q}})]_{\mu^Q} \\ &= \operatorname{seed}^{Q,W'}_{\mathbf{D}} \\ &= j^{Q\otimes W, Q\otimes W'}(\operatorname{seed}^{Q,W}_{\mathbf{D}}). \end{aligned}$$

□

Lemma 4.35. *Suppose π factors finite level ≤ 2 trees (Q,T) and W is a finite level-1 tree, all trees are finite. Then $\pi^T \restriction j^Q \circ j^W(\omega_1+1) = (\pi \otimes W)^{Q\otimes W, T\otimes W} \restriction j^{Q\otimes W}(\omega_1+1)$.*

Proof. Apply Lemma 4.32 and use the fact that $\pi^T(\operatorname{seed}^{Q,W}_{\mathbf{D}}) = \operatorname{seed}^{T,W}_{\pi\otimes W(\mathbf{D})}$ for $\mathbf{D} \in \operatorname{desc}(Q,W)$. □

Lemma 4.36. *Suppose Q is a finite level ≤ 2 tree. Then*

1. *$j^Q \restriction \{u_n : n < \omega\}$ is Δ^1_3, uniformly in Q.*
2. *$j^Q(u_\omega) = u_\omega$.*
3. *$j^Q \restriction u_\omega$ is Δ^1_3, uniformly in Q.*
4. *Suppose P, P' are finite level-1 trees and π factors (P, P'). Then $j^Q(\pi^{P'} \restriction u_\omega)$ is Δ^1_3, uniformly in (Q, P, P', π).*
5. *$j^Q(T_2)$ is Δ^1_3, uniformly in Q.*

Proof. 1 and 2. By Lemma 4.32.

3. $j^Q(\tau^{L[z]}(u_1, \ldots, u_n)) = \tau^{L[z]}(j^Q(u_1), \ldots, j^Q(u_n))$.

4. By Lemma 4.34.

5. by 4. □

The following lemma refines Corollary 4.15.

Lemma 4.37. *Assume $\boldsymbol{\Delta}^1_2$-determinacy. Suppose $x \in \mathbb{R}$. Then for any finite level ≤ 2 tree Q, $j^Q(\kappa^x_3, \lambda^x_3) = (\kappa^x_3, \lambda^x_3)$. Moreover, $S_3 \restriction \kappa^x_3$ and $S_3 \restriction \lambda^x_3$ are both uniformly Δ_1-definable over $L_{\kappa^x_3}[T_2, x]$ from $\{T_2, x\}$.*

Proof. By elementarity, $j^Q(\kappa^x_3)$ is the least γ for which $L_\gamma[j^Q(T_2), x]$ is admissible. But $j^Q(T_2) \in L_{\kappa^x_3}[T_2, x]$ by Lemma 4.36. Consequently, $L_{\kappa^x_3}[j^Q(T_2), x]$ is admissible. Since j^Q is non-decreasing on ordinals, we must have $j^Q(\kappa^x_3) = \kappa^x_3$. Similarly, λ^x_3, being the sup of the ordinals Δ_1-definable over $L_{\kappa^x_3}[T_2, x]$ from $\{T_2, x\}$, is also fixed by j^Q.

To define $S_3 \restriction \kappa^x_3$, it is of course enough to establish a uniformly Δ_1 definition of $j^{Q,Q'} \restriction \kappa^x_3$ over $L_{\kappa^x_3}[T_2, x]$, for Q a level ≤ 2 subtree of Q'. Note that every element of $L_{\kappa^x_3}[T_2, x]$ is Δ_1-definable over $L_{\kappa^x_3}[T_2, x]$ from $u_\omega \cup \{T_2, x\}$, and hence by Łoś, every ordinal in $j^Q(L_{\kappa^x_3}[T_2, x])$ is Δ_1-definable over $j^Q(L_{\kappa^x_3}[T_2, x])$ from parameters in $u_\omega \cup \{j^Q(T_2), x\}$. The lemma follows immediately from Lemma 4.37 and Łoś:

$$j^{Q,Q'}(\gamma) = \gamma'$$

iff for some $\xi < \kappa^x_3$, some Σ_1-formula φ, some ordinal $\alpha < u_\omega$,

$$L_{j^Q(\xi)}[j^Q(T_2), x] \models \forall\delta\ (\delta = \gamma \leftrightarrow \varphi(\delta, j^Q(T_2), x, \alpha)),$$

and

$$L_{j^{Q'}(\xi)}[j^{Q'}(T_2), x] \models \forall\delta\ (\delta = \gamma' \leftrightarrow \varphi(\delta, j^{Q'}(T_2), x, j^{Q,Q'}(\alpha))).$$

This is a Σ_1 definition of $j^{Q,Q'}(\gamma) = \gamma'$ over $L_{\kappa^x_3}[T_2, x]$ from $\{T_2, x\}$. In a similar way, we can write down a Σ_1 definition of $j^{Q,Q'}(\gamma) \neq \gamma'$. The definition of $S_3 \restriction \lambda^x_3$ is similar. □

In light of Lemma 4.37, $L_{\kappa_3^x}[S_3 \restriction \kappa_3^x]$ is regarded as the "lightface core" of $L_{\kappa_3^x}[T_2, x]$, analogous to $L_{\omega_1^x}$ versus $L_{\omega_1^x}[x]$. In parallel to Guaspari-Kechris-Sacks in [5, 20, 40], if C_3 is the largest countable Π^1_3 set of reals, then $x \in C_3$ iff $x \in L_{\kappa_3^x}[S_3 \restriction \kappa_3^x]$ iff $x \in L_{\lambda_3^x}[S_3 \restriction \lambda_3^x]$. A related result about C_3 is in [6] which follows the same line. An inner model theoretic characterization of C_3 is still unknown.

Recall that the set of uncountable $\mathbb{L}$-regular cardinals below u_ω is $\{u_n : 1 \leq n < \omega\}$. The scenario in the AD world suggests that the set of uncountable $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$-regular cardinals should be $\{u_1, u_2\}$. For a finite level ≤ 2 tree Q, by Lemma 4.36, $\mathbb{L}_{u_\omega} \subseteq \mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)] \subseteq \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$, so the set of uncountable $\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]$-regular cardinals should be $\{u_n : n \in A\}$ for some set $\{1, 2\} \subseteq A \subseteq \omega \setminus 1$. Which u_n is $\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]$-regular? The answer to this is an abstraction of Jackson's uniform cofinality analysis on functions $F : [\omega_1]^{Q\uparrow} \to \boldsymbol{\delta}^1_3$ that lie in $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$, originally in [12]. In particular, we confirm that the set of uncountable $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$-regular cardinals is indeed $\{u_1, u_2\}$.

Theorem 4.38. *Suppose Q is a finite level ≤ 2 tree, W is a finite level-1 tree. Suppose $\mathbf{D} = (d, \mathbf{q}, \sigma) \in \operatorname{desc}(Q, W)$. Then*

$$\operatorname{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]}(\operatorname{seed}^{Q,W}_{\mathbf{D}}) = \operatorname{seed}^Q_{\operatorname{ucf}^W_2(\mathbf{D})}.$$

In particular, the set of $\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]$-regular cardinals is exactly

$$\{\operatorname{seed}^Q_{(d,\mathbf{q})} : (d, \mathbf{q}) \in \operatorname{desc}^*(Q) \text{ is regular}\}.$$

Proof. Put $(d, \mathbf{r}) = \operatorname{ucf}^W_2(\mathbf{D})$. Firstly, we prove $\operatorname{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]}(\operatorname{seed}^{Q,W}_{\mathbf{D}}) = \operatorname{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]}(\operatorname{seed}^Q_{(d,\mathbf{r})})$. There is nothing to prove for $d = 1$. Suppose now $d = 2$.

Case 1: q is of continuous type.

Subcase 1.1: $\mathbf{D}$ is of level-2 W-continuous type.

In this case, σ is continuous at ${}^2\beta_{\mathbf{q}}$. For any $\vec{\beta} \in [\omega_1]^{Q\uparrow}$, $\sigma^W({}^2\beta_{\mathbf{q}}) = \sup(\sigma^W \circ j^{P^-,P})''({}^2\beta_{q^-})$. So $\operatorname{cf}^{L_{\kappa_3}[T_2]}(\sigma^W({}^2\beta_{\mathbf{q}})) = \operatorname{cf}^{L_{\kappa_3}[T_2]}({}^2\beta_{q^-})$. Note that $\operatorname{seed}^Q_{(2,\mathbf{r})}$ is represented by the function $\vec{\beta} \mapsto {}^2\beta_{q^-}$. By Łoś, $\operatorname{cf}^{L_{\kappa_3}(j^Q(T_2))}(\operatorname{seed}^{Q,W}_{\mathbf{D}}) = \operatorname{cf}^{L_{\kappa_3}(j^Q(T_2))}(\operatorname{seed}^Q_{(2,\mathbf{r})})$.

Subcase 1.2: $\mathbf{D}$ is of level-2 W-discontinuous type.

Then $\operatorname{pred}_{\prec^W}(\sigma(p_{k-1}))$ exists and is not in $\operatorname{ran}(\sigma)$. Put $P^- = P \setminus \{p_{k-1}\}$. Let σ' factor (P, W) where σ and σ' agree on P^- and $\sigma'(p_{k-1}) = \operatorname{pred}_{\prec^W}(\sigma(p_{k-1}))$. By Lemma 3.5, $\sigma^W({}^2\beta_{\mathbf{q}}) = (\sigma')^W_{\sup}(j^{P^-,P}({}^2\beta_{q^-}))$. So $\operatorname{cf}^{L_{\kappa_3}[T_2]}(\sigma^W({}^2\beta_{\mathbf{q}})) = \operatorname{cf}^{L_{\kappa_3}[T_2]}(j^{P^-,P}({}^2\beta_{q^-}))$. Note that $\operatorname{seed}^Q_{(2,\mathbf{r})}$ is represented by the function $\vec{\beta} \mapsto j^{P^-,P}({}^2\beta_{q^-})$. By Łoś, $\operatorname{cf}^{L_{\kappa_3}(j^Q(T_2))}(\operatorname{seed}^{Q,W}_{\mathbf{D}}) = \operatorname{cf}^{L_{\kappa_3}(j^Q(T_2))}(\operatorname{seed}^Q_{(2,\mathbf{r})})$.

Case 2: q is of discontinuous type.

Subcase 2.1: $\mathbf{D}$ is of level-2 W-continuous type.

Then σ is continuous at ${}^2\beta_{\mathbf{q}}$. Proceed as in Subcase 1.1.

Subcase 2.2: $\mathbf{D}$ is of level-2 W-discontinuous type.

Then $\text{pred}_{\prec^W}(\sigma(p_k^-))$ exists and is not in $\text{ran}(\sigma)$. Put $P^+ = P \cup \{p_k\}$. Let σ' factor (P^+, W) where $\sigma' \supseteq \sigma$ and $\sigma'(p_k) = \text{pred}_{\prec^W}(\sigma(p_k^-))$. By Lemma 3.6, $\sigma^W({}^2\beta_{\mathbf{q}}) = (\sigma')^W_{\sup}(j^{P,P^+}({}^2\beta_{\mathbf{q}}))$. Proceed as in Subcase 1.2.

Note that by Lemma 4.33, each u_n $(1 \le n < \omega)$ is of the form $\text{seed}^{Q,W}_{\mathbf{D}}$ for some finite W and $\mathbf{D} \in \text{desc}(Q, W)$. In summary, we have proved that every $\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]$-regular cardinal must be of the form $\text{seed}^Q_{(d,\mathbf{q})}$, where $(d, \mathbf{q}) \in \text{desc}^*(Q)$ is regular.

Secondly, we prove that if $(d, \mathbf{q}) \in \text{desc}^*(Q)$ is regular, then $\text{seed}^Q_{(d,\mathbf{q})}$ is regular in $\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]$.

Suppose towards a contradiction that $\text{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]}(\text{seed}^Q_{(d,\mathbf{q})}) = \text{seed}^Q_{(e,\mathbf{r})}$, where $(e, \mathbf{r}) \prec^Q (d, \mathbf{q})$. Let $g \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]$ be a cofinal map from $\text{seed}^Q_{(e,\mathbf{r})}$ into $\text{seed}^Q_{(d,\mathbf{q})}$. Let $g = [G]_{\mu^Q}$, where $G \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. By Łoś, for μ^Q-a.e. $\vec{\beta}$, $g(\vec{\beta}) \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$ is a cofinal map from ${}^e\beta_{\mathbf{r}}$ into ${}^d\beta_{\mathbf{q}}$.

We prove the case when $d = e = 2$, the other cases being similar. Put $\mathbf{r} = (r, Z, \vec{z})$. Let Q_1 be a level ≤ 2 tree which extends Q such that $\text{dom}(Q') \setminus \text{dom}(Q) = \{(2, q')\}$, and

1. if $(2, \mathbf{q}) \in \text{desc}(Q)$, then $q' = q^{-\frown}(a)$, $\emptyset \lhd_1^{2Q\{q^-\}} a$, $Q'(q') = Q(q)$;
2. if $(2, \mathbf{q}) \notin \text{desc}(Q)$, then $q' = q^\frown(a)$, $\emptyset \lhd_1^{2Q\{q\}} a$, $Q'_{\text{tree}}(q') = P$.

Let Q_2 be the level ≤ 2 tree defined in a similar way with (q, q') replaced by (r, r'). Let Q' be the tree extending both Q_1 and Q_2 and $\text{dom}(Q') = \text{dom}(Q_1) \cup \text{dom}(Q_2)$. Put $\mathbf{q}' = (q', P, \vec{p})$, $\mathbf{r}' = (r', Z, \vec{z})$. So $\mathbf{q}' \lhd_2^{Q_2} \mathbf{q}$, $\mathbf{r}' \lhd_2^{Q_1} \mathbf{r}$. We partition functions $f \in \omega_1^{Q'\uparrow}$ according to whether $g([f \restriction \text{rep}(Q)]^Q)({}^2[f]^{Q'}_{\mathbf{r}'}) < {}^2[f]^{Q'}_{\mathbf{q}'}$. By Theorem 4.10, we obtain a club $C \in \mu_{\mathbb{L}}$ which is homogeneous for this property. Let $\eta \in C'$ iff $\eta \in C$ and $C \cap \eta$ has order type η. Let $\eta \in C''$ iff $\eta \in C'$ and $C' \cap \eta$ has order type η. If the homogeneous side satisfies $g([f \restriction \text{rep}(Q)]^Q)({}^2[f]^Q_{\mathbf{r}'}) < {}^2[f]^Q_{\mathbf{q}'}$, then every function $f \in (C'')^{Q\uparrow}$ extends to some $f' \in (C')^{Q_1\uparrow}$ by Lemma 3.18, and $\{{}^2[f'']^{Q'}_{\mathbf{r}'} : \exists f'' \in C^{Q'\uparrow}(f' \subseteq f'')\}$ is cofinal in ${}^2[f]^Q_{\mathbf{r}}$ by Lemma 4.28. Hence, $\sup(g([f \restriction \text{rep}(Q)]^Q))''({}^2[f]^Q_{\mathbf{r}}) \le {}^2[f']^{Q_1}_{\mathbf{q}'} < {}^2[f]^Q_{\mathbf{q}}$, contradicting to our assumption. If the homogeneous side satisfies $g([f \restriction \text{rep}(Q)]^Q)({}^2[f]^Q_{\mathbf{r}'}) \ge [f]^Q_{\mathbf{q}'}$, a similar arguments yields $\sup(g([f \restriction \text{rep}(Q)]^Q))''({}^2[f]^Q_{\mathbf{r}}) > {}^2[f]^Q_{\mathbf{q}}$, contradiction again. □

It is easy to deduce the following corollary using Łoś:

Corollary 4.39. *Suppose $\beta < \boldsymbol{\delta}^1_3$ is a limit ordinal. Then β has Q-uniform cofinality $(d, \mathbf{q})$ iff $\mathrm{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]}(\beta) = \mathrm{seed}^Q_{(d,\mathbf{q})}$. In particular, the Q-uniform cofinality of β exists and is unique.*

If π factors finite level ≤ 2 trees (Q, T), then $\pi^T(u_n) = u_m \to \pi^T_{\sup}(u_{n+1}) = u_{m+1}$. Therefore, the continuity of π^T is decided by $\pi^T \restriction \{u_n : n < \omega\}$. If $(d, \mathbf{q}) \in \mathrm{desc}^*(Q)$ is regular, π is *continuous at* $(d, \mathbf{q})$ iff one of the following holds:

1. $d = 1$, either ${}^1\pi(\mathbf{q}) = \min(\prec^{{}^1T})$ or $\mathrm{pred}_{\prec^{{}^1T}}({}^1\pi(q)) \in \mathrm{ran}({}^1\pi)$.
2. $d = 2$, $\mathbf{q} = (\emptyset, \emptyset, ((0)))$, either ${}^1T = \emptyset$ or $\max(\prec^{{}^1T}) \in \mathrm{ran}({}^1\pi)$.
3. $d = 2$, $\mathbf{q} = (q, P, \vec{p}) \in \mathrm{desc}({}^2Q)$, and letting $t' = \max_{<_{BK}} {}^2T\{{}^2\pi(q), -\}$, then either $t' = {}^2\pi(q^-)^\frown(-1)$ or $t' \in \mathrm{ran}({}^2\pi)$.
4. $d = 2$, $\mathbf{q} = (q, P, \vec{p}) \notin \mathrm{desc}({}^2Q)$, and letting $a = \max_{<_{BK}}({}^2T\{{}^2\pi(q)\} \cup \{-1\})$, then either $a = -1$ or ${}^2\pi(q)^\frown(a) \in \mathrm{ran}({}^2\pi)$.

Thus, π is continuous at $(d, \mathbf{q})$ iff π^T is continuous at $\mathrm{seed}^Q_{(d,\mathbf{q})}$. We obtain the following lemma discussing the continuity behavior of π^T. It is the level-2 version of Lemma 3.4.

Lemma 4.40. *Suppose π factors finite level ≤ 2 trees (Q, T), $\gamma < \boldsymbol{\delta}^1_3$ is a limit ordinal, $\mathrm{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]}(\gamma) = \mathrm{seed}^Q_{(d,\mathbf{q})}$, $(d, \mathbf{q}) \in \mathrm{desc}^*(Q)$ is regular. Then*

1. *$\pi^T(\gamma) = \pi^T_{\sup}(\gamma)$ iff π is continuous at $(d, \mathbf{q})$.*
2. *Suppose π is not continuous at $(d, \mathbf{q})$. Let Q^+ be a level ≤ 2 tree and let π^+ factor (Q^+, π) so that Q^+ extends Q, π^+ extends π, and*
   (a) *if $d = 1$, then $\mathrm{dom}(Q^+) \setminus \mathrm{dom}(Q) = \{(1, q^+)\}$, $q \lhd^{{}^1Q}_1 q^+$, ${}^1\pi^+(q^+) = \mathrm{succ}_{\prec^{{}^1T}}({}^1\pi(q^-))$;*
   (b) *if $d = 2$ and $\mathbf{q} = (\emptyset, \emptyset, ((0)))$, then $\mathrm{dom}(Q^+) \setminus \mathrm{dom}(Q) = \{(1, q^+)\}$, $\emptyset \lhd^{{}^1Q}_1 q^+$, ${}^1\pi^+(q^+) = \min_{\prec^{{}^1T}}\{a : \forall r \in \mathrm{dom}({}^1Q)\ {}^1\pi(r) \prec^{{}^1T} a\}$;*
   (c) *if $d = 2$ and $\mathbf{q} = (q, P, \vec{p}) \in \mathrm{desc}({}^2Q)$, then $\mathrm{dom}(Q^+) \setminus \mathrm{dom}(Q) = \{(2, q^+)\}$, $q^+ = \max_{<_{BK}} {}^2Q^+\{q, -\}$, and ${}^2\pi(q^+) = \pi(q^-)^\frown(a)$, $a = \min_{<_{BK}}\{b : \forall r \in {}^2Q(q, -) \setminus \{q^{-\frown}(-1)\}\ {}^2\pi(r) <_{BK} {}^2\pi(q^-)^\frown(b)\}$;*
   (d) *if $d = 2$ and $\mathbf{q} = (q, P, \vec{p}) \notin \mathrm{desc}({}^2Q)$, then $\mathrm{dom}(Q^+) \setminus \mathrm{dom}(Q) = \{(2, q^+)\}$, $q^+ = q^\frown(\max_{<_{BK}} {}^2Q^+\{q\})$, ${}^2\pi^+(q^+) = {}^2\pi(q)^\frown(a)$, $a = \min_{<_{BK}}\{b : \forall c \in {}^2Q\{q\}\ {}^2\pi(q^\frown(c)) <_{BK} {}^2\pi(q)^\frown(b)\}$.*

Then $\pi^T_{\sup}(\gamma) = (\pi^+)^T \circ j^{Q,Q^+}_{\sup}(\gamma)$.

If π factors finite level ≤ 2 trees (Q, T) and π is discontinuous at $(d, \mathbf{q})$, then $\text{pred}(\pi, T, (d, \mathbf{q}))$ is a node in $\text{dom}(T)$ defined as follows:

1. If $d = 1$, then $\text{pred}(\pi, T, (d, \mathbf{q})) = (1, \text{pred}_{\prec^1 T}({}^1\pi(\mathbf{q})))$.
2. If $d = 2$ and $\mathbf{q} = (\emptyset, \emptyset, ((0)))$, then $\text{pred}(\pi, T, (d, \mathbf{q})) = (1, \max_{<_{BK}} {}^1T)$.
3. If $d = 2$ and $\mathbf{q} = (q, P, \vec{p}) \in \text{desc}(Q)$, $q \neq \emptyset$, then $\text{pred}(\pi, T, (d, \mathbf{q})) = (2, \max_{<_{BK}} {}^2T\{{}^2\pi(q), -\})$.
4. If $d = 2$ and $\mathbf{q} = (q, P, \vec{p}) \notin \text{desc}(Q)$, $q \neq \emptyset$, then $\text{pred}(\pi, T, (d, \mathbf{q})) = (2, q^\frown(a))$, $a = \max_{<_{BK}} {}^2T\{{}^2\pi(q)\}$.

If $(2, \mathbf{q}) = (2, (q, P, \vec{p})) \in \text{desc}(Q)$ then put $\text{pred}(\pi, T, (2, q)) = \text{pred}(\pi, T, (2, \mathbf{q}))$. The next lemma is the level-2 version of Lemma 3.5, whose proof is similar.

Lemma 4.41. *Suppose $(Q^-, (d, q, P))$ is a partial level ≤ 2 tree, T is a finite level ≤ 2 tree, π factors (Q, T), and π is discontinuous at (d, q). Let τ factor (Q, T) where τ and π agree on* $\text{dom}(Q) \setminus \{(d, q)\}$, $\tau(d, q) = \text{pred}(\pi, T, (d, q))$. *Suppose* $\text{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^{Q^-}(T_2)]}(\gamma) = \text{seed}^{Q^-}_{\text{ucf}(Q^-, (d,q,P))}$. *Then*

$$\pi^T \circ j^{Q^-,Q}_{\sup}(\gamma) = \tau^T_{\sup} \circ j^{Q^-,Q}(\gamma).$$

The level-2 version of Lemma 3.6 is similarly proved.

Lemma 4.42. *Suppose $(Q, (d, q, P))$ is a partial level ≤ 2 tree,* $\text{ucf}(Q, (d, q, P)) = (d^*, \mathbf{q}^*)$, *$T$ is a finite level ≤ 2 tree, π factors (Q, T), and π is discontinuous at $(d^*, \mathbf{q}^*)$. Let Q^+ be a completion of $(Q, (d, q, P))$ and let τ factor (Q^+, T) so that τ extends π, $\tau(d, q) =$* $\text{pred}(\pi, T, (d^*, \mathbf{q}^*))$. *Suppose* $\text{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^Q(T_2)]}(\gamma) = \text{seed}^Q_{(d^*, \mathbf{q}^*)}$. *Then*

$$\pi^T(\gamma) = \tau^T_{\sup} \circ j^{Q,Q^+}(\gamma).$$

Note that in Lemma 4.42, the completion Q^+ is decided by $\text{pred}(\pi, T, (d^*, \mathbf{q}^*))$. There is no freedom in choosing Q^+.

In the same spirit as Lemmas 3.15-3.20 we will obtain a concrete way of deciding whether a tuple $\vec{\gamma}$ respects a level-3 tree R.

Suppose E is a club in ω_1. For a partial level ≤ 2 tree $(Q, (d, q, P))$, put $\vec{\alpha} = ({}^e\alpha_t)_{(e,t) \in \text{dom}(Q, (d,q,P))} \in [E]^{(Q,(d,q,P))\uparrow}$ iff $\vec{\alpha}$ respects $(Q, (d, q, P))$,

$({}^e\alpha_t)_{(e,t)\in\mathrm{dom}(Q)} \in [E]^{Q\uparrow}$, and $d=1 \to {}^1\alpha_q \in E$, $d=2 \to {}^2\alpha_q \in j^P(E)$. For a level-3 tree R, put

$$\begin{aligned}\mathrm{rep}(R)\restriction E = &\{\vec\beta \oplus_R r : r \in \mathrm{dom}(R), \vec\beta \in [E]^{R_{\mathrm{tree}}(r)\uparrow}\}\\ &\cup \{\vec\beta \oplus_R r^\frown(-1) : r \in \mathrm{dom}(R), \vec\beta \in [E]^{R(r)\uparrow}\}.\end{aligned}$$

Then $\mathrm{rep}(R)\restriction E$ is a closed subset of $\mathrm{rep}(R)$ (in the order topology of $<^R$).

Lemma 4.43. *Suppose R is a finite level-3 tree, $B \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$ is a closed set of ordinals. Then $\vec\gamma \in [B]^{R\uparrow}$ iff there is $F \in (\boldsymbol{\delta}^1_3)^{R\uparrow}$ and $E \in \mu_{\mathbb{L}}$ such that $\vec\gamma = [F]^R$ and for any $r \in \mathrm{dom}(R)$, for any $\vec\beta \in [E]^{R(r)\uparrow}$, $F_r(\vec\beta)$ is a limit point of B.*

Proof. The nontrivial direction is $\Leftarrow$. Suppose $F \in (\boldsymbol{\delta}^1_3)^{R\uparrow}$ and $E \in \mu_{\mathbb{L}}$ are as given. For $r \in \mathrm{dom}(R)$, let $R(r) = (Q_r, (d_r, q_r, P_r))$, and let r^* be the $<_{BK}$-greatest member of $R\{r,-\}$. In parallel to Claim 3.16, by Theorem 4.38 and cofinality considerations in $\mathbb{L}_{\boldsymbol{\delta}^1_3}[j^{Q_r}(T_2)]$, we have

Claim 4.44. *There is $E' \in \mu_{\mathbb{L}}$ such that $E' \subseteq E$ and for any $r \in \mathrm{dom}(R)$, for any $\vec\beta \in [E']^{P_q\uparrow}$,*

1. *if $d_r = 1$ then $B \cap (F_{r^*}(\vec\beta), F_r(\vec\beta))$ has order type ${}^1\beta_{q_r^-}$;*
2. *if $d_r = 2$, $\mathrm{ucf}(R[r]) = (2, \mathbf{q}_{r,*})$, $\mathbf{q}_{r,*} = (q_{r,*}, P_{r,*}, \vec p_{r,*})$, then $B \cap (F_{r^*}(\vec\beta), F_r(\vec\beta))$ has order type ${}^2\beta_{q_{r,*}}$.*

The rest proceeds as in the proof of Lemma 3.15. □

Lemma 4.45. *Suppose R is a finite level-3 tree, $R[r] = (Q_r, (d_r, q_r, P_r))$ for $r \in \mathrm{dom}(R)$, $E \in \mu_{\mathbb{L}}$ is a club. Suppose $f : \mathrm{rep}(R)\restriction E \to \boldsymbol{\delta}^1_3$ satisfies*

1. *if $r \in \mathrm{dom}(R)$, then the Q_r-potential partial level ≤ 2 tower induced by F_r is $R[r]$, the approximation sequence of F_r is $(F_{r\restriction i})_{1\leq i\leq \mathrm{lh}(q)}$, and the uniform cofinality of F_r on $[E]^{Q_r\uparrow}$ is witnessed by $F_{r^\frown(-1)}$, i.e., if $\vec\beta \in [E]^{Q_r\uparrow}$, then $F_r(\vec\beta) = \sup\{F_{r^\frown(-1)}(\vec\beta^\frown(\gamma)) : \vec\beta^\frown(\gamma) \in \mathrm{rep}(R)\restriction E\}$, and the map $\vec\beta \mapsto F_{r^\frown(-1)}(\vec\beta^\frown(\gamma))$ is continuous, order preserving.*
2. *if $R_{\mathrm{tree}}(r^\frown(a)) = R_{\mathrm{tree}}(r^\frown(b))$, and $a <_{BK} b$, then $[F_{r^\frown(a)}]_{\mu^{Q_{r^\frown(a)}}} < [F_{r^\frown(b)}]_{\mu^{Q_{r^\frown(b)}}}$.*

Then there is $E' \in \mu_{\mathbb{L}}$ such that $E' \subseteq E$ and $f \restriction (\mathrm{rep}(R)\restriction E')$ is order preserving.

Proof. Put $\mathrm{ucf}(R[r]) = (d_r, \mathbf{q}_{r,*})$, and if $d_r = 2$ then $\mathbf{q}_{r,*} = (q_{r,*}, P_{r,*}, \vec{p}_{r,*})$. We know by assumption that for μ^{Q_r}-a.e. $\vec{\beta}$, $F_r(\vec{\beta}) = \sup\{F_{r^\frown(a)}(\vec{\beta}^\frown(\gamma)) : r^\frown(a) \in \mathrm{dom}(R), \gamma < {}^{d_r}\beta_{q_{r,*}}\}$. Fix for the moment r such that $d_r \neq 0$. Similarly to the proof of Lemma 3.17, we need a club $E' \in \mu_{\mathbb{L}}$ such that for any $\vec{\beta} \in [E']^{Q_r\uparrow}$, for any $\gamma < \gamma'$ both in $j^{P_r}(E')$, if $R_{\mathrm{tree}}(r^\frown(a)) = R_{\mathrm{tree}}(r^\frown(b))$ then $F_{r^\frown(a)}(\vec{\beta}^\frown(\gamma)) < F_{r^\frown(b)}(\vec{\beta}^\frown(\gamma'))$.

If $\vec{\beta}$ respects Q_r and $\vec{\beta}^\frown(\gamma)$ respects $R(r)$, let $g(\vec{\beta}^\frown(\gamma))$ be the least γ' satisfying that whenever $r^\frown(a), r^\frown(b) \in \mathrm{dom}(R)$, $\delta \leq \gamma$, $\delta' \geq \gamma'$, $\vec{\beta}^\frown(\delta)$ respects $R(r^\frown(a))$, $\vec{\beta}^\frown(\delta')$ respects $R(r^\frown(b))$, we have $F_{r^\frown(a)}(\vec{\beta}^\frown(\delta)) < F_{r^\frown(b)}(\vec{\beta}^\frown(\delta'))$. If Q^+ is a completion of $R(r)$, then for μ^{Q^+}-a.e. $\vec{\xi}$, $g(\vec{\xi}) < {}^{d_r}\xi_{\mathbf{q}_{r,*}}$. By Lemma 4.31, there is $h^{Q^+} : \omega_1 \to \omega_1$ and $E^{Q^+} \in \mu_{\mathbb{L}}$ such that $h \in \mathbb{L}$ and for any $\vec{\xi} \in [E^{Q^+}]^{Q^+\uparrow}$, $g(\vec{\xi}) < j^{P_r}(h^{Q^+})({}^{d_r}\xi_{q_r})$. There are only finitely many completions of $R(r)$. Let $h : \omega_1 \to \omega_1$ where $h(\alpha) = \sup\{h^{Q^+}(\alpha) : Q^+$ is a completion of $R(r)\}$. Let $E' = \cap\{E^{Q^+} : Q^+$ is a completion of $R(r)\}$. Let $\eta \in E''$ iff $h''(\eta \cap E') \subseteq \eta$. E'' is as desired. □

As corollaries of Lemmas 4.43 and 4.45, we obtain:

Lemma 4.46. *Suppose that R is a level-3 tree and $\vec{\gamma} = (\gamma_r)_{r \in \mathrm{dom}(R)}$ is a tuple of ordinals in $\boldsymbol{\delta}^1_3$. Then $\vec{\gamma}$ respects R iff the following holds:*

1. *For any $r \in \mathrm{dom}(R)$, the $R_{\mathrm{tree}}(r)$-potential partial level ≤ 2 tower induced by γ_r is $R[r]$, and the $R_{\mathrm{tree}}(r)$-approximation sequence of γ_r is $(\gamma_{r \restriction l})_{1 \leq l \leq \mathrm{lh}(r)}$.*
2. *If $R_{\mathrm{tree}}(r^\frown(a)) = R_{\mathrm{tree}}(r^\frown(b))$ and $a <_{BK} b$ then $\gamma_{r^\frown(a)} < \gamma_{r^\frown(b)}$.*

Moreover, if $B \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$ is a closed set, B' is the set of limit points of B, then $\vec{\gamma} \in [B]^{R\uparrow}$ iff $\vec{\gamma}$ respects R and for each $r \in \mathrm{dom}(R)$, $\gamma_r \in j^{R_{\mathrm{tree}}(r)}(B')$.

In particular, if $\vec{\gamma}$ respects R, then $\mathrm{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]}(\gamma) = u_{\mathrm{cf}(R(r))}$ for $r \in \mathrm{dom}(R)$, where $u_0 = \omega$.

Lemma 4.47. *Suppose R and R' are level-3 trees with the same domain. Suppose $\vec{\gamma}$ respects both R and R'. Then $R = R'$.*

4.6 Factoring maps between level-2 trees

Definition 4.48. Suppose $I < \omega$. Suppose for each $i < I$, $\bar{J}_i \leq J_i < \omega$ and $A_i = (a_{i,j})_{\bar{J}_i \leq j < J_i}$ is a finite sequence of sets. Then the *contraction* of $(A_i)_{i<I}$ is $(b_k)_{k<K}$ such that

1. $\{a_{i,j} : i < I, \bar{J}_i < j < J_i\} = \{b_k : k < K\}$.

2. For each $k < K$, letting (i^k, j^k) be the $<_{BK}$-least (i,j) such that $a_{i,j} = b_k$, then the map $k \mapsto (i^k, j^k)$ is order preserving with respect to $<$ and $<_{BK}$.

Definition 4.49. Suppose T, Q are level ≤ 2 trees. A $(T, Q, -1)$-*description* is of the form

$$\mathbf{C} = (1, (\mathbf{t}, \emptyset))$$

such that $\mathbf{t} \in {}^1T$. Suppose $(W, \vec{w})$ is a potential partial level ≤ 1 tower of discontinuous type, $\vec{w} = (w_i)_{i \leq m}$. If $w = 0$, the only $(T, Q, (W, \vec{w}))$-*description* is $(2, ((\emptyset, \emptyset, \emptyset), \tau))$, where τ factors $(\emptyset, Q, \emptyset)$, which is called *the constant* $(T, Q, *)$-*description.* If $w > 0$, a $(T, Q, (W, \vec{w}))$-*description* is of the form

$$\mathbf{C} = (2, (\mathbf{t}, \tau))$$

such that

1. $\mathbf{t} \in \operatorname{desc}({}^2T)$ and $\mathbf{t} \neq (\emptyset, \emptyset, \emptyset)$. Let $\mathbf{t} = (t, S, \vec{s})$, $\operatorname{lh}(t) = k$, $\vec{s} = (s_i)_{i < \operatorname{lh}(\vec{s})}$.
2. τ factors (S, Q, W).
3. The contraction of $(\operatorname{sign}_1(\tau(s_i)))_{i<k}$ is $(w_i)_{i<m}$.
4. If t is of continuous type and w_{m-1} does not appear in the contraction of $(\operatorname{sign}_1(\tau(s_i)))_{i<k-1}$ then $\tau(s_{k-1})$ is of level-1 discontinuous type.
5. Either $\operatorname{ucf}(S, \vec{s}) = w_m = -1$ or $\operatorname{ucf}_1(\tau(\operatorname{ucf}(S, \vec{s}))) = \operatorname{ucf}(W, \vec{w})$.

A $(T, Q, *)$-description is either a $(T, Q, -1)$-description or a $(T, Q, (W', \vec{w}'))$-description for some potential partial level ≤ 1 tower $(W', \vec{w}')$ of discontinuous type. For a level-1 tree W, a (T, Q, W)-description is a $(T, Q, (W, \vec{w}'))$-description for some $\vec{w}'$. $\operatorname{desc}(T, Q, -1)$, $\operatorname{desc}(T, Q, (W, \vec{w}))$, $\operatorname{desc}(T, Q, *)$, $\operatorname{desc}(T, Q, W)$ denote the sets of relevant descriptions. We sometimes abbreviate $(d, \mathbf{t}, \tau)$ for $(d, (\mathbf{t}, \tau)) \in \operatorname{desc}(T, Q, *)$ without confusion.

Recalling our notation of $Q \otimes W$, we may regard $\operatorname{desc}(T, Q, -1) \subseteq \operatorname{desc}(T, \emptyset)$, $\operatorname{desc}(T, Q, W) \subseteq \operatorname{desc}(T, Q \otimes W)$. $T \otimes (Q \otimes W)$ is also a "level-1 tree", whose nodes consist of non-constant $(T, Q \otimes W)$-descriptions, so that $\operatorname{desc}(T, Q \otimes W) = \operatorname{desc}(T \otimes (Q \otimes W))$. Every non-constant (T, Q, W)-description is a member of $T \otimes (Q \otimes W)$. The constant $(T, Q, *)$-description $\mathbf{C}_0$ is regarded as the constant $T \otimes (Q \otimes W)$-description, to make sense of $\operatorname{seed}^{T \otimes (Q \otimes W)}_{\mathbf{C}_0}$. The *degree* of $(d, \mathbf{t}, \tau) \in \operatorname{desc}(T, Q, *)$ is d. In fact, if $\mathbf{C} \in \operatorname{desc}(T, Q, *)$ is of degree 2, then there is a unique potential partial level ≤ 1 tower $(W, \vec{w})$ for which $\mathbf{C} \in \operatorname{desc}(T, Q, (W, \vec{w}))$.

Suppose now $\mathbf{C} = (d, \mathbf{t}, \tau) \in \mathrm{desc}(T, Q, *)$, and if $d = 2$, then $\mathbf{C} \in \mathrm{desc}(T, Q, (W, \vec{w}))$, $\mathbf{t} = (t, S, \vec{s})$, $\mathrm{lh}(t) = k$, $\vec{s} = (s_i)_{i<\mathrm{lh}(\vec{s})}$, $\vec{w} = (w_i)_{i \le m}$. The *level-2 signature* of $\mathbf{C}$ is

$$\mathrm{sign}_2(\mathbf{C}) = \begin{cases} \emptyset & \text{if } d = 1, \\ \text{the contraction of } (\mathrm{sign}_2^W(\tau(s_i)))_{i<k} & \text{if } d = 2. \end{cases}$$

$\mathbf{C}$ is *of level-2 continuous type* iff $d = 2$, t is of continuous type, and $\tau(s_{k-1})$ is of level-2 W-continuous type; $\mathbf{C}$ is *of level-2 discontinuous type* otherwise. The *level-2 uniform cofinality* of $\mathbf{C} = (d, \mathbf{t}, \tau)$ is

$$\mathrm{ucf}_2(\mathbf{C})$$

defined as follows. If $d = 1$, then $\mathrm{ucf}_2(\mathbf{C}) = (0, -1)$. If $d = 2$ then

1. if $\mathrm{ucf}(S, \vec{s}) = -1$, then $\mathrm{ucf}_2(\mathbf{C}) = (0, -1)$;
2. if $\mathrm{ucf}(S, \vec{s}) = s_* \neq -1$, then $\mathrm{ucf}_2(\mathbf{C}) = \mathrm{ucf}_2^W(\tau(s_*))$.

If $w_m \neq -1$, $\mathbf{C}$ is said to be of *level-2+ discontinuous type*, and put

$$\mathrm{ucf}_2^+(\mathbf{C}) = \mathrm{ucf}_2^{W^+}(\tau(\mathrm{ucf}(S, \vec{s}))),$$

where W^+ is the completion of (W, w_m). The *level-*$2*$ *signature* of $\mathbf{C}$ is

$$\mathrm{sign}_{2*}(\mathbf{C}) = \begin{cases} ((1, \mathbf{t})) & \text{if } d = 1, \\ ((2, t \restriction i))_{1 \le i \le k-1} & \text{if } d = 2, t \text{ is of continuous type}, \\ ((2, t \restriction i))_{1 \le i \le k} & \text{if } d = 2, t \text{ is of discontinuous type}. \end{cases}$$

$\mathbf{C}$ is *of level-*$2*$ *Q-continuous type* iff $d = 2$ and if $\mathrm{ucf}(S, \vec{s}) \neq -1 \wedge \tau(\mathrm{ucf}(S, \vec{s})) \neq \min(\prec^{Q,W})$), then $\mathrm{pred}_{\prec^{Q,W}}(\tau(\mathrm{ucf}(S, \vec{s}))) \in \mathrm{ran}(\tau)$. Otherwise, $\mathbf{C}$ is *of level-*$2*$ *Q-discontinuous type*. The *level-*$2*$ *Q-uniform cofinality* of $\mathbf{C}$ is

$$\mathrm{ucf}_{2*}^Q(\mathbf{C})$$

defined as follows. If $d = 1$, then $\mathrm{ucf}_{2*}^Q(\mathbf{C}) = (1, \mathbf{t})$. If $d = 2$, t is of continuous type,

1. if $\mathbf{C}$ is of Q-continuous type, then $\mathrm{ucf}_{2*}^Q(\mathbf{C}) = (2, (t^-, S \setminus \{s_{k-1}\}, \vec{s}))$;
2. if $\mathbf{C}$ is of Q-discontinuous type, then $\mathrm{ucf}_{2*}^Q(\mathbf{C}) = (2, (t^-, S, \vec{s}))$.

If $d = 2$, t is of discontinuous type,

1. if $\mathbf{C}$ is of Q-continuous type, then $\mathrm{ucf}_{2*}^Q(\mathbf{C}) = (2, \mathbf{t})$;

2. if $\mathbf{C}$ is of Q-discontinuous type, then $\mathrm{ucf}^Q_{2*}(\mathbf{C}) = (2, (t, S \cup \{s_k\}, \vec{s}))$.

For $h \in \omega_1^{T\uparrow}$, if $\mathbf{C} = (1, \mathbf{t}, \emptyset)$ is a $(T, Q, -1)$-description, then

$$h^Q_{\mathbf{C}} : [\omega_1]^{Q\uparrow} \to \omega_1$$

is the function sending $\vec{\alpha}$ to ${}^1[h]^T_{\mathbf{t}}$ if $\min(\vec{\alpha}) > {}^1[h]^T_{\mathbf{t}}$, sending $\vec{\alpha}$ to $\|(1, \mathbf{t})\|_{<T}$ otherwise; if W is a (possibly empty) level-1 tree, $\mathbf{C} = (2, \mathbf{t}, \tau)$ is a (T, Q, W)-description, $\mathbf{t} = (t, S, \vec{s})$, then

$$h^Q_{\mathbf{C}} : [\omega_1]^{Q\uparrow} \to j^W(\omega_1).$$

is the function that sends $[g]^Q$ to $[{}^2h_{\mathbf{t}} \circ g^W_\tau]_{\mu^W}$. Note here that ${}^2h_{\mathbf{t}} \circ g^W_\tau$ has signature $\mathrm{sign}(W, \vec{w})$, is essentially discontinuous, and has uniform cofinality $\mathrm{ucf}(W, \vec{w})$. In either case, when Q is finite, we have the following: the signature of $h^Q_{\mathbf{C}}$ is $\mathrm{sign}_2(\mathbf{C})$; $h^Q_{\mathbf{C}}$ is essentially continuous iff $\mathbf{C}$ is of level-2 continuous type; the uniform cofinality of $h^Q_{\mathbf{C}}$ is $\mathrm{ucf}_2(\mathbf{C})$. If W^+ is the completion of (W, w_m), then $j^{W,W^+} \circ h^Q_{\mathbf{C}}$ is of discontinuous type and has cofinality $\mathrm{ucf}^+_2(\mathbf{C})$. Moreover, $\mathrm{ran}(h^Q_{\mathbf{C}}) \subseteq \mathrm{ran}(h)$ if $d = 1$, $\mathrm{ran}(h^Q_{\mathbf{C}}) \subseteq j^W(\mathrm{ran}(h))$ if $d = 2$. When T, Q are both finite, $\mathbf{C} = (d, \mathbf{t}, \tau) \in \mathrm{desc}(T, Q, *)$,

$$\mathrm{id}^{T,Q}_{\mathbf{C}}$$

is the function $[h]^T \mapsto [h^Q_{\mathbf{C}}]_{\mu^Q}$, or equivalently, $\vec{\xi} \mapsto {}^1\xi_{\mathbf{t}}$ when $d = 1$, $\vec{\xi} \mapsto \tau^{Q,W}({}^2\xi_{\mathbf{t}})$ when $d = 2$ and $\mathbf{C} \in \mathrm{desc}(T, Q, W)$. The signature of $\mathrm{id}^{T,Q}_{\mathbf{C}}$ is $\mathrm{sign}^Q_2(\mathbf{C})$; $\mathrm{id}^{T,Q}_{\mathbf{C}}$ is essentially continuous iff $\mathbf{C}$ is of level-2$*$ Q-continuous type; the uniform cofinality of $\mathrm{id}^{T,Q}_{\mathbf{C}}$ is $\mathrm{ucf}^Q_{2*}(\mathbf{C})$.

$$\mathrm{seed}^{T,Q}_{\mathbf{C}} \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[j^T \circ j^Q(T_2)]$$

is the element represented modulo μ^T by $\mathrm{id}^{T,Q}_{\mathbf{C}}$. Using Łoś, it is clear that if $d = 1$, then for any $A \in \mu_{\mathbb{L}}$, $\mathrm{seed}^{T,Q}_{\mathbf{C}} \in j^T \circ j^Q(A)$; if $d = 2$, then for any $A \in \mu^{(W,\vec{w})}$, $\mathrm{seed}^{T,Q}_{\mathbf{C}} \in j^T \circ j^Q(A)$. Using Lemma 4.32, we can see that $\mathrm{seed}^{T,Q}_{\mathbf{C}} \in \{u_n : n < \omega\}$, and $\mathrm{seed}^{T,Q}_{\mathbf{C}}$ can be computed in the following concrete way:

- If $d = 1$, then $\mathrm{seed}^{T,Q}_{\mathbf{C}} = \mathrm{seed}^{T,\emptyset}_{\mathbf{C}} = \mathrm{seed}^{T\otimes\emptyset}_{\mathbf{C}}$.
- If $d = 2$ and $\mathbf{C} \in \mathrm{desc}(T, Q, W)$, then $\mathrm{seed}^{T,Q}_{\mathbf{C}} = \mathrm{seed}^{T,Q\otimes W}_{\mathbf{C}} = \mathrm{seed}^{T\otimes(Q\otimes W)}_{\mathbf{C}}$.

If $\mathbf{C} = (1, \mathbf{t}, \emptyset)$, let

$$\mathbf{C}^{T,Q} : \mathbb{L}_{\boldsymbol{\delta}^1_3}[j_{\mu_{\mathbb{L}}}(T_2)] \to \mathbb{L}_{\boldsymbol{\delta}^1_3}[j^T \circ j^Q(T_2)]$$

where $\mathbf{C}^{T,Q}(j_{\mu_{\mathbb{L}}}(h)(\omega_1)) = j^T \circ j^Q(\mathrm{seed}^{T,Q}_{\mathbf{C}})$. If $\mathbf{C} = (2, \mathbf{t}, \tau)$, let

$$\mathbf{C}^{T,Q} : \mathbb{L}_{\boldsymbol{\delta}^1_3}[j^{(W,\vec{w})}(T_2)] \to \mathbb{L}_{\boldsymbol{\delta}^1_3}[j^T \circ j^Q(T_2)]$$

where $\mathbf{C}^{T,Q}(j^{(W,\vec{w})}(h)(\mathrm{seed}^{(W,\vec{w})})) = j^T \circ j^Q(\mathrm{seed}^{T,Q}_{\mathbf{C}})$.

Suppose $(\vec{W}, \vec{w}) = (W_i, w_i)_{i \le m}$ is a potential partial level-1 tower. If $\mathbf{C} = (2, \mathbf{t}, \tau) \in \mathrm{desc}(T, Q, (W_m, \vec{w}))$, $\mathbf{t} = (t, S, \vec{s})$, define $\mathrm{lh}(\mathbf{C}) = m$. If $\bar{m} < m$, then

$$\mathbf{C} \restriction \bar{m} \in \mathrm{desc}(T, Q, (W_l, (w_i)_{i \le l}))$$

is defined by the following: letting l be the least such that $\tau(s_l) \notin \mathrm{desc}(Q, W_{\bar{m}})$, and $\mathbf{D} \in \mathrm{desc}(Q, W_{\bar{m}})$ be such that $\mathbf{D} \lhd_1^{Q, W_{\bar{m}}} \tau(s_l)$, then

1. if $\mathbf{D} \neq \tau(s_l^-)$, then $\mathbf{C} \restriction \bar{m} = (2, \mathbf{t} \restriction l^\frown(-1), \bar{\tau})$, where $\bar{\tau}$ and τ agree on ${}^2T_{\mathrm{tree}}(t \restriction l)$, $\bar{\tau}(s_l) = \mathbf{D}$;
2. if $\mathbf{D} = \tau(s_l^-)$, then $\mathbf{C} \restriction \bar{m} = (2, \mathbf{t} \restriction l, \tau \restriction {}^2T_{\mathrm{tree}}(t \restriction l))$.

Define

$$\mathbf{C} \lhd \mathbf{C}'$$

iff $\mathbf{C} = \mathbf{C}' \restriction \bar{m}$ for some $\bar{m} < \mathrm{lh}(\mathbf{C}')$. Define $\prec^{T,Q} = \prec \restriction \mathrm{desc}(T, Q, *)$. As a corollary to Lemma 4.27, $\lhd^{T,Q}$ inherits the following continuity property.

Lemma 4.50. *Suppose T, Q are finite level ≤ 2 trees, W is a level-1 proper subtree of W'. Suppose $E \in \mu_{\mathbb{L}}$ is a club, E' is the set of limit points of E. Suppose $\mathbf{C} = (2, \mathbf{t}, \tau) \in \mathrm{desc}(T, Q, W)$, $\mathbf{C}' = (2, \mathbf{t}', \tau') \in \mathrm{desc}(T, Q, W')$, $\mathbf{C} \lhd^{T,Q} \mathbf{C}'$. Then for any $h \in \omega_1^{T\uparrow}$, for any $g \in \omega_1^{Q\uparrow}$, for any $\vec{\alpha} \in [E']^{W\uparrow}$,*

$$h_{\mathbf{t}} \circ g^W_\tau(\vec{\alpha}) = \sup\{h_{\mathbf{t}'} \circ g^{W'}_{\tau'}(\vec{\beta}) : \vec{\beta} \in [E]^{W'\uparrow}, \vec{\beta} \text{ extends } \vec{\alpha}\}.$$

Hence, the signature and approximation sequence of $h_{\mathbf{t}} \circ g^W_\tau$ are proper initial segments of those of $h_{\mathbf{t}'} \circ g^{W'}_{\tau'}$ respectively.

Let

$$\langle \mathbf{C} \rangle = \begin{cases} (1, \mathbf{t}) & \text{if } d = 1, \\ (2, \tau \oplus \mathbf{t}) & \text{if } d = 2. \end{cases}$$

Define

$$\mathbf{C} \prec \mathbf{C}'$$

iff $\langle \mathbf{C} \rangle <_{BK} \langle \mathbf{C}' \rangle$, the ordering on subcoordinates in $\mathrm{desc}(Q, *) \cup \mathrm{desc}(Q', *)$ according to $\prec$ acting on $\mathrm{desc}(Q, *) \cup \mathrm{desc}(Q', *)$. The constant $(T, Q, *)$-description, $\mathbf{C}_0$, is the $\prec$-greatest, and we have $\langle \mathbf{C}_0 \rangle = (2, \emptyset)$. Define $\prec^{T,Q} = \prec \restriction \mathrm{desc}(T, Q, *)$. $\prec^{T,Q}$ inherits the following ordering property as a corollary to Lemma 4.25.

Lemma 4.51. *Suppose* $(\vec{W},\vec{w}) = (W_i,w_i)_{i\le m}$ *is a partial level* ≤ 1 *tower. Suppose* $\mathbf{C} = (2,\mathbf{t},\tau) \in \operatorname{desc}(T,Q,W_k)$, $\mathbf{C}' = (2,\mathbf{t}',\tau') \in \operatorname{desc}(T,Q,W_{k'})$, $k \le m$, $k' \le m$, $\mathbf{C} \prec^{T,Q} \mathbf{C}'$. *Then for any* $h \in \omega_1^{T\uparrow}$, *for any* $g \in \omega_1^{Q\uparrow}$, *for any* $\vec{\alpha} \in \omega_1^{W_m\uparrow}$, $h_{\mathbf{t}} \circ g_\tau^{W_m}(\vec{\alpha}) < h_{\mathbf{t}'} \circ g_{\tau'}^{W_m}(\vec{\alpha})$.

Definition 4.52. Suppose X,T,Q are level ≤ 2 trees. Suppose $\pi : \operatorname{dom}(X) \to \operatorname{desc}(T,Q,*)$ is a function. π is said to *factor* (X,T,Q) iff

1. If $(1,x) \in \operatorname{dom}(X)$, then $\pi(1,x) \in \operatorname{desc}(T,Q,-1) \cup \operatorname{desc}(T,Q,{}^2X[\emptyset])$.
2. If $(2,x) \in \operatorname{dom}(X)$, then $\pi(2,x) \in \operatorname{desc}(T,Q,{}^2X[x])$.
3. $\pi(2,\emptyset)$ is the constant $(T,Q,*)$-description.
4. For any $(d,x),(d',x') \in \operatorname{dom}(X)$, if $(d,x) <_{BK} (d',x')$ then $\pi(d,x) \prec^{T,Q} \pi(d',x')$.
5. For any $x \in \operatorname{dom}({}^2X) \setminus \{\emptyset\}$, $\pi(2,x^-) \lhd^{T,Q} \pi(2,x)$.

π is said to *factor* $(X,T,*)$ iff π factors (X,T,Q') for some level ≤ 2 tree Q'.

Suppose T is a level ≤ 2 tree.

$$\mathrm{id}_{T,*}$$

factors (T,T,Q^0) where $\mathrm{id}_{T,*}(1,t) = (1,t,\emptyset)$, $\mathrm{id}_{T,*}(2,t) = (2,(t,S,\vec{s}),\mathrm{id}_{*,S})$ when ${}^2T[t] = (S,\vec{s})$.

$$\mathrm{id}_{*,T}$$

factors (T,Q^0,T) where $\mathrm{id}_{*,T}(1,t) = (2,\mathbf{q}_0,\tau_t^1)$, $\mathbf{q}_0 = ((-1),\{(0)\},((0)))$, τ_t^1 factors $(\{(0)\},T,\emptyset)$, $\tau_t^1((0)) = (1,t,\emptyset)$, $\tau_{*,T}(2,t) = (2,\mathbf{q}_0,\tau_t^2)$ when ${}^2T[t] = (S,\vec{s})$, τ_t^2 factors $(\{(0)\},T,S)$, $\tau_t^2((0)) = (2,(t,S,\vec{s}),\mathrm{id}_S)$.

If $\pi \ne \emptyset$ factors (X,T,Q), T is Π^1_2-wellfounded and $h \in \omega_1^{T\uparrow}$, let

$$h_\pi^Q : [\omega_1]^{Q\uparrow} \to [\omega_1]^{X\uparrow}$$

be the function that sends $\vec{\beta}$ to $(h^Q_{\pi(d,x)}(\vec{\beta}))_{(d,x)\in\operatorname{dom}(X)}$. The fact that $h_\pi^Q(\vec{\beta}) \in [\omega_1]^{X\uparrow}$ follows from Lemmas 3.18, 4.51-4.50. Moreover, for any $\vec{\beta} \in [\omega_1]^{Q\uparrow}$, $h_\pi^Q(\vec{\beta}) \in [\operatorname{ran}(h)]^{X\uparrow}$. In particular, if $Q = X$ then $h^Q_{\mathrm{id}_{*,X}}$ is the identity function on $[\omega_1]^{X\uparrow}$. If T is finite, let

$$\mathrm{id}_\pi^{T,Q}$$

be the function $[h]^T \mapsto [h_\pi^Q]_{\mu^Q}$, or equivalently, $\vec{\xi} \mapsto (\mathrm{id}^{T,Q}_{\pi(d,x)}(\vec{\xi}))_{(d,x)\in\operatorname{dom}(X)}$. Let

$$\mathrm{seed}_\pi^{T,Q} = [\mathrm{id}_\pi^{T,Q}]_{\mu^T}.$$

By Łoś and Lemmas 3.18, 4.51-4.50, it is clear that for any $A \in \mu^X$, $\mathrm{seed}^{T,Q}_\pi \in j^T \circ j^Q(A)$. Define

$$\pi^{T,Q} : \mathbb{L}_{\boldsymbol{\delta}^1_3}[j^X(T_2)] \to \mathbb{L}_{\boldsymbol{\delta}^1_3}[j^T \circ j^Q(T_2)]$$

by sending $j^X(h)(\mathrm{seed}^X)$ to $j^T \circ j^Q(h)(\mathrm{seed}^{T,Q}_\pi)$.

Suppose T, Q are both level ≤ 2 trees. *A representation of* $T \otimes Q$ is a pair (X, π) such that

1. X is a level ≤ 2 tree;
2. π factors (X, T, Q);
3. $\mathrm{ran}(\pi) = \mathrm{desc}(T, Q, *)$;
4. $(d, x) <_{BK} (d', x')$ iff $\pi(d, x) \prec^{T,Q} \pi(d', x')$.

Representations of $T \otimes Q$ are clearly mutually isomorphic. We shall regard

$$T \otimes Q$$

itself as a "level $\leq$ 2 tree" whose level-d component is $(\mathbf{t}, \tau)$ for which $(d, (\mathbf{t}, \tau)) \in \mathrm{desc}(T, Q, *)$, and whose level-2 component sends $(\mathbf{t}, \tau)$ to (W, w_m) if $(2, (\mathbf{t}, \tau)) \in \mathrm{desc}(T, Q, (W, (w_i)_{i \leq m}))$. In this way, π is a "level $\leq$ 2 tree isomorphism" between X and $T \otimes Q$. All the relevant terminologies of level $\leq$ 2 trees carry over to $T \otimes Q$ in the obvious ways. In particular, if W is a finite level-1 tree, a $(T \otimes Q, W)$-description takes one of the following forms (recall that $(d, \mathbf{t}, \tau)$ is simply an abbreviation of $(d, (\mathbf{t}, \tau))$):

1. $(1, (\mathbf{t}, \emptyset), \emptyset)$ for $(1, \mathbf{t}, \emptyset) \in \mathrm{desc}(T, Q, -1)$;
2. $(2, ((\mathbf{t}, \tau), Z, \vec{z}), \psi)$ for $(2, \mathbf{t}, \tau) \in \mathrm{desc}(T, Q, (Z, \vec{z}))$ and ψ factoring (Z, W);
3. $(2, ((\mathbf{t}, \tau)^\frown(-1), Z^+, \vec{z}), \psi)$ for $(2, \mathbf{t}, \tau) \in \mathrm{desc}(T, Q, (Z, \vec{z}))$, $\vec{z} = (z_i)_{i \leq l}$, $Z^+ = Z \cup \{z_l\}$ and ψ factoring (Z^+, W).

$(T \otimes Q) \otimes W$ is thus regarded as a "level-1 tree" whose nodes consists of non-constant $(T \otimes Q, W)$-descriptions. There is a natural isomorphism

$$\iota_{T,Q,W}$$

between "level-1 trees" $(T \otimes Q) \otimes W$ and $T \otimes (Q \otimes W)$, defined as follows.

1. $\iota_{T,Q,W}(1, (\mathbf{t}, \emptyset), \emptyset) = (1, \mathbf{t}, \emptyset)$.

2. If $(2,\mathbf{t},\tau) \in \operatorname{desc}(T,Q,(Z,\vec{z}))$, $\mathbf{t} = (t,S,\vec{s})$, ψ factors (Z,W), define $\iota_{T,Q,W}(2,((\mathbf{t},\tau),Z,\vec{z}),\psi) = (2,\mathbf{t},(Q\otimes\psi)\circ\tau)$.

3. If $(2,\mathbf{t},\tau) \in \operatorname{desc}(T,Q,(Z,\vec{z}))$, $\mathbf{t} = (t,S,\vec{s})$, $\vec{z} = (z_i)_{i\le l}$, $\vec{s} = (s_i)_{i\le k}$, $Z^+ = Z \cup \{z_l\}$, ψ factors (Z^+,W),

   (a) if t is of discontinuous type, define $\iota_{T,Q,W}(2,((\mathbf{t},\tau),Z^+,\vec{z}),\psi) = (2,\mathbf{t}^\frown(-1),\psi*_0\tau)$, where $\psi*_0\tau$ factors $(S\cup\{s_k\},Q,W)$, $\psi*_0\tau$ extends $(Q\otimes\psi)\circ\tau$, $\psi*_0\tau(s_k) = (2,(\mathbf{q}_0,\sigma))$, $\mathbf{q}_0 = ((-1),\{(0)\},((0)))$, $\sigma((0)) = \psi(z_l)$;

   (b) if t is of continuous type, define $\iota_{T,Q,W}(2,((\mathbf{t},\tau),Z^+,\vec{z}),\psi) = (2,\mathbf{t},\psi*_1\tau)$, where $\psi*_1\tau$ factors (S,Q,W), $\psi*_1\tau$ extends $(Q\otimes\psi)\circ(\tau\restriction(S\setminus\{s_k\}))$, $\psi*_1\tau(s_k) = (2,\mathbf{q}^\frown(-1),\sigma^+)$ where $\tau(s_k) = (2,\mathbf{q},\sigma)$, $\mathbf{q} = (q,P,(p_i)_{i\le m})$, σ^+ extends σ, $\sigma^+(p_m) = \psi(z_l)$.

The reason why $\iota_{T,Q,W}$ is a surjection is the following. Suppose $\mathbf{C} \in T\otimes(Q\otimes W)$ is of degree 2. Put $\mathbf{C} = (2,\mathbf{t},\tau)$, $\mathbf{t} = (t,S,\vec{s})$, $\vec{s} = (s_i)_{i<\operatorname{lh}(\vec{s})}$, $k = \operatorname{lh}(t)$. Let $(w_i)_{i<m}$ be the contraction of $(\operatorname{sign}_1(\tau(s_i)))_{i<k}$. Then w_0 is the $<_{BK}$-maximum of $\{w_i : i<k\}$. Let $(Z,\vec{z}) = (Z,(z_i)_{i<m})$ be the potential partial level ≤ 1 tower of continuous type and $\psi : Z \to W$ be the level-1 tree isomorphism such that $\psi(z_i) = w_i$ for any $i<m$. If $\mathbf{t}$ is of continuous type, $\tau(s_{k-1})$ is of level-1 continuous type, but w_{m-1} does not appear in the contraction of $(\operatorname{sign}_1(\tau(s_i)))_{i<k-1}$, then

$$\mathbf{C} = \iota_{T,Q,W}(2,((\mathbf{t},\tau)^\frown(-1),Z,\vec{z}),\psi).$$

Otherwise,

$$\mathbf{C} = \iota_{T,Q,W}(2,((\mathbf{t},\tau),Z,\vec{z}^\frown(z_*)),\psi),$$

where (Z,z_*) is a partial level ≤ 1 tree, $z_* = -1$ if $\operatorname{ucf}(S,\vec{s}) = -1$, $z_*^- = \operatorname{ucf}_1(\tau(\operatorname{ucf}(S,\vec{s})))$ if $\operatorname{ucf}(S,\vec{s}) \ne -1$. $\iota_{T,Q,W}$ justifies the associativity of the $\otimes$ operator acting on level $(\le 2,\le 2,1)$-trees.

The identity function $\operatorname{id}_{T\otimes Q}$ factors $(T\otimes Q,T,Q)$. By definitions and Lemmas 4.32, 3.11,

$$(\operatorname{id}_{T\otimes Q})^{T,Q}(\operatorname{seed}^{(T\otimes Q)\otimes W}_{\mathbf{C}^*}) = \operatorname{seed}^{T\otimes(Q\otimes W)}_{\iota_{T,Q,W}(\mathbf{C}^*)}$$

for any $\mathbf{C}^* \in (T\otimes Q)\otimes W$. Hence, $(\operatorname{id}_{T\otimes Q})^{T,Q}(u_n) = u_n$ for any $n<\omega$. As $(\operatorname{id}_{T\otimes Q})^{T,Q}$ is elementary from $L_{\kappa_3^x}[j^{T\otimes Q}(T_2),x]$ to $L_{\kappa_3^x}[j^T\circ j^Q(T_2),x]$ for any $x\in\mathbb{R}$, $(\operatorname{id}_{T\otimes Q})^{T,Q}$ is the identity map on $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$.

Suppose π factors level ≤ 2 trees (X,T) and Q is another level ≤ 2 tree.

- $\pi\otimes Q$ factors $(X\otimes Q,T\otimes Q)$, defined as follows: $\pi\otimes Q(d,\mathbf{x},\tau) = (d,{}^d\pi(\mathbf{x}),\tau)$.

- $Q \otimes \pi$ factors $(Q \otimes X, Q \otimes T)$, defined as follows: $Q \otimes \pi(d, \mathbf{q}, \tau) = (d, \mathbf{q}, (\pi \otimes W) \circ \tau)$, where τ factors $(P, X \otimes W)$.

We effectively obtain the following lemma which reduces finite iterations of level ≤ 2 ultrapowers to a single level ≤ 2 ultrapower. The proof is in parallel to Lemma 4.34.

Lemma 4.53. *Suppose X, T, Q are finite level ≤ 2 trees. Then*

1. $j^T \circ j^Q = j^{T \otimes Q}$.
2. *π factors (X, T, Q) iff π factors $(X, T \otimes Q)$. If π factors (X, T, Q) then* $\pi^{T,Q} = \pi^{T \otimes Q}$.
3. *If π factors X, T, then*
   (a) $j^Q(\pi^T \restriction a) = (Q \otimes \pi)^{Q \otimes T} \restriction j^Q(a)$ *for any* $a \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$;
   (b) $\pi^T \restriction \mathbb{L}_{\boldsymbol{\delta}^1_3}[j^{X \otimes Q}(T_2)] = (\pi \otimes Q)^{T \otimes Q}$.

Suppose T, Q, U are level ≤ 2 trees. There is a natural "level ≤ 2 tree isomorphism"

$$\iota_{T,Q,U}$$

between $(T \otimes Q) \otimes U$ and $T \otimes (Q \otimes U)$ defined as follows. Suppose $\mathbf{B} \in \operatorname{desc}(T \otimes Q, U, *)$.

1. If $\mathbf{B} = (1, (t, \emptyset), \emptyset)$, $\mathbf{C} = (1, t, \emptyset) \in \operatorname{desc}(T, Q, -1)$, then $\mathbf{C} \in \operatorname{desc}(T, Q \otimes U, -1)$ and $\iota_{T,Q,U}(\mathbf{B}) = \mathbf{C}$.
2. If $\mathbf{B} = (2, ((\mathbf{t}, \tau), Z, \vec{z}), \psi) \in \operatorname{desc}(T \otimes Q, U, W)$, $\mathbf{C} = (2, \mathbf{t}, \tau) \in \operatorname{desc}(T, Q, (Z, \vec{z}))$, $\mathbf{t} = (t, S, \vec{s})$, ψ factors (Z, U, W), then $\iota_{T,Q,U}(\mathbf{B}) = (2, \mathbf{t}, \iota^{-1}_{Q,U,W} \circ (Q \otimes \psi) \circ \tau)$.
3. If $\mathbf{B} = (2, ((\mathbf{t}, \tau)^\frown(-1), Z^+, \vec{z}), \psi) \in \operatorname{desc}(T \otimes Q, U, W)$, $\mathbf{C} = (2, \mathbf{t}, \tau) \in \operatorname{desc}(T, Q, (Z, \vec{z}))$, $\mathbf{t} = (t, S, (s_i)_{i \leq k})$, $\vec{z} = (z_i)_{i \leq l}$,
   (a) if t is of discontinuous type, then $\iota_{T,Q,U}(\mathbf{B}) = (2, \mathbf{t}^\frown(-1), \psi *_0 \tau)$, where $\psi *_0 \tau$ factors $(S \cup \{s_k\}, Q \otimes U, W)$, $\psi *_0 \tau$ extends $\iota^{-1}_{Q,U,W} \circ (Q \otimes \psi) \circ \tau$, $\psi *_0 \tau(s_k) = \iota^{-1}_{Q,U,W}(2, \mathbf{q}_0, \sigma)$, $\mathbf{q}_0 = ((-1), \{(0)\}, ((0)))$, $\sigma((0)) = \psi(z_l)$.
   (b) if t is of continuous type, then $\iota_{T,Q,U}(\mathbf{B}) = (2, \mathbf{t}, \psi *_1 \tau)$, where $\psi *_1 \tau$ factors $(S, Q \otimes U, W)$, $\psi *_1 \tau$ extends $\iota^{-1}_{Q,U,W} \circ (Q \otimes \psi) \circ (\tau \restriction (S \setminus \{s_k\}))$, $\psi *_1 \tau(s_k) = \iota^{-1}_{Q,U,W}(2, \mathbf{q}^\frown(-1), \sigma^+)$ where $\tau(s_k) = (2, \mathbf{q}, \sigma)$, $\mathbf{q} = (q, P, (p_i)_{i \leq m})$, σ^+ extends σ, $\sigma^+(p_m) = \psi(z_l)$.

$\iota_{T,Q,U}$ justifies the associativity of the $\otimes$ operator acting on level $(\leq 2, \leq 2, \leq 2)$ trees.

Lemma 4.54. *Suppose X, T are level ≤ 2 trees, $\theta : \operatorname{rep}(X) \to \operatorname{rep}(T)$ is a function in $\mathbb{L}$, order-preserving and continuous. Then there exists a triple*

$$(Q, \pi, \vec{\gamma})$$

such that Q is a level ≤ 2 tree, π factors (X, T, Q), $\vec{\gamma}$ respects Q, and

$$\forall h \in \omega_1^{T\uparrow}\ h_\pi^{T,Q}(\vec{\gamma}) = [h \circ \theta]^X.$$

Proof. For $d \in \{1, 2\}$, let $A^d = \{x \in {}^1X : \theta(1, (x)) \in \{d\} \times \operatorname{rep}({}^dT)\}$. By order preservation and continuity of θ, A^1 is a $\prec^{^1X}$-initial segment of ^1X. For $x \in A^1$, let $t_x^1 \in {}^1T$ be such that

$$\theta(1, (x)) = (1, (t_x^1)).$$

The existence of t_x^1 follows from the fact that $(1, (x))$ has cofinality ω in $\operatorname{rep}(X)$. For $x \in A^2$, let $\mathbf{t}_x^2 = (t_x^2, S_x^2, \vec{s}_x^2) \in \operatorname{desc}({}^2T)$, $\vec{s}_x^2 = (s_{x,i}^2)_{i<\operatorname{lh}(\vec{s}_x^2)}$ and $\vec{\beta}_x^2 = (\beta_{x,s}^2)_{s \in S_x^2 \cup \{\emptyset\}}$ be such that

$$\theta(1, (x)) = (2, \vec{\beta}_x^2 \oplus_{^2T} t_x^2).$$

For $x \in \operatorname{dom}({}^2X)$, let ${}^2X(x) = (W_x, w_x)$. By order preservation and continuity of θ, we can find $\mathbf{t}_x = (t_x, S_x, \vec{s}_x) \in \operatorname{desc}(T)$ and $\theta_x \in \mathbb{L}$ such that for μ^{W_x}-a.e. $\vec{\alpha}$,

$$\theta(2, \vec{\alpha} \oplus_{^2X} x) = (2, \theta_x(\vec{\alpha}) \oplus_{^2T} t_x).$$

Let $\vec{s}_x = (s_{x,i})_{i<\operatorname{lh}(\vec{s}_x)}$, $s_x = s_{x,\operatorname{lh}(\vec{s}_x)-1}$. Let $[\theta_x]_{\mu^{W_x}} = \vec{\beta}_x = (\beta_{x,s})_{s \in S_x \cup \{\emptyset\}}$, $\theta_x(\vec{\alpha}) = (\theta_{x,s}(\vec{\alpha}))_{s \in S_x \cup \{\emptyset\}}$, so that $\beta_{x,s} = [\theta_{x,s}]_{\mu^{W_x}}$. In particular, $t_\emptyset = \emptyset$, $\beta_{\emptyset,\emptyset} = \omega_1$, and $t_x \neq \emptyset$ when $x \neq \emptyset$. Fixing x, the map $s \mapsto \beta_{x,s}$ is order preserving with respect to $\prec^{S_x}$ and $<$. Let

$$B_x = \{s \in S_x : \beta_{x,s} < \omega_1\}.$$

So B_x is closed under $\prec^{S_x}$. For $s \in S_x \setminus B_x$, let $(P_{x,s}, \vec{p}_{x,s})$ be the potential partial level ≤ 1 tower induced by $\beta_{x,s}$, $\vec{p}_{x,s} = (p_{x,s,i})_{i<\operatorname{lh}(\vec{p}_{x,s})}$, $p_{x,s} = p_{x,s,\operatorname{lh}(\vec{p}_{x,s}-1)}$, let $(\operatorname{seed}^{W_x}_{w_{x,s,i}})_{i<v_{x,s}}$ be the signature of $\beta_{x,s}$, let $(\gamma_{x,s,i})_{i\leq v_{x,s}}$ be the approximation sequence of $\beta_{x,s}$, and let $\operatorname{cf}^{\mathbb{L}}(\beta_{x,s}) = \operatorname{seed}^{W_x}_{w_{x,s}}$ if $\operatorname{cf}^{\mathbb{L}}(\beta_{x,s}) > \omega$. Let $\sigma_{x,s}$ factor $(P_{x,s}, W_x)$, where $\sigma_{x,s}(p_{x,s,i}) = w_{x,s,i}$ for $i < v_{x,s}$. Let

$$D_x = \{s \in S_x \setminus B_x : \beta_{x,s} \text{ is essentially continuous}\},$$
$$E_x = S_x \setminus (B_x \cup D_x).$$

Thus, $v_{x,s} = \text{card}(P_{x,s})$. For $s \in D_x$, $v_{x,s} = \text{lh}(\vec{p}_{x,s})$; for $s \in E_x$, $v_{x,s} = \text{lh}(\vec{p}_{x,s}) - 1$.

By order preservation and continuity of θ, we can see that for $x \in \text{dom}({}^2X)$,

1. If t_x is of continuous type, then θ_{x,s_x} has uniform cofinality $\text{ucf}({}^2X[x])$.
2. If t_x is of continuous type and $x = \emptyset \vee w_{x^-}$ does not appear in $\text{sign}(\theta_{x,s})$ for any $s \in S_x \setminus \{s_x\}$, then θ_{x,s_x} is essentially discontinuous and thus $s_x \in E_x$.
3. If t_x is of discontinuous type then
   (a) if $w_x = -1$, then $s_x = -1$;
   (b) if $w_x \neq -1$, then $s_x \neq -1$, θ_{x,s_x^-} has uniform cofinality w_x^-, and thus $w_{x,s_x^-} = w_x^-$.

Claim 4.55. *Suppose $x, x' \in \text{dom}({}^2X)$, $x = (x')^-$, t_x is of continuous type, and the contraction of $((w_{x,s_{x,j},i})_{i<v_{x,s_{x,j}}})_{j<\text{lh}(t_x)}$ is $(w_{x\restriction i})_{i<\text{lh}(x)}$. Then*

1. *$t_x = t_{x'}$.*
2. *For any $s \in S_x \setminus \{s_x\}$, $\beta_{x,s} = \beta_{x',s}$.*
3. *$(w_{x,s_x,i}, \gamma_{x,s_x,i})_{i<v_{x,s_x}}$ is a proper initial segment of $(w_{x',s_x,i}, \gamma_{x',s_x,i})_{i<v_{x',s_x}}$. Hence, P_{x,s_x} is a proper subtree of P_{x',s_x} and $\vec{p}_{x,s_x}$ is an initial segment of $\vec{p}_{x',s_x}$.*
4. *$\sigma_{x',s_x}(p_{x,s_x}) = w_x$. In particular, the contraction of $((w_{x',s_{x',j},i})_{i<v_{x',s_{x',j}}})_{j<\text{lh}(t_x)}$ is $(w_{x\restriction i})_{i\leq\text{lh}(x)}$.*

Proof. By order preservation and continuity of θ, $\mathbf{t}_x = \mathbf{t}_{x'}$ and for μ^{W_x}-a.e. $\vec{\alpha}$,

1. for any $s \in S_x \setminus \{s_x\}$, if $\vec{\alpha}' \in [\omega_1]^{W_{x'}\uparrow}$ extends $\vec{\alpha}$ then $\theta_{x,s}(\vec{\alpha}) = \theta_{x',s}(\vec{\alpha}')$;
2. $\theta_{x,s_x}(\vec{\alpha}) = \sup\{\theta_{x',s_x}(\vec{\alpha}') : \vec{\alpha}' \in [\omega_1]^{W_{x'}\uparrow} \text{ extends } \vec{\alpha}\}$.

Thus, $\beta_{x,s} = \beta_{x',s}$ for any $s \in S_x \setminus \{s_x\}$, and $j_{\text{sup}}^{W_x,W_{x'}}(\beta_{x,s_x}) \leq \beta_{x',s_x} < j^{W_x,W_{x'}}(\beta_{x,s_x})$. As $t_x = t_{x'}$ is of continuous type and w_x does not appear in $\text{sign}(\theta_{x',s})$ for any $s \in S_{x'} \setminus \{s_{x'}\}$, θ_{x',s_x} is essentially discontinuous, giving $j_{\text{sup}}^{W_x,W_{x'}}(\beta_{x,s_x}) \neq \beta_{x',s_x}$. We can then apply Lemma 3.14 to show that the partial finite level ≤ 1 tower induced by β_{x,s_x} is a proper initial segment of that induced by β_{x',s_x}, and $w_{x',s_x,v_{x,s}} = w_x$. $\square$

Claim 4.56. *Suppose $x, x' \in \mathrm{dom}(^2X)$, $x = (x')^-$, t_x is of discontinuous type, and the contraction of $((w_{x,s_{x,j},i})_{i<v_{x,s_{x,j}}})_{j<\mathrm{lh}(t_x)}$ is $(w_{x|i})_{i<\mathrm{lh}(x)}$. Then*

1. *$t_x \subsetneq t_{x'}$.*
2. *for any $s \in S_x$, $\beta_{x,s} = \beta_{x',s}$.*
3. *$(w_{x,s_x^-,i}, \gamma_{x,s_x^-,i})_{i<v_{x,s_x^-}}$ is a proper initial segment of $(w_{x',s_x,i}, \gamma_{x',s_x,i})_{i<v_{x',s_x}}$. Hence, P_{x,s_x^-} is a proper subtree of P_{x',s_x} and $\vec{p}_{x,s_x^-}$ is an initial segment of $\vec{p}_{x',s_x}$.*
4. *$\sigma_{x',s_x}(p_{x,s_x^-}) = w_x$. In particular, the contraction of $((w_{x',s_{x',j},i})_{i<v_{x',s_{x',j}}})_{j<\mathrm{lh}(t_{x'})}$ is $(w_{x|i})_{i\le\mathrm{lh}(x)}$.*

Proof. By order preservation and continuity of θ, $t_x \subsetneq t_{x'}$ and for μ^{W_x}-a.e. $\vec{\alpha}$,

1. for any $s \in S_x$, if $\vec{\alpha}'$ extends $\vec{\alpha}$ then $\theta_{x,s}(\vec{\alpha}) = \theta_{x',s}(\vec{\alpha}')$;
2. $\theta_{x,s_x^-}(\vec{\alpha}) = \sup\{\theta_{x',s_x}(\vec{\alpha}) : \vec{\alpha}' \text{ extends } \vec{\alpha}\}$.

The rest is similar to the proof of Claim 4.55. □

Let

$$\phi^1 : \{\beta^2_{x,s} : x \in A^2, s \in S^2_x\} \cup \{\beta_{x,s} : x \in \mathrm{dom}(^2X), s \in B_x\} \to Z^1$$

be a bijection such that Z^1 is a level-1 tree and $v < v' \leftrightarrow \phi^1(v) \prec^{Z^1} \phi^1(v')$. Let

$$\phi^2 : \{(w_{x,s,i}, \gamma_{x,s,i})_{i<l} : x \in \mathrm{dom}(^2X), s \in D_x \cup E_x, l < \mathrm{lh}(\vec{p}_{x,s})\} \to Z^2 \cup \{\emptyset\}$$

be a bijection such that Z^2 is a level-1 tree and $v \subseteq v' \leftrightarrow \phi^2(v) \subseteq \phi^2(v')$, $v <_{BK} v' \leftrightarrow \phi^2(v) <_{BK} \phi^2(v')$. Let

$$Q = (^1Q, {^2Q}),$$

where $^1Q = Z^1$, ^2Q is a level-2 tree, $\mathrm{dom}(^2Q) = Z^2$,

$$^2Q[\phi^2((w_{x,s,i}, \gamma_{x,s,i})_{i<\mathrm{lh}(\vec{p}_{x,s})-1})^\frown(-1)] = (P_{x,s}, \vec{p}_{x,s}) \text{ for } s \in D_x,$$
$$^2Q[\phi^2((w_{x,s,i}, \gamma_{x,s,i})_{i<\mathrm{lh}(\vec{p}_{x,s})-1})] = (P_{x,s}, \vec{p}_{x,s}) \text{ for } s \in E_x.$$

Let

$$\vec{\gamma} = (^d\gamma_q)_{(d,q)\in\mathrm{dom}(Q)}$$

where ${}^1\gamma_q = (\phi^1)^{-1}(q)$, ${}^2\gamma_\emptyset = \omega_1$, ${}^2\gamma_q = \gamma_{x,s,l}$ when $q = \phi^2((w_{x,s,i}, \gamma_{x,s,i})_{i\le l})$. For $x \in A^1$, let

$$\pi(1,x) = (1, t^1_x, \emptyset).$$

For $x \in A^2$, let

$$\pi(1,x) = (2, \mathbf{t}^2_x, \tau^2_x),$$

where τ^2_x factors $(S^2_x, Q, \emptyset)$, $\tau^2_x(1,s) = (1, \phi^1(\beta^2_{x,s}), \emptyset)$. For $x \in \mathrm{dom}({}^2X)$, let

$$\pi(2,x) = (2, \mathbf{t}_x, \tau_x),$$

where τ_x factors (S_x, Q, W_x), defined as follows:

$$\tau_x(s) = \begin{cases} (1, \phi^1(\beta_{x,s}), \emptyset) & \text{if } s \in B_x, \\ (2, (\phi^2((w_{x,s,i}, \gamma_{x,s,i})_{i<\mathrm{lh}(\vec{p}_{x,s})-1})^\frown(-1), P_{x,s}, \vec{p}_{x,s}), \sigma_{x,s}) & \text{if } s \in D_x, \\ (2, (\phi^2((w_{x,s,i}, \gamma_{x,s,i})_{i<\mathrm{lh}(\vec{p}_{x,s})-1}), P_{x,s}, \vec{p}_{x,s}), \sigma_{x,s}) & \text{if } s \in E_x. \end{cases}$$

It is easy to check that $(Q, \pi, \vec{\gamma})$ works for the lemma. □

Note that if π factors Π^1_2-wellfounded trees (X,T), then $[\![d,x]\!]_X \le [\![\pi(d,x)]\!]_T$ for any $(d,x) \in \mathrm{dom}(X)$. We say that π *minimally factors* (X,T) iff π factors (X,T), X,T are both Π^1_2-wellfounded and $[\![d,x]\!]_X = [\![\pi(d,x)]\!]_T$ for any $(d,x) \in \mathrm{dom}(X)$. In particular, if T,Q are both Π^1_2-wellfounded, then $\mathrm{id}_{T,*}$ minimally factors $(T, T\otimes Q)$. In the assumption of Lemma 4.54, if X,T are Π^1_2-wellfounded and the map θ is a bijection between $\mathrm{rep}(X)$ and $\mathrm{rep}(T)$, its proof constructs π which minimally factors $(X, T\otimes Q)$. This entails the comparison theorem between Π^1_2-wellfounded trees.

Theorem 4.57. *Suppose X, T are Π^1_2-wellfounded level ≤ 2 trees. Then there exists (Q,π) such that Q is Π^1_2-wellfounded and π minimally factors $(X, T\otimes Q)$.*

We shall see in Section 5 that the minimally of factoring maps between Π^1_2-wellfounded trees corresponds exactly to the Dodd-Jensen property of iterations of mice.

Suppose Q, Q' are finite level ≤ 2 trees, Q is a proper subtree of Q', $(W_i, w_i)_{i\le m'}$ is a partial level ≤ 1 tower, $m \le m'$, $\mathbf{C} \in \mathrm{desc}(T, Q, (W_m, (w_i)_{i\le m}))$, $\mathbf{C}' \in \mathrm{desc}(T, Q', (W_{m'}, (w_i)_{i\le m'})) \setminus \mathrm{desc}(T, Q, (W_m, (w_i)_{i\le m}))$. Define

$$\mathbf{C} \lhd^{T,Q}_2 \mathbf{C}'$$

iff $\mathbf{C}' \prec \mathbf{C}$ and $\bigcup_{m\le k\le m'} \{\mathbf{C}^* \in \mathrm{desc}(T, Q, (W_k, (w_i)_{i\le k})) : \mathbf{C}' \prec \mathbf{C}^* \prec \mathbf{C}\} = \emptyset$. A purely combinatorial argument shows that $\mathbf{C} \lhd^{T,Q}_2 \mathbf{C}'$ iff $\mathbf{C}$ and $\mathbf{C}'$ are both of degree 2 and putting $\mathbf{C} = (2, \mathbf{t}, \tau)$, $\mathbf{C}' = (2, \mathbf{t}', \tau')$, $\mathbf{t} = (t, S, \vec{s})$, $\vec{s} = (s_i)_{i<\mathrm{lh}(\vec{s})}$, $\mathbf{t}' = (t', S', \vec{s}')$, $k = \mathrm{lh}(t)$, then either

1. t is of continuous type, $\mathbf{t} \restriction k-1 = \mathbf{t}' \restriction k-1$, $\tau \restriction (S \setminus \{s_{k-1}\}) \subseteq \tau'$, $\tau(s_{k-1}) \lhd_2^{Q,W_m} \tau'(s_{k-1})$, or

2. t is of discontinuous type, $\mathbf{C} \lhd \mathbf{C}'$, $\tau(s_k^-) \lhd_2^{Q,W_m} \tau(s_k)$.

As a corollary to Lemma 4.26 and Lemma 4.29, $\lhd_2^{T,Q}$ inherits the following continuity property.

Lemma 4.58. *Suppose Q, Q', W, W' are finite, Q is a level ≤ 2 proper subtree of Q', W is a (not necessarily proper) level-1 subtree of W'. Suppose $\mathbf{C} = (2, \mathbf{t}, \tau) \in \mathrm{desc}(T, Q, W)$, $\mathbf{C}' = (2, \mathbf{t}', \tau') \in \mathrm{desc}(T, Q', W')$, $\mathbf{C} \lhd_2^{T,Q} \mathbf{C}'$. Suppose $E \in \mu_{\mathbb{L}}$ is a club, $\eta \in E'$ iff $\eta \in E$ and $E \cap \eta$ has order type η. Then for any $h \in \omega_1^{T\uparrow}$, for any $\vec{\beta} \in [E']^{Q\uparrow}$,*

$$j^{W,W'} \circ h^Q_{\mathbf{C}}(\vec{\beta}) = \sup\{h^{Q'}_{\mathbf{C}'}(\vec{\gamma}) : \vec{\gamma} \in [E]^{Q'\uparrow}, \vec{\gamma} \textit{ extends } \vec{\beta}\}.$$

4.7 Level-3 description analysis

Definition 4.59. Suppose R is a level-3 tree. The *constant R-description* is $\emptyset$. An *R-description* is either the constant R-description or a triple $(r, Q, \overrightarrow{(d,q,P)})$ such that either $r \in \mathrm{dom}(R) \wedge (Q, \overrightarrow{(d,q,P)}) = R[r]$ or $r = r^{-\frown}(-1) \wedge r^- \in \mathrm{dom}(R) \wedge Q$ is a completion of $R(r^-) \wedge (Q, \overrightarrow{(d,q,P)}) = R[r, Q]$. $\mathrm{desc}(R)$ is the set of R-descriptions. $(r, Q, \overrightarrow{(d,q,P)})$ is *of discontinuous type* if $r \in \mathrm{dom}(R)$, *of continuous type* otherwise. An *extended R-description* is either an R-description or a triple $(r, Q, \overrightarrow{(d,q,P)})$ such that $(r^\frown(-1), Q, \overrightarrow{(d,q,P)})$ is an R-description of continuous type. $\mathrm{desc}^*(R)$ is the set of extended R-descriptions. An extended R-description $\mathbf{r}$ is *regular* iff either $\mathbf{r} \in \mathrm{desc}(R)$ of discontinuous type or $\mathbf{r} \notin \mathrm{desc}(R)$. A *generalized R-description* is either $(\emptyset, \emptyset, \emptyset)$ or of the form

$$\mathbf{A} = (\mathbf{r}, \pi, T)$$

so that $\mathbf{r} = (r, Q, \overrightarrow{(d,q,P)}) \in \mathrm{desc}(R) \setminus \{\emptyset\}$, T is a finite level ≤ 2 tree, π factors (Q, T). $\mathrm{desc}^{**}(R)$ is the set of generalized R-descriptions.

Suppose $(Q, (d, q, P))$ is a partial level ≤ 2 tree. We define

$$\mathrm{ucf}^*(Q,(d,q,P)) = \begin{cases} (0,-1,\emptyset) & \text{if } \mathrm{ucf}(Q,(d,q,P)) = (0,-1), \\ (1, q^*, \emptyset) & \text{if } \mathrm{ucf}(Q,(d,q,P)) = (1, q^*), \\ (2, \mathbf{q}^*, \mathrm{id}_{{}^2Q_{\mathrm{tree}(q^*)}}) & \text{if } \mathrm{ucf}(Q,(d,q,P)) = (2, \mathbf{q}^*), \\ & \quad \mathbf{q}^* = (q^*, P^*, \vec{p}^*). \end{cases}$$

Thus, $\mathrm{ucf}^*(Q,(d,q,P)) \in \{(0,-1,\emptyset)\} \cup \mathrm{desc}(Q,P)$, and $\mathrm{cf}(Q,(d,q,P)) = 1$ iff $\mathrm{ucf}^*(Q,(d,q,P)) = \min(\prec^{Q,P})$. If $\mathrm{cf}(Q,(d,q,P)) = 2$, let

$$\mathrm{ucf}^-(Q,(d,q,P)) = \mathrm{pred}_{\prec^{Q,P}}(\mathrm{ucf}^*(Q,(d,q,P))).$$

$\mathrm{ucf}^-(Q,(d,q,P))$ can be computed in the following concrete way. If $d=1$, then $\mathrm{ucf}^-(Q,(1,q,\emptyset)) = (1, \mathrm{pred}_{\prec^{1}Q\cup\{q\}}(q), \emptyset)$; if $d=2$, then $\mathrm{ucf}^-(Q,(2,q,P)) = (2, \mathbf{q}', \mathrm{id}_P)$, where $\mathbf{q}' = (q', P, \vec{p}) \in \mathrm{desc}(Q)$, q' is the $<_{BK}$-maximum of ${}^2Q\{q,-\}$. If Q^* is a completion of $(Q,(d,q,P))$ and $\mathbf{D} = (1,q,\emptyset)$ if $d=1$, $\mathbf{D} = (2,(q,P,\vec{p}),\mathrm{id}_P)$ if $d = 2 \wedge {}^2Q^*[q] = (P,\vec{p})$, then

$$\mathbf{D} = \mathrm{pred}_{\prec^{Q^*,P}}(\mathrm{ucf}^*(Q,(d,q,P)))$$

and

$$\mathrm{ucf}^-(Q,(d,q,P)) = \mathrm{pred}_{\prec^{Q^*,P}}(\mathbf{D}).$$

Suppose $\mathbf{r} = (r, Q, \overrightarrow{(d,q,P)}) \in \mathrm{desc}^*(R)$, $\mathrm{lh}(r) = k$, $\overrightarrow{(d,q,P)} = (d_i,q_i,P_i)_{1\le i\le \mathrm{lh}(\vec{q})}$. For $F \in (\boldsymbol{\delta}^1_3)^{R\uparrow}$, define $F_{\mathbf{r}}$ to be a function on $\omega_1^{Q\uparrow}$: if $\mathbf{r} \in \mathrm{desc}(R)$, then $F_{\mathbf{r}} = F_r$; if $\mathbf{r} \notin \mathrm{desc}(R)$, then $F_{\mathbf{r}}(\vec{\beta}) = F_r(\vec{\beta}\restriction R_{\mathrm{tree}}(r))$. If $\vec{\gamma} = (\gamma_r)_{r\in\mathrm{dom}(R)} \in [\boldsymbol{\delta}^1_3]^{R\uparrow}$, put $\gamma_{\mathbf{r}} = [F_{\mathbf{r}}]_{\mu^Q}$. If $\mathbf{r} \in \mathrm{desc}(R)$ and $\mathbf{A} = (\mathbf{r},\pi,T) \in \mathrm{desc}^{**}(R)$, put $\gamma_{\mathbf{A}} = \pi^T(\gamma_{\mathbf{r}})$. Put $\gamma_\emptyset = \gamma_{(\emptyset,\emptyset,\emptyset)} = \boldsymbol{\delta}^1_3$. Thus, if $\mathbf{r} \in \mathrm{desc}(R)$ is of discontinuous type, then $\gamma_{\mathbf{r}} = \gamma_r$; if $\mathbf{r} \notin \mathrm{desc}(R)$, then $\gamma_{\mathbf{r}} = j^{R_{\mathrm{tree}}(r),Q}(\gamma_r) = \gamma_{(\mathbf{r},\mathrm{id}_{R_{\mathrm{tree}}(r)},Q)}$. The next lemma computes the remaining case when $\mathbf{r} \in \mathrm{desc}(R)$ is of continuous type, justifying that $\gamma_{\mathbf{r}}$ does not depend on the choice of F.

Lemma 4.60. *Suppose R is a level-3 tree, $\vec{\gamma} \in [\boldsymbol{\delta}^1_3]^{R\uparrow}$, $\mathbf{r} = (r,Q,\overrightarrow{(d,q,P)}) \in \mathrm{desc}(R)$ is of continuous type. Then $\gamma_{\mathbf{r}} = j_{\sup}^{R_{\mathrm{tree}}(r^-),Q}(\gamma_{r^-})$.*

Proof. Suppose $\vec{\gamma} = [F]^R$, $F \in (\boldsymbol{\delta}^1_3)^{R\uparrow}$. Put $\mathrm{lh}(r) = k+1$, $\overrightarrow{(d,q,P)} = (d_i,q_i,P_i)_{1\le i\le k}$. We prove the case when $\mathrm{cf}(R(r^-)) = 2$, the other case being similar. Put $R(r^-) = (Q^-,(d,q,P))$, $\pi^- = \pi\restriction\mathrm{dom}(Q^-)$, so that Q is a completion of $R(r^-)$, $(d,q,P) = (d_k,q_k,P_k)$. Put $\mathrm{ucf}(R(r^-)) = (d^*,\mathbf{q}^*)$, $\mathrm{ucf}^-(R(r^-)) = (e,\mathbf{z},\mathrm{id}_P)$.

We firstly show the $\geq$ direction. Suppose $\delta = [G]_{\mu^{Q^-}} < \gamma_{r^-}$, $G \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. By Łoś, for μ^{Q^-}-a.e. $\vec{\beta}$, $G(\vec{\beta}) < F_{r^-}(\vec{\beta}) = \sup_{\xi<{}^{d^*}\beta_{\mathbf{q}^*}} F_r(\vec{\beta}^\frown(\xi))$, where $\vec{\beta}^\frown(\xi)$ is a tuple extending $\vec{\beta}$ whose entry indexed by (d,q) is ξ. Let $H(\vec{\beta})$ be the least $\xi < {}^d\beta_{\mathbf{q}^*}$ satisfying $G(\vec{\beta}) < F_r(\vec{\beta}^\frown(\xi))$. By Lemmas 4.31 and 4.25, there is $h:\omega_1\to\omega_1$ such that $h\in\mathbb{L}$ and for μ^{Q^-}-a.e. $\vec{\beta}$, $H(\vec{\beta}) < j^P(h)({}^e\beta_{\mathbf{z}}) < {}^d\beta_q$. Thus, for μ^Q-a.e. $\vec{\beta}$, $G(\vec{\beta}\restriction\mathrm{dom}(Q^-)) < F_r(\vec{\beta})$. Thus, $j^{Q^-,Q}(\delta) < [F_{\mathbf{r}}]_{\mu^Q}$.

We secondly show the $\leq$ direction. Suppose $\delta = [G]_{\mu^Q} < [F_{\mathbf{r}}]_{\mu^Q}$, $G \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. Then for μ^Q-a.e. $\vec{\beta}$, $G(\vec{\beta}) < F_r(\vec{\beta}) = \sup_{\xi<{}^d\beta_q} F_r(\vec{\beta}\restriction\mathrm{dom}(Q^-)^\frown(\xi))$.

Let $H(\vec\beta)$ be the least $\xi < {}^2\beta_v$ satisfying $G(\vec\beta) < F_r(\vec\beta \restriction \mathrm{dom}(Q^-)^\frown(\xi))$. By Lemmas 4.32 and 4.25 again, there is $h : \omega_1 \to \omega_1$ such that $h \in \mathbb{L}$ and for μ^Q-a.e. $\vec\beta$, $H(\vec\beta) < j^P(h)({}^e\beta_{\mathbf{z}}) < {}^d\beta_q$. Thus, for μ^Q-a.e. $\vec\beta$, $G(\vec\beta) < j^{Q^-,Q}(\eta)$, where η is represented modulo μ^{Q^-} by the function $\vec\beta \mapsto F_r(\vec\beta^\frown j^P(h)({}^e\beta_{\mathbf{z}}))$. Since $\eta < \gamma_{r^-}$, we have $\delta < j^{Q^-,Q}_{\sup}(\gamma_{r^-})$. □

Define

$$C^* = \{\xi < \boldsymbol{\delta}^1_3 : \text{for any finite level} \le 2 \text{ tree } Q, j^Q_{\sup}(\xi) = \xi\}.$$

Assuming $\boldsymbol{\Delta}^1_2$-determinacy, Lemma 4.37 implies that $C^* \cap \kappa^x_3$ has order type κ^x_3, and hence C^* has order type $\boldsymbol{\delta}^1_3$. A tuple $\vec\gamma$ is said to *strongly respect* R iff $\vec\gamma \in [C^*]^{R\uparrow}$. In most applications, we are only concerned with $\vec\gamma$ strongly respecting R. In that case, the techniques in Section 4.6 helps to decide the ordering of $\gamma_{\mathbf{r}}$ for different $\mathbf{r} \in \mathrm{desc}^*(R)$. The results are in parallel to Lemma 3.22.

Define $\langle(\emptyset,\emptyset,\emptyset)\rangle = \langle\emptyset\rangle = \emptyset$. For $\mathbf{A} = (\mathbf{r},\pi,T) \in \mathrm{desc}^{**}(R)$, $\mathbf{r} = (r, Q, \overrightarrow{(d,q,P)})$, $\mathrm{lh}(r) = k$, define

$$\langle\mathbf{A}\rangle = \begin{cases} (r(0), [\![\pi(d_1,q_1)]\!]_T, r(1), \dots, [\![\pi(d_{k-2},q_{k-2})]\!]_T, r(k-2), -1) \\ \qquad \text{if } r \text{ is of continuous type}, \pi \text{ is continuous at } (d_{k-1},q_{k-1}), \\ (r(0), [\![\pi(d_1,q_1)]\!]_T, r(1), \dots, [\![\pi(d_{k-2},q_{k-2})]\!]_T, r(k-2), [\![\mathrm{pred}(\pi,T,(d_{k-1},q_{k-1}))]\!]_T) \\ \qquad \text{if } r \text{ is of continuous type}, \pi \text{ is discontinuous at } (d_{k-1},q_{k-1}), \\ (r(0), [\![\pi(d_1,q_1)]\!]_T, r(1), \dots, [\![\pi(d_{k-1},q_{k-1})]\!]_T, r(k-1), -1) \\ \qquad \text{if } r \text{ is of discontinuous type}, \pi \text{ is continuous at } \mathrm{ucf}(R(r)), \\ (r(0), [\![\pi(d_1,q_1)]\!]_T, r(1), \dots, [\![\pi(d_{k-1},q_{k-1})]\!]_T, r(k-1), [\![\mathrm{pred}(\pi,T,\mathrm{ucf}(R(r)))]\!]_T) \\ \qquad \text{if } r \text{ is of discontinuous type}, \pi \text{ is discontinuous at } \mathrm{ucf}(R(r)). \end{cases}$$

and define $\langle\mathbf{r}\rangle = \langle(\mathbf{r}, Q, \mathrm{id}_Q)\rangle$. If $\mathbf{r}$ is of discontinuous type and Q^+ is a completion of Q, define $\langle(r, Q^+, \overrightarrow{(d,q,P)})\rangle = \langle(\mathbf{r}, Q^+, \mathrm{id}_Q)\rangle$. For $\mathbf{A}, \mathbf{A}' \in \mathrm{desc}^{**}(R)$, define

$$\mathbf{A} \prec \mathbf{A}'$$

iff $\langle\mathbf{A}\rangle <_{BK} \langle\mathbf{A}'\rangle$; define

$$\mathbf{A} \sim \mathbf{A}'$$

iff $\langle\mathbf{A}\rangle = \langle\mathbf{A}'\rangle$. For $\mathbf{r}, \mathbf{r}' \in \mathrm{desc}^*(R)$, define

$$\mathbf{r} \prec \mathbf{r}'$$

iff $\langle\mathbf{r}\rangle <_{BK} \langle\mathbf{r}'\rangle$; define

$$\mathbf{r} \sim \mathbf{r}'$$

iff $\langle \mathbf{r} \rangle = \langle \mathbf{r}' \rangle$. All relations are effective. Define $\prec^R_* = \prec \restriction \operatorname{desc}^{**}(R)$, $\sim^R_* = \sim \restriction \operatorname{desc}^{**}(R)$ $\prec^R = \prec \restriction \operatorname{desc}^{*}(R)$, $\sim^R = \sim \restriction \operatorname{desc}^{*}(R)$. For $r, r' \in \operatorname{dom}(R)$, define $r \prec^R r'$ iff $(r)^\frown R[r] \prec (r')^\frown R[r']$.

Lemma 4.61. *Suppose R is a level-3 tree, $\mathbf{A}, \mathbf{A}' \in \operatorname{desc}^*(R)$, $\vec{\gamma}$ strongly respects R. Then $\mathbf{A} \prec^R \mathbf{A}'$ iff $\gamma_{\mathbf{A}} < \gamma_{\mathbf{A}'}$; $\mathbf{A} \sim^R \mathbf{A}'$ iff $\gamma_{\mathbf{A}} = \gamma_{\mathbf{A}'}$.*

Proof. Put $\mathbf{A} = (\mathbf{r}, T, \pi)$, $\mathbf{A}' = (\mathbf{r}', T', \pi')$. Recall our convention that $\gamma_{(\emptyset,\emptyset,\emptyset)} = \boldsymbol{\delta}^1_3$. The lemma is trivial if $\mathbf{r} = \emptyset$ or $\mathbf{r}' = \emptyset$. Assume now $\mathbf{r}, \mathbf{r}' \neq \emptyset$. Put $\mathbf{r} = (r, Q, \overrightarrow{(d,q,P)})$, $\overrightarrow{(d,q,P)} = (d_i, q_i, P_i)_{1 \leq i \leq \operatorname{lh}(\vec{q})}$, $k = \operatorname{lh}(r)$, $\mathbf{r}' = (r', Q', \overrightarrow{(d',q',P')})$, $\overrightarrow{(d',q',P')} = (d'_i, q'_i, P'_i)_{1 \leq i \leq \operatorname{lh}(\vec{q}')}$, $k' = \operatorname{lh}(r')$. Assume $\vec{\gamma} = [F]^R$, $F \in (C^*)^{R\uparrow}$.

Firstly, we prove that $\mathbf{A} \sim \mathbf{A}'$ implies $\gamma_{\mathbf{A}} = \gamma_{\mathbf{A}'}$.

Case 1: $r = r'$ is of continuous type.

Put $Q^- = R_{\text{tree}}(r^-)$.

Subcase 1.1: π is continuous at (d_{k-1}, q_{k-1}).

Then $[\![\pi(d,q)]\!]_T = [\![\pi'(d,q)]\!]_{T'}$ for any $(d,q) \in \operatorname{dom}(Q^-)$. Put $\tau = \pi \restriction \operatorname{dom}(Q^-)$, $\tau' = \pi' \restriction \operatorname{dom}(Q^-)$. By Lemma 4.60, $\gamma_{\mathbf{A}} = \pi^T_{\sup}(\gamma_{\mathbf{r}}) = \tau^T_{\sup}(\gamma_{r^-})$ and $\gamma_{\mathbf{A}'} = (\tau')^{T'}_{\sup}(\gamma_{r^-})$. Given $\delta = [G]_{\mu^{Q^-}} < \gamma_{r^-}$, we need to show that $\tau^T(\delta) < \gamma_{\mathbf{A}'}$. By Theorem 4.57, there exist X and ψ minimally factoring $(T, T' \otimes X)$. So $\psi \circ \pi(d,q) = \operatorname{id}_{T',*} \circ \pi'(d,q)$ for any $(d,q) \in \operatorname{dom}(Q^-)$. We shall actually show that $\psi^{T',X} \circ \tau^T(\delta) < \gamma_{\mathbf{A}'}$, i.e., $(\psi \circ \tau)^{T',X}(\delta) < \gamma_{\mathbf{A}'}$. By Łoś, it suffices to show that for $\mu^{T'}$-a.e. $\vec{\beta}$, $j^X(G)(\operatorname{id}^{T',X}_{\psi\circ\tau}(\vec{\beta})) < F_{r'}(\vec{\beta}_{\tau'})$. The minimality of ψ implies that $\operatorname{id}^{T',X}_{\psi\circ\tau}(\vec{\beta}) = j^X(\vec{\beta}_\tau)$. It suffices to show that for $\mu^{T'}$-a.e. $\vec{\beta}$, $j^X(G(\vec{\beta}_\tau)) < F_{r'}(\vec{\beta}_{\tau'})$. Hence, it suffices to show that μ^{Q^-}-a.e. $\vec{\beta}$, $j^X(G(\vec{\beta})) < F_{r'}(\vec{\beta})$. As $F_{r'}(\vec{\beta}) \in C^*$, this inequality is a consequence of $G(\vec{\beta}) < F_{r'}(\vec{\beta})$, which holds true for μ^{Q^-}-a.e. $\vec{\beta}$ by assumption.

Subcase 1.2: π is discontinuous at (d_{k-1}, q_{k-1}).

Then $[\![\pi(d,q)]\!]_T = [\![\pi'(d,q)]\!]_{T'}$ for any $(d,q) \in \operatorname{dom}(Q)$. Let τ factor (Q,T) where τ and π agree on $\operatorname{dom}(Q^-)$ and $\tau(d_{k-1}, q_{k-1}) = \operatorname{pred}(\pi, T, (d_{k-1}, q_{k-1}))$, and likewise define τ' which factors (Q', T). By Lemma 4.41, $\gamma_{\mathbf{A}} = \tau^T_{\sup} \circ j^{Q^-,Q}(\gamma_{r^-})$ and $\gamma_{\mathbf{A}'} = (\tau')^T_{\sup} \circ j^{Q^-,Q}(\gamma_{r^-})$. Work with X and ψ minimally factoring $(T, T' \otimes X)$ and argue similarly to Subcase 1.1.

Case 2: $r = r'$ is of discontinuous type.

Subcase 2.1: π is continuous at $\operatorname{ucf}(R(r))$.

Then $[\![\pi(d,q)]\!]_T = [\![\pi'(d,q)]\!]_{T'}$ for any $(d,q) \in \operatorname{dom}(Q)$ and $\gamma_{\mathbf{A}} = \pi^T_{\sup}(\gamma_r)$, $\gamma_{\mathbf{A}'} = (\pi')^T_{\sup}(\gamma_r)$. Argue similarly to Case 1.

Subcase 2.2: π is discontinuous at $\operatorname{ucf}(R(r))$.

Let Q^+ be a completion of $R(r)$ and let τ factor (Q^+, T) so that τ extends π, $\tau(d_k, q_k) = \operatorname{pred}(\pi, T, (d_k, q_k))$, and likewise define τ' which factors (Q^+, T). Then $[\![\tau(d,q)]\!]_T = [\![\tau'(d,q)]\!]_{T'}$ for any $(d,q) \in \mathrm{dom}(Q^+)$. By Lemma 4.42, $\gamma_{\mathbf{A}} = \tau^T_{\sup} \circ j^{Q,Q^+}(\gamma_r)$ and $\gamma_{\mathbf{A}'} = (\tau')^{T'}_{\sup} \circ j^{Q,Q^+}(\gamma_r)$. Argue similarly to Case 1.

Case 3: $r \neq r'$. Assume $r = r'^\frown(-1)$.

Subcase 3.1: π is continuous at (d_{k-1}, q_{k-1}).

It follows from Subcase 1.1 and Subcase 2.1 that $\gamma_{\mathbf{A}} = \pi^T_{\sup}(\gamma_{r'})$ and $\gamma_{\mathbf{A}'} = (\pi')^{T'}_{\sup}(\gamma_{r'})$. Argue similarly as before.

Subcase 3.2: π is discontinuous at (d_{k-1}, q_{k-1}).

Use a combination of Subcase 1.2 and Subcase 2.2.

Secondly, we prove that $\mathbf{A} \prec \mathbf{A}'$ implies $\gamma_{\mathbf{A}} < \gamma_{\mathbf{A}'}$.

Case 1: $\langle \mathbf{A}' \rangle$ is a proper initial segment of $\langle \mathbf{A} \rangle$.

Then $\langle \mathbf{A}' \rangle$ does not end with -1. We prove the typical case when r' is of discontinuous type. So $r' \subsetneq r$. Let $(Q')^+$ be a completion of $R(r')$ and let τ' factor $((Q')^+, T')$ so that τ' extends π', $\tau'(d_k, q_k) = \mathrm{pred}(\pi, T, \mathrm{ucf}(R(r')))$. Then $(Q')^+ = R_{\mathrm{tree}}(r \restriction k')$. We get ψ minimally factoring $(T, T' \otimes X)$ so that $\psi \circ \pi(d,q) = \mathrm{id}_{T',*} \circ \tau'(d,q)$ for any $(d,q) \in \mathrm{dom}((Q')^+)$. We shall actually show that $\psi^{T',X}(\gamma_{\mathbf{A}}) < \gamma_{\mathbf{A}'}$. By Łoś, it suffices to show that for $\mu^{T'}$-a.e. $\vec{\beta}$, $j^X(F_{\mathbf{r}})(\mathrm{id}^{T',X}_{\psi\circ\pi}(\vec{\beta})) < F_{r'}(\vec{\beta}_{\pi'})$. The minimality of ψ implies that $\mathrm{id}^{T',X}_{\psi\circ\pi}(\vec{\beta})$ agrees with $j^X(\vec{\beta}_{\tau'})$ on $\mathrm{dom}((Q')^+)$. It suffices to show that for μ^Q-a.e. $\vec{\beta}$, $j^X(F_{\mathbf{r}}(\vec{\beta})) < F_{r'}(\vec{\beta} \restriction \mathrm{dom}(Q'))$. As $\mathrm{ran}(F) \subseteq C^*$, this would be a consequence of $F_{\mathbf{r}}(\vec{\beta}) < F_{r'}(\vec{\beta} \restriction \mathrm{dom}(Q'))$, which follows from order preservation of F.

Case 2: $\langle \mathbf{A}' \rangle$ is not a proper initial segment of $\langle \mathbf{A} \rangle$.

Similar to Case 1, using the following fact: Suppose X, X' are level ≤ 2 trees and $[\![d_i, x_i]\!]_X = [\![d'_i, x'_i]\!]_{X'}$ for $1 \leq i < n$, $[\![d_n, x_n]\!]_X < [\![d'_n, x'_n]\!]_{X'}$. Then there exist U and ψ minimally factoring $(X, X' \otimes U)$, which implies that for any $\vec{\beta} \in [\omega_1]^{X\uparrow}$, if $\mathrm{id}^{X',U}_{\psi}(\vec{\beta}) = \vec{\delta}$, then δ and $j^U(\vec{\beta})$ agree on $\{(d_i, x_i) : 1 \leq i < n\}$ and ${}^{d_n}\delta_{x_n} < j^U({}^{d'_n}\beta_{x'_n})$. □

4.8 Factoring maps between level-3 trees

Put $\pi \oplus \emptyset = \emptyset$. Suppose Y is a level-3 tree, $\mathbf{y} = (y, X, \overrightarrow{(e, x, W)}) \in \mathrm{desc}(Y)$, $\mathrm{lh}(y) = k$, $\overrightarrow{(e, x, W)} = (e_i, x_i, W_i)_{1 \leq i \leq \mathrm{lh}(\vec{y})}$, π is a function whose domain contains $\mathrm{dom}(X)$, we put

$$\pi \oplus \mathbf{y} = \pi \oplus_Y y = (y(0), \pi(e_1, x_1), y(1), \dots, \pi(e_{k-1}, x_{k-1}), y(k-1)).$$

If $l < \mathrm{lh}(y)$, then $\mathbf{y} \restriction l = (y \restriction l, Y_{\mathrm{tree}}(y \restriction l), (e_i, x_i, W_i)_{1 \leq i \leq l})$.

Definition 4.62. Suppose Y is a level-3 tree, T is a level ≤ 2 tree. The only $(Y, T, \emptyset)$-*description* is $(\emptyset, \emptyset)$, which is called the *constant* $(Y, T, *)$-

description. Suppose $(Q, \overrightarrow{(d,q,P)}) = (Q, (d_i, q_i, P_i))_{1 \leq i \leq k}$ is a potential partial level ≤ 2 tower of discontinuous type. A $(Y, T, (Q, \overrightarrow{(d,q,P)}))$*-description* is of the form

$$\mathbf{B} = (\mathbf{y}, \pi)$$

with the following properties:

1. $\mathbf{y} \in \operatorname{desc}(Y) \setminus \{\emptyset\}$. Put $\mathbf{y} = (y, X, \overrightarrow{(e,x,W)})$, $\operatorname{lh}(y) = l$, $\overrightarrow{(e,x,W)} = (e_i, x_i, W_i)_{1 \leq i \leq \operatorname{lh}(\vec{x})}$.
2. π factors (X, T, Q).
3. The contraction of $(\operatorname{sign}_2(\pi(e_i, x_i)))_{1 \leq i < l}$ is $((d_i, q_i))_{1 \leq i < k}$.
4. If y is of continuous type and (e_{l-1}, x_{l-1}) does not appear in the contraction of $(\operatorname{sign}_2(\pi(e_i, x_i)))_{1 \leq i < l}$, then $\pi(x_{l-1})$ is of level-2 discontinuous type.
5. Put $\operatorname{ucf}(X, \overrightarrow{(e,x,W)}) = (e_*, \mathbf{x}_*)$.
   (a) If $e_* = 0$ then $d_k = 0$.
   (b) If $e_* = 1$ then $\operatorname{ucf}_2(\pi(1, \mathbf{x}_*)) = \operatorname{ucf}(Q, \overrightarrow{(d,q,P)})$.
   (c) If $e_* = 2$, $\mathbf{x}_* = (x_*, W_*, \vec{w}_*) \in \operatorname{desc}(X)$, then $\operatorname{ucf}_2(\pi(2, x_*)) = \operatorname{ucf}(Q, \overrightarrow{(d,q,P)})$.
   (d) If $e_* = 2$, $\mathbf{x}_* = (x_*, W_*, \vec{w}_*) \notin \operatorname{desc}(X)$, then $\operatorname{ucf}_2^+(\pi(2, x_*)) = \operatorname{ucf}(Q, \overrightarrow{(d,q,P)})$.

A (Y, T, Q)-description is a $(Y, T, (Q, \overrightarrow{(d', q', P')}))$-description for some potential partial level ≤ 2 tower $(Q, \overrightarrow{(d', q', P')})$ of discontinuous type. A $(Y, T, *)$-description is a (Y, T, Q')-description for some level ≤ 2 tree Q' or $Q' = \emptyset$. $\operatorname{desc}(Y, T, (Q, \overrightarrow{(d,q,P)}))$, $\operatorname{desc}(Y, T, Q)$, $\operatorname{desc}(Y, T, *)$ denote the sets of relevant descriptions.

Similarly to Definition 4.49, if $\mathbf{B} \in \operatorname{desc}(Y, T, Q)$, then there is at most one $(Q, \overrightarrow{(d,q,P)})$ for which $\mathbf{B} \in \operatorname{desc}(Y, T, (Q, \overrightarrow{(d,q,P)}))$. Suppose that $\mathbf{B} = (\mathbf{y}, \pi)$ is a $(Y, T, (Q, \overrightarrow{(d,q,P)}))$-description, $F \in (\boldsymbol{\delta}^1_3)^{Y \uparrow}$. Then

$$F^T_{\mathbf{B}} : [\omega_1]^{T \uparrow} \to \boldsymbol{\delta}^1_3$$

is the function that sends $[h]^T$ to $[F_{\mathbf{y}} \circ h^Q_\pi]_{\mu^Q}$. Note that $F_{\mathbf{y}} \circ h^Q_\pi$ has signature $\operatorname{sign}(Q, \overrightarrow{(d,q,P)})$, is essentially discontinuous, and has uniform cofinality $\operatorname{ucf}(Q, \overrightarrow{(d,q,P)})$. Of course, $F^T_{\mathbf{B}}$ is meaningful only when T is Π^1_2-wellfounded.

Assuming $\mathbf{\Pi}^1_3$-determinacy, the $\mathbb{L}[T_3]$-measure μ^Y will be defined, and $[F]^Y \to [F^T_{\mathbf{B}}]_{\mu^T}$ will represent an element in $\mathbb{L}[j^Y(T_3)]$ modulo μ^Y. Such kind of results related to level-3 ultrapowers are parallel to Section 4.6. They will be handled in Section 6.

Suppose $(\vec{Q}, \overrightarrow{(d,q,P)}) = (Q_i, (d_i, q_i, P_i))_{1 \leq i \leq k}$ is a potential partial level ≤ 2 tower and $\mathbf{B} = (\mathbf{y}, \pi) \in \operatorname{desc}(Y, T, (Q_k, \overrightarrow{(d,q,P)}))$. Define $\operatorname{lh}(\mathbf{B}) = k$. $B \restriction 0$ is the constant $(Y, T, *)$-description. Suppose $\mathbf{y} = (y, X, \overrightarrow{(e,x,W)})$, $0 < \bar{k} < k$. Then

$$\mathbf{B} \restriction \bar{k} \in \operatorname{desc}(Y, T, (Q_{\bar{k}}, (d_i, q_i, P_i)_{1 \leq i \leq \bar{k}}))$$

is defined by the following: letting l be the least such that $\pi(e_l, x_l) \notin \operatorname{desc}(T, Q_{\bar{k}}, *)$, $\mathbf{C} \in \operatorname{desc}(T, Q_{\bar{k}}, *)$ be such that $\mathbf{C} \lhd_2^{T, Q_{\bar{k}}} \pi(e_l, x_l)$, then

1. if $\mathbf{C} \neq \pi(e_l, x_l)$, then $\mathbf{B} \restriction \bar{k} = (\mathbf{y} \restriction l^\frown(-1), \bar{\pi})$, where $\bar{\pi}$ and π agree on $Y_{\text{tree}}(y \restriction l)$, $\bar{\pi}(e_l, x_l) = \mathbf{C}$;
2. if $\mathbf{C} = \pi(e_l, x_l)$, then $\mathbf{B} \restriction \bar{k} = (\mathbf{y} \restriction l, \pi \restriction Y_{\text{tree}}(y \restriction l))$.

Define

$$\mathbf{B} \lhd \mathbf{B}'$$

iff $\mathbf{B} = \mathbf{B}' \restriction \bar{k}$ for some $\bar{k} < \operatorname{lh}(\mathbf{B}')$. Define $\lhd^{Y,T} = \lhd \restriction \operatorname{desc}(Y, T, *)$. As a corollary to Lemma 4.58, $\lhd^{Y,T}$ inherits the following continuity property.

Lemma 4.63. *Suppose Y is a level-3 tree, T is a level ≤ 2 tree, Q is a level ≤ 2 proper subtree of Q'. Suppose $\mathbf{B} = (\mathbf{y}, \pi) \in \operatorname{desc}(Y, T, Q)$ and $\mathbf{B}' = (\mathbf{y}', \pi') \in \operatorname{desc}(Y, T, Q')$, $\mathbf{B} \lhd^{Y,T} \mathbf{B}'$. Suppose $E \in \mu_{\mathbb{L}}$ is a club, $\eta \in E'$ iff $\eta \in E$ and $E \cap \eta$ has order type η. Then for any $F \in (\boldsymbol{\delta}^1_3)^{Y\uparrow}$, for any $h \in \omega_1^{T\uparrow}$, for any $\vec{\beta} \in [E']^{Q\uparrow}$,*

$$F_{\mathbf{y}} \circ h^Q_\pi(\vec{\beta}) = \sup\{F_{\mathbf{y}'} \circ h^{Q'}_{\pi'}(\vec{\gamma}) : \vec{\gamma} \in [E]^{Q'\uparrow}, \vec{\gamma} \text{ extends } \vec{\beta}\}.$$

Hence, the signature and approximation sequence of $F_{\mathbf{y}} \circ h^Q_\pi$ are proper initial segments of those of $F_{\mathbf{y}'} \circ h^{Q'}_{\pi'}$ respectively.

Given a $(Y, T, *)$-description $\mathbf{B} = (\mathbf{y}, \pi)$, define

$$\langle \mathbf{B} \rangle = \pi \oplus \mathbf{y}.$$

Define

$$\mathbf{B} \prec \mathbf{B}'$$

iff $\langle \mathbf{B} \rangle <_{BK} \langle \mathbf{B}' \rangle$, the ordering on coordinates in $\operatorname{desc}(T, Q, *)$ for some T, Q again according to $\prec$. The constant $(Y, T, *)$-description $\mathbf{B}_0$ is the $\prec$-greatest, and we have $\langle \mathbf{B}_0 \rangle = \emptyset$. Define $\prec^{Y,T} = \prec \restriction \operatorname{desc}(Y, T, *)$. As a corollary to Lemma 4.51, $\prec^{Y,T}$ inherits the following ordering property on F^T_B.

Lemma 4.64. *Suppose* $(Q_i,(d_i,q_i,P_i))_{1\le i\le m}$ *is a partial level* ≤ 2 *tower,* $\mathbf{B}\in \operatorname{desc}(Y,T,Q_k)$*,* $\mathbf{B}'\in \operatorname{desc}(Y,T,Q_{k'})$*,* $k\le m$*,* $k'\le m$*,* $\mathbf{B}\prec^{Y,T}\mathbf{B}'$*. Then for any* $F\in(\boldsymbol{\delta}^1_3)^{Y\uparrow}$*, for any* $\vec{\beta}\in[\omega_1]^{T\uparrow}$*,* $j^{Q_k,Q_m}\circ F^T_{\mathbf{B}}(\vec{\beta})<j^{Q_{k'},Q_m}\circ F^T_{\mathbf{B}'}(\vec{\beta})$*.*

Definition 4.65. Suppose R,Y are level-3 trees, T is a level ≤ 2 tree. Suppose $\rho:\operatorname{dom}(R)\cup\{\emptyset\}\to \operatorname{desc}(Y,T,*)$ is a function. ρ *factors* (R,Y,T) iff

1. $\rho(\emptyset)$ is the constant $(Y,T,*)$-description.
2. For any $r\in\operatorname{dom}(R)$, $\rho(r)\in\operatorname{desc}(Y,T,R[r])$.
3. For any $r^\frown(a),r^\frown(b)\in\operatorname{dom}(R)$, if $a<_{BK}b$ and $R_{\text{tree}}(r^\frown(a))=R_{\text{tree}}(r^\frown(b))$ then $\rho(r^\frown(a))\prec\rho(r^\frown(b))$.
4. For any $r\in\operatorname{dom}(R)$, $\rho(r^-)\lhd^{Y,T}\rho(r)$.

If Y is a level-3 tree, then

$$\operatorname{id}_{Y,*}$$

factors (Y,Y,Q^0) where $\operatorname{id}_{Y,*}(y)=((y,X,\overrightarrow{(e,x,W)}),\operatorname{id}_{*,X})$ for $Y[y]=(X,\overrightarrow{(e,x,W)})$.

For level-3 trees R,Y, we say that $\rho:\operatorname{dom}(R)\to\operatorname{dom}(Y)$ *factors* (R,Y) iff

1. If $r\in\operatorname{dom}(R)$ then $R(r)=Y(\rho(r))$.
2. If $r,r'\in\operatorname{dom}(R)$ and $r\subseteq r'$, then $\rho(r)\subseteq\rho(r')$.
3. If $R_{\text{tree}}(r^\frown(a))=R_{\text{tree}}(r^\frown(b))$ and $a<_{BK}b$, then $\rho(r^\frown(a))<_{BK}\rho(r^\frown(b))$.

If in addition, ρ is onto $\operatorname{dom}(Y)$, then ρ is called a *level-3 tree isomorphism* between R and Y. If ρ factors (R,Y) and $\vec{\gamma}=(\gamma_y)_{y\in\operatorname{dom}(Y)}\in[\boldsymbol{\delta}^1_3]^{Y\uparrow}$, let $\vec{\gamma}_\rho=(\gamma_{\rho,r})_{r\in\operatorname{dom}(R)}\in[\boldsymbol{\delta}^1_3]^{R\uparrow}$ where $\gamma_{\rho,r}=\gamma_{\rho(r)}$.

If ρ factors (R,Y,T) and $F\in(\boldsymbol{\delta}^1_3)^{Y\uparrow}$, let

$$F^T_\rho:[\omega_1]^{T\uparrow}\to[\boldsymbol{\delta}^1_3]^{R\uparrow}$$

be the function that sends $\vec{\xi}$ to $(F^T_{\rho(r)}(\vec{\xi}))_{r\in\operatorname{dom}(R)}$. The fact that $F^T_\rho(\vec{\xi})\in[\boldsymbol{\delta}^1_3]^{R\uparrow}$ follows from Lemmas 4.64-4.63.

Suppose Y is a level-3 tree, T is a level ≤ 2 tree. *A representation of* $Y\otimes T$ is a pair (R,ρ) such that

1. R is a level-3 tree;

2. ρ factors (R, Y, T);

3. $\mathrm{ran}(\rho) = \mathrm{desc}(Y, T, *)$;

4. If $R_{\mathrm{tree}}(r^\frown(a)) = R_{\mathrm{tree}}(r^\frown(b))$, then $a <_{BK} b$ iff $\pi(r^\frown(a)) \prec^{Y,T} \pi(r^\frown(b))$.

Representations of $Y \otimes T$ are clearly mutually isomorphic. As before, we shall regard

$$Y \otimes T$$

itself as a "level-3 tree" whose domain is the set of non-constant $(Y, T, *)$-descriptions and sends $\mathbf{B} \in \mathrm{desc}(Y, T, (Q, (d_i, q_i, P_i)_{1 \leq i \leq k}))$ to $(Q, (d_k, q_k, P_k))$. If Q is a level ≤ 2 tree, then $(Y \otimes T) \otimes Q$ is a "level-3 tree" whose domain consists of non-constant $(Y \otimes T, Q, *)$-descriptions. There is a natural isomorphism

$$\iota_{Y,T,Q}$$

between "level-3 trees" $(Y \otimes T) \otimes Q$ and $Y \otimes (T \otimes Q)$, defined as follows:

1. If $\mathbf{A} = ((\mathbf{B}, Z, \overrightarrow{(d, z, N)}), \psi) \in \mathrm{desc}(Y \otimes T, Q, U)$, $\mathbf{B} = (\mathbf{y}, \pi) \in \mathrm{desc}(Y, T, (Z, \overrightarrow{(d, z, N)}))$, $\mathbf{y} = (y, X, \overrightarrow{(e, x, W)})$, then $\iota_{Y,T,Q}(\mathbf{A}) = (\mathbf{y}, \iota^{-1}_{T,Q,U} \circ (T \otimes \psi) \circ \pi)$.

2. If $\mathbf{A} = ((\mathbf{B}^\frown(-1), Z^+, \overrightarrow{(d, z, N)}), \psi) \in \mathrm{desc}(Y \otimes T, Q, U)$, $\overrightarrow{(d, z, N)} = (d_i, z_i, N_i)_{1 \leq i \leq l}$, $\mathbf{B} = (\mathbf{y}, \pi) \in \mathrm{desc}(Y, T, (Z, \overrightarrow{(d, z, N)}))$, $\mathbf{y} = (y, X, (e_i, x_i, W_i)_{1 \leq i \leq k})$, then

   (a) if y is of discontinuous type, then $\iota_{Y,T,Q}(\mathbf{A}) = (\mathbf{y}^\frown(-1), \psi *_0 \pi)$, where $\psi *_0 \pi$ factors $(X^+, T \otimes Q, U)$, X^+ is a completion of $(X, (e_i, x_i))$, $\psi *_0 \pi$ extends $\iota^{-1}_{T,Q,U} \circ (T \otimes \psi) \circ \pi$, $\psi *_0 \pi(e_k, x_k) = \iota^{-1}_{T,Q,U}(2, \mathbf{t}_0, \tau)$, $\mathbf{t}_0 = ((-1), \{(0)\}, ((0)))$, $\tau((0)) = \psi(d_l, z_l)$;

   (b) if y is of continuous type, then $\iota_{Y,T,Q}(\mathbf{A}) = (\mathbf{y}, \psi *_1 \pi)$, where $\psi *_1 \pi$ factors $(X, T \otimes Q, U)$, $\psi *_1 \pi$ extends $\iota^{-1}_{T,Q,U} \circ (T \otimes \psi) \circ \pi \restriction (\mathrm{dom}(X) \setminus \{(e_k, x_k)\})$, $\psi *_1 \pi(e_k, x_k) = \iota^{-1}_{T,Q,U}(2, \mathbf{t}^\frown(-1), \tau^+)$, where $\pi(e_k, x_k) = (2, \mathbf{t}, \tau)$, $\mathbf{t} = (t, S, (s_i)_{i \leq m})$, τ^+ extends τ, $\tau^+(s_m) = \psi(d_l, z_l)$.

$\iota_{Y,T,Q}$ justifies the associativity of the $\otimes$ operator acting on level $(3, \leq 2, \leq 2)$ trees.

The identity map $\mathrm{id}_{Y \otimes T}$ factors $(Y \otimes T, Y, T)$. ρ factors (R, Y, T) iff ρ factors $(R, Y \otimes T)$. If $y \in \mathrm{dom}(Y)$, $\mathbf{y} = (y, X, \overrightarrow{(e, x, W)}) \in \mathrm{desc}(Y)$,

$$Y \otimes_y T$$

is the level-3 subtree of $Y \otimes T$ whose domain is $\mathrm{dom}(Y \otimes Q^0)$ plus all the $(Y,T,*)$-descriptions of the form $(\mathbf{y},\tau)$. If π factors level ≤ 2 trees (T,Q), then

$$Y \otimes \pi$$

factors $(Y \otimes T, Y \otimes Q)$, where $Y \otimes \pi(\mathbf{y},\psi) = (\mathbf{y},(\pi \otimes U) \circ \psi)$ for $(\mathbf{y},\psi) \in \mathrm{desc}(Y,T,U)$.

If ρ factors finite trees (R,Y,T), then ρ induces

$$\tilde{\rho}^T : \mathrm{desc}^{**}(R) \to \mathrm{desc}^{**}(Y)$$

as follows:

1. If $\mathbf{A} = (\emptyset,\emptyset,\emptyset)$, then $\tilde{\rho}^T(\mathbf{A}) = \mathbf{A}$.
2. If $\mathbf{A} = (\mathbf{r},\psi,U)$, $\mathbf{r} = (r,Q,\overrightarrow{(d,q,P)})$ is of discontinuous type, $\rho(r) = (\mathbf{y},\pi)$, then $\tilde{\rho}^T(\mathbf{A}) = (\mathbf{y},(T \otimes \psi) \circ \pi)$.
3. If $\mathbf{A} = (\mathbf{r},\psi,U)$, $\mathbf{r} = (r,Q,\overrightarrow{(d,q,P)})$ is of continuous type, $\overrightarrow{(d,q,P)} = (d_i,q_i,P_i)_{1\leq i\leq l}$, $\rho(r^-) = (\mathbf{y},\pi)$, $\mathbf{y} = (y,X,(e_i,x_i,W_i)_{1\leq i\leq k})$,
   (a) if y is of discontinuous type, then $\tilde{\rho}^T(\mathbf{A}) = (\mathbf{y}^\frown(-1),\psi *_0 \pi)$, where $\psi *_0 \pi$ factors $(X^+, T \otimes U)$, $\psi *_0 \pi$ extends $(T \otimes \psi) \circ \pi$, $\psi *_0 \pi(e_k,x_k) = (2,\mathbf{t}_0,\tau)$, $\mathbf{t}_0 = ((-1),\{(0)\},((0)))$, $\tau((0)) = \psi(d_l,p_l)$;
   (b) if y is of continuous type, then $\tilde{\rho}^T(\mathbf{A}) = (\mathbf{y},\psi *_1 \pi)$, where $\psi *_1 \pi$ factors $(X, T \otimes U)$, $\psi *_1 \pi$ extends $(T \otimes \psi) \circ (\pi \restriction \mathrm{dom}(X) \setminus \{(e_k,x_k)\})$, $\psi *_1 \pi(e_k,x_k) = (2,\mathbf{t}^\frown(-1),\tau^+)$, where $\pi(e_k,x_k) = (2,\mathbf{t},\tau)$, $\mathbf{t} = (t,S,(s_i)_{i\leq m})$, τ^+ extends τ, $\tau^+(s_m) = \psi(d_l,p_l)$.

$\mathbf{A} \prec^R_* \mathbf{A}'$ iff $\tilde{\rho}^T(\mathbf{A}) \prec^Y_* \tilde{\rho}^T(\mathbf{A}')$; $\mathbf{A} \sim^R_* \mathbf{A}'$ iff $\tilde{\rho}^T(\mathbf{A}) \sim^Y_* \tilde{\rho}^T(\mathbf{A}')$. A purely combinatorial argument shows that if $R = Y \otimes T$, then for any $\mathbf{B} \in \mathrm{desc}^{**}(Y)$ there is $\mathbf{A} \in \mathrm{desc}^{**}(R)$ such that $\tilde{\rho}^T(\mathbf{A}) \sim^Y_* \mathbf{B}$.

Lemma 4.66. *Suppose Q is a finite level ≤ 2 tree, W is a finite level-1 tree, $\theta : [\omega_1]^{Q\uparrow} \to j^W(\omega_1)$ is a function in $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. Suppose $\mathrm{cf}^{\mathbb{L}}([\theta]_{\mu^Q}) = \mathrm{seed}^{Q,W}_{\mathbf{D}}$, $\mathbf{D} = (d,\mathbf{q},\sigma) \in \mathrm{desc}(Q,W)$.*

1. *The uniform cofinality of θ is $\mathrm{ucf}^W_2(\mathbf{D})$.*
2. *$\mathrm{ucf}_1(\mathbf{D}) = -1$ iff $\mathrm{cf}^{\mathbb{L}}(\theta(\vec{\xi})) = \omega$ for μ^Q-a.e. $\vec{\xi}$.*
3. *Fix $w \in W$. Then $\mathrm{ucf}_1(\mathbf{D}) = w$ iff $\mathrm{cf}^{\mathbb{L}}(\theta(\vec{\xi})) = \mathrm{seed}^W_w$ for μ^Q-a.e. $\vec{\xi}$.*

Proof. Let $g \in \mathbb{L}$ be a strictly increasing function from $\mathrm{seed}^{Q,W}_{\mathbf{D}}$ to $[\theta]_{\mu^Q}$ cofinally. Find $G \in \mathbb{L}$ such that $[G]_{\mu^Q} = g$. We have $[\theta]_{\mu^Q} = \sup [G]_{\mu^Q}{}'' \mathrm{seed}^{Q,W}_{\mathbf{D}}$. By Łoś, for μ^Q-a.e. $\vec{\xi}$, $\theta(\vec{\xi}) = \sup G(\vec{\xi})''(\sigma^W({}^d\xi_{\mathbf{q}}))$. (Recall our convention that $\emptyset^W = j^W$.) This shows part 1. Also, for μ^Q-a.e. $\vec{\xi}$, $\mathrm{cf}^{\mathbb{L}}(\theta(\vec{\xi})) = \mathrm{cf}^{\mathbb{L}}(\sigma^W({}^d\xi_{\mathbf{q}}))$, which equals to ω when $\mathrm{ucf}_1(\mathbf{D}) = -1$, equals to $\mathrm{seed}^W_{\mathrm{ucf}_1(\mathbf{D})}$ otherwise. This shows parts 2-3. $\square$

Lemma 4.67. *Suppose R, Y are level-3 trees, $\theta : \mathrm{rep}(R) \to \mathrm{rep}(Y)$ is continuous and order preserving, $\theta \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. Then there exists a triple*

$$(T, \rho, \vec{\delta})$$

such that T is a level ≤ 2 tree, ρ factors (R, Y, T), $\vec{\delta}$ respects T, and

$$\forall F \in (\boldsymbol{\delta}^1_3)^{Y\uparrow}\ F^T_\rho(\vec{\delta}) = [F \circ \theta]^R.$$

Proof. For $r \in \mathrm{dom}(R)$, let $R(r) = (Q_r, (d_r, q_r, P_r))$. For $q \in \mathrm{dom}({}^2Q_r)$, let ${}^2Q_r(q) = (P_{r,q}, p_{r,q})$. Thus, when $d_r = 2$, P_r is the completion of $(P_{r,q_r^-}, p_{r,q_r^-})$. Let $E \in \mu_{\mathbb{L}}$, $\mathbf{y}_r = (y_r, X_r, \overrightarrow{(e_r, x_r, W_r)}) \in \mathrm{desc}(Y)$ and $\theta_r \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$ be such that for any $\vec{\beta} \in [E]^{Q_r\uparrow}$, $\theta_r(\vec{\beta}) \in [\omega_1]^{X_r\uparrow}$ and

$$\theta(\vec{\beta} \oplus_R r) = \theta_r(\vec{\beta}) \oplus_Y y_r.$$

Let $\overrightarrow{(e_r, x_r, W_r)} = (e_{r,i}, x_{r,i}, W_{r,i})_{1 \le i \le \mathrm{lh}(\vec{x}_r)}$. For $x \in \mathrm{dom}({}^2X_r)$, let ${}^2X_r(x) = (W_{r,x}, w_{r,x})$. Thus, when $e_{r,i} = 2$, $W_{r,i}$ is the completion of $(W_{r,x_r^-}, w_{r,x_r^-})$. Let $[\theta_r]_{\mu^{Q_r}} = \vec{\gamma}_r = ({}^e\gamma_{r,x})_{(e,x) \in \mathrm{dom}(X_r)}$, $\theta_r(\vec{\beta}) = ({}^e\theta_{r,x}(\vec{\beta}))_{(e,x) \in \mathrm{dom}(X_r)}$. So ${}^e\gamma_{r,x} = [{}^e\theta_{r,x}]_{\mu^{Q_r}}$. For $e \in \{1, 2\}$, let

$$B^e_x = \{x \in \mathrm{dom}({}^eX_r) : {}^e\gamma_{r,x} < \omega_1\}.$$

So B^1_r is closed under $\prec^{{}^1X_r}$ and $B^2_x = \emptyset$. For $x \in {}^eX_r \setminus B^e_r$, let $(S^e_{r,x}, \vec{s}^e_{r,x})$ be the potential partial level ≤ 1 tower induced by ${}^e\gamma_{r,x}$, $\vec{s}^e_{r,x} = (s^e_{r,x,i})_{i < \mathrm{lh}(\vec{s}^e_{r,x})}$, $s^e_{r,x} = s^e_{r,x,\mathrm{lh}(\vec{s}^e_{r,x})-1}$, let $(\mathrm{seed}^{Q_r, W_{r,x}}_{\mathbf{D}^e_{r,x,i}})_{i < v^e_{r,x}}$ be the signature of ${}^e\gamma_{r,x}$, let $(\delta^e_{r,x,i})_{i \le v^e_{r,x}}$ be the approximation sequence of ${}^e\gamma_{r,x}$, and let $\mathrm{cf}^{\mathbb{L}}({}^e\gamma_{r,x}) = \mathrm{seed}^{Q_r, W_{r,x}}_{\mathbf{D}^e_{r,x}}$ if $\mathrm{cf}^{\mathbb{L}}({}^e\gamma_{r,x}) > \omega$. The existence of $(q^e_{r,x,i})_{i < v^e_{r,x}}$ and $q^e_{r,x}$ follows from Lemma 4.32. Let $\mathbf{D}^e_{r,x,i} = (c^e_{r,x,i}, \mathbf{q}^e_{r,x,i}, \sigma^e_{r,x,i})$, $\mathbf{D}^e_{r,x} = (c^e_{r,x}, q^e_{r,x}, \sigma^e_{r,x})$. Let

$$\tau^e_{r,x}$$

factor $(S^e_{r,x}, Q_r, *)$, where $\tau^e_{r,x}(s^e_{r,x,i}) = \mathbf{D}^e_{r,x,i}$ for $i < v^e_{r,x}$. Let

$$D^e_r = \{x \in {}^eX_r \setminus B^e_r : {}^e\gamma_{r,x} \text{ is essentially continuous}\},$$
$$E^e_r = \mathrm{dom}({}^eX_r) \setminus (B^e_r \cup D^e_r).$$

Thus, $v^e_{r,x} = \operatorname{card}(S^e_{r,x})$. For $x \in D^e_r$, $v^e_{r,x} = \operatorname{lh}(\vec{s}^e_{r,x})$; for $x \in E^e_r$, $v^e_{r,x} = \operatorname{lh}(\vec{s}^e_{r,x}) - 1$.

Put $\operatorname{ucf}(R[r]) = (d^*_r, \mathbf{q}^*_r)$, $\operatorname{ucf}(Y[y_r]) = (e^*_r, \mathbf{x}^*_r)$, if $e^*_r = 2$ then put $\mathbf{x}^*_r = (x^*_r, W^*_r, \vec{w}^*_r)$.

By order preservation and continuity of θ, we can see that for $r \in \operatorname{dom}(R)$,

1. if y_r is of continuous type, then ${}^{e_r}\theta_{r,x_r}$ has uniform cofinality $\operatorname{ucf}(R[r])$;
2. if y_r is of continuous type and $\operatorname{lh}(r) = 1 \vee (d_{r^-}, q_{r^-})$ does not appear in the contraction of $(\operatorname{sign}_2^{Q_r}({}^e\theta_{r,x}))$ for any $(e,x) \in \operatorname{dom}(X_r) \setminus \{(e_r, x_r)\}$, then ${}^{e_r}\theta_{r,x_r}$ is essentially discontinuous;
3. if y_r is of discontinuous type,
   (a) if $d^*_r = 0$, then $e^*_r = 0$;
   (b) if $d^*_r = 1$, then $e^*_r = 1$, ${}^1\theta_{r,\mathbf{x}^*_r}$ has uniform cofinality $(1, \mathbf{q}^*_r)$, and thus by Lemma 4.66, $\mathbf{D}^1_{r,x^*_r} = (1, \mathbf{q}^*_r, \emptyset)$;
   (c) if $d^*_r = 2$ and $\mathbf{q}^*_r \in \operatorname{desc}({}^2Q_r)$, then $e^*_r = 2$ and $\mathbf{x}^*_r \in \operatorname{desc}({}^2X_r)$, ${}^2\theta_{r,x^*_r}$ has uniform cofinality $(2, \mathbf{q}^*_r)$, and thus by Lemma 4.66, $\operatorname{ucf}_2^{W^*_r}(\mathbf{D}^2_{r,x^*_r}) = (2, \mathbf{q}^*_r)$;
   (d) if $d^*_r = 2$ and $\mathbf{q}^*_r \notin \operatorname{desc}({}^2Q_r)$, then $e^*_r = 2$ and $\mathbf{x}^*_r \in \operatorname{desc}({}^2X_r)$, $j^{W_{r,x^*_r}, W^*_r}({}^2\theta_{r,x^*_r})$ has uniform cofinality $(2, \mathbf{q}^*_r)$, and thus by Lemma 4.66, $\operatorname{ucf}_2^{W^*_r}(\mathbf{D}^2_{r,x^*_r}) = (2, \mathbf{q}^*_r)$.

Claim 4.68. *Suppose $r \in \operatorname{dom}(R)$, $x, x' \in \operatorname{dom}({}^2X_r)$, $x = (x')^-$. Suppose the contraction of $(\operatorname{sign}_1(\mathbf{D}^2_{r,x,i}))_{i<v^2_{r,x}}$ is $(w_{r,x\restriction i})_{i<\operatorname{lh}(x)}$. Then*

1. *For any $i < v^2_{r,x}$, $\delta^2_{r,x,i} = \delta^2_{r,x',i}$.*
2. *$(\mathbf{D}^2_{r,x,i}, \delta^2_{r,x,i})_{i<v^2_{r,x}}$ is a proper initial segment of $(\mathbf{D}^2_{r,x',i}, \delta^2_{r,x',i})_{i<v^2_{r,x'}}$. Hence, $S^2_{r,x}$ is a proper subtree of $S^2_{r,x'}$ and $\vec{s}^2_{r,x}$ is an initial segment of $\vec{s}^2_{r,x'}$.*
3. *$\operatorname{sign}_1(\mathbf{D}^2_{r,x',v^2_{r,x}}) = w_{r,x}$. In particular, the contraction of $(\operatorname{sign}_1(\mathbf{D}^2_{r,x',i}))_{i<v^2_{r,x'}}$ is $(w_{r,x\restriction i})_{i\le\operatorname{lh}(x)}$.*
4. *$\mathbf{D}^2_{r,x} \lhd_1^{Q_r, W_{r,x}} \mathbf{D}^2_{r,x'}$.*
5. *If $x \in D^2_r \cup E^2_r$, $x^\frown(c), x^\frown(d) \in \operatorname{dom}({}^2X)$, $c <_{BK} d$, then $\delta^2_{r,x^\frown(c),v^2_{r,x}} < \delta^2_{r,x^\frown(d),v^2_{r,x}}$.*
6. *If $x \in D^2_r \cup E^2_r$, $[h]_{\mu^{S^2_{r,x}}} = \delta^2_{r,x,v^2_{r,x}}$, then for any $g \in E^{Q_r\uparrow}$,*

$$[h \circ g^{Q_r, W_{r,x}}_{\tau^2_{r,x}}]_{\mu^{W_{r,x}}} = \theta_r([g]^{Q_r}).$$

Proof. By Lemma 4.34, $j^{Q_r}(j^{W_{r,x},W_{r,x'}} \restriction j^{W_{r,x}}(\omega_1+1)) = j^{Q_r\otimes W_{r,x},Q_r\otimes W_{r,x'}} \restriction j^{Q_r\otimes W_{r,x}}(\omega_1+1)$ and $j^{Q_r}_{\sup}(j^{W_{r,x},W_{r,x'}} \restriction j^{W_{r,x}}(\omega_1+1)) = j_{\sup}^{Q_r\otimes W_{r,x},Q_r\otimes W_{r,x'}} \restriction j^{Q_r\otimes W_{r,x}}(\omega_1+1)$. Since θ_r takes values in $[\omega_1]^{X_r\uparrow}$ on a μ^{Q_r}-measure one set, for μ^{Q_r}-a.e. $\vec{\xi}$, we have

$$j^{W_{r,x},W_{r,x'}}({}^2\theta_{r,x}(\vec{\xi})) < {}^2\theta_{r,x'}(\vec{\xi}) < j^{W_{r,x},W_{r,x'}}({}^2\theta_{r,x}(\vec{\xi}))$$

and

$$\operatorname{cf}^{\mathbb{L}}({}^2\theta_{r,x}(\vec{\xi})) = \operatorname{seed}^{W_{r,x}}_{w^-_{r,x}}.$$

Hence by Łoś,

$$j_{\sup}^{Q_r\otimes W_{r,x},Q_r\otimes W_{r,x'}}({}^2\gamma_{r,x}) < {}^2\gamma_{r,x'} < j^{Q_r\otimes W_{r,x},Q_r\otimes W_{r,x'}}({}^2\gamma_{r,x})$$

and by Lemma 4.66,

$$\operatorname{ucf}_1(\mathbf{D}_{r,x}) = w^-_{r,x}.$$

We are in a position to apply Lemma 3.14 with

$$A = \{l : \exists \mathbf{D} \in \operatorname{desc}(Q_r, W_{r,x})\ u_l = \operatorname{seed}^{Q_r,W_{r,x'}}_{\mathbf{D}}\},$$

leading to parts 1-4. Part 5 also follows from Lemma 3.14, using the fact that $\gamma_{r,x^\frown(c)} < \gamma_{r,x^\frown(d)}$. We now prove part 6. Note that $\tau^2_{r,x}$ factors $(S^2_{r,x}, Q_r \otimes W_{r,x})$ and in fact, $(\tau^2_{r,x})^{Q_r\otimes W_{r,x}}(\delta^2_{r,x,v_{r,x}}) = \gamma^2_{r,x}$. Suppose we are given h with $[h]_{\mu^{S^2_{r,x}}} = \delta^2_{r,x,v^2_{r,x}}$. Define h_* on $[E]^{Q_r\uparrow}$ by $h_*([g]^{Q_r}) = [h \circ g^{Q_r,W_{r,x}}_{\tau^2_{r,x}}]_{\mu^{W_{r,x}}}$. By Łoś, it suffices to show that $[h_*]_{\mu^{Q_r}} = \gamma^2_{r,x}$. But this follows from Lemma 4.32. This finishes the proof of Claim 4.68. □

In parallel to Claim 4.55, we have

Claim 4.69. *Suppose $r, r' \in \operatorname{dom}(R)$, $r = (r')^-$, y_r is of continuous type, and the contraction of $((\operatorname{sign}_2(\mathbf{D}^{e_{r,j}}_{r,x_{r,j},i}))_{i<v^{e_{r,j}}_{r,x_{r,j}}})_{1\le j<\operatorname{lh}(x_r)}$ is $((d_{r\restriction i}, q_{r\restriction i}))_{1\le i<\operatorname{lh}(r)}$. Then*

1. *$y_r = y_{r'}$.*
2. *For any $(e,x) \in \operatorname{dom}(X_r) \setminus \{(e_r, x_r)\}$, ${}^e\gamma_{r,x} = {}^e\gamma_{r',x}$.*
3. *$(\mathbf{D}^{e_r}_{r,x_r,i}, \delta^{e_r}_{r,x_r,i})_{i<v^{e_r}_{r,x_r}}$ is a proper initial segment of $(\mathbf{D}^{e_r}_{r',x_r,i}, \delta^{e_r}_{r',x_r,i})_{i<v^{e_r}_{r',x_r}}$. Hence S_{r,x_r} is a proper subtree of S_{r',x_r}, and $\vec{s}_{r,x_r}$ is an initial segment of $\vec{s}_{r',x_r}$.*
4. *The level-2 signature of $\tau^{e_r}_{r',x_r}(s_{r,x_r})$ is $((1,q_r))$ if $d_r = 1$, $((2, q_{r\restriction i}))_{1\le i\le \operatorname{lh}(q_r)}$ if $d_r = 2$. In particular, the contraction of $((\operatorname{sign}_2(\mathbf{D}^{e_{r',j}}_{r',x_{r',j},i}))_{i<v^{e_{r',j}}_{r',x_{r',j}}})_{1\le j<\operatorname{lh}(x_r)}$ is $((d_{r\restriction i}, q_{r\restriction i}))_{1\le i\le\operatorname{lh}(r)}$.*

Proof. By order preservation and continuity of θ, $y_r = y_{r'}$ and for μ^{Q_r}-a.e. $\vec{\beta}$,

1. for any $(e,x) \in \operatorname{dom}(X_r) \setminus \{(e_r, x_r)\}$, if $\vec{\beta}'$ extends $\vec{\beta}$ then ${}^{e}\theta_{r,x}(\vec{\beta}) = {}^{e}\theta_{r',x}(\vec{\beta}')$;

2. ${}^{e_r}\theta_{r,x_r}(\vec{\beta}) = \sup\{{}^{e_r}\theta_{r',x_r}(\vec{\beta}') : \vec{\beta}' \text{ extends } \vec{\beta}\}$.

Thus, ${}^{e}\gamma_{r,x} = {}^{e}\gamma_{r',x}$ for any $(e,x) \in \operatorname{dom}(X_r) \setminus \{(e_r, x_r)\}$, and $j_{\sup}^{Q_r,Q_{r'}}({}^{e}\gamma_{r,x}) \leq {}^{e}\gamma_{r',x} < j^{Q_r,Q_{r'}}({}^{e}\gamma_{r,x})$. As $t_x = t_{x'}$ is of continuous type and (d_r, q_r) does not appear in $\operatorname{sign}({}^{e}\theta_{r',x})$ for any $(e,x) \in \operatorname{dom}(X_{r'}) \setminus \{(e_{r'}, x_{r'})\}$, θ_{r',x_r} is essentially discontinuous, giving $j_{\sup}^{Q_r,Q_{r'}}({}^{e}\gamma_{r,x}) \neq {}^{e}\gamma_{r',x}$. With the help of Lemma 4.32 again, we can find level-1 trees $M_r, M_{r'}$ such that M_r is a subtree of $M_{r'}$ and $j_{\sup}^{M_r,M_{r'}}({}^{e}\gamma_{r,x}) < {}^{e}\gamma_{r',x} < j^{M_r,M_{r'}}({}^{e}\gamma_{r,x})$. The claim then follows from Lemma 3.14. $\square$

In parallel to Claim 4.56, we have

Claim 4.70. *Suppose $r, r' \in \operatorname{dom}(R)$, $r = (r')^-$, y_r is of discontinuous type, and the contraction of $((\operatorname{sign}_2(\mathbf{D}^{e_{r,j}}_{r,x_{r,j},i}))_{i<v^{e_{r,j}}_{r,x_{r,j}}})_{1\leq j<\operatorname{lh}(x_r)}$ is $((d_{r\upharpoonright i}, q_{r\upharpoonright i}))_{1\leq i<\operatorname{lh}(r)}$. Put $x_r^* = x_r^-$ if $\operatorname{ucf}(R(r)) \notin \operatorname{desc}(Q_r)$, $x_r^* = x_r^- {}^\frown ((x_r(\operatorname{lh}(x_r)-1))^-)$ if $\operatorname{ucf}(R(r)) \in \operatorname{desc}(Q_r)$. Then*

1. *$y_r \subsetneq y_{r'}$.*

2. *For any $(e,x) \in \operatorname{dom}(X_r)$, ${}^{e}\gamma_{r,x} = {}^{e}\gamma_{r',x}$.*

3. *$(\mathbf{D}^{e_r}_{r,x_r,i}, \delta^{e_r}_{r,x_r,i})_{i<v^{e_r}_{r,x_r}}$ is a proper initial segment of $(\mathbf{D}^{e_r}_{r',x_r^*,i}, \delta^{e_r}_{r',x_r^*,i})_{i<v^{e_r}_{r',x_r^*}}$. The signature and approximation sequence of ${}^{e_r}\gamma_{r,x_r^*}$ are proper initial segments of those of ${}^{e_r}\gamma_{r',x_r}$. Hence S_{r,x_r^*} is a proper subtree of S_{r',x_r}, and $\vec{s}_{r,x_r^*}$ is an initial segment of $\vec{s}_{r',x_r}$.*

4. *The level-2 signature of $\tau^{e_r}_{r',x_r}(s_{r,x_r})$ is $((1, q_r))$ if $d_r = 1$, $((2, q_{r\upharpoonright i}))_{1\leq i\leq \operatorname{lh}(q_r)}$ if $d_r = 2$. In particular, the contraction of $((\operatorname{sign}_2(\mathbf{D}^{e_{r',j}}_{r',x_{r',j},i}))_{i<v^{e_{r',j}}_{r',x_{r',j}}})_{1\leq j<\operatorname{lh}(x_{r'})}$ is $((d_{r\upharpoonright i}, q_{r\upharpoonright i}))_{1\leq i\leq \operatorname{lh}(r)}$.*

Proof. By order preservation and continuity of θ, $y_r \subsetneq y_{r'}$ and for μ^{Q_r}-a.e. $\vec{\beta}$,

1. for any $(e,x) \in \operatorname{dom}(X_r)$, if $\vec{\beta}'$ extends $\vec{\beta}$ then ${}^{e}\theta_{r,x}(\vec{\beta}) = {}^{e}\theta_{r',x}(\vec{\beta}')$;

2. if $\operatorname{ucf}(R(r)) \notin \operatorname{desc}(Q_r)$ then $j^{X_r,X_r^*}({}^{e_r}\theta_{r,x_r^*}(\vec{\beta})) = \sup\{{}^{e_r}\theta_{r',x_r}(\vec{\beta}') : \vec{\beta}' \text{ extends } \vec{\beta}\}$, where $X_r^* = Y_{\text{tree}}(y_{r'} \upharpoonright \operatorname{lh}(y_r)+1)$;

3. if $\operatorname{ucf}(R(r)) \in \operatorname{desc}(Q_r)$ then ${}^{e_r}\theta_{r,x_r^*}(\vec{\beta}) = \sup\{{}^{e_r}\theta_{r',x_r}(\vec{\beta}') : \vec{\beta}' \text{ extends } \vec{\beta}\}$.

The rest is similar to the proof of Claim 4.69. $\square$

Let
$$\phi^1 : \{{}^1\gamma_{r,x} : r \in \mathrm{dom}(R), x \in B^1_r\} \to Z^1$$
be a bijection such that Z^1 is a level-1 tree and $v < v' \leftrightarrow \phi^1(v) \prec^{Z^1} \phi^1(v')$. Let
$$\phi^2 : \{(\mathbf{D}^e_{r,x,i}, \delta^e_{r,x,i})_{i<l} : r \in \mathrm{dom}(R), e \in \{1,2\}, x \in D^e_x \cup E^e_x, l < \mathrm{lh}(\vec{s}_{r,x})\} \to Z^2 \cup \{\emptyset\}$$
be a bijection such that Z^2 is a level-1 tree and $v \subseteq v' \leftrightarrow \phi^2(v) \subseteq \phi^2(v')$, $v <_{BK} v'$ iff $\phi^2(v) <_{BK} \phi^2(v)$, where the ordering of subcoordinates $\mathbf{D}^e_{r,x,i}$ is according to $\prec$. Let
$$T = ({}^1T, {}^2T)$$
where ${}^1T = Z^1$, ${}^2T$ is a level-2 tree, $\mathrm{dom}({}^2T) = Z^2$,
$$ {}^2T[\phi^2((\mathbf{D}^e_{r,x,i}, \delta^e_{r,x,i})_{i<\mathrm{lh}(\vec{s}_{r,x})-1})^\frown(-1)] = (S_{x,r}, \vec{s}_{x,r}) \text{ for } x \in D^e_r,$$
$$ {}^2T[\phi^2((\mathbf{D}^e_{r,x,i}, \delta^e_{r,x,i})_{i<\mathrm{lh}(\vec{s}_{r,x})-1})] = (S_{x,r}, \vec{s}_{x,r}) \text{ for } x \in E^e_r.$$

Let
$$\vec{\delta} = ({}^c\delta_t)_{(c,t)\in\mathrm{dom}(T)}$$
where ${}^1\delta_t = (\phi^1)^{-1}(t)$, ${}^2\delta_\emptyset = \omega_1$, ${}^2\delta_t = \delta^e_{r,x,l}$ where $t = \phi^2((\mathbf{D}^e_{r,x,i}, \delta^e_{r,x,i})_{i\le l})$. For $r \in \mathrm{dom}(R)$, let
$$\rho(r) = (\mathbf{y}_r, \pi_r)$$
where π_r factors (X_r, T, Q_r), defined as follows:
$$\pi_r(e,x) = \begin{cases} (1, \phi^1(\gamma^e_{r,x}), \emptyset) & \text{if } x \in B^e_r, \\ (2, (\phi^2((\mathbf{D}^e_{r,x,i}, \delta^e_{r,x,i})_{i<\mathrm{lh}(\vec{s}_{r,x})-1})^\frown(-1), S^e_{r,x}, \vec{s}^e_{r,x}), \tau^e_{r,x}) & \text{if } x \in D^e_r, \\ (2, (\phi^2((\mathbf{D}^e_{r,x,i}, \delta^e_{r,x,i})_{i<\mathrm{lh}(\vec{s}_{r,x})-1}), S^e_{r,x}, \vec{s}^e_{r,x}), \tau^e_{r,x}) & \text{if } x \in E^e_r. \end{cases}$$

It is easy to check that $(T, \rho, \vec{\delta})$ works for the lemma. $\square$

Put $[\![\emptyset]\!]_R = \mathrm{o.t.}(<^R)$. For $\mathbf{r} = (r, Q, \overrightarrow{(d,q,P)}) \in \mathrm{desc}^*(R)$, put
$$[\![\mathbf{r}]\!]_R = [\vec{\beta} \mapsto \|\vec{\beta} \oplus_R r\|_{<^R}]_{\mu^Q}.$$

If $\mathbf{r} \in \mathrm{desc}(R)$ is of discontinuous type, put $[\![r]\!]_R = [\![\mathbf{r}]\!]_R$. Note that if ρ factors Π^1_3-wellfounded trees (R, Y), then $[\![r]\!]_R \le [\![\rho(r)]\!]_Y$ for any $r \in \mathrm{dom}(R)$. We say that ρ *minimally factors* (R, Y) iff ρ factors (R, Y), R, Y are both Π^1_3-wellfounded and $[\![r]\!]_R = [\![\rho(r)]\!]_Y$ for any $r \in \mathrm{dom}(R)$. In particular, if Y is Π^1_3-wellfounded and T is Π^1_2-wellfounded, then $\mathrm{id}_{Y,*}$ minimally factors $(Y, Y \otimes T)$. In the assumption of Lemma 4.67, if R, Y are Π^1_3-wellfounded and $\mathrm{ran}(\theta)$ is a $<^Y$-initial segment of $\mathrm{rep}(Y)$, its proof constructs ρ which minimally factors $(R, Y \otimes T)$. This entails the comparison theorem between Π^1_3-wellfounded trees.

Theorem 4.71. *Suppose R, Y are Π^1_3-wellfounded level-3 trees and $[\![\emptyset]\!]_R \leq [\![\emptyset]\!]_Y$. Then there exists (T, ρ) such that T is Π^1_2-wellfounded and ρ minimally factors $(R, Y \otimes T)$. Furthermore, if $[\![\emptyset]\!]_R < [\![\emptyset]\!]_Y$, we further obtain $\mathbf{B} \in \mathrm{dom}(Y \otimes T)$ such that $\mathrm{lh}(\mathbf{B}) = 1$ and $[\![\emptyset]\!]_R = [\![\mathbf{B}]\!]_{Y \otimes T}$.*

4.9 Representations of ordinals in $\boldsymbol{\delta}^1_3$

We introduce a coding system for ordinals in $\boldsymbol{\delta}^1_3$ which is the higher level analog of WO. The coding system is guided by Corollary 2.12. Identifying u_ω with $(V_\omega \cup u_\omega)^{<\omega}$, we shall assume X is a Δ^1_3 subset of $\mathbb{R} \times (V_\omega \cup u_\omega)^{<\omega}$ so that the map $v \mapsto X_v$ is a surjection from $\mathbb{R}$ onto $\mathcal{P}((V_\omega \cup u_\omega)^{<\omega})$.

For a finite level-3 tree R and a tuple $\vec{\beta} \oplus_R t \in \mathrm{rep}(R)$, put

$$v \in \mathrm{LO}^R_{\vec{\beta} \oplus_R t}$$

iff for each $\vec{\gamma} \oplus_R s \leq^R \vec{\beta} \oplus_R t$,

$$(X_v)_{\vec{\gamma} \oplus_R s} =_{\mathrm{DEF}} \{(\xi, \eta) : (v, \vec{\gamma} \oplus_R s, \xi, \eta) \in X_v\}$$

is a linear ordering on u_ω. Put

$$v \in \mathrm{LO}^R$$

iff $v \in \mathrm{LO}^R_{\vec{\beta} \oplus_R t}$ for all $\vec{\beta} \oplus_R t \in \mathrm{rep}(R)$. The relations "$v \in \mathrm{LO}^R_{\vec{\beta} \oplus_R t}$" and "$v \in \mathrm{LO}^R$" are Δ^1_3. Put

$$v \in \mathrm{WO}^{R\uparrow}_{\vec{\beta} \oplus_R t}$$

iff for each $\vec{\gamma} \oplus_R s \leq^R \vec{\beta} \oplus_R t$, $(X_v)_{\vec{\gamma} \oplus_R s}$ is a wellordering on u_ω, and the map $\vec{\gamma} \oplus_R s \mapsto \mathrm{o.t.}((X_v)_{\vec{\gamma} \oplus_R s})$ is continuous, order preserving for $\vec{\gamma} \oplus_R s \leq^R \vec{\beta} \oplus_R t$. Put

$$v \in \mathrm{WO}^{R\uparrow}$$

iff $v \in \mathrm{WO}^{R\uparrow}_{\vec{\beta} \oplus_R t}$ for all $\vec{\beta} \oplus_R t \in \mathrm{rep}(R)$. The relations "$v \in \mathrm{WO}^{R\uparrow}_{\vec{\beta} \oplus_R t}$" and "$R$ is a finite level-3 tree $\wedge\, v \in \mathrm{WO}^{R\uparrow}$" are Π^1_3. If $(X_v)_{\vec{\beta} \oplus_R t}$ is a wellordering on u_ω, its order type is denoted by $\|v\|_{\vec{\beta} \oplus_R t}$. A member $v \in \mathrm{WO}^{R\uparrow}$ codes a tuple of ordinals $[v]^R$ that respects R:

$$[v]^R = [\vec{\beta} \oplus_R t \mapsto \|v\|_{\vec{\beta} \oplus_R t}]^R.$$

Clearly, if $v \in \mathrm{WO}^{R\uparrow}$, then $[v]^R \in L_{\kappa_3^{v,R}}[T_2, v, R]$ and is Δ_1-definable in $L_{\kappa_3^{v,R}}[T_2, v, R]$ from $\{T_2, v, R\}$. Put $[v]^R = ([v]^R_t)_{t \in \mathrm{dom}(R)}$. So $[v]^R_t = [\vec{\beta} \mapsto \|v\|^R_{\vec{\beta} \oplus_R t}]_{\mu^{R_{\mathrm{tree}(t)}}}$.

Observe the simple fact that for any finite level-1 tree W, for any $\vec{\alpha} = (\alpha_w)_{w \in W}$ respecting W, there is a Π^1_1-wellfounded level-1 tree W' extending W such that $\alpha_w = \|(w)\|_{<W'}$ for any $w \in W$. Intuitively, W' "represents" $\vec{\alpha}$ in the sense that $\vec{\alpha}$ extends to a tuple $\vec{\alpha}'$ respecting W' and if $\vec{\beta}$ respects W', then $\forall w \in W\ \alpha_w \leq \beta_w$. It is implicitly used in proving that $0^{\#}$ is the unique wellfounded remarkable EM blueprint. Likewise, its higher level analog will be an ingredient in the level-3 EM blueprint formulation of $0^{3\#}$. The goal of the remaining of this section is to prove Lemma 4.79, which states that every $\vec{\gamma}$ respecting a finite level-3 tree R is "representable". Lemma 4.79 will essentially be a strengthening of [13, Theorem 5.3].

The next lemma is an easy corollary of Lemma 3.18. In its statement, $(T, \vec{\gamma})$ is the "amalgamation" of $(Q, \vec{\beta})$ and $(Q', \vec{\beta}')$.

Lemma 4.72. *Suppose Q, Q' are level ≤ 2 trees, $\vec{\beta} = ({}^d\beta_q)_{(d,q) \in \mathrm{dom}(Q)}$ respects Q, $\vec{\beta}' = ({}^d\beta'_q)_{(d,q) \in \mathrm{dom}(Q')}$ respects Q'. Then there are a level ≤ 2 tree T, a tuple $\vec{\gamma} = ({}^d\gamma_t)_{(d,t) \in \mathrm{dom}(T)}$ and maps π, π' factoring (Q,T), (Q',T) respectively such that $\mathrm{dom}(T) = \mathrm{ran}(\pi) \cup \mathrm{ran}(\pi')$, ${}^d\gamma_{{}^d\pi(q)} = {}^d\beta_q$ for any $(d,q) \in \mathrm{dom}(Q)$, ${}^d\gamma_{{}^d\pi'(q)} = {}^d\beta'_q$ for any $(d,q) \in \mathrm{dom}(Q')$.*

Amalgamation of level-3 trees is similar, using Lemma 4.46 instead.

Lemma 4.73. *Suppose R, R' are level-3 trees, $\vec{\gamma} = (\gamma_r)_{r \in \mathrm{dom}(R)}$ respects R, $\vec{\gamma}' = (\gamma'_r)_{r \in \mathrm{dom}(R')}$ respects R'. Then there are a level-3 tree Y, a tuple $\vec{\delta} = (\delta_y)_{y \in \mathrm{dom}(Y)}$ and maps ρ, ρ' factoring (R, Y), (R', Y) respectively such that $\mathrm{dom}(Y) = \mathrm{ran}(\rho) \cup \mathrm{ran}(\rho')$, $\delta_{\rho(r)} = \gamma_r$ for any $r \in \mathrm{dom}(R)$, $\delta_{\rho'(r)} = \gamma'_r$ for any $r \in \mathrm{dom}(R')$.*

Lemma 4.74. *For any $a \in \omega^{<\omega}$, $\{[\![2, (a)]\!]_Q : Q$ is a Π^1_2-wellfounded level ≤ 2 tree, $(a) \in \mathrm{dom}(Q)\}$ is a cofinal subset of u_2.*

Proof. Note that $u_2 = \boldsymbol{\delta}^1_2$ is the sup of ranks of $\boldsymbol{\Sigma}^1_2$ wellfounded relations on $\mathbb{R}$. Given $<^*$, a $\boldsymbol{\Sigma}^1_2$ wellfounded on $\mathbb{R}$, we need to find a Π^1_2-wellfounded level ≤ 2 tree Q such that $\mathrm{rank}(<^*) \leq [\![2, ((0))]\!]_Q$. This suffices for the Lemma by rearranging the nodes in a level ≤ 2 tree in a suitable way. Put $x <^* x' \leftrightarrow \exists y\ x \oplus x' \oplus y \in A$, where A is $\boldsymbol{\Pi}^1_1$. Let $(P_s)_{s \in \omega^{<\omega}}$ be a regular level-1 system such that $P_{x \oplus x' \oplus y}$ is Π^1_1-wellfounded iff $x \oplus x' \oplus y \in A$. Fix an effective bijection $\phi : \omega^{<\omega} \leftrightarrow (\omega^{<\omega})^{<\omega}$. If $(W_n)_{n<\omega}$ is a sequence of nonempty level-1 trees, their join is $\oplus_{n<\omega} W_n = \{(n)^\frown w : w \in W_n\}$. Let Q^* be an infinite level-2 tree whose domain is $\{((0))^\frown q : q \in (\omega^{<\omega})^{<\omega}\}$, and for any real v, $\cup_{n<\omega} Q^*_{\mathrm{tree}}(((0))^\frown \phi(v \restriction n)) = \oplus_{n<\omega} P_{(v)_{2n+2} \oplus (v)_{2n} \oplus (v)_{2n+1}}$. Then Q^* is Π^1_2-wellfounded. Let $Q = (\emptyset, Q^*)$. The proof of Kunen-Martin shows that $\mathrm{rank}(<^*) \leq [\![2, ((0))]\!]_Q$. □

Lemma 4.75. *Suppose Q is a Π^1_2-wellfounded level ≤ 2 tree, $q^* \in \mathrm{dom}({}^2Q)$, P^* is the completion of ${}^2Q(q^*)$. Then $\{[\![2,q']\!]_{Q'} : Q'$ is Π^1_2-wellfounded, $[\![2,q^*]\!]_Q = [\![2,(q')^-]\!]_{Q'}$, ${}^2Q'_{\mathrm{tree}}(q') = P^*\}$ is a cofinal subset of $j^{{}^2Q_{\mathrm{tree}}(q^*),P^*}([\![2,q^*]\!]_Q)$.*

Proof. If $q^* = \emptyset$, we are reduced to Lemma 4.74. Suppose now $q^* \neq \emptyset$. Put ${}^2Q(q^*) = (P^-, p^*)$, so P^* is the completion of (P^-, p^*).

Let $p^{**} = \mathrm{pred}_{\prec^{P^*}}(p^*)$. By remarkability of the level-1 sharps, letting $f(\beta) = [\vec{\alpha} \mapsto \|(2, \vec{\alpha} \upharpoonright P^- {}^\frown g(\alpha_{p^*}) \oplus_{{}^2Q} q^* {}^\frown (-1))\|_{<^Q}]_{\mu^{P^-}}$ for $\beta = [g]_{\mu_{\mathbb{L}}} < u_2$, then $\sup f''u_2 = j^{P^-,P^*}([\![2,q^*]\!]_Q)$. Fix $\beta = [g]_{\mu_{\mathbb{L}}} < u_2$, and we try to find Q', q' such that ${}^2Q'[(q')^-] = {}^2Q[q^*]$, ${}^2Q'_{\mathrm{tree}}(q') = P^*$, and $f(\beta) < [\![2,q']\!]_{Q'}$. Let U be a Π^1_2-wellfounded level ≤ 2 tree obtained by Lemma 4.74 such that $\beta < [\![2,((0))]\!]_U$. Let (X,π) be a representation of $Q \otimes U$, and let $\theta : \mathrm{rep}(X) \to \mathrm{rep}(Q)$ be the order preserving bijection. Let $\mathbf{C} = (2, \mathbf{q}, \tau) \in \mathrm{desc}(Q, U, *)$, where $\mathbf{q} = (q^* {}^\frown (-1), P^*, \vec{p})$, τ extends id_{*,P^-}, $\tau(p^*) = (2, ((0), \{(0)\}, ((0))), \sigma)$, $\sigma((0)) = p^{**}$. Let $(2,x) = \pi^{-1}(\mathbf{C})$. Then for μ^P-a.e. $\vec{\alpha}$, $\theta(2, \vec{\alpha} \oplus_{{}^2X} x) = (2, \vec{\alpha} \upharpoonright P^- {}^\frown (g(\alpha_{p^*})) \oplus_{{}^2Q} q^* {}^\frown (-1))$. Therefore, $[\![2,x]\!]_Q = f([\![2,((0))]\!]_U) > f(\beta)$. (X,x) plays the role of the desired (Q',q'). □

Suppose Q is a level ≤ 2 tree and $\vec{\epsilon} = ({}^d\epsilon_t)_{(d,t) \in \mathrm{dom}(Q)}$ is a tuple of ordinals indexed by $\mathrm{dom}(Q)$. We say that $\vec{\epsilon}$ *is represented by* Q' iff Q is a subtree of Q', Q' is Π^1_2-wellfounded and $\vec{\epsilon} = ([\![d,t]\!]_{Q'})_{(d,t) \in \mathrm{dom}(Q')}$.

Lemma 4.76. *Suppose Q is a finite level ≤ 2 tree and $\vec{\beta} = ({}^d\beta_q)_{(d,q) \in \mathrm{dom}(Q)}$ respects Q. Then $\vec{\beta}$ is represented by some level ≤ 2 tree Q'.*

Proof. By rearranging the nodes in $\mathrm{dom}(Q')$ in a suitable way, it suffices to produce a level ≤ 2 tree Q' and a map π factoring (Q,Q') such that for any $(d,q) \in \mathrm{dom}(Q)$, ${}^d\beta_q = [\![\pi(d,q)]\!]_{Q'}$. By a repeated application of Lemma 4.72, it suffices to show that for any $(d^*, q^* {}^\frown (a)) \in \mathrm{dom}(Q)$,

1. if $d^* = 1$, then there is a Π^1_2-wellfounded level ≤ 2 tree Q' and $q' \in {}^1Q'$ such that ${}^1\beta_{q^* {}^\frown (a)} = [\![1,q']\!]_{Q'}$.
2. if $d^* = 2$ and $P^* = {}^2Q_{\mathrm{tree}}(q^* {}^\frown (a))$, then there is a Π^1_2-wellfounded level ≤ 2 tree Q' and $q' \in \mathrm{dom}({}^2Q')$ such that ${}^2\beta_{q^* {}^\frown (a)} = [\![2,q']\!]_{Q'}$, ${}^2Q'[(q')^-] = {}^2Q[q^*]$, ${}^2Q'_{\mathrm{tree}}(q') = P^*$.

The case $d^* = 1$ is obvious. We assume now $d^* = 2$.

Lemma 4.75 gives us a Π^1_2-wellfounded level ≤ 2 tree T and $t \in \mathrm{dom}({}^2T)$ such that $[\![2,t]\!]_T \geq {}^2\beta_{q^* {}^\frown (a)}$, ${}^2T[t^-] = {}^2Q[q^*]$, ${}^2T_{\mathrm{tree}}(t) = P^*$. Minimizing $[\![2,t]\!]_T$, we may further assume that for any Π^1_2-wellfounded T' and any t' such that $[\![2,t']\!]_{T'} \geq {}^2\beta_{q^* {}^\frown (a)}$, ${}^2T'[(t')^-] = {}^2Q[q^*]$, ${}^2T'_{\mathrm{tree}}(t') = P^*$, we have

$[\![2,t']\!]_{T'} \geq [\![2,q^*\frown(a)]\!]_T$. We claim that $[\![2,t]\!]_T = {}^2\beta_{q^*\frown(a)}$. Suppose otherwise. Put $p^* = {}^2T_{\text{node}}(t)$.

Case 1: $\text{cf}^T(2,t) = 0$.

If ${}^2T\{t,-\}$ has a $<_{BK}$-maximum t', then $[\![2,t]\!]_T = [\![2,t']\!]_T + \omega$. So ${}^2\beta_{q^*\frown(a)} \leq [\![2,t']\!]_T < [\![2,t]\!]_T$, contradicting the minimization assumption. If ${}^2T\{t,-\}$ has $<_{BK}$-limit order type, then $[\![2,t]\!]_T = \sup\{[\![2,t']\!]_T : t' \in {}^2T\{t,-\}\}$, so there is t' satisfying ${}^2\beta_{q^*\frown(a)} \leq [\![2,t']\!]_T < [\![2,t]\!]_T$, contradiction again.

Case 2: $\text{cf}^T(2,t) = 1$.

For $\beta = \omega_1$, put $f(\beta) = [\vec{\alpha} \mapsto \|(2,\vec{\alpha}\frown(\beta) \oplus_{{}^2T} t\frown(-1))\|_{<^T}]_{\mu^{P^*}}$. Then $[\![2,t]\!]_T = \sup\{f(\beta) : \beta < \omega_1\}$. For each limit $\beta < \omega_1$, we shall find a Π^1_2-wellfounded T' and a node t' such that $[\![2,t']\!]_{T'} = f(\beta)$, ${}^2T'[(t')^-] = {}^2Q[q^*], {}^2T'_{\text{tree}}(t') = P^*$, contradicting to the minimization assumption. Fix a limit ordinal $\beta < \omega_1$. Let U be a Π^1_2-wellfounded level ≤ 2 tree such that $[\![1,(0)]\!]_U = \beta$. Let (X,π) be a representation of $T \otimes U$ and let $\theta :$ $\text{rep}(X) \to \text{rep}(T)$ be the order preserving bijection. Let $\mathbf{C} = (2,\mathbf{t},\tau) \in$ $\text{desc}(T,U,*)$, $\mathbf{t} = (t\frown(-1),S,\vec{s})$, τ extends $\text{id}_{*,S}$, $\tau(s_{\text{lh}(\vec{s})-1}) = (1,(0),\emptyset)$. Let $(2,x) = \pi^{-1}(\mathbf{C})$. Then for μ^{P^*}-a.e. $\vec{\alpha}$, $\theta(2,\vec{\alpha}\oplus_{{}^2X} x) = (2,\vec{\alpha}\frown(\beta)\oplus_{{}^2T} t\frown(-1))$. Therefore, $[\![2,x]\!]_X = f(\beta)$. (X,x) plays the role of the desired (T',t').

Case 3: $\text{cf}^T(2,t) = 2$.

Let p^{**} be the $<_{BK}\restriction P^* \cup \{p^*\}$-predecessor of p^*. For $\beta = [g]_{\mu_{\mathbb{L}}} < u_2$, put $f(\beta) = [\vec{\alpha} \mapsto \|(2,\vec{\alpha}\frown g(\alpha_{p^{**}}) \oplus_{{}^2T} t\frown(-1))\|_{<^T}]_{\mu^{P^*}}$. Then $[\![2,t]\!]_T = \sup\{f(\beta) :$ $\beta < u_2\}$. For each limit $\omega_1 < \beta < u_2$, we shall find a Π^1_2-wellfounded T' and a node t' such that $[\![2,t']\!]_{T'} = f(\beta)$, ${}^2T'[(t')^-] = {}^2Q[q^*], {}^2T'_{\text{tree}}(t') = P^*$. Fix a limit ordinal $\omega_1 < \beta < u_2$. By Case 1 and Case 2 of this lemma applied to $(2,((0)))$ in place of $(d^*,q^*\frown(a))$, we can find a Π^1_2-wellfounded level ≤ 2 tree U such that $[\![2,(0)]\!]_U = \beta$. Let (X,π) be a representation of $T \otimes U$ and let $\theta : \text{rep}(X) \to \text{rep}(T)$ be the order preserving bijection. Let $\mathbf{C} = (2,\mathbf{t},\tau)$, $\mathbf{t} = (t\frown(-1),S,\vec{s})$, τ extends $\text{id}_{*,S}$, $\tau(s_{\text{lh}(\vec{s})-1}) = (2,((0),\{(0)\},((0))),\sigma)$, $\sigma((0)) = p^{**}$. Let $(2,x) = \pi^{-1}(\mathbf{C})$. (X,x) plays the role of the desired (T',t'). □

The level-3 version of Lemmas 4.74-4.76 are similarly proved.

Lemma 4.77. *For any $a \in \omega^{<\omega}$, $\{[\![(a)]\!]_R : R$ is a Π^1_3-wellfounded level-3 tree, $(a) \in \text{dom}(R)\}$ is a cofinal subset of $\boldsymbol{\delta}^1_3$.*

Proof. It is possible to imitate the proof of Lemma 4.74. We give an alternative proof using the prewellordering property of the pointclass $\boldsymbol{\Pi}^1_3$. Let G be a good universal Π^1_3-set and let $(R_s)_{s\in\omega^{<\omega}}$ be an effective level-3 system satisfying $x \in G$ iff $R_x =_{\text{DEF}} \cup_{n<\omega} R_{x\restriction n}$ is Π^1_3-wellfounded. G is equipped with the Π^1_3-norm $\varphi(x) = [\![\emptyset]\!]_{R_x}$, the complexity from Theorem 2.1. By

Moschovakis [36, 4C.14], o.t.(ran(φ)) $= \boldsymbol{\delta}^1_3$. The rest of the proof is simple. □

Lemma 4.78. *Suppose R is a Π^1_3-wellfounded level-3 tree, $r^* \in \mathrm{dom}(R)$, Q^* is a completion of $R(r^*)$. Then $\{[\![r']\!]_{R'} : R'$ is Π^1_3-wellfounded, $R'[(r')^-] = R[r^*]$, $R'_{\mathrm{tree}}(r') = Q^*\}$ is a cofinal subset of $j^{R_{\mathrm{tree}}(r^*),Q^*}([\![r^*]\!]_R)$.*

Proof. Put $R(r^*) = (Q^-, (d^*, q^*, P^*))$, $R[r^*] = (Q^-, \overrightarrow{(d, q, P)})$.

Case 1: $\mathrm{cf}(R(r^*)) = 1$.

By Lemma 4.31, letting $f(\xi) = [\vec{\beta} \mapsto \|\vec{\beta}^\frown(\xi) \oplus_R r^*\|_{<^R}]_{\mu^{Q^-}}$ for $\xi < \omega_1$, then $\sup f''\omega_1 = j^{Q^-,Q^*}([\![r^*]\!]_R)$. Fix $\beta < \omega_1$, and we try to find R' and r' such that $R'[(r')^-] = R[r^*]$, $R'(r') = Q^*$, and $f(\beta) < [\![r']\!]_{R'}$. Let U be a Π^1_2-wellfounded level ≤ 2 tree such that $\beta < [\![1,(0)]\!]_U$. Let (Z, ρ) be a representation of $R \otimes U$, and let $\theta : \mathrm{rep}(Z) \to \mathrm{rep}(U)$ be the order preserving bijection. Let $\mathbf{B} = (\mathbf{r}, \pi) \in \mathrm{desc}(R, U, *)$, where $\mathbf{r} = (r^{*\frown}(-1), Q^*, \overrightarrow{(d, q, P)})$, π extends id_{*,Q^-}, $\pi(d^*, q^*) = (1, (0), \emptyset)$. Let $z = \rho^{-1}(\mathbf{B})$. Similarly to Case 1 of the proof of Lemma 4.75, (Z, z) plays the role of the desired (R', r').

Case 2: $\mathrm{cf}(R(r^*)) = 2$.

Put $\mathbf{E} = (e, \mathbf{z}, \mathrm{id}_{P^*}) = \mathrm{ucf}^-(R(r^*))$. By Lemma 4.31, letting $f(\xi) = [\vec{\beta} \mapsto \|\vec{\beta}^\frown(j^{P^*}(g)({}^e\beta_{\mathbf{z}})) \oplus_R r^*\|_{<^R}]_{\mu^{Q^-}}$ for $\xi = [g]_{\mu_{\mathbb{L}}} < u_2$, then $\sup f''u_2 = j^{Q^-,Q^*}([\![r^*]\!]_R)$. Fix $\beta < u_2$ and we try to find R', r' as in Case 1. Let U be a Π^1_2-wellfounded level ≤ 2 tree such that $\beta < [\![2,(0)]\!]_U$, obtained by Lemma 4.74. Let (Z, ρ, θ) be as in Case 1. Let $\mathbf{B} = (\mathbf{r}, \pi) \in \mathrm{desc}(R, U, *)$, where $\mathbf{r} = (r^{*\frown}(-1), Q^*, \overrightarrow{(d, q, P)})$, π extends id_{*,Q^-}, $\pi(d^*, q^*) = (2, ((0), \{(0)\}, ((0))), \tau)$, $\tau(0) = \mathbf{E}$. Let $z = \rho^{-1}(\mathbf{B})$. (Z, z) plays the role of the desired (R', r'). □

Lemma 4.79. *Suppose R is a finite level-3 tree and $\vec{\gamma} = (\gamma_r)_{r \in \mathrm{dom}(R)}$ respects R. Then there is a Π^1_3-wellfounded level-3 tree R' such that $R \subseteq R'$ and for any $r \in \mathrm{dom}(R)$, $\gamma_r = [\![r]\!]_{R'}$.*

Proof. It suffices to produce a level-3 tree R' and a map ρ factoring (R, R') such that for any $r \in \mathrm{dom}(R)$, $\gamma_r = [\![r]\!]_{R'}$. By Lemma 4.73, it suffices to show that for any $r^{*\frown}(a) \in \mathrm{dom}(R)$, letting $Q^* = R_{\mathrm{tree}}(r^{*\frown}(a))$, there is a Π^1_3-wellfounded level-3 tree R' and $r' \in \mathrm{dom}(R')$ such that $\gamma_r = [\![r']\!]_{R'}$, $R'[(r')^-] = R[r^*]$, $R_{\mathrm{tree}}(r') = Q^*$.

Lemma 4.75 gives us a Π^1_3-wellfounded level-3 tree Y and $y \in \mathrm{dom}(Y)$ such that $[\![y]\!]_Y \geq \gamma_{r^{*\frown}(a)}$, $Y[y^-] = R[r^*], Y_{\mathrm{tree}}(y) = Q^*$. Minimizing $[\![y]\!]_Y$, we may further assume that for any Π^1_2-wellfounded Y' and any y' such that $[\![y']\!]_{Y'} \geq \gamma_{r^{*\frown}(a)}$, $Y'[(y')^-] = R[r^*], Y'_{\mathrm{tree}}(y') = Q^*$, we have $[\![y']\!]_{T'} \geq [\![r^{*\frown}(a)]\!]_Y$. We claim that $[\![t]\!]_Y = \gamma_{r^{*\frown}(a)}$. Suppose otherwise. Put $Y(y) = (Q^*, (d^*, q^*, P^*))$.

Case 1: $\mathrm{cf}(Y(y)) = 0$.

Argue as in Case 1 in the proof of Lemma 4.76 to obtain a contradiction.

Case 2: $\mathrm{cf}(Y(y)) = 1$.

For $\beta < \omega_1$, put $f(\beta) = [\vec{\alpha} \mapsto \|\vec{\alpha}^\frown(\beta) \oplus_{^2Q^*} y^\frown(-1)\|_{<^Y}]_{\mu^{Q^*}}$. So $[\![t]\!]_Y = \sup\{f(\beta) : \beta < \omega_1\}$. For each limit ordinal $\beta < \omega_1$, we shall find a Π^1_3-wellfounded Y' and a node $y' \in \mathrm{dom}(Y')$ such that $[\![y']\!]_{Y'} = f(\beta)$, contradicting to the minimization assumption. Fix a limit ordinal $\beta < \omega_1$. Let U be a Π^1_2-wellfounded level ≤ 2 tree such that $[\![1,(0)]\!]_U = \beta$. Let (Z, ρ) be a representation of $Y \otimes U$ and let $\theta : \mathrm{rep}(Z) \to \mathrm{rep}(Y)$ be the order preserving bijection. Let $\mathbf{B} = (\mathbf{y}, \pi) \in \mathrm{desc}(Y, U, *)$, $\mathbf{y} = (y^\frown(-1), X, \overrightarrow{(e, x, W)})$, π extends $\mathrm{id}_{*,X}$, $\pi(e_{\mathrm{lh}(\vec{x})}, x_{\mathrm{lh}(\vec{x})}) = (1, (0), \emptyset)$. Let $z = \rho^{-1}(\mathbf{B})$. Similarly to Case 2 of the proof of Lemma 4.76, (Z, z) constitutes a counterexample.

Case 3: $\mathrm{cf}(Y(y)) = 2$.

Let $\mathbf{E} = (e, \mathbf{z}, \mathrm{id}_{P^*}) = \mathrm{ucf}^-(Y(y))$. For $\beta = [g]_{\mu^{\mathbb{L}}} < u_2$, put $f(\beta) = [\vec{\alpha} \mapsto \|\vec{\alpha}^\frown j^{P^*}(g)({}^e\alpha_{\mathbf{z}}) \oplus_{^2Q^*} y^\frown(-1)\|_{<^Y}]_{\mu^{Q^*}}$. So $[\![t]\!]_Y = \sup\{f(\beta) : \beta < u_2\}$. For each limit ordinal $\omega_1 < \beta < u_2$, we shall find a Π^1_3-wellfounded Y' and a node $y' \in \mathrm{dom}(Y')$ such that $[\![y']\!]_{Y'} = f(\beta)$. Let U be a Π^1_2-wellfounded level ≤ 2 tree such that $[\![2,((0))]\!]_U = \beta$. Let Z, ρ, θ be as in Case 2. Let $\mathbf{B} = (\mathbf{y}, \pi)$, where $\mathbf{y} = (y^\frown(-1), X, \overrightarrow{(e, x, W)})$, π extends $\mathrm{id}_{*,X}$, $\pi(e_{\mathrm{lh}(\vec{x})}, x_{\mathrm{lh}(\vec{x})}) = (2, ((0), \{(0)\}, ((0))), \tau)$, $\tau((0)) = \mathbf{E}$. Let $z = \rho^{-1}(\mathbf{B})$. Similarly to Case 3 of the proof of Lemma 4.76, (Z, z) constitutes a counterexample. □

5 The lightface level-3 sharp

Sections 5.1-5.3 defines a Π^1_4 singleton $0^{3\#}$ which is many-one equivalent to $M_2^{\#}$, under boldface $\mathbf{\Pi}^1_3$-determinacy. The assumption of $\mathbf{\Pi}^1_3$-determinacy is very likely not optimal. Section 5.4 formulates the existence of $0^{3\#}$ as a purely syntactical large cardinal axiom based on the weaker assumption of $\mathbf{\Delta}^1_2$-determinacy.

Recall that $\mathbb{L}[T_3] = \bigcup_{x \in \mathbb{R}} L[T_3, x]$, $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_3] = \bigcup_{x \in \mathbb{R}} L_{\boldsymbol{\delta}^1_3}[T_3, x]$. By Steel, $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_3] = V_{\boldsymbol{\delta}^1_3} \cap \mathbb{L}[T_3]$.

Lemma 5.1. *Assume $\mathbf{\Pi}^1_3$-determinacy.*

1. $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2] = \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_3]$.

2. *Every subset of $\boldsymbol{\delta}^1_3$ in $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_3]$ is definable over $M^-_{2,\infty}(x)$ from $\{x\}$ for some $x \in \mathbb{R}$.*

Proof. 1. T_2 is a Δ^1_3 subset of u_ω, and of course a Σ^1_4 subset of $\boldsymbol{\delta}^1_3$, and hence $T_2 \in L[T_3]$ by Becker-Kechris [3]. This gives the $\subseteq$ inclusion.

If $A \in L_{\boldsymbol{\delta}^1_3}[T_3, x]$, by Theorem 2.18, there must be $\xi < \boldsymbol{\delta}^1_3$ such that $A \in L_\xi[T_3 \restriction \xi, x]$. Pick $y \geq_T x$ such that $\kappa^y_3 > \xi$. Then $A \in L_{\kappa^y_3}[T_2, y]$ by Lemma 4.37. This gives the $\supseteq$ inclusion.

2. By Theorem 2.18, every subset of $\boldsymbol{\delta}^1_3$ in $\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_3]$ is in $M^{\#}_{2,\infty}(x)$ for some $x \in \mathbb{R}$. If $A \subseteq \boldsymbol{\delta}^1_3$ is definable over $M^{\#}_{2,\infty}(x)$ from $\{\gamma, x\}$, $\gamma < \boldsymbol{\delta}^1_3$, letting $y \geq_T M^{\#}_2(x)$ such that γ is definable over $L_{\kappa^y_3}[T_2, y]$, then A is definable over $M^-_{2,\infty}(y)$ from $\{y\}$. □

Caution that Lemma 5.1 does not give a real x for which $T_3 \in L[T_2, x]$. $L_{\kappa^x_3}[T_2, x]$ computes a proper initial segment of T_3, and by varying x, these proper initial segments are cofinal in T_3. However, there is not a single x with $T_3 \in L[T_2, x]$.

5.1 Level-3 boundedness

Recall in Corollary 2.10 that the rank of a $\Sigma^1_3(<u_\omega, x)$ wellfounded relation is bounded by κ^x_3. We would like to strengthen this fact by allowing a suitable code for an arbitrary ordinal in $\boldsymbol{\delta}^1_3$. The strengthening is based on an inner model theoretic characterization of u_ω in $L[T_3, x]$. We say that

$$\delta \text{ is an } L\text{-Woodin cardinal}$$

iff $L(V_\delta) \models \delta$ is Woodin.

Theorem 5.2 (Woodin, [41, Theorem 3.21])**.** *Assume $\boldsymbol{\Pi}^1_3$-determinacy. Let $\kappa = u_\omega$. For $x \in \mathbb{R}$, $M^-_{2,\infty}(x) \models \kappa$ is the least L-Woodin cardinal.*

Corollary 5.3 (Level-3 boundedness)**.** *Assume $\boldsymbol{\Pi}^1_3$-determinacy. Suppose $x \in \mathbb{R}$, $\mathcal{N} \in \mathcal{F}_{2,x}$, η is a cardinal and strong cutpoint of $\mathcal{N}$, $\xi = \pi_{\mathcal{N},\infty}(\eta)$. Suppose g is Coll(ω, η)-generic over $\mathcal{N}$, $r \in \mathbb{R} \cap \mathcal{N}[g]$. Let λ be the least L-Woodin cardinal in $M^-_{2,\infty}(x)$ above ξ. Suppose G is a $\Pi^1_3(r, <u_\omega)$ set equipped with a regular $\Pi^1_3(r, <u_\omega)$ norm φ. Suppose A is a $\Sigma^1_3(r, <u_\omega)$ subset of G. Then*

$$\sup\{\varphi(y) : y \in A\} < (\lambda^+)^{M_{2,\infty}(x)}.$$

Proof. Put $x = 0$ for simplicity. Put

$$\mathcal{G}^{\mathcal{N},\eta}_2 = \{\mathcal{P} \in \mathcal{F}_2 : \mathcal{P} \text{ is a nondropping iterate of } \mathcal{N} \text{ above } \eta\}.$$

$\mathcal{G}^{\mathcal{N},\eta}_2$ is a subsystem of $\mathcal{F}_2$. Let $M^{\mathcal{N},\eta,\#}_{2,\infty}$ be the direct limit of $\mathcal{G}^{\mathcal{N},\eta}_2$. The inclusion map of direct systems induces an embedding between direct limits

$$\pi^{\mathcal{N},\eta}_x : M^{\mathcal{N},\eta,\#}_{2,\infty} \to M^{\#}_{2,\infty}.$$

Let $r_g \in \mathbb{R}$ be the real coding $(g, \mathcal{N}|\eta)$. Every mouse $\mathcal{P} \in \mathcal{G}_2^{\mathcal{N},\eta}$ corresponds to an r_g-mouse $\mathcal{P}[g] \in \mathcal{F}_{2,r_g}$ (converted into an r_g-mouse in the obvious way, cf. [42]). So in the direct limit,

$$M_{2,\infty}^{\mathcal{N},\eta,\#}[g] = M_{2,\infty}^{\#}(r_g).$$

By Corollary 2.10,

$$\sup\{\varphi(y) : y \in A\} < \kappa_3^r,$$

which in turn is smaller than the successor of u_ω in $M_{2,\infty}^{\#}(r_g)$, as $\{T_2, r\} \in M_{2,\infty}^{\#}(r_g)$. By Theorem 5.2, u_ω is the least L-Woodin cardinal of $M_{2,\infty}^{\#}(r_g)$, hence the least L-Woodin cardinal of $M_{2,\infty}^{\mathcal{N},\eta,\#}$ above η. By elementarity, $\pi_x^{\mathcal{N},\eta}(u_\omega) = \lambda$. So $\pi_x^{\mathcal{N},\eta}(\kappa_3^r) < (\lambda^+)^{M_{2,\infty}}$. This finishes the proof. □

We also need to code ordinals in $\boldsymbol{\delta}_3^1$ by direct limits of iterations of Π_3^1-iterable mice. Suppose $x \in \mathbb{R}$ and z codes a Π_3^1-iterable x-mouse $\mathcal{P}_z$. Then

$$\pi_{\mathcal{P}_z,\infty} : \mathcal{P}_z \to (\mathcal{P}_z)_\infty$$

is the direct limit map of all the nondropping iterates of $\mathcal{P}_z$. $o((\mathcal{P}_z)_\infty)$ is the length of a $\Delta_3^1(z)$-prewellordering, namely the one induced by iterations of $\mathcal{P}_z$. By Corollary 2.15, $\pi_{\mathcal{P}_z,\infty}$ and $(\mathcal{P}_z)_\infty$ are both in $L_{\kappa_3^{M_1^\#(z)}}[T_2, M_1^\#(z)]$ and Δ_1-definable over $L_{\kappa_3^{M_1^\#(z)}}[T_2, M_1^\#(z)]$ from $\{T_2, M_1^\#(z)\}$.

5.2 Putative level-3 indiscernibles

The higher level analog of the type of L with n indiscernibles is the type of $M_{2,\infty}^-$ realized by an appropriate $[F]^R$, where $F \in (\boldsymbol{\delta}_3^1)^{R\uparrow}$. Such functions F are coded by subsets of u_ω in $\mathbb{L}_{\boldsymbol{\delta}_3^1}[T_2]$. The coding system is provided by Corollary 2.12.

$\mathcal{L} = \{\underline{\in}\}$ is the language of set theory. For a level-3 tree R, $\mathcal{L}^R$ is the expansion of $\mathcal{L}$ which consists of additional constant symbols $\underline{c_r}$ for each $r \in \mathrm{dom}(R)$. For a level-3 tree R and a tuple of ordinals $\vec{\gamma} = (\gamma_r)_{r \in \mathrm{dom}(R)}$, the $\mathcal{L}$-structure $M_{2,\infty}^-$ expands to the $\mathcal{L}^R$-structure

$$(M_{2,\infty}^-; \vec{\gamma})$$

whose constant $\underline{c_r}$ is interpreted as γ_r.

Definition 5.4. $C \subseteq \boldsymbol{\delta}_3^1$ is said to be *firm* iff every member of C is additively closed, the set $\{\xi : \xi = \text{o.t.}(C \cap \xi)\}$ has order type $\boldsymbol{\delta}_3^1$ and $C \cap \xi \in \mathbb{L}_{\boldsymbol{\delta}_3^1}[T_2]$ for all $\xi < \boldsymbol{\delta}_3^1$.

Definition 5.5. $C \subseteq \boldsymbol{\delta}^1_3$ is called a set of *potential level-3 indiscernibles for* $M^-_{2,\infty}$ iff for any level-3 tree R, for any $F, G \in C^{R\uparrow} \cap \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$,

$$(M^-_{2,\infty}; [F]^R) \equiv (M^-_{2,\infty}; [G]^R).$$

A firm set of potential level-3 indiscernibles for $M^-_{2,\infty}$ is the higher level analog of a set of order indiscernibles for L. Note that the successor elements of C don't really play a part in computing $[F]^R = ([F_r]_{\mu^{R_{\text{tree}(r)}}})_{r\in\text{dom}(R)}$, as the relevant ultrapowers $\mu^{R_{\text{tree}(r)}}$ concentrate on tuples of limit ordinals, hence the prefix "potential".

Lemma 5.6. *Assume $\boldsymbol{\Pi}^1_3$-determinacy. Then there is a firm set of potential level-3 indiscernibles for $M^-_{2,\infty}$.*

Proof. Suppose R is a finite level-3 tree. Let φ be an $\mathcal{L}^R$-sentence. Consider the game $G^{R;\varphi}$ where I produces reals v, x, c and a natural number p, II produces reals v', x', c' and a natural number p'. The payoff is decided according to the following priority list:

1. I and II must take turns to ensure that $v \in \text{WO}^{R\uparrow}$ and $v' \in \text{WO}^{R\uparrow}$. If one of them fails to do so, and $w \in \text{rep}(R)$ is $<^R$-least for which $v \notin \text{WO}^{R\uparrow}_w \vee v' \notin \text{WO}^{R\uparrow}_w$, then I loses iff $v \notin \text{WO}^{R\uparrow}_w$, and II loses iff $v \in \text{WO}^{R\uparrow}_w$.

2. If 1 is satisfied, put $\vec{\gamma} = (\gamma_r)_{r\in\text{dom}(R)}$, where $\gamma_r = \max([v]^R_r, [v']^R_r)$. I must ensure

   (a) x codes a 2-small premouse $\mathcal{P}_x$ which satisfies "I am closed under the $M^\#_1$-operator";

   (b) c codes a strictly increasing, cofinal-in-$o(\mathcal{P}_x)$ sequence of ordinals $(c_n)_{n<\omega}$ relative to x such that each c_n is a cardinal cutpoint of $\mathcal{P}_x$;

   (c) $\mathcal{P}_x|c_1$ is a Π^1_3-iterable mouse;

   (d) p codes a tuple of ordinals $\vec{\alpha} = (\alpha_r)_{r\in\text{dom}(R)}$ in $\mathcal{P}_x|c_0$ relative to x;

   (e) For each $r \in \text{dom}(R)$, $\pi_{\mathcal{P}_x|c_0,\infty}(\alpha_r) = \gamma_r$;

   (f) $(\mathcal{P}_x; \vec{\alpha}) \models \varphi$.

   Otherwise he loses.

3. If 1-2 are satisfied, II must ensure 2(a)-(f) with $(x, c, (c_n)_{n<\omega}, p, \vec{\alpha}, \varphi)$ replaced by $(x', c', (c'_n)_{n<\omega}, p', \vec{\alpha'}, \neg\varphi)$, otherwise he loses.

4. If 1-3 are satisfied, I and II must take turns to ensure for all $2 \leq n < \omega$,

   (a) $\mathcal{P}_x|c_n$ is a Π^1_3-iterable mouse and $\mathcal{P}_{x'}|c'_{n-1} <_{DJ} \mathcal{P}_x|c_n$;

   (b) $\mathcal{P}_{x'}|c'_n$ is a Π^1_3-iterable mouse and $\mathcal{P}_x|c_n <_{DJ} \mathcal{P}_{x'}|c'_n$.

   If one of them fails to do so, and n is least for which (a) or (b) fails at n, then I loses iff (a) fails at n, and II loses iff (a) holds at n.

5. It is impossible that both players obey all the rules, due to a successful comparison between $\mathcal{P}_x$ and $\mathcal{P}_{x'}$. The definition of $G^{R;\varphi}$ is finished.

The payoff of $G^{R;\varphi}$ has complexity $([\![\emptyset]\!]_R + \omega)$-$\Pi^1_3$ for both players. The nontrivial part about the complexity is that 2(e) is Δ^1_3, shown as follows. According to rules 2(a)-(c), $\mathcal{P}_x|c_1$ is Π^1_3-iterable and closed under the (genuine) $M_1^{\#}$-operator, $c_0 < c_1$, and therefore $M_1^{\#}(\mathcal{P}_x|c_0)$ is canonically coded in x. $\pi_{\mathcal{P}_x|c_0,\infty}(\alpha_s)$ is the length of a $\Delta^1_3(\mathcal{P}_x|c_0)$ prewellordering, induced by iterations. By Corollary 2.15, $\pi_{\mathcal{P}_x|c_0,\infty}(\alpha_s)$ is Δ_1-definable over $L_{\kappa_3^x}[T_2, x]$ from $\{T_2, x\}$. $\vec{\gamma}$ is clearly Δ_1-definable over $L_{\kappa_3^v}[T_2, v]$ from $\{T_2, v\}$. So 2(e) is expressed into a Δ_1 statement over $L_{\kappa_3^{v,x}}[T_2, v, x]$ from $\{T_2, v, x, c\}$, or equivalently, $\Delta^1_3(v, x, c)$ by Theorem 2.1.

Hence $G^{R;\varphi}$ is determined. Suppose for definiteness II has a winning strategy σ in $G^{R;\varphi}$. Let C be the set of L-Woodin cardinal cutpoints of $M^-_{2,\infty}(\sigma)$ and their limits. We show that

$$\forall F \in C^{R\uparrow}\ (M^-_{2,\infty}; [F]^R) \models \neg\varphi$$

Suppose towards a contradiction that $F \in C^{R\uparrow}$ but $(M^-_{2,\infty}; [F]^R) \models \varphi$. As $\boldsymbol{\delta}^1_3$ is inaccessible in $M^{\#}_{2,\infty}$, there is a club $D \in M^{\#}_{2,\infty}$ in $\boldsymbol{\delta}^1_3$ so that $M^-_{2,\infty}|\lambda \prec M^-_{2,\infty}$ for any $\lambda \in D$. There is thus a continuous, order preserving $G : \omega + 1 \to C \setminus \sup \operatorname{ran}(F)$ for which $(M^-_{2,\infty}|G(\omega); [F]^R) \models \varphi$. Pick $\mathcal{P} \in \mathcal{F}_2$ and ordinals $(c_n)_{n<\omega}$, $(\alpha_r)_{r \in \operatorname{dom}(R)}$ in $\mathcal{P}$ such that $\pi_{\mathcal{P},\infty}(c_n) = G(n)$ for any $n < \omega$ and $\pi_{\mathcal{P},\infty}(\alpha_r) = [F]^R_r$ for any $r \in \operatorname{dom}(R)$. Thus, $(\mathcal{P}| \sup_{n<\omega} c_n; \vec{\alpha}) \models \varphi$. Let Player I play (v, x, c, p), where $v \in \mathrm{WO}^{R\uparrow}$, $\|v\|^R_w = F(w)$ for any $w \in \operatorname{rep}(R)$, x codes $\mathcal{P}| \sup_{n<\omega} c_n$, c codes $(c_n)_{n<\omega}$ relative to x, p codes $(\alpha_r)_{r\in\operatorname{dom}(R)}$. The response according to σ is denoted by $(v', x', c', p') = (v, x, c, p) * \sigma$. We shall derive a contraction by showing neither player breaks the rules, using $\boldsymbol{\Sigma}^1_3$-boundedness.

As σ is a winning strategy, Player II is not the first person to break the rules. So $v \in \mathrm{WO}^{R\uparrow}$ implies $v' \in \mathrm{WO}^{R\uparrow}$. For each $w \in \operatorname{rep}(R)$ which is either the $<^R$-minimum or a $<^R$-successor, if $\mathcal{N} \in \mathcal{F}_{2,\sigma}$, $\eta \in \mathcal{N}$, $\pi_{\mathcal{N},\infty}(\eta) = F(w)$, g is $\operatorname{Coll}(\omega, \eta)$-generic over $\mathcal{N}$, $r_g \in \mathbb{R}$ being the real coding $(g, \mathcal{N}|\eta)$, then (v', x', c', p') belongs to the set

$$A_w = \{(\bar{v}, \bar{x}, \bar{c}, \bar{p}) * \sigma : \bar{v} \in \mathrm{WO}^{R\uparrow}_w \restriction \xi\}$$

which is $\Sigma^1_3(M_1^{\#}(r_g), {<}u_\omega)$ by Corollary 2.15 and Theorem 2.1. Since σ is a winning strategy, A_w is a subset of

$$B_w = \{(\bar{v}', \bar{x}', \bar{c}', \bar{p}') : \bar{v}' \in \mathrm{WO}_w^{R\uparrow}\}$$

B_w is a $\Pi^1_3({<}u_\omega)$ set, equipped with the $\Pi^1_3({<}u_\omega)$ prewellordering $(\bar{v}', \bar{x}', \bar{c}', \bar{p}') \mapsto \|\bar{v}'\|^R_w$. By Corollary 5.3, $\|v'\|^R_w < \min(C \setminus (F(w)+1))$. By continuity, if w has $<^R$-limit order type, then $\|v'\|^R_w \leq \|v\|^R_w$. Consequently, for $r \in \mathrm{dom}(R)$, $[v']^R_r \leq [v]^R_r$, so if $\vec{\gamma}$ is defined from v, v' as in Rule 2, then $\gamma_r = [v]^R_r$.

By our choice of F and G, Rule 2 is satisfied. Let $\mathcal{P}_x, (c_n)_{n<\omega}, \vec{\alpha}, \mathcal{P}_{x'}, (c'_n)_{n<\omega}, \vec{\alpha}'$ be defined as in Rules 2 and 3. For each $1 \leq n < \omega$, using the Π^1_3-prewellordering on codes of Π^1_3-iterable mice, a similar boundedness argument shows that $\|\mathcal{P}_{x'}|c'_n\|_{<_{DJ}} < \min(C\setminus(G(n)+1))$, and hence $\mathcal{P}_{x'}|c'_n <_{DJ} \mathcal{P}_x|c_{n+1}$. So Rule 4 is satisfied. This is impossible. □

Definition 5.7. Assume $\boldsymbol{\Pi}^1_3$-determinacy. Let C be a firm set of potential level-3 indiscernibles for $M^-_{2,\infty}$. Then

$$0^{3\#}$$

is a map sending a level-3 tree R to the complete consistent $\mathcal{L}^R$-theory $0^{3\#}(R)$, where $\ulcorner\varphi\urcorner \in 0^{3\#}(R)$ iff φ is an $\mathcal{L}^R$-formula and for all $\vec{\gamma} \in [C]^{R\uparrow}$,

$$(M^-_{2,\infty}; \vec{\gamma}) \models \varphi.$$

$0^{3\#}$ is the higher level analog of $0^{\#}$. Each individual $0^{3\#}(R)$ is the higher level analog of the n-type that is realized in L by n indiscernibles. As with the level-1 sharps, we shall give a Π^1_4 axiomatization of $0^{3\#}$ in Section 5.4.

The proof of Lemma 5.6 shows

Lemma 5.8. *Assume $\boldsymbol{\Pi}^1_3$-determinacy. For a finite level-3 tree R, $0^{3\#}(R)$ is a $\Game(\llbracket\emptyset\rrbracket_R + \omega)$-$\Pi^1_3$) real.*

In fact, the complexity of $0^{3\#}(R)$ relies only on $\llbracket\emptyset\rrbracket_R$.

Lemma 5.9. *If Q is a finite level ≤ 2 tree, then $j^Q(M^-_{2,\infty}), j^Q \restriction M^-_{2,\infty}$ are definable over $M^-_{2,\infty}$, uniformly in Q. If X is another finite level ≤ 2 trees and π factors (X, Q), then $\pi^Q \restriction j^X(M^-_{2,\infty})$ is definable over $M^-_{2,\infty}$, uniformly in (π, X, Q).*

Proof. By Theorem 2.18, $j^Q(M^-_{2,\infty}) = L[j^Q(T_3)]$, and every Σ^1_4 subset of $\boldsymbol{\delta}^1_3$ is definable over $M^-_{2,\infty}$. It suffices to show that $j^Q(T_3)$, $j^Q \restriction \boldsymbol{\delta}^1_3$, $\pi^Q \restriction \boldsymbol{\delta}^1_3$ are all Σ^1_4 in the codes.

Let G be a good universal Π^1_3 set and let $\varphi : G \to \boldsymbol{\delta}^1_3$ be a regular Π^1_3 norm. Then $x \in G \wedge y \in G \wedge j^Q(\varphi(x)) = \varphi(y)$ iff there exists $z \in \mathbb{R}$ such that $z = M_1^\#(x, y)$ and $L_{\kappa_3^z}[T_2, z] \models j^Q(\varphi(x)) = \varphi(y)$. Here, the statement $j^Q(\varphi(x)) = \varphi(y)$ is Δ_1 over $L_{\kappa_3^z}[T_2, z]$ from $\{T_2, z\}$ by Corollary 2.15 and Lemma 4.37. Similarly, using Lemma 4.35, $\pi^Q \restriction \kappa_3^x$ and $j^Q(T_3) \restriction \kappa_3^x$ are Δ_1-definable over $L_{\kappa_3^x}[T_2, x]$ from $\{T_2, x\}$, uniformly in x. So $\pi^Q \restriction \boldsymbol{\delta}^1_3$ and $j^Q(T_3)$ are Σ^1_4 using similar arguments. □

Based on Theorem 2.18 and Σ^1_4-absoluteness of iterates of $M_2^\#$, a Σ^1_4 set $A \subseteq \boldsymbol{\delta}^1_3$ has the following alternative definition: $\alpha \in A$ iff $M^-_{2,\infty}$ satisfies that

> for any $\xi > \alpha$ cardinal cutpoint, in the $\text{Coll}(\omega, \xi)$-generic extension, $\pi_{K|\xi,\infty}(\alpha) \in A$.

We introduce the following informal symbols arising from the proof of Lemma 5.9 that will occur in $\mathcal{L}$-formulas or $\mathcal{L}^R$-formulas for a level-3 tree R:

1. If Q is a finite level ≤ 2 tree, $\underline{j^Q}$ is the informal symbol so that $\underline{j^Q}(a) = b$ iff for any ξ cardinal cutpoint such that $\{a, b\} \in K|\xi$, the $\text{Coll}(\omega, \xi)$-generic extension satisfies $j^Q(\pi_{K|\xi,\infty}(a)) = \pi_{K|\xi,\infty}(b)$.

2. If π factors finite level ≤ 2 trees (X, T), $\underline{\pi^T}$ is the informal symbol so that $\underline{\pi^T}(a) = b$ iff iff for any ξ cardinal cutpoint such that $\{a, b\} \in K|\xi$, the $\text{Coll}(\omega, \xi)$-generic extension satisfies $\pi^T(\pi_{K|\xi,\infty}(a)) = \pi_{K|\xi,\infty}(b)$.

3. If Q is a level ≤ 2 subtree of Q', Q' is finite, then $\underline{j^{Q,Q'}} = \underline{(\text{id}_Q)^{Q,Q'}}$, where id_Q factors (Q, Q'), $\text{id}_Q(d, q) = (d, q)$.

4. Corresponding to items 1-3, $\underline{j^Q_{\text{sup}}}, \underline{\pi^T_{\text{sup}}}, \underline{j^{Q,Q'}_{\text{sup}}}$ stand for functions on ordinals that send α to $\sup(\underline{j^Q})''\alpha$, $\sup(\underline{\pi^T})''\alpha$, $\sup(\underline{j^{Q,Q'}})''\alpha$ respectively.

5. $\underline{S_3}$ is the informal symbol such that $(\emptyset, \emptyset) \in \underline{S_3}$ and $((R_i)_{i \leq n}, (\alpha_i)_{i \leq n}) \in \underline{S_3}$ iff $\vec{R}$ is a finite regular level-3 tower and letting $r_i \in \text{dom}(R_{i+1}) \setminus \text{dom}(R_i)$, then $r_k = (r_l)^- \to \alpha_k < \underline{j^{(R_n)_{\text{tree}}(r_k),(R_n)_{\text{tree}}(r_l)}}(\alpha_l)$.

6. For $1 \leq n \leq \omega$, $\underline{u_n}$ is the symbol so that for any $\xi > \underline{u_n}$ cardinal cutpoint, the $\text{Coll}(\omega, \xi)$-generic extension satisfies $\pi_{K|\xi,\infty}(\underline{u_n}) = u_n$.

7. Suppose T is a finite level ≤ 2 tree. If $\mathbf{D} \in \text{desc}(T, U)$, $\|\mathbf{D}\|_{\prec^{T,U}} = n$, then $\underline{\text{seed}^{T,U}_{\mathbf{D}}} = \underline{u_{n+1}}$. If $(1, t) \in \text{dom}(T)$, then $\underline{\text{seed}^T_{(1,t)}} = \underline{\text{seed}^{T,\emptyset}_{(1,t,\emptyset)}}$. If $(2, t) \in \text{dom}(T)$, and ${}^2T_{\text{tree}}[t] = (S, \vec{s})$, then $\underline{\text{seed}^T_{(2,t)}} = \underline{\text{seed}^{T,S}_{(2,(t,S,\vec{s}),\text{id}_S)}}$. $\underline{\text{seed}^T} = (\underline{\text{seed}^T_{(d,t)}})_{(d,t) \in \text{dom}(T)}$.

8. If k is a definable class function and W is a definable class, then $k(W) = \bigcup\{k(W \cap V_\alpha) : \alpha \in \mathrm{Ord}\}$.

9. If X, T, T' are finite level ≤ 2 trees, T is a subtree of T', $a \in \underline{j^X}(V)$, $d \in \{1,2\}$, then

   (a) $\underline{B^T_{X,a}} = \{\underline{\pi^{T\otimes Q}}(a) : Q$ finite level ≤ 2 tree, π factors $(X, T \otimes Q)\}$;

   (b) $\underline{H^T_{X,a}}$ is the transitive collapse of the Skolem hull of $\underline{B^T_{X,a}} \cup \mathrm{ran}(\underline{j^T})$ in $\underline{j^T}(V)$ and $\underline{\phi^T_{X,a}} : \underline{H^T_{X,a}} \to \underline{j^T}(V)$ is the associated elementary embedding;

   (c) $\underline{j^T_{X,a}} = (\underline{\phi^T_{X,a}})^{-1} \circ \underline{j^T}$;

   (d) $\underline{j^{T,T'}_{X,a}} = (\underline{\phi^{T'}_{X,a}})^{-1} \circ \underline{j^{T,T'}} \circ \underline{\phi^T_{X,a}}$;

   (e) $\underline{B^T_{1,a}} = \underline{B^T_{Q^0,a}} \cup \underline{B^T_{Q^1,a}}$, $\underline{B^T_{2,a}} = \underline{B^T_{Q^0,a}} \cup \underline{B^T_{Q^{20},a}} \cup \underline{B^T_{Q^{21},a}}$;

   (f) $\underline{H^T_{d,a}}$ is the transitive collapse of the Skolem hull of $\underline{B^T_{d,a}} \cup \mathrm{ran}(\underline{j^T})$ in $\underline{j^T}(V)$ and $\underline{\phi^T_{d,a}} : \underline{H^T_{d,a}} \to \underline{j^T}(V)$ is the associated elementary embedding;

   (g) $\underline{j^T_{d,a}} = (\underline{\phi^T_{d,a}})^{-1} \circ \underline{j^T}$;

   (h) $\underline{j^{T,T'}_{d,a}} = (\underline{\phi^{T'}_{d,a}})^{-1} \circ \underline{j^{T,T'}} \circ \underline{\phi^T_{d,a}}$.

10. Suppose R is a level-3 tree.

   (a) If $\mathbf{r} = (r, Q, \overrightarrow{(d, q, P)}) \in \mathrm{desc}^*(R)$, $\underline{c_\mathbf{r}}$ is the informal $\mathcal{L}^R$-symbol whose interpretation is

$$\underline{c_\mathbf{r}} = \begin{cases} \underline{j_{\mathrm{sup}}^{R_{\mathrm{tree}}(r^-),Q}}(\underline{c_{r^-}}) & \text{if } \mathbf{r} \in \mathrm{desc}(R) \text{ of continuous type,} \\ \underline{c_r} & \text{if } \mathbf{r} \in \mathrm{desc}(R) \text{ of discontinuous type,} \\ \underline{j^{R_{\mathrm{tree}}(r),Q}}(\underline{c_r}) & \text{if } \mathbf{r} \notin \mathrm{desc}(R). \end{cases}$$

   (b) If T, U are finite level ≤ 2 trees and $\mathbf{B} = (\mathbf{r}, \pi) \in \mathrm{desc}(R, T, U)$, $\mathbf{r} \neq \emptyset$, then $\underline{c^T_\mathbf{B}}$ is the informal $\mathcal{L}^R$-symbol which stands for $\underline{\pi^{T,U}}(\underline{c_\mathbf{r}})$.

   (c) If $\mathbf{A} = (\mathbf{r}, \pi, T) \in \mathrm{desc}^{**}(R)$, $\mathbf{r} \neq \emptyset$, then $\underline{c_\mathbf{A}}$ is the informal $\mathcal{L}^R$-symbol which stands for $\underline{\pi^T}(\underline{c_\mathbf{r}})$.

Put $\mathcal{L}^{\underline{x}} = \{\underline{\in}, \underline{x}\}$; for a level-3 tree R, put $\mathcal{L}^{R,\underline{x}} = \mathcal{L}^R \cup \{\underline{x}\}$, where $\underline{x}$ is a constant symbol. All of the above informal symbols work in $\mathcal{L}^{\underline{x}}$ or $\mathcal{L}^{R,\underline{x}}$, which are intended to be interpreted in $M^-_{2,\infty}(x)$ for $x \in \mathbb{R}$. In particular, for $x \in \mathbb{R}$,

$$(L[\underline{S_3}])^{M^-_{2,\infty}(x)} = M^-_{2,\infty}.$$

Lemma 5.10. *Assume $\mathbf{\Pi}^1_3$-determinacy. Suppose R, Y are finite level-3 trees, T is a finite level ≤ 2 tree, ρ factors (R, Y, T). Then*

$$\ulcorner \varphi(\underline{c_{r_1}}, \ldots, \underline{c_{r_n}}) \urcorner \in 0^{3\#}(R)$$

iff

$$\ulcorner \underline{j^T}(V) \models \varphi(\underline{c^T_{\rho(r_1)}}, \ldots, \underline{c^T_{\rho(r_n)}}) \urcorner \in 0^{3\#}(Y).$$

Proof. Put $\rho(r) = (\mathbf{y}_r, \pi_r)$. Put $R_{\text{tree}}(r) = Q_r$. Suppose $\varphi(v_1, \ldots, v_n)$ is an $\mathcal{L}$-formula, $r_1, \ldots, r_n \in \text{dom}(R)$, and

$$\ulcorner \varphi(\underline{c_{r_1}}, \ldots, \underline{c_{r_n}}) \urcorner \in 0^{3\#}(R).$$

Let C be a firm set of potential level-3 indiscernibles. Suppose $F \in C^{Y\uparrow}$. By Lemma 4.46, F^T_ρ is a function from $\omega_1^{T\uparrow}$ to $[C]^{R\uparrow}$. Recall that $F^T_\rho(\vec{\delta}) = (F^T_{\rho(r)}(\vec{\delta}))_{r \in \text{dom}(R)}$. Hence for any $\vec{\delta} \in \omega_1^{T\uparrow}$,

$$M^-_{2,\infty} \models \varphi(F^T_{\rho(r_1)}(\vec{\delta}), \ldots, F^T_{\rho(r_n)}(\vec{\delta})).$$

For each $r \in \text{dom}(R)$, by definition of ${\pi_r}^{T,Q_r}$, ${\pi_r}^{T,Q_r}([F]^Y_{\mathbf{y}_r}) = [F^T_{\rho(r)}]_{\mu^T}$. Hence by Łoś,

$$j^T(M^-_{2,\infty}) \models \varphi({\pi_{r_1}}^{T,Q_{r_1}}([F]^Y_{\mathbf{y}_{r_1}}), \ldots, {\pi_{r_n}}^{T,Q_{r_n}}([F]^Y_{\mathbf{y}_{r_n}})).$$

Finally, by Lemma 4.60, for $\mathbf{y} = (y, X) \in \text{desc}(Y)$, if y is of discontinuous type then $[F]^Y_{\mathbf{y}} = [F]^Y_y$; if y is of continuous type then $[F]^Y_{\mathbf{y}} = j_{\sup}^{Y_{\text{tree}}(y^-),X}([F]^Y_{y^-})$. Hence,

$$\ulcorner \underline{j^T}(V) \models \varphi(\underline{c^T_{\rho(r_1)}}, \ldots, \underline{c^T_{\rho(r_n)}}) \urcorner \in 0^{3\#}(Y).$$

□

As a corollary to Lemma 5.10, Lemma 5.9 and Theorem 4.71, we obtain:

Corollary 5.11. *Assume $\mathbf{\Pi}^1_3$-determinacy. Suppose R and Y are finite level-3 trees and $[\![\emptyset]\!]_R = [\![\emptyset]\!]_Y$. Then $0^{3\#}(R) \equiv_m 0^{3\#}(Y)$.*

5.3 The equivalence of $x^{3\#}$ and $M_2^\#(x)$

For the other direction of the reduction, we want to compute $\Game(<u_\omega\text{-}\Pi^1_3)$ truth using $0^{3\#}$ as an oracle.

Lemma 5.12. *Assume $\mathbf{\Pi}^1_3$-determinacy. For a finite level-3 tree R, the universal $\Game([\![\emptyset]\!]_R\text{-}\Pi^1_3)$ real is many-one reducible to $0^{3\#}(R)$, uniformly in R.*

Proof. Let $B \subseteq [\![\emptyset]\!]_R \times \mathbb{R}$ be Π^1_3. Let θ be a Σ_1 formula such that

$$(\xi, x) \in B \leftrightarrow L_{\kappa_3^x}[T_2, x] \models \theta(\xi, x).$$

G is the game with output Diff B. We need to decide the winner of G from $0^{3\#}(R)$. B is equipped with the Π^1_3-norm

$$\psi(\xi, x) = \text{the least } \alpha < \kappa_3^x \text{ such that } L_\alpha[T_2, x] \models \theta(\xi, x).$$

If $E \in \mu_{\mathbb{L}}$ is a club, let $\rho^E : [\![\emptyset]\!]_R \to \text{rep}(R) \restriction E$ be the order preserving bijection. For $\vec{\gamma}$ respecting R, let $\theta^I(\vec{\gamma})$ be the following formula:

> There exist $H \in (\boldsymbol{\delta}^1_3)^{R\uparrow}$ and a strategy τ for Player I such that $[H]^R = \vec{\gamma}$ and for any club $E \in \mu_{\mathbb{L}}$, if x is an infinite run according to τ, then for any even $\alpha < [\![\emptyset]\!]_R$, $\forall \beta < \alpha((\beta, x) \in B \wedge \psi(\beta, x) < H(\rho^E(\beta+1)))$ implies $(\alpha, x) \in B \wedge \psi(\alpha, x) < H(\rho^E(\alpha+1))$, and there is $\alpha < [\![\emptyset]\!]_R$ such that $(\alpha, x) \notin B$.

Let $\theta^{II}(\vec{\gamma})$ be the following formula:

> There exist $K \in (\boldsymbol{\delta}^1_3)^{R\uparrow}$ and a strategy σ for Player II such that $[K]^R = \vec{\gamma}$ and for any club $E \in \mu_{\mathbb{L}}$, if x is an infinite run according to σ, then for any odd $\alpha < [\![\emptyset]\!]_R$, $\forall \beta < \alpha((\beta, x) \in B \wedge \psi(\beta, x) < K(\rho^E(\beta+1)))$ implies $(\alpha, x) \in B \wedge \psi(\alpha, x) < K(\rho^E(\alpha+1))$.

Let C be a firm set of level-3 indiscernibles for $M^-_{2,\infty}$. Suppose firstly Player I has a winning strategy τ in G. Let D be the subset of C consisting of L-Woodin cardinals in $M_{2,\infty}(\sigma)$ and their limits. By Corollary 5.3, if x is a consistent run according to σ, then $(0, x) \in B \wedge \psi(0, x) < \min(D)$, for any odd $\alpha < [\![\emptyset]\!]_R$, $(\alpha, x) \in B$ implies $(\alpha+1, x) \in B \wedge \psi(\alpha+1, x) < \min(D \setminus (\psi(\alpha, x)+1))$, and there is $\alpha < [\![\emptyset]\!]_R$ such that $(\alpha, x) \notin B$. Let $H \in D^{R\uparrow}$. Then (H, τ) witnesses $\theta^I([H]^R)$. Let $\mathcal{P} \in \mathcal{F}_2$ and $\vec{\eta} \in \mathcal{P}$ such that $\pi_{\mathcal{P},\infty}(\vec{\eta}) = [H]^R$. Let $\xi_{\vec{\eta}}$ be the least successor cardinal cutpoint of $\mathcal{P}$ above $\max(\vec{\eta})$ and let g be $\text{Coll}(\omega, \xi)$-generic over $\mathcal{P}$. Let $r_{g,\vec{\eta}}$ be the real coding $(g, \vec{\eta})$. Then $\theta^I([H]^R)$ is equivalent to a $\Sigma^1_4(r_{g,\vec{\eta}})$ statement $\bar{\theta}^I(r_{g,\vec{\eta}})$, hence true in $\mathcal{P}[g]$. Hence,

$$\mathcal{P}^{\text{Coll}(\omega,\xi_{\vec{\eta}})} \models \bar{\theta}^I(\dot{r}_{g,\vec{\eta}})$$

By elementarity,

$$(M^-_{2,\infty})^{\text{Coll}(\omega,\xi_{\vec{\gamma}})} \models \bar{\theta}^I(\dot{r}_{g,[H]^R}).$$

By Lemma 5.6, for any $\vec{\gamma} \in [C]^{R\uparrow}$,

$$(M^-_{2,\infty})^{\text{Coll}(\omega,\xi_{\vec{\gamma}})} \models \bar{\theta}^I(\dot{r}_{g,\vec{\gamma}}).$$

By a symmetrical argument, if Player II has a winning strategy in G, then for any $\vec{\gamma} \in [C]^{R\uparrow}$,

$$(M_{2,\infty}^{-})^{\mathrm{Coll}(\omega,\xi_{\vec{\gamma}})} \models \bar{\theta}^{II}(\dot{r}_{g,\vec{\gamma}}).$$

Finally, there does not exist $\vec{\gamma}$ such that

$$(M_{2,\infty}^{-})^{\mathrm{Coll}(\omega,\xi_{\vec{\gamma}})} \models \bar{\theta}^{I}(\dot{r}_{g,\vec{\gamma}}) \wedge \bar{\theta}^{II}(\dot{r}_{g,\vec{\gamma}}).$$

Otherwise, by absoluteness, $\theta^{I}(\vec{\gamma}) \wedge \theta^{II}(\vec{\gamma})$ holds. Let (H,τ) witness $\theta^{I}(\vec{\gamma})$ and let (K,σ) witness $\theta^{II}(\vec{\gamma})$. Let $E \in \mu_{\mathbb{L}}$ be a club such that $H \restriction (\mathrm{rep}(R) \restriction E) = K \restriction (\mathrm{rep}(R) \restriction E)$. Let x be the infinite run according to both τ and σ. Then inductively we can see that for any $\alpha < [\![\emptyset]\!]_R$, $(\alpha, x) \in B \wedge \psi(\alpha, x) < H(\rho^E(\alpha+1))$, but there is $\alpha < [\![\emptyset]\!]_R$ such that $(\alpha, x) \notin B$, which is impossible.

In conclusion, Player I has a winning strategy in B iff for any $\vec{\gamma} \in [C]^{R\uparrow}$, $(M_{2,\infty}^{-})^{\mathrm{Coll}(\omega,\xi_{\vec{\gamma}})} \models \bar{\theta}^{I}(\dot{r}_{g,\vec{\gamma}})$. □

For a real x, $x^{3\#}$ is the obvious relativization of $0^{3\#}$. Combining Lemmas 5.8 and 5.12, Theorem 4.5 and Neeman [37,38], we obtain the equivalence of $x^{3\#}$ and $M_2^{\#}(x)$.

Theorem 5.13. *Assume $\boldsymbol{\Pi}_3^1$-determinacy. For $x \in \mathbb{R}$, $x^{3\#}$ is many-one equivalent to $M_2^{\#}(x)$, the many-one reduction being independent of x.*

By Theorem 5.13 and Moschovakis third periodicity, the winner of the game in the proof of Lemma 5.6 has a winning strategy recursive in $0^{3\#}$. Hence, the set of L-Woodin cardinals in $M_{2,\infty}^{-}(0^{3\#})$ and their limits form a firm set of potential level-3 indiscernibles for $M_{2,\infty}^{-}$.

5.4 Syntactical properties of $0^{3\#}$

Suppose $\mathcal{M}, \mathcal{N}$ are countable Π_3^1-iterable mice. A map $\pi : \mathcal{M} \to \mathcal{N}$ is *essentially an iteration map* iff there are $\mathcal{P}$ and iteration maps $\psi_{\mathcal{M}} : \mathcal{M} \to \mathcal{P}$, $\psi_{\mathcal{N}} : \mathcal{N} \to \mathcal{P}$ such that $\psi_{\mathcal{M}} = \psi_{\mathcal{N}} \circ \pi$. For $\alpha \in \mathcal{M}$, $\beta \in \mathcal{N}$, say that $(\mathcal{M}, \alpha) <_{DJ} (\mathcal{N}, \beta)$ iff either $\mathcal{M} <_{DJ} \mathcal{N}$ or there exist $\mathcal{P}$ and iteration maps $\psi_{\mathcal{M}} : \mathcal{M} \to \mathcal{P}$, $\psi_{\mathcal{N}} : \mathcal{N} \to \mathcal{P}$ such that $\psi_{\mathcal{M}}(\alpha) < \psi_{\mathcal{N}}(\beta)$.

Definition 5.14 (Level-3 EM blueprint)**.** A *pre-level-3 EM blueprint* is a function Γ sending any finite level-3 tree Y to a complete consistent $\mathcal{L}^Y$-theory $\Gamma(Y)$ which contains all of the following additional axioms:

1. ZFC + there is no inner model with two Woodin cardinals $+V = K+$ there is no strong cardinal $+V$ is closed under the $M_1^{\#}$-operator.

2. Suppose X, T, Q, Z are finite level ≤ 2 trees, π factors (X, T), ψ factors (T, Z).

(a) $\underline{j^T} : V \to \underline{j^T}(V)$ is $\mathcal{L}$-elementary. $\underline{j^{Q^0}}$ is the identity map on V.

(b) $\underline{\pi^T} : \underline{j^X}(V) \to \underline{j^T}(V)$ is $\mathcal{L}$-elementary. $\underline{j^{Q^0,T}} = \underline{j^T}$. $\underline{j^{T,T}}$ is the identity map on $\underline{j^T}(V)$.

(c) $\underline{(\psi \circ \pi)^Z} = \underline{\psi^Z} \circ \underline{\pi^T}$.

(d) $\underline{j^T} \circ \underline{j^Q} = \underline{j^{T \otimes Q}}$.

(e) $\underline{j^Q}(\underline{\pi^T}) = \underline{(Q \otimes \pi)^{Q \otimes T}}$.

(f) $\underline{\pi^T} \restriction \underline{j^{X \otimes Q}}(V) = \underline{(\pi \otimes Q)^{T \otimes Q}}$.

3. If ξ is a cardinal and strong cutpoint, then $V^{\mathrm{Coll}(\omega,\xi)}$ satisfies the following: If U is a Π^1_2-wellfounded level ≤ 2 tree, then $K|\xi$ and $(\underline{j^U})^K(K|\xi)$ are countable Π^1_3-iterable mice and $(\underline{j^U})^K \restriction (K|\xi)$ is essentially an iteration map from $K|\xi$ to $(\underline{j^U})^K(K|\xi)$. Here $(\underline{j^U})^K$ stands for the direct limit map of $(\underline{j^{Z,Z'}})^K$ for Z, Z' finite subtrees of U, Z a finite subtree of Z'.

4. For any $y \in \mathrm{dom}(Y)$, "$\underline{c_y} \in \mathrm{Ord}$" is an axiom.

5. If $\mathbf{y} \prec^Y \mathbf{y}'$, then "$\underline{c_{\mathbf{y}}} < \underline{c_{\mathbf{y}'}}$" is an axiom; if $\mathbf{y} \sim^Y \mathbf{y}'$, then "$\underline{c_{\mathbf{y}}} = \underline{c_{\mathbf{y}'}}$" is an axiom.

A level-3 EM blueprint is a pre-level-3 EM blueprint satisfying the *coherency property*: if R, Y, T are finite, ρ factors (R, Y, T), then for each $\mathcal{L}$-formula $\varphi(v_1, \ldots, v_n)$, for each $r_1, \ldots, r_n \in \mathrm{dom}(R)$,

$$\ulcorner \varphi(\underline{c_{r_1}}, \ldots, \underline{c_{r_n}}) \urcorner \in \Gamma(R)$$

iff

$$\ulcorner \underline{j^T}(V) \models \varphi(\underline{c^T_{\rho(r_1)}}, \ldots, \underline{c^T_{\rho(r_n)}}) \urcorner \in \Gamma(Y).$$

In particular, if Γ is a level-3 EM blueprint, ρ_0 factors (R, Y), then $\mathrm{id}_{Y,*} \circ \rho_0$ factors (R, Y, Q^0), so by coherency, $\ulcorner \varphi(\underline{c_{r_1}}, \ldots) \urcorner \in \Gamma(R)$ iff $\ulcorner \varphi(\underline{c_{\rho_0(r_1)}}, \ldots) \urcorner \in \Gamma(Y)$. This degenerates to the usual indiscernability of the (level-1) EM blueprint.

Lemma 5.15. *Assume* $\mathbf{\Pi}^1_3$*-determinacy. Then* $0^{3\#}$ *is a level-3 EM blueprint.*

Proof. We verify Axioms 1-5 in Definition 5.14. Axiom 1 follows from Theorem 2.18. Axiom 2 follows from Lemma 4.53. Axioms 4-5 follow from Lemma 4.61.

Axiom 3 is shown as follows. Let ξ be a cardinal strong cutpoint of $M_{2,\infty}^-$. Let $\mathcal{P} \in \mathcal{F}_2$ and $\eta \in \mathcal{P}$ such that $\pi_{\mathcal{P},\infty}(\eta) = \xi$. Let g be $\mathrm{Coll}(\omega, \eta)$-generic over $\mathcal{P}$. Suppose T is a Π^1_2-wellfounded tree in $\mathcal{P}[g]$. The direct limit of j^T is wellfounded by Proposition 4.19. We need to show that in $\mathcal{P}[g]$, $(\underline{j^T})^{\mathcal{P}}(\mathcal{P}|\eta) : \mathcal{P}|\eta \to (\underline{j^T})^{\mathcal{P}}(\mathcal{P}|\eta)$ is essentially an iteration map, where $(\underline{j^T})^{\mathcal{P}}$ is the direct limit map of $(\underline{j^{T'}})^{\mathcal{P}}$ for finite subtrees T' of T. Since $\mathcal{P}[g]$ is Σ^1_4-correct, we need to show the same fact in V.

Note that $M_{2,\infty}^-$ is definable over $M_{2,\infty}^-(g)$. In fact, $M_{2,\infty}^- = (L[\underline{S_3}])^{M_{2,\infty}^-(g)}$. Let $\mathcal{Q} \in \mathcal{F}_{2,g}$ and ν so that $\pi_{Q,\infty}(\nu) = \xi$. The maps from $\mathcal{P}$ to $\pi^{-1}_{\mathcal{Q},\infty}(M_{2,\infty}^-|\xi)$ and from $\pi^{-1}_{\mathcal{Q},\infty}(M_{2,\infty}^-|\xi)$ to $L_\xi[S_3]$ plus Dodd-Jensen implies that $\mathcal{P}|\eta \sim_{DJ} \pi^{-1}_{\mathcal{Q},\infty}(M_{2,\infty}^-|\xi)$. By Σ^1_4-correctness of set-generic extensions of $\mathcal{Q}$, $\mathcal{Q}$ thinks that "$\mathcal{P}|\eta \sim_{DJ} L_\nu[S_3]$". By elementarity, $(\underline{j^T}(V))^{\mathcal{Q}}$ thinks "$\mathcal{P}|\eta \sim_{DJ} \underline{j^T}(L_\nu[\underline{S_3}])$". We claim that $(\underline{j^T}(V))^{\mathcal{Q}}$ is also Σ^1_3-correct in set-generic extensions. To see this, it suffices to show $p[(\underline{j^T}(\underline{S_3}))^{\mathcal{Q}}] \subseteq p[S_3]$. We know that $(\underline{j^T}(\underline{S_3}))^{\mathcal{Q}}$ embeds into $j^T(S_3)$, so $p[(\underline{j^T}(\underline{S_3}))^{\mathcal{Q}}] \subseteq p[j^T(S_3)]$. But $x \in p[j^T(S_3)]$ implies $x \in p[S_3]$ by absoluteness of wellfoundedness and elementarity of j^T acting on $L[S_3, x]$. Hence, in reality we have $\mathcal{P}|\eta \sim_{DJ} (\underline{j^T}(L_\nu[\underline{S_3}]))^{\mathcal{Q}}$. But $(\underline{j^T})^{\mathcal{P}}(\mathcal{P}|\eta)$ embeds into $(\underline{j^T}(L_\nu[\underline{S_3}]))^{\mathcal{Q}}$, implying that $(\underline{j^T})^{\mathcal{P}}(\mathcal{P}|\eta) \leq_{DJ} \mathcal{P}|\eta$.

Of course, $\mathcal{P}|\eta \leq_{DJ} (\underline{j^T})^{\mathcal{P}}(\mathcal{P}|\eta)$. So $\mathcal{P} \sim_{DJ} (\underline{j^T})^{\mathcal{P}}(\mathcal{P}|\eta)$. A similar argument shows that for any $\alpha \in \mathcal{P}$, $(\mathcal{P}, \alpha) \sim_{DJ} ((\underline{j^T})^{\mathcal{P}}(\mathcal{P}|\eta), (\underline{j^T})^{\mathcal{P}}(\alpha))$. This finishes verifying Axiom 3 of Definition 5.14.

Finally, the coherency property of $0^{3\#}$ is a consequence of Lemma 5.10.

□

We say that the *upward closure* of $A \subseteq (\omega^{<\omega})^{<\omega}$ is

$$\{r \in (\omega^{<\omega})^{<\omega} : \exists a \in A (r \subseteq a)\}.$$

The upward closure does not apply to subcoordinates of $a \in A$. For instance, $b \subsetneq a(\mathrm{lh}(a) - 1)$ does not imply that $a^- {}^\frown(b)$ is in the upward closure of A. For a level-3 tree R and nodes $s_1, \dots, s_n, s'_1, \dots, s'_n$ in $\mathrm{dom}(R)$,

$$\vec{s'} \text{ is an } R\text{-}shift \text{ of } \vec{s}$$

iff there are a level-3 tree S and maps ρ, ρ' factoring (S, R) such that $\mathrm{ran}(\rho)$ is the upward closure of $\vec{s}$, $\mathrm{ran}(\rho')$ is the upward closure of $\vec{s'}$, and $\rho^{-1}(s_i) = (\rho')^{-1}(s'_i)$ for any i.

Lemma 5.16 (Level-3 indiscernability)**.** *Suppose Γ is a level-3 EM blueprint. Suppose R is a level-3 tree and $s_1, \dots, s_n, s'_1, \dots, s'_n$ are nodes in* $\mathrm{dom}(R)$.

Suppose that $\vec{s'}$ is a shift of $\vec{s}$ with respect to R. Then for each formula φ, $\Gamma(R)$ contains the formula

$$\varphi(\underline{c_{s_1}}, \dots, \underline{c_{s_n}}) \leftrightarrow \varphi(\underline{c_{s'_1}}, \dots, \underline{c_{s'_n}}).$$

Proof. Let S be a level-3 tree and ρ, ρ' both factor (S, R) such that $\mathrm{ran}(\rho)$ is upward closure of $\{s_1, \dots, s_n\}$, $\mathrm{ran}(\rho')$ is the upward closure of $\{s'_1, \dots, s'_n\}$. Let $\rho^{-1}(s_i) = t_i = (\rho')^{-1}(s'_i)$. Applying coherency of Γ to ρ, ρ',

$$\begin{aligned} \ulcorner\varphi(\underline{c_{s_1}}, \dots, \underline{c_{s_n}})\urcorner \in \Gamma(R) &\leftrightarrow \ulcorner\varphi(\underline{c_{t_1}}, \dots, \underline{c_{t_n}})\urcorner \in \Gamma(S) \\ &\leftrightarrow \ulcorner\varphi(\underline{c_{s'_1}}, \dots, \underline{c_{s'_n}})\urcorner \in \Gamma(R). \end{aligned}$$

□

Of course, there is extra information in the coherency property beyond Lemma 5.16.

As with the usual treatment of $0^{\#}$, a level-3 EM blueprint Γ admits an $\mathcal{L}$-Skolemized conservative extension. That means, since ZFC $+V = K$ is a part of the axioms, so is "there is a $\Sigma_1^{\mathcal{L}}$-definable wellordering of the universe". Thus, to each $\mathcal{L}$-formula $\varphi(v, w_1, \dots, w_n)$ we may attach a definable $\mathcal{L}$-Skolem term $\tau_\varphi(w_1, \dots, w_n)$ so that the formula $\forall w_1 \dots w_n$ $(\exists v\ \varphi(v, w_1, \dots, w_n) \to \varphi(\tau_\varphi(w_1, \dots, w_n), w_1, \dots, w_n))$ belongs to $\Gamma(R)$, for any R.

If Y is an infinite level-3 tree, put

$$\Gamma(Y) = \bigcup\{\Gamma(R) : R \text{ is a finite level-3 subtree of } Y\}.$$

By coherency, $\Gamma(R) \subseteq \Gamma(R')$ whenever $R \subseteq R'$ are finite. Hence by compactness, $\Gamma(Y)$ is a complete consistent $\mathcal{L}^Y$-theory. The usual argument of EM models with order indiscernibles carries over to obtain a unique up to isomorphism $\mathcal{L}^Y$-structure

$$\mathcal{M}_{\Gamma,Y} = (M; \underline{\in}^M, \underline{c_t}^M : t \in \mathrm{dom}(Y)).$$

such that $\mathcal{M}_{\Gamma,Y}$ is $\mathcal{L}$-Skolem generated by $\{\underline{c_t}^M : t \in \mathrm{dom}(Y)\}$, and

$$\mathcal{M}_{\Gamma,Y} \models \Gamma(Y).$$

$\mathcal{M}_{\Gamma,Y}$ is called the *EM model* associated to Γ and Y. When $\underline{\in}^{\mathcal{M}_{\Gamma,Y}}$ is wellfounded, $\mathcal{M}_{\Gamma,Y}$ is identified with its transitive collapse. Since $\mathcal{M}_{\Gamma,Y}$ is a model of $V = K$, the extender sequence on $K^{\mathcal{M}_{\Gamma,Y}}$ is definable over $\mathcal{M}_{\Gamma,Y}$, this allows us to sometimes treat $\mathcal{M}_{\Gamma,Y}$ as a structure in the language of premice.

If $\mathcal{L}^*$ is a first-order language expanding $\mathcal{L}$, $\mathcal{N}$ is an $\mathcal{L}^*$-structure satisfying axioms 1-3 in Definition 5.14, we make the following notations:

1. If T is a finite level ≤ 2 tree, then $j^T_{\mathcal{N}} = (\underline{j^T})^{\mathcal{N}}$, $\mathcal{N}^T = (\underline{j^T}(V))^{\mathcal{N}}$ is an $\mathcal{L}^*$-structure so that $j^T_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}^T$ is $\mathcal{L}^*$-elementary.

2. If π factors finite level ≤ 2 trees (X,T), then $\pi^T_{\mathcal{N}} = (\underline{\pi^T})^{\mathcal{N}}$. If T, T' are finite level ≤ 2 trees, T is a subtree of T', then $j^{T,T'}_{\mathcal{N}} = (\underline{j^{T,T'}})^{\mathcal{N}}$.

3. If T is a level ≤ 2 tree, then $\mathcal{N}^T$ is the direct limit of $(\mathcal{N}^{T'}, j^{T',T''}_{\mathcal{N}} : T', T''$ finite subtrees of T, T' a finite subtree of $T'')$ and $j^T_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}^T$ is the direct limit map; if T' is a finite subtree of T, then $j^{T',T}_{\mathcal{N}} : \mathcal{N}^{T'} \to \mathcal{N}^T$ is the tail of the direct limit map. The wellfounded part of $\mathcal{N}^T$ is always assumed to be transitive.

4. If π factors level ≤ 2 trees (X,T), then $\pi^T_{\mathcal{N}} : \mathcal{N}^X \to \mathcal{N}^T$ is the factor map between direct limits.

5. If X is a finite level ≤ 2 tree, $a \in \mathcal{N}^X$, $d \in \{1,2\}$
   (a) if T, T' are finite level ≤ 2 trees, T is a subtree of T', then $\mathcal{N}^T_{X,a} = (\underline{H^T_{X,a}})^{\mathcal{N}}$, $j^T_{X,a,\mathcal{N}} = (\underline{j^T_{X,a}})^{\mathcal{N}}$, $\phi^T_{X,a,\mathcal{N}} = (\underline{\phi^T_{X,a}})^{\mathcal{N}}$, $j^{T,T'}_{X,a,\mathcal{N}} = (\underline{j^{T,T'}_{X,a}})^{\mathcal{N}}$, $\mathcal{N}^T_{d,a} = (\underline{H^T_{d,a}})^{\mathcal{N}}$, $j^T_{d,a,\mathcal{N}} = (\underline{j^T_{d,a}})^{\mathcal{N}}$, $\phi^T_{d,a,\mathcal{N}} = (\underline{\phi^T_{d,a}})^{\mathcal{N}}$, $j^{T,T'}_{d,a,\mathcal{N}} = (\underline{j^{T,T'}_{d,a}})^{\mathcal{N}}$;
   (b) if T is a level ≤ 2 tree, then $\mathcal{N}^T_{X,a}$ is the natural direct limit, $j^T_{X,a,\mathcal{N}} : \mathcal{N} \to \mathcal{N}^T_{X,a}$ is the direct limit map, $\phi^T_{X,a,\mathcal{N}} : \mathcal{N}^T_{X,a} \to \mathcal{N}^T$ is the natural factoring map between direct limits; if T' is a finite subtree of T, then $j^{T',T}_{X,a,\mathcal{N}} : \mathcal{N}^{T'}_{X,a} \to \mathcal{N}^T_{X,a}$ is the tail of the direct limit map; similarly define $\mathcal{N}^T_{d,a}$, $j^T_{d,a,\mathcal{N}}$, $\phi^T_{d,a,\mathcal{N}}$, $j^{T',T}_{d,a,\mathcal{N}}$.

If Γ is a level-3 EM blueprint and R is Π^1_3-wellfounded, we make further notations:

1. If T is a level ≤ 2 tree, then $\mathcal{M}^T_{\Gamma,Y} = (\mathcal{M}_{\Gamma,Y})^T$, $j^T_{\Gamma,Y} = j^T_{\mathcal{M}_{\Gamma,Y}}$.

2. If T is a finite subtree of T', then $j^{T,T'}_{\Gamma,Y} = j^{T,T'}_{\mathcal{M}_{\Gamma,Y}}$.

3. If π factors (X,T), then $\pi^T_{\Gamma,Y} = \pi^T_{\mathcal{M}_{\Gamma,Y}}$.

4. If T is a finite subtree of T', $y \in \mathrm{dom}(Y)$, $X = Y_{\mathrm{tree}}(y)$, $d \in \{1,2\}$, then $c_{\Gamma,Y,y} = (\underline{c_y})^{\mathcal{M}_{\Gamma,Y}}$, $\mathcal{M}^T_{\Gamma,Y,y} = (\mathcal{M}_{\Gamma,Y})^T_{X,c_{\Gamma,Y,y}}$, $j^T_{\Gamma,Y,y} = j^T_{X,c_{\Gamma,Y,y},\mathcal{M}_{\Gamma,Y}}$, $\phi^T_{\Gamma,Y,y} = \phi^T_{X,c_{\Gamma,Y,y},\mathcal{M}_{\Gamma,Y}}$, $j^{T,T'}_{\Gamma,Y,y} = j^{T,T'}_{X,c_{\Gamma,Y,y},\mathcal{M}_{\Gamma,Y}}$, $\mathcal{M}^T_{\Gamma,R^d,*} = (\mathcal{M}_{\Gamma,R^d})^T_{d,c_{\Gamma,R^d,((0))}}$, $j^T_{\Gamma,R^d,*} = j^T_{d,c_{\Gamma,R^d,((0))},\mathcal{M}_{\Gamma,R^d}}$, $\phi^T_{\Gamma,R^d,*} = \phi^T_{d,c_{\Gamma,R^d,((0))},\mathcal{M}_{\Gamma,R^d}}$, $j^{T,T'}_{\Gamma,R^d,*} = j^{T,T'}_{d,c_{\Gamma,R^d,((0))},\mathcal{M}_{\Gamma,R^d}}$.

5. If $\mathbf{B} \in \operatorname{desc}(Y, T', *)$ and T' is a finite subtree of T, then $c^T_{\Gamma,Y,\mathbf{B}} = j^{T',T}_{\Gamma,Y}(\underline{c^{T'}_{\mathbf{B}}})^{\mathcal{M}_{\Gamma,Y}}$.

By coherency, if ρ factors (R, Y, T), then ρ induces an elementary embedding

$$\rho^{Y,T}_\Gamma : \mathcal{M}_{\Gamma,R} \to \mathcal{M}^T_{\Gamma,Y}$$

where

$$\rho^{Y,T}_\Gamma(\tau^{\mathcal{M}_{\Gamma,R}}(c_{\Gamma,R,r_1}, \ldots)) = \tau^{\mathcal{M}^T_{\Gamma,Y}}(c^T_{\Gamma,Y,\rho(r_1)}, \ldots).$$

If ρ factors (R, Y), then ρ induces

$$\rho^Y_\Gamma : \mathcal{M}_{\Gamma,R} \to \mathcal{M}_{\Gamma,Y}$$

where $\rho^Y_\Gamma(\tau^{\mathcal{M}_{\Gamma,R}}(c_{\Gamma,R,r_1}, \ldots)) = \tau^{\mathcal{M}_{\Gamma,Y}}(c_{\Gamma,Y,\rho(r_1)}, \ldots)$.

Recall that wellfoundedness of a (level-1) EM blueprint is a Π^1_2 condition, stating that for every countable ordinal α, the EM model generated by order indiscernibles of order type α is wellfounded. Its higher level analog is called iterability, which is a Π^1_4 condition.

Definition 5.17. Let Γ be a level-3 EM blueprint. Γ is *iterable* iff for any Π^1_3-wellfounded level-3 tree Y, $\mathcal{M}_{\Gamma,Y}$ is a Π^1_3-iterable mouse.

Lemma 5.18. *Assume $\boldsymbol{\Pi}^1_3$-determinacy. Then $0^{3\#}$ is iterable.*

Proof. Let Y be any Π^1_3-wellfounded level-3 tree. Let $F \in [C]^{Y\uparrow}$, where C is a firm set of potential level-3 indiscernibles for $M^-_{2,\infty}$. Then $\mathcal{M}_{0^{3\#},Y}$ elementarily embeds into $M^-_{2,\infty}$, the map being generated by $c_{0^{3\#},Y,s} \mapsto [F]^Y_s$. Therefore, $\mathcal{M}_{0^{3\#},Y}$ is iterable. □

Lemma 5.19. *Assume $\boldsymbol{\Delta}^1_2$-determinacy.*

1. *Suppose $\mathcal{N}$ is a countable Π^1_3-iterable mouse satisfying Axioms 1-3 in Definition 5.14.*
   - (a) *If T is a Π^1_2-wellfounded level ≤ 2 tree, then $\mathcal{N}^T$ is a Π^1_3-iterable mouse and $j^T_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}^T$ is essentially an iteration map.*
   - (b) *If ψ minimally factors level ≤ 2 trees (T, X), then $\psi^X_{\mathcal{N}} : \mathcal{N}^T \to \mathcal{N}^X$ is essentially an iteration map.*
2. *Suppose Γ is an iterable level-3 EM blueprint and Y is a Π^1_3-wellfounded level-3 tree. If ψ minimally factors level-3 trees (Y, R) and $[\![\emptyset]\!]_Y = [\![\emptyset]\!]_R$, then $\psi^R_\Gamma : \mathcal{M}_{\Gamma,Y} \to \mathcal{M}_{\Gamma,R}$ is essentially an iteration map.*

Proof. 1(a). By Axiom 1 in Definition 5.14, there are cofinally many cardinal strong cutpoints in $\mathcal{N}$. $j^T_{\mathcal{N}}$ is cofinal in $\mathcal{N}^T$ by definition and a direct limit argument. By Dodd-Jensen, it suffices to show that for any cardinal strong cutpoint ξ of $\mathcal{N}$, $j^T_{\mathcal{N}}(\mathcal{N}|\xi)$ is Π^1_3-iterable and $j^T_{\mathcal{N}} \restriction (\mathcal{N}|\xi)$ is essentially an iteration map from $\mathcal{N}|\xi$ to $\mathcal{N}^T|j^T_{\mathcal{N}}(\xi)$. Fix such ξ. Let g be $\mathrm{Coll}(\omega,\xi)$-generic over $\mathcal{N}$. The statement "for any Π^1_2-wellfounded level ≤ 2 tree T', $j^{T'}_{\mathcal{N}}(\mathcal{N}|\xi)$ is a Π^1_3-iterable mouse, and $j^{T'}_{\mathcal{N}} \restriction (\mathcal{N}|\xi)$ is essentially an iteration map from $\mathcal{N}|\xi$ to $j^{T'}_{\mathcal{N}}(\mathcal{N}|\xi)$" is Π^1_3 in a real $z \in \mathcal{N}[g]$ coding $(\mathcal{N}|\xi, j^{T'}_{\mathcal{N}} \restriction (\mathcal{N}|\xi))_{T' \text{ finite level } \leq 2 \text{ tree}}$. This statement is true in $\mathcal{N}[g]$ by Level ≤ 2 ultrapower invariance axiom in Definition 5.14. It suffices to show $\mathcal{N}[g] \prec_{\Sigma^1_3} V$. But by Axiom 1 in Definition 5.14, $\mathcal{N} \models$"I am closed under the $M_1^{\#}$-operator". Since $\mathcal{N}$ is a Π^1_3-iterable mouse, the $M_1^{\#}$-operators are correctly computed in $\mathcal{N}$. Using genericity iterations [51], $M_1^{\#}$-operators figure out Σ^1_3-truth. Hence, $\mathcal{N}[g] \prec_{\Sigma^1_3} V$.

1(b). By Theorem 4.57, there is a Π^1_2-wellfounded Q and π minimally factoring $(X, T \otimes Q)$. So $\mathrm{id}_{T,*} = \pi \circ \psi$. By Axiom 2 in Definition 5.14, $j^Q_{\mathcal{N}^T} = \pi^{T \otimes Q}_{\mathcal{N}} \circ \psi^X_{\mathcal{N}}$, which is essentially an iteration from $\mathcal{N}^T$ to $\mathcal{N}^{T\otimes Q}$ by part 1(a). By Dodd-Jensen, $\psi^X_{\mathcal{N}}$ is essentially an iteration map.

2. By Theorem 4.71, there is a Π^1_2-wellfounded T and ρ minimally factoring $(R, Y \otimes T)$. So $\mathrm{id}_{Y,*} = \rho \circ \psi$. By Axiom 2 in Definition 5.14 and part 1, $j^T_{\Gamma,Y} = \rho^{Y,T}_{\Gamma} \circ \psi^R_{\Gamma}$ is essentially an iteration from $\mathcal{M}_{\Gamma,Y}$ to $\mathcal{M}^T_{\Gamma,Y}$. By Dodd-Jensen, ψ^R_{Γ} is essentially an iteration map. □

We start to introduce the remarkability property of a level-3 EM blueprint.

For $r, s \in \omega^{<\omega}$, define $r <_0 s$ iff $r(0) <_{BK} s(0)$, $r \leq^R_0 s$ iff $r(0) \leq_{BK} s(0)$. If $\vec{r} = (r_i)_{1 \leq i \leq n}$ is a tuple of nodes in $\omega^{<\omega}$, define $\vec{r} <_0 s$ iff $r_i <_0 s$ for any i. Similarly define $\vec{r} \leq_0 s$, $\vec{r} <_0 \vec{s}$, etc.

Definition 5.20 (Unboundedness)**.** A level-3 EM blueprint Γ is *unbounded* iff for any level-3 tree R, if τ is an $\mathcal{L}$-Skolem term, $\{t, r_1, \dots, r_m\} \subseteq \mathrm{dom}(R)$, $\vec{r} <_0 t$, then $\Gamma(R)$ contains the formula

$$\tau(\underline{c_{r_1}}, \dots, \underline{c_{r_m}}) \in \mathrm{Ord} \rightarrow \tau(\underline{c_{r_1}}, \dots, \underline{c_{r_m}}) < \underline{c_t}.$$

Lemma 5.21. *Assume* $\boldsymbol{\Pi}^1_3$*-determinacy. Then* $0^{3\#}$ *is unbounded.*

Proof. Let C be a firm club of potential level-3 indiscernibles for $M^-_{2,\infty}$. Let $\eta \in D$ iff $C \cap \eta$ has order type $u_\omega \xi$ for some ordinal ξ.

We may further assume that $\mathrm{dom}(R)$ is the upward closure of $\vec{r} \cup \{t\}$ and $R^- =_{\mathrm{DEF}} R \restriction$(the upward closure of $\vec{r}$) is a level-3 subtree of R. The reason is because we can find level-3 trees S^-, S , ρ^- factoring (S^-, R), ρ factoring (S, R) so that S^- is a subtree of S, $\rho^- = \rho \restriction S^-$, $\mathrm{ran}(\rho^-)$ is the upward closure of $\vec{r}$, $\mathrm{ran}(\rho)$ is the upward closure of $\vec{r} \cup \{t\}$. We then work with S and $\rho^{-1}(\vec{r}, t)$ instead, and finally apply the coherency of $0^{3\#}$.

Suppose $\Gamma(R)$ contains the formula "$\tau(\underline{c}_{r_1}, \dots, \underline{c}_{r_m}) \in \mathrm{Ord}$". Then for any $\vec{\gamma} \in [C]^{R\uparrow}$,

$$\tau^{M^-_{2,\infty}(x)}(\gamma_{r_1}, \dots, \gamma_m) < \boldsymbol{\delta}^1_3.$$

Our assumption $\vec{r} <_0 t$ ensures the existence of $\vec{\delta} \in [D]^{R\uparrow}$ extending $\vec{\gamma} \restriction \mathrm{dom}(R^-)$ such that

$$\tau^{M^-_{2,\infty}}(\gamma_{r_1}, \dots, \gamma_{r_m}) < \delta_t.$$

Hence, $\Gamma(R)$ contains the formula "$\tau(\underline{c_{r_1}}, \dots, \underline{c_{r_m}}) < \underline{c_t}$". □

Definition 5.22 (Weak remarkability)**.** A level-3 EM blueprint Γ is *weakly remarkable* iff Γ is unbounded and for any level-3 tree R, if τ is an $\mathcal{L}$-Skolem term, $\vec{r} \cup \vec{s} \cup \vec{s'} \cup \{t\} \subseteq \mathrm{dom}(R)$, $\vec{r} <_0 t \leq_0 \vec{s} \leq_0 \vec{s'}$, $\vec{s'}$ is an R-shift of $\vec{s}$, $\mathrm{lh}(t) = 1$, then $\Gamma(R)$ contains the formula

$$\tau(\underline{c_{r_1}}, \dots, \underline{c_{r_m}}, \underline{c_{s_1}}, \dots, \underline{c_{s_n}}) < \underline{c_t} \rightarrow \\ \tau(\underline{c_{r_1}}, \dots, \underline{c_{r_m}}, \underline{c_{s_1}}, \dots, \underline{c_{s_n}}) = \tau(\underline{c_{r_1}}, \dots, \underline{c_{r_m}}, \underline{c_{s'_1}}, \dots, \underline{c_{s'_n}}).$$

Lemma 5.23. *Assume $\boldsymbol{\Pi}^1_3$-determinacy. Then $0^{3\#}$ is weakly remarkable.*

Proof. Again, we may assume that $\mathrm{dom}(R)$ is the upward closure of $\vec{r} \cup \vec{s} \cup \vec{s'} \cup \{t\}$.

Suppose $0^{3\#}(R)$ contains the formula "$\tau(\underline{c_{r_1}}, \dots, \underline{c_{s_1}}, \dots) < \underline{c_t}$". We need to show that $0^{3\#}(R)$ contains the formula "$\tau(\underline{c_{r_1}}, \dots, \underline{c_{s_1}}, \dots) = \tau(\underline{c_{r_1}}, \dots, \underline{c_{s'_1}}, \dots)$". By Axiom 5 in Definition 5.14, we may further assume that t is in the upward closure of $\vec{s}$. Let S be a level-3 tree and ρ, ρ' both factor (S, R) such that $\mathrm{ran}(\rho)$ is the upward closure of $\vec{r} \cup \vec{s}$, $\mathrm{ran}(\rho')$ is the upward closure of $\vec{r} \cup \vec{s'}$. Put $\rho^{-1}(r_i, s_j, t) = (\bar{r}_i, \bar{s}_j, \bar{t}) = (\rho')^{-1}(r'_i, s'_j, t')$.

Let C be a firm set of potential level-3 indiscernibles for $M^-_{2,\infty}$. Let $C = \bigcup_{\xi < \boldsymbol{\delta}^1_3} C_\xi$ be a disjoint partition of C such that for any $\xi < \boldsymbol{\delta}^1_3$, $\mathrm{o.t.}(C_\xi) = u_\omega$, and for any $\xi < \eta < \boldsymbol{\delta}^1_3$, any member of C_ξ is smaller than any member of C_η. Let D be a club in $\boldsymbol{\delta}^1_3$ where $\nu \in D$ iff $\sup \bigcup_{\xi < \nu} C_\xi = \nu$. As C is firm, D has order type $\boldsymbol{\delta}^1_3$.

If X, Y are subsets of ordinals, define $X \sqsubseteq Y$ iff $X \subseteq Y$ and $X = Y \cap \alpha$ for some α. For each $0 < \xi < \boldsymbol{\delta}^1_3$, let $F^\xi \in D^{S\uparrow}$ so that $(F^\xi)'' \mathrm{rep}(U) \sqsubseteq C_0$, $(F^\xi)''(\mathrm{rep}(S) \setminus \mathrm{rep}(U)) \sqsubseteq C_\xi$. Define $\vec{\gamma}^\xi = (\gamma^\xi_x)_{x \in \mathrm{dom}(S)} = [F^\xi]^S$. Define

$$\epsilon_\xi = \tau^{M^-_{2,\infty}}(\gamma^\xi_{\bar{r}_1}, \dots, \gamma^\xi_{\bar{s}_1}, \dots).$$

Hence,

$$\epsilon_\xi < \min(C_1).$$

For $0 < \eta < \xi < \boldsymbol{\delta}^1_3$, define $\vec{\gamma}^{\eta\xi} = (\gamma^{\eta\xi}_y)_{y \in \mathrm{dom}(R)}$ where $\gamma^{\eta\xi}_{\rho(x)} = \gamma^{\eta}_x$ and $\gamma^{\eta\xi}_{\rho'(x)} = \gamma^{\xi}_x$ for any $x \in \mathrm{dom}(S)$. By Lemma 4.46, $\gamma^{\eta\xi} \in [D]^{R\uparrow}$. Suppose towards a contradiction.

Case 1: $0^{3\#}(R)$ contains the formula "$\tau(\underline{c_{r_1}}, \ldots, \underline{c_{s_1}}, \ldots) > \tau(\underline{c_{r_1}}, \ldots, \underline{c_{s'_1}}, \ldots)$".

Then $\vec{\gamma}^{\xi\eta}$ witnesses that $\epsilon_\eta > \epsilon_\xi$ whenever $0 < \eta < \xi < \boldsymbol{\delta}^1_3$. This is an infinite descending chain of ordinals.

Case 2: $0^{3\#}(R)$ contains the formula "$\tau(\underline{c_{r_1}}, \ldots, \underline{c_{s_1}}, \ldots) < \tau(\underline{c_{r_1}}, \ldots, \underline{c_{s'_1}}, \ldots)$".

Then $\epsilon_\eta < \epsilon_\xi$ whenever $0 < \eta < \xi < \boldsymbol{\delta}^1_3$, contradicting to $\epsilon_\xi < \min(C_1)$. □

If R is a level-3 tree, $t \in \mathrm{dom}(R)$, $\mathrm{lh}(t) = 1$, let

$$R \restriction t = R \restriction \{r \in \mathrm{dom}(R) : r <_0 t\}.$$

Lemma 5.24. *Suppose Γ is a weakly remarkable level-3 EM blueprint. Suppose R is a level-3 tree, $t \in \mathrm{dom}(R)$, $\mathrm{lh}(t) = 1$.*

1. *If τ is an $\mathcal{L}$-Skolem term, $\vec{r} \cup \vec{s} \cup \vec{s'} \cup \{t\} \subseteq \mathrm{dom}(R)$, $\vec{r} <_0 t \leq_0 \vec{s}^\frown \vec{s'}$, $\vec{s'}$ is an R-shift of $\vec{s}$, then $\Gamma(R)$ contains the formula*

$$\tau(\underline{c_{r_1}}, \ldots, \underline{c_{r_m}}, \underline{c_{s_1}}, \ldots, \underline{c_{s_n}}) < \underline{c_t} \rightarrow$$
$$\tau(\underline{c_{r_1}}, \ldots, \underline{c_{r_m}}, \underline{c_{s_1}}, \ldots, \underline{c_{s_n}}) = \tau(\underline{c_{r_1}}, \ldots, \underline{c_{r_m}}, \underline{c_{s'_1}}, \ldots, \underline{c_{s'_n}}).$$

2. *$\Gamma(R)$ contains the scheme "$K|\underline{c_t} \prec V$". In particular, $\Gamma(R)$ contains the formula "$\underline{c_t}$ is inaccessible and there are cofinally many cardinal strong cutpoints below $\underline{c_t}$".*

Proof. 1. Assume without loss of generality that $\mathrm{dom}(R)$ is the upward closure of $\vec{r} \cup \vec{s} \cup \vec{s'} \cup \{t\}$. Suppose $\Gamma(R)$ contains the formula "$\tau(\underline{c_{r_1}}, \ldots, \underline{c_{s_1}}, \ldots) < \underline{c_t}$". Expand R to the level-3 tree S where $\mathrm{dom}(S)$ is the upward closure of $\mathrm{dom}(R) \cup \{s''_i : 1 \leq i \leq n\}$, each $s''_i \notin \mathrm{dom}(R)$, $\vec{s''}$ is an R-shift of $\vec{s}$, $\vec{s} <_0 \vec{s''}$. By coherency and weak remarkability, $\Gamma(S)$ contains the formula $\tau(\underline{c_{r_1}}, \ldots, \underline{c_{s_1}}, \ldots) = \tau(\underline{c_{r_1}}, \ldots, \underline{c_{s''_1}}, \ldots)$. But $\vec{r}^\frown \vec{s}^\frown \vec{s''}$ is a shift of $\vec{r}^\frown \vec{s'}^\frown \vec{s''}$. By indiscernability, $\Gamma(S)$ contains the formula $\tau(\underline{c_{r_1}}, \ldots, \underline{c_{s'_1}}, \ldots) = \tau(\underline{c_{r_1}}, \ldots, \underline{c_{s''_1}}, \ldots)$. Hence, $\Gamma(S)$ contains the formula $\tau(\underline{c_{r_1}}, \ldots, \underline{c_{s_1}}, \ldots) = \tau(\underline{c_{r_1}}, \ldots, \underline{c_{s'_1}}, \ldots)$.

2. Put $\mathcal{N} = \mathcal{M}_{\Gamma,R}$. By coherency of Γ, we may assume that $A = \{s \in \mathrm{dom}(R) : s <_0 t\}$ has $<_{BK}$-limit order type. By Tarski's criterion, we need to show that if $w = \tau^{\mathcal{N}}(z_1, \ldots, z_k) \in \mathrm{Ord}$, $z_1, \ldots, z_k < \underline{c_t}^{\mathcal{N}}$, then $w < \underline{c_t}^{\mathcal{N}}$. To save notations, let $k = 1$, $z_1 = \sigma^{\mathcal{N}}(\underline{c_{r_1}}^{\mathcal{N}}, \ldots, \underline{c_{s_1}}^{\mathcal{N}}, \ldots) < \underline{c_t}^{\mathcal{N}}$, $\vec{r} <_0 t \leq_0 \vec{s}$.

Pick t^* of length 1 such that $\vec{r} <_0 t^* <_0 t$. Build a level-3 tree S that extends R in which there are nodes $t', \vec{s'} \in \text{dom}(S)$ such that $\vec{r} <_0 (t')^\frown \vec{s'} <_0 r^*$ and $(t')^\frown \vec{s'}$ is an S-shift of $(t)^\frown \vec{s}$. Put $\mathcal{P} = \mathcal{M}_{\Gamma,S}$. By weakly remarkability,

$$\sigma^{\mathcal{P}}(\underline{c_{r_1}}^{\mathcal{P}}, \ldots, \underline{c_{s_1}}^{\mathcal{P}}) = \sigma^{\mathcal{P}}(\underline{c_{r_1}}^{\mathcal{P}}, \ldots, \underline{c_{s'_1}}^{\mathcal{P}}, \ldots).$$

By unboundedness of Γ,

$$\tau^{\mathcal{P}}(\sigma^{\mathcal{P}}(\underline{c_{r_1}}^{\mathcal{P}}, \ldots, \underline{c_{s'_1}}^{\mathcal{P}}, \ldots)) < \underline{c_{t^*}}^{\mathcal{P}}.$$

By coherency of Γ, $w < \underline{c_{t^*}}^{\mathcal{N}}$. By Axiom 5 in Definition 5.14, $\underline{c_{r^*}}^{\mathcal{N}} < \underline{c_t}^{\mathcal{N}}$. $\square$

A level-3 tree R is said to be *universal above* t iff $t \in \text{dom}(R)$, $\text{lh}(t) = 1$, and for any level-3 tree S, if $S \restriction t'$ is isomorphic to $R \restriction t$ via π and $\text{dom}(S) \setminus \text{dom}(S \restriction t')$ is finite, then there is a map ρ factoring (S, R) that extends π. Clearly, for any R, there is (R', t) such that $R' \restriction t$ is isomorphic to R and R' is universal above t. If R is Π^1_3-wellfounded, we may further demand that R' is Π^1_3-wellfounded.

Lemma 5.25. *Suppose Γ is a weakly remarkable level-3 EM blueprint, R is universal above t, R' is universal above t', $R \restriction t$ is isomorphic to $R' \restriction t'$. Then*

$$(K|\underline{c_t})^{\mathcal{M}_{\Gamma,R}} \cong (K|\underline{c_{t'}})^{\mathcal{M}_{\Gamma,R'}}.$$

Proof. To begin with, we build an isomorphism $\psi : (\underline{c_t})^{\mathcal{M}_{\Gamma,R}} \to (\underline{c_{t'}})^{\mathcal{M}_{\Gamma,R'}}$ which preserves membership relations in the respective EM models. Given $a \in \mathcal{M}_{\Gamma,R}$ such that $\mathcal{M}_{\Gamma,R} \models a < \underline{c_t}$, find a Skolem term τ and nodes $\vec{r}, \vec{s}$ such that $\vec{r} \prec^R t \preceq^R \vec{s}$ and

$$a = (\tau(\underline{c_{r_1}}, \ldots, \underline{c_{s_1}}, \ldots))^{\mathcal{M}_{\Gamma,R}}.$$

Let S be a level-3 tree and ρ factor (S, R) such that $\text{ran}(S)$ is the upward closure of $\text{dom}(R \restriction t) \cup \vec{s} \cup \{t\}$. By universality, pick ρ' factoring (S, R') which extends π. By coherency of Γ, $(\tau(\underline{c_{\pi(r_1)}}, \ldots, \underline{c_{\rho' \circ \rho^{-1}(s_1)}}, \ldots))^{\mathcal{M}_{\Gamma,R'}} <^{\mathcal{M}_{\Gamma,R'}} \underline{c_{t'}}$. Define

$$\psi(a) = (\tau(\underline{c_{\pi(r_1)}}, \ldots, \underline{c_{\rho' \circ \rho^{-1}(s_1)}}, \ldots))^{\mathcal{M}_{\Gamma,R'}}.$$

ψ is well-defined and preserves membership. For this, we firstly show that $\psi(a)$ does not depend on the choice of ρ'. Suppose ρ'' is another candidate for ρ'. Then $\rho'' \circ \rho^{-1}(\vec{s})$ is an R'-shift of $\rho' \circ \rho^{-1}(\vec{s})$. By Lemma 5.24, $\mathcal{M}_{\Gamma,R'} \models \tau(\underline{c_{\pi(r_1)}}, \ldots, \underline{c_{\rho' \circ \rho^{-1}(s_1)}}, \ldots) = \tau(\underline{c_{\pi(r_1)}}, \ldots, \underline{c_{\rho'' \circ \rho^{-1}(s_1)}}, \ldots)$. Secondly, the reason why $\psi(a)$ does not depend on the choice of τ and $\vec{r}, \vec{s}$ is because of coherency of Γ. In the same spirit, we can show that ψ preserves membership. A completely symmetrical argument gives $\psi' : (\underline{c_{t'}})^{\mathcal{M}_{\Gamma,R'}} \to (\underline{c_t})^{\mathcal{M}_{\Gamma,R}}$.

By Lemma 5.16, $\psi \circ \psi'$ and $\psi' \circ \psi$ are both identity functions. So ψ is an isomorphism between $(\underline{c_t})^{\mathcal{M}_{\Gamma,R}}$ and $(\underline{c_{t'}})^{\mathcal{M}_{\Gamma,R'}}$.

$\mathcal{M}_{\Gamma,R}$ is a model of $V = K$. Working in $\mathcal{M}_{\Gamma,R}$, $K|\underline{c_t}$ has a canonical wellordering of order type $\omega\underline{c_t}$, and similarly for $\mathcal{M}_{\Gamma,R'}$. ψ extends to ψ^*, acting on $(K|\underline{c_t})^{\mathcal{M}_{\Gamma,R}}$ according to these canonical wellorderings. Using the same argument as before, ψ^* is an isomorphism from $(K|\underline{c_t})^{\mathcal{M}_{\Gamma,R}}$ to $(K|\underline{c_{t'}})^{\mathcal{M}_{\Gamma,R'}}$. □

A level-3 tree R is *universal based on* Y iff there is $t \in \operatorname{dom}(R)$ such that $\operatorname{lh}(t) = 1$, R is universal above t and $R \restriction t$ is isomorphic to Y. Suppose Γ is a weakly remarkable level-3 EM blueprint. For a level-3 tree Y, if R is universal based on Y, $t \in \operatorname{dom}(R)$, $\operatorname{lh}(t) = 1$, $R \restriction t$ is isomorphic to Y, put

$$\mathcal{M}^*_{\Gamma,Y} = (K|\underline{c_t})^{\mathcal{M}_{\Gamma,R}}.$$

$\mathcal{M}^*_{\Gamma,Y}$ is well-defined up to an isomorphism. Its wellfounded part is transitivized. By Lemma 5.24, there are cofinally many cardinal strong cutpoints in $\mathcal{M}^*_{\Gamma,Y}$. Similarly, for a level ≤ 2 tree T, define

$$\mathcal{M}^{*,T}_{\Gamma,Y} = (K|\underline{c_t})^{\mathcal{M}^T_{\Gamma,R}}.$$

Hence, $\mathcal{M}^{*,T}_{\Gamma,Y} = (\mathcal{M}^*_{\Gamma,Y})^T$. If ρ factors (Y, Y'), R' is universal above R, then $\rho^{*,Y'}_\Gamma = \rho^{R'}_\Gamma \restriction \mathcal{M}^*_{\Gamma,Y}$. If ρ factors (Y, Y', T), R' is universal above R, then $\rho^{*,Y',T}_\Gamma = \rho^{Y',T}_\Gamma \restriction \mathcal{M}^*_{\Gamma,Y}$.

A Π^1_3-iterable mouse $\mathcal{P}$ is *full* iff for any strong cutpoint η of $\mathcal{P}$, for any Π^1_3-iterable mouse $\mathcal{Q}$ extending $\mathcal{P}|\eta$ which is sound and projects to η, $\mathcal{Q} \trianglelefteq \mathcal{P}$.

Lemma 5.26. *Assume $\boldsymbol{\Delta}^1_2$-determinacy. Suppose Γ is an iterable, weakly remarkable level-3 EM blueprint.*

1. *Suppose Y, Y' are Π^1_3-wellfounded level-3 trees. Then $[\![\emptyset]\!]_Y = [\![\emptyset]\!]_{Y'}$ iff $\mathcal{M}_{\Gamma,Y} \sim_{DJ} \mathcal{M}_{\Gamma,Y'}$; $[\![\emptyset]\!]_Y < [\![\emptyset]\!]_{Y'}$ iff $\mathcal{M}_{\Gamma,Y} <_{DJ} \mathcal{M}_{\Gamma,Y'}$.*

2. *Suppose Y is a Π^1_3-wellfounded level-3 tree. Then $\mathcal{M}^*_{\Gamma,Y}$ is full.*

3. *Suppose Y, Y' are Π^1_3-wellfounded level-3 trees. Then $[\![\emptyset]\!]_Y = [\![\emptyset]\!]_{Y'}$ iff $\mathcal{M}^*_{\Gamma,Y} \sim_{DJ} \mathcal{M}^*_{\Gamma,Y'}$; $[\![\emptyset]\!]_Y < [\![\emptyset]\!]_{Y'}$ iff $\mathcal{M}^*_{\Gamma,Y} <_{DJ} \mathcal{M}^*_{\Gamma,Y'}$.*

Proof. 1. If $[\![\emptyset]\!]_Y \leq [\![\emptyset]\!]_{Y'}$, by Theorem 4.71, there exist a Π^1_3-wellfounded Z and ρ minimally factoring (Y, Z), ρ' minimally factoring (Y', Z) so that $[\![\emptyset]\!]_Y = [\![\emptyset]\!]_Z$. By Lemma 5.19, $\mathcal{M}_{\Gamma,Y} \leq_{DJ} \mathcal{M}_{\Gamma,Z} \sim_{DJ} \mathcal{M}_{\Gamma,Y'}$.

If $[\![\emptyset]\!]_Y < [\![\emptyset]\!]_{Y'}$, we further obtain $t \in \operatorname{dom}(R)$ so that $\operatorname{lh}(t) = 1$ and $[\![\emptyset]\!]_Y = [\![t]\!]_Z$. By unboundedness of Γ, $\operatorname{ran}(\rho^{Y',T}_\Gamma) \subseteq \mathcal{M}_{\Gamma,Z}|c_{\Gamma,Z,t}$. Hence, $\rho^{Y,T}_\Gamma$ is Σ_1-elementary from $\mathcal{M}_{\Gamma,Y}$ into $\mathcal{M}_{\Gamma,Z}|c_{\Gamma,Z,t}$. Hence $\mathcal{M}_{\Gamma,Y} <_{DJ} \mathcal{M}_{\Gamma,Z}$.

2. Recall that there are cofinally many cardinal strong cutpoints in $\mathcal{M}^*_{\Gamma,Y}$. Suppose η is a strong cutpoint of $\mathcal{M}^*_{\Gamma,Y}$ and $\mathcal{M}^*_{\Gamma,Y}|\eta \lhd \mathcal{P}$, $\mathcal{P}$ is a sound Π^1_3-iterable mouse, $\rho_\omega(\mathcal{P}) \leq \eta$. Let Y' be a Π^1_3-wellfounded level-3 tree such that $\|\mathcal{P}\|_{DJ} < [\![\emptyset]\!]_{Y'}$ and Y' is universal based on Y. By part 2, $\mathcal{P} <_{DJ} \mathcal{M}_{\Gamma,Y'}$. Since $\mathcal{M}^*_{\Gamma,Y}|\eta \lhd \mathcal{M}_{\Gamma,Y'}$ and η is a strong cutpoint of $\mathcal{M}_{\Gamma,Y'}$, the comparison between $\mathcal{P}$ and $\mathcal{M}_{\Gamma,Y'}$ is above η. It follows that $\mathcal{P} \lhd \mathcal{M}_{\Gamma,Y'}$. Hence $\mathcal{P} \lhd \mathcal{M}^*_{\Gamma,Y}$.

3. By parts 1-2 and remarkability of Γ. □

Assume $\mathbf{\Delta}^1_2$-determinacy. Suppose Γ is an iterable, weakly remarkable level-3 EM blueprint. Suppose Y is a Π^1_3-wellfounded level-3 tree.

For $s \in \text{dom}(Y)$, let

$$c^*_{\Gamma,Y,s} = \underline{c_s}^{\mathcal{M}^*_{\Gamma,Y}}$$

and

$$c_{\Gamma,Y,s,\infty} = \pi_{\mathcal{M}^*_{\Gamma,Y},\infty}(c^*_{\Gamma,Y,s}).$$

In fact, $c_{\Gamma,Y,s,\infty}$ depends only on $([\![s]\!]_Y, Y_{\text{tree}}(s))$, shown as follows. Suppose Y' is another Π^1_3-wellfounded level-3 tree and $([\![s]\!]_Y, Y_{\text{tree}}(s)) = ([\![s']\!]_{Y'}, Y'_{\text{tree}}(s'))$. By Lemma 4.47, $Y[s] = Y'[s']$. By Theorem 4.71, we can find Π^1_3-wellfounded level-3 trees R, R' which are universal based on Y, Y' respectively, a Π^1_3-wellfounded Z and ρ minimally factoring (R, Z), ρ' minimally factoring (R', Z). In particular, $\rho(s) = \rho'(s')$. By Lemma 5.19, $\rho^Z_\Gamma : \mathcal{M}_{\Gamma,R} \to \mathcal{M}_{\Gamma,Z}$ is essentially an iteration map, sending $c^*_{\Gamma,Y,s}$ to $c_{\Gamma,Z,\rho(s)}$, and similarly on the ρ'-side. Hence $c_{\Gamma,Y,s,\infty} = c_{\Gamma,Y',s',\infty}$. We can safely define

$$c_{\Gamma,Q,\gamma} = c_{\Gamma,Y,s,\infty}$$

for $Y_{\text{tree}}(s) = Q$ and $\gamma = [\![s]\!]_Y$.

If $(Q, \overrightarrow{(d,q,P)}) = (Q, (d_i, q_i, P_i)_{1 \leq i \leq k})$ is a potential partial level ≤ 2 tower, let $F \in B^{(Q,\overrightarrow{(d,q,P)})\uparrow}$ iff $F : [\omega_1]^{Q\uparrow} \to B$ is an order preserving function and

1. if $(Q, \overrightarrow{(d,q,P)})$ is of continuous type, then the signature of F is $(d_i, q_i)_{1 \leq i \leq k}$, F is essentially continuous;

2. if $(Q, \overrightarrow{(d,q,P)})$ is of discontinuous type, then the signature of F is $(d_i, q_i)_{1 \leq i < k}$, F is essentially discontinuous, F has uniform cofinality $\text{ucf}(Q, \overrightarrow{(d,q,P)})$.

Let $\gamma \in [B]^{(Q,\overrightarrow{(d,q,P)})\uparrow}$ iff $\gamma = [F]_{\mu^Q}$ for some $F \in B^{(Q,\overrightarrow{(d,q,P)})\uparrow}$. γ is said to *respect* $(Q, \overrightarrow{(d,q,P)})$ iff $\gamma \in (\boldsymbol{\delta}^1_3)^{(Q,\overrightarrow{(d,q,P)})\uparrow}$. γ is said to *respect* Q if γ respects some potential partial level ≤ 2 tower $(Q, \overrightarrow{(d', q', P')})$. By Lemma 4.79, γ respects Q iff there is a Π^1_3-wellfounded Y and s such that $Y_{\text{tree}}(s) = Q$ and $\gamma = [\![s]\!]_Y$. Hence, $c_{\Gamma,Q,\gamma}$ is defined whenever γ respects Q and the map $\gamma \mapsto c_{\Gamma,Q,\gamma}$ is order preserving. Define

$$c_{\Gamma,\gamma} = c_{\Gamma,Q^0,\gamma}.$$

$c_{\Gamma,\gamma}$ is defined whenever $\gamma < \boldsymbol{\delta}^1_3$ is a limit ordinal. Remarkability will ensure that the map $\gamma \mapsto c_{\Gamma,\gamma}$ is continuous. Assuming $\boldsymbol{\Delta}^1_3$-determinacy, define

$$\begin{aligned} c^{(3)}_{Q,\gamma} &= c_{0^{3\#},Q,\gamma}, \\ c^{(3)}_{\gamma} &= c_{0^{3\#},\gamma}, \\ I^{(3)} &= \{c^{(3)}_{Q,\gamma} : \gamma \text{ respects } Q\}. \end{aligned}$$

$I^{(3)}$ is the higher analog of Silver indiscernibles for L.

Lemma 5.27. *Assume $\boldsymbol{\Pi}^1_3$-determinacy. Then there is a club $C \subseteq \boldsymbol{\delta}^1_3$ such that $C \in L[T_3, 0^{3\#}]$ and for any potential partial level ≤ 2 tree $(Q, \overrightarrow{(d, q, P)})$, for any $\gamma \in [C]^{(Q,\overrightarrow{(d,q,P)})\uparrow}$,*

$$\gamma = c^{(3)}_{Q,\gamma}.$$

Proof. Let D be a firm set of potential level-3 indiscernibles for $M^-_{2,\infty}$ and let $\eta \in C$ iff $\eta \in D$ and $D \cap \eta$ has order type η. C works for the lemma. □

Recall Definition 4.23 for the definition of R^d. An ordinal $\alpha < \omega_1$ is *ω_1-represented by T* iff $(1, (0)) \in \text{dom}(T)$ and $[\![1, (0)]\!]_T = \alpha$. $\alpha < u_2$ is *u_2-represented by T* iff $(2, ((0))) \in \text{dom}(T)$ and $[\![2, ((0))]\!]_T = \alpha$.

Definition 5.28 (Remarkability)**.** A weakly remarkable level-3 EM blueprint Γ is *remarkable* iff

1. $\Gamma(R^0)$ contains the axiom "$\underline{c_{((0))}}$ is not measurable".

2. $\Gamma(R^1)$ contains the following axiom: if ξ is a cardinal and strong cutpoint, $c = \underline{c_{((0))}}$, $b = (\underline{\phi^{Q^1}_{1,c}})^{-1}(c)$, then $V^{\text{Coll}(\omega,\xi)}$ satisfies the following:

   (a) If α is ω_1-represented by both T and T', then $((\underline{j^T_{1,c}})^K(K|\xi), (\underline{j^{Q^1,T}_{1,c}}(b)) \sim_{DJ} ((\underline{j^{T'}_{1,c}})^K(K|\xi), (\underline{j^{Q^1,T'}_{1,c}}(b))$. Here $(\underline{j^U_{1,c}})^K$ stands for the direct limit of $(\underline{j^{Z,Z'}_{1,c}})^K$ for Z, Z' finite subtrees of U, Z a finite subtree of Z', and $(\underline{j^{Q^1,U}_{1,c}})^K$ stands for the tail of the direct limit map from $(\underline{j^{Q^1}_{1,c}})^K(K)$ to $(\underline{j^U_{1,c}})^K(K)$.

(b) Let $F(\alpha) = \pi_{(\underline{j^T_{1,c}})^K(K|\xi),\infty}((\underline{j^{Q^1,T}_{1,c}})^K(b))$ for α represented by T. Then $\sup_{\alpha<\omega_1} F(\alpha) = \pi_{K|\xi,\infty}(c)$.

3. $\Gamma(R^2)$ contains the following axiom: if ξ is a cardinal and strong cutpoint, $e \in \{0,1\}$, $c = \underline{c_{((0))}}$, $b = (\underline{\phi^{Q^{2e}}_{1,c}})^{-1}(c)$, then $V^{\mathrm{Coll}(\omega,\xi)}$ satisfies the following:

   (a) If α is u_2-represented by both T and T', then $((\underline{j^T_{2,c}})^K(K|\xi), (\underline{j^{Q^{2e},T}_{2,c}}(b)) \sim_{DJ} ((\underline{j^{T'}_{2,c}})^K(K|\xi), (\underline{j^{Q^{2e},T'}_{2,c}}(b))$. Here $(\underline{j^U_{2,c}})^K$ stands for the direct limit of $(\underline{j^{Z,Z'}_{2,c}})^K$ for Z, Z' finite subtrees of U, Z a finite subtree of Z', and $(\underline{j^{Q^{2e},U}_{2,c}})^K$ stands for the tail of the direct limit map from $(\underline{j^{Q^{2e}}_{2,c}})^K(K)$ to $(\underline{j^U_{2,c}})^K(K)$.

   (b) Let $F(\alpha) = \pi_{(\underline{j^T_{2,c}})^K(K|\xi),\infty}((\underline{j^{Q^{2e},T}_{2,c}})^K(b))$ for α represented by T. Then $\sup_{\alpha<u_2} F(\alpha) = \pi_{K|\xi,\infty}(c)$.

In the next lemma, we denote $\mathbf{y}^1 = (((0),-1), Q^1, ((1,(0),\emptyset))) \in \mathrm{desc}(R^1)$, $\mathbf{B}^1 = (\mathbf{y}^1, \mathrm{id}_{Q^1,*}) \in \mathrm{desc}(R^1, Q^1, Q^0)$, $\mathbf{y}^{2e} = (((0),-1), Q^{2e}, ((2,((0)),\{(0)\}))) \in \mathrm{desc}(R^2)$, $\mathbf{B}^{2e} = (\mathbf{y}^{2e}, \mathrm{id}_{Q^{2e},*}) \in \mathrm{desc}(R^2, Q^{2e}, Q^0)$ for $e \in \{1,2\}$. Note that if Γ is a level-3 EM blueprint, $d \in \{0,1\}$, then $\Gamma(R^1)$ contains the axiom

$$\underline{c_{((0))}} = \underline{c_{\mathbf{y}^1}} = \underline{c^{Q^1}_{\mathbf{B}^1}}$$

and $\Gamma(R^2)$ contains the axiom

$$\underline{c_{((0))}} = \underline{c_{\mathbf{y}^{20}}} = \underline{c_{\mathbf{y}^{21}}} = \underline{c^{Q^{20}}_{\mathbf{B}^{20}}} = \underline{c^{Q^{21}}_{\mathbf{B}^{21}}}.$$

Lemma 5.29. *Suppose Γ is a level-3 EM blueprint. Suppose $d \in \{1,2\}$, T is a finite level ≤ 2 tree.*

1. $\mathcal{M}^T_{\Gamma,R^d,*} = \mathcal{M}_{\Gamma,R^d\otimes T}$, $\phi^T_{\Gamma,R^d,*} = (\mathrm{id}_{R^d\otimes T})^{R^d,T}_\Gamma$.

2. *If Q^1 is a subtree of T, then* $j^{Q^1,T}_{\Gamma,R^1} \circ (\phi^{Q^1}_{\Gamma,R^1,*})^{-1}(c_{\Gamma,R^1,((0))}) = c_{\Gamma,R^1\otimes T,\mathbf{B}^1}$.

3. *For $e \in \{1,2\}$, if Q^{2e} is a subtree of T, then* $j^{Q^{2e},T}_{\Gamma,R^2} \circ (\phi^{Q^{2e}}_{\Gamma,R^2,*})^{-1}(c_{\Gamma,R^2,((0))}) = c_{\Gamma,R^2\otimes T,\mathbf{B}^{2e}}$.

Proof. 1. Put $Y = R^d$, $c = c_{\Gamma,Y,((0))}$, $R = Y \otimes T$, $\rho = \mathrm{id}_R$ factoring (R,Y,T), $\psi = \mathrm{id}_{Y,*}$ factoring (Y,R). We only prove the typical case when $d = 2$. Put $\mathbf{y} = (((0)), Q^0, \overrightarrow{(e,x,W)}) \in \mathrm{desc}(Y)$. We have to show that

$$\mathrm{ran}(\phi^T_{\Gamma,Y,*}) = \mathrm{ran}(\rho^{Y,T}_\Gamma).$$

The $\subseteq$ direction: If $a \in \mathcal{M}_{\Gamma,Y}$, then $j^T_{\Gamma,Y}(a) = \rho^{Y,T}_\Gamma \circ \psi^R_\Gamma(a)$. If Q is finite, π factors $(Q^0, T \otimes Q)$, then $\pi' =_{\text{DEF}} \text{id}_{T\otimes Q,*} \circ \pi$ factors $(Q^0, (T \otimes Q) \otimes Q^0)$ and $(\mathbf{y}, \pi') \in \text{dom}(Y \otimes (T \otimes Q))$. Hence,

$$(\underline{\pi^{T\otimes Q}}(\underline{c_{((0))}}))^{\mathcal{M}_{\Gamma,Y}} = c_{\Gamma,Y,(\mathbf{y},\pi')} = \rho^{Y,T}_\Gamma(c_{\Gamma,R,\iota^{-1}_{Y,T,Q}(\mathbf{y},\pi')}).$$

If π factors $(Q^{2e}, T\otimes Q)$, $e \in \{0,1\}$, then $\pi' =_{\text{DEF}} \text{id}_{T\otimes Q,*} \circ \pi$ factors $(Q^{2e}, (T\otimes Q) \otimes Q^{2e})$ and $(\mathbf{y}^{2e}, \pi') \in \text{dom}(Y \otimes (T \otimes Q))$. Argue similarly.

The $\supseteq$ direction: By definition.

2,3. Simple computation. □

Lemma 5.30. *Assume $\mathbf{\Delta}^1_2$-determinacy. Suppose Γ is an iterable, weakly remarkable level-3 EM blueprint. The following are equivalent:*

1. *Γ is remarkable.*
2. *The map $\gamma \mapsto c_{\Gamma,\gamma}$ is continuous.*
3. *There exist $\gamma_0, \gamma_1, \gamma_2$ such that for $d \in \{0,1,2\}$, $\text{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]}(\gamma_d) = u_d$ and*

$$c_{\Gamma,\gamma_d} = \{c_{\Gamma,\beta} : \beta < \gamma_d\}.$$

In particular, if $\mathbf{\Pi}^1_3$-determinacy holds, then $0^{3\#}$ is remarkable, and hence the map $\gamma \mapsto c^{(3)}_\gamma$ is continuous.

Proof. $1 \Rightarrow 2$: Suppose $\gamma < \boldsymbol{\delta}^1_3$ is a limit of limit ordinals. By Lemma 4.79, there exists a Π^1_3-wellfounded tree Y such that $\gamma = [\![((0))]\!]_Y$.

Case 1: $\text{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]}(\gamma) = \omega$.

Then R^0 is a subtree of Y and $A =_{\text{DEF}} \{a \in \omega^{<\omega} : a <_{BK} ((0)), (a) \in \text{dom}(Y)\}$ has limit order type. By indiscernability, $c^*_{\Gamma,Y,((0))} = \sup_{a\in A} c^*_{\Gamma,Y,(a)}$. By weak remarkability, for $a \in A$, $\mathcal{M}^*_{\Gamma,Y} \restriction c^*_{\Gamma,Y,(a)} = \mathcal{M}^*_{\Gamma,Y\restriction(a)}$. By remarkability, $\pi_{\mathcal{M}^*_{\Gamma,Y},\infty}$ is continuous at $c^*_{\Gamma,Y,((0))}$. It follows that $c_{\Gamma,\gamma} = \sup_{\beta<\gamma} c_{\Gamma,\beta}$.

Case 2: $\text{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]}(\gamma) = u_d$, $d \in \{1,2\}$.

Then R^d is a subtree of Y. Let $F(\alpha) = [\![\mathbf{B}^1]\!]_{Y\otimes T}$ for $\alpha < u_d$ represented by T. Then $\sup_{\alpha<u_d} F(\alpha) = \gamma$. By remarkability, Lemma 5.29 and absoluteness, $\sup_{\alpha<u_d} c_{\Gamma,F(\alpha)} = c_\gamma$.

$2 \Rightarrow 3$: Trivial.

$3 \Rightarrow 1$: By Lemma 4.79, there exist Π^1_3-wellfounded trees Y^d for $d \in \{0,1,2\}$ such that $\gamma_d = [\![((0))]\!]_Y$. Reverse the argument in $1 \Rightarrow 2$. □

Definition 5.31 (Level ≤ 2 correctness)**.** A level-3 EM blueprint Γ is *level ≤ 2 correct* iff for each finite level-3 tree Y, for each $y \in \mathrm{dom}(Y)$, putting $X = Y_{\mathrm{tree}}(y)$, $\Gamma(Y)$ contains the following axiom:

If $c = \underline{c_y}$, $b = (\underline{\phi^X_{X,c}})^{-1}(c)$, $\xi > c$ is a cardinal and strong cutpoint, then $V^{\mathrm{Coll}(\omega,\xi)}$ satisfies the following:

1. If $\vec{\alpha} = ({}^d\alpha_x)_{(d,x)\in\mathrm{dom}(X)}$ is represented by both T and T', then $((\underline{j^T_{X,c}})^K(K|\xi), (\underline{j^{X,T}_{X,c}})^K(b)) \sim_{DJ} ((\underline{j^{T'}_{X,c}})^K(K|\xi), (\underline{j^{X,T'}_{X,c}})^K(b))$. Here $(\underline{j^U_{X,c}})^K$ stands for the direct limit of $(\underline{j^{Z,Z'}_{X,c}})^K$ for Z, Z' finite subtrees of U, Z a finite subtree of Z', and $(\underline{j^{X,U}_{X,c}})^K$ stands for the tail of the direct limit map from $(\underline{j^X_{X,c}})^K(K)$ to $(\underline{j^U_{X,c}})^K(K)$.
2. Let $F(\vec{\alpha}) = \pi_{(j^T_{X,c})^K(K|\xi),\infty}((\underline{j^{X,T}_{X,c}})^K(b))$ for $\vec{\alpha}$ represented by T. Then $[F]_{\mu^X} = \pi_{K|\xi,\infty}(c)$.

Lemma 5.32. *Suppose Γ is a level-3 EM blueprint, Y is a finite level-3 tree, T is a finite level ≤ 2 tree, $y \in \mathrm{dom}(Y)$, $\mathbf{y} = (y, X, \overrightarrow{(e, x, W)}) \in \mathrm{desc}(Y)$, $\mathbf{B} = (\mathbf{y}, \mathrm{id}_{X,*}) \in \mathrm{desc}(Y, X, Q^0)$. Then*

1. *$\mathcal{M}^T_{\Gamma,Y,y} = \mathcal{M}_{\Gamma,Y\otimes_y T}$, $\phi^T_{\Gamma,Y,y} = (\mathrm{id}_{Y\otimes_y T})^{Y,T}_\Gamma$, where $\mathrm{id}_{Y\otimes_y T}$ factors $(Y \otimes_y T, Y, T)$.*
2. *If X is a subtree of T, then $j^{X,T}_{\Gamma,Y} \circ (\phi^X_{\Gamma,Y,y})^{-1}(c_{\Gamma,Y,y}) = c_{\Gamma,Y\otimes_y X,\mathbf{B}}$.*

Proof. 1. Put $c = c_{\Gamma,Y,y}$, $R = Y \otimes_y T$, $\rho = \mathrm{id}_R$, $\psi = \mathrm{id}_{Y,*}$ factoring (Y, R), $Y[y] = (X, \overrightarrow{(e, x, W)})$, $\mathbf{y} = (y, X, \overrightarrow{(e, x, W)})$. We have to show that

$$\mathrm{ran}(\phi^T_{\Gamma,Y,y}) = \mathrm{ran}(\rho^{Y,T}_\Gamma).$$

The $\subseteq$ direction: If $a \in \mathcal{M}_{\Gamma,Y}$, then $j^T_{\Gamma,Y}(a) = \rho^{Y,T}_\Gamma \circ \psi^R_\Gamma(a)$. If Q is finite, π factors $(X, T \otimes Q)$, then $\pi' =_{\mathrm{DEF}} \mathrm{id}_{T\otimes Q,*} \circ \pi$ factors $(X, (T \otimes Q) \otimes X)$ and $(\mathbf{y}, \pi') \in \mathrm{dom}(Y \otimes (T \otimes Q))$. $\iota^{-1}_{Y,T,Q}(\mathbf{y}, \pi')$ is of the form $(\mathbf{B}, \tau)$ where $\mathbf{B} = (\mathbf{y}, \varphi) \in \mathrm{dom}(Y \otimes_y T)$. Hence,

$$(\underline{\pi^{T\otimes Q}}(\underline{c_y}))^{\mathcal{M}_{\Gamma,Y}} = c_{\Gamma,Y,(\mathbf{y},\pi')} = \rho^{Y,T}_\Gamma(c_{\Gamma,R,(\mathbf{B},\tau)}).$$

The $\supseteq$ direction: If $\mathbf{B} \in \mathrm{dom}(Y \otimes Q^0)$, then $c^T_{\Gamma,Y,\mathbf{B}} = j^T_{\Gamma,Y}(c_{\Gamma,Y,\psi^{-1}(\mathbf{B})})$. If $\mathbf{B} = (\mathbf{y}, \pi) \in \mathrm{dom}(R)$, then $c^T_{\Gamma,Y,\mathbf{B}} \in \mathrm{ran}(\phi^T_{\Gamma,Y,y})$ by definition.

2. Set $X = T$ in part 1. □

It is straightforward to compute that if $Y, y, \mathbf{y}, \mathbf{B}$ are as in the assumption of Lemma 5.32, then

1. if $\vec{\alpha} = ({}^d\alpha_x)_{(d,x)\in\mathrm{dom}(X)}$ is represented by both T and T', then $[\![\mathbf{B}]\!]_{Y\otimes_y T} = [\![\mathbf{B}]\!]_{Y\otimes_y T'}$;
2. letting $G(\vec{\alpha}) = [\![\mathbf{B}]\!]_{Y\otimes_y T}$ for $\vec{\alpha}$ represented by T, then $[\![y]\!]_Y = [G]_{\mu^X}$.

From Lemmas 5.32, 5.27 and absoluteness, we conclude:

Lemma 5.33. *Assume $\boldsymbol{\Delta}^1_2$-determinacy. Suppose Γ is an iterable level-3 EM blueprint. Then the following are equivalent.*

1. *Γ is level ≤ 2 correct.*
2. *For any potential partial level ≤ 2 tower $(X, \overrightarrow{(e,x,W)})$ of continuous type, if $F \in (\boldsymbol{\delta}^1_3)^{(X,\overrightarrow{(e,x,W)})\uparrow}$, then*

$$c_{\Gamma,X,[F]_{\mu^X}} = [\vec{\alpha} \mapsto c_{\Gamma,F(\vec{\alpha})}]_{\mu^X}.$$

3. *For any potential partial level ≤ 2 tower $(X, \overrightarrow{(e,x,W)})$ of continuous type, there exists $F \in (\boldsymbol{\delta}^1_3)^{(X,\overrightarrow{(e,x,W)})\uparrow}$ satisfying*

$$c_{\Gamma,X,[F]_{\mu^X}} = [\vec{\alpha} \mapsto c_{\Gamma,F(\vec{\alpha})}]_{\mu^X}.$$

In particular, if $\boldsymbol{\Pi}^1_3$-determinacy holds, then $0^{3\#}$ is level ≤ 2 correct, and hence, if $F \in (\boldsymbol{\delta}^1_3)^{(X,\overrightarrow{(e,X,W)})\uparrow}$, then

$$c^{(3)}_{X,[F]_{\mu^X}} = [\vec{\alpha} \mapsto c^{(3)}_{F(\vec{\alpha})}]_{\mu^X}.$$

Theorem 5.34. *Assume $\boldsymbol{\Pi}^1_3$-determinacy. Then $0^{3\#}$ is the unique iterable, remarkable, level ≤ 2 correct level-3 EM blueprint.*

Proof. It remains to show uniqueness. Suppose Γ, Γ' are both iterable remarkable level-3 EM blueprints. We carry out a "comparison" between Γ and Γ'. By Corollary 2.15, the function $\gamma \mapsto (c_{\Gamma,\gamma}, c_{\Gamma',\gamma})$ is $\Sigma^1_4(\Gamma,\Gamma')$ in the codes, and hence belongs to $L[T_3,\Gamma,\Gamma']$. By Lemma 5.30, there is a club $C \in L[T_3,\Gamma,\Gamma']$ such that $\gamma = c_{\Gamma,\gamma} = c_{\Gamma',\gamma}$ for any $\gamma \in C$. By Lemma 5.33, if $\gamma \in [C]^{(Q,(d,q,P))\uparrow}$, then $\gamma = c_{\Gamma,Q,\gamma} = c_{\Gamma',Q,\gamma}$.

Suppose R is a finite level-3 tree. Let $\vec{\gamma} \in [C]^{R\uparrow}$. By Lemma 4.79, we can find a Π^1_3-wellfounded Y extending R so that $[\![\emptyset]\!]_Y \in C$ and for any $r \in \mathrm{dom}(R)$, $\gamma_r = [\![r]\!]_Y$. Then $(\mathcal{M}^*_{\Gamma,Y})_\infty = (\mathcal{M}^*_{\Gamma',Y})_\infty = M^-_{2,\infty}|c^{(3)}_{[\![\emptyset]\!]_Y+\omega}$ and for any $r \in \mathrm{dom}(R)$, $\pi_{\mathcal{M}^*_{\Gamma,Y},\infty}(c^*_{\Gamma,Y,r}) = \pi_{\mathcal{M}^*_{\Gamma',Y},\infty}(c^*_{\Gamma',Y,r}) = \gamma_r$. This ensures that $(\mathcal{M}^*_{\Gamma,Y}; (c^*_{\Gamma,Y,r})_{r\in\mathrm{dom}(R)})$ is elementarily equivalent to $(\mathcal{M}^*_{\Gamma',Y}; (c^*_{\Gamma',Y,r})_{r\in\mathrm{dom}(R)})$. Hence, $\Gamma(R) = \Gamma'(R)$. □

The existence of an iterable, remarkable, level ≤ 2 correct level-3 EM blueprint is a purely syntactical definition of a large cardinal. The minimum background assumption to make sense of it is $\mathbf{\Delta}^1_2$-determinacy. However, its existence and uniqueness is proved under boldface $\mathbf{\Pi}^1_3$-determinacy. It is unclear if the assumption of boldface $\mathbf{\Pi}^1_3$-determinacy can be weakened, at least to $\mathbf{\Delta}^1_3$-determinacy+Π^1_3-determinacy. To draw a complete analogy with the level-1 sharp, one would naturally ask

Question 5.35. Assume $\mathbf{\Delta}^1_2$-determinacy. Are the following equivalent?

1. There is an iterable, remarkable, level ≤ 2 correct level-3 EM blueprint.
2. There is an (ω_1, ω_1)-iterable $M_2^{\#}$.
3. Π^1_3-determinacy.

Theorem 5.36. *Assume $\mathbf{\Delta}^1_2$-determinacy. If there is an iterable, remarkable, level ≤ 2 correct level-3 EM blueprint, then Π^1_3-determinacy holds.*

Proof. Let Γ be an iterable, remarkable, level ≤ 2 correct level-3 EM blueprint. Suppose $A \subseteq \mathbb{R}$ is Π^1_3 and G is the game on ω with payoff set A. Let $(R_s)_{s \in \omega^{<\omega}}$ be an effective regular level-3 system such that $x \in A \leftrightarrow R_x =_{\mathrm{DEF}} \cup_{n<\omega} R_{x \restriction n}$ is Π^1_3-wellfounded. By iterability of Γ, $\mathcal{M}^*_{\Gamma,R^0}$ is a Π^1_3-iterable mouse. Working in $\mathcal{M}^*_{\Gamma,R^0}$, define the auxiliary game $H(\underline{c_{((0))}})$ where in rounds $2n$ and $2n+1$, I plays $x(2n) \in \omega, \gamma_n \in \underline{c_{((0))}}$, II plays $x(2n+1)$. Player I is said to follow the rules at stage k iff letting $r_n \in \mathrm{dom}(R_{x \restriction n+1}) \setminus \mathrm{dom}(R_{x \restriction n})$ for $n < k$, then for any $n < m < k$, $r_n = (r_m)^- \to \gamma_m < \underline{j^{(R_{x \restriction n+1})_{\mathrm{tree}(r_n)},(R_{x \restriction m+1})_{\mathrm{tree}(r_m)}}}(\gamma_n)$. Players I wins $H(\underline{c_{((0))}})$ iff he follows the rules at every finite stage k. $H(\underline{c_{((0))}})$ is a closed game for Player I, hence determined in $\mathcal{M}^*_{\Gamma,R^0}$.

Case 1: $\mathcal{M}^*_{\Gamma,R^0} \models$"$\sigma$ is a winning strategy for Player I in $H(\underline{c_{((0))}})$".

Let σ^* be the strategy for Player I in G obtained by following σ and ignoring the auxiliary moves γ_n. If x is (in V) a complete run according to σ^*, then $R_x \in p[\underline{S_3}^{\mathcal{M}^*_{\Gamma,R^0}}]$. By Σ^1_4-correctness of set-generic extensions of $\mathcal{M}^*_{\Gamma,Y}$, $p[\underline{S_3}^{\mathcal{M}^*_{\Gamma,R^0}}] \subseteq p[S_3]$. Hence $x \in A$. This shows σ^* is winning for Player I.

Case 2: $\mathcal{M}^*_{\Gamma,R^0} \models$"$\sigma$ is a winning strategy for Player II in $H(\underline{c_{((0))}})$".

We define a strategy σ^* for II in G as follows: if $\mathrm{lh}(s) = 2n+1$, then $\sigma(s) = a$ iff the formula

$$\sigma((s(0), \underline{c_{r_1}}), s(1), \ldots, (s(2n), \underline{c_{r_n}})) = a$$

belongs to $\Gamma(R_{s \restriction n+1})$, where $r_k \in \mathrm{dom}(R_{s \restriction k+1}) \setminus \mathrm{dom}(R_{s \restriction k})$. We claim that σ^* is a winning strategy for Player II. Suppose otherwise and x is a complete run according to σ^* but $x \in A$. Then R_x is Π^1_3-wellfounded. Let $\mathcal{N} = \mathcal{M}^*(\Gamma, R^+_x)$, where R^+_x extends R_x, $\text{dom}(R^+_x) = \text{dom}(R_x) \cup \{((1))\}$, $R^+_x((1))$ has degree 0. By iterability of Γ, $\mathcal{N}$ is a Π^1_3-iterable mouse. By coherency of Γ, $\mathcal{N} \models$"σ is a winning strategy for Player II in $(H(\underline{c_{((1))}}))^{\mathcal{N}}$". However, $x \oplus (\underline{c_{r_k}}^{\mathcal{N}})_{k<\omega}$ is a complete run according to σ which is legal according to the rules of $(H(\underline{c_{((1))}}))^{\mathcal{N}}$. In V, the tree of attempts of building a complete run according to σ which is legal according to the rules of $(H(\underline{c_{((1))}}))^{\mathcal{N}}$ is illfounded. By absoluteness of wellfoundedness, $\mathcal{N}$ can see such a complete run. Contradiction. □

Theorem 5.36 is a generalization of Martin's theorem that $0^{\#}$ implies Π^1_1-determinacy. It proves $1 \Rightarrow 3$ in Question 5.35. For a real x, a level-3 EM blueprint over x is the obvious generalization of Definition 5.14, i.e., a function Γ that sends R to $\Gamma(R)$, a complete consistent $\mathcal{L}^{\underline{x},R}$-theory containing the additional axioms "$\underline{x} \in \mathbb{R}$" and "$\underline{x}(i) = j$" when $x(i) = j$. Assume $\boldsymbol{\Pi}^1_3$-determinacy, $x^{3\#}$ is the unique iterable, remarkable, level ≤ 2 correct level-3 EM blueprint over x. Thus, in combination with Neeman [37, 38] and Woodin [43], we reach an affirmative answer to the boldface version of Question 5.35.

Theorem 5.37. *Assume $\boldsymbol{\Delta}^1_2$-determinacy. The following are equivalent.*

1. *For all $x \in \mathbb{R}$, there is an iterable, remarkable, level ≤ 2 correct level-3 EM blueprint over x.*
2. *For all $x \in \mathbb{R}$, there is an (ω_1, ω_1)-iterable $M^{\#}_2(x)$.*
3. *$\boldsymbol{\Pi}^1_3$-determinacy.*

Recall the basic fact that L is the Skolem hull of the class of Silver indiscernibles. We exploit its higher level analog. Clearly, $M^-_{2,\infty}$ is not the Skolem hull of $\{c^{(3)}_{X,\alpha} : \alpha \text{ respects } X\}$, as by remarkability, the Skolem hull contains only countably many ordinals below $c^{(3)}_\omega$. The missing part will be generated by ordinals below u_ω in a specific way.

Lemma 5.38. *Suppose $\mathcal{N}$ is Π^1_3-iterable and satisfies $0^{3\#}(\emptyset)$. Then for any limit ordinal $\alpha \in \mathcal{N}$,*

$$\pi_{\mathcal{N},\infty}(\alpha) = \sup\{\pi_{\mathcal{N}^T,\infty}(\beta) : T \textit{ is } \Pi^1_2\textit{-wellfounded, } \beta < j^T_{\mathcal{N}}(\alpha)\}.$$

Proof. Let the *universality of level ≤ 2 ultrapowers axiom* be the following:

> If α is a limit ordinal and $\xi > \alpha$ is a cardinal and cutpoint, then $V^{\mathrm{Coll}(\omega,\xi)}$ satisfies $\pi_{K|\xi,\infty}(\alpha) = \sup\{\pi_{(j^T)^K(K|\xi),\infty}(\beta) : T$ is Π^1_2-wellfounded, $\beta < (\underline{j^T})^K(\alpha)\}$, where $(\underline{j^T})^K$ denotes the direct limit of $(\underline{j^{T'}})^K$ for T' a finite subtree of T.

By elementarity and absoluteness, it suffices to show that $M^-_{2,\infty}$ is a model of this axiom. Fix $\alpha < \boldsymbol{\delta}^1_3$.

Case 1: $\mathrm{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]}(\alpha) = \omega$.

Then $M^-_{2,\infty} \models$ "$\mathrm{cf}(\alpha)$ is not measurable". So when $\alpha < \xi < \boldsymbol{\delta}^1_3$, $(M^-_{2,\infty})^{\mathrm{Coll}(\omega,\xi)} \models$ "$\pi_{K|\xi,\infty}$ is continuous at α".

Case 2: $\mathrm{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]}(\alpha) = u_1$.

Let $F : u_1 \to \alpha$ be order preserving and cofinal, $F \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$. Let $z \in \mathbb{R}$ so that F is Δ_1-definable over $L_{\kappa^z_3}[T_2, z]$ from $\{T_2, z\}$. Let $\mathcal{P} \in \mathcal{F}_{2,z}$ and $\bar{\alpha}, \bar{F} \in \mathcal{P}$ so that $\pi_{\mathcal{P},\infty}(\bar{\alpha}, \bar{F}) = (\alpha, F)$. Let $\mathcal{Q} = (L[\underline{S_3}])^{\mathcal{P}}$. So for any Π^1_2-wellfounded T, $\mathcal{Q}^T = (L[\underline{S_3}])^{\mathcal{P}^T}$ and $j^T_{\mathcal{Q}} = j^T_{\mathcal{P}} \restriction \mathcal{Q}^T$. By absoluteness, it suffices to show that

$$\alpha = \sup\{\pi_{\mathcal{P}^T,\infty}(\beta) : T \text{ is } \Pi^1_2\text{-wellfounded}, \beta < j^T_{\mathcal{P}}(\alpha)\}.$$

This would follow from

$$u_1 = \sup\{\pi_{\mathcal{P}^T,\infty}(\beta) : T \text{ is } \Pi^1_2\text{-wellfounded}, \beta < (\underline{u_1})^{\mathcal{P}^T}\}.$$

The last equality is because $\{\pi_{\mathcal{P}^T,\infty} \circ j^{T',T}_{\mathcal{P}}((\underline{\mathrm{seed}^{T'}_{(1,(0))}})^{\mathcal{P}}) : T$ is Π^1_2-wellfounded, T' is a finite subtree of T, $(0) \in {}^1T'\}$ is a subset of the right hand side and has order type ω_1.

Case 3: $\mathrm{cf}^{\mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]}(\alpha) = u_2$.

Similar to Case 2. □

Definition 5.39. If $\mathcal{N}$ is a structure that satisfies Axioms 1-3 in Definition 5.14 and the universality of level ≤ 2 ultrapowers axiom, then

$$\mathcal{G}_{\mathcal{N}}$$

is the direct system consisting of models $\mathcal{N}^T$ for which T is a Π^1_2-wellfounded level ≤ 2 tree and maps $\pi^{T,T'}_{\mathcal{N}} : \mathcal{N}^T \to \mathcal{N}^{T'}$ for π minimally factoring T, T'. Define

$$\mathcal{N}_\infty = \operatorname{dirlim} \mathcal{G}_{\mathcal{N}},$$
$$\pi_{\mathcal{N}^T,\mathcal{N}_\infty} : \mathcal{N}^T \to \mathcal{N}_\infty \text{ is tail of the direct limit map.}$$

If in addition, $\mathcal{N}$ is countable Π^1_3-iterable mouse, then $\mathcal{G}_{\mathcal{N}}$ is a subsystem of $\mathcal{I}_{\mathcal{N}}$. By Lemma 5.38, $\mathcal{G}_{\mathcal{N}}$ is dense in $\mathcal{I}_{\mathcal{N}}$, so there is no ambiguity in the notation $\mathcal{N}_\infty$:

Lemma 5.40. *Suppose $\mathcal{N}$ is a countable Π^1_3-iterable mouse and satisfies Axioms 1-3 in Definition 5.14 and the universality of level ≤ 2 ultrapowers axiom. If $\pi : \mathcal{N} \to \mathcal{P}$ is an iteration map, then there exist a Π^1_2-wellfounded T and $\psi : \mathcal{P} \to \mathcal{N}^T$ such that $\psi \circ \pi = j^T_{\mathcal{N}}$ and ψ is essentially an iteration map.*

The direct system $\mathcal{G}_{\mathcal{N}}$ is useful even when $\mathcal{N}$ is not Π^1_3-iterable. In the proof of the level-4 Kechris-Martin theorem in Section 7, we will inevitably have to deal with partially iterable level-3 EM blueprints. The structure $\mathcal{N}$ will be the EM model built from a partially iterable level-3 EM blueprint. The advantage of the (possibly illfounded) direct limit $\mathcal{N}_\infty$ is that the order type of its ordinals is easily codable by a subset of u_ω. If X is a finite level ≤ 2 tree, $a \in \mathcal{N}$, $\vec{\beta} = ({}^d\beta_x)_{(d,x)\in\mathrm{dom}(X)}$ is represented by both T and T', then $\pi_{\mathcal{N}^T,\infty} \circ j^{X,T}_{\mathcal{N}}(a) = \pi_{\mathcal{N}^{T'},\infty} \circ j^{X,T'}_{\mathcal{N}}(a)$. We can define

$$\pi_{\mathcal{N},X,\vec{\beta},\infty}(a) = \pi_{\mathcal{N}^T,\infty} \circ j^{X,T}_{\mathcal{N}}(a)$$

for $\vec{\beta}$ represented by T. So

$$\mathcal{N}_\infty = \{\pi_{\mathcal{N},X,\vec{\beta},\infty}(a) : a \in \mathcal{N}, X \text{ finite level} \leq 2 \text{ tree}, \vec{\beta} \in [\omega_1]^{X\uparrow}\}.$$

Essentially, the inner model theoretic comparison between mice is replaced by the comparison between Π^1_2-wellfounded level ≤ 2 trees in Theorem 4.57.

A *level* ≤ 3 *code* for an ordinal in $\boldsymbol{\delta}^1_3$ is of the form

$$(R, \vec{\gamma}, X, \vec{\beta}, \ulcorner\sigma\urcorner)$$

such that R is a finite level-3 tree, $\vec{\gamma}$ respects R, X is a finite level ≤ 2 tree, $\vec{\beta}$ respects X, and σ is an $\mathcal{L}^R$-Skolem term for an ordinal. It codes the ordinal

$$\left|(R, \vec{\gamma}, X, \vec{\beta}, \ulcorner\sigma\urcorner)\right| = \pi_{\mathcal{M}^*_{0^{3\#},R},X,\vec{\beta},\infty}(\sigma^{\mathcal{M}^*_{0^{3\#},R}}((\underline{c_r})_{r\in\mathrm{dom}(R)})).$$

By Lemmas 5.40, every ordinal in $\boldsymbol{\delta}^1_3$ has a level ≤ 3 code. The evaluation function on level ≤ 3 codes is $\Sigma^1_4(0^{3\#})$, and hence definable over $M^-_{2,\infty}(0^{3\#})$.

5.5 Level-3 indiscernibles

If $\vec{\gamma}$ respects a level-3 tree R, define

$$c_{\vec{\gamma}} = (c^{(3)}_{R_{\mathrm{tree}}(r),\gamma_r})_{r\in\mathrm{dom}(R)}$$

which strongly respects R. Combined with Lemma 4.61, this leads to the order of level-3 indiscernibles for $M_{2,\infty}^-$: $c^{(3)}_{Q,\gamma} < c^{(3)}_{Q',\gamma'}$ iff letting $(\gamma_i)_{1\le i\le k}$ be the Q-approximation sequence of γ and $(\gamma'_i)_{1\le i\le k'}$ be the Q'-approximation sequence of γ', then $(\gamma_i)_{1\le i\le k}{}^\frown(-1) <_{BK} (\gamma'_i)_{1\le i\le k'}{}^\frown(-1)$. We prove the general remarkability property of $0^{3\#}$ based on this order.

Lemma 5.41 (General remarkability)**.** *Suppose $\vec{\gamma}$ and $\vec{\gamma}'$ both respect a finite level-3 tree R. Suppose $r \in \operatorname{dom}(R)$ and for any $s \prec^R r$, $\gamma_s = \gamma'_s$. Then for any $\mathcal{L}$-Skolem term τ,*

$$M_{2,\infty}^- \models \tau(c_{\vec{\gamma}}) < c^{(3)}_{R_{\rm tree}(r),\gamma_r} \to \tau(c_{\vec{\gamma}}) = \tau(c_{\vec{\gamma}'}).$$

Proof. Assume $\tau^{M_{2,\infty}^-}(c_{\vec{\gamma}}) < c^{(3)}_{R_{\rm tree}(r),\gamma_r}$. If $s \in \operatorname{dom}(R)$ and $r \preceq^R s$ let $l(s)$ be the largest l so that $\langle r \restriction l\rangle^R = \langle s \restriction l\rangle^R$. It is easy to find $\vec{\gamma}^l \in [\boldsymbol{\delta}^1_3]^{R\uparrow}$ for $l \le \operatorname{lh}(r)$ so that $\vec{\gamma}^0 = \vec{\gamma}$, $\vec{\gamma}^{\operatorname{lh}(r)} = \vec{\gamma}'$ and $\gamma^l_s \ne \gamma^{l+1}_s \to (r \preceq^R s \wedge l(s) = l)$. Thus, we may assume a fixed l_0 so that $\gamma_s \ne \gamma'_s$ implies $l(s) = l_0$. The case $l_0 = 0$ is just Lemma 5.24. Assume now $l_0 > 0$. Note that $l(s) = l_0$ also implies that $R_{\rm tree}(s \restriction l_0) = R_{\rm tree}(r \restriction l_0)$. A sliding argument similar to Lemma 5.24 reduces to the special case that $(\operatorname{lh}(s) = \operatorname{lh}(s') = l_0 + 1 \wedge l(s) = l(s') = l_0) \to \gamma_s < \gamma_{s'}$. Let (Y, ρ, ρ') be the amalgamation obtained by Lemma 4.73 so that ρ, ρ' both factor (R, Y) and if $\vec{\delta}, \vec{\delta}' \in [\boldsymbol{\delta}^1_3]^{R\uparrow}$ and $(\operatorname{lh}(s) = \operatorname{lh}(s') = l_0 + 1 \wedge l(s) = l(s') = l_0) \to \delta_s < \delta_{s'}$, then $\vec{\delta} \oplus \vec{\delta}' \in [\boldsymbol{\delta}^1_3]^{Y\uparrow}$, where $\vec{\delta} \oplus \vec{\delta}' = \vec{\epsilon}$, $\epsilon_{\rho(r)} = \delta_r$, $\epsilon_{\rho'(r)} = \delta'_r$. Put $\eta \in D$ iff $c^{(3)}_\eta = \eta$. By indiscernability, we may assume that $\vec{\gamma}, \vec{\gamma}' \in [D]^{R\uparrow}$, so that $c_{\vec{\gamma}} = \vec{\gamma}$, $c_{\vec{\gamma}'} = \vec{\gamma}'$. It is easy to construct $\vec{\delta}^\xi \in [\boldsymbol{\delta}^1_3]^{R\uparrow}$ for $\xi < \gamma_r$ so that $\vec{\gamma} \oplus \vec{\delta}^\xi \in [\boldsymbol{\delta}^1_3]^{Y\uparrow}$ and $\eta < \xi < \gamma_r \to \vec{\delta}^\eta \oplus \vec{\delta}^\xi \in [\boldsymbol{\delta}^1_3]^{Y\uparrow}$. Put $\epsilon^\xi = \tau^{M_{2,\infty}^-}(c_{\vec{\delta}^\xi})$. By indiscernability, it suffices to show that $\epsilon^\eta = \epsilon^\xi$ for some (or equivalently, for any) $\eta < \xi < \gamma_r$. Suppose otherwise. By indiscernability again, either $\eta < \xi < \gamma_r \to \epsilon^\eta > \epsilon^\xi$ or $\eta < \xi < \gamma_r \to \epsilon^\eta < \epsilon^\xi$. The former gives a descending chain of ordinals. The latter implies that $\gamma_r \le \tau^{M_{2,\infty}^-}(c_{\vec{\gamma}})$, contracting to our assumption. □

Recall that if $c < c'$ are consecutive $L[x]$-indiscernibles, then $L[x^\#] \models c' < c^+$. The level-3 version is similar.

Lemma 5.42. *Assume $\boldsymbol{\Pi}^1_3$-determinacy. For any $c^{(3)}_\omega < \xi \in I^{(3)}$, there is an $\mathcal{L}$-Skolem term τ such that $M_{2,\infty}^-(0^{3\#}) \models$ "$\tau(\sup(I^{(3)} \cap \xi), \cdot)$ is a surjection from $\sup(I^{(3)} \cap \xi)$ onto ξ".*

Proof. The evaluation function on level ≤ 3 codes is $\Sigma^1_4(0^{3\#})$, and hence is definable over $M_{2,\infty}^-(0^{3\#})$. If $(R, \vec{\gamma}, X, \vec{\beta}, \ulcorner\sigma\urcorner)$ and $(R, \vec{\gamma}', X, \vec{\beta}, \ulcorner\sigma\urcorner)$ are both level ≤ 3 codes an ordinal below ξ and $\forall r\ (\gamma_r < \xi \to \gamma_r = \gamma'_r)$, then by Lemma 5.41 they must code the same ordinal. This easily defines a surjection from $\sup(I^{(3)} \cap \xi)$ onto ξ in $M_{2,\infty}^-(0^{3\#})$. □

By Lemma 4.61, for any finite level-3 tree R, if $\mathbf{A} \prec \mathbf{A}'$ then "$\underline{c_{\mathbf{A}}} < \underline{c_{\mathbf{A}'}}$" is true in $0^{3\#}(R)$, if $\mathbf{A} \sim \mathbf{A}'$ then "$\underline{c_{\mathbf{A}}} = \underline{c_{\mathbf{A}'}}$" is true in $0^{3\#}(R)$. For notational convenience, if X is a finite level ≤ 2 tree and $\gamma = [F]_{\mu^X}$ is a limit ordinal, define $c^{(3)}_{X,\gamma} = [\vec{\alpha} \mapsto c^{(3)}_{F(\vec{\alpha})}]_{\mu^X}$; define $c^{(3)}_{\emptyset,\boldsymbol{\delta}^1_3} = \boldsymbol{\delta}^1_3$. Ordinals of the form $c^{(3)}_{X,\gamma}$ when $X \neq \emptyset$ are definable from elements in $I^{(3)}$ over $M^-_{2,\infty}$: If the X-approximation sequence of γ is $(\gamma_i)_{1 \leq i \leq k}$, $(Q, (d_i, q_i, P_i)_{i < \mathrm{lh}(\vec{q})})$ is the X-potential partial level ≤ 2 tower induced by γ, $\pi : Q \to X$ is the induced level-2 factoring map, then

1. if γ is of X-discontinuous type, then $c^{(3)}_{X,\gamma} = \pi^X(c^{(3)}_{Q,\gamma_k})$;
2. if γ is of X-continuous type, Q^- is the subtree of Q obtained by removing (d_k, q_k), then $c^{(3)}_{X,\gamma} = \pi^X \circ j^{Q^-,Q}_{\sup}(c^{(3)}_{Q,\gamma_{k-1}})$.

Define $\bar{I}^{(3)}$ =the closure of $I^{(3)}$ under the order topology. Every ordinal in $\bar{I}^{(3)}$ is of the form $c^{(3)}_{X,\gamma}$ where X is finite and $\gamma < \boldsymbol{\delta}^1_3$ is a limit. Thus, if $\mathbf{A} = (\mathbf{r}, \pi, T) \in \mathrm{desc}^{**}(R)$ and $\vec{\gamma}$ strongly respects R, then $c^{(3)}_{T,\gamma_{\mathbf{A}}} \in \bar{I}^{(3)}$ and is a limit point of $I^{(3)}$.

Given $\gamma_0, \ldots, \gamma_n, \gamma'_0, \ldots \gamma'_n \in I^{(3)}$, $\vec{\gamma}$ is a *shift* of $\vec{\gamma}'$ iff there exist a level-3 tree R, nodes $r_0, \ldots, r_n \in \mathrm{dom}(R)$, $\vec{\delta}, \vec{\delta}'$ both respecting R such that $\gamma_i = c^{(3)}_{R_{\mathrm{tree}}(r_i),\delta_i}$, $\gamma'_i = c^{(3)}_{R_{\mathrm{tree}}(r_i),\delta'_i}$ for any $i \leq n$. By indiscernability, if $\vec{\gamma}$ is a shift of $\vec{\gamma}'$, then for any $\mathcal{L}$-formula φ,

$$M^-_{2,\infty} \models \varphi(\vec{\gamma}) \leftrightarrow \varphi(\vec{\gamma}').$$

Lemma 5.43. *Suppose R is a finite level-3 tree, τ is an $\mathcal{L}$-Skolem term, $\vec{\gamma}$ strongly respects R. Suppose $\mathbf{A} = (\mathbf{r}, \pi, T) \in \mathrm{desc}^{**}(R)$. Then*

$$\tau^{M^-_{2,\infty}}(c_{\vec{\gamma}}) < c^{(3)}_{T,\gamma_{\mathbf{A}}} \to \tau^{M^-_{2,\infty}}(c^{(3)}_{\vec{\gamma}}) < \min(I^{(3)} \setminus \sup\{c^{(3)}_{T',\gamma_{\mathbf{A}'}} : \mathbf{A}' = (\mathbf{r}', \pi', T') \prec^R_* \mathbf{A}\}).$$

Proof. Suppose $\tau^{M^-_{2,\infty}}(c_{\vec{\gamma}}) < c^{(3)}_{T,\gamma_{\mathbf{A}}}$. Let $\delta = \min(I^{(3)} \setminus \sup\{c^{(3)}_{T',\gamma_{\mathbf{A}'}} : \mathbf{A}' = (\mathbf{r}', \pi', T') \prec^R_* \mathbf{A}\})$. We shall show that $\{\delta' : c_{\vec{\gamma}}{}^\frown(\delta')$ is a shift of $c_{\vec{\gamma}}{}^\frown(\delta)\}$ is cofinal in $\gamma_{\mathbf{A}}$. From this and indiscernability, $\tau^{M^-_{2,\infty}}(c_{\vec{\gamma}}) < \delta$.

If $\mathbf{r} = \emptyset$, then $\delta = c^{(3)}_{\gamma_{(a)}+\omega}$ where $a = \max_{<_{BK}} R\{\emptyset\}$. So $\{\delta' : c_{\vec{\gamma}}{}^\frown(\delta')$ is a shift of $c_{\vec{\gamma}}{}^\frown(\delta)\}$ is cofinal in $\gamma_{\mathbf{A}} = \boldsymbol{\delta}^1_3$.

Suppose now $\mathbf{r} = (r, Q, \overrightarrow{(d, q, P)}) \neq \emptyset$, $\mathrm{lh}(r) = k$, $\mathrm{ucf}(R(r)) = (d^*, \mathbf{q}^*)$, and if $d^* = 1$ put $q^* = \mathbf{q}^*$, if $d^* = 2$ put $\mathbf{q}^* = (q^*, P^*, \vec{p}^*)$.

Case 1: $\mathbf{r}$ is of discontinuous type, $\langle \mathbf{A} \rangle$ ends with -1.

Let $s = \max_{<_{BK}} R\{r, -\}$ and $\mathbf{s} = (s, Q, \overrightarrow{(d, q, P)})$. Then $\delta = c^{(3)}_{T, \pi^T(\gamma_{\mathbf{s}}) + \omega}$. It is easy to compute that $\{\delta' : c_{\vec{\gamma}} {}^\frown (\delta') \text{ is a shift of } c_{\vec{\gamma}} {}^\frown (\delta)\}$ is cofinal in $\gamma_{\mathbf{A}}$.

Case 2: $\mathbf{r}$ is of discontinuous type, $\langle \mathbf{A} \rangle$ ends with an ordinal.

If either $d_k = 1$ or $d_k = d^* = 2 \wedge \mathbf{q}^* \in \operatorname{desc}({}^2Q)$, let τ factor (Q, T) so that τ agrees with π on $\operatorname{dom}(Q) \setminus \{(d^*, q^*)\}$, $\tau(d^*, q^*) = \operatorname{pred}(\pi, T, (d^*, \mathbf{q}^*))$. Then $\delta = c^{(3)}_{T, \tau^T(\gamma_r) + \omega}$. Otherwise, let U be the subtree of T obtained by removing $\operatorname{pred}(\pi, T, (d^*, \mathbf{q}^*))$ from its domain. Then $\delta = c^{(3)}_{U, \pi^U(\gamma_r) + \omega}$. In either case, $\{\delta' : c_{\vec{\gamma}} {}^\frown (\delta') \text{ is a shift of } c_{\vec{\gamma}} {}^\frown (\delta)\}$ is cofinal in $\gamma_{\mathbf{A}}$.

Case 3: $\mathbf{r}$ is of continuous type.

Similar to Cases 1 and 2. □

Lemma 5.44. *Suppose R is a finite level-3 tree and $\mathbf{A} = (\mathbf{r}, \pi, T) \in \operatorname{desc}^{**}(R)$, $\mathbf{r} \neq \emptyset$, $\vec{\gamma}$ strongly respects R. Then $c^{(3)}_{T, \gamma_{\mathbf{A}}}$ is a cardinal in $M^-_{2,\infty}$.*

Proof. Otherwise, $\tau^{M^-_{2,\infty}}(c_{\vec{\gamma}})$ is a wellordering on $\alpha =_{\text{DEF}} \operatorname{card}^{M^-_{2,\infty}}(c^{(3)}_{T, \gamma_{\mathbf{A}}})$ of order type $c^{(3)}_{T, \gamma_{\mathbf{A}}}$ and $\alpha < c^{(3)}_{T, \gamma_{\mathbf{A}}}$. Put $\beta = \min(I^{(3)} \setminus \sup\{c^{(3)}_{T', \gamma_{\mathbf{A}'}} : \mathbf{A}' = (\mathbf{r}', \pi', T') \prec^R_* \mathbf{A}\})$. By Lemma 5.43, $\alpha < \beta$. By Lemma 5.41, if $\vec{\delta}$ respects R and $\forall s\ (\delta_s < \beta \to \delta_s = \gamma_s)$, then $\tau^{M^-_{2,\infty}}(c_{\vec{\gamma}}) = \tau^{M^-_{2,\infty}}(c_{\vec{\delta}})$, and hence $\gamma_{\mathbf{A}} = \delta_{\mathbf{A}}$. However, it is easy to find such $\vec{\delta}$ satisfying $\delta_{\mathbf{A}} > \gamma_{\mathbf{A}}$. □

6 The boldface level-3 sharp

From now on, we assume $\boldsymbol{\Pi}^1_3$-determinacy. Recall that $\mathbb{L}[T_3] = \bigcup_{x \in \mathbb{R}} L[T_3, x]$. Every subset of $\boldsymbol{\delta}^1_3$ in $\mathbb{L}[T_3]$ is definable over $M^-_{2,\infty}(x)$ for some $x \in \mathbb{R}$. All the results in Section 5 relativize to any given real x. If R is a Π^1_3-wellfounded level-3 tree, $\mathcal{M}_{x^{3\#}, R}$ is the EM model built from $x^{3\#}(R)$. $\mathcal{M}^*_{x^{3\#}, R}$, $\mathcal{M}^{*,T}_{x^{3\#}, R}$, $c^{(3)}_{x, Q, \gamma}$, $c^{(3)}_{x, \gamma}$, $c_{x, \vec{\gamma}}$, $I^{(3)}_x$, $\bar{I}^{(3)}_x$ have obvious meanings. Fixing x, the function $(Q, \gamma) \mapsto c^{(3)}_{x, Q, \gamma}$ is $\Sigma^1_4(x^{3\#})$ in the codes and hence is definable over $M^-_{2,\infty}(x^{3\#})$.

6.1 Homogeneity properties of S_3

A *level ≤ 3 tree* is of the form $R = ({}^0R, {}^1R, {}^2R, {}^3R)$ so that ${}^{\leq 2}R =_{\text{DEF}} ({}^0R, {}^1R, {}^2R)$ is a level ≤ 2 tree and ${}^3R$ is a level-3 tree. If T is a level ≤ 2 tree and Y is a level-3 tree then $T \oplus Y$ denotes the level ≤ 3 tree $({}^0T, {}^1T, {}^2T, Y)$. R is Π^1_3-wellfounded iff ${}^{\leq 2}R$ is Π^1_2-wellfounded and ${}^3R$ is Π^1_3-wellfounded.

Suppose R is a level $\leq$ 3-tree. Define $\operatorname{dom}(R) = \cup_d \{d\} \times \operatorname{dom}({}^dR)$, $\operatorname{desc}(R) = \cup_d \{d\} \times \operatorname{desc}({}^dR)$. If $\vec{\beta}$ respects ${}^{\leq 2}R$ and $\vec{\gamma}$ respects ${}^3R$, define $\vec{\beta} \oplus \vec{\gamma} = \vec{\delta} = ({}^d\delta_t)_{(d,r) \in \operatorname{dom}(R)}$ where ${}^d\delta_r = {}^d\beta_r$ for $(d, r) \in \operatorname{dom}({}^{\leq 2}R)$ and

${}^3\delta_r = \gamma_r$ for $r \in \mathrm{dom}({}^3R)$. Define $A \oplus B = \{\vec{\beta} \oplus \vec{\gamma} : \vec{\beta} \in A, \vec{\gamma} \in B\}$. If $E \subseteq \omega_1$ and $C \subseteq \boldsymbol{\delta}^1_3$, define $[E,C]^{R\uparrow} = [E]^{\leq 2R\uparrow} \oplus [C]^{{}^3R\uparrow}$. $\vec{\delta}$ respects R iff $\vec{\delta} \in [\omega_1, \boldsymbol{\delta}^1_3]^{R\uparrow}$. A finite level ≤ 3 tree R induces a filter μ^R on finite tuples in $\boldsymbol{\delta}^1_3$, originated from the weak partition property of $\boldsymbol{\delta}^1_3$ under AD. μ^R is the higher level analog of the n-fold product of the club filter on ω_1.

Definition 6.1. Assume $\boldsymbol{\Pi}^1_3$-determinacy. Let R be a finite level ≤ 3 tree. We say

$$A \in \mu^R$$

iff there are clubs $E \subseteq \omega_1$, $C \subseteq \boldsymbol{\delta}^1_3$ such that $E, C \in \mathbb{L}[T_3]$ and

$$[E,C]^{R\uparrow} \subseteq A.$$

If Y is a finite level-3 tree, put $A \in \mu^Y$ iff $[\omega_1]^{Q^0\uparrow} \oplus A \in \mu^{Q^0 \oplus Y}$.

μ^R is an $\mathbb{L}[T_3]$-measure, the reason being as follows. Every $A \in \mathbb{L}[T_3]$ is definable over $M^-_{2,\infty}(x)$ from $\{x\}$ for some real x. By indiscernability and remarkability, the section $A^* =_{\mathrm{DEF}} \{\vec{\beta} : \vec{\beta} \oplus c^{(3)}_{x,\vec{\gamma}} \in A\}$ is invariant in $\vec{\gamma} \in [\boldsymbol{\delta}^1_3]^{R\uparrow}$. So $C = \{c^{(3)}_{x,\xi} : \xi < \boldsymbol{\delta}^1_3\}$ and some E deciding the $\mu^{\leq 2R}$-measure of A^* works. μ^R is the product measure on $\mathbb{L}[T_3]$ of $\mu^{\leq 2R}$ and $\mu^{{}^3R}$. Let $j^R = j^{\mu^R}_{\mathbb{L}[T_3]}$ be the ultrapower map from $\mathbb{L}[T_3]$ to $\mathbb{L}[j^R(T_3)]$. For any $x \in \mathbb{R}$, j^R is elementary from $L[T_3, x]$ to $L[j^R(T_3), x]$. By indiscernability and remarkability again, if $\alpha < \boldsymbol{\delta}^1_3$ and $F : [\omega_1, \boldsymbol{\delta}^1_3]^{R\uparrow} \to \alpha$, $F \in \mathbb{L}[T_3]$, then there is $x \in \mathbb{R}$ and $G \in \mathbb{L}_{\boldsymbol{\delta}^1_3}[T_2]$ such that for any $\vec{\gamma} \in [\boldsymbol{\delta}^1_3]^{{}^3R\uparrow}$, for any $\vec{\beta} \in [\omega_1]^{\leq 2R\uparrow}$, $F(\vec{\beta} \oplus \vec{\gamma}) = G(\vec{\beta})$. Therefore,

$$j^R \restriction \boldsymbol{\delta}^1_3 = j^{\leq 2R} \restriction \boldsymbol{\delta}^1_3.$$

For $(d,r) \in \mathrm{dom}(R) \cup \{(3,\emptyset)\}$, let

$$\mathrm{seed}^R_{(d,r)}$$

be the element represented modulo μ^R by the projection map $\vec{\gamma} \mapsto {}^d\gamma_r$. If R is Π^1_3-wellfounded, the direct limit of $j^{R',R''}$ for R' a finite subtree of R'' and R'' a finite subtree of R is wellfounded, and we let $j^R : \mathbb{L}[T_3] \to \mathbb{L}[j^R(T_3)]$ be the direct limit map; if R' is a finite subtree of R then $j^{R',R} : \mathbb{L}[j^{R'}(T_3)] \to \mathbb{L}[j^R(T_3)]$ is the tail of the direct limit map. If $(d,r) \in \mathrm{dom}(R')$, R' is a finite subtree of R, then $\mathrm{seed}^R_{(d,r)} = j^{R',R}(\mathrm{seed}^{R'}_{(d,r)})$. Let

$$\mathrm{seed}^R = (\mathrm{seed}^R_{(d,r)})_{(d,r) \in \mathrm{dom}(R)}.$$

If Y is a Π^1_3-wellfounded level-3 tree, then $j^Y = j^{Q^0 \oplus Y}$, $\mathrm{seed}^Y_y = \mathrm{seed}^{Q^0\oplus Y}_{(3,y)}$, $\mathrm{seed}^Y = (\mathrm{seed}^Y_y)_{y \in \mathrm{dom}(Y)}$.

In particular, $\mathrm{seed}^R_{(3,\emptyset)} = j^R(\boldsymbol{\delta}^1_3)$, $\mathrm{seed}^R_{(d,r)} = \mathrm{seed}^{\le 2R}_{(d,r)}$ when $d \le 2$. If $\mathbf{A} \in \mathrm{desc}^{**}({}^3R')$, $\mathbf{r} \in \mathrm{desc}^*({}^3R')$, R' finite subtree of R, let $\mathrm{seed}^R_{(3,\mathbf{A})} = j^{R',R}([\vec{\gamma} \mapsto {}^3\gamma_{\mathbf{A}}]_{\mu^{R'}})$, $\mathrm{seed}^R_{(3,\mathbf{r})} = j^{R',R}([\vec{\gamma} \mapsto {}^3\gamma_{\mathbf{r}}]_{\mu^{R'}})$. By Lemma 4.61, $\mathrm{seed}^R_{(3,\mathbf{A})} < \mathrm{seed}^R_{(3,\mathbf{A}')}$ iff $\mathbf{A} \prec \mathbf{A}'$; $\mathrm{seed}^R_{(3,\mathbf{A})} = \mathrm{seed}^R_{(3,\mathbf{A}')}$ iff $\mathbf{A} \sim \mathbf{A}'$. $\mathrm{seed}^R_{(3,\mathbf{A})}$ for finite R is the higher level analog of uniform indiscernibles. If Y is a Π^1_3-wellfounded level-3 tree and $\mathbf{A} \in \mathrm{desc}^{**}(Y)$, $\mathbf{y} \in \mathrm{desc}^*(Y)$, let $\mathrm{seed}^Y_{\mathbf{A}} = \mathrm{seed}^{Q^0\oplus Y}_{(3,\mathbf{A})}$, $\mathrm{seed}^Y_{\mathbf{y}} = \mathrm{seed}^{Q^0\oplus Y}_{(3,\mathbf{y})}$. We will show in Section 6.2 that $\mathrm{seed}^R_{(3,\mathbf{A})} = \mathrm{seed}^{3R}_{\mathbf{A}}$ for $\mathbf{A} \in \mathrm{desc}^{**}(R)$.

Under full AD, the set of $\mathrm{seed}^R_{(3,\emptyset)}$ for finite R is exactly $\{\aleph_{\xi+1} : \omega \le \xi < \omega^{\omega^\omega}\}$ by Martin [13, Theorem 4.17] and Jackson [12]. The set of $\mathrm{seed}^R_{(3,\emptyset)}$ for finite R and their limit points will be level-3 indiscernibles. The rest of this paper will contain a thorough analysis of the structure of level-3 uniform indiscernibles.

6.2 Level-3 uniform indiscernibles

Definition 6.2. If R is a finite level ≤ 3 tree, α is an *R-uniform indiscernible* iff $\alpha \in \bigcap_{x\in\mathbb{R}} j^R(\bar{I}^{(3)}_x)$.

By Lemma 5.43, the set of R-uniform indiscernibles is the closure of $\{\mathrm{seed}^R_{(3,\mathbf{A})} : \mathbf{A} \in \mathrm{desc}^{**}({}^3R)\}$, which has order type $\xi + 1$ if $[\![\emptyset]\!]_{^3R} = \widehat{\xi}$. By Lemmas 5.42-5.44, α is an R-uniform indiscernible iff $\alpha \ge \boldsymbol{\delta}^1_3$ is a cardinal in $\mathbb{L}[j^R(T_3)]$. In particular, the least R-uniform indiscernible is $\boldsymbol{\delta}^1_3$.

Recall that if R is a level-3 tree, $s \in \mathrm{dom}(R)$, then $R \restriction s$ the subtree of R whose domain consists of r for which $\langle r \rangle <_{BK} \langle s \rangle$. If R is a level-3 tree and $\mathbf{A} \in \mathrm{desc}^{**}(R)$, we let $R \restriction \mathbf{A}$ be the subtree of R whose domain consists of r for which $\langle r \rangle <_{BK} \langle \mathbf{A} \rangle$. If R is a level ≤ 3 tree and $\mathbf{A} \in \mathrm{desc}^{**}({}^3R)$, $s \in \mathrm{dom}(R)$, let $R \restriction (3, \mathbf{A}) = {}^{\le 2}R \oplus ({}^3R \restriction \mathbf{A})$, $R \restriction (3, s) = {}^{\le 2}R \oplus ({}^3R \restriction s)$.

Lemma 6.3. *Assume $\boldsymbol{\Pi}^1_3$-determinacy. Suppose R is Π^1_3-wellfounded level $\le$ 3 tree and $\mathbf{A} \in \mathrm{desc}^{**}({}^3R)$. Then $j^{R\restriction(3,\mathbf{A}),R}$ is the identity on $\mathbb{L}_{j^{R\restriction(3,\mathbf{A})}(\boldsymbol{\delta}^1_3)}[j^{R\restriction(3,\mathbf{A})}(T_3)]$. Furthermore, if $s \in \mathrm{dom}({}^3R)$ and $\mathrm{lh}(s) = 1$, then $j^{R\restriction(3,s)}(\boldsymbol{\delta}^1_3) = \mathrm{seed}^R_{(3,s)}$.*

Proof. Using a direct limit argument, it suffices to prove the case when R is finite. We prove that $j^{R\restriction(3,\mathbf{A})}(\boldsymbol{\delta}^1_3)$ is contained in the range of $j^{R\restriction(3,\mathbf{A}),R}$. Suppose $\alpha = [G]_{\mu^R} < \mathrm{seed}^R_{(3,\mathbf{A})}$, $x \in \mathbb{R}$, τ is an $\mathcal{L}$-Skolem term such that $G(\vec{\gamma}) = \tau^{M^-_{2,\infty}(x)}(x, \vec{\gamma})$ for any $\vec{\gamma} \in [\omega_1, \boldsymbol{\delta}^1_3]^{R\uparrow}$ and $G(\vec{\gamma}) < {}^3\gamma_{\mathbf{A}}$ for μ^R-a.e. $\vec{\gamma}$. By Lemma 5.41, if $\vec{\beta}$ respects ${}^{\le 2}R$, $\vec{\delta}$ and $\vec{\delta}'$ both strongly respects ${}^3R$ and $\forall r\ ((3, r) \in \mathrm{dom}(R \restriction (3, \mathbf{A})) \to {}^3\delta_r = {}^3\delta'_r)$, then $\tau^{M^-_{2,\infty}(x)}(x, \vec{\beta} \oplus c^{(3)}_{x,\vec{\delta}}) =$

$\tau^{M^-_{2,\infty}(x)}(x, \vec{\beta} \oplus c^{(3)}_{x,\vec{\delta}'})$. Using the fact that $(Q, \gamma) \mapsto c^{(3)}_{x,Q,\gamma}$ is definable over $M^-_{2,\infty}(x^{3\#})$, we can find an $\mathcal{L}$-Skolem term σ such that for μ^R-a.e. $\vec{\gamma}$,

$$G(\vec{\gamma}) = \sigma^{M^-_{2,\infty}(x^{3\#})}(x^{3\#}, ({}^d\gamma_r)_{(d,r)\in\mathrm{dom}(R\restriction(3,\mathbf{A}))}).$$

Hence, $\alpha = j^{R\restriction(3,\mathbf{A}),A}(\beta)$ where $\beta = \sigma^{j^{R\restriction(3,\mathbf{A})}(M^-_{2,\infty}(x^{3\#}))}(x^{3\#}, \mathrm{seed}^{R\restriction(3,\mathbf{A})})$. This also implies that $j^{R\restriction(3,\mathbf{A})}(\boldsymbol{\delta}^1_3) \geq \mathrm{seed}^R_{(3,\mathbf{A})}$. The "furthermore" part is due to unboundedness of level-3 sharps. □

Suppose Y is a level ≤ 3 tree, T is a level ≤ 2 tree. A $(Y, T, *)$-*description* is of the form $\mathbf{B} = (d, (\mathbf{y}, \pi))$ so that either $d = 3 \wedge (\mathbf{y}, \pi) \in \mathrm{desc}({}^3Y, T, *)$ or $d \leq 2 \wedge (d, (\mathbf{y}, \pi)) \in \mathrm{desc}({}^{\leq 2}Y, T, *)$. As usual, $\mathbf{B} = (d, (\mathbf{y}, \pi))$ is abbreviated by $(d, \mathbf{y}, \pi)$. If Q is a finite level ≤ 2 tree, a (Y, T, Q)-description is $(3, (\mathbf{y}, \pi))$ so that $(\mathbf{y}, \pi) \in \mathrm{desc}({}^3Y, T, Q)$. If P is a finite level-1 tree, a (Y, T, P)-description is $(2, (\mathbf{y}, \pi))$ so that $(2, (\mathbf{y}, \pi)) \in \mathrm{desc}({}^{\leq 2}Y, T, P)$. A $(Y, T, -1)$-description is $(1, (\mathbf{y}, \emptyset))$ so that $\mathbf{y} \in \mathrm{dom}({}^1Y)$. $\mathrm{desc}(Y, T, *)$, $\mathrm{desc}(Y, T, Q)$, etc. denote the sets of relevant descriptions. If Y, T are finite,

$$\mathrm{seed}^{Y,T}_{\mathbf{B}} \in \mathbb{L}(j^Y \circ j^T(T_3))$$

is the element represented modulo μ^Y by $\mathrm{id}^{Y,T}_{\mathbf{B}}$.

Suppose that R, Y are level ≤ 3 trees and T is a level ≤ 2 tree. ρ *factors* (R, Y, T) if ρ is a function on $\mathrm{dom}(R)$, ${}^{\leq 2}\rho =_{\mathrm{DEF}} \rho \restriction \mathrm{dom}({}^{\leq 2}R)$ factors $({}^{\leq 2}R, {}^{\leq 2}Y, T)$ and ${}^3\rho =_{\mathrm{DEF}} \rho \restriction \mathrm{dom}({}^3R)$ factors $({}^3R, {}^3Y, T)$. ρ *factors* (R, Y) iff ${}^{\leq 2}\rho$ factors $({}^{\leq 2}R, {}^{\leq 2}Y)$ and ${}^3\rho$ factors $({}^3R, {}^3Y)$. Suppose that ρ factors (R, Y, T). If $F \in (\omega_1, \boldsymbol{\delta}^1_3)^{Y\uparrow}$, then

$$F^T_\rho : [\omega_1]^{T\uparrow} \to [\omega_1, \boldsymbol{\delta}^1_3]^{R\uparrow}$$

is the function that sends $\vec{\xi}$ to $F^T_{{}^{\leq 2}\rho}(\vec{\xi}) \oplus F^T_{{}^3\rho}(\vec{\xi})$. If T is finite,

$$\mathrm{id}^{Y,T}_\rho$$

is the function $[F]^Y \mapsto [F^T_\rho]_{\mu^T}$. If Y is also finite,

$$\mathrm{seed}^{Y,T}_\rho = [\mathrm{id}^{Y,T}_\rho]_{\mu^Y} \in \mathbb{L}(j^Y \circ j^T(T_3)).$$

By Łoś and Lemmas 4.50,4.51,4.63,4.64,3.18,4.46, for any $A \in \mu^R$, $\mathrm{seed}^{Y,T}_\rho \in j^Y \circ j^T(A)$. We can unambiguously define

$$\rho^{Y,T} : \mathbb{L}(j^R(T_3)) \to \mathbb{L}(j^Y \circ j^T(T_3))$$

by sending $j^R(F)(\mathrm{seed}^R)$ to $j^Y \circ j^T(F)(\mathrm{seed}^{Y,T}_\rho)$. In general, if Y,T are Π^1_3-wellfounded and T is Π^1_2-wellfounded, then $\rho^{Y,T} \circ j^{R',R} = j^{Y',Y} \circ j^{(Y,(T',T))} \circ (\rho')^{Y',T'}$ for R',Y',T' finite subtrees of R,Y,T respectively and $\rho' = \rho \restriction \mathrm{dom}(R')$ factoring (R',Y',T'), where $j^{(Y,(T',T))} = \cup_{x\in\mathbb{R}} j^Y(j^{T',T} \restriction L[j^{T'}(T_3),x])$. In particular, $\rho^{Y,T} \circ j^R(\boldsymbol{\delta}^1_3) = j^Y(\boldsymbol{\delta}^1_3)$. If $\mathbf{A} \in \mathrm{desc}^{**}({}^3R)$, then $\rho^{Y,T}(\mathrm{seed}^R_{(3,\mathbf{A})}) = \mathrm{seed}^Y_{(3,\tilde{\rho}^T(\mathbf{A}))}$.

If Y is a level ≤ 3 tree and T is a level ≤ 2 tree, then

$$Y \otimes T = ({}^{\leq 2}Y \otimes T) \oplus ({}^3Y \otimes T)$$

is (modulo an isomorphism) a level ≤ 3 tree. The domain of $Y \otimes T$ consists of $\mathbf{B} = (d,(\mathbf{y},\pi)) \in \mathrm{desc}(Y,T,*)$. So ρ factors (R,Y,T) iff ρ factors $(R, Y\otimes T)$. The identity map $\mathrm{id}_{Y\otimes T} : \mathbf{B} \mapsto \mathbf{B}$ factors $(Y\otimes T, Y, T)$.

Lemma 6.4. *Suppose Y is a Π^1_3-wellfounded level ≤ 3 tree and T is a Π^1_2-wellfounded level ≤ 2 tree. Then $\mathbb{L}_{j^{Y\otimes T}(\boldsymbol{\delta}^1_3)}[j^{Y\otimes T}(T_3)] = \mathbb{L}_{j^Y(\boldsymbol{\delta}^1_3)}[j^Y \circ j^T(T_3)]$ and $(\mathrm{id}_{Y\otimes T})^{Y,T}$ is the identity map on $\mathbb{L}_{j^{Y\otimes T}(\boldsymbol{\delta}^1_3)}[j^{Y\otimes T}(T_3)]$.*

Proof. Assume without loss of generality that Y,T are finite. Put $R = Y\otimes T$ and $\rho = \mathrm{id}_{Y\otimes T}$. Then $\rho^{Y,T}(u^R_{(3,\mathbf{A})}) = u^Y_{(3,\tilde{\rho}^T(\mathbf{A}))}$ for any $\mathbf{A} \in \mathrm{desc}^{**}({}^3R)$ and $\forall \mathbf{B} \in \mathrm{desc}^{**}(Y)\ \exists \mathbf{A} \in \mathrm{desc}^{**}(R)\ \tilde{\rho}^T(\mathbf{A}) \sim^Y_* \mathbf{B}$. Recall that the set of $\mathbb{L}[j^R(T_3)]$-cardinals in the interval $[\boldsymbol{\delta}^1_3, j^R(\boldsymbol{\delta}^1_3)]$ is exactly $\{u^R_{(3,\mathbf{A})} : A \in \mathrm{desc}^{**}({}^3R)\}$. For any $x\in\mathbb{R}$, $\rho^{Y,T} \restriction L[j^R(T_3),x]$ is elementary from $L[j^R(T_3),x]$ to $L[j^Y \circ j^T(T_3),x]$. Hence, it suffices to show that $\rho^{Y,T} \restriction \boldsymbol{\delta}^1_3$ is the identity and whenever $\|\mathbf{A}\|_{\prec^{3R}_*}$ is a successor cardinal, then $\rho^{Y,T}$ is continuous at $u^R_{(3,\mathbf{A})}$.

By Lemma 4.53 and Corollary 4.15, $j^R \restriction \boldsymbol{\delta}^1_3 = j^{\leq 2R} \restriction \boldsymbol{\delta}^1_3 = j^{\leq 2Y} \circ j^T \restriction \boldsymbol{\delta}^1_3 = j^Y \circ j^T \restriction \boldsymbol{\delta}^1_3$. By indiscernability and remarkability, $\rho^{Y,T} \restriction \boldsymbol{\delta}^1_3 = {}^{\leq 2}\rho^{\leq 2Y,T} \restriction \boldsymbol{\delta}^1_3$ is the identity map.

Suppose $\|\mathbf{A}\|_{\prec^{3R}_*}$ is a successor cardinal and we prove that $\rho^{Y,T}$ is continuous at $\mathrm{seed}^R_{(3,\mathbf{A})}$. Suppose $\alpha < \mathrm{seed}^Y_{(3,\tilde{\rho}^T(\mathbf{A}))}$. There is $x \in \mathbb{R}$ such that

$$\alpha < \min(j^Y(\bar{I}^{(3)}_x) \setminus \sup\{u^Y_{(3,\mathbf{B})} : \mathbf{B} \prec^Y_* \tilde{\rho}^T(\mathbf{A})\}).$$

Let $\beta = \min(j^R(I^{(3)}_x) \setminus \sup\{\mathrm{seed}^R_{(3,\mathbf{B})} : \mathbf{B} \prec^Y_* \mathbf{A}\})$. Then $\beta < \mathrm{seed}^R_{(3,\mathbf{A})}$ and by induction and elementarity,

$$\rho^{Y,T}(\beta) = \min(j^Y \circ j^T(\bar{I}^{(3)}_x) \setminus \sup\{u^Y_{(3,\mathbf{B})} : \mathbf{B} \prec^Y_* \tilde{\rho}^T(\mathbf{A})\}) \geq \alpha,$$

the last inequality from $j^T(\bar{I}^{(3)}_x) \subseteq \bar{I}^{(3)}_x$ and elementarity of j^Y. □

Lemma 6.5. *Suppose Y,Y' are Π^1_3-wellfounded level ≤ 3 trees. Then $[\![\emptyset]\!]_{{}^3Y} = [\![\emptyset]\!]_{{}^3Y'}$ iff $j^Y(\boldsymbol{\delta}^1_3) = j^{Y'}(\boldsymbol{\delta}^1_3)$, $[\![\emptyset]\!]_{{}^3Y} < [\![\emptyset]\!]_{{}^3Y'}$ iff $j^Y(\boldsymbol{\delta}^1_3) < j^{Y'}(\boldsymbol{\delta}^1_3)$.*

Proof. If $[\![\emptyset]\!]_{^3Y} \leq [\![\emptyset]\!]_{^3Y'}$ then by Theorems 4.57 and 4.71, there exist a finite T and ρ that factors (Y, Y', T). So $\rho^{Y',T} \circ j^Y(\boldsymbol{\delta}^1_3) = j^{Y'}(\boldsymbol{\delta}^1_3)$, yielding $j^Y(\boldsymbol{\delta}^1_3) \leq j^{Y'}(\boldsymbol{\delta}^1_3)$. So $[\![\emptyset]\!]_{^3Y} = [\![\emptyset]\!]_{^3Y'}$ implies $j^Y(\boldsymbol{\delta}^1_3) = j^{Y'}(\boldsymbol{\delta}^1_3)$. If $[\![\emptyset]\!]_{^3Y} < [\![\emptyset]\!]_{^3Y'}$, we further obtain $\mathbf{B} \in \mathrm{desc}(^3Y', T, *)$ such that $\mathrm{lh}(\mathbf{B}) = 1$ and $[\![\mathbf{B}]\!]_{^3Y' \otimes T} = [\![\emptyset]\!]_{^3Y}$. Put $Z = (Y' \otimes T) \restriction (3, \mathbf{B})$. Then ρ factors (Y, Z). Hence $j^Y(\boldsymbol{\delta}^1_3) \leq j^Z(\boldsymbol{\delta}^1_3)$. By Lemma 6.3, the factor map $j^{Z, Y' \otimes T}$ is the identity on $\mathbb{L}_{j^Z(\boldsymbol{\delta}^1_3)}[j^Z(\boldsymbol{\delta}^1_3)]$ and $j^Z(\boldsymbol{\delta}^1_3) = \mathrm{seed}^{Y' \otimes T}_{(3,\mathbf{B})}$. By Lemma 6.4, $\mathrm{seed}^{Y' \otimes T}_{(3,\mathbf{B})} < j^{Y'} \circ j^T(\boldsymbol{\delta}^1_3) = j^{Y'}(\boldsymbol{\delta}^1_3)$. □

Lemma 6.6. *Suppose Y is a finite level ≤ 3 tree and $\mathbf{A} \in \mathrm{desc}^{**}(^3Y)$. Suppose $\|\mathbf{A}\|_{\prec^{3Y}_*} = \xi$ and R is a finite level ≤ 3 tree such that $[\![\emptyset]\!]_{^3R} = \widehat{\xi}$. Then* $\mathrm{seed}^Y_{(3,\mathbf{A})} = j^R(\boldsymbol{\delta}^1_3)$.

Proof. If $\mathbf{A} = (\emptyset, \emptyset, \emptyset)$, this is exactly Lemma 6.5. Suppose $\mathbf{A} \neq (\emptyset, \emptyset, \emptyset)$. Let T be a finite level ≤ 2 tree and let ρ minimally factor (R, Y, T). Let $\mathbf{B} \in \mathrm{desc}(^3Y, T, *)$ such that $\mathrm{lh}(\mathbf{B}) = 1$ and $[\![\mathbf{B}]\!]_{^3Y \otimes T} = \widehat{\xi}$. Put $\mathbf{B} = (\mathbf{y}, \pi) \in \mathrm{desc}(^3Y, T, Q)$. A routine computation gives $\mathbf{A} \sim^{3Y}_* (\mathbf{y}, \pi, T \otimes Q)$. So $\mathrm{seed}^{Y \otimes T}_{(3,\mathbf{B})} = \mathrm{seed}^{Y \otimes T}_{(3,\mathbf{A})}$. Put $Z = Y \otimes T \restriction (3, \mathbf{B})$. Then $[\![\emptyset]\!]_{^3Z} = [\![\emptyset]\!]_{^3R}$. By Lemma 6.3, $j^Z(\boldsymbol{\delta}^1_3) = \mathrm{seed}^{Y \otimes T}_{(3,\mathbf{B})}$. By Lemma 6.5, $j^R(\boldsymbol{\delta}^1_3) = j^Z(\boldsymbol{\delta}^1_3)$, and we are done. □

Definition 6.7. In view of Lemma 6.6, we define the *level-3 uniform indiscernibles*:

1. $u^{(3)}_{\xi+1} = j^R(\boldsymbol{\delta}^1_3)$ when $\xi < \omega^{\omega^\omega}$, R is a Π^1_3-wellfounded level ≤ 3 tree and $[\![\emptyset]\!]_{^3R} = \widehat{\xi}$.
2. If $0 < \xi \leq \omega^{\omega^\omega}$ is a limit, then $u^{(3)}_\xi = \sup_{\eta<\xi} u^{(3)}_\eta$.

If R is a finite level ≤ 3 tree and $[\![\emptyset]\!]_{^3R} = \widehat{\xi}$, then the set of R-uniform indiscernibles is $\{u^{(3)}_\eta : 0 < \eta \leq \xi + 1\}$ and we have $\mathrm{seed}^R_{(3,\mathbf{A})} = u^{(3)}_{\eta+1}$ for $\|\mathbf{A}\|_{\prec^{3R}_*} = \eta$.

The next lemma is the higher level analog of $\boldsymbol{\delta}^1_2 = u_2$.

Lemma 6.8. *Assume $\mathbf{\Pi}^1_3$-determinacy. Then $\boldsymbol{\delta}^1_4 = u^{(3)}_2$.*

Proof. If W is a $\Sigma^1_4(x)$ wellfounded relation on $\mathbb{R}$, then W is $\boldsymbol{\delta}^1_3$-Suslin via a tree in $L[T_3, x]$, so by Kunen-Martin and Lemma 5.42, $\mathrm{rank}(W) < ((\boldsymbol{\delta}^1_3)^+)^{L[T_3,x]} < \min(j^{R^0}(I^{(3)}_{x^{3\#}}) \setminus (\boldsymbol{\delta}^1_3 + 1)) < u^{(3)}_2$ as $[\![\emptyset]\!]_{R^0} = \omega = \widehat{1}$. If $\alpha < u^{(3)}_2$, pick x such that $\alpha < \min(j^{R^0}(I^{(3)}_x) \setminus (\boldsymbol{\delta}^1_3 + 1))$. Lemma 5.42 gives a surjection $f : \boldsymbol{\delta}^1_3 \twoheadrightarrow \alpha$ which is definable over $j^{R^0}(M^-_{2,\infty}(x))$ from $\{\boldsymbol{\delta}^1_3, x\}$. From f we can define a $\Delta^1_4(x^{3\#})$ prewellordering of length α. □

6.3 The level-4 Martin-Solovay tree

Let R^∞ be the unique (up to an isomorphism) level-3 tree such that

1. for any finite level-3 tree Y, there exists ρ which minimally factors (Y, R);
2. if $r \in \mathrm{dom}(R^\infty)$ then there exist a finite Y and ρ which minimally factors (Y, R) such that $r \in \mathrm{dom}(\rho)$.

In other words, R^∞ is the minimum Π^1_3-wellfounded level-3 tree that is universal for finite level-3 tree in terms of minimal factorings. We fix the following representation of R^∞, whose domain consists of finite tuples of ordinals in ω^{ω^ω}:

1. $(\xi_1) \in \mathrm{dom}(R^\infty)$ iff $0 < \xi_1 < \omega^{\omega^\omega}$. $R^\infty((\xi_1))$ is the Q^0-partial level ≤ 2 tree induced by $\widehat{\xi_1}$.
2. If $r = (\xi_1, \ldots, \xi_{k-1}) \in \mathrm{dom}(R^\infty)$, then $r^\frown(\xi_k) \in \mathrm{dom}(R^\infty)$ iff $\xi_k < \omega^{\omega^\omega}$ and there exists a completion Q^+ of $R^\infty(r)$ such that the Q^+-approximation sequence of $\widehat{\xi_k}$ is $(\widehat{\xi_i})_{1 \leq i \leq k}$; if $r^\frown(\xi_k) \in \mathrm{dom}(R^\infty)$ and Q^+ is the unique such completion, then $R^\infty(r^\frown(\xi_k))$ is the Q^+-partial level ≤ 2 tree induced by $\widehat{\xi_k}$.

Therefore, $[\![\emptyset]\!]_R = u_\omega$ and if $r = (\xi_1, \ldots, \xi_k) \in \mathrm{dom}(R^\infty)$, then $[\![r]\!]_R = \widehat{\xi_k}$. If Y is a finite level-3 tree, then the map $y \mapsto r_y$ minimally factors (Y, R), where if $([\![y \restriction i]\!]_Y)_{1 \leq i \leq \mathrm{lh}(y)} = (\widehat{\xi_1}, \ldots, \widehat{\xi_{\mathrm{lh}(y)}})$ then $r_y = (\xi_1, \ldots, \xi_{\mathrm{lh}(y)})$. If $0 < \xi < \omega^{\omega^\omega}$, let $R^\infty_\xi = R^\infty \restriction (\xi)$. By Lemma 6.3, if $0 < \xi \leq \omega^{\omega^\omega}$, then the factoring map $j^{R^\infty_\xi, R^\infty}$ is the identity on $\mathbb{L}_{j^{R^\infty_\xi}(\boldsymbol{\delta}^1_3)}[j^{R^\infty_\xi}(T_3)]$ and $j^{R^\infty_\xi}(\boldsymbol{\delta}^1_3) = \mathrm{seed}^{R^\infty}_{(\xi)} = u^{(3)}_\xi$. In particular, $j^{R^\infty}(\boldsymbol{\delta}^1_3) = u^{(3)}_{\omega^{\omega^\omega}}$.

R^∞ will be the tree based on which level-3 sharp codes for ordinals below $u^{(3)}_{\omega^{\omega^\omega}}$ are defined.

$\mathrm{LO}^{(3)}$ is the set of $v \in \mathbb{R}$ such that X_v is a linear ordering of u_ω, where $v \mapsto X_v$ is the Δ^1_3 surjection from $\mathbb{R}$ onto $\mathcal{P}((V_\omega \cup u_\omega)^{<\omega})$, defined in Corollary 2.12 and renamed in the beginning of Section 4.9. $\mathrm{WO}^{(3)} = \mathrm{WO}^{(3)}_0$ is the set of $v \in \mathrm{LO}^{(3)}$ such that X_v is a wellordering of u_ω. $\mathrm{WO}^{(3)}_0$ is Π^1_3. For $v \in \mathrm{WO}^{(3)}$, put $\|v\| = \mathrm{o.t.}(X_v)$. Every ordinal in $\boldsymbol{\delta}^1_3$ is of the form $\|v\|$ for some $v \in \mathrm{WO}^{(3)}$.

A *level-3 sharp code* is a pair $\langle \ulcorner \tau \urcorner, x^{3\#} \rangle$ where τ is an $\mathcal{L}^{\underline{x}, R^\infty}$-Skolem term for an ordinal without free variables. For $0 < \xi \leq \omega^{\omega^\omega}$, $\mathrm{WO}^{(3)}_\xi$ is the set of level-3 sharp codes $\langle \ulcorner \tau \urcorner, x^{3\#} \rangle$ such that τ is an $\mathcal{L}^{\underline{x}, R^\infty_\xi}$-Skolem term. $\mathrm{WO}^{(3)}_\xi$ is Π^1_4 for $0 < \xi \le \omega^{\omega^\omega}$. The ordinal coded by $\langle \ulcorner \tau \urcorner, x^{3\#} \rangle$ is

$$\left| \langle \ulcorner \tau \urcorner, x^{3\#} \rangle \right| = \tau^{(j^{R^\infty}(M^-_{2,\infty}); \mathrm{seed}^{R^\infty})}.$$

For each ξ, $\mathrm{WO}^{(3)}_\xi$ is Π^1_4. By Lemma 6.3, if $\langle \ulcorner \tau \urcorner, x^{3\#} \rangle \in \mathrm{WO}^{(3)}_\xi$ and $\tau = \sigma(\underline{x}, \underline{c_{r_1}}, \dots)$, σ is an $\mathcal{L}$-Skolem term, then

$$\left| \langle \ulcorner \tau \urcorner, x^{3\#} \rangle \right| = \sigma^{j^{R^\infty_\xi}(M^-_{2,\infty})}(x, \mathrm{seed}^{R^\infty_\xi}_{r_1}, \dots).$$

Lemma 6.9. *The relations $v, w \in WO^{(3)}_{\omega^{\omega^\omega}} \wedge |v| = |w|$ and $v, w \in WO^{(3)}_{\omega^{\omega^\omega}} \wedge |v| < |w|$ are both Δ^1_5.*

Proof. $\left| \langle \ulcorner \tau \urcorner, x^{3\#} \rangle \right| = \left| \langle \ulcorner \tau' \urcorner, (x')^{3\#} \rangle \right|$ iff $\tau = \sigma(\underline{x}, \underline{c_{r_1}}, \dots, \underline{c_{r_n}})$, $\tau' = \sigma'(\underline{x}, \underline{c_{r'_1}}, \dots, \underline{c_{r'_{n'}}})$, σ, σ' are $\mathcal{L}$-Skolem terms, and for some finite level-3 tree Y and some ρ factoring (Y, R^∞) such that $\vec{r}^\frown \vec{r'} \subseteq \mathrm{ran}(\rho)$, "$\sigma((\underline{x})_{\mathrm{left}}, \underline{c_{\rho^{-1}(r_1)}}, \dots) = \sigma'((\underline{x})_{\mathrm{right}}, \underline{c_{\rho^{-1}(r'_1)}}, \dots)$" is true in $(x \oplus x')^{3\#}(Y)$. $\square$

Recall that WO_ω is the set of (level-1) sharp codes for ordinals below u_ω. The connection between level-3 sharp codes and level-1 sharp codes or WO is also Δ^1_5. For instance, the relation "$v \in \mathrm{WO}^{(3)}_{\omega^{\omega^\omega}} \wedge w \in \mathrm{WO}_\omega \wedge |v| = |w|$" is Δ^1_5.

If Γ is a pointclass, say that $A \subseteq u^{(3)}_{\omega^{\omega^\omega}} \times \mathbb{R}$ is in Γ iff $\{(v, x) : v \in \mathrm{WO}^{(3)}_{\omega^{\omega^\omega}} \wedge (|v|, x) \in A\}$ is in Γ. Γ acting on subsets of product spaces is defined in the obvious way.

Lemma 6.10.

1. *Suppose that $\xi \le \eta < \omega^{\omega^\omega}$ and ρ factors $(R^\infty_\xi, R^\infty_\eta)$. Then $\rho^{R^\infty_\eta} \restriction u^{(3)}_\xi$ is Δ^1_5, uniformly in (ξ, η, ρ).*
2. *Suppose that $\xi < \omega^{\omega^\omega}$ and Q, Q' are finite level ≤ 2 trees, Q is a subtree of Q'. Then $j^{(R^\infty_\xi, (Q, Q'))} \restriction u^{(3)}_\xi$ is Δ^1_5, uniformly in (ξ, Q, Q').*

Proof. 1. $\alpha < u^{(3)}_\xi \wedge \rho^{R^\infty_\eta}(\alpha) = \beta$ iff there exist $x \in \mathbb{R}$, an $\mathcal{L}$-Skolem term τ and $r_1, \dots, r_n$ such that $r_i \in \mathrm{dom}(R^\infty_\xi)$ for any i and $\alpha = \langle \ulcorner \tau(\underline{x}, \underline{c_{r_1}}, \dots, \underline{c_{r_n}}) \urcorner, x^{3\#} \rangle$, $\beta = \langle \ulcorner \tau(\underline{x}, \underline{c_{\rho(r_1)}}, \dots, \underline{c_{\rho(r_n)}}) \urcorner, x^{3\#} \rangle$.

2. $\alpha < u^{(3)}_\xi \wedge j^{(R^\infty_\xi, (Q, Q'))}(\alpha) = \beta$ iff there exist $x \in \mathbb{R}$, an $\mathcal{L}$-Skolem term τ and $r_1, \dots, r_n$ such that $r_i \in \mathrm{dom}(R^\infty_\xi)$ for any i and $\alpha = \langle \ulcorner \tau(\underline{x}, \underline{c_{r_1}}, \dots, \underline{c_{r_n}}) \urcorner, x^{3\#} \rangle$, $\beta = \langle \ulcorner \underline{j^{Q,Q'}}(\tau(\underline{x}, \underline{c_{(r_1)}}, \dots, \underline{c_{(r_n)}})) \urcorner, x^{3\#} \rangle$. $\square$

By Lemma 5.42, the set of uncountable $\mathbb{L}[j^{R^\infty}(T_3)]$-cardinals $\leq u^{(3)}_{\omega^{\omega^\omega}}$ is the closure of

$$\{u_n : 1 \leq n < \omega\} \cup \{\text{seed}^{R^\infty}_{\mathbf{A}} : \mathbf{A} \in \text{desc}^{**}(R^\infty)\}.$$

By Lemma 6.4, if $\mathbf{A} = (\mathbf{r}, \pi, T) \in \text{desc}^{**}(R^\infty)$, $\mathbf{r} = (r, Q, \overrightarrow{(d, q, P)})$, $r = (\xi_i)_{1 \leq i \leq k}$ and $\text{seed}^{R^\infty}_{\mathbf{A}} > \boldsymbol{\delta}^1_3$ is a successor cardinal in $\mathbb{L}[j^{R^\infty}(T_3)]$, then r is of discontinuous type, ξ_k is a successor ordinal, and letting $r' = (\xi_i)_{1 \leq i < k}{}^\frown(\xi_k - 1)$, $\mathbf{r}' = (r', Q, \overrightarrow{(d, q, P)})$, $\mathbf{A}' = (\mathbf{r}', \pi, T)$, then

$$\{\langle x^{3\#}, \ulcorner \tau^{\underline{j^T(V)}}(\underline{x}, \underline{c_{\mathbf{A}'}}) \urcorner \rangle : x \in \mathbb{R}, \tau \text{ is an } \mathcal{L}\text{-Skolem term for an ordinal}\}$$

is a cofinal subset of $\text{seed}^{R^\infty}_{\mathbf{A}}$.

A level-3 EM blueprint over a real Γ is completely decided by $\Gamma(R^\infty)$. Γ is coded into the real $z_\Gamma \in 2^\omega$ where $z(k) = 0 \leftrightarrow k \in \Gamma(R^\infty)$. We shall identify Γ with z_Γ when no confusion occurs. We define the level-4 Martin Solovay tree T_4 which projects to $\{x^{3\#} : x \in \mathbb{R}\}$. T_4 will be Δ^1_5 as a subset of $(\omega \times u^{(3)}_{\omega^{\omega^\omega}})^{<\omega}$, the complexity based on Lemma 6.10.

Let T be a recursive tree so that $z \in [T]$ iff z is a remarkable level-3 EM blueprint over a real. Let $(r_i)_{1 \leq i < \omega}$ be an effective enumeration of $\text{dom}(R^\infty)$ and let $(\tau_k)_{k<\omega}$ be an effective enumeration of all the $\mathcal{L}$-Skolem terms for an ordinal, where τ_k is $f(k) + 1$-ary. T_4 is the tree on $2 \times u^{(3)}_{\omega^{\omega^\omega}}$ where

$$(t, \vec{\alpha}) \in T_4$$

iff $t \in T$ and

1. if $\xi \leq \eta < \omega^{\omega^\omega}$, $r_1, \dots, r_{f(k)} \in \text{dom}(R^\infty_\xi)$, $r_1, \dots, r_{f(l)} \in \text{dom}(R^\infty_\eta)$, ρ factors $(R^\infty_\xi, R^\infty_\eta)$,
   (a) if "$\tau_k(\underline{x}, \underline{c_{\rho(r_1)}}, \dots, \underline{c_{\rho(r_{f(k)})}}) = \tau_l(\underline{x}, \underline{c_{r_1}}, \dots, \underline{c_{r_{f(l)}}})$" is true in t, then $\rho^{R^\infty_\eta}(\alpha_k) = \alpha_l$;
   (b) if "$\tau_k(\underline{x}, \underline{c_{\rho(r_1)}}, \dots, \underline{c_{\rho(r_{f(k)})}}) < \tau_l(\underline{x}, \underline{c_{r_1}}, \dots, \underline{c_{r_{f(l)}}})$" is true in t, then $\rho^{R^\infty_\eta}(\alpha_k) < \alpha_l$;
2. if $\xi < \omega^{\omega^\omega}$, $r_1, \dots, r_{\max(f(k), f(l))} \in \text{dom}(R^\infty \upharpoonright \xi)$, Q, Q' are finite level ≤ 2 trees, Q is a subtree of Q', "$\underline{j^{Q,Q'}}(\tau_k(\underline{x}, \underline{c_{r_1}}, \dots, \underline{c_{r_{f(k)}}})) = \tau_l(\underline{x}, \underline{c_{r_1}}, \dots, \underline{c_{r_{f(l)}}})$" is true in t, then $j^{(R^\infty_\xi, (Q, Q'))}(\alpha_k) = \alpha_l$.

Theorem 6.11. $p[T_4] = \{x^{3\#} : x \in \mathbb{R}\}$. *Furthermore, for any* $x \in \mathbb{R}$, $(\tau_k^{(j^{R^\infty}(M^-_{2,\infty}(x)))}(x, \text{seed}^{R^\infty}_{r_1}, \dots, \text{seed}^{R^\infty}_{r_{f(k)}}))_{k<\omega}$ *is the honest leftmost branch of* $(T_4)_{x^{3\#}}$.

Proof. By definition, for any x, $(x^{3\#}, (\tau_k^{(j^{R^\infty}(M_{2,\infty}^-(x)))}(x, \text{seed}_{r_1}^{R^\infty}, \ldots, \text{seed}_{r_{f(k)}}^{R^\infty}))_{k<\omega}) \in [T_4]$. Suppose now $(z, \vec{\beta}) \in p[T_4]$. Let x be a real so that z codes a remarkable level-3 EM blueprint Γ over x. We need to show that z is iterable and for any k, $\tau_k^{(j^{R^\infty}(M_{2,\infty}^-(x)))}(z, \text{seed}_{r_1}^{R^\infty}, \ldots, \text{seed}_{r_{f(k)}}^{R^\infty}) \leq \beta_k$. For each k, pick a finite subtree Y_k of R^∞ and $F_k : [\boldsymbol{\delta}_3^1]^{Y_k \uparrow} \to \boldsymbol{\delta}_3^1$ such that $\{r_1, \ldots, r_{f(k)}\} \subseteq \text{dom}(Y_k)$, $F_k \in \mathbb{L}[T_3]$ and $j^{Y_k, R^\infty}([F_k]_{\mu^{Y_k}}) = \beta_k$. By $\mathbb{L}[T_3]$-countable completeness of the club filter on $\boldsymbol{\delta}_3^1$, we can find a club C in $\boldsymbol{\delta}_3^1$ such that $C \in \mathbb{L}[T_3]$ and

1. if $\xi \leq \eta < \omega^{\omega^\omega}$, Y_k is a subtree of R_ξ^∞, Y_l is a subtree of R_η^∞, ρ factors $(R_\xi^\infty, R_\eta^\infty)$,
   (a) if "$\tau_k(\underline{x}, \underline{c_{\rho(r_1)}}, \ldots, \underline{c_{\rho(r_{f(k)})}}) = \tau_l(\underline{x}, \underline{c_{r_1}}, \ldots, \underline{c_{r_{f(l)}}})$" is true in $\Gamma(R^\infty)$, $\vec{\gamma} \in [C]^{R_\eta^\infty \uparrow}$, then $F_k(\vec{\gamma}_\rho \restriction \text{dom}(Y_k)) = F_l(\vec{\gamma} \restriction \text{dom}(Y_l))$;
   (b) if "$\tau_k(\underline{x}, \underline{c_{\rho(r_1)}}, \ldots, \underline{c_{\rho(r_{f(k)})}}) < \tau_l(\underline{x}, \underline{c_{r_1}}, \ldots, \underline{c_{r_{f(l)}}})$" is true in $\Gamma(R^\infty)$, $\vec{\gamma} \in [C]^{R_\eta^\infty \uparrow}$, then $F_k(\vec{\gamma}_\rho \restriction \text{dom}(Y_k)) < F_l(\vec{\gamma} \restriction \text{dom}(Y_l))$.
2. if $\xi < \omega^{\omega^\omega}$, Y_k, Y_l are subtrees of R_ξ^∞, Q, Q' are finite level ≤ 2 trees, Q is a subtree of Q', "$\underline{j^{Q,Q'}}(\tau_k(\underline{x}, \underline{c_{r_1}}, \ldots, \underline{c_{r_{f(k)}}})) = \tau_l(\underline{x}, \underline{c_{r_1}}, \ldots, \underline{c_{r_{f(l)}}})$" is true in $\Gamma(R^\infty)$, $\vec{\gamma} \in [C]^{R_\xi^\infty \uparrow}$, then $j^{Q,Q'}(F_k(\vec{\gamma} \restriction \text{dom}(Y_k))) = F_l(\vec{\gamma} \restriction \text{dom}(Y_l))$.

Suppose S is a Π_3^1-wellfounded level-3 tree. We show that $\mathcal{M}_{\Gamma,S}$ is a Π_3^1-iterable x-mouse. Put $\mathcal{N} = \mathcal{M}_{\Gamma,S}$. Put $\eta \in C_0$ iff $C \cap \eta$ has order type η, $\eta \in D$ iff $C_0 \cap \eta$ has order type η. Fix $\vec{\gamma} \in [D]^{S\uparrow}$. We define an embedding

$$\theta : \text{Ord}^{\mathcal{N}} \to \boldsymbol{\delta}_3^1$$

as follows. If σ is an $\mathcal{L}$-Skolem term, $s_1, \ldots, s_n \in \text{dom}(R)$, R is a finite subtree of S, $a = (\sigma(\underline{c_{s_1}}, \ldots, \underline{c_{s_n}}))^{\mathcal{N}}$, ρ factors (R, Y_k), $\tau_k(\underline{c_{r_1}}, \ldots, \underline{c_{r_{f(k)}}})$ is logically equivalent to $\sigma(\underline{c_{\rho(s_1)}}, \ldots, \underline{c_{\rho(s_n)}})$, $\vec{\delta} \in [C_0]^{Y_k\uparrow}$, $\delta_{\rho(s)} = \gamma_s$ for any $s \in \text{dom}(R)$, we put

$$\theta(a) = F_k(\vec{\delta}).$$

θ is well defined: Suppose σ' is another $\mathcal{L}$-Skolem term, $s_1', \ldots, s_{n'}' \in \text{dom}(R')$, R' is a finite subtree of S, "$\sigma(\underline{x}, \underline{c_{s_1}}, \ldots, \underline{c_{s_n}}) = \sigma'(\underline{x}, \underline{c_{s_1'}}, \ldots, \underline{c_{s_{n'}'}})$" is true in $\Gamma(S)$, ρ' factors $(R', Y_{k'})$, $\tau_{k'}(\underline{x}, \underline{c_{r_1}}, \ldots, \underline{c_{r_{f(k')}}})$ is logically equivalent to $\sigma'(\underline{x}, \underline{c_{\rho'(s_1')}}, \ldots, \underline{c_{\rho'(s_{n'}')}})$, $\vec{\delta}' \in [C_0]^{Y_{k'}\uparrow}$, $\delta_{\rho'(s)} = \gamma_s$ for any $s \in \text{dom}(R')$. Pick $\xi, \xi' < \omega^{\omega^\omega}$ such that Y_k is a subtree of R_ξ^∞, $Y_{k'}$ is a subtree of $R_{\xi'}^\infty$. Let $(Y^*, \psi, \psi', \vec{\epsilon})$ be the amalgamation of $(Y_k, \vec{\delta})$ and $(Y_{k'}, \vec{\delta}')$, obtained by Lemma 4.73.

That is, Y^* is a finite level-3 tree, ψ factors (Y_k, Y^*), ψ' factors $(Y_{k'}, Y^*)$, $\vec{\epsilon} \in [C_0]^{Y^*\uparrow}$, $\epsilon_{\psi(y)} = \delta_y$ for $y \in \operatorname{dom}(Y_k)$, $\epsilon_{\psi'(y)} = \delta'_y$ for $y \in \operatorname{dom}(Y_{k'})$. So "$\sigma(\underline{x}, \underline{c_{\psi\circ\rho(s_1)}}, \dots) = \sigma'(\underline{x}, \underline{c_{\psi'\circ\rho'(s'_1)}}, \dots)$" is true in $\Gamma(Y^*)$. Pick $\eta < \omega^{\omega^\omega}$ large enough so that there exist ϕ, ϕ' factoring $(R^\infty_\xi, R^\infty_\eta)$, $(R^\infty_{\xi'}, R^\infty_\eta)$ respectively and ϕ^* factoring (Y^*, R^∞_η) such that $\phi^* \circ \psi = \phi \restriction \operatorname{dom}(Y_k)$, $\phi^* \circ \psi' = \phi' \restriction \operatorname{dom}(Y_{k'})$. So "$\sigma(\underline{x}, \underline{c_{\phi\circ\rho(s_1)}}, \dots) = \sigma'(\underline{x}, \underline{c_{\phi'\circ\rho'(s'_1)}}, \dots)$" is true in $\Gamma(R^\infty)$. Let $\tau_l(\underline{x}, \underline{c_{r_1}}, \dots, \underline{c_{r_{f(l)}}})$ be logically equivalent to $\sigma(\underline{x}, \underline{c_{\phi\circ\rho(s_1)}}, \dots)$. So "$\tau_k(\underline{x}, \underline{c_{\phi(r_1)}}, \dots) = \tau_l(\underline{x}, \underline{c_{r_1}}, \dots)$" and "$\tau_{k'}(\underline{x}, \underline{c_{\phi'(r_1)}}, \dots) = \tau_l(\underline{x}, \underline{c_{r_1}}, \dots)$" are both true in $\Gamma(R^\infty)$. We can find $\vec{\alpha} \in [C]^{R^\infty_\eta\uparrow}$ so that $\alpha_{\phi^*(y)} = \epsilon_y$ for any $y \in \operatorname{dom}(Y^*)$. By assumption, $F_k(\vec{\delta}) = F_l(\vec{\alpha}) = F_{k'}(\vec{\delta}')$.

Similar arguments show that θ is order preserving and $\theta''((\underline{S_3})^{\mathcal{N}}) \subseteq S_3$. So $\mathcal{N}$ is wellfounded and $p[(\underline{S_3})^{\mathcal{N}}] \subseteq p[S_3]$. We then show that $\mathcal{N}$ is Π^1_3-iterable. By Lemma 5.24, $\mathcal{N}$ has cofinally many cardinal strong cutpoints. For each cardinal strong cutpoint ξ of $\mathcal{N}$, by definition of $\underline{S_3}$, $\mathcal{N}^{\operatorname{Coll}(\omega,\xi)} \models$"$\mathcal{N}|\xi$ is Π^1_3-iterable $\wedge$ $\underline{S_3}$ projects to the set of Π^1_3-wellfounded level-3 towers". The fact that $p[(\underline{S_3})^{\mathcal{N}}] \subseteq p[S_3]$ implies that $\mathcal{N}^{\operatorname{Coll}(\omega,\xi)}$ is Σ^1_3-correct. So $\mathcal{N}|\xi$ is genuinely Π^1_3-iterable. By varying ξ, $\mathcal{N}$ is Π^1_3-iterable.

Next, we show that for any k, $\tau_k^{(j^{R^\infty}(M^-_{2,\infty}(x));\operatorname{seed}^{R^\infty})} \leq \beta_k$. We define an embedding

$$\theta : \{\tau_k^{(M^-_{2,\infty}(x);\vec{\gamma})} : \vec{\gamma} \in [D]^{R^\infty\uparrow}, k < \omega\} \to \boldsymbol{\delta}^1_3$$

by $\theta(\tau_k^{(M^-_{2,\infty}(x);\vec{\gamma})}) = F_k(\vec{\gamma} \restriction \operatorname{dom}(Y_k))$. A similar argument shows that θ is well defined and order preserving. In particular, for any $\vec{\gamma} \in [D]^{R^\infty\uparrow}$, $\tau_k^{(M^-_{2,\infty}(x);\vec{\gamma})} \leq F_k(\vec{\gamma} \restriction \operatorname{dom}(Y_k))$. Hence, $\tau_k^{(j^{R^\infty}(M^-_{2,\infty}(x));\operatorname{seed}^{R^\infty})} \leq \beta_k$. □

7 The level-4 sharp

7.1 The level-4 Kechris-Martin theorem

For a countable structure $\mathcal{P}$ in the language of premice that satisfies Axioms 1-3 in Definition 5.14 and the universality of level ≤ 2 ultrapowers axiom, the direct limit $\mathcal{P}_\infty$ is defined in Definition 5.39. The wellfounded part $\mathcal{P}_\infty$ is always transitive. Recall that every ordinal in $\mathcal{P}_\infty$ is of the form $\pi_{\mathcal{P},X,\vec{\beta},\infty}(a)$ where $a \in \operatorname{Ord}^{\mathcal{N}}$, X is a finite level ≤ 2 tree, $\vec{\beta} \in [\omega_1]^{X\uparrow}$. The relation "$v \in \operatorname{LO}^{(3)}$ codes the order type of $\operatorname{Ord}^{\mathcal{P}_\infty}$" is uniformly $\boldsymbol{\Delta}^1_3$ in the code of $\mathcal{P}$.

For $x \in \mathbb{R}$, a putative x-3-sharp is a remarkable level-3 EM blueprint over x that satisfies the universality of level ≤ 2 ultrapowers axiom. Suppose x^*

is a putative x-3-sharp. For any limit ordinal $\alpha < \boldsymbol{\delta}^1_3$, we can build an EM model

$$\mathcal{M}^*_{x^*,\alpha}$$

as follows. Let R be a level-3 tree such that $[\![\emptyset]\!]_R = \alpha$. Then $\mathcal{M}^*_{x^*,\alpha} = (\mathcal{M}^*_{x^*,R})_\infty$. This definition is independent of the choice of R. Suppose R' is another level-3 tree and $[\![\emptyset]\!]_{R'} = \alpha$, and suppose without loss of generality that ρ minimally factors (R, R') by Theorem 4.71. Then $\rho^{*,R'}_{x^*} : \mathcal{M}^*_{x^*,R} \to \mathcal{M}^*_{x^*,R'}$ induces a canonical embedding $\phi : (\mathcal{M}^*_{x^*,R})_\infty \to (\mathcal{M}^*_{x^*,R'})_\infty$. Let T be Π^1_2-wellfounded and let ψ minimally factor $(R', R \otimes T)$. Then $\psi^{*,R,T}_{x^*} : \mathcal{M}^*_{x^*,R'} \to \mathcal{M}^{*,T}_{x^*,R}$ induces a canonical embedding $\phi' : (\mathcal{M}^*_{x^*,R'})_\infty \to (\mathcal{M}^*_{x^*,R})_\infty$. By coherency, $\phi' \circ \phi = \mathrm{id}$ and hence $\phi = \phi' = \mathrm{id}$. We say that x^* is *α-iterable* iff α is in the wellfounded part of $\mathcal{M}^*_{x^*,\alpha}$.

A *putative level-3 sharp code for an increasing function* is $w = \langle \ulcorner \tau \urcorner, x^* \rangle$ such that x^* is a putative x-3-sharp, τ is a unary $\mathcal{L}^{\underline{x}}$-Skolem term and

$$\text{“}\forall v, v'((v, v' \in \mathrm{Ord} \wedge v < v') \to (\tau(v) \in \mathrm{Ord} \wedge \tau(v) < \tau(v')))\text{”}$$

is true in $x^*(\emptyset)$. The statement “ $\langle \ulcorner \tau \urcorner, x^* \rangle$ is a putative level-3 sharp code for an increasing function, x^* is α-iterable, r codes the order type of $\tau^{\mathcal{M}_{x^*,\alpha}}(\alpha)$” about $(\langle \ulcorner \tau \urcorner, x^* \rangle, r)$ is $\boldsymbol{\Sigma}^1_3$ in the code of α. In addition, when $x^* = x^{3\#}$, $\langle \ulcorner \tau \urcorner, x^* \rangle$ is called a *(true) level-3 sharp code for an increasing function.*

Lemma 7.1. *Assume $\boldsymbol{\Delta}^1_4$-determinacy. Suppose $\kappa \leq u^{(3)}_{\omega^{\omega^\omega}}$ is an uncountable cardinal in $\mathbb{L}[j^{R^\infty}(T_3)]$. If A is a $\Sigma^1_5(x)$ subset of κ and $\sup A < \kappa$, then $\exists w \in \Delta^1_5(x) \cap WO^{(3)}_{\omega^{\omega^\omega}}$ $(\sup A < |w| < \kappa)$.*

Proof. Let $x = 0$. The lemma is trivial if κ is a limit cardinal in $\mathbb{L}[j^{R^\infty}(T_3)]$. Suppose now κ is a successor cardinal in $\mathbb{L}[j^{R^\infty}(T_3)]$. Let B be Π^1_4 such that $w \in \mathrm{WO}^{(3)}_{\omega^{\omega^\omega}} \wedge |w| \in A$ iff $\exists y\ (w, y) \in B$.

Case 1: $\omega_1 \leq \kappa < u_\omega$.

The lower level proof in [23] carries over almost verbatim, except the game becomes Σ^1_4 for the winner and hence a Δ^1_5 winning strategy can be found by Moschovakis third periodicity [36].

Case 2: $\kappa = \boldsymbol{\delta}^1_3 = u^{(3)}_1$.

Suppose $A \subseteq \boldsymbol{\delta}^1_3$ is Σ^1_5 and $\sup A < \boldsymbol{\delta}^1_3$. Let B be Π^1_4 such that $w \in \mathrm{WO}^{(3)}_{\omega^{\omega^\omega}} \wedge |w| \in A$ iff $\exists y\ (w, y) \in B$. Consider the game in which I produces v, II produces (w, y). II wins either $v \notin \mathrm{WO}^{(3)}$ or $v, w \in \mathrm{WO}^{(3)} \wedge \|v\| < \|w\| \wedge (w, y) \in B$. I has a winning strategy, and so has a Δ^1_5 winning strategy τ by Moschovakis third periodicity. By boundedness, $\{\|\tau * x\| : x \in \mathbb{R}\}$ has a $\Delta^1_3(\tau)$ bound, hence has a Δ^1_5 bound.

Case 3: $\kappa = \text{seed}^{R^\infty}_{\mathbf{A}} > \boldsymbol{\delta}^1_3$, $\mathbf{A} \in \text{desc}^{**}(R^\infty)$.

Put $\mathbf{A} = (\mathbf{r}, \pi, T)$, $\mathbf{r} = (r, Q, \overrightarrow{(d, q, P)})$, $r = (\xi_i)_{1 \le i \le k}$. Then r is of discontinuous type and ξ_k is a successor ordinal. Put $r' = (\xi_i)_{1 \le i < k}{}^\frown(\xi_k - 1)$, $\mathbf{r}' = (r', Q, \overrightarrow{(d, q, P)})$, $\mathbf{A}' = (\mathbf{r}', \pi, T)$.

Consider the game in which I produces $\langle \ulcorner\tau\urcorner, a^* \rangle$, II produces $(\langle \ulcorner\sigma\urcorner, b^* \rangle, y)$. II wins iff

1. If $\langle \ulcorner\tau\urcorner, a^* \rangle$ is a putative level-3 sharp code for an increasing function, then so is $\langle \ulcorner\sigma\urcorner, b^* \rangle$. Moreover, for any $\eta < \boldsymbol{\delta}^1_3$, if

$$a^* \text{ is } \eta\text{-iterable } \wedge \tau^{\mathcal{M}_{a^*,\eta}}(\eta) \in \text{wfp}(\mathcal{M}_{a^*,\eta})$$

then

$$b^* \text{ is } \eta\text{-iterable } \wedge \sigma^{\mathcal{M}_{b^*,\eta}}(\eta) \in \text{wfp}(\mathcal{M}_{b^*,\eta}) \wedge \tau^{\mathcal{M}_{a^*,\eta}}(\eta) < \sigma^{\mathcal{M}_{b^*,\eta}}(\eta).$$

2. If $\langle \ulcorner\tau\urcorner, a^* \rangle$ is a true level-3 sharp code for an increasing function, $a^* = a^{3\#}$, then $(\langle \ulcorner \sigma^{\underline{j^T(V)}}(\underline{c_{\mathbf{A}'}}) \urcorner, b^* \rangle, y) \in B$.

This game is Σ^1_4 for Player I. Player I has a winning strategy, and so has a Δ^1_5 winning strategy f. Let σ be the $\mathcal{L}^{\underline{x}, R^\infty}$-Skolem term for $c^{(3)}_{y,T,\underline{\pi^T}(\underline{c_{r'}})+\omega}$ where $\underline{x} = y^{3\#}$. Let

$$w = \langle \ulcorner\sigma\urcorner, (\tau^{3\#})^{3\#} \rangle$$

So $w \in \text{WO}^{(3)}_{\omega^{\omega^\omega}}$ is Δ^1_5 and $|w| < \text{seed}^R_{\mathbf{A}}$. We show that $\sup A < |w|$ using a boundedness argument. For each $\eta < \boldsymbol{\delta}^1_3$, Let Z_η be the set of $r \in \text{LO}^{(3)}$ such that there are putative level-3 sharp codes for increasing function on ordinals $\langle \ulcorner\tau\urcorner, a^* \rangle$, $\langle \ulcorner\sigma\urcorner, b^* \rangle$ and an ordinal $\beta \le \eta$ such that

1. $\langle \ulcorner\tau\urcorner, a^* \rangle = f * \langle \ulcorner\sigma\urcorner, b^* \rangle$;

2. for any $\bar\beta < \beta$, b^* is $\bar\beta$-iterable, $\sigma^{M_{b^*,\eta}(\vec{\beta})}(\bar\beta) \in \text{wfp}(\mathcal{M}_{b^*,\eta})$, $\sigma^{\mathcal{M}_{b^*,\eta}}(\bar\beta) \le \eta$;

3. a^* is β-iterable, $\tau^{\mathcal{M}_{a^*,\eta}}(\beta)$ has order type coded in r.

Z_η is a $\boldsymbol{\Sigma}^1_3$ set in the code of η. Since f is a winning strategy for I, $Z_\eta \subseteq \text{WO}^{(3)}$. By Corollary 5.3, $\{\|r\| : r \in Z_\eta\}$ is bounded by $c^{(3)}_{f,\eta+\omega}$. Hence, if $\langle \ulcorner\sigma\urcorner, b^* \rangle$ is a true level-3 sharp code for an increasing function g and $\langle \ulcorner\tau\urcorner, a^* \rangle = f * \langle \ulcorner\sigma\urcorner, b^* \rangle$, then $\langle \ulcorner\tau\urcorner, a^* \rangle$ is a true level-3 sharp code for an increasing function h and for any $\eta < \boldsymbol{\delta}^1_3$ such that $g''\eta \subseteq \eta$, $h(\eta) < c^{(3)}_{f,\eta+\omega}$. Let $\eta \in C$ iff $g''\eta \subseteq \eta$. By Lemma 5.33, for any $\gamma \in j^T(C)$, $j^T(h)(\gamma) < c^{(3)}_{f,T,\gamma+\omega}$. Hence, $\sup A < |w|$. $\square$

Based on Lemma 7.1, the proof of the following theorem is completely in parallel to the level-2 Kechris-Martin theorem in [23] or [39]. It is proved by induction on the $\mathbb{L}[j^{R^\infty}(T_3)]$-cardinality of $\sup(A)$. A key step uses the following observation: by Lemma 5.42, if $|w| < \mathrm{seed}^{R^\infty}_{\mathbf{A}}$ and $\mathbf{A} = \mathrm{succ}_{\prec^{R^\infty}_*}(\mathbf{B})$ then there is a $\Delta^1_5(w)$ surjection from $\mathrm{seed}^{R^\infty}_{\mathbf{B}}$ onto $|w|$.

Theorem 7.2. *Assume $\boldsymbol{\Delta}^1_4$-determinacy. If A is a nonempty $\Pi^1_5(x)$ subset of $u^{(3)}_{\omega^{\omega^\omega}}$, then $\exists w \in \Delta^1_5(x)$ $(|w| \in A)$. So the pointclass Π^1_5 is closed under quantification over $u^{(3)}_{\omega^{\omega^\omega}}$.*

Definition 7.3. κ^x_5 is the least (T_4, x)-admissible ordinal.

Using Theorem 7.2, we obtain the level-4 version of Theorem 2.1. The proof is parallel to [23] and [3], using Moschovakis set induction in one direction and the Becker-Kechris game in the other direction.

Theorem 7.4. *Assume $\boldsymbol{\Delta}^1_4$-determinacy. Then for each $A \subseteq u^{(3)}_{\omega^{\omega^\omega}} \times \mathbb{R}$, the following are equivalent.*

1. *A is Π^1_5.*
2. *There is a Σ_1 formula φ such that $(\alpha, x) \in A$ iff $L_{\kappa^x_5}[T_4, x] \models \varphi(T_4, \alpha, x)$.*

The ordinal κ^x_5 is defined in a different way in [27]:

$$\begin{aligned} \lambda^x_5 &= \sup\{|W| : W \text{ is a } \Delta^1_5(x) \text{ prewellordering on } \mathbb{R}\},\\ \kappa^x_5 &= \sup\{\lambda^{x,y}_5 : M^\#_3(x) \not\leq_{\Delta^1_5} (x,y)\}. \end{aligned}$$

In parallel to [23], these two definitions are equivalent, and in fact,

$$\begin{aligned} \lambda^x_5 &= \sup\{\xi < \kappa^x_5 : \xi \text{ is } \Delta_1\text{-definable over } L_{\kappa^x_5}[T_4, x] \text{ from } \{T_4, x\}\},\\ \kappa^x_5 &= \sup\{\mathrm{o.t.}(W) : W \text{ is a } \Delta^1_5(x, {<}u^{(3)}_{\omega^{\omega^\omega}}) \text{ wellordering on } \mathbb{R}\}. \end{aligned}$$

Moreover,

$$\forall \alpha < u^{(3)}_{\omega^{\omega^\omega}}\ \exists w \in \mathrm{WO}^{(3)}_{\omega^{\omega^\omega}}\ (|w| = \alpha \wedge \lambda^{x,w}_5 < \kappa^x_5).$$

7.2 The equivalence of $x^{4\#}$ and $M^\#_3(x)$

Suppose x is a real and $\beta \leq u^{(3)}_{\omega^{\omega^\omega}}$. A subset $A \subseteq \mathbb{R}$ is β-$\Pi^1_5(x)$ iff there is a $\Pi^1_5(x)$ set $B \subseteq u^{(3)}_{\omega^{\omega^\omega}} \times \mathbb{R}$ such that $A = \mathrm{Diff}\, B$. β-$\Pi^1_5(x)$ acting on product spaces of ω and $\mathbb{R}$ is defined in the obvious way. Lightface β-Π^1_5 and boldface β-$\boldsymbol{\Pi}^1_5$ have the obvious meanings.

In parallel to the proof of Lemma 4.2, we have

Lemma 7.5. *Assume $\mathbf{\Delta}^1_4$-determinacy. Suppose $\xi < \omega^{\omega^\omega}$ and $m < \omega$. If A is $(u^{(3)}_{\xi+1})^m$-$\Pi^1_5(x)$, then A is $\Game^2((\widehat{\xi+1})$-$\Pi^1_3(x))$.*

If S is a finite regular level-3 tree, let S^+ be the level-3 tree extending S where $\text{dom}(S^+) = \text{dom}(S) \cup \{((1))\}$ and $\text{cf}(S((1))) = 0$. Thus, $[\![\emptyset]\!]_{S^+} = [\![\emptyset]\!]_S + \omega$. If $\vec{\xi} = (\xi_s)_{s\in\text{dom}(S^+)}$ respects S^+, put $\vec{\xi}^- = (\xi_s)_{s\in\text{dom}(S)}$.

If $x \in \mathbb{R}$ and $\alpha < \boldsymbol{\delta}^1_3$, let $\mathcal{N}_{\alpha,\infty}(x) = \mathcal{P}_\infty$ where $\|\mathcal{P}\|_{<_{DJ(x)}} = \alpha$. In particular, $\mathcal{N}_{c^{(3)}_{x,\alpha},\infty}(x) = \mathcal{M}^-_{2,\infty}(x)|c^{(3)}_{x,\alpha}$.

Lemma 7.6. *Assume $\mathbf{\Delta}^1_4$-determinacy. Let $\xi < \omega^{\omega^\omega}$. If A is $\Game^2(\widehat{\xi}$-$\Pi^1_3(x))$, then A is $u^{(3)}_{\xi+2}$-$\Pi^1_5(x)$.*

Proof. Let S be a regular level-3 tree such that $[\![\emptyset]\!]_S = \widehat{\xi}$. By Lemma 5.12, if $(y,r) \in \mathbb{R}^2$, $C \subseteq \mathbb{R}$ is $\widehat{\xi}$-$\Pi^1_3(y,r)$, then we can effectively find a formula φ such that Player I has a winning strategy in $G(C)$ iff

$$\ulcorner\varphi(y,r)\urcorner \in (y,r)^{3\#}(S).$$

Suppose $A = \Game B$, where $B \subseteq \mathbb{R}^2$ is $\Game(\widehat{\xi}$-$\Pi^1_3)$. Suppose φ is an $\mathcal{L}$-formula such that

$$(y,r) \in B \leftrightarrow \ulcorner\varphi(y,r,(\underline{c_s})_{s\in\text{dom}(S)})\urcorner \in (y,r)^{3\#}(S).$$

For ordinals $\vec{\xi}$ respecting S^+, say that M is a Kechris-Woodin non-determined set with respect to $(y,\vec{\xi})$ iff

1. M is a countable subset of $\mathbb{R}$.
2. M is closed under join and Turing reducibility.
3. $\forall\sigma \in M\ \exists v \in M\ \mathcal{N}_{\xi_{((1))},\infty}(y,\sigma\otimes v) \models \neg\varphi(y,\sigma\otimes v,\vec{\xi}^-)$.
4. $\forall\sigma \in M\ \exists v \in M\ \mathcal{N}_{\xi_{((1))},\infty}(y,v\otimes\sigma) \models \varphi(y,v\otimes\sigma,\vec{\xi}^-)$.

Say that z is $(y,\vec{\xi})$-stable iff z is not contained in any Kechris-Woodin non-determined set with respect to $(y,\vec{\xi})$. z is y-stable iff z is $(y,\vec{\xi})$-stable for all $\vec{\xi}$ respecting S^+. The set of (y,z) such that z is y-stable is Π^1_4. By the proof of Kechris-Woodin [29], for all $y \in \mathbb{R}$, there is $z \in \mathbb{R}$ which is y-stable. Let $<^{\vec{\xi}}_y$ be the following wellfounded relation on the set of z which is $(y,\vec{\xi})$-stable:

$$\begin{aligned} z' <^{\vec{\xi}}_y z \leftrightarrow\ & z \text{ is } (y,\vec{\xi})\text{-stable} \wedge z \leq_T z' \wedge \\ & \forall\sigma \leq_T z\ \exists v \leq_T z'\ \mathcal{N}_{\xi_{((1))},\infty}(y,\sigma\otimes v) \models \neg\varphi(y,\sigma\otimes v,\vec{\xi}^-) \\ & \forall\sigma \leq_T z\ \exists v \leq_T z'\ \mathcal{N}_{\xi_{((1))},\infty}(y,v\otimes\sigma) \models \varphi(y,v\otimes\sigma,\vec{\xi}^-). \end{aligned}$$

If z is y-stable, let f^z_y be the function that sends $\vec{\xi}$ to the rank of z in $<^{\vec{\xi}}_y$. Then f^z_y is a function into $\boldsymbol{\delta}^1_3$. By Σ^1_4-absoluteness between V and $\mathcal{N}^{\text{Coll}(\omega,\eta)}$, where $\mathcal{N} \in \mathcal{F}_{2,\infty}(y,z)$ and $\pi_{\mathcal{N},\infty}(\eta) = \xi_{((1))}$, we can see $f^z_y \restriction \{\vec{\xi} \in [\boldsymbol{\delta}^1_3]^{S\uparrow} : \xi_{((1))}$ is a cardinal cutpoint of $M^-_{2,\infty}(y,z)\}$ is definable over $M^-_{2,\infty}(y,z)$ in a uniform way, so there is a $\mathcal{L}^S$-Skolem term τ such that for all $(y,z) \in \mathbb{R}^2$, if z is y-stable, $\vec{\xi}$ respects S, $\xi_{((1))}$ is a cardinal cutpoint of $M^-_{2,\infty}(y,z)$, then

$$f^z_y(\vec{\xi}) = \tau^{M^-_{2,\infty}(y,z)}(y,z,\vec{\xi}).$$

Let

$$\beta^z_y = \tau^{j^{S^+}(M^-_{2,\infty}(y,z))}(y,z,\text{seed}^{R^+}).$$

The function

$$(y,z) \mapsto \beta^z_y$$

is Δ^1_5 in the level-3 sharp codes. The rest is in parallel to the proof of Lemma 4.3. □

Lemma 7.5 and Lemma 7.6 are concluded in a simple equality between pointclasses.

Theorem 7.7. *Assume $\boldsymbol{\Delta}^1_4$-determinacy. Then for $x \in \mathbb{R}$,*

$$\Game^2(<u_\omega\text{-}\Pi^1_3(x)) = <u^{(3)}_{\omega^{\omega^\omega}}\text{-}\Pi^1_5(x).$$

Hence by Theorem 4.5,

$$\Game^4(<\omega^2\text{-}\Pi^1_1(x)) = <u^{(3)}_{\omega^{\omega^\omega}}\text{-}\Pi^1_5(x).$$

The level-4 sharp is defined in parallel to the end of Section 4.1.

Definition 7.8.

$$\mathcal{O}^{T_4,x} = \{(\ulcorner\varphi\urcorner,\alpha) : \varphi \text{ is a } \Sigma_1\text{-formula}, \alpha < u^{(3)}_{\omega^{\omega^\omega}}, L_{\kappa^x_5}[T_4,x] \models \varphi(T_4,x,\alpha).\}$$

Definition 7.9.

$$x^{4\#}_\xi = \{(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner) : \exists\alpha < u^{(3)}_\xi((\ulcorner\varphi\urcorner,\alpha) \notin \mathcal{O}^{T_4,x} \wedge \forall\eta < \alpha(\ulcorner\psi\urcorner,\eta) \in \mathcal{O}^{T_4,x})\}.$$

$$x^{4\#} = \{(n,\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner) : n < \omega \wedge (\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner) \in x^{4\#}_{\omega^{\omega^n}}\}.$$

Applying Theorem 7.7 to the space ω, in combination with Theorem 7.4, we reach the equivalence between $x^{4\#}$ and $M^\#_3(x)$.

Theorem 7.10. *Assume $\boldsymbol{\Delta}^1_4$-determinacy. Then $x^{4\#}$ is many-one equivalent to $M^\#_3(x)$, the many-one reductions being independent of x.*

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