

THE HALPERN-LÄUCHLI THEOREM AT A MEASURABLE CARDINAL

NATASHA DOBRINEN AND DAN HATHAWAY

ABSTRACT. Several variants of the Halpern-Läuchli Theorem for trees of uncountable height are investigated. For κ weakly compact, we prove that the various statements are all equivalent, and hence, the strong tree version holds for one tree on any weakly compact cardinal. For any finite $d \geq 2$, we prove the consistency of the Halpern-Läuchli Theorem on d many normal κ -trees at a measurable cardinal κ , given the consistency of a $\kappa + d$ -strong cardinal. This follows from a more general consistency result at measurable κ , which includes the possibility of infinitely many trees, assuming partition relations which hold in models of AD.

1. INTRODUCTION

Halpern and Läuchli proved their celebrated theorem regarding Ramsey theory on products of trees in [11] as a necessary step for the construction in [12] a model of ZF in which the Boolean Prime Ideal Theorem holds but the Axiom of Choice fails. Over the years, many variations have been investigated and applied. One of the earliest of these is due to Milliken, who in [15], extended theorem of Halpern and Läuchli to colorings of finite products of strong trees. As the two are equivalent in ZFC, this version is often used synonymously with Halpern and Läuchli's original version. Milliken further proved in [16] that the collection of strong trees forms, in modern terminology, a topological Ramsey space. Further variations and applications include ω many perfect trees in [14]; partitions of products in [2]; the density version in [4]; the dual version in [20]; canonical equivalence relations on finite strong trees in [21]; and applications to colorings of subsets of the rationals in [1] and [22], finite Ramsey degrees of the Rado graph in [17] which in turn was applied to show the Rado graph has the rainbow Ramsey property in [3], to name just a few.

The first result generalizing the Halpern-Läuchli Theorem to an uncountable height tree was given by Shelah in [18]. There, in Lemma 4.1, Shelah proved that it is consistent with ZFC that given any finite m and any coloring of the m -sized subsets of the levels of the tree ${}^{<\kappa}2$, (that is, a coloring of $\bigcup_{\zeta < \kappa} [{}^\zeta 2]^m$), into less than κ many colors for κ measurable, then there is a strong subtree (see Definition 2.1) $T \subseteq {}^{<\kappa}2$ on which the coloring takes only finitely many colors. Shelah's proof builds on and extends Harrington's forcing argument for the Halpern-Läuchli Theorem on ${}^{<\omega}2$, and assumes that κ is a cardinal which is measurable after adding λ many

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Cohen subsets of κ , where λ is large enough that the partition relation $\lambda \rightarrow (\kappa^+)_{2^\kappa}^{2^m}$ holds. Thus, his result also holds for any κ which is supercompact after a Laver treatment.

In [5], Dzamonja, Larson and Mitchell extended Shelah's proof to include colorings of all antichains in the tree ${}^{<\kappa}2$ of a fixed finite size m , rather than just colorings of subsets of the same levels. They proved that given a coloring of m -sized antichains of the tree ${}^{<\kappa}2$ with less than κ many colors, there is a strong subtree T isomorphic to ${}^{<\kappa}2$ on which the set of m -sized antichains takes on only finitely many colors. These finitely many colors are classified by the embedding types of the trees induced by the antichains. They then applied this result to find the Ramsey degrees of $< \kappa$ colorings of $[\mathbb{Q}_\kappa]^m$, m any finite integer, and of $< \kappa$ colorings of the copies of a fixed finite graph inside the Rado graph on κ many vertices in [5] and [6], respectively.

This work has left open the following questions.

Question 1.1. For which uncountable cardinals κ can the Halpern-Läuchli Theorem hold for trees of height κ , either in ZFC or consistently?

Question 1.2. What is the consistency strength of the Halpern-Läuchli Theorem for κ uncountable; in particular for κ a measurable cardinal?

Question 1.3. Given a fixed number of trees, are the weaker (somewhere dense) and stronger (strong tree) forms of the Halpern-Läuchli Theorem equivalent, for κ uncountable?

In Section 3, we answer Question 1.3 for the case of weakly compact cardinals: Theorem 3.7 shows that when κ is weakly compact, all the various weaker and stronger forms of the Halpern-Läuchli Theorem on δ many regular κ trees are equivalent, where δ can be any cardinal less than κ . In Fact 2.4, we show that the somewhere dense version for finitary colorings on one tree on any cardinal is a ZFC result. Hence, if κ is weakly compact, then the strong tree version of the Halpern-Läuchli Theorem holds for finitary colorings of one regular subtree tree of ${}^{<\kappa}\kappa$, thus answering Question 1.1 for the case of one tree and finitely many colors.

In Section 4, for any fixed positive integer d , Theorem 4.6 shows that it is consistent for the Halpern-Läuchli Theorem to hold on d many regular trees on a measurable cardinal κ . Further, this theorem yields that a $\kappa + d$ -strong cardinal is an upper bound for the consistency strength of the Halpern-Läuchli Theorem for d many regular trees on a measurable cardinal κ , thus partially answering Questions 1.1 and 1.2. This is weaker than the upper bound of a cardinal κ which is $\kappa + 2d$ -strong, that would be needed if one simply lifted the standard version of Harrington's forcing proof to a measurable cardinal. This result also presents an interesting contrast to the $(\kappa + 2d + 2)$ -strong cardinal mentioned as an upper bound for the consistency strength of Lemma 4.1 in [18] and its strengthening Theorem 2.5 in [5], which colors antichains of size d in one tree on κ . We conjecture in Section 5 that the consistency strength of the Halpern-Lauchli Theorem on d many regular trees for κ measurable is in fact a $\kappa + d$ -strong cardinal. Theorem 4.6 follows from a more general result, Theorem 4.3, which includes the possibility of infinitely many trees assuming certain partition relations which hold assuming AD. We provide this generalization with the aim of future applications.

Finally, we consider when forcing will preserve the Halpern-Läuchli Theorem. In Section 5, Theorem 5.5 shows that whenever κ is measurable and the Halpern-Läuchli Theorem holds at κ , then this is preserved by every $< \kappa$ -closed forcing. Along the way, we prove that for κ measurable, the Halpern-Läuchli Theorem holds at κ if and only if it holds on a stationary set of ordinals below κ (see Propositions 5.3 and 5.4). Section 6 concludes with open problems and their relationships with previous work.

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2. VARIANTS OF THE HALPERN-LÄUCHLI THEOREM AND SIMPLE IMPLICATIONS

This section contains the relevant definitions, various versions of the Halpern-Läuchli Theorem, and the immediate implications between them. A tree $T \subseteq {}^{<\kappa}\kappa$ is a κ -tree if T has cardinality κ and every level of T has cardinality less than κ . T is *perfect* if for each node t in T there is an extension of t which splits in T . We shall call a tree $T \subseteq {}^{<\kappa}\kappa$ *regular* if it is a perfect κ -tree in which every maximal branch has cofinality κ . Given any subset $T \subseteq {}^{<\kappa}\kappa$ and $\zeta < \kappa$, we write

$$T(\zeta) := T \cap {}^\zeta\kappa$$

for the set of nodes on the ζ -th level of T . Given $t \in T$,

$$T[t] := \{s \in T : s \sqsubseteq t \text{ or } t \sqsubseteq s\}.$$

Next, we define the notion of strong subtree, originally defined by Milliken in [15].

Definition 2.1. Let $T \subseteq {}^{<\kappa}\kappa$ be regular. A tree $T' \subseteq T$ is a *strong subtree* of T as witnessed by some set $A \subseteq \kappa$ cofinal in κ if and only if T' is regular and for each $t \in T'(\zeta)$ for $\zeta < \kappa$, $\zeta \notin A$ implies that t has a unique successor in T' on level $\zeta + 1$, and $\zeta \in A$ implies every successor of t in T is also in T' .

Given an ordinal $\delta > 0$ and a sequence $\langle X_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$, define

$$\bigotimes_{i < \delta} X_i := \{ \langle x_i : i < \delta \rangle : (\exists \zeta < \kappa) (\forall i < \delta) x_i \in X_i(\zeta) \},$$

the *level-product* of the X_i 's. We will call a set of nodes all on the same level a *level set*. Similarly, we will call a sequence of nodes or a sequence of sets of nodes a *level sequence* if all nodes are on the same level.

The following is the strong tree version of the Halpern-Läuchli Theorem, which we shall denote by $\text{HL}(\delta, \sigma, \kappa)$.

Definition 2.2. For $\delta, \sigma > 0$ ordinals and κ an infinite cardinal, $\text{HL}(\delta, \sigma, \kappa)$ is the following statement: Given any sequence $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ of regular trees and a coloring $c : \bigotimes_{i < \delta} T_i \rightarrow \sigma$, there exists a sequence of trees $\langle T'_i : i < \delta \rangle$ such that

- (1) each T'_i is a strong subtree of T_i as witnessed by the same set $A \subseteq \kappa$ independent of i , and
- (2) there is some $\sigma' < \sigma$ such that for each $\zeta \in A$, $c \upharpoonright \bigotimes_{i < \delta} T'_i(\zeta) = \{\sigma'\}$.

Given $t \in {}^{<\kappa}\kappa$, we define

$$\text{Cone}(t) := \{s \in {}^{<\kappa}\kappa : s \sqsupseteq t\}.$$

We say that $X \subseteq {}^{<\kappa}\kappa$ *dominates* $Y \subseteq {}^{<\kappa}\kappa$ if and only if

$$(\forall y \in Y)(\exists x \in X) y \sqsubseteq x.$$

We say $X \subseteq {}^{<\kappa}\kappa$ *dominates* $y \in {}^{<\kappa}\kappa$ just when X dominates $\{y\}$. We now give the definition of the Somewhere-Dense Halpern-Läuchli Theorem.

Definition 2.3. For nonzero ordinals δ and σ , and an infinite cardinal κ , $\text{SDHL}(\delta, \sigma, \kappa)$ is the statement that given any sequence $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ of regular trees and any coloring

$$c : \bigotimes_{i < \delta} T_i \rightarrow \sigma,$$

there exist $\zeta < \zeta' < \kappa$, $\langle t_i \in T_i(\zeta) : i < \delta \rangle$, and $\langle X_i \subseteq T_i(\zeta') : i < \delta \rangle$ such that each X_i dominates $T_i(\zeta + 1) \cap \text{Cone}(t_i)$ and

$$|c^{\langle \bigotimes_{i < \delta} X_i \rangle}| = 1.$$

We point out the following simple fact.

Fact 2.4. For every infinite cardinal κ and each positive integer k , $\text{SDHL}(1, k, \kappa)$ holds.

Proof. If $k = 1$, the result is immediate, so assume $k \geq 2$. To prove $\text{SDHL}(1, k, \kappa)$, let T be a regular subtree of ${}^{<\kappa}\kappa$ and c be a coloring from the nodes in T into k . If there exist a node $t \in T$ and a level set $X \subseteq T$ dominating the immediate successors of t such that every node in X has color 0, then we are done. Otherwise, for each node $t \in T$ and each level set $X \subseteq T$ dominating the immediate successors of t , $c^{\langle X \cap \{1, \dots, k-1\} \rangle} \neq \emptyset$. If there exist a node $t \in T$ and a level set $X \subseteq T$ dominating the immediate successors of t such that every node in X has color 1, then we are done. Otherwise, for each node $t \in T$ and each level set $X \subseteq T$ dominating the immediate successors of t , $c^{\langle X \cap \{2, \dots, k-1\} \rangle} \neq \emptyset$. Continuing in this manner, either there is a $j < k-1$ and some node t with a level set X dominating the immediate successors of t each with c -color j , or else, for each node $t \in T$ and each level set $X \subseteq T$ dominating the immediate successors of t , $c^{\langle X \cap \{k-1\} \rangle} \neq \emptyset$. In this case, choose any node $t \in T$ and list the immediate successors of t in T as $\langle s_i : i < \eta \rangle$, where $\eta < \kappa$. For each $i < \eta$, let Y_i denote the set of all the immediate successors of s_i . Then Y_i forms a level set dominating the immediate successors of s_i , so $c^{\langle Y_i \cap \{k-1\} \rangle} \neq \emptyset$. For each $i < \eta$, take one $t_i \in Y_i$ such that $c(t_i) = k-1$. Then the set $\{t_i : i < \eta\}$ is a level set dominating every immediate successor of t and is monochromatic in color $k-1$. \square

Definition 2.5. Given a regular tree T and a level ζ less than the height of T , a set $X \subseteq T$ is ζ -dense if and only if X dominates $T(\zeta)$. Given a sequence of regular trees $\langle T_i : i < \delta \rangle$ and $\vec{x} = \langle x_i : i < \delta \rangle \in \bigotimes_{i < \delta} T_i$, a sequence of sets $X_i \subseteq T_i$ for $i < \delta$ is ζ - \vec{x} -dense if and only if for each $i < \delta$, X_i dominates $T_i(\zeta) \cap \text{Cone}(x_i)$.

The definition of $\text{SDHL}(\delta, \sigma, \kappa)$ can be weakened by not requiring the sets X_i for $i < \delta$ to be level sets; in this case the colorings must color the full product $\prod_{i < \delta} T_i$. Call this weakening $\text{SDHL}'(\delta, \sigma, \kappa)$. Following the terminology in [19], a *somewhere dense matrix* for a sequence of regular trees $\langle T_i : i < \delta \rangle$ is a sequence

of sets $\langle X_i \subseteq T_i : i < \delta \rangle$ such that there are nodes $t_i \in T_i$ for $i < \delta$ all of the same length such that each X_i dominates all successors of t_i in T_i . Thus, $\text{SDHL}'(\delta, \sigma, \kappa)$ is the statement that any coloring $c : \prod_{i < \delta} T_i \rightarrow \sigma$ is constant on $\prod_{i < \delta} X_i$ for some somewhere dense matrix $\langle X_i \subseteq T_i : i < \delta \rangle$. Certainly $\text{SDHL}(\delta, \sigma, \kappa)$ implies $\text{SDHL}'(\delta, \sigma, \kappa)$.

Remark 2.6. The $\text{SDHL}'(\delta, \sigma, \kappa)$ is in fact equivalent to the statement obtained by making the following two modifications. The first modification is to allow the nodes t_i to come from different levels of T_i , while requiring that the supremum of the length of the members of $\{t_i : i < \delta\}$ to be strictly less than the length of any member of $\bigcup\{X_i : i < \delta\}$. It is routine to check that the statement under this modification is equivalent to $\text{SDHL}'(\delta, \sigma, \kappa)$. The second modification is to allow for each X_i to instead dominate all members of $T_i(\zeta) \cap \text{Cone}(t_i)$ for some $\zeta > \text{lh}(t_i)$ (but this implies that X_i dominates $T_i(\text{lh}(t_i) + 1) \cap \text{Cone}(t_i)$ so the statement is not affected).

By the definitions and previous discussions, the following implications are immediate.

Fact 2.7. *The following statements are arranged from strongest to weakest, where δ and σ are assumed to be strictly less than κ .*

- (1) $HL(\delta, \sigma, \kappa)$;
- (2) $SDHL(\delta, \sigma, \kappa)$;
- (3) $SDHL'(\delta, \sigma, \kappa)$;
- (4) $SDHL'(\delta, \sigma, \kappa)$ where the t_i need not be from the same level.

The remark above shows that in fact (4) implies (3). In the next section will show that (3) implies (1). Thus, all four statements are equivalent.

3. THE VARIOUS FORMS OF HALPERN-LÄUCHLI ARE EQUIVALENT, FOR κ WEAKLY COMPACT

This section provides proofs that all the versions of the Halpern-Läuchli Theorem stated in the previous section are equivalent, provided that κ is weakly compact.

First, we prove that if κ is weakly compact and $\delta, \sigma < \kappa$, then $\text{SDHL}'(\delta, \sigma, \kappa)$ implies $\text{SDHL}(\delta, \sigma, \kappa)$. The proof proceeds via the next three lemmas. Recall that a cardinal κ is *weakly compact* if κ is strongly inaccessible and satisfies the Tree Property at κ ; that is, every κ -tree has a cofinal branch.

Lemma 3.1. *Suppose κ is weakly compact. Let $\delta, \sigma < \kappa$ be non-zero ordinals, and let $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ be a sequence of regular trees. Let U be the tree of partial colorings of $\prod_{i < \delta} T_i$ into σ many colors, where a node of U on level α corresponds to a coloring of all tuples $\langle t_i \in T_i : i < \delta \rangle$ where $\sup\{\text{lh}(t_i) : i < \delta\} < \alpha$.*

Suppose $Z \subseteq U$ has size κ and each $z \in Z$ corresponds to a coloring c_z which is not monochromatic on any somewhere dense matrix contained in its domain. Then there is a coloring c of all of $\prod_{i < \delta} T_i$ such that c is not monochromatic on any somewhere dense matrix.

Proof. Note that if $\langle u_\alpha : \alpha < \beta \rangle$ is a strictly increasing sequence of nodes of U for some limit ordinal β , then there is a unique node on level $\sup_{\alpha < \beta} \text{lh}(u_\alpha)$ that extends each u_α . Furthermore, note that U is a κ tree. Fix a set $Z \subseteq U$ as in the hypotheses.

Let S be the set of all predecessors of elements of Z . Then S is a κ -tree. By the weak compactness of κ , fix a length κ branch through S . This branch corresponds to a coloring c of all of $\prod_{i<\delta} T_i$. Now, for each $\alpha < \kappa$ there is a node $z \in Z$ such that the domain of c_z includes $\langle t_i \in \prod_{i<\delta} \bigcup_{\zeta<\alpha} T_i(\zeta) \rangle$ and both c and c_z color those sequences the same way. For each $\alpha < \kappa$, choose one such node and label it z_α .

Now, pick any somewhere dense $\langle X_i \subseteq T_i : i < \delta \rangle$ such that there is some $\alpha < \kappa$ satisfying $(\forall i < \delta)(\forall x \in X_i) \text{lh}(x) < \alpha$. Fix such an $\alpha < \kappa$. We must show that $|c^{\prod_{i<\delta} X_i}| > 1$. We have that for all $\langle x_i \in X_i : i < \delta \rangle$,

$$c(\langle x_i : i < \delta \rangle) = c_{z_\alpha}(\langle x_i : i < \delta \rangle).$$

By the hypothesis on c_{z_α} , we have $|c_{z_\alpha}^{\prod_{i<\delta} X_i}| > 1$. Thus, $|c^{\prod_{i<\delta} X_i}| > 1$ as desired. \square

Lemma 3.2. *Suppose κ is weakly compact. Let $\sigma, \delta > 0$ be ordinals and assume $\delta, \sigma < \kappa$. Assume $\text{SDHL}'(\delta, \sigma, \kappa)$ holds. Let $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ be a sequence of regular trees. Then there is a level $1 \leq \zeta < \kappa$ such that for every coloring*

$$c : \prod_{i<\delta} \bigcup_{\xi<\zeta} T_i(\xi) \rightarrow \sigma,$$

there is a somewhere dense matrix $\langle X_i \subseteq \bigcup_{\xi<\zeta} T_i(\xi) : i < \delta \rangle$ such that c is constant on $\prod_{i<\delta} X_i$.

Proof. Assume there is no such level ζ . Then for each $\zeta < \kappa$ we may pick a coloring $c_\zeta : \prod_{i<\delta} \bigcup_{\xi<\zeta} T_i(\xi) \rightarrow \sigma$ which is not constant on any somewhere dense matrix in the domain of c_ζ . Letting U be the tree of partial colorings described in Lemma 3.1, the set $Z = \{c_\zeta : \zeta < \kappa\}$ is a subset of U of size κ . Applying Lemma 3.1, we obtain a coloring $c : \prod_{i<\delta} T_i \rightarrow \sigma$ for which there is no somewhere dense matrix $\langle X_i \subseteq T_i : i < \delta \rangle$ such that $|c^{\prod_{i<\delta} X_i}| = 1$. Hence, $\text{SDHL}'(\delta, \sigma, \kappa)$ fails, which is a contradiction. \square

The following bounded version is the analogue of the Finite Halpern-Läuchli Theorem for finitely many trees on ω (see, for instance, Theorem 3.9 in [19]).

Lemma 3.3. *Suppose κ is weakly compact. Let $\sigma, \delta < \kappa$ be nonzero ordinals and assume $\text{SDHL}'(\delta, \sigma, \kappa)$ holds. Let $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ be a sequence of regular trees. Then there is a level $1 \leq \zeta < \kappa$ such that for every coloring*

$$\bar{c} : \bigotimes_{i<\delta} T_i(\zeta) \rightarrow \sigma,$$

there is a somewhere dense matrix $\langle Y_i \subseteq T_i(\zeta) : i < \delta \rangle$ such that \bar{c} is constant on $\bigotimes_{i<\delta} Y_i$.

Proof. Fix some $\zeta < \kappa$ for which the Lemma 3.2 holds. For each $i < \delta$ and each node $t \in \bigcup_{\xi<\zeta} T_i(\xi)$, associate a node $f_i(t) \in T_i(\zeta)$ such that $f_i(t) \supseteq t$. Now fix a coloring $\bar{c} : \bigotimes_{i<\delta} T_i(\zeta) \rightarrow \sigma$. Let $c : \prod_{i<\delta} \bigcup_{\xi<\zeta} T_i(\xi) \rightarrow \sigma$ be defined by

$$c(\langle t_i : i < \delta \rangle) := \bar{c}(\langle f_i(t_i) : i < \delta \rangle).$$

By the property of ζ , there is a somewhere dense matrix $\langle X_i \subseteq \bigcup_{\xi<\zeta} T_i(\xi) : i < \delta \rangle$ such that $|c^{\prod_{i<\delta} X_i}| = 1$. For each $i < \delta$, let

$$Y_i := \{f_i(t) : t \in X_i\}.$$

The sequence $\langle Y_i \subseteq T_i(\zeta) : i < \delta \rangle$ is a somewhere dense matrix, and by the definition of c we have $|c'' \bigotimes_{i < \delta} Y_i| = 1$. \square

Remark 3.4. We point out that the set of $\zeta < \kappa$ satisfying Lemma 3.2 is closed upwards, and hence Lemma 3.3 also holds for ζ in a final segment of κ .

By Lemmas 3.1, 3.2, and 3.3, and the remarks in this section, we have shown that the variants in this section are equivalent, when κ is weakly compact.

Theorem 3.5. *Suppose κ is weakly compact. Let $\sigma, \delta > 0$ be ordinals and assume $\delta, \sigma < \kappa$. Then $SDHL'(\delta, \sigma, \kappa)$ holds if and only if $SDHL(\delta, \sigma, \kappa)$ holds.*

It is not known for an arbitrary κ whether or not the somewhere dense version $SDHL(\delta, \sigma, \kappa)$ implies the strong tree version $HL(\delta, \sigma, \kappa)$. However, if κ weakly compact, they are in fact equivalent. The proof of this involves the following lemma.

Lemma 3.6. *Let $\kappa > 0$ be a cardinal and let $\delta, \sigma < \kappa$ be non-zero ordinals. Let $\langle T_i : i < \delta \rangle$ be a sequence of regular trees. Fix $c : \bigotimes_{i < \delta} T_i \rightarrow \sigma$. Suppose there is a level sequence $\vec{x} = \langle x_i \in T_i : i < \delta \rangle$ such that for each $\xi < \kappa$, there is a level sequence $\langle X_i \subseteq T_i(\zeta) : i < \delta \rangle$ for some ζ such that $\langle X_i : i < \delta \rangle$ is ξ - \vec{x} -dense and $|c'' \bigotimes_{i < \delta} X_i| = 1$. Then the conclusion of $HL(\delta, \sigma, \kappa)$ holds.*

Proof. We will construct a sequence of strong subtrees $\langle T'_i \subseteq T_i : i < \delta \rangle$, as witnessed by the same set $A \subseteq \kappa$ independent of i , such that for each $\zeta \in A$, there is some $\sigma_\zeta < \sigma$ such that $c'' \bigotimes_{i < \delta} T'_i(\zeta) = \{\sigma_\zeta\}$. Then, by the pigeonhole principal, there will be κ many σ_ζ that are equal to the same ordinal, call it σ' . Let $\tilde{A} \subseteq A$ be the set of levels associated to the color σ' . We may then thin each T'_i to a strong subtree T''_i as witnessed by \tilde{A} (independent of i) such that for each $\zeta \in \tilde{A}$, $c'' \bigotimes_{i < \delta} T''_i(\zeta) = \{\sigma'\}$. This is the conclusion of $HL(\delta, \sigma, \kappa)$.

For each $i < \delta$, instead of directly constructing T'_i , we will construct a subset S_i of T'_i and T'_i will be the set of all initial segments of elements of S_i . Each S_i will be the union of level sets: $S_i := \bigcup_{\zeta \in A} L_{i,\zeta}$ where each $L_{i,\zeta}$ will be a subset of $T_i(\zeta)$. At the same time, we will construct $A \subseteq \kappa$. We will have it so $(\forall \zeta \in A) |c'' \bigotimes_{i < \delta} L_{i,\zeta}| = 1$. Initially the set A is empty and no $L_{i,\zeta}$'s have been defined.

Now assume we are at some stage of the construction. There are three cases:

Case 1: No $L_{i,\zeta}$'s have been constructed so far (and A is empty). In this case, set ξ to be the level of the x_i and for each $i < \delta$, let $U_{i,\xi+1} := T_i(\xi+1) \cap \text{Cone}(x_i)$.

Case 2: Some $L_{i,\zeta}$'s have been constructed and there is a largest $\xi < \kappa$ for which some $L_{i,\xi}$ exists. Fix this ξ . For each $i < \delta$, define $U_{i,\xi+1} := T_i(\xi+1) \cap (\bigcup_{t \in L_{i,\xi}} \text{Cone}(t))$.

Case 3: The set of η 's for which the $L_{i,\eta}$'s exist is below but cofinal in some fixed $\xi < \kappa$. Let $W_i \subseteq T_i(\xi)$ be the set of $t \in T_i(\xi)$ such that the set of $\eta < \xi$ such that $t \upharpoonright \eta \in L_{i,\eta}$ is cofinal in ξ . For each $i < \delta$, define $U_{i,\xi+1} := T_i(\xi+1) \cap (\bigcup_{t \in W_i} \text{Cone}(t))$.

Assuming one of the three cases above holds, we have both an ordinal ξ and a set $U_{i,\xi+1} \subseteq T_i(\xi+1)$ for each $i < \delta$. Apply the hypothesis of the lemma to get a level sequence $\langle X_i : i < \delta \rangle$ that is $(\xi+1)$ - \vec{x} -dense and $|c'' \bigotimes_{i < \delta} X_i| = 1$. For each $i < \delta$, let $X'_i \subseteq X_i$ be such that each $t \in X'_i$ extends some $u \in U_{i,\xi+1}$, every $u \in U_{i,\xi+1}$ is extended by some $t \in X'_i$, and no two elements of X'_i extend the same element of $U_{i,\xi+1}$. Let ζ be the level of the X'_i . Add ζ to the set A and set $L_{i,\zeta} := X'_i$ for each $i < \delta$. This completes the construction and the proof. \square

Theorem 3.7. *Let $\kappa > 0$ be a weakly compact cardinal and let $\delta, \sigma < \kappa$ be non-zero ordinals. Then $\text{SDHL}(\delta, \sigma, \kappa)$ implies $\text{HL}(\delta, \sigma, \kappa)$.*

Proof. Fix $c : \bigotimes_{i < \delta} T_i \rightarrow \sigma$. Using Lemma 3.3, we claim that there is a level sequence $\vec{x} = \langle x_i \in T_i : i < \delta \rangle$ such that for each $\zeta < \kappa$, there is a level sequence $\langle X_i \subseteq T_i : i < \delta \rangle$ that is ζ - \vec{x} -dense and $|c^{\llbracket \bigotimes_{i < \delta} X_i \rrbracket}| = 1$. From this, the conclusion of $\text{HL}(\delta, \sigma, \kappa)$ follows from Lemma 3.6. To show the claim, suppose towards a contradiction that it is false. For each $\vec{x} \in \bigotimes_{i < \delta} T_i$ we can associate the first $\zeta(\vec{x}) < \kappa$ for which we cannot find a $\zeta(\vec{x})$ - \vec{x} -dense level sequence $\langle X_i : i < \delta \rangle$ such that $|c^{\llbracket \bigotimes_{i < \delta} X_i \rrbracket}| = 1$. We can build a strictly increasing sequence $\langle \eta_\alpha \in \kappa : \alpha < \kappa \rangle$ of ordinals such that for every $\alpha < \kappa$,

$$\eta_{\alpha+1} > \zeta(\vec{x}) \text{ for all } \vec{x} \in \bigcup_{\eta \leq \eta_\alpha} \bigotimes_{i < \delta} T_i(\eta).$$

For $i < \delta$, set

$$T_i^* := \bigcup_{\alpha < \kappa} T_i(\eta_\alpha).$$

Applying $\text{SDHL}(\delta, \sigma, \kappa)$ to the restriction of c to $\bigotimes_{i < \delta} T_i^*$, we find a somewhere dense matrix $\langle X_i^* : i < \delta \rangle$ of the sequence of trees $\langle T_i^* : i < \delta \rangle$ on which c is constant. This means that there exists $\alpha < \alpha' < \kappa$ and $\vec{x} = \langle x_i : i < \delta \rangle \in \bigotimes_{i < \delta} T_i(\eta_\alpha)$ such that for all $i < \delta$,

$$X_i^* \text{ dominates } \text{Cone}(x_i) \cap T_i(\eta_{\alpha'}).$$

It follows that $\langle X_i : i < \delta \rangle$ is an $\eta_{\alpha'}$ - \vec{x} -dense c -monochromatic level sequence, contradicting the definition of $\zeta(\vec{x})$ and the fact that

$$\eta_{\alpha'} \geq \eta_{\alpha+1} > \zeta(\vec{x}).$$

This finishes the proof. \square

Fact 2.4 and Theorem 3.7 yield the following.

Theorem 3.8. *$\text{HL}(1, k, \kappa)$ holds for each positive integer k and each weakly compact cardinal κ .*

4. THE HALPERN-LÄUCHLI THEOREM AT A MEASURABLE CARDINAL

In Theorem 4.6, we prove that for any positive integer d , assuming the existence of a cardinal κ which is $\kappa + d$ -strong cardinal (see Definition 4.5), it is consistent that $\text{HL}(d, \sigma, \kappa)$ holds at a measurable cardinal κ , for all $\sigma < \kappa$. Actually, in Theorem 4.3 we prove an asymmetric version of the $\text{SDHL}(\delta, \sigma, \kappa)$ which holds for finitely or infinitely many trees on a measurable cardinal. The requisites for that theorem are that certain partition relations hold in a model M of ZF and that κ remains measurable after forcing over M with $\text{Add}(\kappa, \lambda)$, where $\text{Add}(\kappa, \lambda)$ denotes the forcing which adds λ many κ -Cohen subsets of κ via partial functions from $\lambda \times \kappa$ into 2 of size less than κ and λ is large enough for a certain partition relation to hold. For infinitely many trees, the relevant partition relations hold assuming AD, and hence in $L(\mathbb{R})$ assuming in V the existence of a limit of Woodin cardinals with a measurable above. However, we do not currently know of such a model of ZF over which forcing with many Cohen subsets of a measurable cardinal preserves the measurability (see Question 6.6 in the next section). Nevertheless, we prove Theorem 4.3 in this generality with the optimism that such a model will be found.

We point out that Harrington's original forcing proof of $\text{HL}(d, k, \omega)$ can be lifted, with minor modifications, to obtain $\text{HL}(d, \sigma, \kappa)$ for κ measurable and $\sigma < \kappa$, provided, as in [18], we work in a model of ZFC in which κ remains measurable after forcing with $\text{Add}(\kappa, \lambda)$, where λ satisfies the partition relation $\lambda \rightarrow (\kappa^+)_2^{2d}$. In order for this to hold via methods of Woodin, one needs to assume the existence of a cardinal κ which is $\kappa + 2d$ -strong. Thus, in a somewhat straightforward manner, combining known results, one may arrive at a $\kappa + 2d$ -strong cardinal as an upper bound for the consistency strength of $\text{HL}(d, \sigma, \kappa)$ for κ measurable and $\sigma < \kappa$.

However, we are interested in the actual consistency strength of $\text{HL}(d, \sigma, \kappa)$ for κ measurable. The version of Harrington's forcing proof for perfect subtrees of ${}^{<\omega}2$ given by Todorćević in Theorem 8.15 of [8] uses the weaker partition relation $\lambda \rightarrow (\aleph_0)_2^d$, rather than the usual $\lambda \rightarrow (\aleph_0)_2^{2d}$. If we could lift his argument to a measurable cardinal, this would bring the consistency strength down to a d -strong cardinal. One of the key lemmas in Todorćević's proof, though, relies strongly on the fact that \aleph_0 is the least infinite cardinal and could not be generalized to uncountable κ . However, using that κ is measurable, we found a way around this. We modified Harrington's argument and proved Lemma 4.2 which uses partition relations which are satisfied when κ is measurable (assuming $\delta < \omega$). This lemma aids in bringing the upper bound of the consistency strength of $\text{HL}(d, \sigma, \kappa)$ for κ measurable down to a $\kappa + d$ -strong cardinal in Theorem 4.6.

Definition 4.1. Temporarily let \mathbb{P} be the forcing to add λ Cohen subsets of κ . A collection $\mathcal{X} \subseteq \mathbb{P}$ is called *image homogenized* if for all $p_1, p_2 \in \mathcal{X}$ and $\xi, \alpha, \beta \in \text{Ord}$, if α is the ξ -th element of $\text{Dom}(p_1)$ and β is the ξ -th element of $\text{Dom}(p_2)$, then $p_1(\alpha) = p_2(\beta)$.

Suppose that we have parameterized conditions in \mathbb{P} according to $[\lambda]^\delta$. That is, for each $\vec{\alpha} \in [\lambda]^\delta$ we have some $p_{\vec{\alpha}}$. For each $\vec{\alpha} \in [\lambda]^\delta$, let

$$\langle \nu(\vec{\alpha}, \xi) : \xi < \text{ot}(\text{Dom}(p_{\vec{\alpha}})) \rangle$$

be the increasing enumeration of the elements of $\text{Dom}(p_{\vec{\alpha}})$.

Lemma 4.2. *Let $1 \leq \delta < \kappa \leq \lambda$ be ordinals with κ and λ infinite cardinals and κ strongly inaccessible. Assume that for all $\mu_1, \mu_2 < \kappa$,*

$$\kappa \rightarrow (\mu_1)_{\mu_2}^{\delta \cdot 2}.$$

For each $\vec{\alpha} \in [\kappa]^\delta$, let $p_{\vec{\alpha}}$ be a condition in the forcing to add λ many Cohen subsets of κ . Assume that the set of conditions $p_{\vec{\alpha}} \in [\kappa]^\delta$ are image homogenized. Then for each $\gamma < \kappa$ there is a sequence

$$\langle H_i \subseteq \kappa : i < \delta \rangle$$

such that $(\forall i < \delta) \text{ot}(H_i) \geq \gamma$, $(\forall i < j < \delta)$ every element of H_i is less than every element of H_j , and the conditions $p_{\vec{\alpha}}$ for $\vec{\alpha} \in \bigotimes_{i < \delta} H_i$ are pairwise compatible.

Proof. Consider a function ι from $\delta \cdot 2$ to $\delta \cdot 2$ such that $\iota \upharpoonright [0, \delta)$ is an increasing sequence of elements in $\delta \cdot 2$ and $\iota \upharpoonright [\delta, \delta \cdot 2)$ is an increasing sequence of elements in $\delta \cdot 2$. Given a set $A \in [\text{Ord}]^{\delta \cdot 2}$, let $\iota[A] := \langle \alpha_{\iota(i)} : i < \delta \cdot 2 \rangle$, where A enumerated in increasing order is $\langle \alpha_i : i < \delta \cdot 2 \rangle$. If $\vec{\alpha}, \vec{\beta} \in [\text{Ord}]^\delta$ are such that $(\exists A) \iota[A] = \langle \vec{\alpha}, \vec{\beta} \rangle$, then we say that $\vec{\alpha}$ is ι -related to $\vec{\beta}$. The idea is that ι codes the relationship between $\vec{\alpha}$ and $\vec{\beta}$. Given any $\vec{\alpha}, \vec{\beta} \in [\text{Ord}]^\delta$, there is some ι such that $\vec{\alpha}$ is ι -related to $\vec{\beta}$.

Call a function ι from $\delta \cdot 2$ to $\delta \cdot 2$ *acceptable* iff for all $i < \delta \cdot 2$,

$$\iota(i), \iota(\delta + i) \in \{2 \cdot i, 2 \cdot i + 1\}.$$

The idea is that when we have our final sequence $\langle H_i : i < \delta \rangle$, if $\vec{\alpha}, \vec{\beta} \in \bigotimes_{i < \delta} H_i$, then $\vec{\alpha}$ will be ι -related to $\vec{\beta}$ for some ι that is acceptable. Also, as can be easily verified by the reader, given any acceptable ι , there are $\vec{\alpha}, \vec{\beta}, \vec{\sigma} \in \bigotimes_{i < \delta} H_i$ such that $\vec{\alpha}$ is ι -related to $\vec{\beta}$, $\vec{\alpha}$ is ι -related to $\vec{\sigma}$, and $\vec{\beta}$ is ι -related to $\vec{\sigma}$.

For an acceptable $\iota : \delta \cdot 2 \rightarrow \delta \cdot 2$, let c_ι be a coloring of subsets of κ of size $\delta \cdot 2$ which encodes the following information. Given $A \in [\kappa]^{\delta \cdot 2}$, let $\vec{\alpha}$ and $\vec{\beta}$ be such that $\iota[A] = \langle \vec{\alpha}, \vec{\beta} \rangle$ (so $\vec{\alpha}$ is ι -related to $\vec{\beta}$). The value of $c_\iota(A)$ should encode

- 1) the relative ordering of the elements of $\text{Dom}(p_{\vec{\alpha}})$ and $\text{Dom}(p_{\vec{\beta}})$, and finally
- 2) whether or not $p_{\vec{\alpha}}$ and $p_{\vec{\beta}}$ are compatible.

This coloring c_ι can be seen to use strictly less than κ colors. Now fix $\gamma < \kappa$. By our assumption that

$$(\forall \mu_1, \mu_2 < \kappa) \kappa \rightarrow (\mu_1)_{\mu_2}^{\delta \cdot 2},$$

get a set $H \subseteq \kappa$ of ordertype at least $\gamma \cdot \delta$ which is homogeneous for each c_ι .

Partition H into δ pieces, from left to right, of size at least γ to get our sequence $\langle H_i : i < \delta \rangle$. We claim that for any two $\vec{\zeta}, \vec{\eta} \in \bigotimes_{i < \delta} H_i$, $p_{\vec{\zeta}}$ and $p_{\vec{\eta}}$ are compatible. Suppose, towards a contradiction, that there are $\vec{\zeta}$ and $\vec{\eta}$ such that $p_{\vec{\zeta}}$ and $p_{\vec{\eta}}$ are incompatible. There is an ι such that $\vec{\zeta}$ is ι -related to $\vec{\eta}$ and ι is acceptable. Since $p_{\vec{\zeta}}$ and $p_{\vec{\eta}}$ are incompatible, fix $\sigma_1 \neq \sigma_2$ such that $\nu(\vec{\zeta}, \sigma_1) = \nu(\vec{\eta}, \sigma_2)$ but $p_{\vec{\zeta}}(\nu(\vec{\zeta}, \sigma_1)) \neq p_{\vec{\eta}}(\nu(\vec{\eta}, \sigma_2))$. The fact that it must be that $\sigma_1 \neq \sigma_2$ follows from the fact that the $p_{\vec{\alpha}}$ are image homogenized. Since whether or not $p_{\vec{\zeta}}$ and $p_{\vec{\eta}}$ are compatible was encoded into c_ι , any two $\vec{\alpha}, \vec{\beta} \in \bigotimes_{i < \delta} H_i$ such that $\vec{\alpha}$ is ι -related to $\vec{\beta}$ will be such that $p_{\vec{\alpha}}$ and $p_{\vec{\beta}}$ are incompatible. Since ι is acceptable, fix $\vec{\alpha}, \vec{\beta}, \vec{\sigma} \in \bigotimes_{i < \delta} H_i$ such that $\vec{\alpha}$ is ι -related to $\vec{\beta}$, $\vec{\alpha}$ is ι -related to $\vec{\sigma}$, and $\vec{\beta}$ is ι -related to $\vec{\sigma}$. By 1) of the definition of c_ι and the fact that the conditions are image homogenized, we have $\nu(\vec{\alpha}, \sigma_1) = \nu(\vec{\beta}, \sigma_2)$. Also, $\nu(\vec{\alpha}, \sigma_1) = \nu(\vec{\sigma}, \sigma_2)$ and $\nu(\vec{\beta}, \sigma_1) = \nu(\vec{\sigma}, \sigma_2)$. Thus, $\nu(\vec{\alpha}, \sigma_1) = \nu(\vec{\beta}, \sigma_1)$. Now we have $\nu(\vec{\beta}, \sigma_1) = \nu(\vec{\beta}, \sigma_2)$, which is impossible because ν is an increasing enumeration of the elements of $\text{Dom}(p_{\vec{\beta}})$ and $\sigma_1 \neq \sigma_2$. \square

In the next theorem, we consider ω to be a measurable cardinal. Notice that 1) and 2) of the hypothesis of the theorem follow from $\lambda \rightarrow (\kappa)_{\kappa}^{\delta \cdot 2}$. The conclusion of this theorem clearly implies $\text{SDHL}(\delta, \sigma, \kappa)$, and hence also $\text{HL}(\delta, \sigma, \kappa)$ by Theorem 3.7. We point out that this proves the analogue for trees on κ of the asymmetric Dense Set Version, Theorem 8.15 in [8], for trees on ω .

Theorem 4.3. *Let κ be an infinite cardinal, $\delta < \kappa$ a non-zero ordinal, and $\lambda \geq \kappa$ a cardinal for which the following partition relations hold:*

- 1) $\lambda \rightarrow (\kappa)_{\kappa}^{\delta}$, and
- 2) $\kappa \rightarrow (\mu_1)_{\mu_2}^{\delta \cdot 2}$, for all pairs $\mu_1, \mu_2 < \kappa$.

Assume that κ is a measurable cardinal in the forcing extension to add λ many Cohen subsets of κ . Let $T_i \subseteq {}^{<\kappa}\kappa$ for $i < \delta$ be regular trees, and let $\sigma < \kappa$ be non-zero and

$$c : \bigotimes_{i < \delta} T_i \rightarrow \sigma.$$

Then there are $\zeta < \kappa$ and $\langle t_i \in T_i(\zeta) : i < \delta \rangle$, such that for all ζ' satisfying $\zeta \leq \zeta' < \kappa$, there are ζ'' satisfying $\zeta' \leq \zeta'' < \kappa$ and $\langle X_i \subseteq T_i(\zeta'') : i < \delta \rangle$ such that, for each $i < \delta$, X_i dominates $T_i(\zeta') \cap \text{Cone}(t_i)$, and

$$|c'' \bigotimes_{i < \delta} X_i| = 1.$$

Furthermore, if $c'' \bigotimes_{i < \delta} X_i = \{0\}$, then $(\forall i < \delta) t_i = \emptyset$.

In particular, $HL(\delta, \sigma, \kappa)$ holds.

Proof. Note that κ is strongly inaccessible. Let \mathbb{P} be the poset to add $\delta \times \lambda$ many Cohen subsets of κ presented in the following way: $p \in \mathbb{P}$ iff p is a size $< \kappa$ partial function from $\lambda \times \delta$ to ${}^{<\kappa}\kappa$ and for all $(\alpha, i) \in \text{Dom}(p)$, $p(\alpha, i) \in T_i$. For the ordering of \mathbb{P} , $q \leq p$ iff

$$\text{Dom}(q) \supseteq \text{Dom}(p) \text{ and } (\forall x \in \text{Dom}(p)) q(x) \supseteq p(x).$$

If each $T_i = {}^{<\kappa}\kappa$, then it is trivial to see that this forcing is equivalent to adding λ Cohen subsets of κ . If not, then as long as each T_i does not have any isolated branches or leaf nodes (which we assume), then because no path leaves tree T_i at a limit level $< \kappa$, one show that the forcings are still equivalent. Without loss of generality, $(\forall p \in \mathbb{P})$ the size of $\text{Dom}(p) \cap (\lambda \times \{i\})$ does not depend on i . Similarly, we may assume $(\forall p \in \mathbb{P})$ the length of the sequence $p(x)$ does not depend on $x \in \text{Dom}(p)$.

Let \dot{G} be the canonical name for the generic object. In particular, $1 \Vdash \dot{G} : \check{\lambda} \times \check{\delta} \rightarrow \check{\kappa}$. Since $1 \Vdash (\check{\kappa} \text{ is a measurable cardinal})$, let $\dot{\mathcal{U}}$ be such that $1 \Vdash (\dot{\mathcal{U}} \text{ is a } \check{\kappa}\text{-complete ultrafilter on } \check{\kappa})$.

For each $\vec{\alpha} \in [\lambda]^\delta$, let $p_{\vec{\alpha}} \in \mathbb{P}$ be a condition and $\sigma_{\vec{\alpha}} < \sigma$ be a color such that if $\langle \alpha_i : i < \delta \rangle$ is the increasing enumeration of $\vec{\alpha}$, then

$$p_{\vec{\alpha}} \Vdash \{ \zeta < \check{\kappa} : c(\langle \dot{G}(\alpha_i, i) \upharpoonright \zeta : i < \delta \rangle) = \check{\sigma}_{\vec{\alpha}} \} \in \dot{\mathcal{U}}.$$

We may assume that if $\sigma_{\vec{\alpha}} = 0$, then $p_{\vec{\alpha}} = 1$. This is because if 1 does not force the color of $\dot{\mathcal{U}}$ many levels to be 0, then there by the nature of the forcing relation and since $\dot{\mathcal{U}}$ is an ultrafilter, there is some condition p such that

$$p \Vdash \{ \zeta < \check{\kappa} : c(\langle \dot{G}(\alpha_i, i) \upharpoonright \zeta : i < \delta \rangle) \neq 0 \} \in \dot{\mathcal{U}}.$$

Then, since $\sigma < \kappa$ and $\dot{\mathcal{U}}$ is κ -complete (in the extension), there is $p' \leq p$ and σ such that

$$p' \Vdash \{ \zeta < \check{\kappa} : c(\langle \dot{G}(\alpha_i, i) \upharpoonright \zeta : i < \delta \rangle) = \sigma \} \in \dot{\mathcal{U}}.$$

Then set $p_{\vec{\alpha}} = p'$ and $\sigma_{\vec{\alpha}} = \sigma$ and we are done. We may also assume, by possibly making the conditions $p_{\vec{\alpha}}$ stronger, that for each $\vec{\alpha} \in [\lambda]^\delta$ and $i < \delta$, that

$$(\alpha_i, i) \in \text{Dom}(p_{\vec{\alpha}}).$$

There is a coloring $\tilde{c} : [\lambda]^\delta \rightarrow \kappa$ such that after we apply the partition relation $\lambda \rightarrow (\kappa)_{\check{\kappa}}^\delta$, we get a set $H \in [\lambda]^\kappa$ such that the following are satisfied:

- 1) the sequence $\langle (\xi, p_{\vec{\alpha}}(\nu(\vec{\alpha}, \xi))) : \xi < \text{ot}(\text{Dom}(p_{\vec{\alpha}})) \rangle$ does not depend on $\vec{\alpha} \in [H]^\delta$ and therefore the set of $p_{\vec{\alpha}}$ for $\vec{\alpha} \in [H]^\delta$ are image homogenized,
- 2) the value $\sigma_{\vec{\alpha}}$ does not depend on $\vec{\alpha} \in [H]^\delta$, and
- 3) for a fixed $i < \delta$, the sequence $\langle p_{\vec{\alpha}}(\alpha_i, i) \in T_i : i < \delta \rangle$ does not depend on $\vec{\alpha} \in [H]^\delta$.

For each $i < \delta$, let $t_i \in {}^{<\kappa}\kappa$ be the unique value of the $p_{\vec{\alpha}}(i, \alpha_i)$'s for $\vec{\alpha} = \langle \alpha_i : i < \delta \rangle \in [H]^\delta$. Let ζ be the common length of each t_i . Let $\sigma' < \sigma$ be the unique value of $\sigma_{\vec{\alpha}}$ for $\vec{\alpha} \in [H]^\delta$. Note that if $\sigma' = 0$, then each $p_{\vec{\alpha}}$ for $\vec{\alpha} \in [H]^\delta$ equals 1, so $(\forall i < \delta) t_i = \emptyset$.

Now pick an arbitrary ζ' satisfying $\zeta \leq \zeta' < \kappa$. Let $\gamma < \kappa$ be a cardinal such that for each $i < \delta$, the cardinality of the set $T(\zeta') \cap \text{Cone}(t_i)$ is $\leq \gamma$. Now apply Lemma 4.2 to get a sequence $\langle H_i \subseteq H : i < \delta \rangle$ such that $(\forall i < \delta) \text{ot}(H_i) \geq \gamma$, $(\forall i < j < \delta)$ every element of H_i is less than every element of H_j , and the conditions $p_{\vec{\alpha}}$ for $\vec{\alpha} \in \bigotimes_{i < \delta} H_i$ are pairwise compatible. To apply this lemma, we needed hypothesis 2) of this theorem (to hold in the ground model). We could instead apply Lemma 4.2 in the forcing extension as long as 2) holds in the extension, and the sequence $\langle H_i : i < \delta \rangle$ would be in the ground model because the forcing is $<\kappa$ -closed.

For each $i < \delta$, let $S_i := T_i(\zeta') \cap \text{Cone}(t_i)$, the set of immediate successors of t_i in T_i . For each $i < \delta$ and $t \in S_i$, pick $\alpha_{i,t} \in H_i$ so that every element of S_i is mapped to a unique element of H_i . This is possible because the cardinality of S_i is $\leq \gamma$ and γ is \leq the ordertype of H_i . For $i < \delta$, let $A_i := \{\alpha_{i,t} : t \in S_i\} \subseteq H_i$. Let $\mathcal{X} \subseteq \bigotimes_{i < \delta} H_i$ be the set $\mathcal{X} := \bigotimes_{i < \delta} A_i$. The conditions $p_{\vec{\alpha}}$ for $\vec{\alpha} \in \mathcal{X}$ are pairwise compatible.

Let $p \in \mathbb{P}$ be a condition which extends $\bigcup\{p_{\vec{\alpha}} : \vec{\alpha} \in \mathcal{X}\}$ and for all $i < \delta$ and $t \in S_i$, $p(\alpha_{i,t}, i) = t$. To see that there exists such a p , first note that $|\mathcal{X}| < \kappa$ and the conditions $p_{\vec{\alpha}} \in \mathcal{X}$ are pairwise compatible, therefore by the nature of \mathbb{P} , $\bigcup_{\vec{\alpha} \in \mathcal{X}} p_{\vec{\alpha}}$ is an element of \mathbb{P} . Second, since $(\forall \vec{\alpha} \in \mathcal{X})(\forall i < \delta) p_{\vec{\alpha}}(\alpha_i, i) = t_i$, we may define p so that $p(i, \alpha_{i,t}) = t$ for all $i < \delta$ and $t \in S_i$ and this will not clash with the $p_{\vec{\alpha}}$.

Since $|\mathcal{X}| < \kappa$, $1 \Vdash (\mathcal{U} \text{ is a } \check{\kappa}\text{-complete ultrafilter})$, and by the hypothesis on each pair $(p_{\vec{\alpha}}, \sigma_{\vec{\alpha}})$, we have that $p \Vdash$ there are arbitrarily large levels $\zeta'' < \check{\kappa}$ such that for each $\vec{\alpha} \in \check{\mathcal{X}}$, we have

$$\check{c}(\langle \dot{G}(\alpha_i, i) \upharpoonright \zeta'' : i < \check{\delta} \rangle) = \check{\sigma}'.$$

We may extend p to a condition $p' \leq p$ as well as get a level $\zeta'' \in [\zeta', \kappa)$ such that the following are satisfied:

- 1) $p' \Vdash$ for each $\vec{\alpha} = \langle \alpha_i : i < \delta \rangle \in \check{\mathcal{X}}$, we have

$$\check{c}(\langle \dot{G}(\alpha_i, i) \upharpoonright \zeta'' : i < \check{\delta} \rangle) = \check{\sigma}'.$$

- 2) for each $i < \delta$ and each $t \in S_i$, there is a unique $\tilde{t} \in T_i(\zeta'')$ such that $p' \Vdash \dot{G}(\check{\alpha}_{i,t}, \check{i}) \supseteq \tilde{t}$.

For each $i < \delta$, let

$$X_i := \{\tilde{t} : t \in S_i\}.$$

We have that each X_i dominates S_i and

$$c'' \bigotimes_{i < \delta} X_i = \{\sigma'\}$$

(because the coloring c is in the ground model, so it is absolute). \square

The following theorem of Erdős and Rado provides many examples of cardinals λ which satisfy the hypothesis in Theorem 4.3, when δ is finite.

Theorem 4.4 (Erdős-Rado, [7]). *For $r \geq 0$ finite and κ an infinite cardinal, $\beth_r(\kappa)^+ \rightarrow (\kappa^+)_\kappa^{r+1}$.*

In particular, if GCH holds, then for finite $d \geq 2$, $\kappa^{+d} \rightarrow (\kappa^+)_\kappa^d$.

Definition 4.5 ([13]). Given a cardinal κ and an ordinal d , κ is $\kappa + d$ -strong if there is an elementary embedding $j : V \rightarrow M$ with critical point κ such that $V_{\kappa+d} = M_{\kappa+d}$.

Theorem 4.6. *Let $d \geq 1$ be any finite integer and suppose κ is a $\kappa + d$ -strong cardinal in a model V of ZFC satisfying GCH. Then there is a forcing extension in which κ remains measurable and $\text{HL}(d, \sigma, \kappa)$ holds, for all $\sigma < \kappa$.*

Proof. Let $\langle \kappa_\alpha : \alpha < \kappa \rangle$ enumerate all the strongly inaccessible cardinals in V below κ . Let \mathbb{P}_κ denote the κ -length reverse Easton support iteration of $\text{Add}(\kappa_\alpha, \kappa_\alpha^{+d})$, and let G_κ be \mathbb{P}_κ -generic over V . Then in $V[G_\kappa]$, κ is still measurable by a standard lifting of the embedding, and GCH holds at and above κ since $|\mathbb{P}_\kappa| = \kappa$. Thus, in $V[G_\kappa]$, the partition relation $\kappa^{+d} \rightarrow (\kappa)_\kappa^d$ holds. Since κ is measurable in $V[G_\kappa]$, the partition relations $\kappa \rightarrow (\mu_1)_{\mu_2}^{d \cdot 2}$ holds for all pairs $\mu_1, \mu_2 < \kappa$. Let \mathbb{Q} denote $\text{Add}(\kappa, \kappa^{+d})$ in $V[G_\kappa]$, and let H be \mathbb{Q} -generic over $V[G_\kappa]$. By an unpublished result of Woodin (see for instance [9] for a proof) κ remains measurable in $V[G_\kappa][H]$, since κ is $\kappa + d$ -strong in V . Thus, the hypotheses of Theorem 4.3 are satisfied in $V[G_\kappa]$, and therefore $\text{HL}(d, \sigma, \kappa)$ holds in $V[G_\kappa]$. \square

We conclude this section by pointing out that, in place of $\text{Add}(\kappa_\alpha, \kappa_\alpha^{+d})$, one could use a reverse Easton iteration of κ_α^{+d} -products with $< \kappa_\alpha$ -support of κ_α -Sacks forcing, as in [9], to achieve a model with κ measurable and $\text{HL}(d, \sigma, \kappa)$ holding. However, the homogeneity argument in the body of Theorem 4.3 would use κ^+ colors, and so we would need to start with a cardinal κ which is $(\kappa + d + 1)$ -strong in the ground model to get an analogue of Theorem 4.6. It should be noted though that for trees of height ω , Harrington's original forcing proof can also be modified to use Sacks forcing; the larger λ needed to accommodate the homogeneity argument would need to satisfy $\lambda \rightarrow (\omega)_{2^\omega}^{2d}$.

5. PRESERVING $\text{SDHL}(\delta, \sigma, \kappa)$ UNDER FORCING

If a forcing has size less than κ , it almost preserves the statement that $\text{SDHL}(\delta, \sigma, \kappa)$ holds for all $\sigma < \kappa$ by the following argument. For this section, fix κ and $1 \leq \delta, \sigma < \kappa$.

Proposition 5.1. *Let \mathbb{P} be a forcing of size $< \kappa$. Let $\langle T_i \subseteq {}^{< \kappa} \kappa : i < \delta \rangle$ be a sequence of regular trees, let \dot{c} be a name for a coloring, and let p be a condition such that $p \Vdash \dot{c} : \bigotimes_{i < \delta} T_i \rightarrow \check{\sigma}$. If $\text{SDHL}(\delta, |\mathbb{P}| \cdot \sigma, \kappa)$ holds, then, in the ground model, there is a somewhere dense matrix $\langle X_i \subseteq T_i : i < \delta \rangle$ and a condition $p' \leq p$ such that $p' \Vdash \dot{c} \restriction \bigotimes_{i < \delta} \check{X}_i = 1$.*

Proof. Let $c' : \bigotimes_{i < \delta} T_i \rightarrow \mathbb{P} \times \sigma$ be any coloring where given any $\vec{x} \in \bigotimes_{i < \delta} T_i$, $c'(\vec{x})$ equals some (q, σ') satisfying $q \Vdash \dot{c}(\vec{x}) = \check{\sigma}'$. Apply $\text{SDHL}(\delta, |\mathbb{P}| \cdot \sigma, \kappa)$ to the coloring c' to get a somewhere dense matrix $\langle X_i \subseteq T_i : i < \delta \rangle$ and a color (p', σ') such that $p' \Vdash \dot{c} \restriction \bigotimes_{i < \delta} \check{X}_i = \{\check{\sigma}'\}$. This finishes the proof. \square

The reason why this last proposition doesn't give us the full preservation of $(\forall \sigma < \kappa) \text{SDHL}(\delta, \sigma, \kappa)$ is because there is the requirement that the sequence of trees be in the ground model. The following remains open.

Question 5.2. Is there a forcing of size $< \kappa$ which destroys $\text{SDHL}(\delta, \sigma, \kappa)$ for any $\delta, \sigma < \kappa$ (where $\delta \geq 2$)?

Now, a forcing which is $\leq\kappa$ -closed will not add any new regular trees or colorings. Hence, $\text{SDHL}(\delta, \sigma, \kappa)$ will be preserved. For the rest of the section, we will show that $\text{SDHL}(\delta, \sigma, \kappa)$ is preserved by $<\kappa$ -closed forcings assuming κ is measurable.

Proposition 5.3. *Assume κ is measurable and $\text{SDHL}(\delta, \sigma, \kappa)$ holds. Then also $\text{SDHL}(\delta, \sigma, \alpha)$ holds for a stationary set of $\alpha < \kappa$.*

Proof. Let \mathcal{U} be a normal ultrafilter on κ . Let $M = \text{Ult}_{\mathcal{U}}(V)$. Consider any length δ sequence of regular trees and any coloring (using $\leq \sigma$ colors) of those trees in M . The sequence of trees and the coloring are also in V , and since $\text{SDHL}(\delta, \sigma, \kappa)$ holds there, there is some somewhere dense matrix in V on which the coloring is constant. Since $V_{\kappa} \subseteq M$, the somewhere dense matrix is in fact in M . Thus, $\text{SDHL}(\delta, \sigma, \kappa)$ holds in M . By Łos's theorem, $\text{SDHL}(\delta, \sigma, \alpha)$ holds for a set of $\alpha < \kappa$ in \mathcal{U} . Thus, $\text{SDHL}(\delta, \sigma, \alpha)$ holds for a stationary set of $\alpha < \kappa$. \square

Proposition 5.4. *Assume that $\text{SDHL}(\delta, \sigma, \alpha)$ holds for a stationary set S of $\alpha < \kappa$. Then $\text{SDHL}(\delta, \sigma, \kappa)$ holds.*

Proof. Let $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ be a sequence of regular trees and let $c : \bigotimes_{i < \delta} T_i \rightarrow \sigma$ be a coloring. If we can find an $\alpha < \kappa$ such that each $T_i \cap {}^{<\alpha}\kappa$ is an α -tree and $\text{SDHL}(\delta, \sigma, \alpha)$ holds, then we will be done. A standard argument shows that for each $i < \delta$, there is a club $C_i \subseteq \kappa$ such that $T_i \cap {}^{<\alpha}\kappa$ is an α -tree for each $\alpha \in C_i$. The set $\bigcap_{i < \delta} C_i$ is a club, so it must intersect S . An $\alpha < \kappa$ in the intersection is as desired. \square

It is well known that if a forcing is $<\kappa$ -closed, then it preserves stationary subsets of κ . Thus, we have the following.

Theorem 5.5. *If κ is measurable and $\text{SDHL}(\delta, \sigma, \kappa)$ holds, then $\text{SDHL}(\delta, \sigma, \kappa)$ also holds in any forcing extension by a forcing which is $<\kappa$ -closed.*

Proof. Let \mathbb{P} be a forcing which is $<\kappa$ -closed. Let $S \subseteq \kappa$ be the set of $\alpha < \kappa$ such that $\text{SDHL}(\delta, \sigma, \alpha)$ holds. The set S is stationary, since κ is measurable. After forcing with \mathbb{P} , since \mathbb{P} is $<\kappa$ -closed, $\text{SDHL}(\delta, \sigma, \alpha)$ still holds for each $\alpha \in S$. Since \mathbb{P} preserves stationary subsets of κ , S is stationary in the extension. Since in the extension κ is the stationary limit of a set of α satisfying $\text{SDHL}(\delta, \sigma, \alpha)$, it follows that $\text{SDHL}(\delta, \sigma, \kappa)$ holds in the extension. \square

Since for κ measurable, $\text{SDHL}(\delta, \sigma, \kappa)$ holds if and only if $\text{HL}(\delta, \sigma, \kappa)$ holds, the strong tree version of Halpern-Läuchli at a measurable cardinal κ is preserved by $<\kappa$ -closed forcing.

6. CLOSING COMMENTS AND OPEN PROBLEMS

The following conjectures and questions are motivated by the results in the previous sections and their comparisons with results in [18], [10] and [5]. As mentioned before, we are interested in the exact consistency strength of the Halpern-Läuchli Theorem at a measurable cardinal.

Conjecture 6.1. For finite $d \geq 2$, κ measurable and $\sigma < \kappa$, the consistency strength of $\text{HL}(d, \sigma, \kappa)$ is a $\kappa + d$ -strong cardinal.

If the conjecture turns out to be true, then there must be a positive answer to the next question.

Question 6.2. Given $d \geq 1$, is there a model of ZFC in which there is measurable cardinal κ such that $\text{HL}(d, \sigma, \kappa)$ holds for all $\sigma < \kappa$, but $\text{HL}(d + 1, \sigma, \kappa)$ fails for some $2 \leq \sigma < \kappa$?

In Section 8 of [5], it is mentioned that a model satisfying the hypotheses of Theorem 2.5 in [5] can be constructed, assuming the existence of a measurable cardinal κ such that $o(\kappa) = \kappa^{+2m+2}$. We conjecture that the form of Halpern-Läuchli in [18] and [5] is strictly stronger than the form $\text{HL}(d, \sigma, \kappa)$.

Conjecture 6.3. Let $d \geq 2$ be a finite number. The consistency strength of $\text{HL}(d, \sigma, \kappa)$ for κ measurable is strictly less than the consistency strength of Theorem 2.5 in [5] for coloring d -sized antichains.

Further, Dzamonja, Larson, and Mitchell point out that Theorem 2.5 in [5] is not a consequence of any large cardinal assumption. This follows from results of Hajnal and Komjáth in [10], a consequence of which is that there is a forcing of size \aleph_1 after which there is a coloring of the pairsets on ${}^{<\kappa}2$ for which there is no strong subtree homogeneous for the coloring. However, that theorem of Hajnal and Komjáth does not seem to immediately provide a counterexample to $\text{HL}(d, 2, \kappa)$, and so we ask the following.

Question 6.4. For $d \geq 2$, is there a large cardinal assumption on κ which implies $\text{HL}(d, 2, \kappa)$ holds?

Or is there an analogue of the Hajnal and Komjath result which would preclude $\text{SDHL}(2, \sigma, \kappa)$ from holding? That is, is the answer to Question 5.2 ‘yes’?

In Section 3 we showed that all the variants considered in this paper are equivalent as long as κ is weakly compact, and showed that $\text{HL}(1, k, \kappa)$ holds when κ is weakly compact and k is finite. However, we have not shown that $\text{HL}(d, 2, \kappa)$ holds for κ weakly compact when $d \geq 2$. In Section 8 of [5], an argument is provided showing that any cardinal κ satisfying their Theorem 2.5 for $m \geq 2$ must be weakly compact. However, that argument does not immediately seem to apply to our situation and so we ask the following.

Question 6.5. For $d \geq 2$, is there an uncountable cardinal κ below the least measurable for which $\text{HL}(d, 2, \kappa)$?

Lastly, we ask for a model of ZF in which the hypotheses of Theorem 4.3 hold for δ infinite.

Question 6.6. Is there a model of ZF satisfying the partition relations stated in the hypotheses of Theorem 4.3 for some infinite $\delta < \kappa$ such that after forcing with $\text{Add}(\kappa, \lambda)$, κ is measurable?

REFERENCES

1. Denis Devlin, *Some partition theorems and ultrafilters on ω* , Ph.D. thesis, Dartmouth College, 1979.
2. C. A. DiPrisco, J. Llopis, and S. Todorcevic, *Parametrized partitions of products of finite sets*, *Combinatorica* **24** (2004), no. 2, 209–232.
3. Natasha Dobrinen, Claude Laflamme, and Norbert Sauer, *Rainbow Ramsey simple structures*, *Discrete Mathematics* (2016), 13 pp, To appear.
4. Pandelis Dodos, Vassilis Kanellopoulos, and Konstantinos Tyros, *Dense subsets of products of finite trees*, *International Mathematics Research Notices* (2013), no. 4, 924–970.

5. M. Džamonja, J. Larson, and W. J. Mitchell, *A partition theorem for a large dense linear order*, Israel Journal of Mathematics **171** (2009), 237–284.
6. ———, *Partitions of large Rado graphs*, Archive for Mathematical Logic **48** (2009), no. 6, 579–606.
7. Paul Erdős and Richard Rado, *A partition calculus in set theory*, Bulletin of the American Mathematical Society **62** (1956), 427–489.
8. Ilijas Farah and Stevo Todorčević, *Some applications of the method of forcing*, Yenisei Series in Pure and Applied Mathematics.
9. Sy-David Friedman and Katherine Thompson, *Perfect trees and elementary embeddings*, Journal of Symbolic Logic **73** (2008), no. 3, 906–918.
10. András Hajnal and Péter Komjáth, *A strongly non-Ramsey order type*, Combinatorica **17** (1997), no. 2, 363–367.
11. J. D. Halpern and H. Läuchli, *A partition theorem*, Transactions of the American Mathematical Society **124** (1966), 360–367.
12. J. D. Halpern and A. Lévy, *The Boolean prime ideal theorem does not imply the axiom of choice*, Axiomatic Set Theory, Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967, American Mathematical Society, 1971, pp. 83–134.
13. Akihiro Kanamori, *The Higher Infinite*, 2nd ed., Springer, 2003.
14. Richard Laver, *Products of infinitely many perfect trees*, Journal of the London Mathematical Society (2) **29** (1984), no. 4, 385–396.
15. Keith R. Milliken, *A ramsey theorem for trees*, Journal of Combinatorial Theory, Series A **26** (1979), 215–237.
16. ———, *A partition theorem for the infinite subtrees of a tree*, Transactions of the American Mathematical Society **263** (1981), no. 1, 137–148.
17. Norbert Sauer, *Coloring subgraphs of the Rado graph*, Combinatorica **26** (2006), no. 2, 231–253.
18. Saharon Shelah, *Strong partition relations below the power set: consistency – was Sierpinski right? II*, Sets, Graphs and Numbers (Budapest, 1991), vol. 60, Colloq. Math. Soc. János Bolyai, North-Holland, 1991, pp. 637–688.
19. Stevo Todorčević, *Introduction to Ramsey Spaces*, Princeton University Press, 2010.
20. Stevo Todorčević and Konstantinos Tyros, *Subsets of products of finite sets of positive upper density*, Journal of Combinatorial Theory, Series A **120** (2013), 183–193.
21. Dimitris Vlitas, *A canonical partition theorem for uniform families of finite strong subtrees*, Discrete Mathematics **335** (2014), 45–65.
22. Vojkan Vuksanović, *Canonical equivalence relations on \mathbb{Q}^n* , Order **20** (2003), no. 4, 373–400.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DENVER, 2280 S VINE ST, DENVER, CO 80208 U.S.A.

E-mail address: Natasha.Dobrinen@du.edu

URL: <http://web.cs.du.edu/~ndobrine>

MATHEMATICS DEPARTMENT, UNIVERSITY OF DENVER, 2280 S VINE ST, DENVER, CO 80208 U.S.A.

E-mail address: Daniel.Hathaway@du.edu

URL: <http://mysite.du.edu/~dhathaw2/>