DERIVED TOPOLOGIES ON ORDINALS AND STATIONARY REFLECTION

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Abstract. We study the transfinite sequence of topologies on the ordinal numbers that is obtained through successive closure under Cantor's derivative operator on sets of ordinals, starting form the usual interval topology. We characterize the non-isolated points in the ξ-th topology as those ordinals that satisfy a strong iterated form of stationary reflection, which we call ξ-simultaneous-reflection. We prove some properties of the ideals of non-ξ-simultaneous-stationary sets and identify their tight connection with indescribable cardinals. We then introduce a new natural notion of Π^1_ξ-indescribability, for any ordinal ξ, which extends to the transfinite the usual notion of Π^1_n-indescribability, and prove that in the constructible universe L, a regular cardinal is $(\xi+1)$-simultaneously-reflecting if and only if it is Π^1_ξ-indescribable, a result that generalizes to all ordinals ξ previous results of Jensen [28] in the case $\xi = 2$, and Bagaria-Magidor-Sakai [5] in the case $\xi = n$. This yields a complete characterization in L of the non-discreteness of the ξ-topologies, both in terms of iterated stationary reflection and in terms of indescribability.

1. Introduction

Cantor's derivative operator on a topological space (X, τ) is the map d_τ that assigns to every subset A of X the set of its limit points. By declaring the sets $d_\tau(A)$ to be open one generates a finer topology. Through successive applications of this process, and by taking unions at limit stages, one obtains a polytopological space $(X, \tau_0, \tau_1, \ldots, \tau_\xi, \ldots)$, where $\tau_0 = \tau$ and $\tau_\zeta \subseteq \tau_\xi$ whenever $\zeta < \xi$. The study of such spaces has been of great interest in recent years, not so much in topology but, perhaps surprisingly, in proof theory and modal logic. When (X, τ) is scattered, the derived polytopological spaces $(X, \tau_0, \tau_1, \ldots, \tau_\xi, \ldots)$ are known in the literature as **GLP**-spaces, for they provide a natural topological interpretation of the logic **GLP** (Japaridze [24]), namely the polymodal extension of the classical Gödel-Löb provability logic **GL** to infinitely-many modal operators $[n]$,

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$n < \omega$. The logic **GL** yields a complete axiomatization of the arithmetical properties of Gödel's provability operator *Prov* for Peano Arithmetic (PA) (Solovay [33]), whereas **GLP** completely axiomatizes the arithmetical properties of the operators $Prov_n$, for $n < \omega$, where $Prov_n$ stands for n-provability, i.e., being provable in PA together with all Π_n true aritmetical sentences (Japaridze [24, 25]; but see also [23, 11]). See [3] and [15] for more on the history and results on **GLP**; and also [6] and [7] for applications in proof theory and ordinal analysis of arithmetic. The main obstacle in the study of **GLP** has been its incompleteness with respect to any class of Kripke frames. But Beklemishev and Gabelaia [12] finally showed in 2011 that **GLP** is complete under topological semantics with respect to the class of all **GLP**-spaces. The question remained, however, about the completeness of **GLP** when one restricts the semantics to ordinal spaces, i.e., spaces (δ, τ), where δ is a limit ordinal and τ is the usual interval topology.

Andreas Blass [14] proved in 1990 the consistency with ZFC of the completeness of **GL** under topological semantics for ordinal spaces greater than or equal to $\aleph_\omega$ with the usual topology τ_0 (a result obtained, independently, by Abashidze [1]), and also with the topology τ_1, known as the club topology. In 2011, Lev Beklemishev [10] was able to combine Blass' results to prove the completeness of $\mathbf{GLP}_2$, the fragment of **GLP** with only two modal operators, with respect to ordinal topological semantics. For $\mathbf{GLP}_3$, not to mention the general $\mathbf{GLP}_n$ case, or even the full **GLP**, this is not yet known. The problem is set-theoretic, for as shown by Blass, even for $\mathbf{GLP}_2$ the non-completeness is also consistent with ZFC, relative to the existence of a Mahlo cardinal.

For the completeness of **GLP**, a necessary requirement is that the topologies τ_n, $n < \omega$, are all non-discrete. But already for the τ_2 topology on ordinals, the non-isolated points α must reflect simultaneously pairs of stationary sets, i.e., for all pairs S T of stationary subsets of α, there is some $\beta < \alpha$ such that $S \cap \beta$ and $T \cap \beta$ are both stationary in β. Any regular α with this property is a large cardinal in the constructible universe L, namely a weakly-compact, i.e., Π^1_1-indescribable, cardinal (Magidor [31]). It follows that the non-discreteness of the τ_2 topology is equiconsistent with the existence of a weakly-compact cardinal. Beklemishev [8] and the author showed, independently, that the non-isolated points in the τ_n topology are exactly those ordinals that are n-simultaneously-reflecting (see definition 2.8 and theorem 2.11 below). Moreover, Bagaria, Magidor, and Sakai [5] recently showed that in L the non-isolated points of the τ_{n+1} topology are exactly those ordinals whose cofinality is a Π^1_n-indescribable cardinal. Thus, in the constructible universe there is an exact correspondence between the properties of $(n + 1)$-simultaneous-reflection, being a non-isolated point in the τ_{n+1} topology, and having Π^1_n-indescribable cofinality.

In the last few years there has been interest in exploring further extensions of **GLP** obtained by adding an arbitrary number of modal operators $[\xi]$, for $\xi < \Lambda$, where Λ is some ordinal, or even for the whole proper class of ordinals. The resulting logics $\mathbf{GLP}_\Lambda$ and their fragments have been intensively studied in [2, 13, 19, 20, 21, 22]. In this paper we are chiefly interested in the ordinal **GLP**-spaces associated to $\mathbf{GLP}_\Lambda$, namely, the polytopological spaces $(\delta, \{\tau_\xi : \xi < \Lambda\})$, where δ is a limit ordinal, or the class OR of all ordinal numbers. Particularly, we are interested in determining the conditions for the topologies τ_ξ to be non-discrete. As we have remarked above, this is a set-theoretic question involving strong forms of stationary reflection, a well-studied property that implies the consistency of large cardinals (see, e.g., [16, 26, 27, 31, 32]). In this paper we define some notions of iterated stationary reflection and simultaneous stationary reflection, which we call ξ-stationarity and ξ-simultaneous-stationarity, respectively, and show that the latter gives the exact property for a point in the τ_ξ topology to be non-isolated. As it turns out, these notions correspond precisely to some large cardinals in the constructible universe L, namely Π^1_ξ-indescribable cardinals. Given the existing vast amount of literature on stationary reflection, both in pure set theory and in its applications to other areas, e.g., Abelian groups and modules (see [18]), we believe our theory of ξ-stationary sets opens up new avenues for research both for the extension to the ξ-stationary case of known results on stationary sets and for applications to other areas.

In section 2 of this paper we give a detailed exposition of the theory of ξ-stationary and ξ-simultaneously-stationary sets and prove that an ordinal is non-isolated in the τ_ξ topology if and only if it is ξ-simultaneously-reflecting (theorem 2.11). In section 3 we consider the ideal of non-ξ-simultaneously-stationary sets on a given ordinal α, and the corresponding dual filter, and we prove that the ideal is proper if and only if α is ξ-simultaneously-reflecting (theorem 3.1), thus giving yet another characterization of the non-isolated points of the τ_ξ topology. Sections 2 and 3 are the revised version of our circulated manuscript [4]. Section 4 is devoted to indescribable cardinals. We introduce a new notion of Π^1_ξ-indescribability, for any ordinal ξ, and prove (proposition 4.3) that every Π^1_ξ-indescribable cardinal is $(\xi+1)$-simultaneously-reflecting. We also consider the associated Π^1_ξ-indescribable filters and analyze their connection with the non-ξ-simultaneously-stationary ideals. In particular, we prove (proposition 4.4) that if κ is Π^1_ξ-indescribable, then the Π^1_ξ-indescribable filter and the non-ξ-simultaneous-stationary ideal are normal. Finally, in section 5 we prove that in L a regular cardinal is $(\xi+1)$-reflecting if and only if it is Π^1_ξ-indescribable, and therefore if and only if it is $(\xi+1)$-simultaneously-reflecting. We finish by showing, modulo suitable large cardinals, that it is consistent that the τ_ξ topology on ordinals is non-discrete while $\tau_{\xi+1}$ is discrete.

2. Derived topologies on ordinals

The *interval topology* on a non-zero ordinal number δ (or on $\delta = OR$, the proper class of all ordinal numbers) is the topology generated by the set $\mathcal{B}_0$ consisting of $\{0\}$ and the open intervals (α, β), for $\alpha < \beta \leq \delta$.

We shall define a transfinite sequence of topologies $\langle \tau_\xi : \xi \in OR \rangle$ on δ, with τ_0 being the interval topology. Notice that $\mathcal{B}_0$ is actually a base for τ_0. Also note that τ_0 is a Hausdorff scattered topology in which 0 and all successor ordinals less than δ are isolated points and every non-zero limit ordinal is a limit point.

Given τ_ξ, let $d_\xi : \mathcal{P}(\delta) \to \mathcal{P}(\delta)$ be the Cantor's derivative operator, defined by:

$$d_\xi(A) = \{\alpha < \delta : \alpha \text{ is a limit point of } A \text{ in the } \tau_\xi \text{ topology}\}.$$

Thus, $\alpha \in d_\xi(A)$ if and only if every set in τ_ξ that contains α contains also some element of A different from α. Observe that $d_\xi(A)$ is a closed subset of δ in the topology τ_ξ, for every A and ξ. Hence $d_\xi(d_\xi(A)) \subseteq d_\xi(A)$. Also notice that if $\xi' \leq \xi$, then $d_\xi(d_{\xi'}(A)) \subseteq d_{\xi'}(A)$.

We then define $\tau_{\xi+1}$ as the topology generated by

$$\mathcal{B}_{\xi+1} := \mathcal{B}_\xi \cup \{d_\xi(A) : A \subseteq \delta\}.$$

If λ is a limit ordinal, then let τ_λ be the union $\bigcup_{\xi<\lambda} \tau_\xi$, which is the topology on δ generated by $\mathcal{B}_\lambda := \bigcup_{\xi<\lambda} \mathcal{B}_\xi$.

Clearly, for every $\xi \leq \xi'$ we have $\tau_\xi \subseteq \tau_{\xi'}$, and so for every $A \subseteq \delta$, $d_{\xi'}(A) \subseteq d_\xi(A)$. Notice that the sets of the form

$$I \cap d_{\xi_0}(A_0) \cap \ldots \cap d_{\xi_{n-1}}(A_{n-1})$$

where $I \in \mathcal{B}_0$, $n < \omega$, $\xi_i < \xi$ and $A_i \subseteq \delta$, all $i < n$, form a base for τ_ξ.

Proposition 2.1. *For every $\xi' < \xi$ and every $A, B \subseteq \delta$,*

$$d_{\xi'}(A) \cap d_\xi(B) = d_\xi(d_{\xi'}(A) \cap B).$$

Proof. Suppose $\gamma \in d_{\xi'}(A) \cap d_\xi(B)$. If $U \in \tau_\xi$ with $\gamma \in U$, then $\gamma \in U \cap d_{\xi'}(A) \in \tau_\xi$. And since $\gamma \in d_\xi(B)$, and $\xi' < \xi$, we have that $(U \cap d_{\xi'}(A)) \cap B - \{\gamma\} \neq \varnothing$. Since U was arbitrary, this implies that $\gamma \in d_\xi(d_{\xi'}(A) \cap B)$. We have thus shown that $d_{\xi'}(A) \cap d_\xi(B) \subseteq d_\xi(d_{\xi'}(A) \cap B)$.

Now suppose $\gamma \in d_\xi(d_{\xi'}(A) \cap B)$. Then clearly $\gamma \in d_\xi(B)$. Also, since $d_\xi(d_{\xi'}(A) \cap B) \subseteq d_\xi(d_{\xi'}(A)) \subseteq d_{\xi'}(A)$, we have that $\gamma \in d_{\xi'}(A) \cap d_\xi(B)$. And this shows that $d_\xi(d_{\xi'}(A) \cap B) \subseteq d_{\xi'}(A) \cap d_\xi(B)$. □

Corollary 2.2. *For every ordinal ξ, the sets of the form*

$$I \cap d_{\xi'}(A_0) \cap \ldots \cap d_{\xi'}(A_{n-1})$$

where $I \in \mathcal{B}_0$, $n < \omega$, $\xi' < \xi$, and $A_i \subseteq \delta$, all $i < n$, form a base for τ_ξ.

Proof. Fix a basic set $I \cap d_{\xi_0}(A_0) \cap \ldots \cap d_{\xi_{n-1}}(A_{n-1})$ of τ_ξ, with $I \in \mathcal{B}_0$ and $\xi_0 \leq \ldots \leq \xi_{n-1} < \xi$. If $n = 0$, or if $n > 0$ and $\xi_i = \xi_{n-1}$, for all $0 \leq i < n$, then there is nothing to prove. If i is the largest such that $\xi_i < \xi_{n-1}$, then by the proposition above we may replace $d_{\xi_i}(A_i) \cap d_{\xi_{i+1}}(A_{i+1})$ by $d_{\xi_{i+1}}(d_{\xi_i}(A_i) \cap A_{i+1}) = d_{\xi_{n-1}}(d_{\xi_i}(A_i) \cap A_{i+1})$. By repeating this replacing operation finitely-many times we end up with a set of the form $d_{\xi_{n-1}}(B_0) \cap \ldots \cap d_{\xi_{n-1}}(B_m)$ that is equal to $d_{\xi_0}(A_0) \cap \ldots \cap d_{\xi_n}(A_{n-1})$. □

We shall investigate the conditions on δ under which the topologies τ_ξ are non-discrete.

In the case of τ_0, a point is not isolated if and only if it is a limit ordinal. Thus τ_0 is non-discrete if and only if $\delta > \omega$.

Notice that $d_0(A)$ is the set of limit points of A in the ordinal ordering. Thus, if the cofinality[1] of α is uncountable and $\alpha \in d_0(A)$, then $d_0(A) \cap \alpha$ is a club (i.e., closed and unbounded) subset of α, in the usual set-theoretic

[1] For all undefined set-theoretic notions and standard set-theoretic results see, e.g., [28].

sense. Thus, not surprisingly, τ_1 is known in the literature as the *club topology* and, as the proof of the next Proposition shows, $\alpha \in \delta$ is a non-isolated point in this topology if and only if α has uncountable cofinality. Hence τ_1 is non-discrete if and only if $\delta > \omega_1$.

Proposition 2.3. *The set* $\mathcal{B}_1 := \mathcal{B}_0 \cup \{d_0(A) : A \subseteq \delta\}$ *is a base for* τ_1.

Proof. Suppose α belongs to the basic set $I \cap d_0(A_0) \cap \ldots \cap d_0(A_{n-1})$, where $I \in \mathcal{B}_0$. If $n = 0$, then there is nothing to prove. So suppose $n > 0$. Then α must be a limit ordinal. If α has cofinality ω, then we can pick a sequence C of order-type ω cofinal in α so that $\{\alpha\} = d_0(C)$. Thus, $\alpha \in d_0(C) \subseteq I \cap d_0(A_0) \cap \ldots d_0(A_{n-1})$. If α has uncountable cofinality, then $I \cap d_0(A_0) \cap \ldots \cap d_0(A_{n-1}) \cap \alpha$ is club in α, and so $\alpha \in d_0(I \cap d_0(A) \cap \ldots \cap d_0(A_{n-1}) \cap \alpha) \subseteq I \cap d_0(A) \cap \ldots \cap d_0(A_n)$. ☐

2.1. **Stationary reflection.** As a warm-up for the general case, let us look first into the necessary conditions on δ for the τ_2 topology to be non-discrete.

Recall that for α an ordinal of uncountable cofinality, a set S of ordinals is called *stationary in* α if $S \cap C \neq \varnothing$, for every club $C \subseteq \alpha$.

One can easily check that for every $A \subseteq \delta$,

$$d_1(A) = \{\alpha : A \text{ is stationary in } \alpha\}.$$

If S is a stationary subset of α, then $d_1(S)$ is known in the literature as the *trace* of S. The operation d_1 is also known as the *Mahlo operation* (see [26] or [27]).

If α is an ordinal of uncountable cofinality and S is a stationary subset of α, one says that S *reflects at* $\beta < \alpha$ if S is stationary in β. And S is said to be *reflecting* if it reflects at some $\beta < \alpha$. Finally, we say that α is *stationary-reflecting* if every stationary subset of α is reflecting.

If α has uncountable cofinality and is not stationary-reflecting, then there must exist some stationary subset S of α such that $d_1(S) = \{\alpha\}$, in which case α is an isolated point of $\mathcal{B}_2$. So, for τ_2 to be a non-discrete topology on δ we need, at least, that some $\alpha < \delta$ be stationary-reflecting.

One can easily see that an ordinal α of uncountable cofinality is stationary-reflecting if and only if $cof(\alpha)$, the cofinality of α, is stationary-reflecting. Also, observe that if a cardinal κ is stationary-reflecting, then κ cannot be the successor of a regular cardinal. For if λ is regular and $\kappa = \lambda^+$, then the set $E^\kappa_\lambda := \{\beta < \kappa : cof(\beta) = \lambda\}$ is stationary. But E^κ_λ cannot reflect at any $\beta < \kappa$, because any such β has a club subset consisting of ordinals of cofinality less than λ. Therefore, the first stationary-reflecting ordinal, if it exists, must be either a weakly inaccessible cardinal or the successor of a singular cardinal. Thus, if, e.g., $\delta \leq \aleph_{\omega+1}$, then τ_2 is discrete on δ.

But for τ_2 to be non-discrete we need more than just the existence of a stationary-reflecting cardinal $\alpha < \delta$. What is needed is some $\alpha < \delta$ of uncountable cofinality such that every pair A, B of stationary subsets of α *simultaneously reflect*, meaning that there exists some $\beta < \alpha$ such that $\beta \in d_1(A) \cap d_1(B)$. Let us call such an α *simultaneoulsy stationary-reflecting*, or *s-reflecting* for short.

Proposition 2.4. *If α is s-reflecting, then $d_1(A_0) \cap \ldots \cap d_1(A_{n-1})$ is stationary, for all stationary $A_0, \ldots, A_{n-1} \subseteq \alpha$. Also, and in particular, if α is stationary-reflecting, then for every stationary $A \subseteq \alpha$, $d_1(A)$ is stationary.*

Proof. Suppose first that $n = 2$. So, fix A_0 and A_1, and fix a club subset C of α. The sets $C \cap A_0$ and $C \cap A_1$ are stationary in α, and therefore they simultaneously reflect at some $\beta < \alpha$. Thus $\beta \in C \cap d_1(A_0) \cap d_1(A_1)$.

Now, assuming the proposition holds for n, let us show it holds for $n+1$. So, fix $A_0, \ldots, A_n$ stationary, and suppose $C \subseteq \alpha$ is club. By the inductive hypothesis, $C \cap d_1(A_0) \cap \ldots \cap d_1(A_{n-1})$ is stationary. So, since the proposition holds for $n = 2$, the set $d_1(C \cap d_1(A_0) \cap \ldots \cap d_1(A_{n-1})) \cap d_1(A_n)$ is also stationary, hence nonempty. But clearly $d_1(C \cap d_1(A_0) \cap \ldots \cap d_1(A_{n-1})) \cap d_1(A_n) \subseteq C \cap d_1(A_0) \cap \ldots \cap d_1(A_n)$. □

Proposition 2.5.

(1) *An ordinal $\alpha < \delta$ is not isolated in the τ_2 topology on δ if and only if α is s-reflecting. Thus, $\mathcal{B}_2$ generates a non-discrete topology on δ if and only if some $\alpha < \delta$ is s-reflecting.*

(2) *$\mathcal{B}_2$ is a base for the τ_2 topology on δ if and only if every stationary-reflecting $\alpha < \delta$ is s-reflecting.*

Proof. (1): If α has uncountable cofinality and is not s-reflecting, then there are stationary $A, B \subseteq \alpha$ such that $d_1(A) \cap d_1(B) = \{\alpha\}$, hence α is an isolated point. Now suppose α is s-reflecting. If α belongs to an arbitrary basic open set $U = I \cap d_i(A_0) \cap \ldots d_i(A_{n-1})$, where $i < 2$ and $I \in \mathcal{B}_0$, then we claim that U contains some ordinal other than α, which will show that α is not an isolated point. If $n = 0$, then $U = I$ and so this is clear. If $n > 0$ and $i = 0$, then U is closed and unbounded in α. If $i = 1$, then by proposition 2.4, $I \cap d_i(d_i(A_0) \cap \ldots \cap d_i(A_{n-1}))$ is stationary in α, and clearly

$$I \cap d_i(d_i(A_0) \cap \ldots \cap d_i(A_{n-1})) \subseteq U.$$

(2): Suppose α is stationary-reflecting but not s-reflecting. So, there exist A and B stationary subsets of α such that $d_1(A) \cap d_1(B) = \{\alpha\}$. But $\{\alpha\} \notin \mathcal{B}_2$, and so $\mathcal{B}_2$ fails to be a base.

For the converse, suppose α belongs to an arbitrary basic open set $U = I \cap d_i(A_0) \cap \ldots d_i(A_{n-1})$, where $i < 2$ and $I \in \mathcal{B}_0$. We need to show that $\alpha \in U' \subseteq U$, for some $U' \in \mathcal{B}_2$. If $\alpha = 0$ or α is a successor, then $\alpha \in \{\alpha\} \in \mathcal{B}_0$. If α has countable cofinality, then for any set C of order-type ω and cofinal in α we have $\alpha \in d_0(C) = \{\alpha\} \in \mathcal{B}_1$. If α has uncountable cofinality and is not stationary-reflecting, then for some stationary $A \subseteq \alpha$ we have $\alpha \in d_1(A) = \{\alpha\} \in \mathcal{B}_2$. So suppose α is stationary-reflecting, hence s-reflecting. If $n = 0$, then $U = I$ and so there is nothing to show. If $n > 0$ and $i = 0$, then U is closed and unbounded in α and so $\alpha \in d_i(\alpha \cap U) \subseteq U$. So, suppose $i = 1$. Then by proposition 2.4 we get that $I \cap d_i(d_i(A_0) \cap \ldots \cap d_i(A_{n-1}))$, is stationary in α, and so

$$\alpha \in d_i(\alpha \cap I \cap d_i(d_i(A_0) \cap \ldots \cap d_i(A_{n-1}))) \subseteq U.$$

□

It is easy to see, using the characterization of weakly-compact cardinals in terms of Π^1_1 indescribability (see [28], [29], or Section 4 below), that every weakly compact cardinal is s-reflecting. Thus, from proposition 2.5, (1) it follows that in every model of set theory where there exists a weakly compact cardinal less than some ordinal δ, the topology τ_2 on δ is non-discrete. Further, R. Jensen [28] shows that in the constructible universe L a regular cardinal κ is stationary-reflecting if and only if it is weakly compact, hence if and only if it is s-reflecting. Thus, proposition 2.5, (2) implies that in L the set $\mathcal{B}_2$ is a base for the τ_2 topology, on any given δ.

2.2. **ξ-Stationarity.** We are ready now to investigate the general conditions that δ must satisfy for the topologies τ_ξ to be non-discrete. We begin with some definitions that generalize the notions of stationary set and stationary reflection.

Definition 2.6. *We say that $A \subseteq \delta$ is* 0-stationary in α *if and only if A is unbounded in α (i.e., $A \cap \alpha \neq \varnothing$ and for every $\beta < \alpha$ there is $\gamma \in A \cap \alpha$ greater than β).*

For $\xi > 0$, we say that A is ξ-stationary in $\alpha < \delta$ *if and only if for every $\zeta < \xi$, every subset S of δ that is ζ-stationary in α* ζ-reflects *to some $\beta \in A$, i.e., S is ζ-stationary in β.*

We say that an ordinal α is ξ-reflecting *if it is ξ-stationary in α.*

Note that A is 1-stationary in α if and only if $A \cap \alpha$ is stationary in α. Clearly, if A is ξ-stationary in α, then A is also ζ-stationary in α, for all $\zeta < \xi$. And if ξ is a limit ordinal, then A is ξ-stationary in α if and only if A is ζ-stationary in α, for all $\zeta < \xi$.

Notice also that every limit ordinal α is 0-reflecting, and α is 1-reflecting if and only if it has uncountable cofinality. Moreover, α is 2-reflecting if and only if it is stationary-reflecting.

Finally, let us observe that there is no ordinal α such that α is $\alpha+1$-reflecting. For suppose, to the contrary, that α is the least such ordinal. Then there is $\beta < \alpha$ such that $\alpha \cap \beta = \beta$ is α-stationary in β. In particular, β is $\beta + 1$-stationary in β, contradicting the minimality of α. However, as we shall see in the next section, it is possible for a (large) cardinal α to be α-reflecting.

Proposition 2.7. *For every $\xi > 0$, if A is ξ-stationary in α and C is a club subset of α, then $A \cap C$ is also ξ-stationary in α. Hence, if α is ξ-reflecting, then every club subset of α is ξ-stationary.*

Proof. This is clear for $\xi = 1$, and also clear for ξ a limit ordinal, provided it holds for all $0 < \zeta < \xi$. So suppose it holds for some $\xi > 0$ and let us see that it also holds for $\xi + 1$. Fix A $\xi + 1$-stationary in α and a club $C \subseteq \alpha$. By the induction hypothesis, $A \cap C$ is ξ-stationary in α. If S is a ξ-stationary subset of α, then by the induction hypothesis, $S \cap C$ is ξ-stationary in α. So, since A is $\xi + 1$-stationary in α, there exists $\beta \in A$ such that $S \cap C \cap \beta$ is ξ-stationary in β. Hence $\beta \in C$. $\square$

As we shall see (proposition 2.10), similarly as in the case of τ_2, for a point α to be non-isolated in the τ_ξ topology we need more than it being ξ-reflecting; what we need is that α satisfies the following property of simultaneous ξ-reflection.

Definition 2.8. *We say that $A \subseteq \delta$ is* 0-simultaneously-stationary in α *(0-s-stationary in α, for short), if and only if A is unbounded in α. For $\xi > 0$, we say that A is* ξ-simultaneously-stationary in α *(ξ-s-stationary in α, for short) if and only if for every $\zeta < \xi$, every pair of subsets S and T of δ that are ζ-s-stationary in α* simultaneously ζ-s-reflect *to some $\beta \in A$, i.e., S and T are both ζ-s-stationary in β.*

We say that an ordinal α is ξ-s-reflecting *if it is ξ-s-stationary in α.*

Note that if A is ξ-s-stationary in α, then A is also ζ-s-stationary in α, for all $\zeta < \xi$. And if ξ is a limit ordinal, then A is ξ-s-stationary in α if and only if A is ζ-s-stationary in α, for all $\zeta < \xi$.

Notice also that for $\xi \in \{0, 1\}$, A is ξ-s-stationary in α if and only if A is ξ-stationary in α. However, the existence of a 2-s-reflecting cardinal has higher consistency strength than the existence of a 2-reflecting cardinal, by results of Magidor [31] and Mekler-Shelah [32].

Proposition 2.9. *For every $\xi > 0$, if A is ξ-s-stationary in α and C is a club subset of α, then $A \cap C$ is also ξ-s-stationary in α. Hence, if α is ξ-s-reflecting, then every club subset of α is ξ-s-stationary.*

Proof. Similar to the proof of 2.7. □

The next proposition establishes an exact correspondance between limit points of the τ_ξ topology and ξ-s-stationarity.

Proposition 2.10.

(1) *For every ξ,*

$$d_\xi(A) = \{\alpha : A \text{ is } \xi\text{-s-stationary in } \alpha\}.$$

(2) *For every ξ and α, A is $\xi+1$-s-stationary in α if and only if $A \cap d_\zeta(S) \cap d_\zeta(T) \cap \alpha \neq \varnothing$ (equivalently, if and only if $A \cap d_\zeta(S) \cap d_\zeta(T)$ is ζ-s-stationary in α) for every $\zeta \leq \xi$ and every pair S, T of subsets of α that are ζ-s-stationary in α.*

(3) *For every ξ and α, if A is ξ-s-stationary in α and A_i is ζ_i-s-stationary in α for some $\zeta_i < \xi$, all $i < n$, then $A \cap d_{\zeta_0}(A_0) \cap \ldots \cap d_{\zeta_{n-1}}(A_{n-1})$ is ξ-s-stationary in α.*

Proof. We prove (1), (2), and (3) by simultaneous induction on ξ. (1) is clear for $\xi \leq 1$, and by Proposition 2.4, (2) holds for $\xi \leq 1$ and (3) holds for $\xi \leq 2$.

(1), (2), and (3) also hold for ξ a limit ordinal, assuming they hold for all $\zeta < \xi$. In the case of (1) and (3) this is clear. As for (2), suppose A is $\xi+1$-s-stationary in α, and S and T are ξ-s-stationary in α. So, there exists $\beta \in A \cap \alpha$ such that S and T are ξ-s-stationary in β, hence since (1) holds for ξ, we have that $\beta \in A \cap d_\xi(S) \cap d_\xi(T)$. To see that $A \cap d_\xi(S) \cap d_\xi(T)$ is in fact ξ-s-stationary, suppose X is ζ-s-stationary in α, for some $\zeta < \xi$. Then since by (3) for ξ the set $T \cap d_\zeta(X)$ is ξ-s-stationary in α, we have

$$A \cap d_\xi(S) \cap d_\xi(T) \cap d_\zeta(X) = A \cap d_\xi(S) \cap d_\xi(T \cap d_\zeta(X)) \neq \varnothing.$$

Suppose now that (1) and (2) hold for ξ and let us see that (1) holds for $\xi + 1$. If $\alpha \in d_{\xi+1}(A)$, then $\alpha \in d_\xi(A)$, hence by the induction hypothesis $A \cap \alpha$ is ξ-s-stationary in α. Now suppose $S, T \subseteq \alpha$ are ξ-s-stationary in α. Again by the induction hypothesis $\alpha \in d_\xi(S) \cap d_\xi(T)$, and since $\alpha \in d_{\xi+1}(A)$ and $d_\xi(S) \cap d_\xi(T)$ is a basic neighborhood of α in the $\tau_{\xi+1}$ topology, $A \cap ((d_\xi(S) \cap d_\xi(T)) \setminus \{\alpha\}) \neq \varnothing$. If $\beta \in A \cap ((d_\xi(S) \cap d_\xi(T)) \setminus \{\alpha\})$, then by the inductive hypothesis both S and T are ξ-s-stationary in β. This shows A is $\xi + 1$-s-stationary in α.

Now suppose A is $\xi + 1$-s-stationary in α. To see that $\alpha \in d_{\xi+1}(A)$, let U be a basic open set in the $\tau_{\xi+1}$ topology, with $\alpha \in U$. We may assume $U = I \cap d_\zeta(A_0) \cap \ldots d_\zeta(A_{n-1})$, where $I \in \mathcal{B}_0$ and $\zeta \leq \xi$ (see Corollary 2.2). If $n = 0$, then $U = I$ is an open interval, and so $A \cap I$ contains ordinals other than α. So suppose $n > 0$. If $\zeta < \xi$, then U is a basic open set of τ_ξ, and since by induction hypothesis $\alpha \in d_\xi(A)$, we have $A \cap (U \setminus \{\alpha\}) \neq \varnothing$. So, suppose $\zeta = \xi$. By the induction hypothesis, A_i is ξ-s-stationary in α, for all $i < n$. So, since A is $\xi + 1$-s-stationary, by the induction hypothesis on (2) for ξ, we have that $A \cap d_\xi(d_\xi(A_0) \cap d_\xi(A_1))$ is ξ-s-stationary in α. Similarly, we get that $A \cap d_\xi(d_\xi(d_\xi(A_0) \cap d_\xi(A_1)) \cap d_\xi(A_2))$ is ξ-s-stationary in α. And so on. Finally, we get that the set

$$X := A \cap I \cap d_\xi(\ldots d_\xi(d_\xi(A_0) \cap d_\xi(A_1)) \cap \ldots) \cap d_\xi(A_{n-1}))$$

is ξ-s-stationary in α. Hence, since $X \subseteq A \cap U$, we have $A \cap (U \setminus \{\alpha\}) \neq \varnothing$. This proves (1) for $\xi + 1$.

Next, assume (1), (2), and (3) hold for ξ and we will show that (3) holds for $\xi + 1$.

By (1) we only need to show that

$$A \cap d_{\zeta_0}(A_0) \cap \ldots \cap d_{\zeta_{n-1}}(A_{n-1}) \cap d_\zeta(S) \cap d_\zeta(T) \neq \varnothing$$

for all ζ-stationary subsets S and T of α, with $\zeta \leq \xi$.

Suppose first that $n = 1$. If $\zeta_0 = \zeta$, then $A \cap d_{\zeta_0}(A_0) \cap d_\zeta(d_\zeta(S) \cap d_\zeta(T))$ is contained in $A \cap d_{\zeta_0}(A_0) \cap d_\zeta(S) \cap d_\zeta(T)$ and is ζ-s-stationary in α, by (2). If $\zeta_0 < \zeta$, then by induction hypothesis on (3), $d_{\zeta_0}(A_0)$ is ζ-s-stationary in α. Hence, again by (2), $A \cap d_\zeta(d_{\zeta_0}(A_0)) \cap d_\zeta(d_\zeta(S) \cap d_\zeta(T)) \neq \varnothing$ and is contained in $A \cap d_{\zeta_0}(A_0) \cap d_\zeta(S) \cap d_\zeta(T)$. Finally, if $\zeta_0 > \zeta$, then by (2) the set $A \cap d_{\zeta_0}(A_0)$ is ζ_0-s-stationary in α, and hence $A \cap d_{\zeta_0}(A_0) \cap d_\zeta(S) \cap d_\zeta(T) \neq \varnothing$.

Now suppose $n > 1$. By (2) and the inductive hypothesis for (3), the set $d_\zeta(d_\zeta(S) \cap d_\zeta(T))$ is ξ-s-stationary in α. Also, by induction hypothesis on n, the set $A \cap d_{\zeta_0}(A_0) \cap \ldots \cap d_{\zeta_{n-2}}(A_{n-2})$ is $\xi + 1$-s-stationary in α. Therefore, letting $\eta := max\{\zeta_{n-1}, \zeta\}$, we have that the set

$$A \cap d_{\zeta_0}(A_0) \cap \ldots \cap d_{\zeta_{n-2}}(A_{n-2}) \cap d_\eta(d_{\zeta_{n-1}}(A_{n-1})) \cap d_\eta(d_\zeta(S) \cap d_\zeta(T))$$

which is contained in

$$A \cap d_{\zeta_0}(A_0) \cap \ldots d_{\zeta_{n-2}}(A_{n-2}) \cap d_{\zeta_{n-1}}(A_{n-1}) \cap d_\zeta(S) \cap d_\zeta(T)$$

is non-empty.

Finally, assuming that (1) and (3) hold for $\xi + 1$, let us see that (2) also holds for $\xi + 1$. Assume first that A is $\xi + 2$-s-stationary in α. By the induction hypothesis we only need to show that $A \cap d_{\xi+1}(S) \cap d_{\xi+1}(T)$ is $\xi + 1$-s-stationary in α, for every S and T that are $\xi + 1$-s-stationary in α.

So fix S and T $\xi+1$-s-stationary in α, and take any ζ-s-stationary subsets X and Y of α, for some $\zeta\leq\xi$. By (1) for ζ it is sufficient to check that

$$A\cap d_{\xi+1}(S)\cap d_{\xi+1}(T)\cap d_\zeta(X)\cap d_\zeta(Y)\neq\varnothing.$$

By (3) for $\xi+1$, the sets $S\cap d_\zeta(X)$ and $T\cap d_\zeta(Y)$ are $\xi+1$-s-stationary in α. Hence, since A is $\xi+2$-s-stationary in α, by (1) for $\xi+1$, there exists some $\beta\in A$ such that

$$\beta\in d_{\xi+1}(S\cap d_\zeta(X))\cap d_{\xi+1}(T\cap d_\zeta(Y))=d_{\xi+1}(S)\cap d_{\xi+1}(T)\cap d_\zeta(X)\cap d_\zeta(Y).$$

Now suppose $A\cap d_\zeta(S)\cap d_\zeta(T)\cap\alpha\neq\varnothing$ for every $\zeta\leq\xi+1$ and S,T that are ζ-s-stationary in α. We claim that A is $\xi+2$-s-stationary. By the induction hypothesis we only need to check that every S,T that are $\xi+1$-s-stationary in α $\xi+1$-s-reflect to some $\beta\in A$. So fix such S and T. By our assumption, there is $\beta\in A\cap d_{\xi+1}(S)\cap d_{\xi+1}(T)\cap\alpha$. Thus, by (1) for $\xi+1$, both S and T are $\xi+1$-s-stationary in β. □

Theorem 2.11. *For every ξ, an ordinal $\alpha<\delta$ is not isolated in the τ_ξ topology on δ if and only if α is ξ-s-reflecting. Thus $\mathcal{B}_\xi$ generates a non-discrete topology on δ if and only if some $\alpha<\delta$ is ξ-s-reflecting.*

Proof. We have already proved this in the previous sections for $0\leq\xi\leq 2$ (propositions 2.3 and 2.5). Notice that the theorem holds for limit ξ, provided it holds for all $\zeta<\xi$. So let us assume that it holds for some fixed $\xi>1$ and we will see that it holds for $\xi+1$.

If α is not $\xi+1$-s-reflecting, then for some $\zeta\leq\xi$ and some A,B that are ζ-s-stationary in α, $d_\zeta(A)\cap d_\zeta(B)$ is not ζ-s-stationary in α (by proposition 2.10 (2)). So, for some $\zeta'<\zeta$ and some S,T that are ζ'-stationary in α, $d_\zeta(A)\cap d_\zeta(B)\cap d_{\zeta'}(S)\cap d_{\zeta'}(T)=\{\alpha\}$, hence α is an isolated point in the topology $\tau_{\xi+1}$.

Now suppose that α is $\xi+1$-s-reflecting. By the induction hypothesis, α is not isolated in the τ_ξ topology. Let us see that α is not isolated in the $\tau_{\xi+1}$ topology either. So suppose that $\alpha\in U$, where U is a basic open set in $\tau_{\xi+1}$, which may be assumed to be of the form $U=I\cap d_\zeta(A_0)\cap\ldots\cap d_\zeta(A_{n-1})$, with $\zeta\leq\xi$ and I an open interval. But since $I\cap\alpha$ is a club subset of α, propositions 2.9 and 2.10 imply that U is $\xi+1$-s-stationary in α. □

Proposition 2.12. *If A is ξ-s-stationary in α, then $A\setminus d_\xi(A)$ is also ξ-s-stationary in α.*

Proof. This follows directly from proposition 2.10 (1) and the general fact that a topological space (X,τ) is scattered if and only if $d_\tau(A)=d_\tau(A\setminus d_\tau(A))$, for every $A\subseteq X$, where d_τ is Cantor's derivative operator on (X,τ). □

The following proposition shows that ξ-reflection and ξ-s-reflection are, essentially, properties of regular cardinals.

Proposition 2.13.

(1) *α is ξ-reflecting if and only if $cof(\alpha)$ is ξ-reflecting.*
(2) *α is ξ-s-reflecting if and only if $cof(\alpha)$ is ξ-s-reflecting.*

Hence, the first ξ-reflecting and the first ξ-s-reflecting ordinals, whenever they exist, are regular cardinals.

Proof. Fix a club $C \subseteq \alpha$ of order-type $cof(\alpha)$, and let $i : C \to cof(\alpha)$ be the unique continuous order-isomorphism. Then, for every $S \subseteq \alpha$ and every $0 < \zeta \leq \xi$, S is ζ-stationary (ζ-s-stationary) in α if and only if $S \cap C$ is ζ-stationary (ζ-s-stationary) in α (Propositions 2.7 and 2.9), if and only if the image of $S \cap C$ under i is ζ-stationary (ζ-s-stationary) in $cof(\alpha)$. The last equivalence may be shown by an easy induction on ζ. □

3. The ideal of non-ξ-s-stationary sets

For each limit ordinal α and each ordinal ξ, let $\mathcal{I}^\xi_\alpha$ be the set of non-ξ-s-stationary subsets of α, and let $\mathcal{F}^\xi_\alpha = (\mathcal{I}^\xi_\alpha)^* := \{A \subseteq \alpha : \alpha - A \in \mathcal{I}^\xi_\alpha\}$. Thus, $\mathcal{I}^0_\alpha$ is the set of bounded subsets of α, and $\mathcal{F}^0_\alpha$ is the set of tail subsets of α. If α has uncountable cofinality, then $\mathcal{I}^1_\alpha$ is the ideal of non-stationary subsets of α and $\mathcal{F}^1_\alpha$ is the club filter over α. Notice that $\zeta \leq \xi$ implies $\mathcal{I}^\zeta_\alpha \subseteq \mathcal{I}^\xi_\alpha$ and $\mathcal{F}^\zeta_\alpha \subseteq \mathcal{F}^\xi_\alpha$. Also note that $A \subseteq \alpha$ belongs to $\mathcal{F}^\xi_\alpha$ if and only if there for some $\zeta < \xi$ and some ζ-s-stationary sets $S, T \subseteq \alpha$, the set $d_\zeta(S) \cap d_\zeta(T) \cap \alpha$ is contained in A. In particular, if $S \subseteq \alpha$ is ζ-s-stationary, with $\zeta < \xi$, then $d_\zeta(S) \cap \alpha \in \mathcal{F}^\xi_\alpha$.

Theorem 3.1. *For every ξ, an ordinal α is ξ-s-reflecting if and only if $\mathcal{I}^\xi_\alpha$ is a proper ideal, hence if and only if $\mathcal{F}^\xi_\alpha$ is a proper filter.*

Proof. Assume first that α is ξ-s-reflecting (hence $\alpha \notin \mathcal{I}^\xi_\alpha$), and let us show that $\mathcal{I}^\xi_\alpha$ is an ideal. For $\xi = 0$ this is clear. So, suppose $\xi > 0$ and $A, B \in \mathcal{I}^\xi_\alpha$. There exist $\zeta_A, \zeta_B < \xi$, and there exist sets $S_A, T_A \subseteq \alpha$ that are ζ_A-s-stationary in α, and sets $S_B, T_B \subseteq \alpha$ that are ζ_B-s-stationary in α, such that $d_{\zeta_A}(S_A) \cap d_{\zeta_A}(T_A) \cap A = d_{\zeta_B}(S_B) \cap d_{\zeta_b}(T_B) \cap B = \varnothing$. Hence,

$$(d_{\zeta_A}(S_A) \cap d_{\zeta_A}(T_A) \cap d_{\zeta_B}(S_B) \cap d_{\zeta_B}(T_B)) \cap (A \cup B) = \varnothing.$$

By proposition 2.10 (3), the set $X := d_{\zeta_A}(S_A) \cap d_{\zeta_A}(T_A) \cap d_{\zeta_B}(S_B) \cap d_{\zeta_B}(T_B)$ is $max\{\zeta_A, \zeta_B\}$-s-stationary in α. Now notice that

$$d_{max\{\zeta_A,\zeta_B\}}(X) \subseteq X$$

and so we have

$$d_{max\{\zeta_A,\zeta_B\}}(X) \cap \alpha \cap (A \cup B) = \varnothing$$

which witnesses that $A \cup B \in \mathcal{I}^\xi_\alpha$.

For the converse, assume $\mathcal{I}^\xi_\alpha$ is a proper ideal, hence $\mathcal{F}^\xi_\alpha$ is a proper filter. Suppose that A and B are ζ-s-stationary subsets of α, for some $\zeta < \xi$. Then $d_\zeta(A) \cap \alpha$ and $d_\zeta(B) \cap \alpha$ are in $\mathcal{F}^\xi_\alpha$. Moreover, if $S, T \subseteq \alpha$ are any ζ'-s-stationary sets, for some $\zeta' < \xi$, then also $d_{\zeta'}(S) \cap \alpha$ and $d_{\zeta'}(T) \cap \alpha$ belong to $\mathcal{F}^\xi_\alpha$. Hence, since $\mathcal{F}^\xi_\alpha$ is a filter,

$$d_\zeta(A) \cap d_\zeta(B) \cap d_{\zeta'}(S) \cap d_{\zeta'}(T) \cap \alpha \in \mathcal{F}^\xi_\alpha$$

which implies, since $\mathcal{F}^\xi_\alpha$ is proper, that $d_\zeta(A) \cap d_\zeta(B) \cap d_{\zeta'}(S) \cap d_{\zeta'}(T) \cap \alpha \neq \varnothing$. This shows that $d_\zeta(A) \cap d_\zeta(B)$ is ξ-s-stationary in α. Since A and B were arbitrary, this implies that α is ξ-s-reflecting. □

If $\mathcal{F} \subseteq \mathcal{P}(A)$, for some set A, we say that $S \subset A$ *has positive $\mathcal{F}$-measure*, or *is $\mathcal{F}$-positive*, if $S \cap B \neq \varnothing$ for every $B \in \mathcal{F}$. Thus, if $A \subseteq \alpha$ *has positive $\mathcal{F}^\xi_\alpha$-measure*, then A is a ξ-s-stationary subset of α. Let us denote by $(\mathcal{F}^\xi_\alpha)^+$

the set of all subsets of α of positive $\mathcal{F}^\xi_\alpha$-measure. Notice that $(\mathcal{F}^\xi_\alpha)^+$ is non-empty if and only if $\mathcal{F}^\xi_\alpha$ is proper, i.e., it does not contain the empty set, in which case $(\mathcal{F}^\xi_\alpha)^+$ is exactly the set of ξ-s-stationary subsets of α.

We shall next address the issue of the consistency of $\mathcal{B}_\xi$ being a base or a sub-base for a non-discrete topology on δ. By Theorem 2.11 this reduces to the consistency of the existence of an ξ-s-reflecting cardinal.

4. Indescribable cardinals

Recall that a formula of second-order logic is Σ^1_0 (or Π^1_0) if it does not have quantifiers of second order, but it may have any finite number of first-order quantifiers and free first-order and second-order variables. In general, a formula is Σ^1_n, for $n > 0$, if it is of the form

$$\exists X_0, \dots, X_k \varphi(X_0, \dots, X_k)$$

where $k < \omega$, the variables $X_0, \dots, X_k$ are of second order, and $\varphi(X_0, \dots, X_k)$ is Π^1_{n-1}.

And a formula is Π^1_n, for $n > 0$, if it is of the form

$$\forall X_0, \dots, X_k \varphi(X_0, \dots, X_k)$$

where $\varphi(X_0, \dots, X_k)$ is Σ^1_{n-1}.

The notion of n-stationarity is Π^1_n expressible (see the proof of proposition 4.3 below). However, to express ξ-stationarity for $\xi \geq \omega$ we need to extend the definition of Π^1_n and Σ^1_n formulas to include the limit case.

Definition 4.1. *For ξ any ordinal, we say that a formula is $\Sigma^1_{\xi+1}$ if it is of the form*

$$\exists X_0, \dots, X_k \varphi(X_0, \dots, X_k)$$

where $\varphi(X_0, \dots, X_k)$ is Π^1_ξ.

And a formula is $\Pi^1_{\xi+1}$ if it is of the form

$$\forall X_0, \dots, X_k \varphi(X_0, \dots, X_k)$$

where $\varphi(X_0, \dots, X_k)$ is Σ^1_ξ.

If ξ is a limit ordinal, then we say that a formula is Π^1_ξ if it is of the form

$$\bigwedge_{\zeta<\xi} \varphi_\zeta$$

where φ_ζ is Π^1_ζ, all $\zeta < \xi$, and it has only finitely-many free second-order variables. And we say that a formula is Σ^1_ξ if it is of the form

$$\bigvee_{\zeta<\xi} \varphi_\zeta$$

where φ_ζ is Σ^1_ζ, all $\zeta < \xi$, and it has only finitely-many free second-order variables.

The indescribability of rank-initial segments of the set-theoretic universe with respect to Π^1_ξ sentences yields the following notion of Π^1_ξ-indescribable cardinal. (Other notions of Π^1_ξ-indescribability have been considered in the literature, e.g., by Jensen [17].)

Definition 4.2. *A cardinal* κ *is* Π^1_ξ-indescribable *if for all subsets* $A \subseteq V_\kappa$ *and every* Π^1_ξ *sentence* φ*, if*

$$\langle V_\kappa, \in, A\rangle \models \varphi$$

then there is some $\lambda < \kappa$ *such that*

$$\langle V_\lambda, \in, A \cap V_\lambda\rangle \models \varphi.$$

One may also define, similarly, the notion of Σ^1_ξ-indescribable cardinal. However, it is easily seen that a cardinal κ is Π^1_ξ-indescribable if and only if it is $\Sigma^1_{\xi+1}$-indescribable.

Note that if κ is Π^1_ξ-indescribable, then necessarily $\xi < \kappa$. One can easily sen that if κ is a measurable cardinal, then κ is Π^1_ζ-indescribable for all $\zeta < \kappa$. Also, if $j : M \to M$ is an elementary embedding, with M a transitive model of ZFC, and with critical point κ, then κ is Π^1_ζ-indescribable in M, for every $\zeta < \kappa$. It follows that the consistency of the existence of the ω-Erdös cardinal η_ω implies the consistency of the existence of a cardinal κ that is Π^1_ζ-indescribable, for all $\zeta < \kappa$ (by arguments as in [26] Theorem 17.33 and Exercise 17.29). Also, if κ is Π^1_ξ-indescribable, then κ is Π^1_ξ-indescribable in the constructible universe L (by an argument similar to [29] 6.7).

Proposition 4.3. *Every* Π^1_ξ*-indescribable cardinal is* $(\xi+1)$*-s-reflecting. Hence, if* ξ *is a limit ordinal and a cardinal* κ *is* Π^1_ζ*-indescribable for all* $\zeta < \xi$*, then* κ *is* ξ*-s-reflecting.*

Proof. Let κ be an infinite cardinal. Clearly, the fact that a set $A \subseteq \kappa$ is 0-s-stationary (i.e., unbounded) in κ can be expressed as a Π^1_0 sentence $\varphi_0(A)$ over $\langle V_\kappa, \in, A\rangle$. Inductively, one can now show that for every $\xi > 0$, the fact that a set $A \subseteq \kappa$ is ξ-s-stationary in κ can be expressed by a Π^1_ξ sentence φ_ξ over $\langle V_\kappa, \in, A\rangle$. Namely, by the sentence

$$\bigwedge_{\zeta<\xi} (A \text{ is } \zeta\text{-s-stationary})$$

in the case ξ is a limit ordinal, and by the sentence

$$\bigwedge_{\zeta<\xi-1} (A \text{ is } \zeta\text{-s-stationary}) \wedge$$
$$\forall S, T (S, T \text{ are } (\xi-1)\text{-s-stationary in } \kappa \to$$
$$\exists \beta \in A (S \text{ and } T \text{ are } (\xi-1)\text{-s-stationary in } \beta))$$

which is easily seen to be equivalent to a Π^1_ξ sentence, in the case ξ is a successor ordinal.

Now suppose κ is Π^1_ξ-indescribable, and suppose that A and B are ζ-s-stationary subsets of κ, for some $\zeta \leq \xi$. Thus,

$$\langle V_\kappa, \in, A, B\rangle \models \varphi_\zeta[A] \wedge \varphi_\zeta[B].$$

By the Π^1_ζ-indescribability of κ there exists $\beta < \kappa$ such that

$$\langle V_\beta, \in, A \cap \beta, B \cap \beta\rangle \models \varphi_\zeta[A \cap \beta] \wedge \varphi_\zeta[B \cap \beta]$$

which implies that A and B are ζ-s-stationary in β. Hence κ is $(\xi+1)$-s-reflecting. □

Thus, if there exists a Π^1_ξ-indescribable cardinal κ below some ordinal δ, then κ, and also all ordinals less than δ of cofinality κ, are limit points in the $\tau_{\xi+1}$ topology on δ.

4.1. **Indescribable filters.** Suppose φ is a Π^1_ξ sentence, and $A \subseteq \kappa$ is such that $\langle V_\kappa, \in, A\rangle \models \varphi$. Then we let

$$R_{A,\varphi} := \{\alpha < \kappa : \langle V_\alpha, \in, A \cap \alpha\rangle \models \varphi\}.$$

If κ is a Π^1_ξ-indescribable cardinal, then the *Π^1_ξ-indescribable filter F^ξ_κ on κ* is the proper filter generated by the sets $R_{A,\varphi}$. Notice that if $\zeta \leq \xi$, then $F^\zeta_\kappa \subseteq F^\xi_\kappa$.

Recall that that if $\mathcal{X} = \langle X_\alpha : \alpha < \kappa\rangle$ is a sequence of subsets of a cardinal κ, then the *diagonal intersection* of $\mathcal{X}$ is the set

$$\Delta_{\alpha<\kappa} X_\alpha := \{\beta < \kappa : \beta \in \bigcap_{\alpha<\beta} X_\alpha\}.$$

Observe that for every ξ,

$$d_\xi(\Delta_{\alpha<\kappa} X_\alpha) \subseteq \Delta_{\alpha<\kappa} d_\xi(X_\alpha).$$

For suppose $\Delta_{\alpha<\kappa} X_\alpha$ is ξ-s-stationary in β. If $\alpha_0 < \beta$, then $(\Delta_{\alpha<\kappa} X_\alpha \cap \beta) \setminus \alpha_0 \subseteq X_{\alpha_0} \cap \beta$, which implies that X_{α_0} is ξ-s-stationary in β, i.e., $\beta \in d_\xi(X_{\alpha_0})$. Thus, $\beta \in \bigcap_{\alpha<\beta} d_\xi(X_\alpha)$, hence $\beta \in \Delta_{\alpha<\kappa} d_\xi(X_\alpha)$.

Also, recall that a filter $\mathcal{F}$ on some cardinal κ is *normal* if it is closed under diagonal intersections. Equivalently, if every regressive function on an $\mathcal{F}$-positive set S is constant on an $\mathcal{F}$-positive subset of S.

If $\mathcal{F}$ is a normal filter on κ and it contains all tail sets, i.e., sets of the form $\kappa - \lambda$, some $\lambda < \kappa$, then it contains all club subsets of κ. Moreover, $\mathcal{F}$ is *κ-complete*, i.e., the intersection of less than κ-many elements of the filter is also in the filter. For if $\langle X_\alpha : \alpha < \lambda\rangle$, some $\lambda < \kappa$, is a sequence of members of the filter, put $X_\alpha = \kappa$ for all $\lambda \leq \alpha < \kappa$. Then by normality, $\Delta_{\alpha<\kappa} X_\alpha$ is in the filter, and since $\kappa - \lambda$ is also in the filter,

$$(\Delta_{\alpha<\kappa} X_\alpha) - \lambda = (\bigcap_{\alpha<\lambda} X_\alpha) - \lambda \in \mathcal{F}$$

and therefore $\bigcap_{\alpha<\lambda} X_\alpha \in \mathcal{F}$.

Proposition 4.4. *If κ is a Π^1_ξ-indescribable cardinal, and $\xi > 0$, then the filter F^ξ_κ is normal and κ-complete.*

Proof. The proof of normality is similar to the one by Levy [30] in the case $0 < \xi < \omega$ (see also [26], 6.11), using the fact that there is a universal Π^1_ξ formula. Namely, for each $\xi > 0$ there is a Π^1_ξ formula $\psi_\xi(X, y_\xi, z_\xi)$, with X a second order variable and y_ξ and z_ξ first-order variables, such that for every Π^1_ξ formula $\varphi(X)$ there is $k_\varphi \subseteq \xi$ such that for every limit ordinal α greater than ξ and every $A \subseteq V_\alpha$,

$$\langle V_\alpha, \in\rangle \models \varphi[A] \quad \text{if and only if} \quad \langle V_\alpha, \in\rangle \models \psi_\xi[A, k_\varphi, \xi].$$

When ξ is a successor ordinal, the universal Π^1_ξ formula may be obtained as in [30]. When ξ is a limit ordinal, $\psi_\xi(X, y_\xi, z_\xi)$ may be taken as the formula

$$z_\xi \text{ is a limit ordinal } \wedge\ y_\xi \subseteq \xi \text{ codes } \langle y_\zeta : \zeta < z_\xi\rangle \wedge \bigwedge_{\zeta<\xi} \psi_\zeta(X, y_\zeta, z_\zeta)$$

which is clearly equivalent to a Π^1_ξ formula. Then, given any Π^1_ξ formula $\varphi(X) = \bigwedge_{\zeta<\xi} \varphi_\zeta(X)$ we let k_φ to be a subset of ξ coding the sequence $\langle k_{\varphi_\zeta} : \zeta < \xi\rangle$, so that for every limit ordinal α greater than ξ and every $A \subseteq V_\alpha$,

$$\langle V_\alpha, \in\rangle \models \varphi[A] \text{ iff } \langle V_\alpha, \in\rangle \models \bigwedge_{\zeta<\xi} \psi_\zeta[A, k_{\varphi_\zeta}, \zeta] \text{ iff } \langle V_\alpha, \in\rangle \models \psi_\xi[A, k_\varphi, \xi].$$

To prove normality, suppose $X \subseteq \kappa$ is F^ξ_κ-positive. Without loss of generality, every element of X is a limit ordinal greater than ξ. Suppose $f : X \to \kappa$ is regressive and, towards a contradiction, assume that f is not constant on any positive set. So, for each $\alpha < \kappa$ there is some $A_\alpha \subseteq V_\kappa$ and $k_\alpha \subseteq \xi$ such that

$$\langle V_\kappa, \in\rangle \models \psi_\xi[A_\alpha, k_\alpha, \xi]$$

yet

$$\langle V_\lambda, \in\rangle \models \neg\psi_\xi[A_\alpha \cap V_\lambda, k_\alpha, \xi]$$

for every $\lambda \in X$ such that $f(\lambda) = \alpha$. Let $\Gamma : \kappa \times \kappa \to \kappa$ be the standard definable bijection (see [26] 3.5), and let $S := \{\Gamma(\alpha, \beta) : \alpha < \kappa \wedge \beta \in A_\alpha\}$ and $T := \{\Gamma(\alpha, \beta) : \alpha < \kappa \wedge \beta \in k_\alpha\}$. Let θ be the sentence "For every ordinal x there is a bigger ordinal y". Let $\varphi(S, T, \xi)$ be the sentence

$$\theta \wedge \forall\alpha\forall Y\forall v(Y = \{\beta : \Gamma(\alpha, \beta) \in S\} \wedge v = T[\alpha] \to \psi_\xi(Y, v, \xi)),$$

which is equivalent to a Π^1_ξ sentence: If ξ is a successor, then this is clear. And if ξ is a limit, then it is also clear because it is equivalent to the sentence

$$\bigwedge_{\zeta<\xi}(\theta \wedge \forall\alpha\forall Y\forall v(Y = \{\beta : \Gamma(\alpha, \beta) \in S\} \wedge v = T[\alpha] \to \psi_\zeta(Y, v(\zeta), \zeta))).$$

Since $\langle V_\kappa, \in, S, T, \xi\rangle \models \varphi[S, T, \xi]$ and X is positive, there exists $\lambda \in X$ such that

$$\langle V_\lambda, \in, S \cap V_\lambda, T \cap V_\lambda, \xi\rangle \models \varphi[S \cap V_\lambda, T \cap V_\lambda, \xi].$$

But since f is regressive, $\alpha := f(\lambda) < \lambda$, so $A_\alpha \cap V_\lambda$ and $T[\alpha]$ belong to V_λ, and therefore

$$\langle V_\lambda, \in\rangle \models \psi_\xi[A_\alpha \cap V_\lambda, k_\alpha, \xi]$$

thus yielding a contradiction.

Finally, since F^ξ_κ is normal and contains all the tail subsets of κ, it is κ-complete. □

Observe that if $X \in F^\xi_\kappa$, then X is $(\xi + 1)$-s-stationary. For suppose that $R_{A,\varphi} \subseteq X$, and S, T are ζ-s-stationary subsets of κ, for some $\zeta \leq \xi$. Then, letting $\psi(S, T)$ be the Π^1_ζ sentence asserting that S and T are ζ-s-stationary (proposition 4.3), we have that $R_{S,\psi}, R_{T,\psi} \in F^\xi_\kappa$ and $R_{A,\varphi} \cap R_{S,\psi} \cap R_{T,\psi} \subseteq X \cap d_\zeta(S) \cap d_\zeta(T)$. Hence, since F^ξ_κ is proper and $R_{A,\varphi} \cap R_{S,\psi} \cap R_{T,\psi} \in F^\xi_\kappa$, we have that $X \cap d_\zeta(S) \cap d_\zeta(T) \cap \kappa$ is nonempty.

Also, notice that if κ is Π^1_ξ-indescribable, then for every $\zeta \leq \xi$ and every pair S, T of ζ-s-stationary subsets of κ, the set $d_\zeta(S) \cap d_\zeta(T) \cap \kappa$ belongs to F^ξ_κ. Hence, $\mathcal{F}^\xi_\kappa \subseteq F^\xi_\kappa$.

Proposition 4.5. *If κ is a Π^1_ξ-indescribable cardinal, and $\xi > 0$, then the filter $\mathcal{F}^{\xi+1}_\kappa$ is normal and κ-complete.*

Proof. Let $\langle X_\alpha : \alpha < \kappa \rangle$ be a sequence of of members of $\mathcal{F}^{\xi+1}_\kappa$. Without loss of generality, we may assume $X_\alpha = d_{\zeta_\alpha}(S_\alpha) \cap d_{\zeta_\alpha}(T_\alpha) \cap \kappa$, where S_α and T_α are ζ_α-stationary subsets of κ, some $\zeta_\alpha \leq \xi$. Since the filter F^ξ_κ contains all the sets X_α, and is normal, we have that $\Delta_{\alpha<\kappa} X_\alpha$ is also in the filter, and therefore, by our observation above, it is $(\xi+1)$-s-stationary. Thus, $d_\xi(\Delta_{\alpha<\kappa} X_\alpha) \cap \kappa \in \mathcal{F}^{\xi+1}_\kappa$. Now we have the following inclusions:

$$d_\xi(\Delta_{\alpha<\kappa} X_\alpha) \cap \kappa \subseteq \Delta_{\alpha<\kappa} d_\xi(X_\alpha) \cap \kappa = \Delta_{\alpha<\kappa} d_\xi(d_{\zeta_\alpha}(S_\alpha) \cap d_{\zeta_\alpha}(T_\alpha) \cap \kappa) \cap \kappa \subseteq$$
$$\subseteq \Delta_{\alpha<\kappa}(d_{\zeta_\alpha}(S_\alpha) \cap d_{\zeta_\alpha}(T_\alpha) \cap \kappa) = \Delta_{\alpha<\kappa} X_\alpha$$

and so $\Delta_{\alpha<\kappa} X_\alpha \in \mathcal{F}^{\xi+1}_\kappa$.

The κ-completeness follows from normality since all tail subsets of κ belong to $\mathcal{F}^{\xi+1}_\kappa$. □

5. Reflection and indescribability in L

As shown in [5], in the constructible universe L the converse to proposition 4.3 also holds for $n > 0$, and so a regular cardinal is $(n+1)$-reflecting if and only if it is Π^1_n-indescribable, and therefore if and only if it is $(n+1)$-s-reflecting. Hence, a regular cardinal is ω-reflecting, if and only if it is ω-s-reflecting, and if and only if it is Π^1_n-indescribable for every $n < \omega$. Thus, it follows from Theorem 2.11 that in L the topology τ_{n+1} on some ordinal δ in non-discrete if and only if there exists a Π^1_n-indescribable cardinal below δ. The non-isolated points are precisely those ordinals whose cofinality is Π^1_n-indescribable. Moreover, in L, for every $n \leq \omega$ the set $\mathcal{B}_n$ is a base for the τ_n topology.

The proof of the main theorem of [5] shows in fact that if $V = L$ and κ is a regular Π^1_{n-1}-indescribable cardinal, then for every Π^1_n formula $\varphi(X)$ and every $A \subseteq \kappa$ such that $\langle V_\kappa, \in, A \rangle \models \varphi(A)$, there exists an n-s-stationary set $S \subseteq \kappa$ such that $d_n(S) \subseteq D_{\varphi,A}$. Hence, if κ is Π^1_n-indescribable (equivalently, regular and $(n+1)$-reflecting), then $\mathcal{F}^n_\kappa = F^n_\kappa$.

With similar arguments as in [5] we will show that, in L, the same holds for every ordinal $\xi > 0$, namely a regular cardinal is $(\xi+1)$-reflecting if and only if it is Π^1_ξ-indescribable, and therefore if and only if it is $(\xi+1)$-s-reflecting. Hence, for every limit ordinal ξ, a regular cardinal is ξ-reflecting if and only if it is ξ-s-reflecting, and if and only if it is Π^1_ζ-indescribable for every $\zeta < \xi$. One direction is given by proposition 4.3. For the other direction it is sufficient to prove the following.

Theorem 5.1. *Assume $V = L$. Suppose $\xi > 0$ and κ is a regular $(\xi+1)$-reflecting cardinal. Then for every $A \subseteq \kappa$ and every Π^1_ξ sentence Ψ, if $\langle L_\kappa, \in, A \rangle \models \Psi$, then there exists a ξ-stationary $S \subseteq \kappa$ such that Ψ reflects to every ordinal λ in $d_\xi(S)$, i.e., $\langle L_\lambda, \in, A \cap \lambda \rangle \models \Psi$.*

Proof. We proceed by induction on $\xi > 0$. The case $\xi = 1$ is due to Jensen ([28] 6.1). So, suppose $\xi > 1$ and the Theorem holds for $0 < \zeta < \xi$. We shall prove it for ξ.

Suppose κ is regular and $(\xi+1)$-reflecting. So, $\xi < \kappa$. By the inductive hypothesis, κ is Π^1_ζ-indescribable, for every $\zeta < \xi$. Hence κ is innaccessible and the set *Reg* of regular cardinals below κ is stationary.

Fix $A \subseteq \kappa$ and a Π^1_ξ sentence Ψ, and suppose

$$\langle L_\kappa, \in, A\rangle \models \Psi.$$

We shall find a ξ-stationary $S \subseteq \kappa$ such that Ψ reflects to every ordinal in $d_\xi(S)$. Since $d_\xi(S)$ intersects every $(\xi+1)$-stationary subset of κ, our proof will also show that for every $(\xi+1)$-stationary $T \subseteq \kappa$, every Π^1_ξ sentence true in $\langle L_\kappa, \in, A\rangle$ reflects to some ordinal in T. Thus, we shall assume, inductively, that

(∗) For every regular ξ-reflecting cardinal λ, every ξ-stationary $T \subseteq \lambda$, and every $\zeta < \xi$, every Π^1_ζ sentence true in $\langle L_\lambda, \in, A\cap\lambda\rangle$ reflects to some element of T.

The case $\xi = 2$ is shown in [28] 6.1, as part of the proof that in L, if a regular cardinal is 2-reflecting, then it is Π^1_1-indescribable. Notice that (∗) holds for a limit ξ if and only if it holds for all (equivalently, for unboundedly many) ordinals $\xi' < \xi$.

The set $R := \{\alpha < \kappa : \xi < \alpha \text{ and } \alpha \text{ is not } \xi\text{-reflecting}\}$ is ξ-stationary in κ, for given any ζ-stationary $T \subseteq \kappa$, some $\zeta < \xi$, the least ordinal greater than ξ where T ζ-reflects belongs to R. If α is a cardinal in R, then α is not Π^1_ζ-indescribable, for some $\zeta < \xi$ (proposition 4.3). So, for each regular $\alpha \in R$, let λ_α be the least ordinal greater than or equal to $\alpha+\omega$ such that L_{λ_α} contains some subset of α that is a witness to the non Π^1_ζ-indescribability of α, for some $\zeta < \xi$. Let λ^-_α be the largest limit ordinal less than or equal to λ_α. Note that $\lambda^-_\alpha > \alpha$.

Since we shall be only interested in the satisfaction of Ψ in structures with universes of the form L_α, with α a limit ordinal, and so sufficient coding apparatus is available, we may assume that all second-order variables appearing in Ψ range over sets of ordinals.

Define

$$S := \{\alpha \in R \cap Reg : A\cap\alpha \in L_{\lambda^-_\alpha} \wedge L_{\lambda^-_\alpha} \models \text{“}\langle L_\alpha, \in, A\cap\alpha\rangle \models \Psi\text{”}\}.$$

We will show that S is ξ-stationary in κ. So fix any ζ_0-stationary subset T of κ, where $\zeta_0 < \xi$.

Fix a large $k < \omega$. Let $N := \langle N_\alpha : \alpha < \kappa\rangle \in L_{\kappa^+}$ be the natural continuous $\subseteq$-increasing $\in$-chain of Σ_k-elementary substructures of L_{κ^+} of size $< \kappa$, such that $|\xi|^+ \cup \{\kappa, T, A\} \subseteq N_0$ (so all Π^1_ξ formulas are in N_0), and $N_\alpha \cap \kappa$ is an ordinal, for every $\alpha < \kappa$. Namely, let M_0 be the Σ_k-Skolem hull in L_{κ^+}, via the standard Σ_k-definable Skolem functions, of the set $|\xi|^+ \cup \{\kappa, T, A\}$, and let $M_{\alpha+1}$ be the Σ_k-Skolem hull in L_{κ^+}, via the standard Σ_k-definable Skolem functions, of $\{M_\alpha\} \cup M_\alpha$. If α is a limit, then let $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. For every $\alpha < \kappa$, let $f(\alpha)$ be the least $\gamma > f(\beta)$, for all

$\beta < \alpha$, such that $M_\gamma \cap \kappa$ is an ordinal. Since κ is regular, $f(\alpha) < \kappa$, for all $\alpha < \kappa$. Then let $N_\alpha = M_{f(\alpha)}$.

For every $\alpha < \kappa$, and $\zeta < \xi$ greater than 0, let

$$R^\zeta_\alpha := \bigcap_{D \in N_\alpha;\ \psi \in \Pi^1_\zeta} R^\zeta_{D,\psi}$$

where

$$R^\zeta_{D,\psi} := \{\alpha < \kappa : \langle L_\alpha, \in, D \cap \alpha\rangle \models \psi\}$$

whenever $D \subseteq \kappa$ and $\langle L_\kappa, \in, D\rangle \models \psi$, and let $R^\zeta_{D,\psi} := \kappa$, otherwise.

Since $|N_\alpha| < \kappa$, and since κ is inaccessible and so there are $< \kappa$-many Π^1_ζ formulas, by the κ-completeness of the normal filter F^ζ_κ we have that $R^\zeta_\alpha \in F^\zeta_\kappa$. Moreover, if α is a limit, then $R^\zeta_\alpha = \bigcap_{\beta<\alpha} R^\zeta_\beta$.

By normality, $E^\zeta := \triangle_{\alpha<\kappa} R^\zeta_\alpha \in F^\zeta_\kappa$. If $\alpha \in E^\zeta$, then L_α reflects all Π^1_ζ sentences, with parameters in N_α, that are true in L_κ. For if $D \subseteq \kappa$, $D \in N_\alpha$, $\psi(X)$ is Π^1_ζ, and $L_\kappa \models \psi[D]$, then $D \in N_\beta$, for some $\beta < \alpha$. Hence, since $\alpha \in \bigcap_{\eta<\alpha} R^\zeta_\eta$, we have that $\alpha \in R^\zeta_\beta$, and therefore $L_\alpha \models \psi[D \cap \alpha]$.

Let $C \subseteq \kappa$ be the club subset of all α such that $N_\alpha \cap \kappa = \alpha$. So, $C \in F^\zeta_\kappa$. Also, $Reg \in F^\zeta_\kappa$, because κ is regular, and this fact is Π^1_1 expressible over $\langle V_\kappa, \in\rangle$.

Suppose α belongs to $F^\zeta := C \cap Reg \cap E^\zeta \in F^\zeta_\kappa$, and let L_γ be the transitive collapse of N_α, via the collapsing isomorphism $\pi : N_\alpha \to L_\gamma$. Note that $\pi(\kappa) = \alpha$, because $N_\alpha \cap \kappa = \alpha$, and therefore $\pi(A) = A \cap \alpha$ and $\pi(T) = T \cap \alpha$. Also note that π is the identity on Π^1_ξ formulas.

Claim 5.2. *L_γ is correct about Π^1_ζ sentences holding in L_α. That is, if $D \in L_\gamma$, $D \subseteq \alpha$, $\psi(X)$ is a Π^1_ζ formula, with X as its only free second-order variable, and $L_\gamma \models$ "$L_\alpha \models \psi[D]$", then $L_\alpha \models \psi[D]$.*

Proof. Suppose $L_\gamma \models$ "$L_\alpha \models \psi[D]$". We have $\pi(D^*) = D$, for some $D^* \subseteq \kappa$. Moreover, $D^* \cap \alpha = D$. Since π is an isomorphism and $\pi(\psi(X)) = \psi(X)$, we have that $N_\alpha \models$ "$L_\kappa \models \psi[D^*]$". Hence, by elementarity, $L_{\kappa^+} \models$ "$L_\kappa \models \psi[D^*]$", and therefore $L_\kappa \models \psi[D^*]$.

Thus, since $\alpha \in R^\zeta_\alpha$ (because $\alpha \in E^\zeta$ and $R^\zeta_\alpha = \bigcap_{\beta<\alpha} R^\zeta_\beta$), we have that $\alpha \in R^\zeta_{D^*,\psi}$, and so $L_\alpha \models \psi[D]$. □

Claim 5.3. *If α is the least element of F^ζ greater than ξ, and L_γ is the transitive collapse of N_α, then $\alpha \in R$ and $\lambda_\alpha \leq \gamma + 1$.*

Proof. Let $\alpha \in F^\zeta \setminus \xi$ be least, and let $\pi : N_\alpha \to L_\gamma$ be the transitive collapse isomorphism. Since the embedding $id \circ \pi^{-1} : L_\gamma \to L_{\kappa^+}$ is Σ_k-elementary, the natural continuous $\subseteq$-increasing $\in$-chain $\langle N'_\eta : \eta < \alpha\rangle$ of Σ_k-elementary substructures of L_γ of size $< \alpha$ such that $|\xi|^+ \cup \{\alpha, T \cap \alpha, A \cap \alpha\} \subseteq N'_0$, and $N'_\eta \cap \alpha$ is an ordinal, is precisely $\langle \pi(N_\eta) : \eta < \alpha\rangle$, hence it belongs to L_γ. Since $N_\alpha \models |N_\eta| < \kappa$, for every $\eta < \alpha$, we have that $L_\gamma \models |N'_\eta| < \alpha$. Hence

$$L_\gamma \models \text{“}|\bigcup_{\eta<\alpha} N'_\eta| = \alpha\text{”}.$$

Thus, in L_γ we can define an enumeration $\bar{D} := \langle\langle D_\beta, \psi_\beta\rangle : \beta < \alpha\rangle$ of all the pairs $\langle D_\beta, \psi_\beta\rangle$ such that $D_\beta \subseteq \alpha$, ψ_β is Π^1_ζ, and $L_\alpha \models \psi_\beta(D_\beta)$. Let $\psi(X, x, \zeta)$ be the universal Π^1_ζ formula, and for each $\beta < \alpha$, let $k_\beta \subseteq \zeta \cup \omega$ code ψ_β. Now let Γ be the canonical definable bijection between κ^3 and κ (see [26] 3.5), and let

$$D := \{\Gamma(\beta, \delta, 0) : \beta < \alpha \wedge \delta \in D_\beta\} \cup \{\Gamma(\beta, \eta, 1) : \beta < \alpha \wedge \eta \in k_\beta\}.$$

Thus, $D \in L_{\gamma+1}$. Note that $D \subseteq \alpha$, because α being an infinite cardinal is closed under Γ. Now letting $\theta(X)$ be a Π^1_ζ formula equivalent to the formula

$$\forall E \forall k(\exists \beta \forall \delta \forall \eta((\delta \in E \leftrightarrow \Gamma(\delta, \beta, 0) \in X) \wedge (\eta \in k \leftrightarrow \Gamma(\beta, \eta, 1) \in X)) \rightarrow \psi(E, k, \zeta))$$

we have that $L_\alpha \models \theta(D)$.

Since π is the identity on α, and $\pi(\kappa) = \alpha$, for every $\alpha' < \alpha$ we have that $N_{\alpha'} \cap \kappa = \alpha'$ if and only if $\pi(N_{\alpha'}) \cap \alpha = \alpha'$. It follows that $C \cap \alpha = \{\alpha' < \alpha : N_{\alpha'} \cap \kappa = \alpha'\} = \{\alpha' < \alpha : \pi(N_{\alpha'}) \cap \alpha = \alpha'\}$, hence $C \cap \alpha$ belongs to L_γ.

We claim that $\theta(D)$ does not reflect to any $L_{\alpha'}$ with $\alpha' \in C \cap \alpha$ and greater than ξ. For, aiming for a contradiction, suppose $\alpha' \in C \cap \alpha$ is the least such that $\xi < \alpha'$ and $L_{\alpha'} \models \theta[D \cap \alpha']$. Then,

$$D \cap \alpha' = \{\Gamma(\delta, \beta, 0) : \beta < \alpha' \wedge \delta \in D_\beta \cap \alpha'\} \cup \{\Gamma(\beta, \eta, 1) : \beta < \alpha' \wedge \eta \in k_\beta\}$$

and therefore $L_{\alpha'} \models \psi_\beta[D_\beta \cap \alpha']$, for all $\beta < \alpha'$. So, $L_{\alpha'}$ reflects all Π^1_ζ sentences, with parameters in $N_{\alpha'}$, that are true in L_κ. Hence, $\alpha' \in R^\zeta_{\alpha'}$, and therefore $\alpha' \in E^\zeta$. Moreover, since $L_{\alpha'}$ reflects the Π^1_1 sentence implying $\alpha' \in Reg$, we have that $\alpha' \in F^\zeta$, thus contradicting the minimality of α.

It follows that $\alpha \in R$, for otherwise, by induction hypothesis α is Π^1_ζ-indescribable, hence L_α must reflect the sentence $\theta(D)$ to some $\eta \in C \cap \alpha$ greater than ξ.

Since $C \cap \alpha, D \in L_{\gamma+1}$, we have that $\lambda_\alpha \leq \gamma + 1$, as $C \cap \alpha$ and D, together with the Π^1_ζ formula $\theta(X)$, witness the non-Π^1_ζ-indescribability of α. $\square$

Let α and L_γ be as in the last claim. Then in L_γ there is no counterexample to α not being ξ-reflecting, because $L_{\kappa^+} \models$ "κ is Π^1_ζ-indescribable", for every $\zeta < \xi$, and so $L_\gamma \models$ "α is Π^1_ζ-indescribable", for all $\zeta < \xi$. Hence, by Claim 5.2, we have that $\gamma < \lambda_\alpha \leq \gamma + 1$. Therefore, $\lambda^-_\alpha = \gamma$. So, since $L_{\kappa^+} \models$ "$\langle L_\kappa, \in, A\rangle \models \Psi$", we have that $L_\gamma \models$ "$\langle L_\alpha, \in, A \cap \alpha\rangle \models \Psi$", and thus $\alpha \in S$.

To complete the proof that S is ξ-stationary in κ, it only remains to show that $T \cap \alpha$ is ζ_0-stationary in α. Note first that $T \cap \alpha \in L_\gamma$. So, since $L_{\kappa^+} \models$ "$L_\kappa \models T$ is ζ_0-stationary in κ", it follows by Σ_k-elementarity that $L_\gamma \models$ "$L_\alpha \models T \cap \alpha$ is ζ_0-stationary in α". Now by Claim 5.2 above, we have that $L_\alpha \models$ "$T \cap \alpha$ is ζ_0-stationary in α", and so $T \cap \alpha$ is indeed ζ_0-stationary in α.

We will next show that Ψ holds in $\langle L_\lambda, \in, A\rangle$ whenever $\lambda \in d_\xi(S)$. Notice first that since κ is $(\xi + 1)$-reflecting, $d_\xi(S) \neq \varnothing$. Also note that since $S \subseteq Reg$, every $\lambda \in d_\xi(S)$ is regular; for if λ were singular, there would be a club subset of λ consisting only of singular cardinals which, since S is stationary in λ, would have to intersect S.

So, take $\lambda \in d_\xi(S)$ and let us show that Ψ holds in $\langle L_\lambda, \in, A\rangle$. Suppose, aiming for a contradiction, that $\langle L_\lambda, \in, A\rangle \models \neg\Psi$. If ξ is a successor ordinal, then we may assume Ψ is of the form $\forall X \varphi(X)$, where $\varphi(X)$ is $\Sigma_{\xi-1}$; and if ξ is a limit, then Ψ is of the form $\bigwedge_{\zeta<\xi}\varphi_\zeta$, where φ_ζ is Π^1_ζ. Thus, in the successor case, for some $B \subseteq \lambda$,

$$L_\lambda \models \neg\varphi[B, A\cap\lambda].$$

And in the limit case,

$$L_\lambda \models \neg\varphi_{\bar\zeta}$$

for some $\bar\zeta < \xi$.

Let $\zeta := \xi - 1$, if ξ is a successor, and $\zeta := \bar\zeta$ if ξ is a limit.

Let $\langle M_\alpha : \alpha < \lambda\rangle$ be the natural $\subseteq$-increasing $\in$-chain of elementary substructures of L_{λ^+}, each of size $< \lambda$, such that $|\zeta|^+ \cup \{\lambda, A\cap\lambda\} \subseteq M_0$, and $M_\alpha \cap \lambda$ is an ordinal, for every α. We also require that $B \in M_0$, in the case ξ is a successor.

Since λ is regular and ξ-reflecting, by induction hypothesis it is Π^1_ζ-indescribable, for every $\zeta < \xi$. Hence the Π^1_ζ-indescribable filter F^ζ_λ is normal and λ-complete, for all $\zeta > 0$.

For each $\alpha < \lambda$, let

$$S_\alpha := \bigcap_{D\in M_\alpha;\ \psi\in\Pi^1_\zeta} S_{D,\psi}$$

where

$$S_{D,\psi} := \{\alpha < \lambda : \alpha \text{ is a limit ordinal } \wedge L_\alpha \models \psi[D\cap\alpha]\}.$$

By λ-completeness, $S_\alpha \in F^\zeta_\lambda$. Moreover, if α is a limit, then $S_\alpha = \bigcap_{\beta<\alpha} S_\beta$. By normality, the sets $C := \{\alpha < \lambda : M_\alpha \cap \lambda = \alpha\}$ and $E := \triangle_{\alpha<\lambda} S_\alpha$ belong to F^ζ_λ. Notice that if $\beta \in C \cap E$, then L_β reflects all Π^1_ζ sentences, with parameters in M_β, that are true in L_λ.

Since S is ξ-stationary in λ, by our inductive hypothesis $(*)$ for ξ, we have that S intersects every element of F^ζ_λ. So, pick $\beta \in C\cap E\cap S$.

Let L_γ be the transitive collapse of M_β, via the collapsing isomorphism $\pi : M_\beta \to L_\gamma$. Note that since $\beta \in C$, $\pi(\lambda) = \beta$ and therefore $A \cap \beta \in L_\gamma$ (and also $B\cap\beta \in L_\gamma$ in the successor case). Then as in Claim 5.2, L_γ is correct about Π^1_ζ sentences holding in L_β.

Thus, in the successor case, since $L_{\lambda^+} \models$ "$L_\lambda \models \neg\varphi[B, A\cap\lambda]$", and therefore

$$L_\gamma \models \text{“}L_\beta \models \neg\varphi[B\cap\beta, A\cap\beta]\text{”}$$

we have that

$$L_\beta \models \neg\varphi[B\cap\beta, A\cap\beta].$$

Hence, since $\beta \in S$, it follows that $\gamma > \lambda^-_\beta$, because $B\cap\beta \in L_\gamma \setminus L_{\lambda^-_\beta}$.

Similarly, in the limit case we have

$$L_\beta \models \neg\psi_\zeta[A\cap\beta]$$

and it again follows that $\gamma > \lambda^-_\beta$, because no witness to $\neg\psi_\zeta[A\cap\beta]$ exists in $L_\gamma \setminus L_{\lambda^-_\beta}$.

But since $L_{\lambda^+} \models$ "λ is Π^1_ζ-indescribable", we have that $L_\gamma \models$ "β is Π^1_ζ-indescribable". Hence, since L_γ is correct about Π^1_ζ sentences holding in L_β, in L_γ there is no counterexample to β being not Π^1_ζ-indescribable, and therefore $\gamma < \lambda_\beta$, which contradicts the fact that $\gamma > \lambda^-_\beta$, because γ is a limit. This completes the proof of theorem 5.1. □

Thus, if $V = L$ and there exists a Π^1_ξ indescribable cardinal, then for every ordinal δ greater than the first Π^1_ξ indescribable cardinal, $\tau_{\xi+1}$ is a non-discrete topology on δ. Moreover, if $V = L$, then for every ξ, the set $\mathcal{B}_\xi$ is a base for the τ_ξ topology.

Assuming the consistency of the appropriate large cardinals (with ZFC) one can easily get a model of ZFC where $\tau_{\xi+1}$ is discrete and τ_ξ isn't.

Theorem 5.4. *$CON(\exists\kappa < \lambda\,(\kappa$ is Π^1_ξ-indescribable $\wedge$ λ is inaccessible$))$ implies $CON(\tau_\xi$ is non-discrete $\wedge\, \tau_{\xi+1}$ is discrete$)$.*

Proof. Let κ be Π^1_ξ-indescribable, and let $\lambda > \kappa$ be inaccessible. In L, κ is Π^1_ξ-indescribable and λ is inaccessible. So, in L, let κ_0 be the least Π^1_ξ-indescribable cardinal, and let λ_0 be the least inaccessible cardinal above κ_0. Then L_{λ_0} is a model of ZFC in which κ_0 is Π^1_ξ-indescribable and no ordinal greater than κ_0 is 2-reflecting. The reason is that if α is a regular cardinal greater than κ_0, then $\alpha = \beta^+$, for some cardinal β. And since Jensen's principle $\Box_\beta$ holds, there exists a stationary subset of α that does not reflect. □

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