Mathias and set theory

Akihiro Kanamori*

Department of Mathematics & Statistics, Boston University, 111 Cummington Mall, Boston MA 02215, United States of America

Abstract

On the occasion of his 70th birthday, the work of Adrian Mathias in set theory is surveyed in its full range and extent.

1 Introduction

Adrian Richard David Mathias (born 12 February 1944) has cut quite a figure on the "surrealist landscape" of set theory ever since it became a modern and sophisticated field of mathematics, and his 70th birthday occasions a commemorative account of his mathematical *oeuvre*. It is of particular worth to provide such an account, since Mathias is a set theorist distinctive in having both established a range of important combinatorial and consistency results as well as in carrying out definitive analyses of the axioms of set theory.

Setting out, Mathias secured his set-theoretic legacy with the *Mathias real*, now squarely in the pantheon of generic reals, and the eventual rich theory of happy families developed in its surround. He then built on and extended this work in new directions including those resonant with the Axiom of Determinacy, and moreover began to seriously take up social and cultural issues in mathematics. He reached his next height when he scrutinized how Nicolas Bourbaki and particularly Saunders Mac Lane attended to set theory from their mathematical perspectives, and in dialectical engagement investigated how their systems related to mainstream axiomatic set theory. Then in new specific research, Mathias made an incisive set-theoretic incursion into dynamics. Latterly, Mathias refined his detailed analysis of the axiomatics of set theory to weaker set theories and minimal axiomatic sufficiency for constructibility and forcing.

We discuss Mathias's mathematical work and writings in roughly chronological order, bringing out their impact on set theory and its development. We describe below how, through his extensive travel and varied working contexts, Mathias has engaged with a range of stimulating issues. For accomplishing

*e-mail: aki@math.bu.edu

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this, Mathias's webpage proved to be a valuable source for articles and details about their contents and appearance. Also, discussions and communications with Mathias provided detailed information about chronology and events. The initial biographical sketch which follows forthwith is buttressed by this information.

Mathias came up to the University of Cambridge and read mathematics at Trinity College, receiving his B.A. in 1965. This was a heady time for set theory, with Paul Cohen in 1963 having established the independence of the Axiom of Choice and of the Continuum Hypothesis with inaugural uses of his method of forcing. With the infusion of new model-theoretic and combinatorial methods, set theory was transmuting into a modern, sophisticated field of mathematics. With the prospect of new investigative possibilities opening up, Mathias with enterprise procured a studentship to study with Ronald Jensen at Bonn, already well on the way to becoming the leading set theorist in Europe.

With Jensen and through reading and study, Mathias proceeded to assimilate a great deal of set theory, and this would lay the groundwork not only for his lifelong research and writing but also for a timely survey of the entire subject. In the summer of 1967, he with Jensen ventured into the New World, participating in a well-remembered conference held at the University of California at Los Angeles from 10 July to 5 August, which both summarized the progress and focused the energy of a new field opening up. There, he gave his first paper [46], a contribution to the emerging industry of independences among consequences of the Axiom of Choice.[1]

At the invitation of Dana Scott, Jensen and Mathias spent the autumn and winter of 1967/68 at Stanford University, and the year flowing into the summer proved to be arguably Mathias's most stimulative and productive in set theory. Mathias benefitted from the presence at Stanford of Harvey Friedman and Kenneth Kunen and from the presence at nearby Berkeley of Jack Silver and Karel Příkrý. With inspirations and initiatives from the U.C.L.A. conference, Mathias worked steadily to complete a survey of all of the quickly emerging set theory, "Surrealist landscape with figures". This was in typescript *samizdat* circulation by the summer, and proved to be of enormous help to aspiring set theorists. Almost as an aside, Mathias and Kopperman established model-theoretic results about groups [32].

What came to occupy center stage were developments stimulated by a seminar at Stanford in the autumn of 1967 and cumulating in dramatic results established back in Bonn in the summer of 1968. As described in § 2, Mathias established decisive results about the partition relation $\omega \longrightarrow (\omega)^{\omega}_{2}$, coming up with the key concept of *capturing* for the Mathias real and establishing that the relation holds in Solovay's model. Mathias quickly announced his results in [41] and in four weeks penned *On a generalization of Ramsey's theorem* [42], submitted in August 1968 to Trinity College for a research fellowship.

Unsuccessful, Mathias forthwith spent the academic year 1968/69 at the University of Wisconsin, Madison, at the invitation of Kunen. There Mathias became familiar with the work of Kunen and David Booth, particularly on combinatorics of ultrafilters over ω. In late 1968, Mathias again applied for research fellowships at Cambridge with [42], and by April 1969, he was notified

[1]Mathias [46] established, with forcing, that Tarski's Order Extension Principle, that every partially ordered set can be extended to a total ordering, is independent of Mostowski's Ordering Principle, that every set has a total ordering.

that he was successful at Peterhouse. June 1969 he spent at Monash University, Australia, and by then he had come to an elaboration of his work, also described in § 2, in terms of happy families, Ramsey ultrafilters generically included in them, and Mathias forcing adapted to the situation.

In October 1969 Mathias duly took up his fellowship at Peterhouse. In 1970, he with [42] was admitted to the Ph.D. by the University of Cambridge; from October 1970 for five years he held a university assistant lectureship; and from 1972 he held a Peterhouse teaching fellowship. Mathias would spend just over two decades as a fellow at Peterhouse, in a position of considerable amenity and stability. In 1971, with energy and enthusiasm he organized the Cambridge Summer School of Logic, August 1 to 21, a conference that arguably rivaled the U.C.L.A. conference, and brought out a proceedings [44]. From 1972 to 1974, Mathias edited the *Mathematical Proceedings of the Cambridge Philosophical Society*. This journal had been published as the *Proceedings of the Cambridge Philosophical Society* since 1843 and had published mathematical papers; Mathias during his tenure managed to attach the "mathematical" to its title.[2] In 1973, Mathias worked at the Banach Center at Warsaw to fully update his 1968 survey. The revision process however got out of hand as set theory was expanding too rapidly, and so he would just publish his survey, with comments added about recent work, as *Surrealist landscape with figures* [51]. In late 1974, Mathias was finally able to complete a mature, seasoned account of his work on $\omega \longrightarrow (\omega)^{\omega}_{2}$ and Mathias reals, and this appeared with a publication delay as *Happy families* [48] in 1977. In 1978, Mathias organized another summer school at Cambridge, August 7 to 25, and brought out a corresponding compendium [52]. The academic year 1979/80 Mathias spent as *Hochschulassistent* to Jensen at the Mathematical Institute at Freiburg; since then, the Freiburg group of logicians have been unfailingly hospitable and encouraging to Mathias.

In ongoing research through this period and into the mid-1980s, Mathias obtained important results on filters and in connection with the Axiom of Determinacy, building on his work on $\omega \to (\omega)^{\omega}_{2}$. Moreover, Mathias began to take up topics and themes in rhetorical pieces about mathematics as set in society and culture. § 2 describes this far-ranging work.

In the 1980s, an unhappy climate developed in Peterhouse, and in 1990, Mathias did not have his fellowship renewed. This was a remarkable turn of events, not the least for setting in motion a memorable journey, a veritable wanderer fantasy that started in the last year of his fellowship: The academic year 1989/90 Mathias spent at the Mathematical Sciences Research Institute, Berkeley, and the Spring of 1991 he was Visiting Professor at the University of California at Berkeley. For 1991/92 Mathias was Extraordinary Professor at the University of Warsaw. For much of 1992/93 he was *Dauergast* at the research institute at Oberwolfach in Germany. The years 1993 to 1996 Mathias spent at Centre de Recerca Matemàtica of the Institut d'Estudis Catalans at Barcelona at the suggestion of Joan Bagaria. 1996/97 he was in Wales. For 1997/98, Mathias was at the Universidad de los Andes at Bogotá, Colombia, at the suggestion of Carlos Montenegro.

At the beginning of these wanderings, Mathias engaged in a controversy with the distinguished mathematician Saunders Mac Lane. This stimulated Mathias to fully take up the investigation of the scope and limits of "Mac Lane set

[2] Mathias, personal communication.

theory", something that he had started to do while still at Peterhouse. Over several years in the mid-1990s, the timing too diffuse to chronicle here, Mathias provided a rich, definitive analysis of weak set theories around Mac Lane and Kripke-Platek. § 3 describes these developments, which established Mathias as a major figure in the fine axiomatization of set theory.

During his three-year period in Barcelona, Mathias was stimulated by colleagues and circumstances to pursue a set-theoretic approach to a basic iteration problem in dynamics. After establishing the subject in 1996, he would elaborate and refine it well into the next millennium. § 4 describes this work, which decisively put the face of well-foundedness on dynamics.

In early 1999, Mathias landed on the island of La Réunion in the Indian Ocean, the most remote of the overseas départements of France, and obtained tenure at the university in 2000. At last there was stability again, though he was retired from the professoriate in 2012, with *France d'outre mer* also being subject to the French university rules for mandatory retirement.

Through these years, Mathias would continue to travel, and at Barcelona during a "set theory year" 2003/04, Mathias newly investigated set theories weaker than Kripke-Platek and intended to be axiomatic bases for developing constructibility, specifically in being able to carry the definition of the truth predicate for bounded formulas. Thus in continuing engagement with axiomatics, Mathias soon addressed the problem of finding the weakest system that would support a smooth, recognizable theory of forcing. He met with remarkable success in the theory of rudimentary recursion and provident sets. § 5 describes this work, his finest and at the same time deepest work on the axiomatics of set theory.

In the fullness of time, Mathias returned to his first tussle with ill-suited set theories, the "ignorance of Bourbaki". After pointing out mathematical pathologies, Mathias gave full vent to his disapproval of Bourbaki's logic and influence in a lengthy piece. Though there is polemic, one also sees both Mathias's passionate advocacy of set theory in the face of detractors, and his hope for a kind of regeneration of mathematics through competition rather than centralization. The last section, § 6, gets at these matters, very much to be considered part of his mathematical *oeuvre*.

On July 18, 2015, Adrian Mathias was admitted to the degree of Doctor of Science by the University of Cambridge.[3] Mathias was honored at the Fifth European Set Theory Conference held at the Isaac Newton Institute at Cambridge by having August 27, 2015 declared as "Mathias Day", a day given over to talks on his work and its influence.

2 $\omega \longrightarrow (\omega)^\omega_2$ and Mathias Reals

Recall that in the Erdős-Rado partition calculus from the 1950s, $[X]^\gamma = \{y \subseteq X \mid y \text{ has order type } \gamma\}$ for X a set of ordinals, and that the partition relation for ordinals

$$\beta \longrightarrow (\alpha)^\gamma_\delta$$

[3]The Doctor of Science is a higher doctorate of the university. According to the *Statutes and Ordinances of the University of Cambridge* (p. 519): "In order to qualify for the degree of Doctor of Science or Doctor of Letters a candidate shall be required to give proof of distinction by some original contribution to the advancement of science and of learning."

asserts that for any partition $f\colon [\beta]^\gamma \to \delta$, there is an $H \in [\beta]^\alpha$ *homogeneous* for f, i.e., $|f``[H]^\gamma| \leq 1$.

Frank Ramsey established Ramsey's Theorem, that for $0 < r, k < \omega$, $\omega \longrightarrow (\omega)^r_k$ and the Finite Ramsey Theorem, that for any $0 < r, k, m < \omega$, there is an $n < \omega$ such that $n \longrightarrow (m)^r_k$ [80].

In 1934, the youthful Paul Erdős, still at university in Hungary, and György Szekeres popularized the Finite Ramsey Theorem with a seminal application to a combinatorial problem in geometry.[4] Erdős forthwith, in a letter to Richard Rado, asked whether the "far-reaching" generalization $\omega \longrightarrow (\omega)^\omega_2$ of Ramsey's Theorem can be established, and by return post Rado provided the now well-known counterexample deploying a well-ordering of the reals—the first result of Ramsey theory after Ramsey's.[5] With this, Ramsey theory and the partition calculus as developed by Erdős and his collaborators would focus on finite exponents, i.e., partitions of $[\beta]^\gamma$ for finite γ.

In the post-Cohen climate of Axiom of Choice independence results, the young Harvey Friedman, in a seminar conducted by Dana Scott at Stanford on partition relations in the autumn of 1967, newly raised the possibility of establishing the *consistency* of $\omega \longrightarrow (\omega)^\omega_2$. Alerted to a related possibility raised by Scott, that *definable* partitions of $[\omega]^\omega$ might have infinite homogeneous sets, Fred Galvin and Karel Příkrý at Berkeley established, by the winter of 1967, the now well-known and widely applied Galvin-Příkrý Theorem [22]. A set $Y \subseteq [\omega]^\omega$ is *Ramsey* if and only if there is an $x \in [\omega]^\omega$ such that $[x]^\omega \subseteq Y$ or $[x]^\omega \subseteq [\omega]^\omega - Y$. Nash-Williams, Cohen, and Ehrenfeucht in early contexts had established that open sets are Ramsey. Reconstruing the classical notion of a set of reals being Borel, Galvin and Příkrý established that Borel partitions have large homogeneous sets: If Y is Borel, then Y is Ramsey. Analytic ($\mathbf{\Sigma}^1_1$) sets are classically the projection of Borel subsets of the plane, and Silver at Berkeley forthwith improved the result to: If Y is analytic, then Y is Ramsey [82].

For Mathias, this past would be a prologue. In June of 1968, back in Bonn, he returned to the possibility of $\omega \longrightarrow (\omega)^\omega_2$, i.e., that *every* $Y \subseteq [\omega]^\omega$ is Ramsey. With ideas, results, and concepts in the air, he would in a few weeks put together the workings of a known pivotal *model* with a newly tailored *genericity* concept to achieve a decisive result.[6]

Mathias had learned from Silver of Solovay's celebrated 1964 model in which every set of reals is Lebesgue measurable [83], and discussions with Jensen led to its further understanding. Solovay's model was remarkable for its early sophistication and revealed what standard of argument was possible with forcing. Starting with an inaccessible cardinal, Solovay first passed to a generic extension given by the Lévy collapse of the cardinal to render it ω_1 and then to a desired inner model, which can most simply be taken to be the constructible closure $\mathbf{L}(\mathbb{R})$ of the reals. The salient point here is that in the generic extension $\mathbf{V}[G]$,

$$\text{if } Y \text{ is a set of reals ordinal definable from a real } r\text{, then there is a formula } \varphi(\cdot,\cdot) \text{ such that: } x \in Y \text{ if and only if } \mathbf{V}[r][x] \models \varphi[r,x]. \qquad (*)$$

Solovay used this to get the Lebesgue measurability of Y with an infusion of

[4] Cf. [21].

[5] Cf. [20].

[6] Mathias [42, § 0] describes the progression, from which much of what follows, as well as the previous paragraph, are drawn.

random reals.

After Příkrý came up with Příkrý forcing for measurable cardinals in the summer of 1967, Mathias started exploring a version of this forcing for ω around the end of 1967. *Mathias forcing* has as conditions $\langle s, A\rangle$, where $s \subseteq \omega$ is finite, $A \subseteq \omega$ is infinite, and $\max(s) < \min(A)$, ordered by:

$$\langle t, B\rangle \leq \langle s, A\rangle \text{ if and only if } s \text{ is an initial segment of } t \text{ and } B \cup (t - s) \subseteq A.$$

A condition $\langle s, A\rangle$ is to determine a new, generic subset of ω through initial segments s, the further members to be restricted to A. A generic subset of ω thus generated is a *Mathias real.*

Mathias opined that $\omega \longrightarrow (\omega)^{\omega}_{2}$ should hold in Solovay's inner model if every infinite subset of a Mathias generic real is also Mathias generic, and that this indeed should be the case from his previous work. In the small hours of July 7, 1968, everything fell into place when Mathias came up with the property of *capturing*, a sort of well-foundedness notion. A Mathias condition $\langle s, A\rangle$ *captures* a dense set Δ of conditions if and only if every infinite subset B of A has a finite initial segment t such that $\langle s \cup t, A - (\max t + 1)\rangle \in \Delta$. Mathias saw that for any $\langle s, A\rangle$ and dense set Δ, there is an infinite subset A' of A such that $\langle s, A'\rangle$ captures Δ. With this, Mathias proved (a) for any condition $\langle s, A\rangle$ and formula ψ of the forcing language, there is an infinite $B \subseteq A$ such that $\langle s, B\rangle$ decides ψ, i.e., s need not be extended to decide formulas, as well as (b) if x is a Mathias real over a model M and $y \subseteq x$ is infinite, then y is a Mathias real over M. Thus, $\omega \longrightarrow (\omega)^{\omega}_{2}$ was confirmed in Solovay's inner model: For a set Y of reals definable from a real r as in (*), by (a) there is in $\mathbf{V}[r]$ a Mathias condition $\langle \varnothing, A\rangle$ that decides $\varphi(r, c)$, where c is the canonical name for a Mathias real. There is surely in $\mathbf{V}[G]$ a real $x \subseteq A$ Mathias generic over $\mathbf{V}[r]$, and by (b) x confirms that Y is Ramsey.

With energy and initiative Mathias, in four weeks starting July 8, penned *On a generalization of a theorem of Ramsey* [42], providing a comprehensive account of Solovay's model, Mathias reals, and the consistency of $\omega \longrightarrow (\omega)^{\omega}_{2}$, as well a forcing proof of Silver's analytic-implies-Ramsey result and a range of results about Mathias reals that exhibited their efficacy and centrality.

By June of 1969, Mathias had uncovered a rich elaboration of his work. As formulated in Booth, an ultrafilter U over ω is *Ramsey* if and only if for any $f : [\omega]^2 \longrightarrow 2$ there is an $H \in U$ homogeneous for f [10]. For a filter F, *F-Mathias forcing* is Mathias forcing with the additional proviso that conditions $\langle s, A\rangle$ are to satisfy $A \in F$. Mathias realized that a real x Mathias over a ground model $\mathbf{V}$ generates a Ramsey ultrafilter F on $\wp(\omega) \cap \mathbf{V}$ given by $F = \{X \in \wp(\omega) \cap \mathbf{V} \mid x - X \text{ is finite}\}$ and that generically adjoining x to $\mathbf{V}$ is *equivalent* to first generically adjoining the corresponding F, without adjoining any reals, and then doing F-Mathias forcing over $\mathbf{V}[F]$ to adjoin x. A *happy family* is, in one formulation, a set A of infinite subsets of ω such that $\wp(\omega) - A$ is an ideal, and: whenever $X_i \in A$ with $X_{i+1} \subseteq X_i$ for $i \in \omega$ there is a $Y \in A$ which diagonalizes the X_i's, i.e., its increasing enumeration f satisfies $f(i+1) \in X_i$ for every $i \in \omega$. The set of infinite subsets of ω is a happy family, as is a Ramsey ultrafilter. A happy family is just the sort through which one can force a Ramsey ultrafilter without adjoining any reals. Mathias thus had a tripartite elaboration: One entertains happy families by overlaying Ramsey ultrafilters, which themselves can be reduced to Mathias reals. With this, Mathias could give systematic generalizations not only of his earlier results but those of Galvin-Příkrý and Silver.

Coordinating this work with Příkrý forcing, Mathias established a corresponding characterization, that if U is a normal ultrafilter over a measurable cardinal κ, then a countable $x \subseteq \kappa$ is Příkrý generic if and only if for any $X \in U$, $x - X$ is finite [45]. This "Mathias property" would become a pivotal feature of all generalized Příkrý forcings (cf. [23]).

Mathias eventually laid out his theory with all its trimmings and trappings in *Happy families* [48], proving the Ramseyness results as well as further, 1969 results in the elaborated context. For H a happy family, say that a set $Y \subseteq [\omega]^\omega$ is *H-Ramsey* if and only if there is an $x \in H$ such that $[x]^\omega \subseteq Y$ or $[x]^\omega \subseteq [\omega]^\omega - Y$. Mathias established that every analytic set is H-Ramsey for every happy family H. At the other end, Mathias established results about Solovay's inner model if one started with a Mahlo cardinal: In the Lévy collapse of a Mahlo cardinal to ω_1, every set of reals in $\mathbf{L}(\mathbb{R})$ is in fact H-Ramsey for every happy family H, and also, in $\mathbf{L}(\mathbb{R})$ there are no maximal almost disjoint families of subsets of ω.

To frame [48] at one end, it is worth mentioning that Mathias in [48], for the happy-families improvement of Silver's analytic-implies-Ramsey, returns to a classical characterization of analytic sets. The Luzin-Sierpiński investigation of analytic sets in the 1910s was the first occasion where well-foundedness was explicit and instrumental, and today one can vouchsafe the sense of analytic sets as given by well-founded relations on finite sequences of natural numbers. Mathias in his researches would continue to engage with the theory of analytic sets, and his work would itself draw out his later contention that set theory itself is ultimately the study of well-foundedness.

To elaborate on the Mahlo cardinal results at the other end of [48], Mathias made a distinctive advance by incorporating elementary substructures into the mix to establish results about the $\mathbf{L}(\mathbb{R})$ of the Lévy collapse of a Mahlo cardinal instead of just an inaccessible cardinal. Mathias's result from [48, § 5] that, in this Lévy collapse, every set of reals in $\mathbf{L}(\mathbb{R})$ is H-Ramsey for every happy family H would eventually be complemented 30 years later in terms of consistency by Todd Eisworth [18], who showed, applying a later Henle-Mathias-Woodin [53] result, that if the Continuum Hypothesis holds and every set of reals in $\mathbf{L}(\mathbb{R})$ is U-Ramsey for every Ramsey ultrafilter U, then ω_1 is Mahlo in $\mathbf{L}$. A prominent open problem addresses Mathias's first, 1968 result that in the Solovay Lévy collapse of an inaccessible to ω_1, every set of reals in $\mathbf{L}(\mathbb{R})$ is Ramsey. Is the consistency strength of having an inaccessible cardinal necessary? Halbeisen and Judah, among others, considered this question and established related results [24].

For Mathias's other Mahlo cardinal result, a family of infinite subsets of ω is *almost disjoint* if distinct members have finite intersection, and is *maximal almost disjoint* (m.a.d.) if moreover no proper extension is almost disjoint. Mathias observed that m.a.d. families generate happy families, and showed that m.a.d. families cannot be analytic. He moreover built on his previous work to show that in the Lévy collapse of a Mahlo cardinal to ω_1, there are no m.a.d. families in $\mathbf{L}(\mathbb{R})$. Almost a half a century later, Asger Törnquist showed that in Solovay's original Lévy collapse of just an inaccessible cardinal to ω_1, there are no m.a.d. families in $\mathbf{L}(\mathbb{R})$ [86]. The question remains whether it is consistent, relative to ZF, whether there are no m.a.d. families.

As to the core of Mathias's [48], the extent to which a piece of mathematics is pursued and extended by others is a measure of both its mathematical significance and depth. Early on, topological proofs of Silver's analytic-implies-Ramsey result were found by Erik Ellentuck [19] and of Mathias's analytic-implies-U-Ramsey for Ramsey ultrafilters U were found by Alain Louveau and Keith Milliken [36, 78]. The Ellentuck Theorem is "an infinite dimensional Ramsey theorem" in what is now considered optimal form, deploying what is now widely known as the *Ellentuck topology*, with open sets of form $O_{\langle a,S\rangle} = \{x \in {}^{\omega}\omega \mid a \subseteq x \subseteq S\}$ for a Mathias condition $\langle a, S\rangle$. Andreas Blass and Claude Laflamme took Mathias's theory from [48] to a next level of generalization, extending it to non-Ramsey ultrafilters and corresponding Mathias-type generic reals [7, 34].

In their [14], Timothy Carlson and Stephen Simpson established "dual" Ramsey theorems. Their Dual Ramsey Theorem asserts that, with $(\omega)^k$ the set of partitions of ω into k cells, if $k < \omega$ and $(\omega)^k = C_1 \cup C_2 \cup \ldots \cup C_n$ with the C_i's Borel, then there is an i and some k-cell partition H such that all its k-cell coarsenings lie in C_i (partitions are taken to be equivalence relations on ω, and the topology here is the product topology on $2^{\omega\times\omega}$). They established a dual Ellentuck theorem and introduced a dual Mathias forcing. Carlson subsequently generalized a large part of the Ramsey theory at the time to Ramsey spaces, structures that satisfy a corresponding Ellentuck theorem [13]. A *Ramsey space* is a space of infinite sequences with a topology such that every set with the property of Baire is Ramsey and no open set is meager. Stevo Todorčević's [85] is a magisterial axiomatic account of abstract Ramsey spaces, with corresponding combinatorial forcing and Ellentuck theorems.

Returning to the original Mathias real, it has taken a fitting place in the pantheon of generic reals. Mathias forcing, in a filter form, was deployed in the classic [40, p. 153] by Martin and Solovay, in a proof that in modern terms can be construed as showing that Martin's Axiom implies $\mathfrak{p} = \mathfrak{c}$, i.e., that the pseudo-intersection number is the cardinality of the continuum. Mathias forcing has since become common fare in the study of cardinal invariants of the continuum. After Richard Laver in [35] famously established the consistency of Borel's Conjecture with a paradigmatic countable support iteration featuring Laver reals, he noted that, after all, Mathias reals could have been deployed instead [35, p. 168]. James Baumgartner worked this out in an incisive account [5] of iterated forcing for the 1978 Cambridge conference organized by Mathias. Mathias reals in various forms would occur in a range of work on ultrafilters, e.g., by Blass and Saharon Shelah on ultrafilters with small generating sets [9].

3 Varia

In his two decades at Peterhouse, Mathias pursued research that built on and resonated with his Ramseyness work as well as forged new directions. Moreover, he began to articulate ways of thinking and points of view about mathematics as set in society and culture.

Following on his incisive incorporation of Ramsey ultrafilters, Mathias made contributions to an emerging theory of filters and ultrafilters. A filter F over ω is a *p-point* if and only if whenever $X_i \in F$ for $i \in \omega$, there is a $Y \in F$

such that $Y - X_i$ is finite for every $i \in \omega$. The *Rudin-Keisler ordering* $\leq_{\mathrm{RK}}$ is defined generally by: For filters $F, G \subseteq \wp(I)$, $F \leq_{\mathrm{RK}} G$ if and only if there is an $f \colon I \to I$ such that $F = f_*(G)$, where $f_*(G) = \{X \subseteq I \mid f^{-1}(X) \in G\}$. These concepts emerged in the study of the Stone-Čech compactification $\beta\mathbb{N}$, identifiable with the ultrafilters over ω. Walter Rudin showed that the Continuum Hypothesis (CH) implies that there is a p-point in $\beta\mathbb{N} - \mathbb{N}$, and since p-points are topologically invariant and there are non-*p*-points, $\beta\mathbb{N} - \mathbb{N}$ is not homogeneous. Answering an explicitly posed question, Mathias showed in [43] that CH implies that there is an ultrafilter U over ω with no p-point below it in the Rudin-Keisler ordering. For this, Mathias used properties of analytic sets.

With a neat observation, Mathias showed in [47] that $\omega \longrightarrow (\omega)^\omega_2$ implies that the filters over ω are as curtailed as they can be. Let $\mathrm{Fr} = \{X \subseteq \omega \mid \omega - X \text{ is finite}\}$—the *Fréchet filter.* A filter over ω is *feeble*, being as far from being an ultrafilter as can be, if and only if there is a finite-to-one $f \colon \omega \to \omega$ such that $f_*(F) = \mathrm{Fr}$. In [47], Mathias pointed out that for filters F over ω extending Fr, F is feeble if and only if a corresponding $P^F \subseteq \wp(\omega)$ is Ramsey. Hence, $\omega \to (\omega)^\omega_2$ implies that every filter over ω extending Fr is feeble! Being feeble would become a pivotal concept for filters,[7] especially as it was soon seen that a filter over ω extending Fr is feeble if and only if it is meager in the usual topology on $\wp(\omega)$. In Solovay's model or under the assumption of the *Axiom of Determinacy*, every filter extending Fr is feeble forthwith because every set of reals has the Baire property.

Already in the early 1970's, whether ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice) implies that there are p-point ultrafilters over ω became a focal question. As an approach to the question, the present author asked whether, if not such an ultrafilter, at least coherent filters exist, where in the above terms, a filter over ω is *coherent* if and only if it extends Fr, is a p-point, and is not feeble [28]. Remarkably, in [50], Mathias quickly established that if $0^\#$ does not exist or $2^{\aleph_0} \leq \aleph_{\omega+1}$, then there are coherent filters. The elegant proof depended on a covering property for families of sets, with Jensen's recent Covering Theorem for $\mathbf{L}$ providing the ballast with $0^\#$. Around 1978, Shelah famously established[8] that it is consistent relative to ZFC that there are no p-point ultrafilters. Still, the question of whether ZFC implies that there are coherent filters has remained. Adding to the grist, the 1990 [27], with Mathias a co-author, contains a range of results about and applications of coherent filters. This paper came about as a result of another co-author, Winfried Just, having rediscovered the main results of Mathias's [50] around 1986.

Continuing his engagement with analytic sets, Mathias with Andrzej Ostazewski and Michel Talagrand in [77] addressed the following proposition queried by Rogers and Jayne: Given a non-Borel analytic set A, there is a compact set K such that $K \cap A$ is not Borel. They showed that Martin's Axiom + ¬CH implies this proposition, and that $\mathbf{V}$=$\mathbf{L}$ implies its negation.

With his work on $\omega \longrightarrow (\omega)^\omega_2$ in hand, Mathias pursued the study of analogous partition properties for uncountable cardinals as they became topical. The situating result would be the Kechris-Woodin characterization of the Axiom of Determinacy, AD [30]. With Θ the supremum of ordinals ξ such that there is a

[7] Cf. [8, § 6].
[8] Cf. [87] or [81, VI§§ 3 & 4].

surjection: $\wp(\omega) \to \xi$ and the demarcating limit of the effect of AD, they showed that assuming $\mathbf{V}{=}\mathbf{L}(\mathbb{R})$, AD is equivalent to Θ being the limit of cardinals κ having the *strong partition property*, i.e., $\kappa \longrightarrow (\kappa)^{\kappa}_{\alpha}$ for every $\alpha < \kappa$.

With James Henle in [25], Mathias lifted results about "smooth" functions from *Happy families* [48, § 6] to the considerably more complex situation of uncountable κ satisfying the strong partition property. They achieved a remarkably strong continuity for functions $[\kappa]^{\kappa} \to [\kappa]^{\kappa}$ and applied it to get results about the Rudin-Keisler ordering in this context.

With Henle and Hugh Woodin in [26], Mathias elaborated on the *Happy families* [48, § 4] forcing for the happy family of infinite subsets of ω, newly dubbed the "Hausdorff extension". They showed that assuming $\omega \to (\omega)^{\omega}_{2}$, the Hausdorff extension not only adjoins no new real but no new sets of ordinals at all. With this, they could show, applying [25] and pointing out that AD + $\mathbf{V}{=}\mathbf{L}(\mathbb{R})$ implies $\omega \to (\omega)^{\omega}_{2}$, the following: if AD, then in the Hausdorff extension of $\mathbf{L}(\mathbb{R})$, Θ is still the limit of cardinals κ having the strong partition property *and* there is a Ramsey ultrafilter over ω. In particular, AD fails, and so $\mathbf{V}{=}\mathbf{L}(\mathbb{R})$ is necessary for the Kechris-Woodin characterization.

In a determinacy capstone of sorts, in [53], Mathias squarely took up a ZF (Zermelo-Fraenkel set theory) issue about ordinals and their subsets that arose in the investigation of AD and provided a penetrating exegesis. An ordinal is *unsound* if and only if it has subsets A_n for $n \in \omega$ such that uncountably many ordinals are realized as ordertypes of sets of form $\bigcup\{A_n \mid n \in a\}$ for some $a \subseteq \omega$. Woodin had asked whether there is an unsound ordinal, and eventually showed that AD implies that there is one less than ω_2. While the issue remains unsettled in ZF, in [53], Mathias showed that (a) if ω_1 is regular, then every ordinal less than $\omega_1^{\omega+2}$ is sound, and (b) $\aleph_1 \leq 2^{\aleph_0}$, i.e., there is a uncountable well-orderable set of reals, if and only if $\omega_1^{\omega+2}$ is exactly the least unsound ordinal. The proofs proceed through an intricate combinatorial analysis of indecomposable ordinals and a generalization of the well-known Milnor-Rado paradox. An interesting open question is the following: in Solovay's model in which every set of reals is Lebesgue measurable, starting specifically from $\mathbf{L}$ as the ground model, is every ordinal sound?

During this period, no doubt with the fellowship of Peterhouse an intellectual stimulus, Mathias became rhetorically engaged with various sociological and cultural aspects of mathematics. An opening shot was an address given at the Logic Colloquium '76, of which a whiff remains in the proceedings [49, p. 543] in which Mathias likened postures in mathematics to stances in religion.[9] In *Logic and terror*, read to the Perne Club[10] on February 12, 1978, Mathias deftly potted some history in the service of drawing out first the surround of the Law of the Excluded Middle $A \vee \neg A$ and of the Law of Contradiction $A \wedge \neg A$, and then in connection with the latter, the tension between (formal) logic and (Hegelian) dialectic in the Soviet Union, eventually resolved by Stalin's backdoor admittance of the former (published as [54] and [55]).

[9]With his recent conversion to Catholicism, Mathias considered that [49, p. 543] "parallels may be drawn between Platonism and Catholicism, which are both concerned with what is true; between intuitionism and Protestant presentations of Christianity, which are concerned with the behaviors of mathematicians and the morality of individuals; between formalism and atheism, which deny any need for postulating external entities; and between category theory and dialectical materialism."

[10]The Perne Club is a club of Peterhouse where papers are read to senior and junior members of the College on historical and philosophical matters.

The ignorance of Bourbaki, read to the Quintics Club[11] on October 29, 1986, would set in motion initiatives in Mathias's later work (published as [56, 57] and translated as [60] and [69]). He argued that Bourbaki, in their setting up of a foundations for mathematics, ignored Gödel's work on incompleteness until much later and that the set theory that they did fix on was inadequate. With more deftly potted history, Mathias suggested that the reason for these was Bourbaki's underlying accentuation of the geometrical over the arithmetical. Extending his reach, Mathias took on the distinguished mathematician Saunders Mac Lane in this view, newly stressing that while weak set theories are adequate for set formation, they are not for recursive definition—a theme to be subsequently much elaborated by Mathias.

Toward the end of what would be his time at Peterhouse, Mathias, exhibiting research breadth and prescience, made a first observation on a conjecture of Erdős, one which would become a focal problem more than two decades later. Erdős around 1932, while still an undergraduate, made one of his earliest conjectures, in number theory: For any sequence $x_1, x_2, x_3, \ldots$ with each x_n either -1 or $+1$ and any integer C, there exist positive m and d such that $C < |\sum_{k=1}^{m} x_{kd}|$. This is a remarkably simple assertion, and the problem of trying to affirm it came to be known as the Erdős Discrepancy Problem. Mathias around 1986 affirmed the conjecture for $C = 1$, and latterly published the proof in the proceedings of a 1993 conference celebrating Erdős' 80th birthday [59]. Much later in 2010, the Polymath Project took up solving the Erdős Discrepancy Problem as a project, Polymath5; Polymath Project was started by Timothy Gowers to carry out collaborative efforts online to solve problems. Polymath5 describes aspects of the collaborative effort on the Erdős conjecture [79], and under Annotated Bibliography, the Mathias paper [59] is annotated: "This one page paper established that the maximal length of [the] sequence for the case where $C = 1$ is 11, and is the starting point for our experimental studies." Extending the polymath project work, Boris Konev and Alexei Lisitsa showed that every sequence of length at least 1161 satisfies the conjecture in the case $C = 2$ [31]. Finally and suddenly, Terence Tao in September 2015 announced a proof of the Erdős conjecture [84].

4 Mac Lane Set Theory

At the beginning of his worldly wanderings, Mathias engaged in a notable controversy—in the classical sense of an exchange carried out in published articles—with Mac Lane. This stimulated Mathias to fully take up the investigation of the scope and limits of "Mac Lane set theory", something that he had started to do while still at Peterhouse.

The publication of Mac Lane's *Mathematics: Form and Function* [37] met with a noticeable lack of response or even acknowledgement. One exception was Mathias's [58], which took issue with Mac Lane's advocacy of a weak set theory and his contention that set theory cannot serve a substantial foundational role because of independence results. Mac Lane's set theory, ZBQC, is ZFC without Replacement and with Separation restricted to the Δ_0 (bounded quantifier) formulas. Mathias pointed out that this is inadequate for iterative constructions.

[11] The Quintics Club is an undergraduate mathematics club for the junior members of five colleges of the University of Cambridge of which Peterhouse is one.

The spatial (geometric) and temporal (arithmetical, iterative) are two modes that posit "an essential *bimodality* of mathematical thought", and set theory buttresses the latter as a study of well-foundedness. In reply, Mac Lane maintained that his ZBQC does better fit what most mathematicians do [38]; that for other work "there are other foundations"; that there is "no need for a single foundation". To Mathias's veering toward set theory as a foundation for mathematics "in ontological terms", Mac Lane opined: "Each mathematical notion is protean, thus deals with different realities, so does not have an ontology".

A decade later, Mathias in a last rhetorical article [62] divided "[o]pponents of a full-blooded set-theoretic account of the foundations of mathematics" among three categories: Those who "may hold, with Mac Lane, that . . . [ZBQC] suffices for 'all important mathematics' "; those who "may hold, with the early Bourbakistes, that ZC [Zermelo set theory, including Foundation and Choice] suffices"; and those who "may accept the axioms of ZFC, but deny the relevance of large cardinals to ordinary mathematics". Focusing on the last, Mathias formulated in palatable terms and described the essential involvement of large cardinal properties in several "strong statements of analysis": $\mathbf{\Sigma}^1_2$ sets have the perfect set property; $\mathbf{\Pi}^1_1$ sets are determined; $\mathbf{\Sigma}^1_2$ sets are universally Baire; and $\mathbf{\Sigma}^1_2$ sets are determined. Moreover, Mathias issued challenges to find various direct proofs, e.g., of $\mathbf{\Sigma}^1_2$ sets being universally Baire implying that $\mathbf{\Pi}^1_1$ sets are determined, that does not proceed through large cardinals—a potent kind of argument insisting on purity of proofs.[12] Mathias's [62] was followed by Mac Lane's [39] in the manner of a reply. He simply wrote contrarily that "Mathias has not produced any counter examples of actual [sic] mathematics which requires the use of a stronger [than Δ_0] separation", and argued for ZBQC being bolstered by an equiconsistency with a suitable categorical foundation recently established with the Mitchell-Benabou language.

Having taken on the question of the adequacy of weak set theories for mathematics in his exchange with Mac Lane, Mathias during a period of relative stability in the mid-1990s at the Centre de Recerca Matemàtica at Barcelona, would become deeply engaged with the analysis of the strength of various set theories. *Slim models of Zermelo set theory* [64] resonates with his initial insistence on adequate set theories supporting recursive definitions. Let Z be Zermelo set theory (without Choice) and KP, Kripke-Platek set theory.[13] To specify for here and later, as Mathias has it, Z has Extensionality, Empty Set, Pairing, Union, Power Set, Foundation, Infinity, and Separation; and KP has Extensionality, Empty Set, Pairing, Union, Π_1 Foundation, Δ_0 Separation, and Δ_0 Collection.

With Foundation for A being the assertion $A \neq \varnothing \longrightarrow \exists x \in A(x \cap A = \varnothing)$, Foundation for sets A—the usual Foundation—and Foundation for classes A are easily seen to be equivalent in the presence of Separation, as first observed by Gödel. The axiom system KP is usually formulated with Foundation for all classes A, but Mathias considers it befitting to have just Π_1 Foundation, which expectedly is Foundation for Π_1-defined classes. The schemes of Δ_0 Separation and Δ_0 Collection are analogously restricted to Δ_0 formulas and so forth for

[12]That every $\mathbf{\Sigma}^1_2$ set is universally Baire is equivalent to every set of ordinals having a sharp, and that every $\mathbf{\Pi}^1_1$ set is determined is equivalent to every real having a sharp. Hence, the first implies the second through large cardinals hypotheses.

[13]Cf. [17] for the historical particulars on Zermelo's axiomatization and [4] for Kripke-Platek set theory.

other classes of formulas in the Lévy hierarchy, where Collection generally is the following the following scheme, a variant of Replacement:

$$\forall x \in a \exists y \varphi(x, y) \longrightarrow \exists b \forall x \in a \exists y \in b \varphi(x, y).$$

Affirming that Z is weak as a vehicle for recursive definitions while KP is "orthogonal" in providing a natural setting for such, Mathias established in [64] that in $\mathsf{Z} + \mathsf{KP}$, one can recursively define a supertransitive (i.e., closed under subsets) inner (i.e., containing all ordinals) model that exhibits evident failures of Replacement, starting with the class $\mathbf{V}_\omega = \mathbf{HF}$ of hereditarily finite sets not being a set. Bringing together some classical ideas, he deployed a simple yet potent scheme for controlling entry into inner models according to growth rates of functions from ω to ω.

In the magisterial *The strength of Mac Lane set theory* [65], Mathias provided a rich and definitive analysis of set theories around Mac Lane's and Kripke-Platek, bringing in the full weight of basic themes and delineating a range of theories according to relative consistency, motivating concepts, and basic set-existence principles and techniques. With remarkable energy and both syntactic and semantic finesse, Mathias set out sharp results in minimal settings, elaborating and refining a half a century of work.

First, Mathias carries out von Neumann's classical construction, with minimal hypotheses, of the inner model of well-founded sets, to effect the relative consistency of the Axiom of Foundation as well as Transitive Containment, viz. that every set is a member of a transitive set. Transitive Containment is a consequence of KP, a first step toward Replacement. Mathias takes MAC to be Mac Lane's ZBQC together with Transitive Containment; takes M to be MAC without Choice; and next takes up what is to become a focal Axiom H:

$$\forall u \exists t (\bigcup t \subseteq t \wedge \forall z (\bigcup z \subseteq z \wedge |z| \leq |u| \longrightarrow z \subseteq t)).$$

That is, for any set u, there is a transitive set t of which every transitive set of size at most $|u|$ is a subset. Axiom H is a second step toward Replacement. Mathias approximates having Mostowski collapses, i.e., transitizations of well-founded, extensional relations, and makes isomorphic identifications to get the relative consistency of having Axiom H, e.g., Con(MAC) implies Con(MAC+H). Drawing out the centrality of Axiom H, Mathias shows that over a minimal set theory H is equivalent to actually having Mostowski collapses, and with a minimal Skolem hull argument, that over MAC, H subsumes KP, being equivalent, quite notably, to Σ_1 Separation together with Δ_0 Collection.

Mathias next makes the steep ascent to the first height, parsimoniously building the constructible hierarchy $\mathbf{L}$ and the relative consistency of AC without a direct recursive definition. Working in M, he simulates Gödel's recursive set-by-set generation along well-orderings, takes transitizations, and makes identifications as before for adjoining H, to define $\mathbf{L}$. Working out condensation, Mathias then gets to his first significantly new result, that

$$\text{Con}(\mathsf{M}) \text{ implies Con}(\mathsf{M} + \mathsf{KP} + \mathbf{V}{=}\mathbf{L}),$$

completing the circle to its first announcement in his [58]. With Z weak for recursive definitions, this approach notably provides the first explicit proof of "Con(Z) implies Con($\mathsf{Z} + \mathsf{AC}$)", once announced by Gödel. This speaks to a historical point: Between 1930 and 1935, Gödel tried to generate **L** recursively along well-orderings, themselves provided along the way in "autonomous progression". It was only after his embrace of Replacement in 1935, giving the von Neumann ordinals as canonical versions of all well-orderings, that he could rigorously get an inner model built on the spine of the ordinals.[14]

Setting out on a further climb, Mathias next layers, with the mediation of **L**, the region between $\mathsf{M}+\mathsf{KP}$ and $\mathsf{Z}+\mathsf{KP}$ in terms of Σ_n Separation. The thrust is that with **V**=**L**, Σ_n Separation implies the consistency of Σ_{n-1} Separation, and that Σ_n Separation implies the same in the sense of **L**. With H a steady pivot, Mathias proceeds in a minimal setting to construct Σ_n hulls based on **L**-least witnesses, and there is a notable refinement of analysis deploying "fine structure" lemmas, adapted from work of Sy Friedman, to control quantifier complexity.

Proceeding in a different direction that draws out a subtle interplay between power set and recursion, Mathias develops, partly with the purpose of getting sharp independence results, a theory subsuming M+KP first isolated for "power-admissible sets" by Harvey Friedman. The germ is to incorporate $\forall x \subseteq y$ and $\exists x \subseteq y$ as part of bounded quantification, setting up a new basis for a Lévy-type hierarchy, the Takahashi hierarchy of $\Sigma_n^{\wp}$ formulas. Working through a subtle syntactical analysis, Mathias develops normal forms and situates $\Delta_0^{\wp}$ Separation. The focus becomes $\mathsf{KP}^{\wp}$, which is $\mathsf{M}+\mathsf{KP}+\Pi_1^{\wp}$ Foundation $+\Delta_0^{\wp}$ Collection. Bringing in the Gandy Basis Theorem, standard parts of admissible sets, and forcing over ill-founded models, Mathias is able to establish the surprising result that, unlike for KP,

$$\mathsf{KP}^{\wp} + \mathbf{V}{=}\mathbf{L} \text{ proves the consistency of } \mathsf{KP}^{\wp},$$

and delimitative results, e.g., even $\mathsf{KP}^{\wp}+\mathsf{AC}+$ "every cardinal has a successor" does not prove H (nor therefore Σ_1 Separation).

Mathias attends, lastly, to systems type-theoretic in spirit, working the theme that MAC with its Power Set and Δ_0 Separation, is latently in this direction. Mathias shows that e.g., in MAC "strong stratifiable Σ_1 Collection" is provable, a narrow bridge to Quine's NF. Also, Mathias provides the first explicit proof of a result implicit in John Kemeny's 1949 thesis, that MAC is equiconsistent with the simple theory of types (together with the Axiom of Infinity).

Although definitively developed with a wide range of themes and concepts and a great deal of detail, one can discern in Mathias's [65] a base line speaking to his earlier engagements about and larger conception of set theory. As he retrospectively wrote [65, 10.7], "The purpose of my paper . . . is to study the relation of Mac Lane's system, which encapsulates in set-theoretic terms his mathematical world, to the Kripke-Platek system that gives a standard formalization of a certain kind of abstract recursion." As a reviewer noted, "Monumental in conception and rich in results, this paper merits the attention not just of set-theorists, but of all mathematicians concerned with the broader foundations of mathematics" [6].

Mathias's later [72] can be seen as extending the sharp deductive analysis of [65] to encompass Replacement and thus full ZF. Consider the scheme

$$\forall y \in u \exists ! z \varphi(y,z) \longrightarrow \exists w \forall y \in u \forall z (\varphi(y,z) \rightarrow z \in w) \qquad \text{(Repcoll)}$$

[14] Cf. [29, § 4].

with passive parameters allowed in φ and y possibly a vector of variables. Loosely speaking, any class function restricted to a set has range included in a set. Instances of this scheme have hypotheses stronger than those of Collection and conclusions weaker than those of Replacement. Replacement implies Collection over ZF, since if $\forall u \exists z \varphi$ and a is a set, one can for each $x \in a$ functionally specify the least rank of a witnessing z. A. K. Simpson had asked about the strength of $\mathsf{Repcoll}$, and Mathias answered his question in [72] by showing that, unexpectedly,

$$\text{the theory } \mathsf{M} + \mathsf{Repcoll} \text{ implies } \mathsf{ZF},$$

where, to recall, M is the relatively weak theory, MAC without Choice. Actually, Infinity does not play a role here, and can be effaced from both sides. Also, if Infinity is retained, then Transitive Containment can be effaced from the left side. The thrust of the proof is to show that in M, every set has a rank, and, using this together with Δ_0 Separation and $\mathsf{Repcoll}$, to work inductively up the Lévy hierarchy of formulas to full Separation and that exact ranges of class functions of sets are sets as well.

We further tuck in here work connected by a thread, of even later vintage. Mac Lane had envisioned his foundational set theory as sufficient for mathematics, inclusive of category theory, and the Mac Lane-Mathias controversy had turned on the possible necessity of strong, even large cardinal, hypotheses. In recent affirmations, the very strong large cardinal principle, Vopěnka's Principle, has been pressed to display categorical consequences, e.g., that all reflective classes in locally presentable categories are small-orthogonality classes. In collaborative work with Joan Bagaria, Carles Casacuberta and Jiří Rosický, Mathias (a) sharpened this result by reducing the hypothesis to having a proper class of supercompact cardinals yet still drawing a substantial conclusion; (b) got categorical equivalents to Vopěnka's Principle; and (c) showed as a consequence that "the existence of cohomological localizations of simplicial sets, a long-standing open problem in algebraic topology, is implied by the existence of arbitrarily large supercompact cardinals" [2, 3].

5 Dynamics

During his time at Barcelona, Mathias was also stimulated by colleagues and circumstances to pursue a set-theoretic approach to a basic iteration problem in dynamics, having once been alerted to such a possibility in the late 1970s. Then, chaotic dynamics, with the stuff of period orbits, strange attractors, and the like, was quite the rage, with the straightforward mathematical context of iterating functions providing simply posed problems. Mathias took up a basic issue cast in general terms; saw the applicability of descriptive set theory; and established substantial results that revealed a remarkable structure for recurrent points and "long delays" in dynamics. With [61] an initial article, Mathias, once established at La Réunion, put together the main account [63] as well as produced [70, 68, 67] containing refinements and further solutions. Taking an initial cue from dynamics, in [63], Mathias started by setting up a transfinite context:

Let χ be a Polish space (complete, separable metric space) and $f \colon \chi \to \chi$ a continuous function. We say that $\langle \chi, f \rangle$ is a *dynamical system*, a topological space with a continuous function acting on it. Define a relation $\curvearrowright_f$ on χ by

$x \curvearrowright_f y$ if and only if there is an increasing $\alpha\colon \omega \to \omega$ with $\lim_{n\to\infty} f^{\alpha(n)}(x) = y$, i.e., y is a cluster point of the f-iterates of x. This is basic to topological dynamics which focuses on recurrent points, point b such that $b \curvearrowright b$. Let

$$\omega_f(x) = \{y \mid x \curvearrowright_f y\} \text{ and}$$
$$\Gamma_f(X) = \bigcup\{\omega_f(x) \mid x \in X\},$$

both being $\curvearrowright_f$-closed as $\curvearrowright_f$ is easily seen to be a transitive relation. For $a \in \chi$, recursively define

$$A^0(a,f) = \omega_f(a)\,,$$
$$A^{\beta+1}(a,f) = \Gamma_f(A^\beta(a,f))\,, \text{and}$$
$$A^\lambda(a,f) = \textstyle\bigcap_{\nu<\lambda} A^\nu(a,f) \text{ for limit } \lambda.$$

Then $A^0(a,f) \supseteq A^1(a,f) \supseteq A^2(a,f) \supseteq \dots$ again by the transitivity of $\curvearrowright_f$. Let $\vartheta(a,f)$ by the least ordinal ϑ such that $A^{\vartheta+1}(a,f) = A^\vartheta(a,f)$, and let $A(a,f) = A^{\vartheta(a,f)}(a,f)$. The thrust of Mathias's work is to investigate the closure ordinal $\vartheta(a,f)$ as providing the dynamic sense of $\curvearrowright_f$.

Mathias first established that $\vartheta(a,f) \le \omega_1$, with the inequality being strict when $A(a,f)$ is Borel. With the result, and context, reflective of familiar paths for analytic sets, he associated to each $x \in \omega_f(a)$ a tree of $\curvearrowright_f$-descending finite sequences, so that $x \notin A(a,f)$ if and only if the tree is well-founded. Then he adapted to $\curvearrowright_f$ the Kunen proof of the Kunen-Martin Theorem on bounding ranks of well-founded trees. Notably, Mathias's argument works for any transitive relation in place of $\curvearrowright_f$, and so it can be seen as a nice incorporation of well-foundedness into the study of transitivity.

Particular to $\curvearrowright_f$ and dynamics, in [63], Mathias established a striking result about recurrent points, points b such that $b \curvearrowright_f b$. With an intricate metric construction of a recurrent point, he showed that $y \in A(a,f)$ if and only if for some z, $a \curvearrowright_f z \curvearrowright_f z \curvearrowright_f y$, so that in particular there are recurrent points in $\omega_f(a)$ exactly when $A(a,f) \neq \varnothing$.

More particular still with χ being Baire space, ${}^\omega\omega$, Mathias showed that if $s\colon {}^\omega\omega \to {}^\omega\omega$ is the (backward) shift function given by $s(g)(n) = g(n+1)$, then for each $\zeta < \omega_1$ there is an $a \in {}^\omega\omega$ such that $\vartheta(a,s) = \zeta$, a "long delay". For this, he carefully embedded countable well-founded trees into the graph of $\curvearrowright_f$. Approaching the issue of whether there can be χ, f and a such that $\vartheta(a,f)$ is actually ω_1, Mathias adapted his embedding apparatus to ill-founded trees and carried out a Cantor-Bendixson analysis on the hyperarithmetic hierarchy to provide an effective answer: there is a recursive $a \in {}^\omega\omega$ such that $\vartheta(a,s) = \omega_1^{\mathrm{CK}}$, the first non-recursive ordinal. For this Mathias was inspired by Kreisel's construction of a recursively coded closed set whose Cantor-Bendixson sequences of derivations stabilizes at ω_1^{CK}.

Mathias's formulations and results, being of evident significance for dynamics, soon attracted those working in the area. In particular, Lluís Alsedà, Moira Chas, and Jaroslav Smítal set off Mathias's results against a known backdrop and established a characterization for the closed unit interval of reals of positive topological entropy [1]. This work led to new observations by Alexander Sharkovskii, well-known for his pioneering work on periodic points for dynamical systems [1, p. 1721].

In [70], Mathias answered questions left open in [63]. Using Baire space ${}^{\omega}\omega$ and the shift function s, he showed that *there is a recursive real a such that $A^1(a, s)$ is not even Borel, and $A^2(a, s)$ is empty.* In [63], Mathias had approached, but did not resolve, whether there could be a some χ, f, and a with the longest delay $\vartheta(a, f) = \omega_1$ and indeed Alsedà, Chas and Smítal conjectured no [1, p. 1720]. In the best result in this subject, certainly the one with the most involved proof, Mathias established that with Baire space and the shift function s, there is a recursive real b giving the longest delay, $\vartheta(b, s) = \omega_1$. In subsequently written papers [68, 67], Mathias set out ways for extending the results of [70] to general χ and f.

Also stimulated by Mathias's work, his colleague Christian Delhommé at La Réunion has developed it in generalizing directions. For instance, he extended Mathias's [63] embedding of countable well-founded trees into the graph of $\curvearrowright_s$ to countable binary relations, appropriately retracted, in a broadly general setting [15].

While these may be the outward landmarks, there is a great deal of details and elaboration of concepts in [63], as further pursued in [70, 68, 67]. The subject that Mathias uncovered from a simple dynamics issue has remarkable depth and richness, as his penetrating results and constructions have shown. As such, it is a testament both to Mathias's mathematical prowess as well as to his insistence on well-foundedness as the bedrock of set theory.

6 Weaker Set Theories

In continuing travels but also with the stability afforded by La Réunion, Mathias, with his definitive work on Mac Lane set theory as providing a broad context, pursued themes in the axiomatics of set theory with renewed energy. He newly illuminated the interstices of deductible possibilities and refined systematic interconnections and minimal axiomatic sufficiency in connection with constructibility and forcing.

Mathias's *Weak systems of Gandy, Jensen, and Devlin* [71], written during the "set theory year" 2003/04 at Barcelona, provides a definitive analysis of set theories weaker than Kripke-Platek (KP) for lack of full Δ_0 Collection. Mathias's earlier [65] was initially stimulated by the question of the adequacy of Mac Lane's set theory for ongoing mathematics, and here he was initially stimulated by the question of the adequacy of a basic set theory in Keith Devlin's *Constructibility* [16] for the investigation of constructibility in terms of Jensen's rudimentary set functions and the $\mathbf{J}_\alpha$ hierarchy.

Mathias first variegates the landscape between ReS, which is KP without Δ_0 Collection, and KP with a range of systems and variants, focal ones being the following. DB "Devlin Basic" is ReS augmented with having Cartesian products. GJ "Gandy-Jensen" is DB augmented with Rudimentary Replacement: for Δ_0 formulas φ,

$$\forall x \exists w \forall v \in x \exists t \in w \forall u (u \in t \longleftrightarrow u \in x \land \varphi(u, v)) .$$

Further augmentations with restricted versions of Δ_0 Collection are formulated, getting closer to full KP. Devlin's original system is DB augmented with Infinity and Foundation for all Classes. GJ axiomatizes the closure under the rudimentary functions for Jensen's fine structure investigations; a transitive set is closed under the rudimentary functions if and only if it models GJ minus Π_1 Foundation.

Mathias next sets out the crucial set formations that can be effected in the various systems, and then establishes independences through over a dozen models. In particular, he applied his "slim model" technique from [64] to get a supertransitive inner model of DB but not GJ, the model showing that DB cannot prove the existence of $[\omega]^3$.

Mathias's distinctive contribution is to incorporate into the various systems an axiom S asserting for all x the existence of $\mathrm{S}(x) = \{y \mid y \subseteq x \text{ is finite}\}$. Proceeding analogously to [65, § 6], Mathias carries out a subtle syntactical analysis of formulas having $\forall y \in \mathrm{S}(x)$ and $\exists y \in \mathrm{S}(x)$ as part of bounded quantification and sets up a corresponding hierarchy of Σ_n^{S} formulas. With that, he augments the various systems with Infinity and S and establishes corresponding Separation, Collection, etc. for formulas in the new hierarchy as well as semantic independences. With this preparation Mathias confirms in exacting detail the flaws in Devlin's book, *Constructibility*, especially the inadequacy of his basic set theory for formulating the satisfaction predicate for Δ_0 formulas. GJ+ Infinity does work, as it axiomatizes the rudimentary functions. Answering a call for doing without the theory of rudimentary functions, Mathias shows that what also works for a parsimonious development of constructibility is Devlin's system (DB + Infinity + Foundation for all Classes) as augmented by S, as well as the particularly enticing subsystem MW "Middle Way": DB + Infinity $+ \forall a \forall k \in \omega([a]^k \in \mathbf{V})$.

Already in the mid-1990s, deeply engaged in axiomatics, Mathias became interested in the problem of finding the weakest system that would support a smooth, recognizable theory of forcing. Through a period of germination in the new millennium proceeding through his contextualizing work on weak systems for constructibility, Mathias developed the concepts of rudimentary recursion and provident set, finally to meet with a remarkable success, in his maturity, which secures the axiomatic and methodological essence of a fundamental technique in set theory. With this work proceeding in forward and circling strides, [73] provides an overview, and then [12, 76] systematically set out the details in full.

The *Rudimentary recursion, gentle functions and provident sets* [12], with Nathan Bowler, worked towards an optimal theory for forcing, drawing on previous axiomatics and getting at the recursions just sufficient for formulating forcing. In KP, one has that if G is a total Σ_1^1 function, then so is F given by $F(x) = G(F{\restriction}x)$, and such recursions handily suffice for forcing. The first move toward purity of method is to work only with *rudimentarily recursive* functions, i.e., those F as given above but defined from rudimentary G.

As described in [12], Mathias systematically developed a theory of rudimentary recursion in weak set theories and toward their use in forcing. A significant complication was that the composition of two rudimentarily recursive functions is not necessarily rudimentarily recursive, and to finesse this, Bowler developed the *gentle functions*, functions $H \circ F$ where H is rudimentary and F is rudimentarily recursive. The composition of gentle functions *is* gentle, and this considerably simplified the formulation of forcing. That formulation also to require having the forcing partial order as a parameter, the general theory was extended to the p-rudimentarily recursive functions, functions F given by

$F(x) = G(p, F{\restriction}x)$ with p as a parameter in the rudimentary recursion.

Forcing is to be done over the provident sets. A set is *provident* if and only if it is non-empty, transitive, closed under pairing and for all $x, p \in A$ and p-rudimentarily recursive F, $F(x) \in A$. Mathias variously characterized the provident sets, and ramified them as cumulative unions of transitive sets in a canonical fashion. The Jensen rudimentary functions are nine, and can be but together into one set formation function T such that for transitive u, $\bigcup_{n\in\omega} T^n(u)$ is rudimentarily closed. With this, given a transitive set c, define c_ν and P^c_ν by simultaneous recursion:

$$\begin{array}{lll} c_0 = \varnothing\,, & c_{\nu+1} = c \cap \{x \mid x \subseteq c_v\}\,, & c_\lambda = \bigcup_{\nu<\lambda} c_\nu \\ P^c_0 = \varnothing\,, & P^c_{\nu+1} = T(P^c_\nu) \cup c_{\nu+1} \cup \{c_\nu\}\,, & P^c_\lambda = \bigcup_{\nu<\lambda} P^c_\nu\,. \end{array}$$

This master recursion gradually builds up rudimentary closed levels relative to c. Mathias showed that for transitive c and indecomposable ordinal ϑ, P^c_ϑ is provident. Moreover, one can define the *provident closure* of any non-empty M by: $\mathrm{Prov}(M) = \bigcup\{P^c_\vartheta \mid c$ is the transitive closure of some finite subset of $M\}$, where ϑ is the least indecomposable ordinal not less than the set-theoretic rank of M. In particular, if M is already provident, $\mathrm{Prov}(M) = M$ exhibits a canonical ramification. With this, one gets a finite set of axioms Prov warranting the recursion so that the transitive models of Prov are exactly the provident sets. The set $\mathbf{J}_\nu$ is provident if and only if $\omega\nu$ is indecomposable, so that $\mathbf{J}_\omega = \mathbf{V}_\omega = \mathbf{HF}$ is provident, the only provident set not satisfying Infinity, and so are $\mathbf{J}_{\omega^2}, \mathbf{J}_{\omega^3} \ldots$.

With the above theory, *Provident sets and rudimentary set forcing* [76] duly carries out a parsimonious development of forcing. The Shoenfield-Kunen approach[15] is taken, carefully tailored to be effected in Prov + Infinity. Although the overall scheme to be followed is thus straightforward, the progress step by step reveals many eddies of deduction and astute choices of terms, drawing out the methodological necessity and sufficiency of provident sets. Mathias eventually shows, in a rich surround of textured results established in his context, that if M is provident, $P \in M$ is a forcing partial order, and G is P-generic over M, then $M[G]$ is provident, with $M[G] = \mathrm{Prov}(M \cup \{G\})$. Moreover, he shows of the various axioms of set theory like Power Set that they persist from M to $M[G]$. Mathias's [76] is a veritable paean to formalism and forcing, one that exhibits an intricate melding of axiomatics and technique in set theory. As such, it together with his [12] is Mathias's arguably most impressive accomplishment in axiomatics.

7 Bourbaki

In the fullness of time and with his remarkable work on axiomatics in hand, Mathias latterly took up the lance once again against the windmill of the "ignorance of Bourbaki" [57], circling back to his first tussle with ill-suited set theories. This time there would be telling mathematical pathologies exposed as well as a remarkable arching argument inset in French history and praxis.

Bourbaki, in their first book, the 1954-1957 *Théorie des Ensembles*, developed a theory of sets in a logical formalism as the axiomatic basis of their

[15] Cf. [33, VII].

structural exposition of mathematics. They adapted for purposes of quantification the Hilbert ε-operator, which for each formula φ introduces a term $\varepsilon x \varphi$ replicating the entire formula. It will be remembered that Hilbert in the 1920's had introduced these terms, presumably motivated by the use of ideal points in mathematics; he had $\varphi(t) \to \varphi(\varepsilon x \varphi)$ for terms t, and defined the quantifiers by $\exists x \varphi \longleftrightarrow \varphi(\varepsilon x \varphi)$ and $\forall x \varphi \longleftrightarrow \varphi(\varepsilon x \neg \varphi)$. In [66], Mathias underlined the cumulating complications in Bourbaki's rendition by showing that their definition of the cardinal number 1, itself awkward, when written out in their formalism would have length 4,523,659,424,929 !

Earlier in 1948, André Weil, on behalf of Bourbaki, gave an invited address to the Association of Symbolic Logic on set theory as a "foundations of mathematics for the working mathematician" (cf. [11]). Bourbaki's system has Extensionality, Separation, Power Set, ordered pair as primitive with axioms to match, and Cartesian products. From Separation and Power set, one gets singletons $\{x\}$. In [74], Mathias showed, to blunt purpose yet with considerable dexterity, that there is a model of Bourbaki's system in which the Pairing Axiom, having $\{x, y\}$, fails!

Hilbert, Bourbaki and the scorning of logic [75] is Mathias's mature criticism of Bourbaki, one that works out rhetorically a line of argument through history, mathematics, and education. There is a grand sweep, but also a specificity of mathematical detail and a sensitivity to systemic influence, no doubt heightened by his teaching years at La Réunion.

Mathias's line of argument is as follows:

(a) Hilbert in 1922 proposed an alternative treatment of first-order logic using his ε-operator,

(b) which, despite its many unsatisfactory aspects, was adopted by Bourbaki for their exposition of mathematics,

(c) and by Godement for his classic *Cours d'Algèbre*, though leading him to express distrust of logic.

(d) It is this distrust, intensified to a phobia by the vehemence of Dieudonné's writings,

(e) and fostered by, e.g., the errors and obscurities of a well-known undergraduate text, Jacqueline Lelong-Ferrand and Jean-Marie Arnaudiès' *Cours de mathèmatiques*,

(f) that has, it is suggested, led to the exclusion of logic from the CAPES examination—"tout exposé de logique formelle est exclu".

(g) Centralist rigidity has preserved the underlying confusion and consequently flawed teaching;

(h) the recovery will start when mathematicians adopt a post-Gödelian treatment of logic.

For (a) and (b), Mathias recounts in detail Hilbert's engagement with logic and his program for establishing the consistency of mathematics and how Bourbaki adopted wholesale the awkward Hilbert ε-operator approach to logic. According to Mathias, with the appearance of Gödel's work on incompleteness "the hope of a single proof of the consistency and completeness of mathematics, in my view the only justification for basing an encyclopaedic account of mathematics on Hilbert's operator, had been dashed". For (c), (d), and (e) to be regarded as case studies, Mathias quotes at considerable length from the sources, revealing most particularly in the last case tissues of confusion about truth vs. provability and metalanguage vs. uninterpreted formalism. For (f) and (g), Mathias is unrestrained about the over-centralized French educational system and how in its rigidity it has perpetuated the Bourbaki hostility to logic and set theory.

In the articulation of (h), Mathias synthetically puts forth layer by layer a larger vision of mathematics and its regeneration, one that straddles his earlier work and writing. He first observes, through inner minutes of Bourbaki, that there was uncertainty and even dissension in the tribe about the adequacy of their adopted set theory. Mathias then recapitulates, with specific examples from his own writings, mathematics that the Bourbakistes would not be able to encompass. Calling Bourbaki as well as Mac Lane structuralists, Mathias regards structuralism and set theory as on opposite sides of the divide between taking equality up to isomorphism as good enough and not, and recalls his exchange with Mac Lane. Mathias then emphasizes how the divide brings out a dual nature of mathematics, and how mathematics is impoverished by the disregard of one side or the other. Mathias personally defines set theory as the study of well-foundedness; as such, it is highly successful in meeting the call for recursive constructions—and it is to be pointed out that his own work is a particular and abiding testament to this. In a concluding peroration, Mathias points to the stultifying effects of centralization and bureaucratization across centuries and cultures and calls for intellectual independence within community, as at Oxford and Cambridge, with all contributing through competition to the regeneration of mathematics and more broadly, culture.

Adrian Mathias is an estimable mathematician with a remarkable range of results addressing problems that emerged in the course of a wide-ranging engagement with mathematics. From his work with Mathias reals to the dynamics of iterated maps, he has uncovered a great deal of structure and brought forth new understanding. But also, Mathias can be seen, quite distinctively, as a fine analyst of social and cultural aspects of mathematics and the axiomatic basis of concepts and methods of set theory. His work on the last attains an exceptional virtuosity, and with that, one can count Mathias—-whose work is going from strength to strength—as one of the very few who through logical axiomatic analysis has contributed to meaning in mathematics.

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