HOMOGENEOUS FAMILIES ON TREES AND SUBSYMMETRIC BASIC SEQUENCES

C. BRECH, J. LOPEZ-ABAD, AND S. TODORCEVIC

Abstract. We study density requirements on a given Banach space that guarantee the existence of subsymmetric basic sequences by extending Tsirelson's well-known space to larger index sets. We prove that for every cardinal κ smaller than the first Mahlo cardinal there is a reflexive Banach space of density κ without subsymmetric basic sequences. As for Tsirelson's space, our construction is based on the existence of a rich collection of homogeneous families on large index sets for which one can estimate the complexity on any given infinite set. This is used to describe detailedly the asymptotic structure of the spaces. The collections of families are of independent interest and their existence is proved inductively. The fundamental stepping up argument is the analysis of such collections of families on trees.

1. Introduction

Recall that a set of *indiscernibles* for a given structure $\mathcal{M}$ is a subset X with a total ordering $<$ such that for every positive integer n every two increasing n-tuples $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$ of elements of X have the same properties in $\mathcal{M}$. A simple way of finding an extended structure on κ without an infinite set of indiscernibles is as follows. Suppose that $\mathcal{F}$ is a family of finite subsets of κ that is compact, as a natural subset of the product space 2^κ, and *large*, that is, every infinite subset of κ has arbitrarily large subsets in $\mathcal{F}$. Let $\mathcal{M}_\mathcal{F}$ be the structure $(\kappa, (\mathcal{F}\cap [\kappa]^n)_n)$ that has κ as universe and that has infinitely many n-ary relations $\mathcal{F}\cap[\kappa]^n \subseteq [\kappa]^n$. It is easily seen that $\mathcal{M}_\mathcal{F}$ does not have infinite indiscernible sets.

While in set theory and model theory indiscernibility is a well-studied and unambiguous notion, in the context of the Banach space theory it has several versions, the most natural one being the notion of a *subsymmetric sequence* or a *subsymmetric set*. In a normed space $(X, \|\cdot\|)$,

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a sequence $(x_\alpha)_{\alpha\in I}$ indexed in an ordered set $(I,<)$ is called *C-subsymmetric* when

$$\|\sum_{j=1}^{n} a_j x_{\alpha_j}\| \leq C\|\sum_{j=1}^{n} a_j x_{\beta_j}\|$$

for every sequence of scalars $(a_j)_{j=1}^n$ and every $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ in I. When $C = 1$ this corresponds exactly to the notion of an indiscernible set and it is easily seen that this can always be assumed by renorming X with an appropriate equivalent norm. Another closely related notion is *unconditionality*. Recall that a sequence $(x_i)_{i\in I}$ in some Banach space is *C-unconditional* whenever

$$\|\sum_{i\in I} \theta_i a_i x_i\| \leq C\|\sum_{i\in I} a_i x_i\|$$

for every sequence of scalars $(a_i)_{i\in I}$ and every sequence $(\theta_i)_{i\in I}$ of signs. Of particular interest are the indiscernible coordinate systems, such as the Schauder basic sequences. The unit bases of the classical sequence spaces ℓ_p, $p \geq 1$ or c_0 (in any density) are subsymmetric and unconditional (in fact, symmetric, i.e. indiscernible by permutations) bases. Moreover, every basic sequence in one of these spaces has a symmetric subsequence. But this is not true in general: there are basic sequences without unconditional subsequences, the simplest example being the summing basis of c_0. However, it is more difficult to find a weakly-null basis without unconditional subsequences (B. Maurey and H. P. Rosenthal [MaRo]). Now we know that there are Banach spaces without unconditional basic sequences. The first such example was given by W. T. Gowers and B. Maurey [GoMa], a space which was moreover reflexive. Concerning subsymmetric sequences, we mention that the unit basis of the Schreier space [Schr] does not have subsymmetric subsequences, and the Tsirelson space [Tsi] is the first example of a reflexive space without subsymmetric basic sequences.

All these are separable spaces so it is natural to ask if large spaces must contain infinite unconditional or subsymmetric sequences, since from the theory of large cardinals we know that infinite indiscernible sets exist in large structures. In general, this is a consequence of certain Ramsey principles (i.e. higher-dimensional versions of the pigeonhole principles). Indeed, it was proved by Ketonen [Ke] that Banach spaces with density bigger than the first ω-Erdős cardinal have subsymmetric sequences. Recall that a cardinal number κ is called ω-Erdős when every countable coloring of the collection of finite subsets of κ has an infinite subset A of κ where the color of a given finite subset F of A depends only on the cardinality of F. Such cardinals are large cardinals, and their existence cannot be proved on the basis of the standard axioms of set theory. It is therefore natural to ask what is the minimal cardinal number $\mathfrak{nc}$ ($\mathfrak{ns}$) such that every Banach space of density at least $\mathfrak{nc}$ (resp. $\mathfrak{ns}$) has an unconditional (respectively, subsymmetric) basic sequence. It is natural to consider also the relative versions of these cardinals restricted to various classes of Banach spaces like, for example, the class of reflexive spaces where we use the notations $\mathfrak{nc}_{\mathrm{refl}}$ and $\mathfrak{ns}_{\mathrm{refl}}$, respectively. Note that it follows from E. Odell's partial unconditionality result [Od2] that every weakly-null subsymmetric basic sequence is unconditional, hence $\mathfrak{nc}_{\mathrm{refl}} \leq \mathfrak{ns}_{\mathrm{refl}}$. Moreover, an easy application of Odell's result and Rosenthal's ℓ_1-dichotomy gives that $\mathfrak{nc} \leq \mathfrak{ns}$.

Concerning lower bounds for these cardinal numbers, it was proved by S. A. Argyros and A. Tolias [ArTo] that $\mathfrak{nc} > 2^{\aleph_0}$, and by E. Odell [Od1] that $\mathfrak{ns} > 2^{\aleph_0}$. For the reflexive case, we know that $\mathfrak{nc}_{\mathrm{refl}} > \aleph_1$ ([ArLoTo]), and in the recent paper [ArMo], S. A. Argyros and P. Motakis proved that $\mathfrak{ns}_{\mathrm{refl}} > 2^{\aleph_0}$. Finally we mention that in [LoTo2] the Erdős cardinals are characterized in terms of the existence of compact and large families and the sequential version of $\mathfrak{ns}_{\mathrm{refl}}$. More precisely it is proved that the first Erdős cardinal $\mathfrak{e}_\omega$ is the minimal cardinal κ such that every long weakly-null basis of length κ has a subsymmetric basic sequence, or equivalently the minimal cardinal κ such that there is no compact and large family of finite subsets of κ.

The study of upper bounds is of different nature and seems to involve more advanced set-theoretic considerations connected to large cardinal principles. This can be seen, for example, from the aforementioned result of Ketonen or from results of P. Dodos, J. Lopez-Abad and S. Todorcevic who proved in [DoLoTo] that $\mathfrak{nc} \le \aleph_\omega$ holds consistently relative to the existence of certain large cardinals and who proved in [LoTo2] that Banach's Lebesgue measure extension axiom implies that $\mathfrak{nc}_{\mathrm{refl}} \le 2^{\aleph_0}$.

In this paper we continue the research on the existence of subsymmetric sequences in a normed space of large density, and we prove that $\mathfrak{ns}_{\mathrm{refl}}$ is rather large, distinguishing thus the cardinals $\mathfrak{ns}_{\mathrm{refl}}$ and $\mathfrak{nc}_{\mathrm{refl}}$. In contrast to the sequential version of $\mathfrak{ns}_{\mathrm{refl}}$, that is closely linked to indiscernibles of *relational* structures $\mathcal{M}_{\mathcal{F}}$ for compact and large families $\mathcal{F}$, the full version of $\mathfrak{ns}_{\mathrm{refl}}$ is more related to the existence of indiscernibles in structures that are not just relational but also have operations, suggesting that not only we need to understand families on finite sets but also "operations" with them. In the separable context, this is well-known and can be observed in the construction of the Tsirelson space, where finite products of the *Schreier family* are used in a crucial way. The natural approach in the non-separable setting would be to generalize Tsirelson's construction using analogues of the Schreier family, certain large compact families, on larger index sets. However, in the uncountable level these families cannot be spreading and therefore, if one just copies Tsirelson's construction on the basis of them, the corresponding non-separable Tsirelson-like spaces will always contain almost isometric copies of ℓ_1 ([LoTo2, Theorem 8.2]). This lead us to change our perspective and use the well-known interpolation technique [LiTz, Example 3.b.10], an approach that appeared recently in the work of Argyros and Motakis mentioned above. In this perspective, a key tool is a suitable operation $\times$, that we call *multiplication*, of compact families of finite sets. In fact, the multiplication is an operation which associates to a family $\mathcal{F}$ on the fixed index set I and family $\mathcal{H}$ on ω, a family $\mathcal{F} \times \mathcal{H}$ on I which has, in a precise sense, many elements of the form $\bigcup_{n\in x} s_n$, where $x \in \mathcal{H}$ and $(s_n)_{n<\omega}$ is an arbitrary sequence of elements of $\mathcal{F}$. It is well known that such multiplication exists in ω and it models in some way the ordinal multiplication on uniform families. It is also the main tool to define the generalized Schreier families on ω, vastly used in modern Banach space theory to study ranks of compact notions (e.g. summability methods), or of asymptotic notions (e.g. spreading models). These are *uniform* families, so that any restriction of them looks like the entire family. We generalize this property to the uncountable level by defining *homogeneous* families, that despite being uncountable families on large index sets, have countable Cantor-Bendixson rank which moreover does not change substantially when passing to restrictions. In particular, if $\mathcal{F}$ is homogeneous, then the structure $\mathcal{M}_{\mathcal{F}}$ does not have infinite sets of indiscernibles, but we also get lower and upper bounds for the rank of the collection of their (finite) sets of indiscernibles.

We then introduce the notion of a *basis* of families, which is a rich collection of homogeneous families admitting a multiplication, and we prove that they exist on quite large cardinal numbers. The existence of such bases is proved inductively. For example, we prove that if κ has a basis then 2^κ has also a basis. This is done by representing 2^κ as the complete binary tree $2^{\leq\kappa}$, and observing that we can use the *height* function $\mathrm{ht} : 2^{\leq\kappa} \to \kappa+1$ to pull back a basis on κ to a restricted version of basis on $2^{\leq k}$, consisting of homogeneous families of finite chains of $2^{\leq\kappa}$. Actually, we prove the following more general equivalence (Theorem 3.1).

Theorem. *For an infinite rooted tree T the following are equivalent.*
(a) *There is a basis of families on T.*
(b) *There is a basis of families consisting of chains of T and there is a basis consisting of antichains of T.*

In particular, we obtain a basis on 2^ω that can be used to build a reflexive space of density 2^ω without subsymmetric basic sequences, giving another proof of the result in [ArMo]. Also, one proves inductively that for every cardinal number κ smaller than the first inaccessible cardinal, there is a basis on κ and a corresponding Banach space of density κ with similar properties. We then use Todorcevic's method of walks on ordinals [To] to build trees on cardinals up to the first Mahlo cardinal number and find examples of reflexive Banach spaces of large densities without subsymmetric basic subsequences. Moreover, as observed above for the structure $\mathcal{M}_\mathcal{F}$, we can bound the complexity of the (finite) subsymmetric basic sequences and obtain the following.

Theorem. *Every cardinal κ below the first Mahlo cardinal has a basis. Consequently, for every such cardinal κ and every $\alpha < \omega_1$, there is a reflexive Banach space $\mathfrak{X}$ of density κ with a long unconditional basis and such that every bounded sequence in $\mathfrak{X}$ has an ℓ_1^α-spreading model subsequence but the space $\mathfrak{X}$ does not have ℓ_1^β-spreading model subsequences for β large enough, only depending on α. In particular, $\mathfrak{X}$ contains no infinite subsymmetric basic sequence.*

The paper is organized as follows. In Section 2 we introduce some basic topological, combinatorial and algebraic facts on families of finite chains of a given partial ordering. We then define homogeneous families and bases of them. We finish this part by proving some upper bounds for the topological rank of a family that uses the well-known Ramsey property of barriers on ω. Section 3 is the main part of this paper. The main motive of study is, given a tree, the collection $\mathcal{A} \odot_T \mathcal{C}$ of all finite subtrees of T whose chains are in a fixed family $\mathcal{C}$ and such that the family of immediate successors of a given node is in another fixed family $\mathcal{A}$. We study $\mathcal{A} \odot_T \mathcal{C}$ both combinatorially and topologically. The combinatorial part is based on the canonical form of a sequence of finite subtrees, and allow us to define a natural multiplication. The topological one consists in finding upper bounds of the rank of the family $\mathcal{A} \odot_T \mathcal{C}$ in terms of the corresponding ranks of the families $\mathcal{A}$ and $\mathcal{C}$, much in the spirit of how one easily bounds the size of a finite tree from its height and splitting number. This operation allows to lift bases on chains and of immediate successors to bases on the whole tree, our main result of this work done in Theorem 3.1. We apply this in Section 4 to prove that cardinal numbers smaller than the first Mahlo cardinal have a basis. To do this, we represent such cardinals as nodes of a tree having bases on chains and on immediate successors. We achieve this last part by proving several principles transference of basis. Finally, we use bases to build reflexive Banach spaces without subsymmetric basic sequences.

2. Basic definitions

Let I be a set. A set $\mathcal{F}$ is called a *family on I* when the elements of $\mathcal{F}$ are finite subsets of I. Let $\mathcal{P}=(P,<)$ be a partial ordering. A *family on chains of $\mathcal{P}$* is a family on P consisting of chains of $\mathcal{P}$. Let $\mathrm{Ch}_<$ be the collection of all chains of $\mathcal{P}$. Given $k\le\omega$, let

$$[I]^k := \{s\subseteq I \,:\, \#s=k\}, \qquad [I]^k_< := [I]^k\cap \mathrm{Ch}_<,$$
$$[I]^{\le k} := \{s\subseteq I \,:\, \#s\le k\}, \qquad [I]^{\le k}_< := [I]^{\le k}\cap \mathrm{Ch}_<.$$

For a family $\mathcal{F}$ on I and $A\subseteq I$, let $\mathcal{F}\upharpoonright A := \mathcal{F}\cap\mathcal{P}(A)$. Recall that a family $\mathcal{F}$ on I is *hereditary* when it is closed under subsets and it is *compact* when it is a closed subset of $2^I:=\{0,1\}^I$, after identifying each set of $\mathcal{F}$ with its characteristic function. In this case, $\mathcal{F}$ is a scattered compact space. Since each element of $\mathcal{F}$ is finite, it is not difficult to see that $\mathcal{F}$ is compact if and only if every sequence $(s_n)_{n\in\omega}$ in $\mathcal{F}$ has a subsequence $(t_n)_{n\in\omega}$ forming a Δ-system with root in $\mathcal{F}$, that is, such that

$$t_{k_0}\cap t_{k_1}=t_{l_0}\cap t_{l_1}\in\mathcal{F} \text{ for every } k_0\neq k_1 \text{ and } l_0\neq l_1.$$

The intersection $t_k\cap t_l$, $k\neq l$ is called the *root* of $(t_n)_n$. By weakening the notion of compactness, we say that $\mathcal{F}$ is *pre-compact* if every sequence in $\mathcal{F}$ has a Δ-subsequence (with root not necessarily in $\mathcal{F}$). It is easy to see then that $\mathcal{F}$ is pre-compact if and only if its $\subseteq$-closure $\{s\subseteq I \,:\, s\subseteq t \text{ for some } t\in\mathcal{F}\}$ is compact.

Recall the *Cantor-Bendixson derivatives* of a topological space X:

$$X^{(0)}:=X,\quad X^{(\alpha)}=\bigcap_{\beta<\alpha}(X^{(\beta)})'$$

where Y' denotes the collection of accumulation points of Y, that is, those points $p\in X$ such that each of its open neighborhoods has infinitely many points in Y. The minimal ordinal α such that $X^{(\alpha+1)}=X^{(\alpha)}$ is called the *Cantor-Bendixson rank* $\mathrm{rk}_{\mathrm{CB}}(X)$ of X. In the case of a compact family $\mathcal{F}$ on an index set I, being scattered, its Cantor-Bendixson index is the first α such that $\mathcal{F}^{(\alpha)}=\emptyset$, and therefore it must be a successor ordinal.

Definition 2.1. *Given a compact family $\mathcal{F}$ on some index set I, let*

$$\mathrm{rk}(\mathcal{F}):=\mathrm{rk}_{\mathrm{CB}}(\mathcal{F})^-$$

where $(\alpha+1)^-=\alpha$. We say that a compact family $\mathcal{F}$ is countably ranked when $\mathrm{rk}(\mathcal{F})$ *is countable. Let $\mathcal{P}$ be a partial ordering. A family $\mathcal{F}$ on chains of $\mathcal{P}$ the* small rank relative to $\mathcal{P}$ *of $\mathcal{F}$ is*

$$\mathrm{srk}_{\mathcal{P}}(\mathcal{F}):=\inf\{\mathrm{rk}(\mathcal{F}\upharpoonright C) \,:\, C \textit{ is an infinite chain of } \mathcal{P}\}.$$

A compact and hereditary family $\mathcal{F}$ on chains of $\mathcal{P}$ is called $(\alpha,\mathcal{P})$-homogeneous *if* $[P]^{\le 1}\subseteq\mathcal{F}$ *and*

$$\alpha=\mathrm{srk}_{\mathcal{P}}(\mathcal{F})\le\mathrm{rk}(\mathcal{F})<\iota(\alpha),$$

where $\iota(\alpha)$ is defined below in Definition 2.4. $\mathcal{F}$ is $\mathcal{P}$-homogeneous *if it is $(\alpha,\mathcal{P})$-homogeneous for some $\alpha<\omega_1$.*

When I is countable, its rank is countable as well. Hence the small rank of a compact family is always countable. In general, $\mathrm{rk}(\mathcal{F})\le \#I$ and the extreme case can be achieved. For total orderings $\mathcal{P}$ we are going to use srk, α-homogeneous and homogeneous instead of $\mathrm{srk}_{\mathcal{P}}$, $(\alpha,\mathcal{P})$-homogeneous and $\mathcal{P}$-homogeneous, respectively.

Definition 2.2. *The* normal Cantor form *of an ordinal α is the unique expression $\alpha=\omega^{\alpha[0]}\cdot n_0[\alpha]+\cdots+\omega^{\alpha[k]}\cdot n_k[\alpha]$ where $\alpha\ge\alpha[0]>\alpha[1]>\cdots>\alpha[k]\ge 0$ and $n_i[\alpha]<\omega$ for every $i\le k$.*

Suppose that $*$ is an operation on countable ordinals and suppose that $\alpha>0$ is a countable ordinal. We say that α is $*$-indecomposable *when $\beta*\gamma<\alpha$ for every $\beta,\gamma<\alpha$.*

Remark 2.3. It it well-known that

(i) α is sum-indecomposable if and only $\alpha=\omega^\beta$.
(ii) $\alpha>1$ is product-indecomposable if and only if $\alpha=\omega^\beta$ for some sum-indecomposable β.
(iii) For $\alpha>\omega$, α is exponential-indecomposable if and only if $\alpha=\omega^\alpha$.
(iv) product-indecomposability imply sum-indecomposability, and exponential-indecomposability imply product and sum indecomposability.

So, $1,\omega,\omega^2$ and $1,\omega,\omega^\omega$ are the first 3 sum-indecomposable, and product-indecomposable ordinals, respectively. If we define, given $\alpha<\omega_1$, $\overline{\alpha}_0:=\alpha$, $\overline{\alpha}_{n+1}:=(\overline{\alpha}_n)^\alpha$ and $\overline{\alpha}_\omega:=\sup_n\overline{\alpha}_n$, then $\omega,\overline{\omega}_\omega,\overline{(\overline{\omega}_\omega)}_\omega$ are the first 3 exponential-indecomposable ordinals. We will use *exp-indecomposable* to refer to exponential-indecomposable ordinals.

Definition 2.4. *Given a countable ordinal α, let*

$$\iota(\alpha)=\min\{\lambda>\alpha\,:\,\lambda\text{ is exp-indecomposable}\}.$$

Let $\mathrm{Fn}(\omega_1,\omega)$ be the collection of all functions $f:\omega_1\to\omega$ such that $\mathrm{supp}\,f:=\{\gamma<\omega_1\,:\,f(\gamma)\ne 0\}$ is finite. When considered the pointwise sum $+$ $(\mathrm{Fn}(\omega_1,\omega),+)$ is an ordered commutative monoid. Let $\nu:\omega_1\to\mathrm{Fn}(\omega_1,\omega)$ be defined by $\nu(\alpha)(\gamma)=n_i[\alpha]$ if and only if $\gamma=\alpha[i]$. Let $\sigma:\mathrm{Fn}(\omega_1,\omega)\to\omega_1$ be defined by $\sigma(f)=\sum_{i\le k}\omega^{\alpha_i}\cdot f(\alpha_i)$, where $\{\alpha_0>\cdots>\alpha_n\ge 0\}=\mathrm{supp}\,f$. In other words, σ is the inverse of ν. Given $\alpha,\beta<\omega_1$, the Hessenberg sum *(see e.g. [Si]) is defined by*

$$\alpha\dot{+}\beta:=\sigma(\nu(\alpha)+\nu(\beta)).$$

It is easy to see that if α is exp-indecomposable, then $\beta\dot{+}\gamma<\alpha$ for every $\beta,\gamma<\alpha$.

Definition 2.5. *Let $\mathcal{F}$ and $\mathcal{G}$ be families on chains of a partial ordering $\mathcal{P}$. Define*

$$\begin{aligned}\mathcal{F}\cup\mathcal{G}:=&\{s\subseteq P\,:\,s\in\mathcal{F}\text{ or }s\in\mathcal{G}\},\\ \mathcal{F}\sqcup_{\mathcal{P}}\mathcal{G}:=&\{s\cup t\,:\,s\cup t\text{ is a chain and }s\in\mathcal{F},\,t\in\mathcal{G}\},\\ \mathcal{F}\sqcup\mathcal{G}:=&\{s\cup t\,:\,s\in\mathcal{F},\,t\in\mathcal{G}\},\\ \mathcal{F}\boxtimes_{\mathcal{P}}(n+1):=&(\mathcal{F}\boxtimes_{\mathcal{P}}n)\sqcup_{\mathcal{P}}\mathcal{F};\quad \mathcal{F}\boxtimes_{\mathcal{P}}1:=\mathcal{F},\\ \mathcal{F}\boxtimes(n+1):=&(\mathcal{F}\boxtimes n)\sqcup_{\mathcal{P}}\mathcal{F};\quad \mathcal{F}\boxtimes 1:=\mathcal{F}.\end{aligned}$$

Observe that when $\mathcal{P}$ is a total ordering the operations $\sqcup_{\mathcal{P}}$ and $\sqcup$ are the same.

Proposition 2.6. *The operations $\cup$, $\sqcup_{\mathcal{P}}$ and $\sqcup$ preserve pre-compactness and hereditariness. Moreover, if $\mathcal{F}$ and $\mathcal{G}$ are countably ranked families on chains of $\mathcal{P}$, then*

(i) $\mathrm{rk}(\mathcal{F}\cup\mathcal{G})=\max\{\mathrm{rk}(\mathcal{F}),\mathrm{rk}(\mathcal{G})\}$,
(ii) $\mathrm{rk}(\mathcal{F}\sqcup\mathcal{G})=\mathrm{rk}(\mathcal{F})\dot{+}\mathrm{rk}(\mathcal{G})$,
(iii) $\mathrm{rk}(\mathcal{F}\sqcup_{\mathcal{P}}\mathcal{G})\le\mathrm{rk}(\mathcal{F})\dot{+}\mathrm{rk}(\mathcal{G})$.

Consequenlty, if $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{P}$-homogeneous, then $\mathcal{F}\cup\mathcal{G}$, $\mathcal{F}\sqcup_{\mathcal{P}}\mathcal{G}$ and $\mathcal{F}\sqcup\mathcal{G}$ are $(\gamma,\mathcal{P})$-homogeneous with $\gamma\ge\max\{\mathrm{srk}_{\mathcal{P}}(\mathcal{F}),\mathrm{srk}_{\mathcal{P}}(\mathcal{G})\}$.

Proof. It is easy to see that if $\mathcal{F}$ and $\mathcal{G}$ are pre-compact, hereditary, then $\mathcal{F}*\mathcal{G}$ is pre-compact, hereditary, for $*\in\{\cup,\sqcup_{\mathcal{P}},\sqcup\}$. Let us see (i): An easy inductive argument shows that $(\mathcal{F}\cup\mathcal{G})^{(\alpha)}=\mathcal{F}^{(\alpha)}\cup\mathcal{G}^{(\alpha)}$ for every countable α. (ii): It is a general fact that for every compact spaces K and L and every α one has that

$$(K\times L)^{(\alpha)}=\bigcup_{\beta\dot{+}\gamma=\alpha}(K^{(\beta)}\times L^{(\gamma)}). \tag{1}$$

When K and L are countable, we have that $\mathrm{rk}(K\times L)=\mathrm{rk}(K)\dot{+}\mathrm{rk}(L)$. The proof of (1) is done by induction on α and by considering the case when α is sum-indecomposable or not. Now let $\mathcal{F}$ and $\mathcal{G}$ be with countable rank. Suppose that $\mathcal{P}$ is a total ordering. Let $\mathcal{F}\times\mathcal{G}\to\mathcal{F}\sqcup\mathcal{G}$, $(s,t)\mapsto s\cup t$. This is clearly continuous, onto and finite-to-one, so $\mathrm{rk}(\mathcal{F}\sqcup\mathcal{G})=\mathrm{rk}(\mathcal{F}\times\mathcal{G})=\mathrm{rk}(\mathcal{F})\dot{+}\mathrm{rk}(\mathcal{G})$. If $\mathcal{P}$ is in general a partial ordering, then it follows from this that $\mathrm{rk}(\mathcal{F}\sqcup_{\mathcal{P}}\mathcal{G})\le\mathrm{rk}(\mathcal{F})\dot{+}\mathrm{rk}(\mathcal{G})$, proving (iii). Now suppose that $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{P}$-homogeneous. We have clearly that

$$\max\{\mathrm{srk}_{\mathcal{P}}(\mathcal{F}),\mathrm{srk}_{\mathcal{P}}(\mathcal{G})\}\le\min\{\mathrm{srk}_{\mathcal{P}}(\mathcal{F}\cup\mathcal{G}),\mathrm{srk}_{\mathcal{P}}(\mathcal{F}\sqcup_{\mathcal{P}}\mathcal{G}),\mathrm{srk}_{\mathcal{P}}(\mathcal{F}\sqcup\mathcal{G})\}. \tag{2}$$

On the other hand, $\max\{\mathrm{rk}(\mathcal{F}\cup\mathcal{G}),\mathrm{rk}(\mathcal{F}\sqcup_{\mathcal{P}}\mathcal{G}),\mathrm{rk}(\mathcal{F}\sqcup\mathcal{G})\le\mathrm{rk}(\mathcal{F})\dot{+}\mathrm{rk}(\mathcal{G})$. Since $\mathrm{rk}(\mathcal{F}),\mathrm{rk}(\mathcal{G})<\lambda:=\max\{\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F})),\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{G})\}$, it follows by the indecomposability of λ that $\mathrm{rk}(\mathcal{F})\dot{+}\mathrm{rk}(\mathcal{G})<\lambda$. This, together with (2) gives the desired result. ☐

2.1. **Bases of homogeneous families.** We recall a well-known generalization of Schreier families on ω, called uniform families. We are going to use them mainly as a tool to compute upper bounds of ranks of operations of compact families. We use the following standard notation: given $M,s,t\subseteq\omega$, we write $s<t$ to denote that $\max s<\min t$ and let $M/s:=\{m\in M\,:\,s<m\}$. Notice that a family $\mathcal{F}$ on ω is pre-compact if and only if every sequence in $\mathcal{F}$ has a *block* Δ-subsequence $(s_n)_{n\in\omega}$, that is, such that $s<s_m\setminus s<s_n\setminus s$ for every $m<n$, where s is the root of $(s_n)_n$. We write $s\sqsubseteq t$ to denote that s is an initial part of t, that is, $s\subseteq t$ and $t\cap(\max s+1)=s$, and $s\sqsubset t$ to denote that $s\sqsubseteq t$ and $s\neq t$.

Definition 2.7. *Given a family $\mathcal{F}$ on ω and $n<\omega$, let*

$$\mathcal{F}_{\{n\}}:=\{s\subseteq\omega\,:\,n<s\text{ and }\{n\}\cup s\in\mathcal{F}\}.$$

Let α be a countable ordinal number, and let $\mathcal{F}$ be a family on an infinite subset $M\subseteq\omega$. The family $\mathcal{F}$ is called an α-uniform family on M *when $\emptyset\in\mathcal{F}$ and*

(a) $\mathcal{F}=\{\emptyset\}$ *if $\alpha=0$;*
(b) $\mathcal{F}_{\{m\}}$ *is β-uniform on M/m for every $m\in M$, if $\alpha=\beta+1$;*

(c) *$\mathcal{F}_{\{m\}}$ is α_m-uniform on M/m for every $m \in M$ and $(\alpha_m)_{m\in M}$ is an increasing sequence such that $\sup_{m\in M} \alpha_m = \alpha$, if α is limit.*

It is important to remark that uniform families are not uniform fronts, which were introduced by P. Pudlak and V. Rödl in [Pu-Ro] following previous works of C. Nash-Williams. Recall that a family $\mathcal{B}$ on M is called an α-uniform front on M when $\mathcal{B} = \{0\}$ if $\alpha = 0$, and if $\alpha > 0$ then $\emptyset \notin \mathcal{B}$, and $\mathcal{B}_{\{n\}}$ is a γ-uniform front on M/n for every $n \in M$, if $\beta = \gamma + 1$, and $\mathcal{B}_{\{n\}}$ is a α_n-uniform front on M/n for every $n \in M$ and $(\alpha_n)_{n\in M}$ is increasing with $\sup_{m\in M} \alpha_m$. In fact, given a uniform family $\mathcal{F}$, the collection of its $\subseteq$-*maximal* elements $\mathcal{F}^{\max}$ is a uniform front:

Proposition 2.8. (a) *Every uniform family is compact.*

(b) *The following are equivalent:*

(b.1) *$\mathcal{F}$ is an α-uniform family on M.*

(b.2) *$\mathcal{F}^{\max}$ is an α-uniform front on M such that $\mathcal{F} = \overline{\mathcal{F}^{\max}} = (\mathcal{F}^{\max})^{\sqsubseteq}$.*

Proof. (a) is proved by a simple inductive argument on α. To prove that (b.1) implies (b.2), one observes first that $(\mathcal{F}^{\max})^{\sqsubseteq} = \mathcal{F}$, because $\mathcal{F}$ is compact, and then again use an inductive argument. The proof of that (b.2) implies (b.1) one uses the well-known fact that if $\mathcal{B}$ is a uniform front, then $\overline{\mathcal{B}} = \overline{\mathcal{B}}^{\sqsubseteq}$ (see for example [ArTod]). $\square$

Definition 2.9. *Given two families $\mathcal{F}$ and $\mathcal{G}$ on ω their sum and product are defined by*

$$\mathcal{F} \oplus \mathcal{G} := \{s \cup t \,:\, s < t,\, s \in \mathcal{G} \text{ and } t \in \mathcal{F}\},$$

$$\mathcal{F} \otimes \mathcal{G} := \{\bigcup_{i<n} s_i \,:\, \{s_i\}_i \subseteq \mathcal{F},\, \max s_i < \min s_{i+1},\, i < n, \text{ and } \{\min s_i\}_i \in \mathcal{G}\}.$$

The following are well-known facts of uniform fronts, and that are extended to uniform families by using the previous proposition. For more information on uniform fronts, we refer to [Lo], [LoTo1] and [ArTod].

Proposition 2.10. (a) *The rank of an α-uniform family is α.*

(b) *The unique n-uniform family on M, $n < \omega$, is $[M]^{\leq n}$.*

(c) *If $\mathcal{F}$ is an α-uniform family on M, then $\mathcal{F} \upharpoonright N$ is an α-uniform family on N for every $N \subseteq M$ infinite. Consequently, α-uniform families on ω are α-homogeneous.*

(d) *If $\mathcal{F}$ is an α-uniform family on M, and $\theta : M \to N$ is an order-preserving bijection, then $\{\theta"(s) \,:\, s \in \mathcal{F}\}$ is an α-uniform family on N.*

(e) *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are α and β uniform families on M, respectively. Then $\mathcal{F} \cup \mathcal{G}$, $\mathcal{F} \oplus \mathcal{G}$, $\mathcal{F} \sqcup \mathcal{G}$ and $\mathcal{F} \otimes \mathcal{G}$ are $\max\{\alpha, \beta\}$, $\alpha + \beta$, $\alpha \dot{+} \beta$ and $\alpha \cdot \beta$-uniform families on M, respectively.*

(f) *Uniform fronts have the* Ramsey property*: if $c : \mathcal{F} \to n$ is a coloring of a uniform front on M, then there is $N \subseteq M$ infinite such that c is constant on $\mathcal{F} \upharpoonright N$.*

(g) *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are uniform families on M. Then there is some $N \subseteq M$ such that either $\mathcal{F} \upharpoonright N \subseteq \mathcal{G}$ or $\mathcal{G} \upharpoonright N \subseteq \mathcal{F}$. Moreover, when $\mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{G})$, the first alternative must hold and in addition $(\mathcal{F} \upharpoonright N)^{\max} \cap (\mathcal{G} \upharpoonright N)^{\max} = \emptyset$.*

(h) *If $\mathcal{F}$ is a uniform family on M then there is $N \subseteq M$ infinite such that $\mathcal{F} \upharpoonright N$ is hereditary.*

(i) *If $\mathcal{F}$ is compact and $\sqsubseteq$-hereditary family on ω, then there is $M \subseteq \omega$ infinite such that $\mathcal{F} \upharpoonright M$ is a uniform family on M.* □

Remark 2.11. (i) The only new observation in the previous proposition is the fact in (e) that states that unions and square unions of uniform families is a uniform family, and that can be easily proved by induction on the maximum of the ranks.

(ii) A simple inductive argument shows that for every countable α and every $M \subseteq \omega$ infinite there is an α-uniform family $\mathcal{F}_\alpha$ on M, and, although uniform families are not necessarily hereditary using (d) and (h) one can build them being hereditary.

We obtain the following consequence for families on an arbitrary partial ordering.

Corollary 2.12. *Suppose that $\mathcal{P} = (P, <_P)$ is a partial ordering, and suppose that $\mathcal{F}$ and $\mathcal{G}$ are compact and hereditary families with $\mathrm{rk}(\mathcal{F}) < \mathrm{srk}(\mathcal{G})$. Then every infinite chain C of $\mathcal{P}$ has an infinite subchain $D \subseteq C$ such that $\mathcal{F} \upharpoonright D \subseteq \mathcal{G}$.*

Proof. Fix $\mathcal{F}$, $\mathcal{G}$ and C as in the statement. By going to a subchain of C, we assume that $(C, <_P)$ has order type ω. Since $(C, <_P)$ is order-isomorphic to ω with its natural order, it follows from Proposition 2.10 (i), (g) that there is some infinite subchain $D \subseteq C$ such that $\mathcal{F} \upharpoonright D \subseteq \mathcal{G}$. □

Among uniform families, the *generalized Schreier families* have been widely studied and used particularly in Banach space theory. They have an algebraic definition and have extra properties, as for example being spreading. Also, they have a sum-indecomposable rank. We recall the definition now.

Definition 2.13. *The* Schreier family *is*

$$\mathcal{S} := \{s \subseteq \omega \,:\, \#s \leq \min s\}.$$

A Schreier sequence *is defined inductively for $\alpha < \omega_1$ by*

(a) $\mathcal{S}_0 := [\omega]^{\leq 1}$,
(b) $\mathcal{S}_{\alpha+1} := \mathcal{S}_\alpha \otimes \mathcal{S}$ *and*
(c) $\mathcal{S}_\alpha := \bigcup_{n<\omega} (\mathcal{S}_{\alpha_n} \upharpoonright \omega \setminus n)$ *where $(\alpha_n)_n$ is such that $\sup_n \alpha_n = \alpha$, if α is limit.*

Note that the family $\mathcal{S}_\alpha$ depends on the choice of (α_n) converging to limit ordinals α.

Definition 2.14 (Spreading families). *A family $\mathcal{F}$ on ω is* spreading *when for every $s = \{m_0 < \cdots < m_k\} \in \mathcal{F}$ and $t = \{n_0 < \cdots < n_k\}$ with $m_i \leq n_i$ for every $i \leq k$ one has that $t \in \mathcal{F}$.*

The following is easy to prove.

Proposition 2.15. *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are spreading. Then $\mathcal{F} \cup \mathcal{G}$, $\mathcal{F} \sqcup \mathcal{G}$, $\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{F} \otimes \mathcal{G}$ are spreading.* □

The generalized Schreier families are uniform families and they have extra properties.

Proposition 2.16. (a) *$\mathcal{S}_\alpha$ is hereditary, spreading and ω^α-uniform.*
(b) *For every $\alpha \leq \beta$ there is $n < \omega$ such that $\mathcal{S}_\alpha \upharpoonright (\omega \setminus n) \subseteq \mathcal{S}_\beta$.*

(c) *For every α, β and γ such that $\alpha+\beta \le \gamma$ there is $n < \omega$ such that $(\mathcal{S}_\alpha \otimes \mathcal{S}_\beta) \upharpoonright (\omega \setminus n) \subseteq \mathcal{S}_\gamma$.*

Proof. (a) The first two properties are well-known. The proof of that $\mathcal{S}_\alpha$ is a ω^α-uniform family is done by induction on α. The case $\alpha = 0$ is trivial, while it is easy to verify that $\mathcal{S}$ is an ω-uniform family, so $\mathcal{S}_{\alpha+1} = \mathcal{S}_\alpha \otimes \mathcal{S}$ is a $\omega^\alpha \cdot \omega = \omega^{\alpha+1}$-uniform family by Proposition 2.10 and inductive hypothesis. Suppose that α is limit. For a given $m, n \in \omega$, let $\alpha_m^n < \omega^{\alpha_n}$ be such that $(\alpha_m^n)_m$ is increasing, $\sup_m \alpha_m^n = \alpha_n$ and $(\mathcal{S}_{\alpha_n})_{\{m\}}$ is a α_m^n uniform family. Since for every $m \in \omega$ we have that

$$(\mathcal{S}_\alpha)_{\{m\}} = \bigcup_{n \le m} (\mathcal{S}_{\alpha_n})_{\{m\}} \upharpoonright (\omega \setminus m)$$

it follows from Proposition 2.10 (e) that $(\mathcal{S}_\alpha)_{\{m\}}$ is a $\beta_m := \max_{n\le m} \alpha_m^n$-uniform family on ω/m. It is easy to see that $(\beta_m)_m$ is increasing and satisfies that $\sup_m \beta_m = \omega^\alpha$. (b) is proved by a simple inductive argument. □

Definition 2.17. *Let $\mathfrak{S}$ be the collection of all hereditary, spreading uniform families on ω.*

Proposition 2.18. *For every $\alpha < \omega_1$ there is a hereditary, spreading α-uniform family on ω.*

Proof. Let $(\mathcal{S}_\alpha)_{\alpha<\omega}$ be a Schreier sequence, and given a countable ordinal α with normal Cantor form $\alpha = \sum_{i \le k} \omega^{\alpha_i} \cdot n_i$ we define

$$\mathcal{F}_\alpha := (\mathcal{S}_{\alpha_0} \otimes [\omega]^{\le n_0}) \oplus \cdots \oplus (\mathcal{S}_{\alpha_k} \otimes [\omega]^{\le n_k}).$$

Then each $\mathcal{F}_\alpha$ is a hereditary, spreading α-uniform family on ω. □

We present now the concept of basis, that intends to generalize the notion of uniform family on ω, and the multiplication $\otimes$ between them. It seems that there is no canonical definition for the multiplication $\mathcal{F} \times \mathcal{G}$ of two families on an index set I. However, when $\mathcal{G}$ is a family on ω we can define it quite naturally as follows.

Definition 2.19. *Let $\mathcal{F}$ be a homogeneous family on chains of a partial ordering $\mathcal{P}$, and let $\mathcal{H}$ be a homogeneous family on ω. We say that a family $\mathcal{G}$ on chains of $\mathcal{P}$ is a* multiplication *of $\mathcal{F}$ by $\mathcal{H}$ when*

(M.1) *$\mathcal{G}$ is homogeneous and $\iota(\mathrm{srk}_\mathcal{P}(\mathcal{G})) = \iota(\mathrm{srk}_\mathcal{P}(\mathcal{F}) \cdot \mathrm{srk}(\mathcal{H}))$.*

(M.2) *Every sequence $(s_n)_{n<\omega}$ in $\mathcal{F}$ such that $\bigcup_n s_n$ is a chain of $\mathcal{P}$ has an infinite subsequence $(t_n)_n$ such that for every $x \in \mathcal{H}$ one has that $\bigcup_{n\in x} t_n \in \mathcal{G}$.*

Example 2.20. (i) $\mathcal{F} \boxtimes_\mathcal{P} n$ is a multiplication of any homogeneous family $\mathcal{F}$ by $[\omega]^{\le n}$.

(ii) If $\mathcal{P}$ does not have any infinite chain, then any homogeneous family $\mathcal{F}$ on chains of $\mathcal{P}$ has finite rank and given any homogeneous family $\mathcal{H}$ on ω, $\mathcal{G} = \mathcal{F}$ satisfies (M.2) .

Notice that always $\mathcal{F} \subseteq \mathcal{G}$ for every multiplication $\mathcal{G}$ of $\mathcal{F}$ by any family $\mathcal{H} \ne \{\emptyset\}$. When the family $\mathcal{F} = [\kappa]^{\le 1}$ and $\mathcal{H}$ is the Schreier family $\mathcal{S}$, then the existence of a family $\mathcal{G}$ satisfying (M.2) is equivalent to κ not being ω-Erdős (see [LoTo2], and the remarks after Theorem 4.1). Let us use the following notation. Given a collection $\mathfrak{C}$ of families on chains of $\mathcal{P}$ and $\alpha < \omega_1$ let $\mathfrak{C}_\alpha := \{\mathcal{F} \in \mathfrak{C} : \mathrm{srk}_\mathcal{P}(\mathcal{F}) = \alpha\}$.

Definition 2.21 (Basis of homogeneous families)**.** *Let $\mathcal{P}=(P,<)$ be a partial ordering with an infinite chain. A* basis (of homogeneous families) on chains of $\mathcal{P}$ *is a pair* $(\mathfrak{B},\times)$ *such that:*

(B.1) $\mathfrak{B}$ *consists of homogeneous families on chains of $\mathcal{P}$, it contains all cubes, and $\mathfrak{B}_\alpha\neq\emptyset$ for all $\omega\le\alpha<\omega_1$.*
(B.2) $\mathfrak{B}$ *is closed under $\cup$ and $\sqcup_{\mathcal{P}}$, and if $\mathcal{F}\subseteq\mathcal{G}\in\mathfrak{B}$ is such that $\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F}))=\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{G}))$ then $\mathcal{F}\in\mathfrak{B}$.*
(B.3) $\times:\mathfrak{B}\times\mathfrak{S}\to\mathfrak{B}$ *is such that for every $\mathcal{F}\in\mathfrak{B}$ and every $\mathcal{H}\in\mathfrak{S}$ one has that $\mathcal{F}\times\mathcal{H}$ is a multiplication of $\mathcal{F}$ by $\mathcal{H}$.*

When $\mathcal{P}=(P,<)$ is a total ordering, we simply say that $\mathfrak{B}$ is a basis of families on P.

Proposition 2.22. *There is basis of families on ω.*

Proof. Let $\mathfrak{B}$ be the collection of all $\mathcal{F}$ homogeneous families on ω such that $\iota(\mathrm{srk}(\mathcal{F}))=\iota(\mathrm{srk}(\langle\mathcal{F}\rangle_{\mathrm{spr}})$, where $\langle\mathcal{F}\rangle_{\mathrm{spr}}$ is the set of all $\{n_1<\cdots<n_k\}$ such that there is $\{m_1,\cdots,m_k\}\in\mathcal{F}$ such that $m_i\le n_i$ for all $i=1,\cdots,k$. Given $\mathcal{F}\in\mathfrak{B}$ and $\mathcal{H}\in\mathfrak{S}$, let

$$\mathcal{F}\times_\omega\mathcal{H}:=(\mathcal{F}\otimes\mathcal{H})\oplus\mathcal{F}.$$

It is routine to check all properties of basis, except (M.2): Suppose that $(s_k)_k$ is a sequence in $\mathcal{F}$. Let $(t_k)_{k<\omega}$ be a Δ-subsequence with root $t\in\mathcal{F}$ such that $t<t_k\backslash t<t_{k+1}\backslash t$ for every k. Suppose that $x\in\mathcal{H}$. Then $\{\min t_k\setminus t\}_{k\in x}\in\mathcal{H}$, because $\mathcal{H}$ is spreading. Hence, $\bigcup_{k\in x}(t_k\setminus t)\in\mathcal{F}\otimes\mathcal{H}$. Since $t<\bigcup_{k\in x}(t_k\setminus t)$, it follows that $\bigcup_{k\in x}t_k=t\cup\bigcup_{k\in x}(t_k\setminus t)\in(\mathcal{F}\otimes\mathcal{G})\oplus\mathcal{F}$. $\square$

The following gives a characterization of the existence of a basis.

Proposition 2.23. *A partial ordering $\mathcal{P}$ with an infinite chain has a basis if and only if there is a pair $(\mathfrak{B},\times)$, called* pseudo-basis *such that (B.3) holds for $(\mathfrak{B},\times)$, and*

(B.1') $\mathfrak{B}$ *consists of homogeneous families on chains of $\mathcal{P}$, it contains all cubes, and for every $\omega\le\alpha<\omega_1$ there is $\mathcal{F}\in\mathfrak{B}$ such that $\alpha\le\mathrm{srk}_{\mathcal{P}}(\mathcal{F})\le\iota(\alpha)$.*
(B.2') $\mathfrak{B}$ *is closed under $\cup$ and $\sqcup_{\mathcal{P}}$.*

Proof. Suppose that $(\mathfrak{B},\times)$ satisfies (B.1'), (B.2') and (B.3). Let $C=\{p_n\}_n$ be an infinite chain of $\mathcal{P}$, of order type ω. Fix a basis $(\mathfrak{B}(\omega),\times_\omega)$ of families on ω. For each $\mathcal{G}\in\mathfrak{B}(\omega)$, let $\overline{\mathcal{G}}:=\{\{p_n\}_{n\in x}\,:\,x\in\mathcal{G}\}$. Then $\overline{\mathcal{G}}$ is homeomorphic to $\mathcal{G}$. Given $\mathcal{F}\in\mathfrak{B}$, let $\widetilde{\mathcal{F}}:=\{s\in\mathcal{F}\,:\,s\cap C=\emptyset\}$. Now let $\mathfrak{B}'$ be the collection of all unions $\widetilde{\mathcal{F}}\cup\overline{\mathcal{G}}$ such that $\mathcal{F}\in\mathfrak{B}$, $\mathcal{G}\in\mathfrak{S}$, and finally let $\mathfrak{B}''$ be the collection of all $\mathcal{P}$-homogeneous families $\mathcal{F}$ such that there is some $\mathcal{G}\in\mathfrak{B}'$ with $\mathcal{F}\subseteq\mathcal{G}$ and $\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F}))=\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{G}))$. For each $\mathcal{F}\in\mathfrak{B}''$ we choose $\mathcal{G}_{\mathcal{F}}\in\mathfrak{B}$ and $\mathcal{H}_{\mathcal{F}}\in\mathfrak{B}(\omega)$ such that $\mathcal{F}\subseteq\widetilde{\mathcal{G}_{\mathcal{F}}}\cup\overline{\mathcal{H}_{\mathcal{F}}}\in\mathfrak{B}'$ and $\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F}))=\iota(\mathrm{srk}_{\mathcal{P}}(\widetilde{\mathcal{G}_{\mathcal{F}}}\cup\overline{\mathcal{H}_{\mathcal{F}}}))$. Then we define $\mathcal{F}\times'\mathcal{H}:=(\widetilde{\mathcal{G}_F\times\mathcal{H}})\cup\overline{(\mathcal{H}_{\mathcal{F}}\times_\omega\mathcal{H})}$. It is easy to check that $(\mathfrak{B}'',\times')$ is a basis on chains of $\mathcal{P}$. $\square$

2.2. **Ranks and uniform families.** The objective of this part is mainly to present two results. The first one, Proposition 2.25, states that the fact that $K^{(\alpha)}\neq\emptyset$ for a compact metrizable space K can be coded by a nice mapping $f:\mathcal{B}\to K$ defined on an α-uniform family $\mathcal{B}$. The second one, Proposition 2.27, gives an upper bound for the rank of a family of finite sets, and its proof uses the *Ramsey* property of uniform barriers.

Definition 2.24. *Given families $\mathcal{F}$ on ω and $\mathcal{G}$ on a partial ordering $\mathcal{P}$, we say that a mapping $f:\mathcal{F}\to\mathcal{G}$ between two families is $(\sqsubseteq,\subseteq)$-increasing when $s\sqsubseteq t$ implies that $f(s)\subseteq f(t)$.*

The fact that a point is in a certain derivative of a compact metrizable space K can be witnessed by a continuous and 1-1 mapping from a uniform family into K.

Proposition 2.25. *Suppose that K is a compact metrizable space and let $\alpha<\omega_1$. Then a point $p\in K$ is such that $p\in K^{(\alpha)}$ if and only if for every α-uniform family $\mathcal{B}$ there is a 1-1 and continuous function $\theta:\mathcal{B}\to K$ such that $\theta(\emptyset)=p$. In case $K=\mathcal{F}$ is a compact family on I, $p\in\mathcal{F}^{(\alpha)}$ if and only if for every α-uniform family $\mathcal{B}$ there is a 1-1 and continuous mapping $\theta:\mathcal{B}\to\mathcal{F}$ such that $p=\theta(\emptyset)$ and such that θ is $(\sqsubseteq,\subseteq)$-increasing.*

Proof. Given $p\in K$ and $\varepsilon>0$, let $B(p,\varepsilon)$ be the open ball around p and radius ε. The proof is by induction on α. Suppose that $p\in K^{(\alpha)}$ and let $\mathcal{B}$ be a α-uniform family on M and $\mathcal{C}$ the collection of $\sqsubseteq$-maximal subsets in $\mathcal{B}$. Without loss of generality we assume that $M=\omega$. Let $\alpha_n<\alpha$ be such that $\mathcal{C}_{\{n\}}$ is α_n-uniform on ω/n. Choose $(p_n)_n$ in $K^{(\alpha_n)}$ converging non-trivially to p such that there are mutually disjoint closed balls B_n around p_n with $\mathrm{diam}(B_n)\downarrow_n 0$. Since each $p_n\in K^{(\alpha_n)}$ it follows by inductive hypothesis that for each n there is a 1-1 and continuous function

$$\theta_n:\overline{\mathcal{B}_{\{n\}}}=\mathcal{C}_{\{n\}}\to B_n$$

with $\theta_n(\emptyset)=p_n$. Let $\theta:\mathcal{B}\to K$ be defined by $\theta(\emptyset)=p$, $\theta(s):=\theta_{\min s}(s\setminus\{\min s\})$, for $s\neq\emptyset$. By the choice of the balls B_n it follows that θ is 1-1. We verify now that θ is continuous: Suppose that $(s_k)_k$ tends to s. Suppose first that $s\neq\emptyset$, let $n:=\min s$. Then there is k_0 such that for every $k\geq k_0$ one has that $\min s_k=n$. It follows that for every $k\geq k_0$, $\theta(s_k)=\theta_n(t_k)$ and $\theta(s)=\theta_n(t)$, where $t_k:=s_k\setminus\{n\}$ and $t:=s\setminus\{n\}$. Hence, $\lim_{k\to\infty}\theta(s_k)=\lim_{k\to\infty}\theta_n(t_k)=\theta_n(t)=\theta(s)$. Suppose now that $s=\emptyset$. Fix $\gamma>0$ and suppose that $d(p,\theta(s_k))\geq\gamma$ for every k belonging to an infinite subset $M\subseteq\omega$. Without loss of generality, we may assume that $(s_k)_{k\in M}$ is a Δ-system with empty root such that $s_k<s_l$ if $k<l$ in M. Since $\theta(s_k)\in B_{n_k}$, for $n_k:=\min s_k$ for every k, and since $(n_k)_{k\in M}$ tends to infinity, $(p_{n_k})_{k\in M}$ converges to p, so that there is some k such that $d(p,\theta(s_k))<\gamma$, a contradiction. The reverse implication is trivial.

Now, if $\mathcal{F}$ is a compact family on I and α is a countable ordinal, then $p\in\mathcal{F}^{(\alpha)}$ if and only if $p\in(\mathcal{F}\upharpoonright I_0)^{\alpha}$ for some countable subset I_0. Now we can apply the first part of the Proposition to the compact and metrizable $K=\mathcal{F}\upharpoonright I_0$ and find, recursively on α a 1-1 and continuous $\theta:\mathcal{B}\to\mathcal{F}\upharpoonright I_0$ which in addition is $(\sqsubseteq,\subseteq)$-increasing. □

The following technical result will be useful to prove result on upper bounds of ranks.

Lemma 2.26. *Suppose that $\mathcal{B}$ and $\mathcal{C}$ are uniform families, $\mathcal{F}$ is a compact family on some index set I with $\mathrm{rk}(\mathcal{F})<\mathrm{rk}(\mathcal{C})$. Suppose that $\lambda:\mathcal{B}\otimes\mathcal{C}\to\mathcal{F}$ is $(\sqsubseteq,\subseteq)$-increasing. Then there is a finite subset x of ω and some infinite set $x<M$ such that $\{x\}\sqcup\mathcal{B}\upharpoonright M\subseteq\mathcal{B}\otimes\mathcal{C}$ and such that λ is constant on $\{x\}\sqcup(\mathcal{B}\upharpoonright M)^{\max}$. If in addition λ is continuous, then λ is constant on $\{x\}\sqcup(\mathcal{B}\upharpoonright M)$.*

Proof. Let $\mathcal{B}_0:=\mathcal{B}^{\max}$, $\mathcal{C}_0:=\mathcal{C}^{\max}$, and for each $s\in\mathcal{B}_0\otimes\mathcal{C}_0$, let $s=\bigcup_{i\leq k_s}s_i$ be the canonical decomposition of s; i.e. $s_i<s_{i+1}$ are in $\mathcal{B}_0$ and $\{\min s_i\}_{i\leq k_s}\in\mathcal{C}_0$. A use of the Ramsey property of uniform fronts gives an infinite subset $M\subseteq\omega$ such that one of the following conditions hold.

(a) For every $s_0 < \cdots < s_{k-1}$ in $\mathcal{B}_0 \restriction M$ with $\{\min s_i\}_{i<k} \in \mathcal{C} \restriction M \setminus \mathcal{C}_0$ and every $s_{k-1} < x < y \in \mathcal{B}_0 \restriction M$ one has that $\lambda(\bigcup_{i<k} s_i \cup x) \neq \lambda(\bigcup_{i<k} s_i \cup y)$.

(b) For every $s = \bigcup_{i \leq k_s} s_i \in \mathcal{B}_0 \restriction M \otimes \mathcal{C}_0 \restriction M$ there is $k < k_s$ such that for every $s_{k-1} < x < y \in \mathcal{B}_0 \restriction M$ one has that $\lambda(\bigcup_{i<k} s_i \cup x) = \lambda(\bigcup_{i<k} s_i \cup y)$.

Suppose that (a) holds. Let $(s_i)_{i<\omega}$ be a sequence in $\mathcal{B}_0 \restriction M$ such that $s_i < s_{i+1}$ for every i. Let $f : \mathcal{C} \restriction M \to \mathcal{F}$ be defined by $f(x) := \lambda(\bigcup_{i \in x} s_i)$.

Claim 2.26.1. *There is $f(\emptyset) \subseteq z \in \mathcal{F}^{(\mathrm{rk}(\mathcal{C}))}$ and consequently,* $\mathrm{rk}(\mathcal{F}) \geq \mathrm{rk}(\mathcal{C})$.

Proof of Claim: This is done by induction on the rank α of $\mathcal{C}$. For each $m \in M$, $\mathcal{C}_m$ is α_m-uniform on $M_m := M/s_m$, and satisfies (a), where $\alpha_m \uparrow \alpha$ if α is limit, and $\alpha_m = \alpha^-$ if α is successor. So, if we define $f_m : \mathcal{C}_m \restriction M_m \to \mathcal{F}$, $f_m(x) = f(s_m \cup \bigcup_{i \in x} s_i)$, then we can find $f_m(\emptyset) \subseteq z_m \in \mathcal{F}^{(\alpha_m)}$ for every m. By (a), it follows that $(z_m)_m$ are pairwise different. Since $\mathcal{F}$ is compact, there is a subsequence $(z_m)_{m \in N}$ which is a non-trivial Δ-system, with root $z \in \mathcal{F}^{(\alpha)}$. Since θ is $(\sqsubseteq, \subseteq)$-increasing, then so is f. Hence $f(\emptyset) \subseteq f(s_m) \subseteq z_m$, so $f(\emptyset) \subseteq z$. □

Suppose that (b) holds. Fix $s = \bigcup_{i \leq k_s} s_i \in \mathcal{B}_0 \restriction M \otimes \mathcal{C}_0 \restriction M$. Let $k < k_s$ be such that, setting $x := \bigcup_{i<k} s_i$, then $\lambda(x \cup y) = \lambda(x \cup z)$ for every $x < y < z$ for every $y, z \in \mathcal{B}_0 \restriction M$. We claim that $\lambda(x \cup y) = \lambda(x \cup z)$ for every $x < y, z \in \mathcal{B}_0 \restriction M$. Find $y, z < w \in \mathcal{B}_0 \restriction M$. Then $\lambda(x \cup y) = \lambda(x \cup w) = \lambda(x \cup z)$.

If we assume that λ is in addition continuous, since $\mathcal{B} \restriction M$ is scattered, the set of isolated points is dense. Hence $\{x\} \sqcup (\mathcal{B} \restriction M)^{\max}$ is dense in $\{x\} \sqcup (\mathcal{B} \restriction M)$. Since λ is constant on $\{x\} \sqcup (\mathcal{B} \restriction M)^{\max}$, it is constant on $\{x\} \sqcup (\mathcal{B} \restriction M)$. □

We obtain the following upper estimation on ranks.

Proposition 2.27. *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are countable ranked families and suppose that $\lambda : \mathcal{F} \to \mathcal{G}$ is $\subseteq$-increasing. Then*

$$\mathrm{rk}(\mathcal{F}) < \sup_{t \in \mathcal{G}} (\mathrm{rk}(\{s \in \mathcal{F} \, : \, \lambda(s) \subseteq t\}) + 1) \cdot (\mathrm{rk}(\mathcal{G}) + 1).$$

If in addition λ is continuous, then we obtain that

$$\mathrm{rk}(\mathcal{F}) < \sup_{t \in \mathcal{G}} (\mathrm{rk}(\{s \in \mathcal{F} \, : \, \lambda(s) = t\}) + 1) \cdot (\mathrm{rk}(\mathcal{G}) + 1).$$

PROOF. Let $\alpha := \sup_{t \in \mathcal{G}} (\mathrm{rk}(\{s \in \mathcal{F} \, : \, \lambda(s) \subseteq t\}) + 1)$, $\beta := \mathrm{rk}(\mathcal{G})$, and suppose that $\mathcal{F}^{(\alpha \cdot (\beta+1))} \neq \emptyset$. Let $\mathcal{B}$ and $\mathcal{C}$ be α and $\beta + 1$ uniform families, let $f : \mathcal{B} \otimes \mathcal{C} \to \mathcal{F}$ be a 1-1, continuous and $(\sqsubseteq, \subseteq)$-increasing function, and let $\theta := \lambda \circ f$. By hypothesis, θ is $(\sqsubseteq, \subseteq)$-increasing, so it follows from Lemma 2.26 that there are $x \subset \omega$ finite and $x < M$ infinite such that $\{x\} \sqcup \mathcal{B} \restriction M \subseteq \mathcal{B} \otimes \mathcal{C}$ and such that θ is constant on $\{x\} \sqcup (\mathcal{B} \restriction M)^{\max}$ with value $t \in \mathcal{G}$. This implies that the mapping $\theta_0(y) := \theta(x \cup y)$ for every $y \in \mathcal{B} \restriction M$ is a 1-1 and continuous mapping $\theta_0 : \mathcal{B} \restriction M \to \{s \in \mathcal{F} \, : \, \lambda(s) \subseteq t\}$, hence $\mathrm{rk}(\{s \in \mathcal{F} \, : \, \lambda(s) \subseteq t\}) \geq \alpha$, and this is impossible.

Suppose that in addition λ is continuous. Then the desired result is proved similarly by changing the definition of α with $\alpha := \sup_{t \in \mathcal{G}} (\mathrm{rk}(\{s \in \mathcal{F} \, : \, \lambda(s) = t\}) + 1)$, and then using the particular case of continuous functions in Lemma 2.26. □

3. Bases of families on trees

A tree T is determined by its chains and antichains. Given two families $\mathcal{A}$ and $\mathcal{C}$ on chains and on antichains of a tree T, respectively, one can define a third family $\mathcal{A} \odot_T \mathcal{C}$ consisting of all subsets of T generating a subtree whose antichains are in $\mathcal{A}$ and its chains are in $\mathcal{C}$. In general, the antichains of a tree are difficult to understand; on the contrary, the particular antichains consisting of immediate successors of a node are typically simpler (e.g. in a complete binary tree), and it makes more sense to define $\mathcal{A} \odot_T \mathcal{C}$ in terms of these particular simpler antichains. This operation on families will allow us, for example, to step up from a basis of families on a cardinal number κ to a basis on its cardinal exponential 2^κ, and more.

Recall that a (set-theoretical) *tree* $T = (T, <)$ is a set of nodes T with a partial order $<$ such that $\{u < t \, : \, u \in T\}$ is well ordered for every $t \in T$. A *rooted* tree is a tree with a minimal element 0, called the *root* of T. All trees we use are rooted, so that whenever we say tree, we mean a rooted tree. Trees are a sort of lexicographical product of two orderings, the one defining the tree order $<$ and the following. Given $t \le u$ in T, let $\mathrm{Is}_t(u)$ be the immediate successor of t which is below u, that is, $\mathrm{Is}_t(u)$ be the smallest $v \le u$ such that $t < v$. Then, given $t \in T$ and $x \subseteq T$, let

$$\mathrm{Is}_t\text{"}x := \{\mathrm{Is}_t(u) \, : \, t < u \in x\}.$$

For simplicity, we write Is_t for $\mathrm{Is}_t\text{"}(T)$, that is, the set of all immediate successors of t in T. For every $t \in T$, fix a total ordering $<_t$ of Is_t. Let $<_a$ be the partial ordering in T defined by $t <_a u$ if and only if there is v such that $t, u \in \mathrm{Is}_v$ and $t <_v u$. Hence, a chain with respect to $<_a$ is a set of immediate successors of a fixed node. Notice that both $<$ and $<_a$ can be extended to a total ordering $\prec$ on T by defining $t_0 \prec t_1$ if and only if $t_0 < t_1$, or if t_0 and t_1 are $<$-incomparable and $\mathrm{Is}_{t_0 \wedge t_1}(t_0) <_a \mathrm{Is}_{t_0 \wedge t_1}(t_1)$. We are now able to state the main result of this section.

Theorem 3.1. *The following are equivalent for an infinite tree T.*

(a) *There is a basis of families on T.*
(b) *There is a basis of families on chains of $(T, <)$, if there is an infinite $<$-chain, and there is a basis of families of families on chains of $<_a$, if there is an infinite $<_a$-chain.*

Notice that it follows from König's Lemma that when T is infinite, either there is a $<$-chain or an infinite $<_a$-chain. We pass now to recall well-known combinatorial principles on trees. Let $T = (T, <)$ be a complete rooted tree with root 0. Recall that a *chain* of a tree is a totally ordered subset of it. Given $t \le u \in T$, let

$$[t, u] := \{v \in T \, : \, t \le v \le u\}.$$

Similarly, one defines the corresponding (semi) open intervals. Given $x \subseteq T$, let $x_{\le t} := x \cap [0, t]$, $x_{\ge t} := \{u \in x \, : \, t \le u\}$, and let $x_{<t}$ and $x_{>t}$ be their open analogues.

Given $t, u \in T$, let

$$t \wedge u := \max(T_{\le t} \cap T_{\le u}),$$

which is well defined by the completeness of T. We say that $s \subseteq T$ is $\wedge$-*closed* when $t \wedge u \in s$ for every $t, u \in s$. Given $s \subseteq T$, let $\langle s \rangle$ be the *subtree generated* by s, that is, the minimal $\wedge$-closed subset of T containing s. We say that a subset $\tau \subseteq T$ is a *subtree* of T when $\langle \tau \rangle = \tau$.

Definition 3.2. *Given a family $\mathcal{F}$ on T, let*

$$\langle \mathcal{F} \rangle := \{x \subseteq \langle s \rangle \,:\, s \in \mathcal{F}\}.$$

The following easy fact will be helpful:

Proposition 3.3. *For every finite set $s \subseteq T$ and every $t \in T$, we have that*

$$\langle s \rangle = \{t_0 \wedge t_1 \,:\, t_0, t_1 \in s\},$$

$$\mathrm{Is}_t\text{"}\langle s \rangle = \mathrm{Is}_t\text{"}s.$$

In particular, $\langle s \rangle$ is finite whenever s is finite. In general, if $(s_i)_{i \in I}$ is a family of subsets of T, then

$$\langle \bigcup_{i \in I} s_i \rangle = \bigcup_{\{i,j\} \in [I]^2} \langle s_i \cup s_j \rangle.$$

Finally, given $s \subseteq T$, let

$$(s)_{\max} := \{\text{maximal elements of } s\}.$$

Definition 3.4. *Let $s = (t_k)_{k \in \omega}$ be a sequence of nodes in T.*

(a) *s is called a* comb *if s is an antichain such that*

$$t_k \wedge t_l = t_k \wedge t_m \text{ and } t_k \wedge t_l < t_l \wedge t_m \text{ for every } k < l < m.$$

The chain $(u_k)_k$, $u_k = t_k \wedge t_l$ $(k < l)$ is called the $\wedge$-chain of the comb $(t_k)_k$.

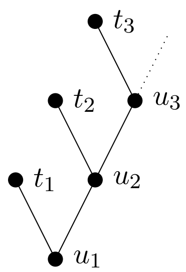

FIGURE 1. A comb $(t_k)_{k \in \omega}$ and its corresponding $\wedge$-chain $(u_k)_k$.

(b) *s is called a* fan *if*

$$t_k \wedge t_l = t_{k'} \wedge t_{l'} \text{ for every } k \neq l \text{ and } k' \neq l'.$$

The node $u := t_k \wedge t_l$ $(k \neq l)$ is called the $\wedge$-root of the fan $(t_k)_k$.

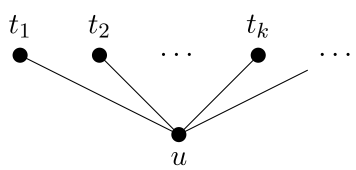

FIGURE 2. A fan $(t_k)_{k \in \omega}$ and its corresponding $\wedge$-root u.

Proposition 3.5. *Every infinite subset of T contains either an infinite chain, or an infinite comb, or an infinite fan.*

Proof. Fix a sequence $(t_k)_{k\in\omega}$ such that $t_k \neq t_l$ for $k \neq l$. By the Ramsey Theorem, there is M_0 such that either $(t_k)_{k\in M_0}$ is a chain or an antichain.

Claim 3.5.1. *If $(t_k)_{k\in M_0}$ is an antichain, then there is $M_1 \subseteq M_0$ such that $t_k \wedge t_l = t_k \wedge t_m$ for every $k < l < m$ in M_1.*

Proof of Claim: If $(t_k)_{k\in M_0}$ is an antichain and since there are no infinite decreasing chains in T, by the Ramsey Theorem, there is $M_1 \subseteq M_0$ such that

(a1) either $t_k \wedge t_l = t_k \wedge t_m$ for ever $k < l < m$ in M_1,
(b1) or else $t_k \wedge t_l < t_k \wedge t_m$ for ever $k < l < m$ in M_1.

Let us see that (b1) cannot happen: Fix $k_0 < k_1 < k_2 < k_3$ in M_1. Then

$$t_{k_0} \wedge t_{k_1} \wedge t_{k_i} = (t_{k_0} \wedge t_{k_1}) \wedge (t_{k_0} \wedge t_{k_i}) = t_{k_0} \wedge t_{k_1},$$

for $i = 2, 3$. On the other hand,

$$t_{k_0} \wedge t_{k_1} = t_{k_0} \wedge t_{k_1} \wedge t_{k_i} = (t_{k_0} \wedge t_{k_i}) \wedge (t_{k_1} \wedge t_{k_i}) \in \{t_{k_0} \wedge t_{k_i}, t_{k_1} \wedge t_{k_i}\},$$

for $i = 2, 3$; so either $t_{k_0} \wedge t_{k_1} = t_{k_0} \wedge t_{k_i}$ for some $i = 2, 3$, or else $t_{k_1} \wedge t_{k_2} = t_{k_0} \wedge t_{k_1} = t_{k_1} \wedge t_{k_3}$. Both cases are impossible since they contradict (b1). $\square$

For each $k \in M_1$, let $u_k := t_k \wedge t_l$ for some (all) $l > k$ in M_1. Yet again, since there are no infinite decreasing chains in T, by the Ramsey Theorem, there is $M_2 \subseteq M_1$ such that

(a2) either $u_k = u_l = \overline{u}$ for every $k < l$ in M_2,
(b2) or $u_k < u_l$ for every $k < l$ in M_2.

If (a2) holds, then $(t_k)_{k\in M_2}$ is a fan with $\wedge$-root $\overline{u}$. If (b2) holds, then $(t_k)_{k\in M_2}$ is a comb with $\wedge$-chain $(u_k)_{k\in M_2}$. $\square$

Corollary 3.6. *Every infinite subtree of T contains either an infinite chain or an infinite fan. Consequently,*

(a) *If $A \subseteq T$ is an infinite accumulation point of a sequence of subtrees of T, then A is a subtree of T that contains either an infinite chain or an infinite fan.*
(b) *For every family $\mathcal{F}$ on T with countable rank one has that*

$$\mathrm{srk}(\mathcal{F}) = \inf\{\mathrm{rk}(\mathcal{F} \upharpoonright X) : X \textit{ is an infinite chain, comb or fan}\}.$$

$\square$

Proposition 3.7. *Suppose that τ_0, τ_1 are subtrees of T. Then*

$$\mathrm{Ch}_a \upharpoonright \langle \tau_0 \cup \tau_1 \rangle = (\mathrm{Ch}_a \upharpoonright \tau_0) \sqcup_a (\mathrm{Ch}_a \upharpoonright \tau_1), \tag{3}$$

$$\mathrm{Ch}_c \upharpoonright \langle \tau_0 \cup \tau_1 \rangle \subseteq (\mathrm{Ch}_c \upharpoonright \tau_0) \sqcup_c (\mathrm{Ch}_c \upharpoonright \tau_1) \sqcup_c [T]^{\leq 1}. \tag{4}$$

Proof. (3) follows from the fact that $\mathrm{Is}_t"(\langle s \rangle) = \mathrm{Is}_t"(s)$. For (4), let c be a chain of $\langle \tau_0 \cup \tau_1 \rangle$, and suppose that $t_0 < t_1$ belong to $c \setminus (\tau_0 \cup \tau_1)$. Then $t_0 = u_0 \wedge v_0$ and $t_1 = u_1 \wedge v_1$ with $u_0, u_1 \in \tau_0$ and $v_0, v_1 \in \tau_1$. Then, either $u_0 \wedge u_1 = t_0$ or $v_0 \wedge v_1 = t_0$, and both are impossible. $\square$

We introduce the operation $\odot_T$.

Definition 3.8 (The operation $\odot_T$)**.** *Let* Ch_a *and* Ch_c *be the collection of all* $<_a$*-chains and of all* $<$*-chains of* T*, respectively. Given two families* $\mathcal{A}$ *and* $\mathcal{C}$ *on* T*, let* $\mathcal{A}\odot_T\mathcal{C}$ *be the family of all finite subsets* s *of* T *such that*

(a) $\langle s\rangle\cap\mathrm{Ch}_a\subseteq\mathcal{A}$*; that is, for every* $t\in T$*, one has that* $\mathrm{Is}_t"\langle s\rangle\in\mathcal{A}$*.*
(b) $\langle s\rangle\cap\mathrm{Ch}_c\subseteq\mathcal{C}$*; that is, every chain in* $\langle s\rangle$ *belongs to* $\mathcal{C}$*.*

Remark 3.9. (i) The family $\mathcal{A}\odot_T\mathcal{C}$ is closed under generated subtrees, that is, $\langle s\rangle\in\mathcal{A}\odot_T\mathcal{C}$ if $s\in\mathcal{A}\odot_T\mathcal{C}$.
(ii) When $[\mathrm{Is}_t]^{\le 1}\subseteq\mathcal{A}$ for all $t\in T$ the condition (a) above is equivalent to
  (a') For every $t\in\langle s\rangle$, one has that $\mathrm{Is}_t"(s)\in\mathcal{A}$.
(iii) $(\mathcal{A}\odot_T\mathcal{C})\cap\mathrm{Ch}_c=\mathcal{C}\cap\mathrm{Ch}_c$, and when $[T]^{\le 2}_c\subseteq\mathcal{C}$, then $(\mathcal{A}\odot_T\mathcal{C})\cap\mathrm{Ch}_a=\mathcal{A}\cap\mathrm{Ch}_a$.

We introduce some notation. We are going to use a and c to refer to $<_a$ and $<$. For example,

$$\sqcup_a,\ \mathrm{srk}_a,\ \sqcup_c,\ \mathrm{srk}_c$$

denote $\sqcup_{(T,<_a)}$, $\mathrm{srk}_{(T,<_a)}$, $\sqcup_{(T,<)}$ and $\mathrm{srk}_{(T,<)}$ respectively.

Proposition 3.10. *Let* $\mathcal{A}$ *and* $\mathcal{C}$ *be two families on chains of* $(T,<_a)$ *and of* $(T,<)$*, respectively.*

(a) *If* $\mathcal{A}$ *and* $\mathcal{C}$ *are compact, hereditary, then so is* $\mathcal{A}\odot_T\mathcal{C}$*.*
(b) $\mathrm{srk}(\mathcal{A}\odot_T\mathcal{C})\le\min\{\mathrm{srk}_a(\mathcal{A}),\mathrm{srk}_c(\mathcal{C})\}$ *when* Ch_a *and* Ch_c *are non-compact.*
(c) *Suppose that* $[T]^{\le 2}_a\subseteq\mathcal{A}$ *and that* $[T]^{\le 2}_c\subseteq\mathcal{C}$*. Then*

$$\mathrm{srk}(\mathcal{A}\odot_T(\mathcal{C}\sqcup_c[T]^{\le 1})=\begin{cases}\mathrm{srk}_a(\mathcal{A}) & \text{if } \mathrm{Ch}_c\subseteq[T]^{<\omega}\\ \mathrm{srk}_c(\mathcal{C}) & \text{if } \mathrm{Ch}_a\subseteq[T]^{<\omega}\\ \min\{\mathrm{srk}_a(\mathcal{A}),\mathrm{srk}_c(\mathcal{C})\},\min\{\mathrm{srk}_a(\mathcal{A}),\mathrm{srk}_c(\mathcal{C})+1\} & \text{otherwise.}\end{cases}$$

Proof. Set $\mathcal{F}:=\mathcal{A}\odot_T\mathcal{C}$. (a): Hereditariness of $\mathcal{F}$ is trivial. Suppose that s is an infinite subset of T which is the limit of a sequence $(\tau_k)_k$ in $\mathcal{F}$. Since $\mathcal{F}$ is, by definition, $\wedge$-closed, we may assume without loss of generality that each τ_k is a subtree. It follows that s is a subtree of T as well. Hence, by Corollary 3.6, s contains either an infinite chain C, or an infinite fan F. In the first case, $\tau_k\cap F\in\mathcal{C}$ for every k and $\tau_k\cap C\to_k C$, which is impossible since $\mathcal{C}$ is compact. If s contains an infinite fan F with $\wedge$-root u, then $\mathrm{Is}_u"(F)$ is an accumulation point of the sequence $(\mathrm{Is}_u"(s_k))_k$ in $\mathcal{A}$, which is impossible by the compactness of $\mathcal{A}$.

(b) is trivial. (c): Set $\mathcal{F}:=\mathcal{A}\odot_T(\mathcal{C}\sqcup_c[T]^{\le 1})$. Clearly $\mathrm{srk}(\mathcal{F})\le\mathrm{srk}(\mathcal{C})$. Since $[T]^{\le 2}_c\subseteq\mathcal{C}$, we have that $\mathrm{srk}(\mathcal{F})\le\mathrm{srk}(\mathcal{A})$. Now we use Corollary 3.6 to compute $\mathrm{srk}(\mathcal{F})$. If X is an infinite chain then clearly $\mathrm{rk}(\mathcal{F}\restriction X)\ge\mathrm{srk}(\mathcal{C})+1$. Suppose that $X=\{t_n\}_{n<\omega}$ is an infinite fan with $\wedge$-root u. For each $n<\omega$, let $v_n:=\mathrm{Is}_u(t_n)$. Since for every $x\subseteq\omega$ one has that $\langle\{t_n\}_{n\in x}\rangle=\{t_n\}_{n\in x}\cup\{u\}$, it follows that the maximal chains of $\langle\{t_n\}_{n\in x}\rangle$ have cardinality 2, so, they belong to $\mathcal{C}$, by hypothesis. This means that $\{t_n\}_{n\in x}\in\mathcal{F}$ if and only if $\{v_n\}_{n\in x}\in\mathcal{A}$, and consequently $\mathrm{rk}(\mathcal{F}\restriction X)=\mathrm{rk}(\mathcal{A}\restriction\{v_n\}_n)\ge\mathrm{srk}_a(\mathcal{A})$. Finally, suppose that $X=\{t_n\}_n$ is a comb with $\wedge$-chain $C:=\{u_k\}_k$. In one hand, given $\{t_k\}_{k\in x}\in\mathcal{F}$, we have by definition that $\{u_k\}_{k\in x}\in\mathcal{C}\sqcup_c[T]^{\le 1}$. Hence, $\mathrm{rk}(\mathcal{F}\restriction \mathrm{X})\le\mathrm{rk}(\mathcal{C}\restriction \mathrm{C})+1$. On the other hand, given $\{u_k\}_{k\in x}\in\mathcal{C}$

we know that $\{u_k\}_{k\in x}\cup\{t_p\}\in C\sqcup_c [T]^{\le 1}$, where $p:=\max x$. Since

$$\langle\{t_k\}_{k\in x}\rangle=\{t_k,u_k\}_{k\in x}$$

it follows that $\langle\{t_k\}_{k\in x}\rangle\in\mathcal{A}\odot_T(\mathcal{C}\sqcup_c[T]^{\le 1})$ whenever $\{u_k\}_{k\in x}\in\mathcal{C}$. So, $\{u_k\}_{k\in x}\in\mathcal{C}\mapsto\{t_k\}_{k\in x}\in\mathcal{F}$ is continuous and 1-1. Hence $\mathrm{rk}(\mathcal{F}\restriction X)\ge\mathrm{srk}_c(\mathcal{C})$. $\square$

Definition 3.11 (The Basis on T)**.** *Let T be an infinite tree, and suppose that for $(T,<_a)$ and $(T,<)$ either they have a basis of families on their chains or they do not have infinite chains. Let $(\mathfrak{B}^a,\times_a)$ be either a basis of families on chains of $(T,<_a)$ or $\mathfrak{B}^a:=\{\mathcal{A}:\mathcal{A}$ is hereditary and $\mathcal{A}\subseteq\mathrm{Ch}_a\}$ and $\mathcal{A}\times_a\mathcal{H}:=\mathcal{A}$ for every $\mathcal{A}\in\mathfrak{B}^a$ and $\mathcal{H}\in\mathfrak{S}$, if there are no infinite $<_a$-chains. We define similarly $(\mathfrak{B}^c,\times_c)$. Let $\mathfrak{B}$ be the collection of all families $\mathcal{F}$ on T such that*

(BT.1) *$\mathcal{F}$, $\langle\mathcal{F}\rangle$ are homogeneous and $\mathrm{rk}(\langle\mathcal{F}\rangle)<\iota(\mathrm{srk}(\mathcal{F}))$.*
(BT.2) *$\langle\mathcal{F}\rangle\cap\mathrm{Ch}_a\in\mathfrak{B}^a$ and $\langle\mathcal{F}\rangle\cap\mathrm{Ch}_c\in\mathfrak{B}^c$.*

Given $\mathcal{F}\in\mathfrak{B}$ and $\mathcal{H}\in\mathfrak{S}$, let

(BT.3) $\mathcal{F}\times\mathcal{H}:=((\mathcal{A}\times_a\mathcal{H})\sqcup_a[T]^{\le 1})\odot_T((\mathcal{C}\times_c\mathcal{H})\boxtimes_c 5)$ *where $\mathcal{A}:=\langle\mathcal{F}\rangle\cap\mathrm{Ch}_a$ and $\mathcal{C}:=\langle\mathcal{F}\rangle\cap\mathrm{Ch}_c$.*

Notice also that since T is infinite it follows from König's Lemma that either there is an infinite $<_a$-chain or an infinite $<$-chain. Notice also that there is no infinite $<_a$-chain if and only if $[T]^{<\omega}_a$ is a compact and hereditary family, and similarly for $<$.

Proposition 3.12. *Suppose that $\mathcal{F}\in\mathfrak{B}$. Then*

(a) $\iota(\mathrm{srk}(\mathcal{F}))=\iota(\mathrm{srk}(\langle\mathcal{F}\rangle))=\iota(\mathrm{srk}_a(\mathcal{F}\cap\mathrm{Ch}_a))=\iota(\mathrm{srk}_a(\langle\mathcal{F}\rangle\cap\mathrm{Ch}_a))=\iota(\mathrm{srk}_c(\mathcal{F}\cap\mathrm{Ch}_c))=\iota(\mathrm{srk}_c(\langle\mathcal{F}\rangle\cap\mathrm{Ch}_c))$ *if both Ch_a and Ch_c are not compact.*
(b) $\iota(\mathrm{srk}(\mathcal{F}))=\iota(\mathrm{srk}(\langle\mathcal{F}\rangle))=\iota(\mathrm{srk}_c(\mathcal{F}\cap\mathrm{Ch}_c))=\iota(\mathrm{srk}_c(\langle\mathcal{F}\rangle\cap\mathrm{Ch}_c))$ *if Ch_a is compact.*
(c) $\iota(\mathrm{srk}(\mathcal{F}))=\iota(\mathrm{srk}(\langle\mathcal{F}\rangle))=\iota(\mathrm{srk}_a(\mathcal{F}\cap\mathrm{Ch}_a))=\iota(\mathrm{srk}_a(\langle\mathcal{F}\rangle\cap\mathrm{Ch}_a))$ *if Ch_c is compact.*

Proof. We only prove (a); (b) and (c) have a similar proof. We know that

$$\mathrm{srk}(\mathcal{F})\le\mathrm{srk}(\langle\mathcal{F}\rangle)\le\mathrm{rk}(\langle\mathcal{F}\rangle)<\iota(\mathrm{srk}(\mathcal{F})),$$

by definition of $\mathcal{B}$. This means that $\iota(\mathrm{srk}(\mathcal{F}))=\iota(\mathrm{srk}(\langle\mathcal{F}\rangle))$. Similarly,

$$\mathrm{srk}_a(\mathcal{F}\cap\mathrm{Ch}_a)\le\mathrm{srk}_a(\langle\mathcal{F}\rangle\cap\mathrm{Ch}_a)\le\mathrm{rk}(\langle\mathcal{F}\rangle)<\iota(\mathrm{srk}(\mathcal{F}))\le\iota(\mathrm{srk}_a(\mathcal{F}\cap\mathrm{Ch}_a)).$$

So, $\iota(\mathrm{srk}_a(\mathcal{F}\cap\mathrm{Ch}_a))=\iota(\mathrm{srk}_a(\langle\mathcal{F}\rangle\cap\mathrm{Ch}_a))$. On the other hand,

$$\mathrm{srk}(\mathcal{F})\le\mathrm{srk}_a(\mathcal{F}\cap\mathrm{Ch}_a)\le\mathrm{rk}(\mathcal{F})<\iota(\mathrm{srk}(\mathcal{F})),$$

hence $\iota(\mathrm{srk}(\mathcal{F}))=\iota(\mathrm{srk}_a(\mathcal{F}\cap\mathrm{Ch}_a))$. Similarly one proves that $\iota(\mathrm{srk}(\mathcal{F}))=\iota(\mathrm{srk}_c(\mathcal{F}\cap\mathrm{Ch}_c))=\iota(\mathrm{srk}_c(\langle\mathcal{F}\rangle\cap\mathrm{Ch}_c))$. $\square$

The next to results are the keys to show Theorem 3.1.

Lemma 3.13. *If $\mathrm{rk}(\mathcal{A}),\mathrm{rk}(\mathcal{C})<\lambda$ with λ exp-indecomposable, then $\mathrm{rk}(\mathcal{A}\odot_T\mathcal{C})<\lambda$.*

Lemma 3.14. *$\times$ is a multiplication.*

The upper bound for the rank of $\mathcal{A} \odot_T \mathcal{C}$ in terms of the ranks of $\mathcal{A}$ and $\mathcal{C}$ is treated in the next Subsection 3.1, and the multiplication $\times$ is studied in the Subsection 3.2, where we find the canonical form of a Δ-sequence of finite subtrees of T.

Proposition 3.15. *Let $\mathcal{A} \in \mathfrak{B}^a$ and $\mathcal{C} \in \mathfrak{B}^c$ be such that $[T]_a^{\leq 2} \subseteq \mathcal{A}$ and $[T]_c^{\leq 2} \subseteq \mathcal{C}$. In addition, suppose that $\iota(\mathrm{srk}(\mathcal{A})) = \iota(\mathrm{srk}(\mathcal{C}))$ if Ch_a and Ch_c are not compact. Then $\mathcal{A} \odot_T \mathcal{C} \in \mathfrak{B}$.*

PROOF. Fix $\mathcal{A}$ and $\mathcal{C}$ as in the statement, and set $\mathcal{F} := \mathcal{A} \odot_T \mathcal{C}$. Then $\langle \mathcal{F} \rangle = \mathcal{F}$. Suppose that $\mathrm{Ch}_a, \mathrm{Ch}_c$ are not compact. We know by Proposition 3.10 that $\iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}_a(\mathcal{A})) = \iota(\mathrm{srk}_c(\mathcal{C})) = \lambda$. Since $\mathcal{A}$ and $\mathcal{C}$ are homogeneous, it follows that $\mathrm{rk}(\mathcal{A}), \mathrm{rk}(\mathcal{C}) < \lambda$. By Lemma 3.13, we obtain that $\mathrm{rk}(\mathcal{F}) < \lambda = \iota(\mathrm{srk}(\mathcal{F}))$. By Remark 3.9, $\mathcal{F} \cap \mathrm{Ch}_a = \mathcal{A} \in \mathfrak{B}^a$ and $\mathcal{F} \cap \mathrm{Ch}_c = \mathcal{C} \in \mathfrak{B}^c$, so $\mathcal{F} \in \mathfrak{B}$. The cases when Ch_a or Ch_c is compact are proved in a similar way. $\square$

We are now ready to prove the main result of this section.

Proof of Theorem 3.1. Let us see that $(\mathfrak{B}, \times)$ defined on Definition 3.11 is a basis of families on T. (B.1): Notice that if τ is a finite tree, then

$$\#\tau \leq \frac{a(\tau)^{c(\tau)+1} - 1}{a(\tau) - 1} \tag{5}$$

where $a(\tau)$ and $c(\tau)$ are the maximal cardinality of a $<_a$-chain and a $<$-chain, respectively. This means that if $\mathcal{F} = [T]^{\leq n}$, then $\langle \mathcal{F} \rangle \subseteq [T]^{n^{n+1}}$. Hence, $\langle \mathcal{F} \rangle$ is homogeneous and $\mathrm{srk}(\mathcal{F}) = \mathrm{srk}(\langle \mathcal{F} \rangle) = \omega$. It is easy to see that (BT.2) holds for $\mathcal{F} = [T]^{\leq n}$. Now let $\omega \leq \alpha < \omega_1$. Suppose that $\mathrm{Ch}_a, \mathrm{Ch}_c \nsubseteq [T]^{<\omega}$. Let $\mathcal{A}_0 \in \mathfrak{B}^a_\alpha$, and $\mathcal{C}_0 \in \mathfrak{B}^c_\alpha$. Then $\mathcal{A} := \mathcal{A}_0 \cup [T]^{\leq 2} \in \mathfrak{B}^a_\alpha$ and $\mathcal{C} := \mathcal{C}_0 \sqcup_c [T]^{\leq 1} \in \mathfrak{B}^c$. We can apply Proposition 3.10 to $\mathcal{F} := \mathcal{A} \odot_T \mathcal{C}$ to conclude that $\mathcal{F} \in \mathfrak{B}_\alpha$. Suppose that Ch_a is compact. Let $\mathcal{C} \in \mathfrak{B}^c_\alpha$, and set

$$\mathcal{F} := \mathcal{C} \cup \{s \subseteq \{t_i\}_{i<n} : \{t_i\}_{i<n} \text{ is a finite comb with chain } \{u_i\}_{i<n} \in \mathcal{C}\}.$$

Notice that $\langle \mathcal{F} \rangle = \mathcal{F} \subseteq [T]_a^{\leq 2} \odot_T (\mathcal{C} \sqcup_c [T]^{\leq 1})$, so it follows that $\mathrm{rk}(\mathcal{F}) < \iota(\mathrm{srk}_a(\mathcal{C})) = \iota(\mathrm{srk}(\mathcal{F})) = \alpha$. This means that $\mathcal{F}$ is homogeneous. On the other hand, $\mathcal{C} \subseteq \mathcal{F} \cap \mathrm{Ch}_c \subseteq \mathcal{C} \sqcup_c [T]^{\leq 1}$, so $\mathcal{F} \cap \mathrm{Ch}_c \in \mathfrak{B}^c$. So, $\mathcal{F} \in \mathfrak{B}_\alpha$. Finally, suppose that Ch_c is compact. Let $\mathcal{A} \in \mathfrak{B}^a_\alpha$, and set $\mathcal{F} := \mathcal{A} \odot_T [T]_c^{\leq 2}$. It is easy to see that $\mathcal{F} \in \mathfrak{B}_\alpha$.

(B.2): Suppose that $\mathcal{F}, \mathcal{G} \in \mathfrak{B}$. It is easy to see that $\mathcal{F} \cup \mathcal{G} \in \mathfrak{B}$. Let us see that $\mathcal{F} \sqcup \mathcal{G} \in \mathfrak{B}$. Then $\mathcal{F} \sqcup \mathcal{G}$ is homogeneous, with infinite rank. On the other hand,

$$\langle \mathcal{F} \sqcup \mathcal{G} \rangle \subseteq \langle \langle \mathcal{F} \rangle \sqcup \langle \mathcal{G} \rangle \rangle \subseteq (\langle \mathcal{F} \rangle \sqcup_a \langle \mathcal{G} \rangle) \odot_T (\langle \mathcal{F} \rangle \sqcup_c \langle \mathcal{G} \rangle \sqcup_c [T]^{\leq 1}),$$

by Proposition 3.7. Now,

$$\begin{aligned} \mathrm{rk}(\langle F \rangle \sqcup_a \langle \mathcal{G} \rangle) \leq & \mathrm{rk}(\langle F \rangle) \dot{+} \mathrm{rk}(\langle G \rangle) < \max\{\iota(\mathrm{srk}(\mathcal{F})), \iota(\mathrm{srk}(\mathcal{G}))\} \leq \iota(\mathrm{srk}(\mathcal{F} \sqcup \mathcal{G})) \\ \mathrm{rk}(\langle F \rangle \sqcup_c \langle \mathcal{G} \rangle \sqcup_c [T]^{\leq 1}) \leq & \mathrm{rk}(\langle F \rangle) \dot{+} \mathrm{rk}(\langle G \rangle) + 1 < \max\{\iota(\mathrm{srk}(\mathcal{F})), \iota(\mathrm{srk}(\mathcal{G}))\} \leq \iota(\mathrm{srk}(\mathcal{F} \sqcup \mathcal{G})). \end{aligned}$$

From this and Lemma 3.13 we obtain that

$$\mathrm{rk}(\langle \mathcal{F} \sqcup \mathcal{G} \rangle) < \iota(\mathrm{srk}(\mathcal{F} \sqcup \mathcal{G}))$$

On the other hand, by Proposition 3.7 and Remark 3.9 (iii) we know that

$$\begin{aligned}\langle \mathcal{F}\sqcup\mathcal{G}\rangle \cap \mathrm{Ch}_a =&(\mathcal{F}\cap \mathrm{Ch}_a)\sqcup_a (\mathcal{G}\cap \mathrm{Ch}_a)\in \mathfrak{B}_a\\ \langle \mathcal{F}\sqcup\mathcal{G}\rangle \cap \mathrm{Ch}_c \subseteq&(\langle\mathcal{F}\rangle\cap \mathrm{Ch}_c)\sqcup_c (\langle\mathcal{G}\rangle\cap \mathrm{Ch}_c)\sqcup_c [T]^{\le 1}\in \mathfrak{B}_c.\end{aligned}$$

Since

$$\begin{aligned}\iota(\mathrm{srk}(\langle \mathcal{F}\sqcup\mathcal{G}\rangle \cap \mathrm{Ch}_c)) =& \max\{\iota(\mathrm{srk}_c(\mathcal{F}\cap \mathrm{Ch}_c)),\iota(\mathrm{srk}_c(\mathcal{G}\cap \mathrm{Ch}_c))\} =\\ =&\iota(\mathrm{srk}_c((\langle\mathcal{F}\rangle\cap \mathrm{Ch}_c)\sqcup_c (\langle\mathcal{G}\rangle\cap \mathrm{Ch}_c)\sqcup_c [T]^{\le 1}))\end{aligned}$$

it follows from the property (B.2) of $\mathfrak{B}_c$ that $\langle \mathcal{F}\sqcup\mathcal{G}\rangle \cap \mathrm{Ch}_c\in\mathfrak{B}_c$, so $\mathcal{F}\sqcup\mathcal{G}\in\mathfrak{B}$.

Suppose that $\mathcal{F}\subseteq\mathcal{G}\in\mathfrak{B}$ is a homogeneous family such that $\iota(\mathrm{srk}(\mathcal{F}))=\iota(\mathrm{srk}(\mathcal{G}))$. Then, $\langle\mathcal{F}\rangle\subseteq\langle\mathcal{G}\rangle$, so,

$$\mathrm{rk}(\langle\mathcal{F}\rangle)\le \mathrm{rk}(\langle\mathcal{G}\rangle)<\iota(\mathrm{srk}(\mathcal{G}))=\iota(\mathrm{srk}(\mathcal{F}))\le\iota(\mathrm{srk}(\langle\mathcal{F}\rangle)). \tag{6}$$

So, $\langle\mathcal{F}\rangle$ is homogeneous. Now, $\langle\mathcal{F}\rangle\cap\mathrm{Ch}_a\subseteq\langle\mathcal{G}\rangle\cap\mathrm{Ch}_a\in\mathfrak{B}_a$ and $\langle\mathcal{F}\rangle\cap\mathrm{Ch}_c\subseteq\langle\mathcal{G}\rangle\cap\mathrm{Ch}_c\in\mathfrak{B}_c$. Suppose that Ch_a and Ch_c are both non-compact. Then, $\mathrm{srk}_a(\langle\mathcal{F}\rangle\cap\mathrm{Ch}_a)\ge\mathrm{srk}(\langle\mathcal{F}\rangle)$. Hence, from (6) we obtain that

$$\mathrm{srk}_a(\langle F\rangle\cap\mathrm{Ch}_a)\le\mathrm{srk}(\langle G\rangle\cap\mathrm{Ch}_a)\le\mathrm{rk}(\langle\mathcal{G}\rangle)<\iota(\mathrm{srk}(\langle\mathcal{F}\rangle))\le\iota(\mathrm{srk}_a(\langle\mathcal{F}\rangle\cap\mathrm{Ch}_a)).$$

This means that $\iota(\mathrm{srk}_a(\langle\mathcal{F}\rangle))=\iota(\mathrm{srk}_a(\langle\mathcal{G}\rangle))$, and consequently, $\langle\mathcal{F}\rangle\cap\mathrm{Ch}_a\in\mathfrak{B}^a$. Similarly, $\langle\mathcal{F}\rangle\cap\mathrm{Ch}_c\in\mathfrak{B}^c$, hence $\mathcal{F}\in\mathfrak{B}$. The cases when Ch_a or Ch_c are compact are proved similarly.

(B.3) is the content of Lemma 3.14. $\square$

3.1. **The operation $\odot_T$ and ranks.** We compute an upper bound of the rank of the family $\mathcal{A}\odot_T\mathcal{C}$ in terms of the ranks of $\mathcal{A}$ and $\mathcal{C}$, respectively. Fix a tree T, a compact and hereditary family $\mathcal{C}$ on chains of T and a compact and hereditary family on immediate successors of nodes of T. As we have observed in (5) for finite trees, it is natural to expect an upper bound of the rank of $\mathcal{A}\odot_T\mathcal{C}$ by an exponential-like function of the rank of $\mathcal{A}$ and the rank of $\mathcal{C}$.

Definition 3.16. *Given a countable ordinal number α, we define a function $f_\alpha:\omega_1\to\omega_1$ as follows:*

$$\begin{aligned}f_\alpha(0):=&1\\ f_\alpha(\xi+1):=&f_\alpha(\xi)\cdot(\max\{\alpha,\xi\}\cdot\omega)\\ f_\alpha(\xi):=&\sup_{\eta<\xi}f_\alpha(\eta),\ \textit{when } \xi \textit{ is limit.}\end{aligned}$$

Remark 3.17. (a) f_α is a continuous strictly increasing mapping such that $f_\alpha(\xi)$ is sum-indecomposable for every α and ξ.

(b) $f_\alpha(\xi)\ge(\alpha\cdot\omega)^\xi$ always, and if $\xi\le\alpha$ then $f_\alpha(\xi)=(\alpha\cdot\omega)^\xi$.

(c) $f_\beta(\xi)<\alpha$ for every $\beta,\xi<\alpha$ if and only if α is exp-indecomposable: Suppose that α is closed under $f_\cdot(\cdot)$. Let $\beta,\xi<\alpha$. Then $\beta^\xi\le f_\beta(\xi)<\alpha$. Suppose that α is exp-indecomposable. Let $\beta,\xi<\alpha$. Since $f_\cdot(\cdot)$ is increasing in both variables, and since α is product-indecomposable, w.l.o.g. we assume that β is sum-indecomposable, i.e. $\beta=\omega^{\beta_0}$ with $\xi\le\beta<\alpha$. Then,

$$f_\beta(\xi)=(\beta\cdot\omega)^\xi=\omega^{(\beta_0+1)\cdot\xi}<\omega^\alpha=\alpha.$$

Lemma 3.13 follows from Remark 3.17 (c) and the following.

Lemma 3.18. $\mathrm{rk}(\mathcal{A} \odot_T \mathcal{C}) < f_{\mathrm{rk}(\mathcal{A})+1}(\mathrm{rk}(\mathcal{C}) + 1)$.

Definition 3.19. *Given a subtree U of T with root 0, let*

$$\mathrm{stem}(U) := \{t \in U \ : \ \textit{every } u \in U \textit{ is comparable with } t\}.$$

Let $(\mathcal{A} \odot_T \mathcal{C})_T$ be the subset of $\mathcal{A} \odot_T \mathcal{C}$ consisting of subtrees of T.

Notice that $\mathrm{stem}(U)$ is a non-empty chain in U, because $0 \in \mathrm{stem}(U)$.

Proposition 3.20. *For every countable ordinal α one has that $x \in (\mathcal{A} \odot_T \mathcal{C})^{(\alpha)}$ if and only if there is some subtree y of T containing x such that $y \in ((\mathcal{A} \odot_T \mathcal{C})_T)^{(\alpha)}$.*

Proof. First of all, by definition $x \in \mathcal{A} \odot_T \mathcal{C}$ if and only if $\langle x \rangle \in \mathcal{A} \odot_T \mathcal{C}$. Also, since $\mathcal{A} \odot_T \mathcal{C}$ is hereditary, it follows that each $(\mathcal{A} \odot_T \mathcal{C})^{(\alpha)}$ is also hereditary. This proves that if $x \subseteq y \in ((\mathcal{A} \odot_T \mathcal{C})_T)^{(\alpha)}$, then $x \in (\mathcal{A} \odot_T \mathcal{C})^{(\alpha)}$. Now we prove that if $x \in (\mathcal{A} \odot_T \mathcal{C})^{(\alpha)}$ then there is there is a subtree $x \subseteq y$ on T belonging to $(\mathcal{A} \odot_T \mathcal{C})^{(\alpha)}$ by induction on α. The case $\alpha = 0$ is treated above. Suppose that α is limit, and let $(\alpha_n)_n$ be an increasing sequence with supremum α. By inductive hypothesis, for every n there is some subtree y_n of T such that $x \subseteq y_n$ and such that $y_n \in ((\mathcal{A} \odot_T \mathcal{C})_T)^{(\alpha_n)}$. By compactness, there is an infinite set M such that $(y_n)_{n \in M}$ is a Δ-sequence with root y. A limit of subtrees is a subtree, hence y is a subtree that contains x and $y \in \bigcap_{n \in M} ((\mathcal{A} \odot_T \mathcal{C})_T)^{(\alpha_n)}$, so $y \in ((\mathcal{A} \odot_T \mathcal{C})_T)^{(\alpha)}$. Suppose that $x \in (\mathcal{A} \odot_T \mathcal{C})^{(\alpha+1)}$. Choose a non-trivial Δ-sequence $(x_n)_n$ in $(\mathcal{A} \odot_T \mathcal{C})^{(\alpha)}$ with limit x, and for each n choose a subtree y_n of T containing x_n and in $((\mathcal{A} \odot_T \mathcal{C})_T)^{(\alpha)}$. Now find an infinite subset $M \subseteq \omega$ such that $(y_n)_n$ is a Δ-sequence with root y. Since $x_n \subseteq y_n$, it follows that $(y_n)_{n \in M}$ is non-trivial, hence $x \in y \in ((\mathcal{A} \odot_T \mathcal{C})_T)^{(\alpha+1)}$. □

Definition 3.21. *Given a chain c of T, let $(\mathcal{A} \odot_T \mathcal{C})_c := \{x \in (\mathcal{A} \odot_T \mathcal{C})_T \ : \ c \sqsubseteq \mathrm{stem}(x)\}$.*

To simplify the notation, let $\mathcal{F} := \mathcal{A} \odot_T \mathcal{C}$. Observe that $\mathcal{F}_c$ is always a compact family, and that $\mathcal{F}_\emptyset = \mathcal{F}$.

Lemma 3.22. *Let c be a chain in $\mathcal{C}$. If $\mathrm{rk}(\mathcal{F}_c) \geq f_{\mathrm{rk}(\mathcal{A})+1}(\xi)$ then $c \in \mathcal{C}^{(\xi)}$.*

Proof. For each countable ordinal ξ, set $\beta_\xi := f_{\mathrm{rk}(\mathcal{A})+1}(\xi)$. Fix a chain $c \in \mathcal{C}$ such that $\mathrm{rk}(\mathcal{F}_c) \geq \beta_\xi$, and we have to prove that $c \in \mathcal{C}^{(\xi)}$. The proof is by induction on ξ. The case $\xi = 0$ or limit are trivial. Suppose that $\xi = \eta + 1$. Since $\mathcal{C}$ is compact, we can assume without loss of generality that c is maximal such that $\mathrm{rk}(\mathcal{F}_c) \geq \beta_\xi$, i.e.

$$\text{if } c \subsetneq c' \in \mathcal{C}, \text{ then } \mathrm{rk}(\mathcal{F}_{c'}) < \beta_\xi. \tag{7}$$

When $c \neq \emptyset$, let Let $t_c := \max c$.

$$\begin{aligned} \mathcal{G} :=& \{x \in \mathcal{F}_c \ : \ \mathrm{Is}_{t_c}"(x) \subseteq x\} \\ \mathcal{H} :=& \{x \in \mathcal{F}_c \ : \ x \cap \mathrm{Is}_{t_c} = \emptyset\}. \end{aligned}$$

It is easy to see that both $\mathcal{G}$ and $\mathcal{H}$ are compact, and since subtrees are closed under $\wedge_{\mathrm{is}}$ one has that

$$\mathcal{F}_c = \mathcal{G} \cup \mathcal{H}.$$

Since β_ξ is sum-indecomposable, it follows that when $c \neq \emptyset$, then $\max\{\mathrm{rk}(\mathcal{G}), \mathrm{rk}(\mathcal{H})\} \geq \beta_\xi$, so we have the following two cases to consider.

Case 1. $c \neq \emptyset$ and $\mathrm{rk}(\mathcal{G}) \geq \beta_\xi$. Let now

$$\begin{aligned} I :=& \{u \in \mathrm{Is}_{t_c} \,:\, \mathrm{rk}(\mathcal{F}_{c\cup\{u\}}) \geq \beta_\eta\} \\ J :=& \mathrm{Is}_{t_c} \setminus I \\ \mathcal{G}_I :=& \{x \in \mathcal{G} \,:\, \mathrm{Is}_{t_c} \in \mathcal{A} \upharpoonright I\} \\ \mathcal{G}_J :=& \{x \in \mathcal{G} \,:\, \mathrm{Is}_{t_c} \in \mathcal{A} \upharpoonright J\}. \end{aligned}$$

Clearly $\mathcal{G} \subseteq \mathcal{G}_I \sqcup \mathcal{G}_J$. So, there are two subcases to consider.

Subcase 1.1 $\mathrm{rk}(\mathcal{G}_I) \geq \beta_\xi$. Notice that I is finite because otherwise, for each $u \in I$ we know by inductive hypothesis that $c \cup \{u\} \in \mathcal{C}^{(\eta)}$, so $c \in \mathcal{C}^{(\xi)}$. Since

$$\mathcal{G}_I \subseteq \bigcup_{K \subseteq I} \bigsqcup_{u \in K} \mathcal{F}_{c\cup\{u\}}$$

it follows that there is some $u \in I$ such that $\mathcal{F}_{c\cup\{u\}}$ has rank at least β_ξ, contradicting (7).

Subcase 1.2 $\mathrm{rk}(\mathcal{G}_J) \geq \beta_\xi$. Let $\lambda : \mathcal{G}_J \to \mathcal{A} \upharpoonright J$ be defined by $\lambda(x) := \mathrm{Is}_{t_c}(x)$. This mapping is $\subseteq$-increasing and since $\lambda(x) = x \cap \mathrm{Is}_{t_c}$ for every $x \in \mathcal{G}_J$, it follows that λ is continuous. By Proposition 2.25, the definition of $f_{\mathrm{rk}(\mathcal{A})+1}(\cdot)$ we obtain that

$$\beta_\eta \cdot (\mathrm{rk}(\mathcal{A}) \cdot \omega) \leq \beta_\xi \leq \mathcal{G}_{\mathrm{J}} < \sup_{y \in \mathcal{A} \upharpoonright J} (\mathrm{rk}\{x \in \mathcal{G}_j \,:\, \lambda(x) = y\} + 1) \cdot \mathrm{rk}(\mathcal{A} \upharpoonright J).$$

So there must be $y \in \mathcal{A} \upharpoonright J$ such that

$$\mathrm{rk}\{x \in \mathcal{G}_J \,:\, \lambda(x) = y\} \geq \beta_\eta.$$

We also have that

$$\{x \in \mathcal{G}_J \,:\, \lambda(x) = y\} \subseteq \bigsqcup_{u \in y} \mathcal{F}_{c\cup\{u\}}. \tag{8}$$

Observe that $y \neq \emptyset$, because $\{x \in \mathcal{G}_J \,:\, \lambda(x) = \emptyset\} = \{c\}$ has rank 0. Hence, it follows from (8) that there must be $u \in y$ such that $\mathcal{F}_{c\cup\{u\}}$ has rank at least β_η, contradicting the fact that $u \in J$.

Case 2. $c \neq \emptyset$ and $\mathrm{rk}(\mathcal{H}) \geq \beta_\xi$, or $c = \emptyset$. Let $\widetilde{\mathcal{H}} = \mathcal{H}$ if $c \neq \emptyset$, and let $\widetilde{\mathcal{H}} = \mathcal{F}$ when $c = \emptyset$. Let $\mu : \widetilde{\mathcal{H}} \to \mathcal{C}_c$ be defined by $\mu(x) := \{\min(x \setminus c)\} \cup c$ when $c \subsetneq x$ and $\mu(c) := c$. Since $\mathrm{rk}(\widetilde{\mathcal{H}}) \geq \beta_\xi$, given a β_ξ-uniform family $\mathcal{B}$, we can find $f : \mathcal{B} \to \widetilde{\mathcal{H}}$ continuous, 1-1 and $(\sqsubseteq, \subseteq)$-increasing. Let $\lambda : \mathcal{B} \to \mathcal{C}_c$ be defined for $x \in \mathcal{B}$ by

$$\lambda(x) := \bigcup_{y \sqsubseteq x} \mu(f(y)).$$

Notice that $\lambda(x) \in \mathcal{C}$ and that λ is $(\sqsubseteq, \subseteq)$-increasing. Moreover λ is continuous: Suppose that $x_n \to_n x$ in $\mathcal{B}$. W.l.o.g. we assume that $(x_n)_n$ is a Δ-sequence with root x such that $x_m \setminus x < x_n \setminus x$ for every $m < n$ and that $\lambda(x_n) \to_n d \in \mathcal{C}$. We have to prove that $\lambda(x) = d$. Since $x \in x_n$ for every n, it follows that $\lambda(x) \subseteq d$. Now suppose that there is $t \in d \setminus \lambda(x)$; by definition this would mean that $t < \min(\lambda(x) \setminus c)$; since $d \subseteq f(x)$ we have that $t \in f(x)$, but then

$$\min(f(x) \setminus c) \leq_T t < \min(\bigcup_{y \sqsubseteq x} \mu(f(y)) \leq_T \min(f(x) \setminus c)$$

which is impossible. For every $x \in \mathcal{B}^{\max}$ let $\varphi(x)$ be the maximal initial part of x such that $\lambda(x) = \lambda(\emptyset)$, and let M be an infinite subset of ω such that $\mathcal{B}_0 := \varphi"(\mathcal{B} \upharpoonright M)$ is a γ-uniform family on M for some $\gamma \leq \beta_\xi$.

SUBCASE 2.1. $\gamma = \beta_\xi$. Since $c \subsetneq \lambda(x)$ for every $x \neq \emptyset$ in $\mathcal{B}$, and since $\mathcal{B}_0 \neq \{\emptyset\}$, it follows that $c \subsetneq \lambda(x) = \mu(\emptyset) = c'$. By the definition of $\mathcal{B}_0$ it follows that the restriction of f to $\mathcal{B}_0$ satisfies that $f(x) \in \mathcal{F}_{c'}$. Consequently, $\mathrm{rk}(\mathcal{F}_{c'}) \geq \gamma = \beta_\xi$, a contradiction with (7).

SUBCASE 2.2. $\gamma < \beta_\xi$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be β_η-uniform and ξ-uniform families on M, respectively. Since by definition $\beta_\xi \geq \beta_\eta \cdot (\xi \cdot \omega)$, it follows that $\gamma, \beta_\eta \cdot \xi < \beta_\xi$. Since β_ξ is sum-indecomposable, it follows that $(\beta_\eta \cdot \xi) + \gamma < \beta_\xi$. Hence, by the properties of the uniform families (Proposition 2.10) we obtain that there is $N \subseteq M$ such that

$$((\mathcal{B}_1 \otimes \mathcal{B}_2) \oplus \mathcal{B}_0) \upharpoonright N \subseteq \mathcal{B}.$$

Fix now $x \in (\mathcal{B}_0 \upharpoonright N)^{\max}$, set $N_x := N/x$, and $\lambda_x : (\mathcal{B}_1 \otimes \mathcal{B}_2) \upharpoonright N_x \to \mathcal{C}_c$, $\lambda_x(y) := \lambda(x \cup y)$. Since $c \notin \mathcal{C}^{(\xi)}$, it follows that $\mathrm{rk}(\mathcal{C}_c) < \xi$. Since λ_x is $(\sqsubseteq, \subseteq)$-increasing and continuous, it follows from Lemma 2.26 that there is some $x < y$ finite and $y < P$ infinite, $y \cup P \subseteq N_x$, such that $\{y\} \sqcup \mathcal{B}_1 \upharpoonright P \subseteq (\mathcal{B}_1 \otimes \mathcal{B}_2) \upharpoonright N$ and such that λ_x is constant on $\{y\} \sqcup \mathcal{B}_1 \upharpoonright P$ with value $d = \lambda(x \cup y) \in \mathcal{C}_c$. Since $\mathcal{B}_1 \upharpoonright P$ contains non-empty elements z, it follows that $x \sqsubset x \cup y \cup z$, so by the maximality of x,

$$c \subseteq \lambda(\emptyset) = \lambda(x) \subsetneq \lambda(x \cup y \cup z) = \lambda(x \cup y).$$

On the other hand, the mapping $f_0 : \mathcal{B}_2 \upharpoonright P \to \mathcal{F}_d$, $f_0(z) := f(x \cup y \cup z)$, witnesses that $\mathrm{rk}(\mathcal{F}_d) \geq \beta_\eta$, so by inductive hypothesis, $d \in \mathcal{C}^{(\eta)}$. In this way we can find $x < y_0 < y_1 < \cdots < y_n < \cdots$ such that $\lambda(x \cup y_n) \in \mathcal{C}^{(\eta)}$ and

$$c \subseteq \lambda(\emptyset) = \lambda(x) \subsetneq \lambda(x \cup y_n) \tag{9}$$

for every n. Since λ is continuous and $x \cup y_n \to_n x$, it follows that $\lambda(x \cup y_n) \to_n \lambda(x) = \lambda(\emptyset)$ and non-trivially, by (9). Hence, $\lambda(\emptyset) \in \mathcal{C}^{(\xi)}$, and so $c \in \mathcal{C}^{(\xi)}$, because $\mathcal{C}$ is hereditary and $c \subseteq \lambda(\emptyset)$. This is impossible. □

PROOF OF LEMMA 3.18. Suppose otherwise that $\mathrm{rk}(\mathcal{A} \odot_T \mathcal{C}) \geq f_{\mathrm{rk}(\mathcal{A})+1}(\mathrm{rk}(\mathcal{C}) + 1)$. Then by Proposition 3.20, it follows that $\mathrm{rk}((\mathcal{A} \odot_T \mathcal{C})_T) \geq f_{\mathrm{rk}(\mathcal{A})+1}(\mathrm{rk}(\mathcal{C}) + 1)$. So, by Lemma 3.22, we have that $\emptyset \in \mathcal{C}^{(\mathrm{rk}(\mathcal{C})+1)}$, a contradiction. □

3.2. **Canonical form of sequences of finite subtrees.** We prove here Lemma 3.14, that is if $\mathcal{F} \in \mathfrak{B}$ and $\mathcal{H} \in \mathfrak{S}$, then for every sequence $(s_n)_{n<\omega}$ in $\mathcal{F}$ there is an infinite subset $M \subseteq \omega$ such that $\bigcup_{n \in x} s_n \in \mathcal{F} \times \mathcal{H}$ for every $x \in \mathcal{H} \upharpoonright M$. The proof is based on a combinatorial analysis of sequences of finite subtrees of T, done in the next Lemma 3.32, that uses crucially the Ramsey property. This relation between the Ramsey theory, uniform fronts and BQO-WQO theory of trees is well studied and has produced fundamental results like Kruskal Theorem [Kr] (see also Nash-Williams paper [Na1]) and Laver Theorem [La].

We start with some simple analysis of the tree generated by two finite subtrees τ_0 and τ_1.

Definition 3.23. *Given τ_0, τ_1 two finite subtrees of T and $t \in \tau_0 \cup \tau_1$, let*

$$\begin{aligned} i(t) :=& \min\{i \in 2 \,:\, t \in \tau_i\}, \\ \pi(\tau_0,\tau_1) :=& \{w \in \langle \tau_0 \cup \tau_1 \rangle \,:\, (\tau_0 \setminus \tau_1)_{\ge w} \neq \emptyset \text{ and } (\tau_1 \setminus \tau_0)_{\ge w} \neq \emptyset\}, \\ \sigma(\tau_0,\tau_1) :=& \pi(\tau_0,\tau_1)_{\max}, \\ \overline{\sigma}(\tau_0,\tau_1) :=& \{t_0 \wedge t_1 \,:\, t_0 \perp t_1 \text{ are in } \tau_0 \cup \tau_1 \text{ and } t_0 \wedge t_1 \notin \tau_0 \cup \tau_1\}. \end{aligned}$$

Definition 3.24. *For each $w \in \sigma(\tau_0,\tau_1)$, fix $t^0(w) \in (\tau_0 \setminus \tau_1)_{\ge w}$ and $t^1(w) \in (\tau_1 \setminus \tau_0))_{\ge w}$ such that $w = t^0(w) \wedge t^1(w) \notin \tau_0 \cup \tau_1$ and whenever $w \in \overline{\sigma}(\tau_0,\tau_1)$, then $t^0(w) \perp t^1(w)$*

Proposition 3.25. *$\overline{\sigma}(\tau_0,\tau_1) \subseteq \sigma(\tau_0,\tau_1)$.*

Proof. Clearly $\overline{\sigma}(\tau_0,\tau_1) \subseteq \pi(\tau_0,\tau_1)$, so given $w \in \overline{\sigma}(\tau_0,\tau_1)$, let us prove that w is maximal there, so that $w \in \sigma(\tau_0,\tau_1)$. Suppose otherwise that there is $w' \in \sigma(\tau_0,\tau_1)$ such that $w < w'$ and let us get a contradiction. If $\mathrm{Is}_w(t^0(w)) = \mathrm{Is}_w(w')$, then $w = t^0(w) \wedge t^1(w) = t^1(w') \wedge t^1(w) \in \tau_1$, a contradiction. Otherwise, $w = t^0(w) \wedge t^1(w) = t^0(w) \wedge t_0(w') \in \tau_0$, which contradicts the hypothesis. □

Given $t, u \in T$, let

$$t \wedge_{\mathrm{is}} u := \begin{cases} \min\{t,u\} & \text{if } t,u \text{ are comparable} \\ \mathrm{Is}_{t\wedge u}(t) & \text{if } t,u \text{ are incomparable.} \end{cases}$$

The following easy fact will be helpful:

Proposition 3.26. *For every finite set $s \subseteq T$, every finite subtree $\tau \subseteq T$ and every $t \in T$, we have that*

$$\begin{aligned} \langle s \rangle_{\mathrm{is}} &= \{t_0 \wedge t_1 : t_0, t_1 \in s\} \cup \{t_0 \wedge_{\mathrm{is}} t_1 : t_0, t_1 \in s, t_0 \perp t_1\}, \\ \langle \tau \rangle_{\mathrm{is}} &= \tau \cup \{t_0 \wedge_{\mathrm{is}} t_1 : t_0, t_1 \in \tau, t_0 \perp t_1\}, \\ \mathrm{Is}_t\text{''} s &= \mathrm{Is}_t\text{''} \langle s \rangle_{\mathrm{is}}. \end{aligned}$$

In particular, if $t \in \langle s \rangle_{\mathrm{is}}$, then $u \in (s)_{\ge t} \neq \emptyset$.

Definition 3.27. *Given two finite subtrees τ_0, τ_1, let*

$$\begin{aligned} \varrho :=& \ \varrho(\tau_0,\tau_1) = \tau_0 \cap \tau_1 \\ \overline{\varrho} :=& \ \overline{\varrho}(\tau_0,\tau_1) := \langle \tau_0 \rangle_{\mathrm{is}} \cap \langle \tau_1 \rangle_{\mathrm{is}} \\ \varrho_0 :=& \ \overline{\varrho} \cup \{0\} \\ (\tau_0,\tau_1)_\infty :=& \ \{u \in (\varrho_0)_{\max} \,:\, (\pi(\tau_0,\tau_1))_{\ge u} \neq \emptyset\}. \end{aligned}$$

Proposition 3.28. *For every $u \in (\tau_0,\tau_1)_\infty$ one has that $\#(\sigma(\tau_0,\tau_1))_{\ge u} = 1$.*

Proof. If $u \in (\tau_0,\tau_1)_\infty$, then clearly $(\sigma(\tau_0,\tau_1))_{\ge u} \neq \emptyset$. Suppose there are $w_0 \neq w_1 \in (\sigma(\tau_0,\tau_1))_{\ge u}$ for some $u \in (\tau_0,\tau_1)_\infty$. Observe that $w_0 \perp w_1$, since both of them are maximal in $\pi(\tau_0,\tau_1)$. Hence,

$$u \le t^0(w_0) \wedge t^0(w_1) = t^1(w_0) \wedge t^1(w_1),$$

so that

$$u < t^0(w_0) \wedge_{\rm is} t^0(w_1) = t^1(w_0) \wedge_{\rm is} t^1(w_1) \in \langle \tau_0 \rangle_{\rm is} \cap \langle \tau_1 \rangle_{\rm is} = \overline{\varrho} \subseteq \varrho_0,$$

contradicting the maximality of u in ϱ_0. □

Definition 3.29. *For every $u \in (\tau_0, \tau_1)_\infty$, let $\varpi_{\tau_0,\tau_1}(u)$ be the unique element of $(\sigma(\tau_0, \tau_1))_{\geq u}$.*

Proposition 3.30. *For every $w \in \pi(\tau_0, \tau_1)$, either there is $u \in (\tau_0, \tau_1)_\infty$ such that $w \leq \varpi_{\tau_0,\tau_1}(u)$, or else there is $u \in (\varrho_0)_{\max}$ such that $w < u$. Consequently, $\overline{\sigma}(\tau_0, \tau_1) \subseteq \mathrm{ran}\,(\varpi_{\tau_0,\tau_1})$.*

Proof. Given $w \in \pi(\tau_0, \tau_1)$, suppose there is no $u \in (\varrho_0)_{\max}$ such that $w < u$ and let

$$u := \max\{v \in \varrho_0 : v \leq w\}.$$

Let us prove that u is maximal in ϱ_0 so that w witnesses that $u \in (\tau_0, \tau_1)_\infty$. Suppose by contradiction that there is $v \in (\varrho_0)_{\max}$ such that $u < v$ and in particular, $v \in \overline{\varrho}$. Notice that the definition of u implies that $v \perp w$. Hence, $u \leq w \wedge v < w$, so that $u < w \wedge_{\rm is} v \leq w$. But $w \wedge_{\rm is} v = t^0(w) \wedge_{\rm is} v = t^1(w) \wedge_{\rm is} v$, so that $w \wedge_{\rm is} v \in \overline{\varrho}$ and we get a contradiction with the maximality of u below w. It follows that $u \in (\tau_0, \tau_1)_\infty$ and $w \leq \varpi_{\tau_0,\tau_1}(u)$.

Finally, suppose that $w \in \overline{\sigma}(\tau_0, \tau_1)$. Then, $w \in \sigma(\tau_0, \tau_1)$, by Proposition 3.25. It is easy to see from the definition of $\overline{\sigma}(\tau_0, \tau_1)$ that there is no $u \in (\varrho_0)_{\max}$ such that $w < u$. Hence $w \leq \varpi_{\tau_0,\tau_1}(u)$ for some $u \in (\tau_0, \tau_1)_\infty$ and it follows from the maximality of w that $w = \varpi_{\tau_0,\tau_1}(u)$. □

The following result guarantees that the new points of the tree generated by two finite subtrees τ_0 and τ_1 are given by the function ϖ_{τ_0,τ_1} and hence, they are controlled by the maximal elements of $(\tau_0, \tau_1)_\infty$.

Corollary 3.31. $\langle \tau_0 \cup \tau_1 \rangle = \tau_0 \cup \tau_1 \cup \{\varpi_{\tau_0,\tau_1}(u) : u \in (\tau_0, \tau_1)_\infty\}$.

Proof. If $w \in \langle \tau_0 \cup \tau_1 \rangle \setminus (\tau_0 \cup \tau_1)$, then there are $t_0 \in \tau_0 \setminus \tau_1$ and $t_1 \in \tau_1 \setminus \tau_0$ such that $w = t_0 \wedge t_1$ and notice that $t_0 \wedge t_1 \in \overline{\sigma}(\tau_0, \tau_1)$. Then, by Proposition 3.30, there is $u \in (\tau_0, \tau_1)_\infty$ such that $t_0 \wedge t_1 = \varpi_{\tau_0,\tau_1}(u)$. The other inclusion follows directly from the definitions. □

To prove that $\times$ is a multiplication we have to deal with the tree generated by a sequence of finite subtrees. Given a sequence $(\tau_k)_k$ of finite subtrees and $M \subseteq \omega$, let τ_M be the subtree generated by $\bigcup_{k \in M} \tau_k$. In order to be able to control the chains and the immediate successors of some τ_M, we will first find some suitable infinite M such that the subsequence $(\tau_k)_{k \in M}$ has some uniformity respective to the new points of τ_M. This is the content of the next result, that can be seen as a generalization of Proposition 3.5, which guarantees the existence of an infinite fan, chain or comb inside any infinite subset of a tree.

If we assume each τ_k to be a singleton $\{t_k\}$ and apply Proposition 3.5 to get an infinite M such that $\{t_k : k \in M\}$ is a fan, a chain or a comb, then the corresponding tree τ_M is given by $\{t_k : k \in M\} \cup \{w\}$, $\{t_k : k \in M\}$ or $\{t_k : k \in M\} \cup \{w_k : k \in M\}$, respectively. The case (2.1) corresponds to $\{t_k : k \in M\}$ being a comb, so that the new points $\{\varpi_k : k \in M\}$ form a chain; case (2.2) corresponds to $\{t_k : k \in M\}$ being a fan with root w which is the only new point; case (2.3) corresponds to $\{t_k : k \in M\}$ being a chain and no new points ($\varpi_k = t_k$); and case (2.4) corresponds to $t_k = t_{k'}$ for all $k, k' \in M$.

In the next result, after refining the sequence to get a fixed τ_∞ and $\varpi_i(u) := \varpi_{i,j}(u) = \varpi_{i,k}(u)$ for $i < j < k$, each of these four cases might happen for each of the sequences of points $(\varpi_k(u))_{k\in M}$.

Theorem 3.32 (Canonical form of sequences of subtrees)**.** *Suppose that $(\tau_k)_k$ is a sequence of finite subtrees of T forming a Δ-system with root ϱ and such that $(\langle\tau_k\rangle_{\mathrm{is}})_k$ forms a Δ-system with finite root $\overline{\varrho}$. Then there is a subsequence $(\tau_k)_{k\in M}$ such that*

(1) *For every $i \neq j$ and $k \neq l$ in M one has that*

$$\tau_\infty :=(\tau_i, \tau_j)_\infty = (\tau_k, \tau_l)_\infty \tag{10}$$

(2) *Let $u \in \tau_\infty$. For each $i < j$ write $\varpi_{i,j}(u) := \varpi_{\tau_i,\tau_j}(u)$. Then $\varpi_i(u) := \varpi_{i,j}(u) = \varpi_{i,k}(u)$ for every $i < j < k$, and $\varpi_i(u) \leq \varpi_j(u)$ for every $i \leq j$.*
Moreover, one of the following holds.
(2.1) *$\varpi_i(u) < \varpi_j(u)$ for every $i < j$ and $\varpi_i(u) \notin \bigcup_k \tau_k$ for every $i < j$.*
(2.2) *$w(u) := \varpi_i(u) = \varpi_j(u) \notin \bigcup_k \tau_k$ for every i.*
(2.3) *$\varpi_i(u) < \varpi_j(u)$ and $\varpi_i(u) \in \tau_i \setminus \varrho$ for every $i < j$.*
(2.4) *$w(u) := u = \varpi_i(u) = \varpi_j(u) \in \varrho$ for every $i < j$.*

Definition 3.33. *We call a Δ-sequence $(\tau_k)_k$ of root ϱ such that $(\langle\tau_k\rangle_{\mathrm{is}})_k$ is a Δ-sequence of root $\overline{\varrho}$ satisfying (1) and (2) of Theorem 3.32 above a* well-placed *sequence. In this case, let τ^0_∞ be the set of those $u \in \tau_\infty$ such that $w(u) = \varpi_i(u) = \varpi_j(u)$ for every $i < j$ in ω and let $\tau^1_\infty := \tau_\infty \setminus \tau^0_\infty$. For each $u \in \tau^1_\infty$, let $z(u) := \sup_{i\in\omega} \varpi_i(u)$.*

Given $I \subseteq \omega$, we use the terminology τ_I to denote $\langle\bigcup_{k\in I} \tau_k\rangle$.

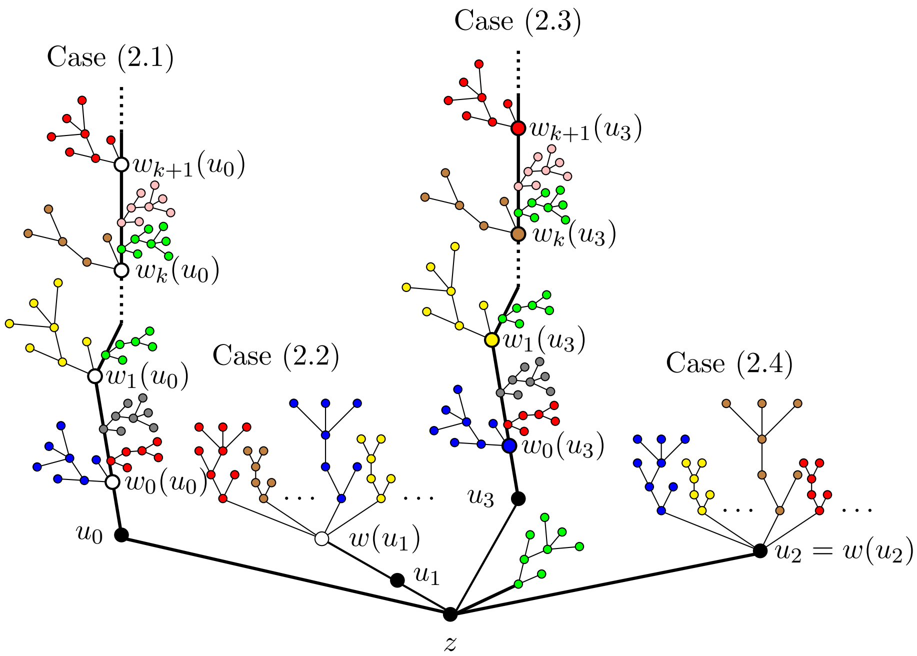

FIGURE 3. A well placed sequence $(\tau_k)_k$.

Each color in the figure corresponds to one of the elements of the sequence: blue nodes belong to the subtree τ_0, yellow nodes to τ_1, and so on. Black is used to denote elements of the extended root $\overline{\varrho}$ and white, to nodes not belonging to any of the subtrees.

Proof of Theorem 3.32. We will apply the Ramsey Theorem and refine the sequence $(\tau_k)_{k\in\omega}$ finitely many times in order to get the desired subsequence $(\tau_k)_{k\in M}$.

First, since $\overline{\varrho}$ is finite and each $(\tau_i,\tau_j)_\infty \subseteq \varrho_0 = \overline{\varrho}\cup\{0\}$, it follows from the Ramsey Theorem that we may assume, by passing to a subsequence $(\tau_k)_{k\in M}$, that (1) holds.

Now fix $u \in \tau_\infty$. Applying the Ramsey Theorem and passing again to a subsequence, we may assume that exactly one of the following holds:

(a1) $\varpi_{i,j}(u) \notin \varrho$ for every $i<j$ in M.
(b1) $\varpi_{i,j}(u) \in \varrho$ for every $i<j$ in M.

If (b1) holds, since ϱ is finite, we may pass to a further subsequence and get that $\varpi_{i,j}(u) = \varpi_{k,l}(u)$ for every $i<j$ and $k<l$ in M. In particular we get (2.4).

From now on we will assume that (a1) holds and prove that we have one of the other three cases (2.1), (2.2) or (2.3).

Claim 3.33.1. *For every $i<j$ in M and $k\in M\setminus\{i,j\}$, one has that $\varpi_{i,j}(u)\notin\tau_k$.*

Proof of Claim: We color a triple $i<j<k$ by 0 if $\varpi_{i,j}(u)\in\tau_k$, by 1 if $\varpi_{i,j}(u)\notin\tau_k$ and $\varpi_{i,k}(u)\in\tau_j$, by 2 if $\varpi_{i,j}(u)\notin\tau_k$, $\varpi_{i,k}(u)\notin\tau_j$ and $\varpi_{j,k}(u)\in\tau_i$ and by 3 otherwise. By the Ramsey Theorem, we may assume that all triples in M are monochromatic. We prove that its color is 3. In the other two cases, there are $i<j$ and $k\neq l$ such that

$$\varpi_{i,j}(u)\in\tau_k\cap\tau_l=\varrho$$

which contradicts (a1). □

For each $i<j$, let $t^i_{i,j}(u)\in\tau_i\setminus\varrho$ and $t^j_{i,j}(u)\in\tau_j\setminus\varrho$ be such that

$$\varpi_{i,j}(u)=t^i_{i,j}(u)\wedge t^j_{i,j}(u).$$

Since each τ_i is finite and $t^i_{i,j}(u)\in\tau_i$, we may assume by the Ramsey Theorem that

$$t_i(u):=t^i_{i,j}(u)=t^i_{i,k}(u) \text{ for every } i<j<k.$$

Hence, for each $i<j<k$ in M, $\varpi_{i,j}(u)$ and $\varpi_{i,k}(u)$ are comparable since they are both below $t_i(u)$.

Claim 3.33.2. *By passing to an infinite subset of M, we may assume that $\varpi_{i,j}(u)=\varpi_{i,k}(u)$ for every $i<j<k$ in M.*

Proof of Claim: By the Ramsey Theorem, we may assume that one of the following holds:

(a2) $\varpi_{i,j}(u)=\varpi_{i,k}(u)$ for every $i<j<k$ in M.
(b2) $\varpi_{i,j}(u)<\varpi_{i,k}(u)$ for every $i<j<k$ in M.
(c2) $\varpi_{i,j}(u)>\varpi_{i,k}(u)$ for every $i<j<k$ in M.

Notice that (c2) is impossible, since trees have no infinite strictly decreasing chains. We claim that (b2) is also impossible and therefore, (a2) holds. Given $i < j < k$, if (b2) holds, then $\varpi_{i,j}(u) \le t^j_{i,j}(u), t^k_{i,k}(u)$, so that $\varpi_{i,j}(u) \in \pi(\tau_j, \tau_k)$. By the maximality of $\varpi_{j,k}(u)$ in $\pi(\tau_j, \tau_k)$, we get that $\varpi_{i,j}(u) \le \varpi_{j,k}(u)$. Then, $\varpi_{i,j}(u) \le \varpi_{i,k}(u) \wedge \varpi_{j,k}(u)$.

If $\varpi_{i,j}(u) < \varpi_{i,k}(u) \wedge \varpi_{j,k}(u)$, then the fact that $\varpi_{i,k}(u) \wedge \varpi_{j,k}(u) \in \pi(\tau_i, \tau_j)$ contradicts the fact that $\varpi_{i,j}(u)$ is maximal in $\pi(\tau_i, \tau_j)$. If $\varpi_{i,j}(u) = \varpi_{i,k}(u) \wedge \varpi_{j,k}(u)$, then we get that $\varpi_{i,j} \in \tau_k$, which is a contradiction with Claim 3.33.1. Therefore, (b2) cannot be true and we conclude that (a2) holds. □

Let now $\varpi_i(u) := \varpi_{i,j}(u)$ for every $i < j$.

Claim 3.33.3. *By passing to an infinite subset of M, we may assume that either $\varpi_i(u) = \varpi_j(u)$ for every $i < j$ in M, or $\varpi_i(u) < \varpi_j(u)$ for every $i < j$ in M.*

Proof of Claim: By the Ramsey Theorem, we may assume that one of the following holds:

(a3) $\varpi_i(u) = \varpi_j(u)$ for every $i < j$ in M.
(b3) $\varpi_i(u) < \varpi_j(u)$ for every $i < j$ in M.
(c3) $\varpi_i(u) > \varpi_j(u)$ for every $i < j$ in M.
(d3) $\varpi_i(u)$ and $\varpi_j(u)$ are incompatible for every $i < j$ in M.

Again (c3) is impossible, since trees have no infinite strictly decreasing chains. We claim that (d3) is also impossible and therefore, either (a3) or (b3) holds. If (d3) holds, then we have that for $i < j < k < l$, $u \le \varpi_i(u) = t^k_{i,k}(u) \wedge t^k_{j,k}(u) < t^k_{i,k}(u) \wedge_{\mathrm{is}} t^k_{j,k}(u) \in \langle \tau_k \rangle_{\mathrm{is}}$ and $t^k_{i,k}(u) \wedge_{\mathrm{is}} t^k_{j,k}(u) = t^l_{i,l}(u) \wedge_{\mathrm{is}} (u) t^l_{j,l} \in \langle \tau_l \rangle_{\mathrm{is}}$, contradicting the maximality of u in ϱ_0. Hence, either (a3) or (b3) holds. □

In any case, by Claim 3.33.1 we may assume that $\varpi_i(u) \notin \tau_k$ for $i \ne k$. Hence, by the Ramsey Theorem, we may assume that one of the following holds:

(a4) $\varpi_i(u) \notin \bigcup_k \tau_k$ for every i in M.
(b4) $\varpi_i(u) \in \tau_i \setminus \varrho$ for every i in M.

Now, if (a3) holds, (b4) cannot hold and we get that u satisfies (2.2). If (b3) and (a4) hold, we get (2.1) and if (b3) and (b4) hold, then we get (2.3). This concludes the proof of the theorem. □

Corollary 3.34. *Given a well-placed sequence $(\tau_k)_{k<\omega}$, $I \subseteq \omega$ and $u \in \tau^1_\infty$, we have that:*

(i) *For every $t \in [u, z(u)]$, there is $i \in I$ such that if $t' \in (\tau_I)_{>t}$ with $\mathrm{Is}_t(t') \ne \mathrm{Is}_t(z(u))$, then $t' \in \tau_i$.*
(ii) $(\tau_I)_{<z(u)} \subseteq \bigcup_{k \in I} (\tau_k \cup \{\varpi_k(u)\})$.

Proof. (i): Let $u \le t \le z(u)$ and suppose there are $i_0 < i_1$ in I and $t_j \in (\tau_{i_j} \setminus \varrho)_{>t}$ such that $\mathrm{Is}_t(t_j) \ne \mathrm{Is}_t(z(u))$, $j = 0, 1$. Let $w = t_0 \wedge t_1$ and notice that $w \in \pi(\tau_{i_0}, \tau_{i_1})$. By Proposition 3.30, either there is $v \in \tau_\infty$ such that $w \le \varpi_{i_0,i_1}(v)$ or there is $v \in (\varrho_0)_{\max}$ such that $w < v$. Since $u \in (\varrho)_{\max}$ and $u \le t \le w$, the second alternative cannot hold and the first alternative holds with $u = v$. Since $w \le \varpi_{i_0,i_1}(u) = \varpi_{i_0}(u) < z(u)$, it follows that $w = t$. But then, $t = \varpi_{i_1}(u) \wedge t_1 \in \tau_{i_1}$, which is impossible both in cases (2.1) and (2.3). Finally, notice that if

$t' \in (\tau_I)_{>t} \setminus \bigcup_{k\in I} \tau_k$ is such that $\mathrm{Is}_t(t') \neq \mathrm{Is}_t(z(u))$, then there are $i_0 < i_1$ in I and $t_j \in (\tau_{i_j} \setminus \varrho)_{>t'}$, so that that $\mathrm{Is}_t(t_j) = \mathrm{Is}_t(t') \neq \mathrm{Is}_t(z(u))$, $j = 0, 1$, which we just proved that cannot happen.

(ii): If $t \in \tau_I \setminus \bigcup_{k\in I} \tau_k$, by Corollary 3.31, there are $i_0 < i_1$ in I such that $t = \varpi_{i_0,i_1}(v)$ for some $v \in \tau_\infty$. If $t \leq z(u)$, the maximality of u and v guarantee that $u = v$. Hence, $t = \varpi_{i_0,i_1}(u) = \varpi_{i_0}(u)$ □

Corollary 3.35. *Given a well-placed sequence $(\tau_k)_{k<\omega}$, $I \subseteq \omega$ and $u \in \tau_\infty^0$, we have that:*

(i) *For every $t \in \mathrm{Is}_{w(u)}$, there is a $i \in I$ such that $(\tau_I)_{\geq t} \subseteq \tau_i$.*
(ii) *For every $t \in [u, w(u)[$, there is $i \in I$ such that if $t' \in (\tau_\omega)_{>t}$ with $\mathrm{Is}_t(t') \neq \mathrm{Is}_t(w(u))$, then $t' \in \tau_i$.*
(iii) $(\tau_I)_{<w(u)} \subseteq \bigcup_{k\in I} \tau_k$.

Proof. (i): Let $t \in \mathrm{Is}_{w(u)}$ and suppose there are $i_0 < i_1$ in I and $t_j \in (\tau_{i_j} \setminus \varrho)_{\geq t}$, $j = 0, 1$. Let $w = t_0 \wedge t_1$ and notice that $w \in \pi(\tau_{i_0}, \tau_{i_1})$. By Proposition 3.30, either there is $v \in \tau_\infty$ such that $w \leq \varpi_{i_0,i_1}(v)$ or there is $v \in (\varrho_0)_{\max}$ such that $w < v$. Since $u \in (\varrho)_{\max}$ and $u \leq w(u) < t \leq w$, the second alternative cannot hold and the first alternative holds with $u = v$, which cannot hold as well, since $\varpi_{i_0}(u) = w(u) < t \leq w$. Finally, notice that if $t' \in (\tau_I)_{>t} \setminus \bigcup_{k<\omega} \tau_k$, then there are $i_0 < i_1$ in I and $t_j \in (\tau_{i_j} \setminus \varrho)_{>t'}$, so that that $t_j > t$, $j = 0, 1$, which we just proved that cannot happen.

(ii): Given $t \in [u, w(u)[$, suppose there are $i_0 < i_1$ in I and $t_j \in (\tau_{i_j} \setminus \varrho)_{>t}$ such that $\mathrm{Is}_t(t_j) \neq \mathrm{Is}_t(w(u))$, $j = 0, 1$. Let $w = t_0 \wedge t_1$ and notice that $w \in \pi(\tau_{i_0}, \tau_{i_1})$. By Proposition 3.30, either there is $v \in \tau_\infty$ such that $w \leq \varpi_{i_0,i_1}(v)$ or there is $v \in (\varrho_0)_{\max}$ such that $w < v$. Since $u \in (\varrho)_{\max}$ and $u \leq t \leq w$, the second alternative cannot hold and the first alternative holds with $u = v$. Since $w \leq \varpi_{i_0,i_1}(u) = \varpi_{i_0}(u) = w(u)$, it follows that $w = t = w(u)$, a contradiction. Finally, notice that if $t' \in (\tau_I)_{>t} \setminus \bigcup_{k\in I} \tau_k$ is such that $\mathrm{Is}_t(t') \neq \mathrm{Is}_t(w(u))$, then there are $i_0 < i_1$ in I and $t_j \in (\tau_{i_j} \setminus \varrho)_{>t'}$, so that that $\mathrm{Is}_t(t_j) = \mathrm{Is}_t(t') \neq \mathrm{Is}_t(w(u))$, $j = 0, 1$, which we just proved that cannot happen.

(iii): If $t \in \tau_I \setminus \bigcup_{k\in I} \tau_k$, by Corollary 3.31, there are $i_0 < i_1$ in I such that $t = \varpi_{i_0,i_1}(v)$ for some $v \in \tau_\infty$. Hence, if $t < w(u)$, the maximality of u and v guarantee that $u = v$. Hence, $t = \varpi_{i_0,i_1}(u) = \varpi_{i_0}(u) = w(u)$, a contradiction. □

Corollary 3.36. *Let $(\tau_k)_{k\in\omega}$ be a well-placed sequence of finite subtrees of T. There are finite sets $\mathfrak{a}, \mathfrak{f} \subseteq T$ and, for each $z \in \mathfrak{f}$, there is a chain $\{\varpi_i(z) : i \in \omega\}$ such that for any finite $\emptyset \neq I \subseteq \omega$, the following hold:*

1. *For every $t \in \tau_I$, there are $z \in \mathfrak{f}$ and $i \in I$ such that*
$$\#\left([t \wedge z, t] \cap (\tau_I \setminus \tau_i)\right) \leq 1.$$
2. *For every $z \in \mathfrak{f}$,*
$$[0, z] \cap (\tau_I \setminus \bigcup_{i\in I} \tau_i) \subseteq \{\varpi_i(z) : i \in I\}.$$
3. *For every $t \in \tau_I \setminus \mathfrak{a}$, there is $i \in I$ such that*
$$\#(\mathrm{Is}_t(\tau_I) \setminus \mathrm{Is}_t(\tau_i)) \leq 1.$$

PROOF. Let

$$\mathfrak{f} :=((\varrho_0)_{\max} \setminus \tau_\infty) \cup \{w(u) \,:\, u \in \tau_\infty^0\} \cup \{z(u) \,:\, u \in \tau_\infty^1\}$$

and let us prove property 1.

Given $t \in \tau_I$, if there is $u \in (\varrho_0)_{\max}$ such that $t \leq u$, then either $u \in \mathfrak{f}$, or $w(u) \in \mathfrak{f}$ or $z(u) \in \mathfrak{f}$, depending on whether $u \in (\varrho_0)_{\max} \setminus \tau_\infty$, $u \in \tau_\infty^0$ or $u \in \tau_\infty^1$, respectively. In any case, there is $z \in \mathfrak{f}$ such that $t \wedge z = t$ so that property 1 holds trivially. Otherwise, there is a unique $u \in (\varrho_0)_{\max}$ such that $u < t$. In case $u \notin \tau_\infty$, it follows from the definition of τ_∞ that $(\pi(\tau_i, \tau_j))_{\geq u} = \emptyset$ for every $i \neq j$ in I, which implies property 1. If $u \in \tau_\infty^1$, then $z(u) \in \mathfrak{f}$ and $t \wedge z(u) \in [u, z(u)]$. Then, Corollary 3.34.(i) guarantees that there is $i \in \omega$ such that $]t \wedge z(u), t] \subseteq \tau_i$, so that property 1 holds. Finally, if $u \in \tau_\infty^0$, then $w(u) \in \mathfrak{f}$ and $u \leq t \wedge w(u) \leq w(u)$. If $t \wedge w(u) < w(u)$, then Corollary 3.35.(ii) guarantees that there is $i \in \omega$ such that $]t \wedge w(u), t] \subseteq \tau_i$, so that condition 1 holds. If $t \wedge w(u) = w(u)$, it follows that $w(u) \leq t$ and the case when $t = w(u)$ is trivial. If $w(u) < t$, Corollary 3.35.(i) applied to $\mathrm{Is}_{w(u)}(t)$ guarantees that there is $i \in \omega$ such that $]t \wedge w(u), t] \subseteq \tau_i$, so that property 1 holds and this concludes the proof of condition 1.

To prove property 2, given $z \in \mathfrak{f}$, let us consider three different cases. If $z = z(u)$ for some $u \in \tau_\infty^1$, let $(\varpi_i(z))_{i\in\omega}$ be the sequence $(\varpi_i(u))_{i\in\omega}$ and if $z = w(u)$ for some $u \in \tau_\infty^0$, let $(\varpi_i(z))_{i\in\omega}$ be the constant sequence equal to $\varpi_i(u) = w(u)$. Finally, if $z \in (\varrho_0)_{\max} \setminus \tau_\infty$, let $(\varpi_i(z))_{i\in\omega}$ be the constant sequence equal to z

Now, given $t \in \tau_I \setminus \bigcup_{i\in I} \tau_i$, by Corollary 3.31 there are $i_0 < i_1$ in I such that $t = \varpi_{i_0,i_1}(v) = \varpi_{i_0}(v)$ for some $v \in \tau_\infty$ since the sequence is well-placed. Then, if $t \leq z$, the maximality of v implies that $z \in \tau_\infty$ and since u is also maximal, $u = v$ and $t = \varpi_{i_0}(v) = \varpi_{i_0}(u)$, which concludes the proof of property 2.

It remains to prove property 3. Let

$$\mathfrak{a} :=\varrho_0 \cup \{w(u) \,:\, u \in \tau_\infty^0\}$$

and let $t \in \tau_I \setminus \mathfrak{a}$ and $u \in (\varrho_0)_{\max}$ be such that $u < t$ or $t < u$ (the equality cannot hold since $\varrho_0 \subseteq \mathfrak{a}$ and $t \notin \mathfrak{a}$).

If $t < u$, and there are $i_0 < i_1$ such that $\mathrm{Is}_t\text{"}(\tau_{i_j}) \setminus \{\mathrm{Is}_t(u)\} \neq \emptyset$ for $j = 0, 1$, then choose $t_j\text{"} \in \mathrm{Is}_t(\tau_{i_j}) \setminus \{\mathrm{Is}_t(u)\}$, $j = 0, 1$. Then, $t_0, t_1 \perp u$, hence $\tau_{i_0} \ni t_0 \wedge u = t = t_1 \wedge u \in \tau_{i_1}$, and so $t \in \varrho$, which is impossible.

Suppose now that $u < t$. If $u \notin \tau_\infty$, then property 2 holds trivially. If $u < t$ and $u \in \tau_\infty^0$, we have to consider that cases $t < w(u)$ and $w(u) < t$ (again the equality cannot hold since $w(u) \in \mathfrak{a}$ and $t \notin \mathfrak{a}$). If $t < w(u)$, then Corollary 3.35.(ii) implies that $\mathrm{Is}_t\text{"}(\tau_I) = \mathrm{Is}_t\text{"}(\tau_i) \cup \{\mathrm{Is}_t(w(u))\}$ for some $i \in I$, so that property 2 holds. If $w(u) < t$, then Corollary 3.35.(i) applied to $\mathrm{Is}_{w(u)}(t)$ implies that $\mathrm{Is}_t\text{"}(\tau_I) = \mathrm{Is}_t(\tau_i)$ for some $i \in I$, so that property 2 holds. If $u < t$ and $u \in \tau_\infty^1$, then Corollary 3.34.(i) implies that $\mathrm{Is}_t\text{"}(\tau_I) = \mathrm{Is}_t(\tau_i) \cup \{\mathrm{Is}_t(z(u))\}$ for some $i \in I$, so that property 2 holds, including in case $z(u) < t$. $\square$

PROOF OF LEMMA 3.14. Fix $\mathcal{F} \in \mathfrak{B}$ and $\mathcal{H} \in \mathfrak{S}$. Set $\mathcal{A} := \langle\mathcal{F}\rangle \cap \mathrm{Ch}_a$, $\mathcal{C} := \langle\mathcal{F}\rangle \cap \mathrm{Ch}_c$ and $\mathcal{G} := \mathcal{F} \times \mathcal{H}$. Then, $\mathcal{F} \subseteq \mathcal{A} \odot_T \mathcal{C}$. Then $\mathcal{G} = ((\mathcal{A} \times_a \mathcal{H}) \sqcup_a [T]^{\leq 1}) \odot_T ((\mathcal{C} \times_c \mathcal{H}) \boxtimes_c 5)$. Suppose that Ch_a and Ch_c are not compact. By Proposition 3.12,

$$\lambda := \iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}(\langle\mathcal{F}\rangle)) = \iota(\mathrm{srk}_a(\mathcal{A})) = \iota(\mathrm{srk}_c(\mathcal{C})). \tag{11}$$

It follows from this, the property (M.1) of $\times_a$ and $\times_c$, and (11) that

$$\begin{aligned}\iota(\mathrm{srk}_a(\mathcal{A}\times_a \mathcal{H})\sqcup_a [T]^{\le 1}) =&\iota(\mathrm{srk}_a(\mathcal{A}\times_a \mathcal{H})) = \iota(\mathrm{srk}_a(\mathcal{A})\cdot \mathrm{srk}(\mathcal{H})) = \max\{\lambda, \iota(\mathrm{srk}(\mathcal{H}))\},\\ \iota(\mathrm{srk}_c(\mathcal{C}\times_c \mathcal{H})\boxtimes_c 5) =&\iota(\mathrm{srk}_a(\mathcal{C}\times_c \mathcal{H})) = \iota(\mathrm{srk}_c(\mathcal{C})\cdot \mathrm{srk}(\mathcal{H})) = \max\{\lambda, \iota(\mathrm{srk}(\mathcal{H}))\},\end{aligned} \quad (12)$$

so, by Proposition 3.15 we obtain that $\mathcal{G}\in\mathfrak{B}$. By Proposition 3.10,

$$\begin{aligned}\mathrm{srk}(\mathcal{F})\cdot\mathrm{srk}(\mathcal{H}) \le& \mathrm{srk}(\mathcal{A}\odot_T \mathcal{C})\cdot \mathrm{srk}(\mathcal{H}) \le \min\{\mathrm{srk}_a(\mathcal{A}), \mathrm{srk}_c(\mathcal{C})\}\cdot \mathrm{srk}(\mathcal{H})\le\\ &\le \min\{\mathrm{srk}_a(\mathcal{A}\times_a\mathcal{H}), \mathrm{srk}_c(\mathcal{C}\times_c \mathcal{H})\}\le \mathrm{srk}(\mathcal{G}).\end{aligned}$$

On the other hand, by Remark 3.9, $\mathcal{G}\cap \mathrm{Ch}_a = (\mathcal{A}\times_a \mathcal{H})\sqcup_a [T]^{\le 1}$, and $\mathcal{G}\cap\mathrm{Ch}_c = (\mathcal{C}\times_c\mathcal{H})\boxtimes_c 5$. By Proposition 3.12 and the above, $\iota(\mathrm{srk}(\mathcal{G})) = \max\{\lambda, \mathrm{srk}(\mathcal{H})\} = \iota(\mathrm{srk}(\mathcal{F})\cdot\mathrm{srk}(\mathcal{H}))$. Hence, $\mathrm{srk}(\mathcal{G}) < \iota(\mathrm{srk}(\mathcal{F})\cdot \mathrm{srk}(\mathcal{H}))$. The cases when Ch_a or Ch_c are compact are proved similarly. We only check that $\mathcal{G}\in\mathfrak{B}$ when Ch_a is compact. We know that $\mathrm{rk}((\mathcal{A}\times_a\mathcal{H})\sqcup_a [T]^{\le 1})\le \mathrm{rk}(\mathcal{A})+1$. Since $\lambda := \iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}(\langle\mathcal{F}\rangle)) = \iota(\mathrm{srk}_c(\mathcal{C}))$, it follows that $\mathrm{rk}((\mathcal{A}\times_a\mathcal{H})\sqcup_a [T]^{\le 1}) < \lambda$. By this, (12), Lemma 3.13 and Proposition 3.12,

$$\mathrm{rk}(\langle\mathcal{G}\rangle) = \mathrm{rk}(\mathcal{G}) < \max\{\lambda, \mathrm{srk}(\mathcal{H})\} = \mathrm{srk}(\mathcal{G}) = \mathrm{srk}(\langle\mathcal{G}\rangle).$$

This ends the proof of property (M.1) for $\times$. Let us prove now (M.2) for $\times$. In any of the three possible cases, depending on the compactness of Ch_a and Ch_c, we have that $\mathcal{F}\times\mathcal{H} = \mathcal{A}\odot_T\mathcal{C}$.

Claim 3.36.1. *The family $\langle\mathcal{F}\rangle_{\mathrm{is}} := \{\langle s\rangle_{\mathrm{is}} : s\in\mathcal{F}\}$ is compact.*

Proof of Claim: It is clear that an accumulation point A of the family $\langle\mathcal{F}\rangle_{\mathrm{is}}$ is a subtree closed under is. Going towards a contradiction, we suppose that A is infinite subtree. Then by Corollary 3.6 it contains an infinite chain or an infinite fan. It cannot contain an infinite fan F because then this would be an accumulation point of $\mathcal{F}\cap\mathrm{Ch}_a$. Suppose that $C = t_{n n<\omega}$ is a chain included in A, $t_n < t_{n+1}$. The only non-trivial case is when $t_n = \mathrm{Is}_{u_n}(t_{n+1})$ for some infinite chain $D = \{u_n\}_{n<\omega}$. Then, D is an accumulation point of $\langle\mathcal{F}\rangle\cap\mathrm{Ch}_c$, and this is impossible. □

Let $(\tau_k)_k$ be a sequence in $\langle\mathcal{F}\rangle$. We may assume by the previous claim that each τ_k is a tree and that the sequence $(\langle\tau_k\rangle_{\mathrm{is}})_k$ is a Δ-sequence. By Corollary 3.36 there is a subsequence $(\sigma_k)_{k<\omega}$ of $(\tau_k)_k$ and there are $\mathfrak{a},\mathfrak{f}$ finite subsets of T such that 1., 2. and 3. there hold. By refining the subsequence $(\nu_k)_{k<\omega}$ finitely many times, we may assume that for every $I\in\mathcal{H}$, we have that:

(i) For all $z\in\mathfrak{f}$, $\{w_i(z) : i\in I\}\in(\langle\mathcal{F}\rangle\cap\mathrm{Ch}_c)\times_c\mathcal{H}$.
(ii) For all $z\in\mathfrak{f}$, $\bigcup_{i\in I}(\nu_i\cap[0,z])\in(\langle\mathcal{F}\rangle\cap\mathrm{Ch}_c)\times_c\mathcal{H}$.
(iii) For all $t\in\mathfrak{a}$, $\bigcup_{i\in I}\mathrm{Is}_t"\nu_i\in(\langle\mathcal{F}\rangle\cap\mathrm{Ch}_a)\times_a\mathcal{H}$.

For (i) we use that $[T]^{\le 1}\subseteq\mathcal{F}$. Fix $I\in\mathcal{H}\restriction M$ and let us prove that $\nu_I\in\mathcal{F}\times\mathcal{H}$, which is enough to guarantee that $\bigcup_{i\in I}\nu_i\in\mathcal{F}\times\mathcal{H}$, since this family is hereditary.

Claim 3.36.2. *For every $t\in\nu_I$ one has that $(\nu_I)_{\le t}\in\mathcal{C}$.*

Proof of Claim: Given $t\in\nu_I$, by property 1. of the sequence $(\nu_k)_{k\in M}$, there are $z\in\mathfrak{f}$, $\bar{t}\in\nu_M$ and $i\in M$ such that

$$(\nu_M)\cap[t\wedge z, t]\subseteq\nu_i\cup\{\bar{t}\}.$$

Then, the property 2. of $(\nu_k)_{k\in M}$ implies that

$$\nu_I \cap [0,z] \subseteq (\bigcup_{i\in I}(\nu_i \cap [0,z])) \cup \{w_i(z) : i \in I\} \cup \{z\}.$$

Hence,

$$\nu_I \cap [0,t] \subseteq \nu_I \cap [t\wedge z, t] \cup (\nu_I \cap [0,z]) \subseteq (\nu_i \cap [t\wedge z, t]) \cup (\bigcup_{i\in I}(\nu_i \cap [0,z])) \cup \{w_i(z) : i \in I\} \cup \{\bar{t}, z\}.$$

Now notice that

- $\nu_i \cap [t\wedge z, t] \in \langle \mathcal{F} \rangle \cap \mathrm{Ch}_c \subseteq (\langle \mathcal{F} \rangle \cap \mathrm{Ch}_c) \times_c \mathcal{H}$;
- $\bigcup_{i\in I}(\nu_i \cap [0,z]) \in (\langle \mathcal{F} \rangle \cap \mathrm{Ch}_c) \times_c \mathcal{H}$ by (ii) above;
- $\{w_i(z) : i \in I\} \in (\langle \mathcal{F} \rangle \cap \mathrm{Ch}_c) \times_c \mathcal{H}$ by (i) above;
- $\{\bar{t}, z\} \in [T]^{\leq 2} \subseteq \mathcal{C} \boxtimes 2$.

Putting all together, we obtain that

$$\nu_I \cap [0,t] \in ((\langle \mathcal{F} \rangle \cap \mathrm{Ch}_c) \times_c \mathcal{H}) \boxtimes_c 5 = \mathcal{C}.$$

□

Claim 3.36.3. *For every $t \in \nu_I$ one has that $\mathrm{Is}_t"(\nu_I) \in \mathcal{A}$.*

Proof of Claim: Given $t \in \nu_I$, if $t \notin \mathfrak{a}$, the property 2. of $(\nu_k)_{k\in M}$ implies that there are $j \in I$ and $\bar{t} \in \mathrm{Is}_t$ such that

$$\mathrm{Is}_t"(\nu_I) \subseteq \mathrm{Is}_t"(\nu_j) \cup \{\bar{t}\} \in (\langle \mathcal{F} \rangle \cap \mathrm{Ch}_a) \sqcup_a [T]^{\leq 1}.$$

If $t \in \mathfrak{a}$, it follows from (iii) above that

$$\mathrm{Is}_t"(\nu_I) \subseteq \bigcup_{i\in I} \mathrm{Is}"(\nu_i) \in (\langle \mathcal{F} \rangle \cap \mathrm{Ch}_a) \times_a \mathcal{H}$$

In any case, we have that $\mathrm{Is}_t"(\nu_I) \in \mathcal{A}$. □

These two claims imply that $\nu_I \in \mathcal{F} \times \mathcal{H}$ for every $I \in \mathcal{H}$. □

4. Bases of families on (not so) large cardinals

The purpose of the section is to use Theorem 3.1 to prove the following.

Theorem 4.1. *Every cardinal θ strictly smaller than the first Mahlo cardinal has a basis of families on θ.*

Recall that κ is ω-Erdős when for every coloring $c : [\kappa]^{<\omega} \to 2$ there is an infinite c-homogeneous subset $A \subseteq \kappa$, that is, for $s, t \in A$, $c(s) = c(t)$ when $\#s = \#t$. A compact and hereditary family on κ is called *large* when $\mathrm{srk}(\mathcal{F}) \geq \omega$, or, equivalently, when $\mathcal{F}$ satisfies (M.2) for $[\kappa]^{\leq 1}$ and the Schreier family $\mathcal{S}$. It is proved in [LoTo2] that the existence of such families in κ is equivalent to κ not being ω-Erdős.

Problem 1. *Characterize when κ has $\geq \omega$-homogeneous families.*

Problem 2. *Characterize the cardinal numbers κ such that that there exists $c : [\kappa]^{<\omega} \to 2$ and an $\geq \omega$-homogeneous family on κ such that every $s \in \mathcal{F}$ is c-homogeneous and such that for every sequence $(s_n)_{n<\omega}$ in $\mathcal{F}$ and every $l < \omega$ there are $n_1 < \cdots < n_l$ such that $\bigcup_{i=1}^{l} s_{n_i}$ is c-homogeneous.*

The first such κ not satisfying this coloring property is at least the first Mahlo cardinal and smaller than the first ω-Erdős cardinal. In order to prove Theorem 4.1, it suffices to find for every cardinal number κ less than the first Mahlo cardinal a tree T on κ such that there is a basis of families on chains of $(T, <)$ and a basis of families on chains of $(T, <_a)$. The proof of the existence of the tree T is done recursively on κ. For example, when κ is not strong limit, there must be $\lambda < \kappa$ such that $2^\lambda \geq \kappa$, so, by inductive hypothesis, there must be a basis on λ, and it is natural that this basis can be lift up to a basis on $<$-chains via the height mapping. The case when κ is not regular is similar. When on the contrary κ is inaccessible, the tree T on κ is substantially more complicated, and in fact relies on the method of walks in ordinals.

4.1. **Binary trees.** We start by analyzing the case of binary trees. Let T be the complete binary tree $2^{\leq\kappa}$, and assume that there is a basis of families on κ. We have the height function $\mathrm{ht} : T \to \kappa + 1$ that preserves chains of T. We are going to see how to use the height function ht to transfer a basis of families on κ to a basis on $<$-chains of T, and prove the following.

Theorem 4.2. *Suppose that κ has a basis. Then 2^κ also has a basis.*

Definition 4.3. *Let $\mathcal{P} = (P, \leq_P)$ and $\mathcal{Q} = (Q, \leq_Q)$ be partial orderings and $\lambda : P \to Q$.*

(i) λ *is* chain-preserving *when $p_0 \leq_P p_1$ implies that $\lambda(p_0) \leq_Q \lambda(p_1)$ or $\lambda(p_1) \leq_Q \lambda(p_0)$.*
(ii) λ *is* 1-1 on chains *when $\lambda \restriction C$ is 1-1 for every chain C of $\mathcal{P}$.*
(iii) λ *is* adequate *when it is chain-preserving and 1-1 on chains.*

In other words, λ is chain-preserving if it is a *graph homomorphism* between the corresponding *comparability graphs.* Observe that λ is chain-preserving if and only if $\lambda"(C)$ is a chain of $\mathcal{Q}$ for every chain C of $\mathcal{P}$. Observe also that when $\mathcal{Q}$ is a total ordering, every mapping $\lambda : P \to Q$ is chain preserving. The main result here is the following.

Theorem 4.4. *Let $\mathcal{P}$ and $\mathcal{Q}$ be partial orderings which have infinite chains, and let $\lambda : \mathcal{P} \to \mathcal{Q}$ be adequate. If $\mathcal{Q}$ has a basis of homogeneous families, then so has $\mathcal{P}$.*

We can prove now the stepping up result from κ to 2^κ.

Proof of Theorem 4.2. Suppose that κ has a basis. Let T be the complete binary tree $2^{\leq\kappa}$. The height mapping $\mathrm{ht} : T \to \kappa + 1$ is strictly monotone, so it follows from Theorem 4.4 that there is a basis of families on chains of T. Each set Is_t has size 2, so it follows from Theorem 3.1 that the cardinal number 2^κ has a basis. $\square$

Remark 4.5. Let ε be the first exp-indecomposable ordinal $> \omega$. Then it is easy to see that $f_1(\alpha) = (\alpha \cdot \omega)^\alpha$. Using this, and the construction of the basis on 2^κ from the one in κ we can give upper bounds of the ranks of ω^α-homogeneous families in small exponential cardinals. For the index set ω we have ω^α-families of rank exactly ω^α (e.g. Schreier families). We obtain in the index set $2^{\aleph_0}$ ω^α-homogeneous families $\mathcal{F}_\alpha$ such that

$$\omega^\alpha \le \mathrm{srk}(\mathcal{F}_\alpha) \le \mathrm{rk}(\mathcal{F}_\alpha) < \begin{cases} \omega^{\omega^{\alpha+1}+\omega} & \text{if } \alpha < \omega \\ \omega^{\omega^\alpha+\alpha\cdot\omega} & \text{if } \omega \le \alpha < \varepsilon. \end{cases}$$

One step up further, we have families on $2^{2^{\aleph_0}}$ such that

$$\omega^\alpha \le \mathrm{srk}(\mathcal{F}_\alpha) \le \mathrm{rk}(\mathcal{F}_\alpha) < \begin{cases} \omega^{\omega^{\omega^{\alpha+1}+\omega}+\omega^{\alpha+2}} & \text{if } \alpha < \omega \\ \omega^{\omega^{\omega^\alpha+\alpha\cdot\omega}+\omega^{\alpha+1}} & \text{if } \omega \le \alpha < \varepsilon, \end{cases}$$

and so on.

Definition 4.6 (preimage)**.** *Given partial orderings $\mathcal{P}$ and $\mathcal{Q}$, $\lambda : P \to Q$ and a family $\mathcal{G}$ on chains of $\mathcal{Q}$, let*

$$\lambda^{-1}(\mathcal{G}) :=\{s \subseteq P \,:\, s \textit{ is a chain of } \mathcal{P} \textit{ and } \lambda\text{"} s \in \mathcal{G}\}.$$

Lemma 4.7. *Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are two partial orderings, $\lambda : \mathcal{P} \to \mathcal{Q}$ is adequate. Suppose also that $\mathcal{G}$ is a family on chains of $\mathcal{Q}$.*

(a) *If $\mathcal{G}$ is pre-compact, hereditary, then so is $\lambda^{-1}(\mathcal{G})$.*
(b) *If $\mathcal{G}$ is countably ranked, then*

$$\mathrm{rk}(\lambda^{-1}(\mathcal{G})) <\omega \cdot (\mathrm{rk}(\mathcal{G}) + 1),. \tag{13}$$

Consequently, if $\mathcal{P}$ has infinite chains, and $\mathcal{G}$ is $(\alpha, \mathcal{Q})$-homogeneous, $\alpha \ge \omega$, then $\lambda^{-1}\mathcal{G}$ is $(\beta, \mathcal{P})$-homogeneous with $\alpha \le \beta < \iota(\beta)$.

Proof. Set $\mathcal{F} := \lambda^{-1}(\mathcal{G})$. It is clear that $\mathcal{F}$ is hereditary when $\mathcal{G}$ is hereditary. Suppose that $\mathcal{G}$ is pre-compact. Let $(x_n)_n$ be a sequence in $\mathcal{F}$. W.l.o.g. we assume that $(x_n)_n$ converges to $A \subseteq P$. Limit of chains are chains, so A is a chain of P. The proof will be finished when we verify that A is finite. We assume that $(\lambda\text{"}x_n)_n$ is a Δ-sequence with root y. It is easy to see that $\lambda\text{"}A \subseteq y$. Since λ is 1-1 on chains, it follows that $\#A \le \#y$, so A is finite. Suppose that $\mathcal{G}$ has countable rank. We apply Proposition 2.27 to $\lambda : \mathcal{F} \to \mathcal{G}$ to conclude that

$$\mathrm{rk}(\mathcal{F}) < \sup_{y\in\mathcal{G}}(\mathrm{rk}(\{x \in \mathcal{F} \,:\, \lambda(x) \subseteq y\}) + 1) \cdot (\mathrm{rk}(\mathcal{G}) + 1). \tag{14}$$

Observe that given $y \in \mathcal{G}$, since λ is 1-1 on chains, it follows that

$$\{x \in \mathcal{F} \,:\, \lambda(x) \subseteq y\} \subseteq [P]^{\le \# y},$$

so from (14) we obtain the desired inequality in (13).

Suppose that $\mathcal{P}$ has infinite chains and let $\mathcal{G}$ be $(\alpha, \mathcal{Q})$-homogeneous. Let us see that $\mathcal{F}$ is $(\beta, \mathcal{P})$-homogeneous with $\beta \ge \alpha$. Let X be an infinite chain of $\mathcal{P}$ such that $\mathrm{srk}_{\mathcal{P}}(\mathcal{F}) = \mathrm{rk}(\mathcal{F} \upharpoonright X) = \beta$. Then $Y := \lambda\text{"}X$ is an infinite chain of $\mathcal{Q}$. Since $h : \mathcal{F} \upharpoonright X \to \mathcal{G} \upharpoonright Y$, $h(s) := \lambda\text{"}s$ is an homeomorphism, it follows that $\mathrm{rk}(\mathcal{G} \upharpoonright Y) = \beta$, hence $\mathrm{srk}(\mathcal{G}) \le \beta$. On the other hand, it follows from (13) and the fact that $\mathcal{G}$ is homogeneous that

$$\mathrm{rk}(\mathcal{F}) < \omega \cdot (\mathrm{rk}(\mathcal{G}) + 1) < \iota(\mathrm{srk}(\mathcal{G})) \le \iota(\mathrm{srk}(\mathcal{G})).$$

□

Proof of Theorem 4.4. Let $(\mathfrak{C}, \times_{\mathcal{Q}})$ be a basis of families on chains of $\mathcal{Q}$. Let $\mathfrak{B}$ be the collection of all $\mathcal{P}$-homogeneous families $\lambda^{-1}\mathcal{G}$ with $\mathcal{G} \in \mathfrak{C}$. For each $\mathcal{F} \in \mathfrak{B}$, choose $\mathcal{G}_{\mathcal{F}}$ such that $\mathcal{F} = \lambda^{-1}(\mathcal{G}_{\mathcal{F}})$, and for $\mathcal{H} \in \mathfrak{S}$, let $\mathcal{F} \times \mathcal{H} := \lambda^{-1}(\mathcal{G}_{\mathcal{F}} \times_{\mathcal{Q}} \mathcal{H})$. We check that $(\mathfrak{B}, \times)$ satisfies (B.1'), (B.2') and (B.3), which is enough to guarantee the existence of a basis on $\mathcal{P}$, by Proposition 2.23. Given $\mathcal{G} \in \mathfrak{C}_\alpha$, $\lambda^{-1}\mathcal{G} \in \mathfrak{B}$ and by Lemma 4.7 we know that $\lambda^{-1}\mathcal{G}$ is β-uniform with $\alpha \le \beta < \iota(\alpha)$. We check now (B.2'). Suppose that $\mathcal{G}_0, \mathcal{G}_0 \in \mathfrak{C}$. Then $\lambda^{-1}\mathcal{G}_0 \sqcup_{\mathcal{P}} \lambda^{-1}(\mathcal{G}_1) = \lambda^{-1}(\mathcal{G}_0 \sqcup_{\mathcal{Q}} \mathcal{G}_1)$, so $\lambda^{-1}\mathcal{G}_0 \sqcup_{\mathcal{P}} \lambda^{-1}(\mathcal{G}_1) \in \mathfrak{B}$, because $\mathcal{G}_0 \sqcup_{\mathcal{Q}} \mathcal{G}_1 \in \mathfrak{C}$. Similarly one shows that $\mathfrak{B}$ is closed under $\cup$.

Finally, we verify (B.3) for $(\mathfrak{B}, \times)$. Fix $\mathcal{F} \in \mathfrak{B}$ and $\mathcal{H} \in \mathfrak{S}$. Then $\mathcal{F} \times \mathcal{H} = \lambda^{-1}(\mathcal{G}_{\mathcal{F}} \times_{\mathcal{Q}} \mathcal{H})$.

$$\begin{aligned}\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F} \times \mathcal{H})) =&\iota(\mathrm{srk}_{\mathcal{Q}}(\mathcal{G}_{\mathcal{F}} \times_{\mathcal{Q}} \mathcal{H})) = \max\{\iota(\mathrm{srk}_{\mathcal{Q}}(\mathcal{G}_{\mathcal{F}})), \iota(\mathrm{srk}(\mathcal{H}))\} \\ =&\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F}) \cdot \mathrm{srk}(\mathcal{H})).\end{aligned}$$

Let now $(s_n)_n$ be a sequence in $\mathcal{F}$ such that $C := \bigcup_n s_n$ is a chain. Then $\lambda\text{"}C = \bigcup_n \lambda\text{"}s_n$ is a $\mathcal{Q}$-chain, and $\lambda\text{"}s_n \in \mathcal{G}_{\mathcal{F}}$. By the property (M.2) of $\times_{\mathcal{Q}}$, we obtain that there is a subsequence $(t_n)_n$ of $(s_n)_n$ such that $\bigcup_{n\in x} \lambda\text{"}t_n \in \mathcal{G}_{\mathcal{F}} \times_{\mathcal{Q}} \mathcal{H}$ for every $x \in \mathcal{H}$. This means that $\bigcup_{n\in x} t_n \in \mathcal{F} \times \mathcal{H}$ for such $x \in \mathcal{H}$. □

4.2. **Trees from walks on ordinals.** We pass now to study certain trees on inaccessible cardinal numbers. They are produced using the method of walks on ordinals. We introduce some basic notions of this. For more details we refer the reader to the monograph [To].

Definition 4.8. *A* C-sequence $\overline{C} := (C_\alpha)_{\alpha<\theta}$ *is a sequence such that* $C_\alpha \subseteq \alpha$ *is a closed and unbounded subset of* α *with* $\mathrm{otp}(C_\alpha) = \mathrm{cof}(\alpha)$*. The* $\overline{C}$-walk *from* β *to* $\alpha < \beta$ *is the finite sequence of ordinals defined recursively by*

$$\begin{aligned}\mathrm{Tr}(\alpha, \beta) :=&(\beta)^\frown \pi(\alpha, \min(C_\beta \setminus \alpha)) \\ \mathrm{Tr}(\alpha, \alpha) :=&(\alpha).\end{aligned}$$

We write then the $\overline{C}$*-walk as* $\beta = \pi_0(\alpha, \beta) > \cdots > \pi_l(\alpha, \beta) = \alpha$*, where* $l + 1 = \mathrm{ht}(\pi(\alpha, \beta)$*, and for each* $i \le l$*,* $\pi_i(\alpha, \beta)$ *is the* i^{th} *term of* $\pi(\alpha, \beta)$*. Let*

$$\varrho_2(\alpha, \beta) := \mathrm{ht}(\mathrm{Tr}(\alpha, \beta)) - 1.$$

We now define the mapping $\varrho_0 : [\theta]^2 \to (\mathcal{P}(\theta))^{<\omega}$ *for* $\alpha \le \beta$ *recursively by*

$$\begin{aligned}\varrho_0(\alpha, \beta) :=&(C_\beta \cap \alpha)^\frown \varrho_0(\alpha, \min(C_\beta \setminus \alpha)) \\ \varrho_0(\alpha, \alpha) :=&\emptyset.\end{aligned}$$

Let $T = T(\varrho_0)$ *be the tree whose nodes are* $\varrho(\cdot, \beta) \upharpoonright \alpha$*,* $\alpha \le \beta$*, ordered by end-extension as functions. Given* $t \in T(\varrho_0)$*, let* $\alpha_t \le \beta_t$ *be such that* $t = \varrho_0(\cdot, \beta_t) \upharpoonright \alpha_t$*. We say that* (α_t, β_t) *represents* t*.*

Proposition 4.9. T *has size* θ*, and if* θ *is strong limit, then for every* $t \in T$ *one has that* $\#\mathrm{Is}_t(T) < \theta$*.*

Proof. This is a tree on a quotient of $[\theta]^{\le 2}$, so it has cardinality θ. Now observe that the immediate successors of $t = \varrho_0(\cdot, \beta) \upharpoonright \alpha$ are extensions u of t whose support is $\alpha + 1$. It follows that the number of them is at most $(2^\alpha)^{<\omega} < \theta$, when we assume that θ is strong limit. □

In other words, the partial ordering $<_{\mathrm{a}}$ is the disjoint union of small partial orderings. This is the content of the next fact. Recall that given a sequence of partial orderings $(\mathcal{P}_i)_{i\in I}$ we denote by $\biguplus_{i\in I}\mathcal{P}_i$ its *disjoint union*, which is the partial ordering on $\bigcup_{i\in I}P_i\times\{i\}$ defined by $(p,i)<(q,j)$ if and only if $i=j$ and $p<_{\mathcal{P}_i}q$.

Proposition 4.10. *Suppose that θ is a regular cardinal number such that $\omega_1^{\omega_1}<\theta$, and suppose that every $\xi<\theta$ has a basis on families on ξ. Suppose that $(\theta_\xi)_{\xi<\theta}$ is a sequence of infinite ordinals such that $\sup_\xi\theta_\xi=\theta$. Then the disjoint union of $\biguplus_{\xi<\theta}(\theta_\xi,<)$ has a basis of families on chains of the disjoint union.*

Proof. The proof is a counting argument. Set $\mathcal{P}:=\biguplus_{\xi<\theta}(\theta_\xi,<)$. First of all, let $C\subseteq\theta$ be such that $\#C=\theta$ and $(\theta_\xi)_{\xi\in C}$ is strictly increasing with supremum θ. For each $\xi\in C$ let $(\mathfrak{C}^\xi,\times'_\xi)$ be a basis on $\theta_\xi\times\{\xi\}$. Let $F:C\to\omega_1^{\omega_1}$ be the mapping that to $\xi\in C$ and $\alpha<\omega_1$ assigns

$$F(\xi)(\alpha):=\min\{\mathrm{rk}(\mathcal{F})\,:\,\mathcal{F}\in\mathfrak{C}^\xi_\alpha\}<\iota(\alpha)<\omega_1.$$

Since $\omega_1^{\omega_1}<\theta$ there must be $D\subseteq C$ of cardinality θ and $f\in\omega_1^{\omega_1}$ such that $F(\xi)=f$ for every $\xi\in D$. Define now for each $\xi<\theta$, $\mu_\xi:=\min\{\gamma\in D\,:\,\theta_\xi\le\theta_\gamma\}$. Fix $\xi<\theta$. Let $\mathfrak{B}^\xi$ be equal to $\mathfrak{C}^\xi$ if $\xi\in D$, and let $\mathfrak{B}^\xi$ be the collection of families $\{x\times\{\xi\}\,:\,x\subseteq\theta_\xi$ and $x\times\{\mu_\xi\}\in\mathcal{F}\}$ for $\mathcal{F}\in\mathfrak{C}^{\mu_\xi}$. For $\xi\in D$, let $\times_\xi=\times'_\xi$. Suppose that $\xi\notin D$. For each $\mathcal{F}\in\mathfrak{B}_\xi$, let $\mathcal{G}_\mathcal{F}$ be such that $\mathcal{F}=\{x\times\{\xi\}\,:\,x\subseteq\theta_\xi$ and $x\times\{\mu_\xi\}\in\mathcal{G}_\mathcal{F}\}$, and define

$$\mathcal{F}\times_\xi\mathcal{H}:=\{x\times\{\xi\}\,:\,x\times\{\mu_\xi\}\in\mathcal{G}_\mathcal{F}\times_{\mu_\xi}\mathcal{H}\}.$$

It is easy to see that $(\mathfrak{B}^\xi,\times_\xi)$ is a basis on $\theta_\xi\times\{\xi\}$ for every $\xi<\theta$. Let $\mathfrak{B}$ be the collection of all $\mathcal{P}$-homogeneous families $\mathcal{F}$ such that $\mathcal{F}\restriction\theta_\xi\times\{\xi\}\in\mathfrak{B}^\xi$. Define for $\mathcal{F}\in\mathfrak{B}$ and $\mathcal{H}\in\mathfrak{S}$

$$\mathcal{F}\times\mathcal{H}:=\bigcup_{\xi<\theta}(\mathcal{F}\restriction(\theta_\xi\times\{\xi\})\times_\xi\mathcal{H}).$$

We check that $(\mathfrak{B},\times)$ is a pseudo-basis on chains of $\mathcal{P}$. It is easy to see that $\mathfrak{B}$ contains all finite cubes. Now let $\omega\le\alpha<\omega_1$ and we prove that $\mathfrak{B}_\alpha\neq\emptyset$. For each $\xi\in D$, let $\mathcal{F}_\xi\in\mathfrak{C}^\xi_\alpha$. Define for each $\xi<\theta$ $\mathcal{G}_\xi=\mathcal{F}_\xi$ if $\xi\in D$, and $\mathcal{G}_\xi:=\{x\times\{\xi\}\,:\,x\subseteq\theta_\xi$ and $x\times\{\mu_\xi\}\in\mathcal{F}_{\mu_\xi}\}$. Notice that each $\mathcal{F}_\xi$ is α_ξ-uniform with $\alpha\le\alpha_\xi<\iota(\alpha)$. Notice also that $\sup_{\xi<\theta}\mathrm{rk}(\mathcal{F}_\xi)=f(\alpha)<\iota(\alpha)$. Now let $\mathcal{G}:=\bigcup_{\xi<\theta}\mathcal{G}_\xi$. Since

$$\mathrm{rk}(\mathcal{G})\le\sup_{\xi<\theta}\mathrm{rk}(\mathcal{G}_\xi)+1=f(\alpha)+1<\iota(\alpha),$$

and $\mathrm{srk}_\mathcal{P}(\mathcal{G})=\alpha$, it follows that $\mathcal{G}$ is $(\alpha,\mathcal{P})$-homogeneous. It is easy to see that $\mathfrak{B}$ is closed under $\cup$ and $\sqcup$. Now we prove that $\times$ is a multiplication. Fix $\mathcal{F}\in\mathfrak{B}$ and $\mathcal{H}\in\mathfrak{S}$, and suppose that $\mathcal{F}$ is $(\alpha,\mathcal{P})$-homogeneous. By definition

$$\begin{aligned}\iota(\mathrm{srk}_\mathcal{P}(\mathcal{F}\times\mathcal{H}))=&\iota(\min_{\xi<\theta}(\mathrm{srk}((\mathcal{F}\restriction(\theta_\xi\times\{\xi\}))\times_\xi\mathcal{H})))=\\=&\max\{\min_{\xi<\theta}\iota(\mathrm{srk}((\mathcal{F}\restriction(\theta_\xi\times\{\xi\})),\iota(\mathrm{srk}(\mathcal{H}))\}\\=&\max\{\iota(\alpha),\iota(\mathrm{srk}(\mathcal{H}))\}=\iota(\mathrm{srk}_\mathcal{P}(\mathcal{F})\cdot\mathrm{srk}(\mathcal{H})).\end{aligned}$$

It is easy to see that $\times$ satisfies (M.2). $\square$

Observe that if $t < u$ in $T(\varrho_0)$, then we can take $(\alpha_t, \beta_t), (\alpha_u, \beta_u)$ representing t and u respectively such that $\alpha_t < \alpha_u$ and $\beta_t \le \beta_u$: Take representatives (α_t, β_t), (α_u, β_u) of t and u respectively. Then $\alpha_t < \alpha_u$ and $\varrho_0(\cdot, \beta_t) \restriction \alpha_t = \varrho_0(\cdot, \beta_u) \restriction \alpha_t$, hence $(\alpha_t, \min\{\beta_t, \beta_u\})$ is a representative of t and satisfies the required condition together with (α_u, β_u). The following is well-known.

Proposition 4.11. *$t < u$ if and only if $\varrho_0(\alpha_t, \beta_t) = \varrho_0(\alpha_t, \beta_u)$.* □

Definition 4.12. *Given a C-sequence $\overline{C}$ on θ, let*

$$\mathcal{I}(\overline{C}) := \{C \subseteq \theta \,:\, C \sqsubseteq C_\alpha \text{ for some } \alpha < \theta\}.$$

We consider $\mathcal{I}(\overline{C})$ ordered by $\sqsubset$.

Proposition 4.13. *$\varrho_0 : (T, <) \to \mathcal{I}(\overline{C})^{<\omega}_{\mathrm{lex}}$ is strictly monotone, and consequently $\varrho_0 : (T, <) \to \mathcal{I}(\overline{C})^{<\omega}_{\mathrm{qlex}}$ is adequate.*

Proof. Suppose that $t < u$ in $T(\varrho_0)$.

Claim 4.13.1. *Suppose that $\varrho_0^i(t) = \varrho_0^i(u)$ for every $i \le k$. Then $\pi_i(\alpha_t, \beta_u) = \pi_i(\alpha_u, \beta_u)$ for every $i \le k+1$ and $\varrho_0^{k+1}(t) \sqsubseteq \varrho_0^{k+1}(u)$.*

Proof of Claim: Induction on $k \ge 0$. Suppose is true for $k-1$. Then $\pi_i(\alpha_t, \beta_u) = \pi_i(\alpha_u, \beta_u)$ for every $i \le k$. It follows that

$$\begin{aligned}\pi_{k+1}(\alpha_t, \beta_u) &= \min(C_{\pi_k(\alpha_t,\beta_u)} \setminus \alpha_t) = \min(C_{\pi_k(\alpha_u,\beta_u)} \setminus \alpha_t)\\ C_{\pi_k(\alpha_t,\beta_t)} \cap \alpha_t &= \varrho_0^k(t) = \varrho_0^k(u) = C_{\pi_k(\alpha_u,\beta_u)} \cap \alpha_u.\end{aligned}$$

In particular, $C_{\pi_k(\alpha_u,\beta_u)} \cap [\alpha_t, \alpha_u[= \emptyset$ hence $\min(C_{\pi_k(\alpha_u,\beta_u)} \setminus \alpha_t) = \min(C_{\pi_k(\alpha_u,\beta_u)} \setminus \alpha_u)$, so

$$\pi_{k+1}(\alpha_t, \beta_u) = \pi_{k+1}(\alpha_u, \beta_u).$$

Finally,

$$\varrho_0^{k+1}(t) = \varrho_0^{k+1}(\alpha_t, \beta_t) = \varrho_0^{k+1}(\alpha_t, \beta_u) = C_{\pi_{k+1}(\alpha_t,\beta_u)} \cap \alpha_t = C_{\pi_{k+1}(\alpha_u,\beta_u)} \cap \alpha_t \sqsubseteq \varrho_0^{k+1}(u).$$

□

It follows that $\varrho_0^0(t) \sqsubseteq \varrho_0^0(u)$, so there must be $k < \varrho_2(\alpha_u, \beta_u)$ such that $\varrho_0^k(t) \sqsubset \varrho_0^k(u)$, since otherwise for every k $\pi_k(\alpha_t, \beta_u) = \pi_k(\alpha_u, \beta_u)$ would imply that $\alpha_t = \alpha_u$. □

So, the mapping ϱ_0 is adequate, hence if $\mathcal{I}(\overline{C})^{<\omega}_{\mathrm{lex}}$ has a basis of families on chains, then ϱ_0 will transfer it to a basis on $<$-chains. We need to analyze then the lexicographical orderings, finite or infinite. This is the content of the next part. Given I and J, let $\pi_I : I \times J \to I$, $\pi_J : I \times J \to J$ be the canonical projections. Given $i \in I$, $j \in J$, and $\Delta \subseteq I \times J$, let

$$\begin{aligned}(\Delta)_i &:= \pi_J"(\pi_I^{-1}(i) \cap \Delta),\\ (\Delta)^j &:= \pi_I"(\pi_J^{-1}(j) \cap \Delta).\end{aligned}$$

be the corresponding sections.

Recall that given two partial orderings $\mathcal{P} = (P, \le_P)$ and $\mathcal{Q} = (Q, \le_Q)$, let $\mathcal{P} \times_{\mathrm{lex}} \mathcal{Q} := (P \times Q, <_{\mathrm{lex}})$ be the lexicographical product of $\mathcal{P}$ and $\mathcal{Q}$ defined by $(p_0, q_0) <_{\mathrm{lex}} (p_1, q_1)$ if and only if $p_0 <_P p_1$, or if $p_0 = p_1$ and $q_0 <_Q q_1$. This generalizes easily to finite products $\mathcal{P}_1 \times_{\mathrm{lex}} \mathcal{P}_2 \times_{\mathrm{lex}} \cdots \times_{\mathrm{lex}} \mathcal{P}_n$, and the corresponding finite powers $\mathcal{P}^n_{\mathrm{lex}}$. One can also define infinite lexicographical products, but they are not going to be used here. Instead we use *quasi-lexicographical power* $\mathcal{P}^{<\omega}_{\mathrm{qlex}}$ on $P^{<\omega}$ defined by $(p_i)_{i<m} <_{\mathrm{qlex}} (q_i)_{i<n}$ if and only if $m < n$ or if $m = n$ and $(p_i)_{i<m} <_{\mathrm{lex}} (q_i)_{i<m}$. Finally, let $\mathrm{lh} : P^{<\omega} \to \omega$ be the length function. The main result here is the following.

Theorem 4.14. *Let $\mathcal{P}$ and $\mathcal{Q}$ be partial orderings.*

(a) *If $\mathcal{P}$ and $\mathcal{Q}$ have bases of families on the corresponding chains, then $\mathcal{P} \times_{\mathrm{lex}} \mathcal{Q}$ also has a basis of families on its chains.*
(b) *If there is a basis on chains of $\mathcal{P}$, then there is also a basis of families on chains of each finite lexicographical power $\mathcal{P}^n_{\mathrm{lex}}$ and there is a basis of families on chains of $\mathcal{P}^{<\omega}_{\mathrm{lex}}$.*

Definition 4.15 (Fubini product of families). *Given I and J, let $\pi_I : I\times J \to I$, and $\pi_J : I\times J \to J$ be the canonical coordinate projections. Given families $\mathcal{F}$ and $\mathcal{G}$ on I and J respectively, let*

$$\mathcal{F} \circledast_{\mathrm{F}} \mathcal{G} :=\{x \subseteq I \times J \,:\, \pi_I"x \in \mathcal{F} \text{ and } (x)_i \in \mathcal{G} \text{ for every } i \in I\}.$$

We call $\mathcal{F} \circledast_{\mathrm{F}} \mathcal{G}$ the Fubini product of $\mathcal{F}$ and $\mathcal{G}$. Given $n \geq 1$, let $\mathcal{F}^{n+1}_{\mathrm{F}} = \mathcal{F}^n_{\mathrm{F}} \circledast_{\mathrm{F}} \mathcal{F}$.

It is easy to see that if $\mathcal{F}$ and $\mathcal{G}$ are families on chains of $\mathcal{P}$ and $\mathcal{Q}$, respectively, then $\mathcal{F} \circledast_F \mathcal{G}$ is a family on chains of $\mathcal{P} \times_{\mathrm{lex}} \mathcal{G}$.

Definition 4.16 (Power operation). *For each $n < \omega$, let $\mathcal{F}_n$ be a family on chains of $\mathcal{P}^n_{\mathrm{lex}}$, and let $\mathcal{G}$ be a family on ω. We define $((\mathcal{F}_n)_n)^{\mathcal{G}}$ as the collection of all $x \subseteq P^{<\omega}$ such that*

(i) *$x \cap [P]^n \in \mathcal{F}_n$ for every $n < \omega$.*
(ii) *$\mathrm{lh}"x \in \mathcal{G}$.*

Given a family $\mathcal{F}$ on chains of $\mathcal{P}$, let $\mathcal{F}^{\mathcal{G}} := ((\mathcal{F}^n_{\mathrm{F}})_n)^{\mathcal{G}}$.

Lemma 4.17. *Let $\mathcal{P}$ be $\mathcal{Q}$ two partial orderings. For each $1 \leq n < \omega$, let $\mathcal{F}_n$ be a family on chains of $\mathcal{P}^n_{\mathrm{lex}}$, and suppose also that $\mathcal{G}$ and $\mathcal{H}$ are families on chains of $\mathcal{Q}$ and ω, respectively. Set $\mathcal{F} := \mathcal{F}_1$.*

(a) *If $\mathcal{F}_n$, $n < \omega$, $\mathcal{G}$, $\mathcal{H}$ are pre-compact, hereditary, then so are $\mathcal{F} \circledast_{\mathrm{F}} \mathcal{G}$, $((\mathcal{F}_n)_n)^{\mathcal{H}}$, and $\mathcal{F}^{\mathcal{H}}$.*
(b) *If $\mathcal{F}_n$, $n < \omega$, $\mathcal{G}$ and $\mathcal{H}$ are countably ranked families, then*
  (b.1) $\mathrm{rk}(\mathcal{F} \circledast_{\mathrm{F}} \mathcal{G}) < (\mathrm{rk}(\mathcal{G}) \cdot \omega)) \cdot (\mathrm{rk}(\mathcal{F}) + 1)$,
  (b.2) $\mathrm{rk}(((\mathcal{F}_n)_n)^{\mathcal{H}}) < \sup_{n<\omega}(\mathrm{rk}(\mathcal{F}_n) + 1) \cdot (\mathrm{rk}(\mathcal{H}) + 1)$,
  (b.3) $\mathrm{rk}(\mathcal{F}^{\mathcal{H}}) < (\mathrm{rk}(\mathcal{F}) \cdot \omega)^{\omega} \cdot (\mathrm{rk}(\mathcal{H}) + 1)$, *if* $\mathrm{rk}(\mathcal{F}) \geq 1$.
(c) *When the corresponding families are countable ranked,*
  (c.1) $\mathrm{srk}_{\mathrm{lex}}(\mathcal{F} \circledast_{\mathrm{F}} \mathcal{G}) = \min\{\mathrm{srk}_{\mathcal{P}}(\mathcal{F}), \mathrm{srk}_{\mathcal{Q}}(\mathcal{G})\}$,
  (c.2) $\mathrm{srk}_{\mathrm{qlex}}(((\mathcal{F}_n)_n)^{\mathcal{H}}) = \min\{\min_{n<\omega} \mathrm{srk}_{\mathcal{P}^n_{\mathrm{lex}}}(\mathcal{F}_n), \mathrm{srk}(\mathcal{H})\}$, *and*
  (c.3) $\mathrm{srk}_{qlex}(\mathcal{F}^{\mathcal{H}}) = \min\{\mathrm{srk}_{\mathcal{P}}(\mathcal{F}), \mathrm{srk}(\mathcal{H})\}$.
(d) *If each $\mathcal{F}_n$ is $(\alpha, \mathcal{P}^n_{\mathrm{lex}})$-homogeneous, $\mathcal{G}$ is $(\alpha, \mathcal{Q})$-homogeneous and $\mathcal{H}$ is α-homogeneous, then $\mathcal{F}\circledast_{\mathrm{F}}\mathcal{G}$ is (α, lex)-homogeneous, and both $((\mathcal{F}_n)_n)^{\mathcal{H}}$ and $\mathcal{F}^{\mathcal{H}}$ is (α, qlex)-homogeneous.*

Proof. *The operation $\circledast_{\mathrm{F}}$*: The hereditariness is easy to prove. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are pre-compact. Let $(x_n)_n$ be a sequence in $\mathcal{F} \circledast_{\mathrm{F}} \mathcal{G}$. Let $M \subseteq \omega$ be infinite and such that $(\pi_I"x_n)_{n\in M}$

is a Δ-sequence with root y. Let $N \subseteq M$ be infinite such that $((x_n)_p)_{n\in N}$ is a Δ-sequence with root z_p for every $p \in y$. Let $x := \bigcup_{p\in y} z_p$. It is easy to see that $(x_n)_{n\in N}$ is a Δ-sequence with root x. Suppose now that $\mathcal{F}$ and $\mathcal{G}$ have countable rank, and set $\mathcal{H} := \mathcal{F} \circledast_{\mathrm{F}} \mathcal{G}$. We apply Proposition 2.27 to the projection $\pi_P : P \times Q \to P$ to conclude that

$$\mathrm{rk}(\mathcal{H}) < \sup_{y\in\mathcal{F}}(\mathrm{rk}(\{x \in \mathcal{H} \,:\, \pi_P"(x) \subseteq y\}) + 1) \cdot (\mathrm{rk}(\mathcal{F}) + 1).$$

Now observe that for a given $y \in \mathcal{F}$ one has that

$$\{x \in \mathcal{H} \,:\, \pi_P"x \subseteq y\} \subseteq \bigsqcup_{p\in y}\{\{p\} \times z \subseteq P \times Q \,:\, z \in \mathcal{G}\},$$

by definition of $\mathcal{H}$. Clearly $\{\{p\} \times z \subseteq P \times Q \,:\, z \in \mathcal{G}\}$ is homeomorphic to $\mathcal{G}$, so by Proposition 2.6 (iii.3.),

$$\mathrm{rk}(\{x \in \mathcal{H} \,:\, \pi_P"x \subseteq y\}) < \mathrm{rk}(\mathcal{G}) \cdot \omega.$$

The power operation: The fact that this operation preserves hereditariness is trivial to prove. Suppose that each $\mathcal{F}_n$ is a pre-compact family on chains of $\mathcal{P}^n_{\mathrm{lex}}$ and that $\mathcal{H}$ is a pre-compact family on ω. Set $\mathcal{Z} := ((\mathcal{F}_n)_n)^{\mathcal{H}}$, and suppose that $(x_k)_k$ is a sequence in $\mathcal{Z}$. We assume that $(\mathrm{lh}(x_n))_n$ is a Δ-sequence in ω with root z. Now for each $n \in z$ and each $k < \omega$, let $y^n_k := \mathrm{lh}^{-1}(n) \cap x_k \in \mathcal{F}_n$. Let $(x_k)_{k\in M}$ be a subsequence of $(x_k)_k$ such that for each $n \in z$ one has that $(y^n_k)_{k\in M}$ is a Δ-sequence with root y_n. It is easy to verify that $(x_k)_{k\in M}$ is a Δ-sequence with root $\bigcup_{n\in z} y_n$. The inequality in (b.2) follows from Proposition 2.27. The properties of $\mathcal{F}^{\mathcal{H}}$ follow from the corresponding properties of the Fubini product and the power operation.

(c) follows from the fact that if $C = \{(p_n, q_n)\}_{n<\omega}$ is a chain of $\mathcal{P} \times_{\mathrm{lex}} \mathcal{Q}$ then there is an infinite $M \subseteq \omega$ such that either $p_m \neq p_m$ for every $m < n \in M$, or $p_m = p_n$ and $q_m \neq q_n$ for every $m < n$ in M. Similarly, given a chain $C = \{\overline{p}^n\}_{n<\omega}$, there is an infinite $M \subseteq \omega$ such that either $\mathrm{lh}(\overline{p}^n) = l$ for every $n \in M$ and $\{\overline{p}^n\}_{n\in M}$ is an infinite chain of $\mathcal{F}^l_{\mathrm{F}}$, or $\mathrm{lh}(\overline{p}^m) \neq \mathrm{lh}(\overline{p}^n)$ for every $m \neq n$ in M. (d) is a consequence of (a) (b) and (c). $\square$

Proof of Theorem 4.14. (a): Suppose that $(\mathfrak{B}^{\mathcal{P}}, \times_{\mathcal{P}})$ and $(\mathfrak{B}^{\mathcal{Q}}, \times_{\mathcal{Q}})$ are bases of families on chains of $\mathcal{P}$ and $\mathcal{Q}$ respectively. Let $\mathfrak{B}$ be the collection of all $\mathcal{P} \times_{\mathrm{lex}} \mathcal{Q}$-homogeneous families $\mathcal{F}$ such that

(i) $\pi_{\mathcal{P}}"(\mathcal{F}) := \{\pi_{\mathcal{P}}"(x) \,:\, x \in \mathcal{F}\} \in \mathfrak{B}_{\mathcal{P}}$.
(ii) $(\mathcal{F})_{\mathcal{P}} := \{(x)_p \,:\, x \in \mathcal{F},\, p \in P\} \in \mathfrak{B}_{\mathcal{Q}}$.

Given $\mathcal{F} \in \mathfrak{B}$ and $\mathcal{H} \in \mathfrak{S}$, let

$$\mathcal{F} \times \mathcal{H} := (\pi_{\mathcal{P}}"(\mathcal{F}) \times_{\mathcal{P}} \mathcal{H}) \circledast_{\mathrm{F}} ((\mathcal{F})_{\mathcal{P}} \times_{\mathcal{Q}} \mathcal{H}).$$

We verify that $(\mathfrak{B}, \times)$ is a pseudo-basis. First of all, each family on $\mathfrak{B}$ is $\mathcal{P} \times_{\mathrm{lex}} \mathcal{Q}$-homogeneous. Next, given $n < \omega$, $\pi_{\mathcal{P}}"([P \times Q]^{\leq n}_{\mathrm{lex}}) = [P]^{\leq n}_{\mathcal{P}}$, and $([P \times Q]^{\leq n}_{\mathrm{lex}})_{\mathcal{P}} = [Q]^{\leq n}_{\mathcal{Q}}$, so $[P \times Q]^{\leq n}_{\mathrm{lex}} \in \mathfrak{B}$. Now given α infinite, we choose $\mathcal{F} \in \mathfrak{B}^{\mathcal{P}}_\alpha$ and $\mathcal{G} \in \mathfrak{B}^{\mathcal{Q}}_\alpha$. Then $\mathcal{Z} := \mathcal{F} \circledast_{\mathrm{F}} \mathcal{G} \in \mathfrak{B}_\alpha$: We know from Lemma 4.17 that $\mathcal{Z}$ is α-homogeneous. Since $\pi_{\mathcal{P}}"(\mathcal{Z}) = \mathcal{F}$ and $(\mathcal{Z})_{\mathcal{P}} = \mathcal{G}$, we have that $\mathcal{Z} \in \mathfrak{B}$. We check now (B.2'): Let $\mathcal{F}_0, \mathcal{F}_1 \in \mathfrak{B}$. Since $\pi_{\mathcal{P}}"(\mathcal{F}_0 \cup \mathcal{F}_1) = \pi_{\mathcal{P}}"(\mathcal{F}_0) \cup \pi_{\mathcal{P}}"(\mathcal{F}_1)$ and $(\mathcal{F}_0 \cup \mathcal{F}_1)_{\mathcal{P}} = (\mathcal{F}_0)_{\mathcal{P}} \cup (\mathcal{F}_1)_{\mathcal{P}}$, we obtain that $\mathfrak{B}$ is closed under $\cup$. Secondly, $\pi_{\mathcal{P}}"(\mathcal{F}_0 \sqcup_{\mathrm{lex}} \mathcal{F}_1) =$

$\pi_{\mathcal{P}}"(\mathcal{F}_0) \sqcup_{\mathcal{P}} \pi_{\mathcal{P}}"(\mathcal{F}_1)$, and $(\mathcal{F}_0)_{\mathcal{P}}, (\mathcal{F}_1)_{\mathcal{P}} \subseteq (\mathcal{F}_0 \sqcup_{\mathrm{lex}} \mathcal{F}_1)_{\mathcal{P}} \subseteq (\mathcal{F}_0)_{\mathcal{P}} \sqcup_{\mathcal{Q}} (\mathcal{F}_1)_{\mathcal{P}}$. This means that

$$\iota(\mathrm{srk}_{\mathcal{Q}}((\mathcal{F}_0 \sqcup_{\mathrm{lex}} \mathcal{F}_1)_{\mathcal{P}})) = \max\{\iota(\mathrm{srk}_{\mathcal{Q}}((\mathcal{F}_0)_{\mathcal{Q}})), \iota(\mathrm{srk}_{\mathcal{Q}}((\mathcal{F}_1)_{\mathcal{Q}}))\} = \iota(\mathrm{srk}_{\mathcal{Q}}((\mathcal{F}_0)_{\mathcal{P}} \sqcup_{\mathcal{Q}} (\mathcal{F}_1)_{\mathcal{P}})),$$

so $(\mathcal{F}_0 \sqcup_{\mathrm{lex}} \mathcal{F}_1)_{\mathcal{P}} \in \mathfrak{B}_{\mathcal{Q}}$. Since in addition $\mathcal{F}_0 \sqcup_{\mathrm{lex}} \mathcal{F}_1$ is lex-homogeneous, we obtain that $\mathcal{F}_0 \sqcup_{\mathrm{lex}} \mathcal{F}_1 \in \mathfrak{B}$, by definition. Finally we check that $\times$ is a multiplication. The property (M.1) of $\times$ follows from Lemma 4.17 (c). Now suppose that $(s_n)_n$ is a sequence in $\mathcal{F} \in \mathfrak{B}$. Let $(t_n)_n$ be a subsequence of $(s_n)_n$ such that

(1) $(\pi_{\mathcal{P}}" t_n)_n$ is a Δ-sequence with root y.
(2) For every $x \in \mathcal{H}$ one has that $\bigcup_{n\in x} \pi_{\mathcal{P}}"(t_n) \in (\pi_{\mathcal{P}}"\mathcal{F}) \times_{\mathcal{P}} \mathcal{H}$.
(3) For every $x \in \mathcal{H}$ and every $p \in y$ one has that $\bigcup_{n\in x} (t_n)_p \in (\mathcal{F})_{\mathcal{P}} \times_{\mathcal{Q}} \mathcal{H}$.

Since $(\mathcal{F})_{\mathcal{P}} \subseteq (\mathcal{F})_{\mathcal{P}} \times \mathcal{H}$, the conditions above imply that given $x \in \mathcal{H}$ one has that $\bigcup_{n\in x} t_n \in \mathcal{F} \times \mathcal{H}$.

(b) Finite lexicographical powers have bases of families on chains by (a). For each $n < \omega$, let $(\mathfrak{B}_n, \times_n)$ be a basis on chains of $\mathcal{P}^n_{\mathrm{lex}}$. Let $\mathfrak{B}$ be the collection of all qlex-homogeneous families on $\mathcal{P}^{<\omega}_{\mathrm{qlex}}$ such that

(i) $\mathcal{F} \upharpoonright [P]^n \in \mathfrak{B}_n$ for every $1 \le n < \omega$.
(ii) $\mathrm{lh}"(\mathcal{F}) := \{\mathrm{lh}"(s) \, : \, s \in \mathcal{F}\} \in \mathcal{H}$.

Given $\mathcal{F} \in \mathfrak{B}$ and $\mathcal{H} \in \mathfrak{S}$, let

$$\mathcal{F} \times \mathcal{H} = (((\mathcal{F} \upharpoonright [P]^n) \times_n \mathcal{H})_n)^{\mathrm{lh}"(\mathcal{F}) \times_\omega \mathcal{H}}.$$

We check that $(\mathfrak{B}, \times)$ is a pseudo-basis. Given $1 \le k < \omega$, set $\mathcal{F} := [P^{<\omega}]^{\le k}_{\mathrm{qlex}}$. Then for each n one has that $\mathcal{F} \upharpoonright [P]^n = [[P]^n]^{\le k}_{\mathrm{lex}} \in \mathfrak{B}_n$, and $\mathrm{lh}"(\mathcal{F}) = [\omega]^{\le k} \in \mathfrak{B}_\omega$, so $\mathcal{F} \in \mathfrak{B}$. One shows as in (a) that $\mathfrak{B}$ is closed under $\cup$ and $\sqcup_{\mathrm{qlex}}$. Finally, we check that $\times$ is a multiplication. That $\times$ satisfies (M.1) it follows from Lemma 4.17 (c). Let now $\mathcal{F} \in \mathfrak{B}$, $\mathcal{H} \in \mathfrak{B}_\omega$, and let $(s_k)_k \in \mathcal{F}$. Let $(t_k)_k$ be a subsequence of $(s_k)_k$ such that

(1) $(\mathrm{lh}" t_k)_k$ is a Δ-sequence with root $y \subseteq \omega$.
(2) For every $x \in \mathcal{H}$ and every $n \in y$ one has that $\bigcup_{k\in x} (s_k \cap [P]^n) \in (\mathcal{F} \upharpoonright [P]^n) \times_n \mathcal{H}$.
(3) For every $x \in \mathcal{H}$ one has that $\bigcup_{k\in x} \mathrm{lh}"(y_k) \in (\mathrm{lh}"\mathcal{F} \times_\omega \mathcal{H})$.

It is easy to verify that $\bigcup_{k\in x} t_k \in \mathcal{F} \times \mathcal{H}$ for every $x \in \mathcal{H}$. □

4.3. Cardinals smaller than the first Mahlo have a basis.

Definition 4.18. *A C-sequence on θ is* small *when there is a function $f : \theta \to \theta$ such that $\mathrm{otp}(C_\alpha) < f(\min C_\alpha)$ for every $\alpha < \theta$.*

Proposition 4.19. *A strong limit cardinal θ has a small C-sequence if and only if θ is smaller than the first Mahlo cardinal.*

Proof. Suppose that θ is smaller than the first Mahlo cardinal. Choose a closed and unbounded set $D \subseteq \theta$ consisting of non-inaccessible cardinals. For each $\alpha < \theta$ let $\lambda(\alpha) \in D$ be the maximal element of D smaller or equal than α, and for each λ in D let λ^+_D be the first element of D bigger than λ. Let $f : \theta \to \theta$, $f(\alpha) = 2^{\lambda(\alpha)^+_D} + 1$. $f(\alpha) < \theta$ because we are assuming that θ is strong limit. Observe also that $2^\alpha < f(\alpha)$. We define now the $\overline{C}$ sequence. Fix $\alpha \in \theta$. Suppose first that $\alpha \notin D$. Write $\alpha = \lambda(\alpha) + \beta$. Since $\mathrm{cof}(\alpha) = \mathrm{cof}(\beta)$, we can choose a club $C_\alpha \subseteq [\lambda(\alpha), \alpha[$ with $\mathrm{otp}(C_\alpha) = \mathrm{cof}(\beta)$. It follows that $\mathrm{otp}(C_\alpha) = \mathrm{cof}(\beta) < \lambda(\alpha)^+_D < f(\lambda(\alpha)) = f(\min C_\alpha)$. Suppose that $\alpha \in D$. If α is singular, then we choose C_α in a way that $\mathrm{otp}(C_\alpha) = \mathrm{cof} < \min C_\alpha$. Observe that then $\mathrm{otp}(C_\alpha) < \min C_\alpha < f(\min C_\alpha)$. Finally, if α is not strong limit, then let $\beta < \alpha$ be such that $2^\beta \geq \alpha$. Let now $C_\alpha \subseteq [\beta, \alpha[$. It follows that $\mathrm{otp}(C_\alpha) = \mathrm{cof}(\alpha) \leq \alpha \leq 2^\beta \leq 2^{\min C_\alpha} < f(\min C_\alpha)$.

Suppose now that θ is bigger or equal to a Mahlo cardinal κ. By pressing down Lemma there is a stationary set S of inaccessible cardinals of κ and $\gamma < \kappa$ such that $\min C_\alpha = \gamma$ for every $\alpha \in S$. Hence $\alpha = \mathrm{cof}(\alpha) = \mathrm{otp}(C_\alpha) < f(\gamma)$ for every α, and this is of course impossible. $\square$

Proof of Theorem 4.1. The proof is by induction on $\theta < \mu_0$. We see that there is a tree $T = (T, <)$ on θ that has bases of families on chains of $(T, <)$ and of $(T, <_a)$. Suppose that θ is not strong limit. Then there is $\kappa < \theta$ such that $\theta \leq 2^\kappa$. By Theorem 4.2, 2^κ has a basis, so θ does.

Suppose that θ is strong limit. Let $T := T(\varrho_0)$ on θ. Let $\overline{C}$ be a small C-sequence on θ, and let $f : \theta \to \theta$ be a witness of it. By inductive hypothesis, and by Proposition 4.10 we know that the disjoint union $\biguplus_{\xi<\theta} f(\xi)$ of $(f(\xi))_{\xi<\theta}$ has a basis of families on chains. Let now $\lambda : \mathcal{I}(\overline{C}) \to \biguplus_{\xi<\theta} f(\xi) = \bigcup_{\xi<\theta} f(\xi) \times \{\xi\}$, $\lambda(C) := (\mathrm{otp}(C), \min C)$. Then $\lambda : (\mathcal{I}(\overline{C}), \sqsubset) \to (\biguplus_{\xi<\theta} f(\xi), <)$ is chain preserving and 1-1 on chains. Hence, by Theorem 4.4, there is a basis of families on chains of $\mathcal{I}(\overline{C})^{<\omega}_{\mathrm{qlex}}$. Since T has infinite chains and Proposition 4.13 tells that ϱ_0 is strictly monotone, it follows again by Theorem 4.4 that there is a basis of families on chains of $(T, <)$.

Observe that $\bigcup_{t\in T} \mathrm{Is}_t$ is a disjoint union, $\#T = \theta$. Since we are assuming that θ is inaccessible, it follows that T is $< \theta$-branching, hence for every $t \in T$ there is a basis of families on Is_t. Hence, by Proposition 4.10, there is a basis of families on chains of $(T, <_a)$. $\square$

5. Subsymmetric sequences and ℓ_1^α-spreading models

We present now new examples of Banach spaces without subsymmetric basic sequences of density κ. Their construction uses bases of families on κ. On one side, the multiplication of families will imply the non existence of subsymmetric basic sequences. On the other side, the fact that the families are homogeneous will allow to bound the complexity of finite subsymmetric basic sequences. In fact, we will give examples of spaces such that every non-trivial sequence on it has a subsequence such that a large family of finite further subsequences behave like ℓ_1^n. Since in addition the spaces are reflexive, we will have, as for the Tsirelson space, that there are no subsymmetric basic sequences.

Definition 5.1. *Recall that a non-constant sequence $(x_n)_n$ in a Banach space $X = (X, \|\cdot\|)$ is called* subsymmetric *when there is a constant $C \geq 1$ such that*

$$\|\sum_{i=1}^n a_i x_{l_i}\| \leq C \|\sum_{i=1}^n a_i x_{k_i}\| \tag{15}$$

for every n, scalars $(a_i)_{i=1}^n$, $l_1 < l_2 < \cdots < l_n$ and $k_1 < k_2 < \cdots < k_n$.

Sometimes it is assumed, not in here, that subsymmetric sequences are unconditional basic sequences. Notice that by Rosenthal's ℓ_1-Theorem and Odell's partial unconditionality, it follows that if $(x_n)_n$ is a subsymmetric basic sequence, then either is equivalent to the ℓ_1 unit basis, or its difference sequence $(x_{2n} - x_{2n-1})_n$ is subsymmetric and unconditional. This is sharp, as it is shown by the summing basis of c_0.

Definition 5.2. *Let $\mathcal{S}_\alpha$ be an α-Schreier family on ω. A bounded sequence $(x_n)_n$ in a normed space $\mathfrak{X}$ is called an* ℓ_1^α-spreading model *when there is a constant $C > 0$ such that*

$$\|\sum_{n\in s} a_n x_n\| \ge C \sum_{n\in s} |a_n| \text{ for every } s \in \mathcal{S}_\alpha.$$

Let us say that a sequence in a Banach space is *non-trivial* when it does not have norm-convergent subsequences.

REMARK 5.3. (a) Suppose that $(\mathcal{S}_\alpha)_{\alpha<\omega_1}$ is a generalized Schreier sequence. If $(x_n)_n$ is a ℓ_1^α-spreading model, and $\beta \le \alpha$, then $(x_n)_n$ is a ℓ_1^β-spreading model: This is a consequence of the fact that for every $\beta < \alpha$ there is some integer n such that $\mathcal{S}_\beta \upharpoonright \omega \setminus n \subseteq \mathcal{S}_\alpha$.

(b) Suppose that a space X does not contain ℓ_1 and it is such that every non-trivial sequence has a ℓ_1-spreading model subsequence. Then X does not have subsymmetric sequences. If in addition X have an unconditional basis, then X is in addition reflexive: Suppose otherwise that $(x_n)_n$ is a subsymmetric sequence $(x_n)_n$. It follows that $(x_n)_n$ is bounded and that $(x_n)_n$ does not have norm-convergent subsequences. So, by hypothesis, there is a ℓ_1-spreading model subsequence $(y_n)_n$. This implies that $(y_n)_n$ is equivalent to the unit basis of ℓ_1, and this is impossible. The latter condition follows from the James criterion of reflexivity.

Definition 5.4. *Recall that given a family $\mathcal{F}$ on κ, we define the corresponding* generalized Schreier space $X_\mathcal{F}$ *as the completion of $c_{00}(\kappa)$ with respect to the norm*

$$\|x\|_\mathcal{F} := \max\{\|x\|_\infty, \max_{s\in\mathcal{F}} \sum_{\xi\in s} |(x)_\xi|\}.$$

It is easy to see that the unit basis of $c_{00}(\kappa)$ is a 1-unconditional basis of $X_\mathcal{F}$, and that $X_\mathcal{F}$ is c_0-saturated if $\mathcal{F}$ is compact, and contains a copy of ℓ_1 otherwise. When the family $\mathcal{F}$ is compact, hereditary and α-homogeneous with α infinite, then every subsequence of the unit basis of $X_\mathcal{F}$ has a ℓ_1-spreading model subsequence, consequently, no subsequence of the unit basis is subsymmetric. These families $\mathcal{F}$ exist on cardinal numbers not being ω-Erdős (see [LoTo2]).

Theorem 5.5. *Suppose that θ is smaller than the first Mahlo cardinal number. Then for every $\alpha < \omega_1$ there is a Banach space X of density θ with a long 1-unconditional basis $(u_\xi)_{\xi<\theta}$ such that every subsequence of $(u_\xi)_{\xi<\theta}$ has a further ℓ_1^α-spreading model subsequence, and no subsequence of $(u_\xi)_{\xi<\theta}$ is a $\ell_1^{\iota(\omega^\alpha)}$-spreading model.*

PROOF. Fix a basis $(\mathfrak{B}, \times)$ on θ, let $\mathcal{F}$ be an $\omega^\alpha + 1$-homogeneous family in $\mathfrak{B}$ and let $X := X_\mathcal{F}$. Let $(u_\xi)_{\xi\in M}$ be an infinite subsequence of the unit basis $(u_\xi)_{\xi<\theta}$ of $X_\mathcal{F}$. Since $\mathrm{rk}(\mathcal{F} \upharpoonright M) > \omega^\alpha$, and $\mathrm{rk}(\mathcal{S}_\alpha) = \omega^\alpha$ there is some infinite subset N of M such that $\{\xi_n\}_{n\in x} \in \mathcal{F}$ for every $x \in \mathcal{S}_\alpha \upharpoonright$

N. Let $N = \{n_k\}_k$ be the increasing enumeration of N, and set $x_k := u_{\xi_{n_k}}$ for every $k < \omega$. We claim that

$$\|\sum_{k\in x} a_k x_k\|_{\mathcal{F}} = \sum_{k\in x} |a_k|$$

for every $x \in \mathcal{S}_\alpha$: Fix $x \in \mathcal{S}_\alpha$. Then $\{n_k\}_{k\in x} \in \mathcal{S}_\alpha \restriction N$, because $\mathcal{S}_\alpha$ is spreading. This means that $\{\xi_{n_k}\}_{k\in x} \in \mathcal{F}$, so

$$\|\sum_{k\in x} a_k x_k\|_{\mathcal{F}} = \|\sum_{k\in x} a_k u_{\xi_{n_k}}\|_{\mathcal{F}} \ge \sum_{k\in x} |a_k|.$$

On the other hand, let given a subsequence $(x_n)_{n<\omega}$ be a subsequence of $(u_\xi)_{\xi<\theta}$, $x_n := u_{\xi_n}$. we assume that $\xi_n < \xi_{n+1}$ for every n. Let $\mathcal{G} := \{x \subseteq \omega \, : \, \{\xi_n\}_{n\in x} \in \mathcal{F}\}$. Then the mapping $x \in \mathcal{G} \mapsto \{\xi_n\}_{n\in x} \in \mathcal{F}$ is continuous and 1-1, hence $\mathrm{rk}(\mathcal{G}) \le \mathrm{rk}(\mathcal{F}) < \iota(\omega^\alpha)$. Since $\iota(\omega^\alpha)$ is exp-indecomposable, $\mathrm{rk}(\mathcal{S}_{\iota(\omega^\alpha)}) = \omega^{\iota(\omega^\alpha)} = \iota(\omega^\alpha)$. It follows by the quantitative version of Ptak's Lemma (see for example [LoTo1, Lemma 4.7]) that for every $\varepsilon > 0$ there is some convex combination $(a_n)_{n\in x}$ supported in $x \in \mathcal{S}_{\iota(\omega^\alpha)}$ such that

$$\sup_{x\in\mathcal{G}} \sum_{n\in x} |a_n| < \varepsilon.$$

This means that $\|\sum_{n\in x} a_n x_n\|_{\mathcal{F}} < \varepsilon \sum_{n\in x} |a_n|$. □

Definition 5.6. *Recall that given an α-Schreier family $\mathcal{S}_\alpha$, let $T_\alpha := T_{\mathcal{S}_\alpha}$ be the α-Tsirelson space defined as the completion of c_{00} under the norm*

$$\|x\|_\alpha := \max\{\|x\|_\infty, \sup_{(E_i)_i} \frac{1}{2} \sum_i \|E_i x\|_\alpha\}$$

where the $\sup$ *above runs over all sequences of sets $(E_i)_i$ such that $E_i < E_{i+1}$ and $\{\min E_i\}_i \in \mathcal{S}_\alpha$, and where $Ex = \sum_{n\in E} (x)_n$.*

An equivalent way of defining $\|\cdot\|_\alpha$ is as follows. Let $K_0 := \{\pm u_n\}_{n<\omega}$ and let

$$K_{n+1} := K_n \cup \{\frac{1}{2}\sum_{i<k} \varphi_i \, : \, \{\varphi_i\}_{i<k} \subseteq K_n, \, \varphi_i < \varphi_{i+1}, \, i < k-1, \text{ and } \{\min \operatorname{supp} \varphi_i\}_{i<k} \in \mathcal{S}_\alpha\}.$$

Let $K := \bigcup_n K_n$. Then $\|x\|_\alpha = \sup_{\varphi\in K} \langle \varphi, x\rangle$. It is easy to see that each $\varphi \in K$ has a decomposition

$$\varphi = \sum_i \frac{1}{2^i} \varphi_i$$

where φ_i is a vector with coordinates -1 or 1, supported in $\mathcal{S}_{\alpha\cdot i}$ and $(f_i)_i$ pairwise disjointly supported. It is well known that every normalized block subsequence of the unit basis $(u_n)_n$ of T_α is equivalent to a subsequence of the unit basis. Since clearly from the definition every subsequence of $(u_n)_n$ is a ℓ_1^α-spreading model, it follows that every non-trivial sequence in T_α has a ℓ_1^α-spreading model subsequence. Now suppose that $(x_n)_n$ is a non-trivial sequence in T_α. W.l.o.g. we assume that $(x_n)_n$ is a subsequence of the unit basis, $x_n = u_{k_n}$. Let $\varepsilon > 0$, and let n be such that $\varepsilon 2^n > 1$. By the quantitative version of Ptak's Lemma, there is some convex combination $(a_n)_{n\in x}$ supported in $x \in \mathcal{S}_{\alpha\cdot\omega}$ such that $\sup_{x\in\bigcup_{i<n}\mathcal{S}_{\alpha\cdot n}} \sum_{k\in x}|a_k| < \varepsilon$. We claim that $\|\sum_{n\in x} a_n x_n\|_\alpha \le 3\varepsilon$: Fix $\varphi \in K$, $\varphi = \sum_i 2^{-i}\varphi_i$ decomposed as above. Then,

$$|\langle\varphi, \sum_{j\in x} a_j u_{k_j}\rangle| \le \sum_{i<n} \frac{1}{2^i}|\langle\varphi_i, \sum_{j\in x} a_j u_{n_j}\rangle| + \frac{1}{2^n}\sum_{j\in x}|a_j| \le \sum_{i<n}\frac{1}{2^i}\sum_{j\in x\cap \operatorname{supp}\varphi_j}|a_j| + \frac{1}{2^n}\sum_{j\in x}|a_j| \le 3\varepsilon.$$

We have just proved the following.

Theorem 5.7. *T_α is a reflexive Banach space whose unit basis is 1-unconditional and such that every non-trivial sequence has a ℓ_1^α-spreading model subsequence but it does not have $\ell_1^{\alpha\cdot\omega}$-spreading models. Consequently, T_α does not have subsymmetric basic sequences.* □

5.1. **The interpolation method.** We recall the following well-known construction, presented in a general, not necessarily separable, context: fix an infinite cardinal number κ, let $(\|\cdot\|_n)_{n\in\omega}$ be a sequence of norms in $c_{00}(\kappa)$ and $\|\cdot\|_X$ be a norm on $c_{00}(\mathbb{N})$ such that $(e_n)_n$ is a 1-unconditional basic sequence of the completion X of $(c_{00}(\mathbb{N}), \|\cdot\|_X)$. Let X_n, $n \in \mathbb{N}$, be the completion of $(c_{00}(\kappa), \|\cdot\|_n)$. For $x \in c_{00}(\kappa)$, define

$$|||x||| := \|\sum_n \frac{\|x\|_n}{2^{n+1}} e_n\|_X.$$

It is not difficult to see that $|||\cdot|||$ is a norm on $c_{00}(\kappa)$ (the fact that $(u_n)_n$ is a 1-unconditional basic sequence of $(c_{00}(\mathbb{N}), \|\cdot\|_X)$ is crucial to prove the triangle inequality). Let $\mathfrak{X}$ be the completion of $(c_{00}, |||\cdot|||)$.

Remark 5.8. Observe that the dual unit ball of $\mathfrak{X}$ is closed under the following operation. Given $x_i^* \in B_{X_i^*}$ for $i = 1, \dots, n$ and $\sum_{i=1}^n b_i e_i^* \in B_{X^*}$, then

$$\sum_{i=1}^n \frac{b_i}{2^{i+1}} x_i^* \in B_{\mathfrak{X}^*}.$$

To see this, there is a simple computation. Given $x \in c_{00}$ we have that

$$|(\sum_{i=1}^n \frac{b_i}{2^{i+1}}x_i^*)(x)| \le \sum_{i=1}^n \frac{|b_i|}{2^{i+1}}\|x\|_i = |(\sum_{i=1}^n |b_i| e_i^*)(\sum_{i=1}^n \frac{1}{2^{i+1}}\|x\|_i e_i)| \le$$
$$\le \left\|\sum_{i=1}^n |b_i| e_i^*\right\|_{X^*}\left\|\sum_{i=1}^n \frac{1}{2^{i+1}}\|x\|_i e_i\right\|_X \le \|x\|_{\mathfrak{X}}$$

The following follows easily from the definition.

Proposition 5.9. *Suppose that $(x_\xi)_{\xi<\lambda}$ is a C-unconditional basic sequence of each X_n. Then $(x_\xi)_{\xi<\lambda}$ is a C-unconditional basic sequence of $\mathfrak{X}$.* □

In our construction, this will be the case, so that we will be able to apply the following result.

Proposition 5.10. *Suppose that X is a space with an unconditional basis and without isomorphic copies of ℓ_1. Then the following are equivalent.*

(a) *Every non trivial bounded sequence in X has an ℓ_1^α-spreading model subsequence.*
(b) *Every non-trivial weakly-convergent sequence in X has ℓ_1^α-spreading model subsequence.*

(c) *Every non-trivial weakly-null sequence in X has ℓ_1^α-spreading model subsequence.*

PROOF. Suppose that (b) holds. It follows that c_0 does not embed into X. Hence, by James' criteria of reflexivity for spaces with an unconditional basis, X is reflexive, whence every bounded sequence has a weakly-convergent subsequence and now (a) follows directly from (b). Now suppose that (c) holds and let us prove (b): Suppose that $(x_k)_k$ is a non-trivial weakly-convergent sequence with limit x. Let $x^* \in S_{X^*}$ be such that $x^*(x) = \|x\|$. Let $y_k := x_k - x$ for every k. By hypothesis, we can find $\varepsilon > 0$ and a subsequence $(z_n)_n$ of $(y_n)_n$ such that $\|\sum_n a_n y_n\| \geq \varepsilon \sum_n |a_n|$ for every sequence of scalars $(a_n)_n$ supported in $\mathcal{S}_\alpha$. Let $(v_n)_n$ be a further subsequence of $(z_n)_n$ such that $|x^*(v_n)| \leq \varepsilon/2$ for every n. We claim that $(v_n + x)_n$ is a subsequence of $(x_n)_n$ which is a ℓ_1^α-spreading model: Fix a sequence $(a_n)_n$ supported in $\mathcal{S}_\alpha$, and let $y^* \in B_{X^*}$ be such that $y^*(\sum_n a_n z_n) \geq \varepsilon \sum_n |a_n|$. Let $z^* := y^* - \lambda x^* \in \mathrm{Ker}(x) \cap 2B_{X^*}$, where $\lambda := y^*(x)/\|x\|$. Then,

$$\|\sum_n a_n(v_n + x)\| \geq \frac{1}{2} z^* \left(\sum_n a_n(v_n + x)\right) = \frac{1}{2} z^* \left(\sum_n a_n v_n\right) \geq \frac{1}{2} y^* \left(\sum_n a_n v_n\right) -$$
$$-\lambda x^* \left(\sum_n a_n v_n\right) \geq \frac{\varepsilon}{4} \sum_n |a_n|.$$

□

Finally, one of the interesting features of the resulting space of the interpolation is given by the following proposition.

Proposition 5.11. *If Y can be isomorphically embedded into $\mathfrak{X}$, then a subspace Z of Y can be isomorphically embedded into some X_n or into X.*

PROOF. Let $Y_0 \subseteq \mathfrak{X}$ be isomorphic to Y. If there is some n such that $I_n : Y_0 \to X_n$ is not strictly singular, then choosing $Y_1 \subseteq Y_0$ such that $I_n \upharpoonright Y_1$ is an isomorphism we obtain that Y_1 embeds into X_n. Suppose that all $I_n \upharpoonright Y_0$ are strictly singular.

Fix a strictly positive summable sequence $(\varepsilon_n)_n$, $\sum_n \varepsilon_n < 1/4$. Let $y_0 \in S_{Y_0}$. Let $n_0 \in \mathbb{N}$ be such that

$$\|\sum_{n > n_0} \frac{1}{2^{n+1}} \|y_0\|_n e_n\|_X \leq \varepsilon_0.$$

Let $y_1 \in S_{Y_0}$ be such that

$$\max_{n \leq n_0} \|y_1\|_n \leq \varepsilon_1$$

Let $n_1 > n_0$ be such that

$$\|\sum_{n > n_1} \frac{1}{2^{n+1}} \|y_1\|_n e_n\|_X \leq \varepsilon_1.$$

Let now $y_2 \in S_{Y_0}$ be such that

$$\max_{n \leq n_1} \|y_2\|_n \leq \varepsilon_2.$$

In this way, we can find an strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of integers normalized vectors $y_n \in Y_0$ such that for every k one has that

$$\|\sum_{n>n_k} \frac{1}{2^{n+1}}\|y_k\|_n e_n\|_X \le \varepsilon_k$$
$$\max_{n\le n_{k-1}} \|y_k\|_n \le \varepsilon_k.$$

Set $w_0 := \sum_{n\le n_0}(\|y_0\|_n/2^{n+1})e_n \in X$, and for each $k\ge 1$, let

$$w_k := \sum_{n=n_{k-1}+1}^{n_k} \frac{\|y_{kn}\|}{2^{n+1}} e_n \in X.$$

Let $(a_k)_k$ be a sequence of scalars with $\max_k |a_k| = 1$. Then, after some computations, one can show that

$$\begin{aligned}\left|\|\sum_k a_k y_k\|_{\mathfrak{X}} - \|\sum_k a_k w_k\|_X\right| \le & |a_0| \|\sum_{n\le n_0} \frac{1}{2^{n+1}}\|y_0\|_n e_n\|_X + \\ & +\sum_{k\ge 1}|a_k|\left(\|\sum_{n\le n_{k-1}} \frac{1}{2^{n+1}}\|y_k\|_n e_n\|_X + \|\sum_{n>n_k} \frac{1}{2^{n+1}}\|y_k\|_n e_n\|_X\right) < \\ & < \frac{1}{2}.\end{aligned}$$

Since $\|w_k\|_X \ge 3/4$, and $(w_k)_k$ is a block subsequence of the basis $(e_n)_n$, it follows that $\|\sum_k a_k w_k\|_X \ge 3/4 \max_k |a_k|$, we obtain that

$$\left|\|\sum_k a_k y_k\|_{\mathfrak{X}} - \|\sum_k a_k w_k\|_X\right| < \frac{2}{3}\|\sum_k a_k w_k\|_X.$$

Hence,

$$\frac{1}{3}\|\sum_k a_k w_k\|_X \le \|\sum_k a_k y_k\|_{\mathfrak{X}} \le \frac{4}{3}\|\sum_k a_k w_k\|_X.$$

□

5.2. **The Banach space $\mathfrak{X}$.** The next result gives the existence of the desired Banach space $\mathfrak{X}$ of density κ, subject to the existence of bases of families on κ.

Theorem 5.12. *Suppose that κ has a basis of families. Then for every $1\le\alpha<\omega_1$ there is a reflexive Banach space $\mathfrak{X}_\alpha$ of density κ with a long unconditional basis such that*

(a) *every bounded sequence without norm convergent subsequences has a ℓ_1^α-spreading model subsequence, and*
(b) *$\mathfrak{X}_\alpha$ does not have $\ell_1^{\iota(\omega^\alpha)+\alpha\cdot\omega}$-spreading models.*

Consequently,

(c) *$\mathfrak{X}_\alpha$ does not have subsymmetric sequences;*
(d) *if $\iota(\omega^\alpha)+\alpha\cdot\omega\le\beta$, then $\mathfrak{X}_\alpha$ and $\mathfrak{X}_\beta$ are totally incomparable, i.e. there is no infinite dimensional subspace of $\mathfrak{X}_\alpha$ isomorphic to a subspace of $\mathfrak{X}_\beta$.*

PROOF. Let $(\mathfrak{B},\times)$ be a basis of families on κ. Let $\mathcal{F}_0 := [\kappa]^{\le 1}$, and for $n<\omega$ let $\mathcal{F}_{n+1} := \mathcal{F}_n \times \mathcal{S}_\alpha$. Notice that $\mathrm{rk}(\mathcal{F}_n) < \iota(\omega^\alpha)$ for every n. Let $\mathfrak{X}$ be the interpolation space from an α-Tsirelson space T_α and the sequence of generalized Schreier spaces $X_{\mathcal{F}_n}$, $n\in\mathbb{N}$. Since each $X_{\mathcal{F}_n}$ is c_0-saturated, and T_α is reflexive, it follows from Proposition 5.11 that $\mathfrak{X}$ does not have isomorphic copies of ℓ_1.

Claim 5.12.1. *Every non-trivial bounded sequence has an ℓ_1^α-spreading model.*

From this claim, and the fact that the unit basis $(u_\gamma)_{\gamma<\kappa}$ is unconditional, we obtain that $\mathfrak{X}$ is reflexive. We pass now to prove that previous claim.

Proof of Claim: Fix such sequence $(x_k)_k$. Since $\mathfrak{X}$ does not have isomorphic copies of ℓ_1, we may assume, by Proposition 5.10, that $(x_k)_k$ is a non-trivial weakly-null sequence. Since $(u_\xi)_{\xi<\kappa}$ is a Schauder basis of $\mathfrak{X}$, by going to a subsequence if needed, we assume that $(x_k)_k$ is disjointly supported, and

$$\|x_k\| \ge \gamma > 0 \text{ for every } k.$$

The proof is now rather similar to that of Proposition 5.11.
CASE 1. There is $\varepsilon>0$, $n\in\mathbb{N}$ and an infinite subsequence $(y_k)_k$ of $(x_k)_k$ such that

$$\|y_k\|_n \ge \varepsilon \text{ for every } k.$$

For each k choose $s_k \in \mathcal{F}_n \upharpoonright \operatorname{supp} y_k$ such that

$$\sum_{\xi\in s_k} |u_\xi^*(y_k)| \ge \varepsilon.$$

By hypothesis, $\mathcal{F}_{n+1} = \mathcal{F}_n \times \mathcal{S}_\alpha$, so there is a subsequence (t_k) of $(s_k)_k$ such that $\bigcup_{k\in v} t_k \in \mathcal{F}_{n+1}$ for every $v\in\mathcal{S}_\alpha$. Let $(z_k)_k$ be the subsequence of $(y_k)_k$ such that $\sum_{\xi\in t_k}|u_\xi^*(z_k)| \ge \varepsilon$ for every k. We claim that $(z_k)_k$ is a ℓ_1^α-spreading model. So, fix a sequence of scalars $(a_k)_{k\in s}$ indexed by $s\in\mathcal{S}_\alpha$. Then $t := \bigcup_{k\in s} t_k \in \mathcal{F}_{n+1}$, hence,

$$\begin{aligned}\|\sum_{k\in s} a_k z_k\| \ge &\frac{1}{2^{n+2}}\|\sum_{k\in s} a_k z_k\|_{n+1} \ge \frac{1}{2^{n+2}}\sum_{\xi\in t}\left|u_\xi^*(\sum_{k\in s} a_k z_k)\right| = \frac{1}{2^{n+2}}\sum_{k\in s}|a_k|\sum_{\xi\in t_k}\left|u_\xi^*(z_k)\right| \ge \\ \ge &\frac{\varepsilon}{2^{n+2}}\sum_{k\in s}|a_k|.\end{aligned}$$

CASE 2. For every $\varepsilon>0$ and every n the set

$$\{k\in\mathbb{N} \,:\, \|x_k\|_n \ge \varepsilon\} \text{ is finite.}$$

So, let $(y_k)_k$ be a subsequence of $(x_k)_k$ such that

$$\|y_k\|_{k-1} \le \frac{1}{2^{k+1}} \text{ for every } k$$

Now we can find $(n_k)_k$ such that for every k one has that

$$\|\sum_{n=n_k}^{n_{k+1}-1}\frac{1}{2^{n+1}}\|y_k\|_n t_n\|_T \geq \frac{\gamma}{2}$$

$$\|\sum_{n\notin[n_k,n_{k+1}[}\frac{1}{2^{n+1}}\|y_k\|_n t_n\|_T \leq \frac{\gamma}{2^{k+4}}$$

For every k choose $\sum_{n=n_k}^{n_{k+1}-1} b_n t_n^* \in S_{T_\alpha^*}$ and $s_n \in \mathcal{F}_n$ with $n\in[n_k,n_{k+1}[$ such that

$$\sum_{n=n_k}^{n_{k+1}-1}\frac{1}{2^{n+1}}b_n\psi_n(y_n)\geq\frac{\gamma}{2} \text{ for every } k,$$

where $\psi_n := \sum_{\xi\in s_n}\varepsilon_\xi u_\xi^* \in \mathcal{B}_{(X_{\mathcal{F}_n})^*}$ and ε_ξ is the sign of $|u_\xi^*(z_k)|$ for every k, every $n\in[n_k,n_{k+1}[$ and every $\xi\in s_n$. For every k let

$$\varphi_k := \sum_{n=n_k}^{n_{k+1}-1}\frac{b_n}{2^{n+1}}\psi_n \in B_{\mathfrak{X}^*},$$

because of Remark 5.8. Notice that $\varphi_k(y_k)\geq\gamma/2$ for every k. Given $s\in\mathcal{S}_\alpha$ we have that $\{n_k\}_{k\in s}\in\mathcal{S}_\alpha$, because $\mathcal{S}_\alpha$ is a spreading family. Hence,

$$\frac{1}{2}\sum_{k\in s}\sum_{n=n_k}^{n_{k+1}-1} b_n t_n^* \in B_{T_\alpha^*}.$$

It follows from this and Remark 5.8 that

$$\frac{1}{2}\sum_{k\in s}\varphi_k = \frac{1}{2}\sum_{k\in s}\sum_{n=n_k}^{n_{k+1}-1}\frac{1}{2^{n+1}}b_n\psi_n \in B_{\mathfrak{X}^*}.$$

Now, fix $s\in\mathcal{S}_\alpha$ and scalars $(a_k)_{k\in s}$, and for each $k\in s$, let σ_k be the sign of a_k. Then,

$$\begin{aligned}\|\sum_{k\in s}a_k y_k\| \geq & |\langle\frac{1}{2}\sum_{k\in s}\sigma_k\varphi_k,\sum_{k\in s}a_k y_k\rangle| \geq \frac{\gamma}{4}\sum_{k\in s}|a_k| - \sum_{k\in s}|\langle\frac{1}{2}\sum_{j\in s\setminus\{k\}}\varphi_j, a_k y_k\rangle| \geq \\ \geq & \frac{\gamma}{4}\sum_{k\in s}|a_k| - \sum_{k\in s}|a_k|\left\|\sum_{n\notin[n_k,n_{k+1}[}\frac{1}{2^{n+1}}\|y_k\|_n t_n\right\|_{T_\alpha} \geq \\ \geq & \frac{\gamma}{4}\sum_{k\in s}|a_k| - \sum_{k\in s}|a_k|\frac{\gamma}{2^{k+4}} \geq \frac{\gamma}{8}\sum_{k\in s}|a_k|.\end{aligned}$$

Hence, $(y_k)_k$ is an ℓ_1^α-spreading model. □

(b): Suppose otherwise that $(x_k)_k$ is a weakly-null sequence such that $\|\sum_{k\in s}a_k x_k\|_{\mathfrak{X}}\geq \varepsilon\sum_{k\in s}|a_k|$ for every sequence of scalars $(a_k)_{k\in s}$ supported in $s\in\mathcal{S}_\beta$, for $\beta := \iota(\omega^\alpha)+\alpha\cdot\omega$. There are two cases to consider.

Case 1. There is some subsequence $(y_k)_k$ of $(x_k)_k$ some n and some $C>0$ such that

$$\frac{1}{2^{n+1}}\|\sum_{k\in s}a_k y_k\|_n \leq \|\sum_{k\in s}a_k y_k\|_{\mathfrak{X}} \leq C\|\sum_{k\in s}a_k y_k\|_n$$

for every $(a_k)_{k\in s}$ supported in $s\in \mathcal{S}_{\iota(\omega^\alpha)}$. Let $\theta:\mathcal{F}_n\to \mathrm{FIN}$, $\theta(s):=\{k\in\omega\,:\,\operatorname{supp} y_k\cap s\neq\emptyset\}$. This is a continuous mapping, so $\mathcal{C}:=\theta"\mathcal{F}_n$ is compact, and $\gamma:=\mathrm{rk}(\mathcal{C})\le \mathrm{rk}(\mathcal{F}_n)<\iota(\omega^\alpha)$, by the choice of $\mathcal{F}_n$. By the quantitative version of by Ptak's Lemma we can find a convex combination $(a_k)_{k\in s}$ supported in $\mathcal{S}_{\iota(\omega^\alpha)}$ such that $\sum_{k\in t}|a_k|<\varepsilon/(C\sup_k\|y_k\|_n)$ for every $t\in\mathcal{C}$. Let $v\in\mathcal{F}_n$ be such that

$$\|\sum_{k\in s}a_k y_k\|_n=\sum_{\xi\in v}|u_\xi^*(\sum_{k\in s}a_k y_k)|.$$

Then

$$\|\sum_{k\in s}a_k y_k\|_{\mathfrak{X}}\le C\|\sum_{k\in s}a_k y_k\|_n\le \sum_{k\in\theta(s)}|a_k|\|y_k\|_n<\varepsilon,$$

and this is impossible.

Case 2. There is a normalized block subsequence $(y_k)_k$ of $(x_k)_k$ with $y_k:=\sum_{i\in s_k}b_i x_i$ and $s_k\in\mathcal{S}_{\iota(\omega^\alpha)}$ and that is 2-equivalent to a block subsequence $(w_k)_k$ of the basis $(t_n)_n$ of T_α. Since for every δ,γ there is an integer n such that $\mathcal{S}_\delta\otimes\mathcal{S}_\gamma\restriction(\omega\setminus n)\subseteq\mathcal{S}_{\delta+\gamma}$, we assume without loss of generality $\bigcup_{k\in x}s_k\in\mathcal{S}_{\iota(\omega^\alpha)+\alpha\cdot\omega}$ for every $x\in\mathcal{S}_{\alpha\cdot\omega}$. Let $K:=\sup_k\|x_k\|_{\mathfrak{X}}$. By Theorem 5.7, let $(a_k)_{k\in v}$ be a sequence supported in $v\in\mathcal{S}_{\alpha\cdot\omega}$ such that $\|\sum_{k\in v}a_k w_k\|_{T_\alpha}<(\varepsilon/2K)\sum_{k\in v}|a_k|$. Hence,

$$\|\sum_{k\in v}a_k\sum_{i\in s_k}b_i x_i\|_{\mathfrak{X}}\le 2\|\sum_{k\in v}a_k w_k\|_{T_\alpha}<\frac{\varepsilon}{K}\sum_{k\in v}|a_k|\le\varepsilon\sum_{k\in v}|a_k|\sum_{i\in s_k}|b_i|,$$

and this is impossible because $\bigcup_{k\in v}s_k\in\mathcal{S}_{\iota(\omega^\alpha)+\alpha\cdot\omega}$. □

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Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, 05314-970, São Paulo, Brazil

E-mail address: brech@ime.usp.br

Instituto de Ciencias Matemáticas (ICMAT), CSIC-UAM-UC3M-UCM, C/ Nicolás Cabrera 13-15, Campus Cantoblanco, UAM, 28049 Madrid, Spain

E-mail address: abad@icmat.es

Institut de Mathématiques de Jussieu, UMR 7586, 2 place Jussieu - Case 247, 75222 Paris Cedex 05, France and Department of Mathematics, University of Toronto, Toronto, Canada, M5S 2E4

E-mail address: stevo@math.toronto.edu