A DIFFERENCE EQUATION INVOLVING FIBONACCI NUMBERS

by

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Abstract: In this short note we solve a non linear difference equation which becomes related to Fibonacci numbers.

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1- A NON LINEAR DIFFERENCE EQUATION

Consider the difference equation given by:

$$x_{n-1} = \bar{\alpha}_n x_n\, x_{n-1} + \bar{\beta}_n x_{n-2} \quad n = 1, 2, \dots \tag{1}$$

where x_n is obtained from the previous values in a non linear way in the form expressed in it. The problem to be solved is to find x_n in terms of the firsts values of it. We will propose a general form for the values of x_n called quadrature and generating function procedure. The values of $\bar{\beta}_n$ will not be arbitrary but in the solution they will depend upon other parameters. Simply the proposal is as follows

$$x_n = \alpha_n\, c(n) + \frac{\beta_n}{c(n)} \tag{2}$$

then

$$x_n\, x_{n-1} = \alpha_n \alpha_{n-1} c(n) c(n-1) + \frac{\beta_n}{c(n)} \alpha_{n-1} c(n-1) + \alpha_n \beta_{n-1} \frac{c(n)}{c(n-1)} + \frac{\beta_n \beta_{n-1}}{c(n) c(n-1)} \tag{3}$$

Introducing this expression in (1) and x_{n-2}, then after identifying terms we obtain the following recursive of difference equations

$$\alpha_{n+1} c(n+1) = \bar{\alpha}_n\, \alpha_n \alpha_{n-1} c(n) c(n-1) \tag{4}$$

$$\frac{\beta_{n+1}}{c(n+1)} = \bar{\alpha}_n \frac{\beta_n \beta_{n-1}}{c(n) c(n-1)} \tag{5}$$

On the other hand,

$$\alpha_n \beta_{n-1} \frac{c(n)}{c(n-1)} = \beta_{n-1} \bar{\alpha}_{n-1}\, \alpha_{n-1} \alpha_{n-2} c(n-2) \tag{6}$$

then

$$\bar{\alpha}_n \beta_{n-1} \bar{\alpha}_{n-1} \alpha_{n-1} + \overline{\beta}_n = 0 \tag{7}$$

In a similar way

$$\beta_{n-1} \alpha_n \frac{1}{c(n-2)} \frac{\beta_n}{\beta_{n-1}} \frac{\alpha_{n-1}}{\bar{\alpha}_{n-1}} \alpha_{n-1} \alpha_{n-2} = \beta_n \alpha_{n-1} \frac{c(n-1)}{c(n)} \tag{8}$$

from which it follows

$$\bar{\alpha}_n \alpha_n \beta_n + \overline{\beta}_n \beta_{n-1} \alpha_{n-1} \alpha_{n-2} = 0 \tag{9}$$

or

$$\bar{\beta}_n = -\frac{\bar{\alpha}_n \, \alpha_n \, \beta_n}{\beta_{n-1}\alpha_{n-1}\alpha_{n-2}} \tag{10}$$

which is a condition to be fulfilled by the coefficient $\bar{\beta}_n$as we already mentioned.

From the equality

$$\bar{\beta}_n = \bar{V}\bar{\alpha}_n \, \beta_{n-1}\bar{\alpha}_{n-1}\alpha_{n-1} = -\frac{\bar{\alpha}_n \, \alpha_n \, \beta_n}{\beta_{n-2}\bar{\alpha}_{n-1} \, \alpha_{n-2}} \tag{11}$$

one derives

$$\alpha_n \, \beta_n = \bar{\alpha}^2{}_{n-1} \, \beta_{n-1} \, \alpha_{n-1} \, \beta_{n-2} \, \alpha_{n-2} \tag{12}$$

This last difference equation is the same as that equation obtained by multiplying **(4)** and **(5).**

Now the equation **(12)** has the general form of

$$\Upsilon_n = \varepsilon_{n-1}\Upsilon_{n-1}\Upsilon_{n-2} \tag{13}$$

Developing the first terms of **(13)** we have

$$\begin{aligned} \Upsilon_n &= \varepsilon_{n-1}\varepsilon_{n-2} \, \Upsilon^2{}_{n-2} \, \Upsilon_{n-3} = \varepsilon_{n-1}\varepsilon_{n-2}\varepsilon^2{}_{n-3} \, \Upsilon^3{}_{n-3} \, \Upsilon_{n-4} \\ &= \varepsilon_{n-1}\varepsilon_{n-2}\varepsilon^2{}_{n-3}\varepsilon^3{}_{n-4} \, \Upsilon^5{}_{n-4}\Upsilon^3{}_{n-5} = \varepsilon_{n-1}\varepsilon_{n-2}\varepsilon^2_{n-3}\varepsilon^3_{n-4}\varepsilon^5_{n-5}\Upsilon^8_{n-5}\Upsilon^5_{n-6} \\ &= \varepsilon_{n-1}\varepsilon_{n-2}\varepsilon^2_{n-3}\varepsilon^3_{n-4}\varepsilon^5_{n-5}\varepsilon^8_{n-6}\Upsilon^{13}_{n-6}\Upsilon^8_{n-7} \end{aligned} \tag{14}$$

Where yon immediately see that Fibonacci numbers appear in the developing of Υ_n.

We define

0	1	1	2	3	5	8	13	21	**(15)**
p(0)	p(1)	p(2)	p(3)	p(4)	p(5)	p(6)	p(7)	p(8)	

as the Fibonacci numbers which are recursively given by

$$p(n+1) = p(n) + p(n-1)$$

with p(0) = 0 and p(1) = 1.

In order to find the solution of **(13)** let us consider some equalities. First we well prove that

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \prod_{s=1}^{n-2} \varepsilon_{n-1-s}^{p(s)} = \prod_{s=2}^{n} \varepsilon_{n-1-s}^{p(s)} \quad \textbf{(16)}$$

In order to see that this equality hold true consider the equality

$$\prod_{s=1}^{n-2} \varepsilon_{n-1-s}^{p(s)} = \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s-1)} \quad \textbf{(17)}$$

which is obtained just by a change of variables $s + 1 = \bar{s}$. Using this last equality then since p(0) = 0 and p(1) + p(0) = p(2) then

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s-1)} = \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s+1)} \, \varepsilon_{n-1}^{p(1)} = \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s+1)} \, \varepsilon_{n-1}^{p(2)} = \prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s+1)} \quad \textbf{(18)}$$

Now with the change of variable $s - 1 = \bar{s}$ we have

$$\prod_{s=2}^{n} \varepsilon_{n+1-s}^{p(s)} = \prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s+1)} \quad \textbf{(19)}$$

and therefore the equality **(16)** is valid.

Now we will claim that

$$\Upsilon_n = \prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \, \Upsilon_1^{p(n)} \Upsilon_0^{p(n-1)} \quad \textbf{(20)}$$

is the solution of the difference equation **(13).**

We will prove it by induction.

For n=2 we have

$$\Upsilon_2 = \prod_{s=1}^{1} \varepsilon_{2-s}^{p(s)} \, \Upsilon_1^{p(2)} \Upsilon_0^{p(1)} = \varepsilon_1 \Upsilon_1 \Upsilon_0 \quad \textbf{(21)}$$

Now assume that the formula is true for $j \leq n - 1$ then we have to prove that it is valid for n. This is equivalent to prove that the next equality is true

$$\Upsilon_n = \prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \, \Upsilon_1^{p(n)} \Upsilon_0^{p(n-1)} \quad \textbf{(22)}$$

We remember that

$$\gamma_n = \varepsilon_{n-1}\,\gamma_{n-1}\,\gamma_{n-2} \tag{13}$$

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)}\ \gamma_1^{p(n)}\gamma_0^{p(n-1)} = \varepsilon_{n-1}\prod_{s=1}^{n-2}\varepsilon_{n-1-s}^{p(s)}\ \gamma_1^{p(n-1)}\gamma_0^{p(n-2)}\prod_{s=1}^{n-3}\varepsilon_{n-2-s}^{p(s)}\ \gamma_1^{p(n-2)}\gamma_0^{p(n-3)} \tag{23}$$

or equivalently

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} = \varepsilon_{n-1}\prod_{s=1}^{n-2}\varepsilon_{n-1-s}^{p(s)}\prod_{s=1}^{n-3}\varepsilon_{n-2-s}^{p(s)} \tag{24}$$

Make the change of variable $s+2=\bar{s}$ in the first product of the right hand of **(24)** and $s+1=\bar{s}$ in the second product of the same hand, we have

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} = \varepsilon_{n-1}\prod_{s=2}^{n-1}\varepsilon_{n-s}^{p(s)}\prod_{s=3}^{n-1}\varepsilon_{n-s}^{p(s-2)} = \varepsilon_{n-1}\,\varepsilon_{n-2}^{p(1)}\prod_{s=3}^{n-1}\varepsilon_{n-s}^{p(s-2)+p(s-1)} = \varepsilon_{n-1}\,\varepsilon_{n-2}^{p(1)}\prod_{s=3}^{n-1}\varepsilon_{n-s}^{p(s)} \tag{25}$$

and since $p(1) \ = \ p(2)$

$$= \varepsilon_{n-1}^{p(1)}\varepsilon_{n-2}^{p(2)}\prod_{s=3}^{n-1}\varepsilon_{n-s}^{p(s)}$$

and this the equality given by **(24)** or **(25)** is true.

Then applying the formula **(22)** to the recurrence equation **(12)**one gets the solution

$$\alpha_n\beta_n = \prod_{s=1}^{n-1}\alpha_{n-s}^{-2p(s)}\ \alpha_1^{p(n)}\beta_1^{p(n)}\alpha_0^{p(n-1)}\,\beta_0^{p(n-1)} \tag{26}$$

On the other hand the equation **(4)** has the solution

$$\alpha_n\ c(n) = \prod_{s=1}^{n-1}\alpha_{n-s}^{-p(s)}\ \alpha_1^{p(n)}c_1^{p(n)}\alpha_0^{p(n-1)}c_0^{p(n-1)} \tag{27}$$

and equation **(5)**

$$\frac{\beta_n}{c(n)} = \prod_{s=1}^{n-1}\alpha_{n-s}^{-p(s)}\,\frac{\beta_1^{p(n)}\,\beta_0^{p(n-1)}}{c_1^{p(n)}\,c_0^{p(n-1)}} \tag{28}$$

Thus the general solution **(2)** is given by

$$x_n = \prod_{s=1}^{n-1} \alpha_{n-s}^{-p(s)} \left[\alpha_1^{p(n)} c_1^{p(n)} \alpha_0^{p(n-1)} c_0^{p(n-1)} + \frac{\beta_1^{p(n)} \beta_0^{p(n-1)}}{c_1^{p(n)} c_0^{p(n-1)}} \right] \quad \textbf{(29)}$$

The equality **(11)** which is a restriction on the coefficients $\overline{\beta_n}$ can be expressed as

$$\bar{\beta}_n = -\bar{\alpha}_n \bar{\alpha}_{n-1} \prod_{s=1}^{n-2} \alpha_{n-s}^{-2p(s)}\, \alpha_1^{p(n-1)} \beta_1^{p(n-1)} \alpha_0^{p(n-2)} \beta_0^{p(n-2)} \quad \textbf{(30)}$$

This giving the values $\alpha_1, \alpha_0, \beta_0, \beta_1$ and c_1, c_0 we have solved in a suitable way the problem of finding a general solution for equation **(1)** by quadrature.

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